# Analytic Foundations of the Quantum Core in a Timeless Euclidean Model: Reduction of Full Cauchy Data and the Free Two-Point Structure

A. N. Smirnov
andreysxxxx@gmail.com

November 1, 2026

**Abstract**

We study the analytic foundations of the free quantum core in a timeless Euclidean model with a single actual configuration of a real harmonic field in $\mathbb{E}^4$ and an internal observer.

For the full elliptic problem $u'' = A^2 u$ with nonnegative self-adjoint $A$, we obtain an exact decomposition on a finite slab into the Poisson extension of the first trace and a boundary remainder. On the sector $A \geq \kappa I > 0$, control of the boundary mismatch yields exponential bounds on the remainder and the deviation from the decaying-branch graph $u' = -Au$. We derive a sufficient criterion for indistinguishability of the solution and its Poisson approximation under the linear readouts of a finite protocol with prescribed tolerances. Subexponential growth on the half-line yields the exact transfer $P_h^E = \mathrm{e}^{-hA}$ on the positive spectral subspace.

The canonical two-point form is constructed as the Green's function of the fundamental Laplacian. A regular equal-slice limit, a positive, spatially homogeneous and isotropic equal-slice form, compatible transfer of one argument, and $O(4)$ invariance determine it uniquely up to normalization and zero components: $\widetilde{C}_E(p) = c/|p|^2$.

Reflection positivity is proved for this kernel. The associated Gaussian measure admits standard Osterwalder–Schrader reconstruction with a free Fock space, a positive generator, and unitary dynamics. The Markov property of the full Gaussian slice field is derived from the covariance. The construction describes a free massless scalar sector without introducing an additional fundamental field. The realization of a complete measurement protocol lies outside the scope of this work.

# Contents

**1 Introduction** **3**

**2 Underlying structure and operational domain of analysis** **6**

**3 The full elliptic Cauchy problem** **13**

**4 Operational stability and quantitative reduction** **18**

**5 Operational meaning and applicability of the reduction** **33**

**6 The canonical free two-point structure** **39**

**7 Reduction to standard free quantum reconstruction** **43**

**8 Operational measurements and affine statistics** **46**

**9 Structure of the proofs and conditions of applicability** **47**

**10 Discussion and conclusions** **50**

**A A detail of the local-instability proof** **51**

**B The zero mode and a finite spatial domain** **51**

**C Changing the observer-dependent representation** **51**

**D Map of standard results and hypotheses to be verified** **52**

# 1 Introduction

## 1.1 The underlying model and the preceding quantum reconstruction

The timeless Euclidean model explores the possibility of describing observable physics without fundamental time or a predefined causal structure. Its basic objects are Euclidean space $(\mathbb{E}^4, \delta_{AB})$ and a single actual configuration of a real scalar field

$$\Phi_0 : \mathbb{E}^4 \longrightarrow \mathbb{R}, \qquad \Delta_{\mathbb{E}^4}\Phi_0 = 0. \tag{1}$$

The observer is a localized structure of the same field. Events, causal links, and effective dynamics belong to the reconstruction based on the data and registration procedures available to the observer; they do not constitute an additional fundamental layer. The immutability of $\Phi_0$ therefore does not imply the absence of observable evolution: the latter characterizes the reconstructed description rather than a change in the fundamental configuration.

Global harmonicity of $\Phi_0$ does not imply global boundedness. Liouville's theorem for harmonic functions states that a function harmonic everywhere and bounded on all of $\mathbb{E}^4$ is constant; global boundedness is not imposed in the present model. The theorem therefore does not exclude nontrivial global harmonic configurations. The observer's operational readouts are localized in bounded working regions, whereas reconstruction stability and accuracy requirements are formulated for specific working ranges.

Reference [1] investigated operational conditions for the emergence of special relativity: observable transformations take the Lorentz form with a finite invariant limiting speed. They constitute consistent re-descriptions of accessible registrations and are not identified with direct Euclidean transformations, which may change the reconstructed event structure. Reference [2] extends this framework to a slowly varying foliation; under the structural conditions specified there, the effective geometry is described by gravitational equations. The present study uses the locally inertial regime of this reconstruction rather than deriving gravitational dynamics anew.

The immediate predecessor of this paper is the study of the quantum core of the effective theory in [3]. Under explicitly specified working conditions, that study constructed a quantum description comprising a Hilbert space, a compatible complex structure, unitary dynamics, a description of measurements, and relativistic closure of free sectors. However, applying standard quantum constructions and establishing the origin of their applicability conditions are distinct tasks. Section 1.4 of that paper explicitly records this distinction: mathematical justification of several working conditions was left open. The present paper addresses the analytic part of this problem.

Other parts of the research program concern reconstruction classes and the status of their parameters [4], the visible Standard Model sector and dark matter [5], and the cosmological constant and dark energy in relation to the limit of stable remote reconstruction [6]. These studies provide a broader context, but their specific conclusions are not used as premises of the analytic theorems proved below.

## 1.2 The problem of mathematically grounding the quantum core

The working conditions of the original quantum reconstruction link three distinct tasks. The first concerns the transfer of effective data: one must determine when a description on a single slice suffices for extension and which operator implements that transfer. The second concerns the free statistical structure: one must determine the two-point form and verify the conditions for constructing a quantum representation. The third concerns measurement: the resulting state space must be related to registrations and stable records of the internal observer.

Sections 1.4 and 3.3–3.4 of [3] include among the working conditions local effective closure, stationarity, a positive sensitivity form, a chosen two-point covariance, and the quasifree character of the state. A compatible positive complex structure is considered in Section 3.8, and the conditions for a measurement regime and representability of its statistics in Sections 3.9–3.10. These conditions play different logical roles and require separate justifications.

Here we formulate sufficient conditions for reducing full Cauchy data, construct a free reference two-point form, and establish its analytic properties. The results concern explicitly specified classes of data and reference forms.

The study focuses on the first two tasks. Completeness of a single boundary trace is established after selecting a sector of the full problem, rather than assumed at the outset. Construction and characterization of the reference form are distinguished from the symmetry of the fundamental equation: symmetry of a law and invariance of a particular statistical state are different properties.

**Relation to classical constructions and the contribution of this work.** Spectral methods for elliptic Cauchy problems and conditional stability estimates under a priori constraints have been developed in the theory of ill-posed problems; for a positive self-adjoint spatial operator, such a setting is considered, in particular, in [7]. In a flat half-space, the Poisson extension and the relation between the boundary trace and the normal derivative are associated with the Dirichlet-to-Neumann operator $\sqrt{-\Delta_\Sigma}$; this classical connection and its generalizations to fractional powers of operators are presented in [8, 9]. The present work studies reduction of the initially full data pair $(f, \pi)$: the boundary mismatch on a finite slab determines the error of the Poisson description and the deviation from the decaying-branch graph. These estimates are related to the sensitivity of a finite set of readouts and to prescribed tolerances. In contrast to recovery from noisy Cauchy data, the task here is to establish a criterion under which a single trace suffices for an approximate description of an already existing solution. Exact reduction is considered separately, under a growth condition on the half-line.

The covariance defined by the Green's function of the Laplacian and the construction of the associated Gaussian measure belong to the standard free-field construction [21]; the passage from a Euclidean hierarchy to a Lorentzian quantum theory is provided by Osterwalder–Schrader (OS) reconstruction [10, 11]. The second part of this work characterizes this free reference structure through compatibility with the reduced transfer, equal-slice regularity, and symmetry. The constructive definition of the reference form is accompanied by verification of the conditions for quantum reconstruction. The specific contribution is to establish sufficient conditions for passing from the full harmonic problem to a reduced free sector and to control quantitatively its application to finite readouts. Throughout, the analytic data, the reference correlation structure, and observable quantum dynamics retain distinct mathematical and operational roles.

## 1.3 Scope and main results

**Reduction of full Cauchy data.** In a fixed flat foliation, the spectral formulation of the elliptic problem is

$$u'' = Bu, \qquad (u(0), u'(0)) = (f, \pi), \qquad B \geq 0, \tag{2}$$

where $u(s)$ is the trace of a harmonic variation, the prime denotes differentiation with respect to the geometric parameter $s$, and $B$ is a nonnegative self-adjoint spatial operator. Both components $(f, \pi)$ are retained initially; no dependence of the normal derivative on the first trace is prescribed in advance.

Theorem 1 expresses the deviation from the Poisson extension in terms of the second boundary trace on a finite slab. On the sector $A \geq \kappa I$, where $A = \sqrt{B}$ and $\kappa > 0$, we obtain

an estimate showing suppression of this contribution with distance from the second boundary, provided the boundary mismatch is controlled. Thus, a quantitative approximate reduction is established first. The exact graph $\pi = -Af$ associated with the decaying spectral branch follows from a separate subexponential growth condition on the half-line (Corollary 4.3), with zero modes treated separately. After this selection, inter-slice transfer for a positive increment $h$ takes the form

$$P_h^E = \mathrm{e}^{-hA}, \qquad A = \sqrt{B}. \tag{3}$$

Its semigroup properties and strong continuity then follow from the spectral construction. Continuity on finite intervals does not suffice for exact branch selection.

**The free two-point structure.** In the flat scalar sector we consider $B = -\Delta_\Sigma$ and $A = |D|$, where $|D|$ is the operator with Fourier multiplier $|k|$, $k \in \mathbb{R}^3$. The Green's function of the fundamental operator $-\Delta_{\mathbb{E}^4}$, associated with the Dirichlet form, is chosen as the canonical mathematical representative of the free two-point structure. Theorem 2 then characterizes it within the class of forms having a regular equal-slice limit, a positive, spatially homogeneous and isotropic equal-slice form, compatible transfer of one argument, and four-dimensional Euclidean invariance. Under these conditions, apart from the separately discussed zero component, one obtains

$$C(\tau_E) = \frac{c}{2A}\mathrm{e}^{-|\tau_E|A}, \qquad \tau_E = s - s', \tag{4}$$

where $c > 0$ specifies the overall normalization. In the full Fourier representation, the corresponding multiplier is $c/|p|^2$, where $p \in \mathbb{R}^4$ is the Euclidean Fourier variable. We use the following convention: $\widehat{C}_E(\tau_E; k)$ denotes the spatial Fourier multiplier of the two-point form, and $\widetilde{C}_E(p_0, k)$ its full four-dimensional Fourier multiplier, where $p = (p_0, k)$ and $p_0$ is conjugate to $s$. The subscript $E$ indicates the Euclidean representation. The inverse operator in (4) is understood on the specified test spaces; infrared and zero modes are treated separately.

Lemma 5.1 characterizes the same spectral weight when the Lorentz-type observable transformations (operational re-descriptions) are represented by a consistent action on the effective scalar quantum algebra and one fixed positive-frequency reference state is invariant under this action. Invariance of the spectral measure on the positive massless shell determines the equal-slice spectral weight $q(r) = c/(2r)$, where $r = |k| > 0$, and hence $\widetilde{C}_E(p) = c/|p|^2$. The conditions on this action and its distinction from direct $O(4)$ rotations of the fundamental field are explained in Section 5.4.

The canonical representative satisfies all the hypotheses of the uniqueness theorem. Relating this reference structure to the statistics of a specific protocol is a condition for applying the results (Section 9.2).

### 1.4 Passage to the quantum representation

Reflection positivity is verified directly for the constructed kernel. The associated Gaussian measure yields a free hierarchy of correlation functions to which the standard Osterwalder–Schrader results [10, 11, 12, 13] apply. This exact reference construction yields a one-particle space, a free Fock space, a positive generator, and a unitary group. The Markov property of the full Gaussian slice field is established after the covariance has been constructed and is not used to select it. Applying OS reconstruction requires verification of the conditions on the entire Euclidean hierarchy, beyond positivity of the functional that suffices for the general Gelfand–Naimark–Segal (GNS) construction.

The direct Euclidean transfer (3) acts on analytic field traces and is not observable time evolution. Its contractivity does not contradict unitarity of the quantum group $U(t) = \exp(-\mathrm{i}tH_{\mathrm{phys}}/\hbar)$, where $H_{\mathrm{phys}}$ is the generator in energy units and $\hbar = \hbar_{\mathfrak{C}} > 0$ is a coefficient with the dimension of action relating the time-translation generator to the energy

normalization of the chosen reconstruction class $\mathfrak{C}$. The parameter of this group must be matched to the observable time of the relativistic sector. Agreement of time normalizations is necessary, but does not by itself identify the observables and their statistics; a detailed distinction is given in Section 5.4.

The reference Gaussian measure represents the free correlation structure, not an ensemble of fundamental configurations. Its mathematical status and its relation to registrations are discussed in Section 7.2.

### 1.5 Domain of applicability

Two regimes of reduction of the full elliptic problem are established: a quantitative approximation on a finite slab with controlled boundary mismatch, and exact selection on the positive spectral subspace under a growth condition. An exact massless scalar quantum representation is constructed for the canonical free two-point form. Each result has its own domain of applicability; using the finite-slab estimates requires accounting for admissible field data, the geometry of the working region, and readout sensitivity.

Section 8 establishes the representation of positive affine probabilities on all normal quantum states by a positive operator-valued measure (POVM). Applying this result to an apparatus presupposes stable registers and a consistent identification of preparations with quantum states. Construction of a measurement protocol is outside the scope of this work; the analytic proofs are independent of such a construction.

Joint physical application of the free quantum representation and a measurement protocol requires their realization by the same channels of a single harmonic configuration, rather than merely the separate existence of the two constructions. Such a realization, as well as interacting fields and the origins of masses, spin, and internal gauge symmetries, is not investigated here.

### 1.6 Organization of the paper

Section 2 specifies the fundamental and operational framework, and Section 3 introduces the full Cauchy data and spectral branches. Section 4 proves quantitative reduction on a slab and exact selection under a growth condition; Section 5 establishes the limits of their operational application. Sections 6 and 7 are devoted to the two-point form and standard free quantum reconstruction. Sections 8 and 9 present affine measurement statistics, the structure of the proofs, and conditions of applicability. Section 10 concludes the paper; the appendices clarify local instability, zero modes, the observer-dependent representation, and the hypotheses of standard theorems.

## 2 Underlying structure and operational domain of analysis

### 2.1 Ontology, foliation, and mathematical variations

Fix a unit normal

$$n \in S^3$$

and the family of parallel hyperplanes

$$\Sigma_s = \{x \in \mathbb{E}^4 : n_A x^A = s\}. \tag{5}$$

The parameter $s$ is fundamentally geometric and orders the slices of the chosen foliation. Its use does not introduce fundamental time. Tangential coordinates on different $\Sigma_s$ are canonically identified by Euclidean translations along $n$.

With a consistent choice of origins, observable time $t$ is related to the length coordinate $x^0$ by the calibration

$$x^0 = v_t t, \tag{6}$$

where $v_t > 0$ is the coefficient calibrating reconstructed observable time against the length coordinate. The previously constructed relativistic regime establishes $v_t = v_{\max}$ [1], where $v_{\max}$ is the invariant limiting speed. Below, $v_t$ is used in time-calibration formulas and in the normalization of time-translation generators, whereas $v_{\max}$ is used in statements concerning the limiting speed and causality.

In the inherited relativistic sector, the difference between slice parameters also specifies the length coordinate along the reconstructed time direction,

$$\Delta x^0_{\mathrm{SR}} = \Delta s = v_t \Delta t, \tag{7}$$

where $t$ is the observable real time of the corresponding inertial frame of reference (IFR). Relation (7) belongs to the reconstructed observable description and does not identify direct Euclidean transfer along $s$ with physical time evolution.

Let

$$u_0(s) := \Phi_0|_{\Sigma_s}, \qquad f_0 := u_0(0),$$

where $\Phi_0$ is a single actual configuration of the fundamental field satisfying

$$\Delta_{\mathbb{E}^4} \Phi_0 = 0.$$

The working variation spaces are used affinely:

$$f = f_0 + \delta f, \qquad \delta f \in (X_{\mathfrak{C}}, \tau_{\mathfrak{C}}). \tag{8}$$

Here $X_{\mathfrak{C}}$ is the real linear space of field variations of the chosen reconstruction class $\mathfrak{C}$, and $\tau_{\mathfrak{C}}$ is its topology. Variations $\delta f$ describe the local analytic structure relative to a single actual configuration.

Linearity of $X_{\mathfrak{C}}$ likewise does not imply that every element corresponds to a global configuration containing the same observer, registers, and protocol. In particular, a local harmonic variation may be admissible analytically without extending to a global configuration in the same reconstruction class that preserves this observer.

Linearity of the fundamental equation ensures linearity of its space of harmonic variations. However, the set of finite configurations preserving a particular observer, its threshold mechanisms, and the implemented protocol may in general be nonlinear. The linear statements below therefore concern the local pre-threshold sector, before threshold registration, and do not require the entire set of physically admissible configurations to be regarded in advance as a smooth linear manifold.

At this stage, a single trace

$$\Phi|_{\Sigma_s}$$

is not assumed to determine extension to other slices. Single-slice closure, a graph relation expressing the normal derivative in terms of the trace, and selection of a spectral branch are not initial conditions. Sufficient conditions for quantitative reduction on a finite slab and for exact selection are established in Section 4; properties of the resulting transfer are derived after the sector has been selected.

## 2.2 Reconstruction class and internal observer

Fix a triple

$$(\mathfrak{C}, O, \Pi),$$

where $\mathfrak{C}$ is a reconstruction class, $O$ is an internal localized observer, and $\Pi$ is a specific physically realizable protocol.

The topological space

$$(X_{\mathfrak{C}}, \tau_{\mathfrak{C}})$$

characterizes the local sector of the reconstruction class under consideration. A specific protocol selects a finite set of accessible readouts and the corresponding operational accuracy within this space, but does not redefine the working space itself or its topology.

The observer's local structure, registers, and memory are realized by the same actual configuration $\Phi_0$.

However, the internal character of the observer does not imply that the restriction

$$\Phi|_{\Sigma_s}$$

without normal-derivative data is a complete state of the harmonic problem. Single-slice determinacy is treated as a result of reduction under the conditions of Section 4.

Here, single-slice determinacy means that the trace fixes the continuation within the chosen reduced sector; single-slice closure refers to a transfer law acting on those traces without independent normal-derivative data. Completeness concerns whether the data specify a state of the analytic problem under consideration. These exact properties are distinct from the sufficiency of one trace for an operational approximation with finite accuracy.

The primary operational fact is a local registration act leaving a record in the observer's internal registers; such an act corresponds to an observable local vertex of the event network. Events outside the region of direct registration enter this network as reconstructed vertices, related to local registrations by admissible rules of causal reconstruction [1]. The full event network is therefore in general observer-dependent and is not assumed a priori to be shared by different observers. Consistency is required for information that becomes shared through their physical interaction, not for the identity of all reconstructed event networks.

For the subsequent analysis, the conditions inherited from the preceding reconstruction of special relativity must be distinguished explicitly from the properties established here. Among the admissibility conditions for configurations used in [1], we retain the existence of a local modal representation suitable for operational readout and the operational interpretability of interactions between field modes and the observer's localized internal modes. In the adopted local modal description, $u_\alpha$ denotes field-mode profiles, $a_\alpha(s)$ their coefficients on the slice $\Sigma_s$, $\chi_\beta$ the observer's internal mode profiles, and $b_\beta(s)$ the corresponding readout coefficients. The coefficients of local linear readout are denoted by $C^{(O)}_{\beta\alpha}$, and the observer's localization region on that slice by $\Omega_O(s) \subset \Sigma_s$. We use the representation

$$\chi_\beta = \sum_\alpha C^{(O)}_{\beta\alpha} u_\alpha, \qquad b_\beta(s) = \sum_\alpha C^{(O)}_{\beta\alpha} a_\alpha(s), \qquad \operatorname{supp} \chi_\beta \subseteq \Omega_O(s). \tag{9}$$

This local modal readout structure is used below as a prequantum characteristic of the internal observer.

Linearity of this pre-threshold layer does not imply linearity of the observer's full response. Registration thresholds, event formation, and memory updates may in general be nonlinear.

Inter-slice transfer, its stationarity, single-slice closure, and spectral-branch selection are not taken as ready-made premises from the preceding reconstruction of special relativity. Sufficient analytic conditions for reduction are established for the full second-order harmonic problem; criteria for their operational application are given in Section 4.5 and Section 5.

Application of the analytic results to a given observer presupposes the existence of configurations realizing its local channels and protocol. A joint global construction of the observer, registers, memory, and full causal-reconstruction protocol lies outside the scope of this work; the analytic theorems below do not establish its existence.

### 2.3 Pre-threshold readouts and a finite protocol

Pre-threshold readouts are quantities arising from the observer's interaction with the field before threshold registration and recording. In the modal sector considered here, they are specified by local linear combinations of field coefficients (9). Linearity concerns these channels; the threshold response and register update may be nonlinear.

After canonically identifying the parallel slices geometrically with the reference slice $\Sigma_0$, a fixed protocol $\Pi$ uses a finite family of continuous real-valued linear channels on each slice $\Sigma_s$; $X'_{\mathfrak{C}}$ denotes the continuous real linear dual:

$$\mathscr{L}_{\Pi,s} = \{\ell_{1,s}, \dots, \ell_{N_\Pi,s}\}, \qquad N_\Pi < \infty, \qquad \ell_{j,s} \in X'_{\mathfrak{C}}. \tag{10}$$

Finiteness of this family characterizes only the specific physically realizable registration procedure. It does not imply finite dimensionality of the fundamental field, a finite number of field modes, finiteness of the set of all possible protocols available to the observer, or an ultraviolet cutoff.

The main Sobolev realization is

$$X_{\mathfrak{C}} = H^r(\mathbb{R}^3) \tag{11}$$

Here $r \in \mathbb{R}$ denotes the Sobolev regularity order, and $H^r(\mathbb{R}^3)$ for field traces means $H^r(\mathbb{R}^3;\mathbb{R})$. The complex Fourier transform of a real trace satisfies $\widehat{f}(-k) = \overline{\widehat{f}(k)}$. Complex modes and spectral calculations use the complexification of this space and complex-linear extensions of the real channels; physical traces and readout outputs remain real. When a modal coefficient is written in complex form, its real and imaginary parts count as separate real outputs if both are accessible to the protocol. The number $N_\Pi$ refers specifically to the chosen real channels. This technical complexification is not identified with the subsequent quantum reconstruction of a complex state space.

We use the standard duality

$$\big(H^r(\mathbb{R}^3)\big)' \simeq H^{-r}(\mathbb{R}^3) \hookrightarrow \mathscr{D}'(\mathbb{R}^3). \tag{12}$$

Here $\mathscr{D}(\mathbb{R}^3) = C_c^\infty(\mathbb{R}^3)$ is the space of smooth compactly supported test functions, and $\mathscr{D}'(\mathbb{R}^3)$ the space of distributions. Every continuous linear channel can therefore be viewed as a distribution. Since $C_c^\infty(\mathbb{R}^3)$ is dense in $H^r(\mathbb{R}^3)$, the action of a continuous functional on test functions uniquely determines it on all of $H^r(\mathbb{R}^3)$. Its support

$$\operatorname{supp} \ell_{j,s}$$

is thus well defined in the standard distributional sense.

In particular,

$$x \notin \operatorname{supp} \ell_{j,s}$$

means that there exists an open neighborhood $U \ni x$ such that

$$\ell_{j,s}(F) = 0 \qquad \text{for all } F \in C_c^\infty(U). \tag{13}$$

Relation (12) and the distributional definition of support are standard facts of functional analysis, not physical premises of the model.

What is specific to this model is the locality of the internal observer's direct access. In the local readout structure inherited from the reconstruction of special relativity, the observer's direct channels are localized within the region occupied by its structure. Thus,

$$\boxed{\operatorname{supp} \ell_{j,s} \subseteq \overline{\Omega_O(s)}.} \tag{14}$$

Here $\Omega_O(s) \subset \Sigma_s$ denotes the operational localization region of the observer's structure on the corresponding slice; when data are represented in the common space $X_{\mathfrak{C}}$, this region is understood after the same canonical geometric identification with $\Sigma_0$.

Condition (14) is not an additional quantum assumption about a channel's spatial profile. It expresses locality of the inherited prequantum readout structure. In particular, no representation of a physical channel by a smooth function

$$\chi \in C_c^\infty(\Omega_O).$$

is assumed. In general,

$$\ell_{j,s} \in H^{-r}$$

may have substantially lower regularity and unbounded spectral support.

Assign a finite operational accuracy to each channel of the fixed protocol:

$$\varepsilon_{j,s} > 0.$$

For two data sets, define

$$d_{\Pi,s}(f,g) := \max_{1\le j\le N_\Pi} \frac{|\ell_{j,s}(f-g)|}{\varepsilon_{j,s}}. \tag{15}$$

The condition

$$d_{\Pi,s}(f,g) \le 1 \tag{16}$$

means that the difference between $f$ and $g$ does not exceed the resolution of any channel of this protocol on the slice $\Sigma_s$.

Relation (16) is used only as a quantitative criterion of operational indistinguishability at finite precision. In general, it is not an equivalence relation because transitivity may fail. Finite precision alone therefore does not define a linear quotient space and cannot replace exact equality in operator-theoretic arguments.

For exact indistinguishability by all channels of a fixed protocol, introduce

$$N_{\Pi,s} := \bigcap_{j=1}^{N_\Pi} \ker \ell_{j,s}. \tag{17}$$

Since the channels $\ell_{j,s}$ are continuous, each $\ker \ell_{j,s}$ is closed; hence $N_{\Pi,s}$ is a closed linear subspace of $X_{\mathfrak{C}}$. The condition

$$f - g \in N_{\Pi,s}$$

means that $f$ and $g$ are exactly indistinguishable by all linear channels of this protocol on the slice $\Sigma_s$. The corresponding quotient space is

$$X_{\mathfrak{C}}/N_{\Pi,s}.$$

For a fixed protocol, this quotient space is in general finite-dimensional. The readout map corresponding to the channel family $\mathscr{L}_{\Pi,s}$,

$$\mathcal{L}_{\Pi,s} : X_{\mathfrak{C}} \longrightarrow \mathbb{R}^{N_\Pi}, \qquad f \longmapsto (\ell_{1,s}(f), \dots, \ell_{N_\Pi,s}(f)) \tag{18}$$

has kernel $N_{\Pi,s}$, so that

$$X_{\mathfrak{C}}/N_{\Pi,s} \simeq \operatorname{Ran} \mathcal{L}_{\Pi,s} \subseteq \mathbb{R}^{N_\Pi}. \tag{19}$$

Consequently,

$$\dim(X_{\mathfrak{C}}/N_{\Pi,s}) \le N_\Pi. \tag{20}$$

This quotient space characterizes only the data exactly distinguishable by the specific protocol. It is not identified with the full field-data space, the collection of all possible readouts of the observer, or the state space of the subsequent quantum reconstruction.

The index $s$ in $N_{\Pi,s}$ is essential: the functional realization of the protocol and its null directions may depend on the slice. Thus, for $s_1 \neq s_2$, one may in general have

$$N_{\Pi,s_1} \neq N_{\Pi,s_2}.$$

The inter-slice transfer obtained in Section 4 on the exactly selected sector acts directly on field data. To represent it on classes exactly indistinguishable by this protocol, the operator

$$T_{s_2,s_1} : X_{\mathfrak{C}} \to X_{\mathfrak{C}}$$

must satisfy

$$T_{s_2,s_1} N_{\Pi,s_1} \subseteq N_{\Pi,s_2}. \tag{21}$$

Only when (21) holds is the induced map

$$[T_{s_2,s_1}] : X_{\mathfrak{C}}/N_{\Pi,s_1} \longrightarrow X_{\mathfrak{C}}/N_{\Pi,s_2},$$

well defined, since the transferred data class is then independent of the choice of representative. The criterion for the resulting semigroup and relevant examples are discussed in Section 5.3.

Accordingly, the following distinctions must be maintained:

(i) the standard functional-analytic realization of continuous channels as elements of $H^{-r} \subset \mathscr{D}'$;
(ii) physical locality of pre-threshold readout;
(iii) finite operational accuracy of a specific protocol;
(iv) exact indistinguishability by its linear channels, defined by $N_{\Pi,s}$;
(v) the finite-dimensional protocol representation $X_{\mathfrak{C}}/N_{\Pi,s}$, which is not the full field-data space;
(vi) possible compatibility of this representation with inter-slice transfer, to be verified only after that transfer has been obtained.

### 2.4 Spatial and operational scales

Let

$$\Omega_O := \Omega_O(0) \subset \Sigma_0$$

denote a chosen bounded region of operational localization of the observer on the reference slice. It contains the local structure realizing the observer, its registers, and the degrees of freedom directly involved in registration. However, $\Omega_O$ is not the support of the harmonic field: the fundamental configuration $\Phi_0$ is in general global.

Define the working Euclidean geometric scale of the observer's structure by

$$L_O^E := \operatorname{diam}_\delta \Omega_O. \tag{22}$$

The precise numerical value of $L_O^E$ depends on the chosen localization region and is therefore not declared a canonical parameter of the reconstruction class. For the analysis below, it is only important that the scale-separation relations used remain valid under admissible changes of $\Omega_O$ that preserve the observer's operational structure.

The size of the observer's entire structure must be distinguished from the spatial extent of a specific protocol. The direct sensitivity of a fixed protocol on the reference slice can be localized in any sufficient region

$$K_{\text{sens}}^{(O,\Pi)} \subseteq \overline{\Omega_O}. \tag{23}$$

Minimality of this region is not required to establish locality: if all protocol channels are localized in a given set, any larger region still contained in $\overline{\Omega_O}$ is also sufficient.

Define the corresponding estimate of geometric extent by

$$L^E_{\mathrm{sens}} := \mathrm{diam}_\delta\, K^{(O,\Pi)}_{\mathrm{sens}} \le L^E_O. \tag{24}$$

Since a sufficient region need not be minimal, $L^E_{\mathrm{sens}}$ is an upper bound on the spatial extent of direct readout, not automatically an intrinsic canonical scale of the protocol.

For a fixed finite linear protocol, one can define a region canonical relative to the chosen channel family:

$$K^{(O,\Pi)}_{\mathrm{sens},*} := \overline{\bigcup_{j=1}^{N_\Pi} \mathrm{supp}\, \ell_{j,0}}. \tag{25}$$

It is the smallest closed set containing the supports of all linear channels of this protocol on the reference slice. The corresponding scale

$$L^E_{\mathrm{sens},*} := \mathrm{diam}_\delta\, K^{(O,\Pi)}_{\mathrm{sens},*} \tag{26}$$

satisfies

$$L^E_{\mathrm{sens},*} \le L^E_{\mathrm{sens}} \le L^E_O. \tag{27}$$

This quantity characterizes the geometric extent of the fixed channel family, but does not by itself determine its spectral resolution.

Neither $L^E_O$ nor $L^E_{\mathrm{sens}}$ characterizes the size of the world reconstructed relative to the observer. For the large working domain of causal reconstruction on the reference slice, we retain the notation

$$\Omega \subset \Sigma_0,$$

used in the preceding papers. The main local analysis is carried out in an intermediate region

$$\Omega_O \Subset K \Subset \Omega. \tag{28}$$

Write

$$L^E_K := \mathrm{diam}_\delta\, K. \tag{29}$$

The local regime considered here requires the separation

$$L^E_O \ll L^E_K. \tag{30}$$

The inclusion $K \Subset \Omega$ alone does not establish smallness of boundary or global corrections. The smallness required for the local theorem must be quantified through their effect on pre-threshold channels and compared with the finite operational accuracy of the protocol.

Thus, $\Omega$ is used here only as a large working domain of the given causal-reconstruction regime. The present paper requires neither its maximum size nor its global shape to be known. A separate analysis of remote reconstruction showed that, under additional conditions, finite accuracy of local data and instability of inverse elliptic continuation can lead to a finite maximum scale of stable reconstruction $L_{\max}$ and a corresponding region $\Omega(L_{\max})$ [6]. Here that global result merely indicates a possible limit of applicability of the local regime and does not enter the proof of the quantum core.

The geometric size $L^E_O$ of the observer's structure and spectral resolution are distinct characteristics of a protocol. Let $L > 0$ denote the length of the working slab along the geometric foliation parameter $s$. In particular, the relation

$$L \gg L^E_O$$

is not by itself a criterion for field modes to behave classically. The quantum-core construction below rests on analytic reduction of the harmonic problem and characterization of the canonical two-point form. The relation between observer size and the quantum-to-classical transition is not investigated here and is not used in the proofs.

Finally, the spatial organization of the observer is accompanied by an operational organization along the foliation parameter $s$. Let $S_{\mathrm{rel}}$ denote the characteristic interval in $s$ over which a stable register state is established, $S_{\Pi}$ the extent in $s$ of a specific protocol, and $S_{\mathrm{mem}}$ the interval of stable record retention. A reproducible registration regime requires

$$S_{\mathrm{rel}} \ll S_{\Pi} \ll S_{\mathrm{mem}}. \tag{31}$$

This hierarchy expresses the ability to form a stable record, use it in subsequent reconstruction, and distinguish register states before and after an event.

In the inherited relativistic sector, intervals of the foliation parameter have a fixed calibration relative to observable time:

$$\Delta t = \frac{\Delta s}{v_t}. \tag{32}$$

The intervals $S_{\mathrm{rel}}$, $S_{\Pi}$, and $S_{\mathrm{mem}}$ therefore correspond to the physical time scales

$$\frac{S_{\mathrm{rel}}}{v_t}, \qquad \frac{S_{\Pi}}{v_t}, \qquad \frac{S_{\mathrm{mem}}}{v_t}$$

in the reconstructed description of the corresponding IFR. The $S$-quantities themselves retain their geometric and operational meaning in this section. In particular, the calibration (32) does not imply that direct inter-slice transfer along $s$ is observable time evolution.

Hierarchy (31) characterizes the stability of the organization of the observer and protocol over the range of $s$ under consideration. It is not identical to stationarity of slice data or of the observer's full response. Homogeneity of free inter-slice transfer in the foliation parameter is established after sector selection in Section 4.7.

Thus, the initial framework distinguishes:

(i) the fundamental harmonic configuration and its mathematical variations;
(ii) the internal observer and the inherited local pre-threshold modal layer;
(iii) the finite set of linear channels of a specific protocol and their finite operational accuracy;
(iv) the spatial size of the observer's structure, the extent of a specific protocol, and the intermediate scale of local analysis;
(v) geometric organization along the foliation parameter $s$ and its inherited calibration against observable time $\Delta t = \Delta s / v_t$.

Neither single-slice completeness, stable inter-slice transfer, a stationary generator, nor the quantum-to-classical transition is included in this initial structure.

# 3 The full elliptic Cauchy problem

## 3.1 Harmonic variations and full data on a slice

Let $\delta\Phi$ be a mathematical harmonic variation of the actual configuration $\Phi_0$ under consideration. By linearity of the fundamental equation,

$$\Delta_{\mathbb{E}^4}\delta\Phi = 0. \tag{33}$$

It is not assumed here that the finite configuration $\Phi_0 + \delta\Phi$ necessarily realizes the same observer, registers, and protocol. The variation $\delta\Phi$ is used as an element of the linear analytic space introduced in Section 2.

In coordinates adapted to the fixed foliation, write

$$u(s) := \delta\Phi|_{\Sigma_s}.$$

Since parallel slices are geometrically identified by Euclidean translation along $n$, their tangential coordinates and corresponding data spaces can be considered in a common representation. This identification is purely geometric and does not by itself specify an operator transferring field data between slices. Sufficient conditions for reduction to single-slice transfer are established in Section 4. The resulting Euclidean transfer differs from observable time evolution in the reconstructed Lorentzian sector.

The observer's finite operational accuracy likewise does not concern this geometric identification: it limits access to field data without changing the exact mathematical form of the harmonic equation.

In this representation, equation (33) becomes

$$\partial_s^2 u + \Delta_\Sigma u = 0. \tag{34}$$

Introduce the positive operator

$$B := -\Delta_\Sigma, \qquad A := B^{1/2} = |D|. \tag{35}$$

The harmonic problem then takes the form

$$u''(s) = Bu(s) = A^2 u(s). \tag{36}$$

The word "local" in the section title refers to the operational domain of application of the subsequent results, not to an artificial restriction of the harmonic problem to the intermediate region $K$. Analytically, variations are considered on the full flat slice

$$\Sigma \simeq \mathbb{R}^3,$$

whereas the observer's pre-threshold channels are localized within

$$\Omega_O \Subset K \Subset \Omega$$

in the sense of Section 2. No artificial boundary conditions on $\partial K$ are therefore introduced at this stage.

On a fixed slice, the two components of the full mathematical data for the second-order problem are

$$f := u(0), \qquad \pi := u'(0). \tag{37}$$

Here "full" means only that the pair $(f, \pi)$ contains both independent second-order data components for the harmonic variation already under consideration. It does not mean that an arbitrary pair of functions of the appropriate regularity necessarily defines a solution of the elliptic Cauchy problem on a nonzero interval in $s$.

No relation of the form

$$\pi = Gf$$

is assumed at this stage. In particular, single-slice closure does not follow from linearity of the Laplace equation alone.

Completeness of the pair $(f, \pi)$ is mathematical rather than operational. It does not mean that the internal observer can determine both functions directly, in full, or with arbitrary accuracy. As in Section 2, a specific physical protocol has access only to finitely many local pre-threshold channels

$$\ell_{j,s}$$

with finite resolutions

$$\varepsilon_{j,s} > 0.$$

Thus, full mathematical Cauchy data and finite operational access to the field belong to different levels of description.

In the next section, stability is formulated with respect to variations of the full pair $(f, \pi)$, while their operationally observable effects are estimated through pre-threshold channels. A finite number of channels in a single protocol does not by itself control every direction in the infinite-dimensional Cauchy-data space. Whenever such control is needed below, the corresponding point-separation or observability condition must be established separately.

The graph relation

$$\pi = Gf,$$

is treated as the result of selecting a sector of the full problem, not as an initial premise. Sufficient analytic conditions for this selection and its quantitative approximation are established in Section 4.

### 3.2 Spectral realization

In the main Sobolev realization

$$X_{\mathfrak{C}} = H^r(\mathbb{R}^3)$$

the operators $A$ and $B$ are regarded as positive self-adjoint operators in $H^r(\mathbb{R}^3)$ with domains

$$D(A) = H^{r+1}(\mathbb{R}^3), \qquad D(B) = H^{r+2}(\mathbb{R}^3), \qquad B = A^2. \tag{38}$$

In the Fourier representation,

$$\widehat{Af}(k) = |k|\widehat{f}(k), \qquad \widehat{Bf}(k) = |k|^2\widehat{f}(k). \tag{39}$$

For the trace and normal derivative, it is natural to use the Cauchy-data phase space in the Sobolev scale

$$\mathcal{X}_r := H^r(\mathbb{R}^3) \times H^{r-1}(\mathbb{R}^3). \tag{40}$$

Thus,

$$f \in H^r, \qquad \pi \in H^{r-1}.$$

Definition (40) specifies the space in which the two data components are compared, but does not assert solvability of the Cauchy problem for every pair in $\mathcal{X}_r$.

Denote the spectral measure of $A$ by $E_A$. For

$$0 < \kappa < M < \infty$$

introduce the spectral projections

$$P_{\kappa,M} := E_A([\kappa, M]), \qquad P_\kappa := E_A([\kappa, \infty)). \tag{41}$$

The notation $P$ is used to distinguish spectral projections from the physical protocol $\Pi$ introduced in Section 2.

The lower bound $\kappa > 0$ is purely technical. It separates the band under consideration from a neighborhood of the zero spectral point and allows $A^{-1}$ to be used without further qualifications. Introducing $\kappa$ does not imply a physical infrared cutoff. Likewise, finite $M$ is used for intermediate spectral estimates and does not introduce a physical ultraviolet cutoff.

The limits

$$M \to \infty \qquad \text{and} \qquad \kappa \downarrow 0$$

are distinct limiting procedures and are taken only where the corresponding estimates suffice to justify them.

On a fixed band $[\kappa, M]$, the operators $A$, $A^{-1}$, and all functions of $A$ used below are bounded on the corresponding spectral subspace. Hence, for spectrally restricted data, the full Cauchy problem has its usual operator-theoretic meaning.

All subsequent operator equalities are initially understood after application of $P_{\kappa,M}$. Statements without spectral restriction are made only after the relevant estimates and limiting arguments have been established. In particular, on every fixed band,

$$(P_{\kappa,M}u)'' = A^2 P_{\kappa,M}u \tag{42}$$

with initial data

$$P_{\kappa,M}f, \qquad P_{\kappa,M}\pi.$$

This notation does not assert well-posedness of the elliptic Cauchy problem in the $H^r$ norm for arbitrary data. On the contrary, its spectral instability is an essential part of the subsequent operational analysis.

### 3.3 The two branches of the full problem

For the formulas below, let

$$h := s_2 - s_1 \geq 0 \tag{43}$$

denote the geometric difference between the parameters of two parallel slices identified as above. With $\Sigma_0$ as the initial slice, one simply has $h = s$. At this stage, $h$ is the Euclidean inter-slice distance along the chosen foliation, not a parameter of observable physical time evolution. In the reconstructed relativistic description, it corresponds to the time interval

$$\Delta t = \frac{h}{v_t}, \tag{44}$$

but this calibration does not identify direct transfer between slices with observable time evolution.

On a fixed spectral band, the solution of (36) with data (37) is

$$u(h) = \cosh(hA)f + A^{-1}\sinh(hA)\pi. \tag{45}$$

Here and throughout this subsection, restriction to a fixed band $[\kappa, M]$ is understood unless stated otherwise.

Define

$$c_+ := \frac{1}{2}\left(f + A^{-1}\pi\right), \qquad c_- := \frac{1}{2}\left(f - A^{-1}\pi\right). \tag{46}$$

Then

$$u(h) = \mathrm{e}^{hA}c_+ + \mathrm{e}^{-hA}c_-. \tag{47}$$

Consequently, the fundamental harmonic problem does not by itself select one of the two branches. Both occur in the general spectrally restricted solution of the second-order problem.

Their functional-analytic status is different, however. The operators

$$S_-(h) := \mathrm{e}^{-hA}, \qquad h \geq 0, \tag{48}$$

form a strongly continuous contraction semigroup on $H^r(\mathbb{R}^3)$:

$$\|S_-(h)\|_{H^r \to H^r} \leq 1. \tag{49}$$

By contrast,

$$\mathrm{e}^{hA}, \qquad h > 0,$$

is not a bounded operator on all of $H^r$. It is well defined on the spectral band $[\kappa, M]$ and satisfies

$$\|\mathrm{e}^{hA} P_{\kappa,M}\|_{H^r \to H^r} \le \mathrm{e}^{hM}, \tag{50}$$

but this bound is not uniform as $M \to \infty$.

These estimates characterize the two mathematical branches of the full problem. Sufficient conditions for quantitative reduction and exact selection are established in Section 4; their application to a fixed protocol is discussed in Section 4.5 and Section 5.

After selection, the operator

$$P_h^E = \mathrm{e}^{-hA}$$

coincides with the previously introduced $S_-(h)$ and describes direct Euclidean inter-slice transfer, distinct from observable unitary evolution. Taking the initial slice as the origin gives the observable-time calibration

$$t = \frac{h}{v_t}.$$

The relation between the two representations is considered in quantum reconstruction and in its matching to the relativistic temporal structure.

### 3.4 Why a single trace is initially incomplete

Formula (45) shows that knowledge of

$$f = u(0)$$

alone does not in general determine $u(h)$ for $h \neq 0$, because the normal derivative

$$\pi = u'(0)$$

remains an independent data component.

This can be seen directly. Let $\mathscr{S}(\mathbb{R}^d)$ denote the Schwartz space of smooth rapidly decreasing test functions; its continuous dual $\mathscr{S}'(\mathbb{R}^d)$ consists of tempered distributions. Complex Fourier calculations use the complexification of the test space. Choose

$$0 \neq \eta \in \mathscr{S}(\mathbb{R}^3)$$

such that

$$\operatorname{supp} \widehat{\eta} \subset \{k : \kappa < |k| < M\}$$

for some $0 < \kappa < M < \infty$, and consider the data

$$f = 0, \qquad \pi = \eta. \tag{51}$$

Then

$$u(0) = 0, \qquad u(h) = A^{-1} \sinh(hA)\eta. \tag{52}$$

For $h \neq 0$, the last expression is nonzero if $\eta \neq 0$. Thus, there are distinct spectrally restricted harmonic variations with the same trace on $\Sigma_0$ but different traces on neighboring slices.

More generally, let $u_1$ and $u_2$ be two such variations, and write

$$w := u_1 - u_2, \qquad \Delta\pi := \pi_1 - \pi_2.$$

If their initial traces agree,

$$w(0) = 0,$$

but their normal derivatives differ,

$$\Delta\pi \neq 0,$$

then on any fixed positive spectral band,

$$w(h) = A^{-1}\sinh(hA)\Delta\pi. \tag{53}$$

For an individual spectral component with $A = \lambda > 0$, this gives

$$w_\lambda(h) = \frac{\sinh(\lambda h)}{\lambda}\Delta\pi_\lambda. \tag{54}$$

Thus, uncertainty in the normal derivative, which is not contained in a single trace, does not automatically remain operationally invisible on other slices. Moreover, for large $\lambda h$, the corresponding deviation contains an exponentially growing component.

Single-slice closure is therefore not a property of the original equation

$$\Delta_{\mathbb{E}^4}\delta\Phi = 0.$$

To pass from the full data pair

$$(f, \pi)$$

to a single-slice description

$$f \longmapsto u(h),$$

one must select a special sector of admissible Cauchy data in which the normal derivative is related to the trace.

The graph

$$\pi = -Af$$

is not included among the premises of its own derivation. The next section considers variations of the full Cauchy-data pair, establishes sufficient analytic conditions for reduction, and formulates criteria for its application at finite readout accuracy.

# 4 Operational stability and quantitative reduction

Operational stability is not treated below as an additional fundamental field law. It is a condition for a local regime to belong to a reconstruction class compatible with a stable internal observer and a reproducible registration protocol. At this stage, it also has no meaning of fundamental causal evolution: the parameter used to compare slices remains the geometric foliation parameter.

## 4.1 Continuity and a fixed error budget

Operational stability means that small admissible changes in the full Cauchy data do not cause uncontrolled changes in the primary signals accessible to the observer within a fixed regime. Write the full data as $z = (f, \pi) \in \mathcal{X}_r$ and let $u_z$ denote the corresponding existing spectral solution. Before single-slice reduction, stability is formulated for these data, while the consequences of their variations are tested operationally through pre-threshold channels.

In general, the realization of a channel may depend on the slice. For an admissible family of continuous channels $\ell_h$, the natural continuity condition is therefore

$$z_j \longrightarrow 0 \quad \Longrightarrow \quad \sup_{0\le h\le L} \left|\ell_h(u_{z_j}(h))\right| \longrightarrow 0 \tag{55}$$

for existing solutions in the chosen working regime. Here convergence $z_j \to 0$ is understood in the topology of the full Cauchy-data space. Formula (55) assumes neither that $u_{z_j}$ is determined by the first component $f_j$ alone nor that the family $\ell_h$ is stationary.

Condition (55) must be distinguished from a quantitative requirement with input uncertainty and operational resolution already fixed. For tolerances $\varepsilon_{\ell,h} > 0$, such a requirement can be written as

$$\sup_{0\le h\le L} \frac{\left|\ell_h(u_{z_1}(h) - u_{z_2}(h))\right|}{\varepsilon_{\ell,h}} \le 1 \quad \text{for a prescribed set of pairs } (z_1, z_2). \tag{56}$$

Continuity allows amplification by any finite factor. Condition (56) additionally fixes the input uncertainty, the application horizon, and the admissible output budget. These quantities must be determined by the protocol, by conditions on the reconstruction class, or by independent field estimates, rather than chosen to produce the desired spectral-branch selection.

### 4.2 What ultraviolet instability excludes

The full growing branch

$$\mathrm{e}^{h|D|}, \qquad h > 0,$$

is not a bounded operator on $H^r(\mathbb{R}^3)$. Even for packets with Fourier support in the bands

$$N < |k| < N + 1$$

the amplification factor is of order $\mathrm{e}^{hN}$ and therefore admits no uniform bound as $N \to \infty$.

This statement is compatible with the local character of the analysis. The intermediate region

$$\Omega_O \Subset K \Subset \Omega$$

restricts the spatial region of the observer's direct operational access but introduces no spectral cutoff of the form

$$|k| \le k_{\max}.$$

As established in Section 2, harmonic variations are considered analytically on the full flat slice $\Sigma \simeq \mathbb{R}^3$, whereas pre-threshold channels are localized within $\Omega_O \Subset K$. Spatial locality of a channel is in general compatible with unbounded spectral support. Boundedness of $K$ alone therefore neither removes the high-frequency directions $|k| \to \infty$ nor acts as an ultraviolet regularizer.

A substantially stronger statement relevant to this model follows directly from the locality of pre-threshold readout. A nonzero channel with compact spatial support cannot remain continuous on all of $H^r$ after composition with the full growing branch.

**Proposition 4.1** (The growing branch and a local channel)**.** *Let $h > 0$, $r \in \mathbb{R}$, and let $\ell$ be a nonzero compactly supported distribution on $\mathbb{R}^3$. Consider the functional*

$$f \longmapsto \ell\Big(\mathrm{e}^{h|D|} f\Big), \tag{57}$$

*initially defined on the dense subspace of functions whose Fourier transforms belong to $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. Then the functional* (57) *has no continuous extension to all of $H^r(\mathbb{R}^3)$.*

*Proof.* Suppose otherwise. Continuity of the functional on $H^r$ then means that the distribution

$$\mathrm{e}^{h|D|}\ell$$

belongs to $H^{-r}(\mathbb{R}^3)$. In the Fourier representation, this requires

$$(1+|k|^2)^{-r/2}\mathrm{e}^{h|k|}\widehat{\ell}(k) \in L^2(\mathbb{R}^3). \tag{58}$$

Fix $0 < a < h$. For any $N \geq 0$, write

$$(1+|k|)^N \mathrm{e}^{a|k|}|\widehat{\ell}(k)| = \Big[(1+|k|^2)^{-r/2}\mathrm{e}^{h|k|}|\widehat{\ell}(k)|\Big]\Big[(1+|k|)^N(1+|k|^2)^{r/2}\mathrm{e}^{-(h-a)|k|}\Big].$$

The first factor belongs to $L^2(\mathbb{R}^3)$ by (58), and the second belongs to $L^2(\mathbb{R}^3)$ by exponential decay. The Cauchy–Schwarz inequality therefore gives

$$(1+|k|)^N \mathrm{e}^{a|k|}\widehat{\ell}(k) \in L^1(\mathbb{R}^3) \qquad \text{for all } N \geq 0,\ 0 < a < h. \tag{59}$$

Condition (59) allows the inverse Fourier transform to be defined at the complex argument

$$z = x + \mathrm{i}y, \qquad |y| < a,$$

by the absolutely convergent integral

$$F(z) := (2\pi)^{-3} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}k\cdot z}\widehat{\ell}(k)\, dk. \tag{60}$$

The polynomial bounds in (59) justify differentiation under the integral sign, so $F$ is holomorphic in the complex tube

$$\{x + \mathrm{i}y :\ |y| < a\}.$$

On the real slice $y = 0$, the function $F$ represents the original distribution $\ell$; in particular, $\ell$ is thus represented by a real-analytic function.

But $\ell$ has compact support. Its real-analytic representative therefore vanishes on the nonempty open set

$$\mathbb{R}^3 \setminus \operatorname{supp} \ell.$$

The identity theorem for real-analytic functions implies that it vanishes identically on $\mathbb{R}^3$, contradicting $\ell \neq 0$.

Hence no continuous extension of (57) to all of $H^r(\mathbb{R}^3)$ exists. □

The argument is a standard consequence of Paley–Wiener-type exponential bounds for compactly supported distributions. The relevant functional-analytic detail is also given in Appendix A and may be compared with the standard results in [16].

Proposition 4.1 essentially uses two features of the analytic regime under consideration: the absence of a postulated ultraviolet cutoff in the harmonic-variation space and spatial locality of the channel. The latter is not introduced specifically to obtain this result, but is inherited from locality of the internal observer:

$$\operatorname{supp} \ell_{j,s} \subseteq \overline{\Omega_O(s)} \Subset K.$$

Since $\Omega_O(s)$ is bounded, the corresponding pre-threshold channels have compact spatial support. Compact support does not, however, bound their spectral support and therefore does not eliminate sensitivity to high-frequency directions of the full harmonic problem.

The proposition excludes the full growing branch as a continuous inter-slice transfer on all of $H^r$ compatible with nonzero local readout. Membership in a specific operational class does not require all high-frequency variations to be admissible. On a bounded spectral sector, the growing transfer is bounded, and the proposition does not exclude such a component. By itself it does not select the graph

$$\pi = -Af.$$

The operator

$$\mathrm{e}^{hA}$$

is also well defined as an unbounded operator on its domain, in particular as the inverse of Poisson transfer in the selected sector. Spatial localization in $K$ is not a spectral cutoff and does not remove the instability established for the full branch. The following counterexamples illustrate the precise limits of this result; sufficient conditions for reduction are formulated thereafter.

### 4.3 Counterexamples to overly strong selection claims

The preceding proposition excludes the global growing branch $\mathrm{e}^{hA}$ as a continuous transfer on all of $H^r$ in the presence of a nonzero local channel. It does not follow, however, that any stability condition on a finite interval automatically selects the pure decaying branch

$$\mathrm{e}^{-hA},$$

or that a stable single-slice transfer must be stationary. The following two examples establish these limitations.

First, stable transfer on finite intervals may contain a growing component within a bounded spectral range. For $\nu_0 > 0$, define the spectral generator

$$G_{\nu_0} = AE_A([0,\nu_0]) - AE_A((\nu_0,\infty)). \tag{61}$$

Since the spectral projections commute with $A$, we have

$$G_{\nu_0}^2 = A^2 \tag{62}$$

on $D(A^2)$. Thus, for $f \in D(A^2)$, the family

$$u(h) = \mathrm{e}^{hG_{\nu_0}} f$$

is a classical solution of

$$u'' = A^2 u.$$

For general $f \in H^r$, the same family is understood as the corresponding spectral solution.

Spectrally, the transfer is

$$\mathrm{e}^{hG_{\nu_0}} = \mathrm{e}^{hA}E_A([0,\nu_0]) + \mathrm{e}^{-hA}E_A((\nu_0,\infty)). \tag{63}$$

Hence, for every finite $L > 0$,

$$\sup_{0\le h\le L} \|\mathrm{e}^{hG_{\nu_0}}\|_{H^r\to H^r} \le \mathrm{e}^{L\nu_0} < \infty. \tag{64}$$

Continuity and uniform boundedness of transfer on a finite interval thus allow a mixed choice of spectral sign: the low-frequency part may follow the growing branch if its spectrum is bounded above, while the high-frequency part follows the decaying branch. This does not contradict Proposition 4.1: that proposition excludes the full growing branch on an unbounded spectrum, whereas in (63) the growing sector is restricted by $A \le \nu_0$.

Moreover, continuity alone does not even require prior reduction of the full Cauchy-data pair to a single graph. On any fixed bounded spectral band, the full two-component transfer for $(f,\pi)$ is also continuous. Continuity on finite intervals alone therefore does not reduce the number of independent Cauchy-data components.

The discontinuity in spectral sign in (61) does not imply discontinuity of the operator with respect to the data. If

$$f_j \longrightarrow f \qquad \text{in } H^r,$$

then (64) implies, for each fixed $0 \le h \le L$,

$$\mathrm{e}^{hG_{\nu_0}} f_j \longrightarrow \mathrm{e}^{hG_{\nu_0}} f.$$

Thus, continuity of the map with respect to field data must not be tacitly strengthened to continuity of the spectral-sign choice as a function of wave number.

The second example shows that even stable, contractive single-slice transfer on a finite slab need not be stationary. For a single spectral mode with $A = \lambda > 0$, set

$$a_L(s) := \frac{\cosh\bigl(\lambda(L-s)\bigr)}{\cosh(\lambda L)}, \qquad u(s) = a_L(s) f, \qquad 0 \le s \le L. \tag{65}$$

Then

$$u'' = \lambda^2 u, \qquad u(0) = f, \qquad u'(L) = 0.$$

For $0 \le s_1 \le s_2 \le L$, define the inter-slice multiplier

$$T_{s_2,s_1} := \frac{a_L(s_2)}{a_L(s_1)} = \frac{\cosh\bigl(\lambda(L-s_2)\bigr)}{\cosh\bigl(\lambda(L-s_1)\bigr)}. \tag{66}$$

Then

$$u(s_2) = T_{s_2,s_1} u(s_1), \qquad 0 < T_{s_2,s_1} \le 1.$$

Forward transfer in the foliation parameter on the finite slab is therefore contractive.

However, its instantaneous generator

$$g_s := \frac{a_L'(s)}{a_L(s)} = -\lambda \tanh\bigl(\lambda(L-s)\bigr) \tag{67}$$

depends on $s$ and, except in the limiting regime far from the right boundary, differs from $-\lambda$. Nonstationarity does not contradict autonomy of the original second-order equation, since

$$\dot g_s + g_s^2 = \lambda^2. \tag{68}$$

Thus, passing to a single-slice description may produce a nonstationary generator even when the original second-order problem is autonomous.

These examples show that, on a finite range of the foliation parameter, stability alone implies neither

$$\pi = -Af, \tag{69}$$

nor the stationary law

$$u(h) = \mathrm{e}^{-hA} u(0). \tag{70}$$

The next step must therefore be quantitative: one must estimate how close the admissible Cauchy data are to the decaying-branch graph for a specified finite horizon and finite error budget. Exact selection of the Poisson branch will then require a separate, stronger limiting condition.

### 4.4 Decomposition and reduction on a finite slab

The following result gives an exact decomposition of an existing harmonic solution into the Poisson extension of its first trace and a remainder carrying the dependence on the second boundary trace. Quantitative reduction occurs when this remainder is small in the region of interest. Thus, finite operational accuracy is not replaced by an unjustified exact exclusion of a branch. The spectral formulation is consistent with the approach to elliptic Cauchy problems used in [7]; here the focus is on controlling the deviation from the Poisson extension through the mismatch of the second boundary trace.

A finite slab here means the interval $0 \le s \le L$ in the geometric foliation parameter. Finiteness of $L$ does not imply boundedness of the spectrum of $A$.

We specify the notion of solution used below. Let $A \ge \kappa I$, $\kappa > 0$, be a self-adjoint operator in the analytic Hilbert space $\mathcal{H}$, and, for $M > \kappa$, set

$$P_{\kappa,M} := E_A([\kappa, M]). \tag{71}$$

By a spectral solution on $[0, L]$ we mean an $\mathcal{H}$-valued family $u(s)$ such that, for every $M > \kappa$, the function $u_M(s) := P_{\kappa,M}u(s)$ belongs to $C^2([0, L]; \mathcal{H})$ and satisfies

$$u_M''(s) = A^2 u_M(s). \tag{72}$$

Derivatives at the endpoints are understood as one-sided derivatives. This definition does not require $u''(s)$ to exist in $\mathcal{H}$ before the spectral restriction is removed, and does not assert solvability of the elliptic Cauchy problem for arbitrary initial data.

In the main realization of the preceding section, the result is applied to the positive spectral sector

$$\mathcal{H} = E_A([\kappa, \infty))H^r(\mathbb{R}^3), \qquad A = |D|,$$

with $A$ restricted to that sector. Here $\mathcal{H}$ is an analytic field-data space, not the state space of the subsequent quantum reconstruction.

**Theorem 1** (Decomposition and quantitative reduction on a finite slab)**.** *Let $A \ge \kappa I$, $\kappa > 0$, be self-adjoint in $\mathcal{H}$, let $L > 0$, and let $u$ be a spectral solution of $u'' = A^2 u$ on $[0, L]$ in the sense specified above. Write*

$$f := u(0) \in \mathcal{H}, \qquad g := u(L) \in \mathcal{H}, \qquad b_L := g - S_-(L)f, \qquad S_-(s) := \mathrm{e}^{-sA}. \tag{73}$$

*For $0 \le s \le L$, define the bounded Borel multipliers*

$$d_{s,L}(\lambda) := \frac{\sinh(s\lambda)}{\sinh(L\lambda)}, \qquad r_{s,L}(\lambda) := \frac{\mathrm{e}^{s\lambda}}{\sinh(L\lambda)}, \qquad \lambda \ge \kappa, \tag{74}$$

*and the corresponding operators by the spectral functional calculus:*

$$D_{s,L} := d_{s,L}(A), \qquad R_{s,L} := r_{s,L}(A). \tag{75}$$

*Then the exact identity*

$$u(s) = S_-(s)f + D_{s,L}b_L. \tag{76}$$

*holds throughout the slab.*

*Moreover, $A^{-1}u$ belongs to $C^1([0, L]; \mathcal{H})$. Introduce the following notation for the normalized normal derivative:*

$$A^{-1}u'(s) := \frac{\mathrm{d}}{\mathrm{d}s}(A^{-1}u(s)). \tag{77}$$

*If $u'(s)$ exists directly in $\mathcal{H}$, this notation agrees with the ordinary application of $A^{-1}$ to the derivative. In general, the meaning in (77) is used, including one-sided derivatives at $s = 0$ and $s = L$. In this sense,*

$$A^{-1}u'(s) + u(s) = R_{s,L}b_L. \tag{78}$$

*In particular,*

$$\|u(s) - S_-(s)f\|_{\mathcal{H}} \le \eta_\kappa(s, L)\|b_L\|_{\mathcal{H}}, \tag{79}$$

$$\|A^{-1}u'(s) + u(s)\|_{\mathcal{H}} \le 2\eta_\kappa(s, L)\|b_L\|_{\mathcal{H}}, \tag{80}$$

*where*

$$\eta_\kappa(s, L) := \frac{\mathrm{e}^{-\kappa(L-s)}}{1 - \mathrm{e}^{-2\kappa L}}. \tag{81}$$

*Proof.* First restrict the spectrum to $[\kappa, M]$. Set

$$u_M := P_{\kappa,M}u, \qquad f_M := P_{\kappa,M}f, \qquad g_M := P_{\kappa,M}g, \qquad b_{L,M} := P_{\kappa,M}b_L.$$

Since $P_{\kappa,M}$ commutes with functions of $A$, we have

$$b_{L,M} = g_M - S_-(L)f_M.$$

On this spectral subspace, $A$ is bounded, and expansion (47) gives

$$u_M(s) = \mathrm{e}^{sA}c_{+,M} + \mathrm{e}^{-sA}c_{-,M}, \qquad f_M = c_{+,M} + c_{-,M}.$$

Consequently,

$$b_{L,M} = \big(\mathrm{e}^{LA} - \mathrm{e}^{-LA}\big)c_{+,M} = 2\sinh(LA)c_{+,M}.$$

Since $\lambda \ge \kappa > 0$, the denominator $\sinh(L\lambda)$ does not vanish. Substituting

$$c_{+,M} = \tfrac{1}{2}\sinh(LA)^{-1}b_{L,M}, \qquad c_{-,M} = f_M - c_{+,M}$$

yields

$$u_M(s) = S_-(s)f_M + D_{s,L}b_{L,M}. \tag{82}$$

At this step, all products of hyperbolic functions of $A$ are considered on the bounded spectral subspace.

For $\lambda \ge \kappa$ and $0 \le s \le L$,

$$0 \le \frac{\sinh(s\lambda)}{\sinh(L\lambda)} = \mathrm{e}^{-(L-s)\lambda}\frac{1 - \mathrm{e}^{-2s\lambda}}{1 - \mathrm{e}^{-2L\lambda}} \le \eta_\kappa(s, L). \tag{83}$$

Furthermore,

$$0 < \frac{\mathrm{e}^{s\lambda}}{\sinh(L\lambda)} = \frac{2\mathrm{e}^{-(L-s)\lambda}}{1 - \mathrm{e}^{-2L\lambda}} \le 2\eta_\kappa(s, L). \tag{84}$$

These estimates prove boundedness of the multipliers (74) and show that the definition (75) is well posed. In particular,

$$\|D_{s,L}\| \le \eta_\kappa(s, L), \qquad \|R_{s,L}\| \le 2\eta_\kappa(s, L).$$

The bounds are uniform in $M$ and bounded throughout the slab, since

$$\eta_\kappa(s, L) \le \frac{1}{1 - \mathrm{e}^{-2\kappa L}}, \qquad 0 \le s \le L.$$

Because $A \geq \kappa I$, the projections $P_{\kappa,M}$ converge strongly to $I$ as $M \to \infty$. Thus, for every $s \in [0, L]$,

$$u_M(s) \to u(s), \qquad f_M \to f, \qquad b_{L,M} \to b_L$$

in $\mathcal{H}$. Boundedness of $S_-(s)$ and $D_{s,L}$ allows passage to the limit in (82), yielding (76). Moreover,

$$\sup_{0 \leq s \leq L} \|u_M(s) - u(s)\|_{\mathcal{H}} \leq \|(I - P_{\kappa,M})f\|_{\mathcal{H}} + \frac{\|(I - P_{\kappa,M})b_L\|_{\mathcal{H}}}{1 - \mathrm{e}^{-2\kappa L}} \longrightarrow 0.$$

In particular, $u$ is continuous on $[0, L]$ as a uniform limit of the continuous functions $u_M$.

For the normal derivative, first differentiate (82) at finite $M$. The term $S_-(s)f_M$ satisfies the decaying-branch graph relation, while for the remainder we use the identity $\cosh(s\lambda) + \sinh(s\lambda) = \mathrm{e}^{s\lambda}$. Hence,

$$A^{-1}u'_M(s) + u_M(s) = R_{s,L}b_{L,M}. \tag{85}$$

Write $v_M(s) := A^{-1}u'_M(s)$. The uniform bounds for $R_{s,L}$ and the already established uniform convergence of $u_M$ imply

$$v_M(s) \longrightarrow v(s) := -u(s) + R_{s,L}b_L$$

uniformly on $[0, L]$. Therefore $v$ is continuous.

On each spectral subspace,

$$A^{-1}(u_M(s) - f_M) = \int_0^s v_M(\sigma)\,\mathrm{d}\sigma.$$

The operator $A^{-1}$ is bounded, and the integrands converge uniformly. Thus, as $M \to \infty$,

$$A^{-1}(u(s) - f) = \int_0^s v(\sigma)\,\mathrm{d}\sigma.$$

It follows that $A^{-1}u \in C^1([0, L]; \mathcal{H})$ and

$$\frac{\mathrm{d}}{\mathrm{d}s}(A^{-1}u(s)) = v(s) = -u(s) + R_{s,L}b_L.$$

This proves (77) and (78), including their one-sided interpretation at the slab endpoints.

Finally, applying the operator bounds obtained above to the exact identities (76) and (78) gives (79) and (80). □

In the Sobolev realization on the sector $|k| \geq \kappa$, the normal derivative exists in $H^{r-1}$, and its normalized value $A^{-1}u'$ belongs to $H^r$. Indeed, on this sector $A : H^r \to H^{r-1}$ and $A^{-1} : H^{r-1} \to H^r$ are continuous, so (77) agrees with the natural scale of full Cauchy data. Applying the theorem does not require $u'(s) \in H^r$ at the slab boundaries.

The quantity $g$ is the second boundary trace of the solution under consideration. Decomposition (76) is exact for any traces satisfying the hypotheses of the theorem. To use it as a Poisson approximation, an independent quantitative bound

$$\|b_L\|_{\mathcal{H}} \leq M_L$$

is imposed on the chosen data class. Without such control, the small factor in (79) and (80) can be offset by a large boundary mismatch. The estimates establish quantitative closeness to the decaying-branch graph when $b_L$ is controlled, not exact exclusion of the second branch on a finite slab.

### 4.5 Operational form of the approximate graph

To apply Theorem 1 to a fixed protocol, consider a class of solutions with boundary mismatch

$$b_L = u(L) - S_-(L)u(0).$$

An independently established uniform bound $M_L < \infty$ is assumed for this class:

$$\|b_L\|_{\mathcal{H}} \le M_L \tag{86}$$

Ways to verify this bound are discussed in Section 5.1.

The remainder of this subsection uses the main Sobolev realization

$$\mathcal{H} = E_A([\kappa, \infty))H^r(\mathbb{R}^3) \subset X_{\mathfrak{C}} = H^r(\mathbb{R}^3) \tag{87}$$

with the norm induced by $H^r(\mathbb{R}^3)$. The operator estimates of Theorem 1 are therefore directly compatible with the duality between the channels $X'_{\mathfrak{C}}$ and field data.

Suppose the interior portion of the slab being used is restricted by

$$0 \le s \le l < L.$$

Then (79) gives

$$\|u(s) - S_-(s)f\|_{\mathcal{H}} \le \eta_\kappa(s, L)M_L \le \frac{\mathrm{e}^{-\kappa(L-l)}}{1 - \mathrm{e}^{-2\kappa L}}\, M_L, \qquad 0 \le s \le l. \tag{88}$$

Thus, for a fixed spectral gap $\kappa > 0$, the effect of the second Cauchy-data component on the interior portion of the slab is exponentially suppressed as the distance $L - l$ from the second boundary increases, provided $M_L$ remains controlled.

We now express this operator estimate in terms of the finite protocol $\Pi$. Recall that on $\Sigma_s$ the protocol is specified by a finite family of continuous channels

$$\ell_{j,s} \in X'_{\mathfrak{C}}, \qquad j = 1, \dots, N_\Pi,$$

and corresponding operational resolutions

$$\varepsilon_{j,s} > 0.$$

Before a stationary regime has been derived, neither $\ell_{j,s}$ nor $\varepsilon_{j,s}$ is assumed independent of $s$.

For each fixed $s \in [0, l]$,

$$\begin{aligned} |\ell_{j,s}(u(s) - S_-(s)f)| &\le \|\ell_{j,s}\|_{X'_{\mathfrak{C}}}\, \|u(s) - S_-(s)f\|_{X_{\mathfrak{C}}} \\ &\le \|\ell_{j,s}\|_{X'_{\mathfrak{C}}}\, \eta_\kappa(s, L)M_L. \end{aligned} \tag{89}$$

For uniform control over the entire interval $0 \le s \le l$, define

$$\Lambda_{j,l} := \sup_{0 \le s \le l} \|\ell_{j,s}\|_{X'_{\mathfrak{C}}}, \tag{90}$$

and

$$\varepsilon_{j,l}^{\min} := \inf_{0 \le s \le l} \varepsilon_{j,s}. \tag{91}$$

If

$$\Lambda_{j,l} < \infty, \qquad \varepsilon_{j,l}^{\min} > 0,$$

then

$$\sup_{0\le s\le l}\bigl|\ell_{j,s}\bigl(u(s)-S_-(s)f\bigr)\bigr|\le\Lambda_{j,l}\frac{\mathrm{e}^{-\kappa(L-l)}}{1-\mathrm{e}^{-2\kappa L}}M_L. \tag{92}$$

Consequently, a sufficient condition for operational indistinguishability of the exact solution and its Poisson approximation for the entire fixed protocol on $[0,l]$ is

$$\boxed{\Lambda_{j,l}\frac{\mathrm{e}^{-\kappa(L-l)}}{1-\mathrm{e}^{-2\kappa L}}M_L\le\varepsilon_{j,l}^{\min},\qquad j=1,\dots,N_\Pi.} \tag{93}$$

Condition (93) provides a sufficient criterion for applying the Poisson approximation to a fixed protocol, given the independent bound (86) and the stated readout-sensitivity bounds.

The parameter $\kappa$ does not introduce a fundamental infrared cutoff either. It selects the positive spectral sector on which the quantitative estimate provides the prescribed accuracy at fixed $L$, $l$, and $M_L$. The limit $\kappa\downarrow 0$, if needed, must be considered separately.

For a nonlinear primary response, the operator estimate requires an additional step. Suppose that, on the data set under consideration, the response on $\Sigma_s$ is given by a map

$$\mathcal{R}_s : X_{\mathfrak{C}}\longrightarrow Y$$

taking values in a metric space $(Y,d_Y)$. Suppose there is a modulus of continuity $\omega_{\mathcal{R},l}:[0,\infty)\to[0,\infty)$, uniform for $0\le s\le l$, nondecreasing, satisfying $\omega_{\mathcal{R},l}(0)=0$ and $\lim_{a\downarrow 0}\omega_{\mathcal{R},l}(a)=0$, such that

$$d_Y\bigl(\mathcal{R}_s(f),\mathcal{R}_s(g)\bigr)\le\omega_{\mathcal{R},l}\bigl(\|f-g\|_{X_{\mathfrak{C}}}\bigr) \tag{94}$$

for all $f,g$ under consideration and $s\in[0,l]$. Then (88) implies

$$d_Y\bigl(\mathcal{R}_s(u(s)),\mathcal{R}_s(S_-(s)f)\bigr)\le\omega_{\mathcal{R},l}\left(\frac{\mathrm{e}^{-\kappa(L-l)}}{1-\mathrm{e}^{-2\kappa L}}M_L\right). \tag{95}$$

Preservation of a discrete threshold decision about an event, however, requires an additional margin from the boundaries of the corresponding classification regions. Such a margin does not follow from the operator estimate and must be controlled by properties of the specific registration protocol.

The next result directly bounds the thickness of the admissible layer of full Cauchy data above a fixed initial trace.

**Corollary 4.2** (Bound on the hidden normal component)**.** *Let $A\ge\kappa I$, $\kappa>0$, and let $u_1,u_2$ be two spectral solutions on $[0,L]$ in the sense of Theorem 1, with the same initial trace,*

$$\Delta f:=u_1(0)-u_2(0)=0.$$

*If their difference at the second boundary satisfies*

$$\|\Delta u(L)\|_{\mathcal{H}}\le M_\Delta,$$

*then the normalized difference of the normal components on the initial slice satisfies*

$$\boxed{\|A^{-1}\Delta\pi(0)\|_{\mathcal{H}}\le\frac{M_\Delta}{\sinh(\kappa L)}.} \tag{96}$$

*Here $A^{-1}\Delta\pi(0)$ is understood in the same normalized Sobolev scale as in Theorem 1.*

*Proof.* Since $\Delta f = 0$, the difference of the solutions on a fixed spectral band has the form

$$\Delta u(s) = A^{-1} \sinh(sA) \Delta\pi(0).$$

At $s = L$, this yields

$$A^{-1}\Delta\pi(0) = \sinh(LA)^{-1}\Delta u(L).$$

On the positive spectral sector $A \geq \kappa I$, the operator $\sinh(LA)^{-1}$ is bounded, and

$$\| \sinh(LA)^{-1} \| \leq \frac{1}{\sinh(\kappa L)}.$$

Removing the upper spectral restriction by the same strong limiting argument as in Theorem 1 yields (96). □

Estimate (96) shows that, when the second boundary is controlled, the admissible layer of normalized normal data above fixed $f$ narrows exponentially as $\kappa L$ increases. For finite $M_\Delta$ and $L$, however, this layer generally has nonzero thickness. A finite error budget therefore does not imply the exact equality

$$\pi = -Af.$$

The rigorously established result is quantitative closeness of the admissible data to the decaying-branch graph, not disappearance of all representatives outside it.

We have thus obtained a sufficient criterion for applying the Poisson approximation to a fixed protocol, given independently specified control of field data and channels. For a specific reconstruction class, the uniform bound (86) and the corresponding readout estimates must be verified.

### 4.6 The exact graph in an independently specified limiting class

The finite-slab Theorem 1 gives quantitative closeness to the decaying-branch graph, but does not eliminate the second branch exactly at finite $L$ and finite error budget. An exact result requires an independently specified, stronger limiting class of solutions.

For a self-adjoint operator $A \geq 0$, a spectral solution on the half-line will mean an $\mathcal{H}$-valued family

$$u : [0, \infty) \longrightarrow \mathcal{H},$$

with the following properties.

For every

$$0 < \kappa < M < \infty$$

and every finite $L > 0$, the spectral projection

$$E_A([\kappa, M])u$$

belongs to

$$C^2([0, L]; E_A([\kappa, M])\mathcal{H})$$

and satisfies on $[0, L]$ the equation

$$\frac{d^2}{ds^2} E_A([\kappa, M])u(s) = A^2 E_A([\kappa, M])u(s).$$

The zero spectral component

$$u_{\ker A}(s) := E_A(\{0\})u(s)$$

is likewise assumed to be locally twice continuously differentiable and to satisfy

$$u''_{\ker A}(s) = 0. \tag{97}$$

This definition covers both the positive spectral sector, where the estimates with a positive gap $\kappa$ apply, and the possible kernel of $A$. The notation $u_{\ker A}$ refers to the spectral solution's component in the zero eigenspace; it is distinct from the trace $u_0$ of the actual configuration introduced in Section 2.1.

Consider the subexponential growth condition:

$$\forall\, \varepsilon > 0 \quad \exists\, C_\varepsilon < \infty : \qquad \|u(s)\|_{\mathcal{H}} \le C_\varepsilon \mathrm{e}^{\varepsilon s}, \qquad s \ge 0. \tag{98}$$

It is formulated directly for the full solution and presupposes neither single-slice completeness, a graph relation between the trace and its normal derivative, nor a choice of spectral-branch sign. Condition (98) is substantially stronger than continuity or boundedness of transfer on each fixed finite interval.

**Corollary 4.3** (Exact reduction under subexponential growth)**.** *Let $A \ge 0$ be self-adjoint in $\mathcal{H}$, and let $u$ be a spectral solution on the half-line in the sense specified above. Suppose $u$ satisfies* (98)*.*

*Then on the positive spectral subspace*

$$\mathcal{H}_+ := E_A((0,\infty))\mathcal{H} \tag{99}$$

*one has the exact reduction*

$$E_A((0,\infty))u(s) = \mathrm{e}^{-sA} E_A((0,\infty))u(0), \qquad s \ge 0. \tag{100}$$

*On the zero spectral subspace* $\operatorname{Ker} A$*, the general solution is*

$$u_{\ker A}(s) = a + sb, \qquad a, b \in \operatorname{Ker} A. \tag{101}$$

*Subexponential growth alone does not require $b = 0$. If, in addition,*

$$\sup_{s \ge 0} \|u(s)\|_{\mathcal{H}} < \infty, \tag{102}$$

*then $b = 0$. Thus, under this additional condition, the zero component is stationary as well.*

*Proof.* Fix $\kappa > 0$ and write

$$P_\kappa := E_A([\kappa, \infty)), \qquad \mathcal{H}_\kappa := P_\kappa \mathcal{H}.$$

On $\mathcal{H}_\kappa$, the restriction $A_\kappa := A|_{\mathcal{H}_\kappa}$ satisfies

$$A_\kappa \ge \kappa I.$$

The function

$$u_\kappa(s) := P_\kappa u(s)$$

is a spectral solution for $A_\kappa$ on every finite slab. Moreover, since $P_\kappa$ is an orthogonal projection,

$$\|u_\kappa(s)\|_{\mathcal{H}} \le \|u(s)\|_{\mathcal{H}} \le C_\varepsilon \mathrm{e}^{\varepsilon s},$$

so the same subexponential growth condition holds uniformly for $u_\kappa$.

Apply Theorem 1 to $u_\kappa$ on $[0, L]$, with $L > s$ for the fixed value of $s$ under consideration. Set

$$f_\kappa := u_\kappa(0), \qquad b_{\kappa,L} := u_\kappa(L) - \mathrm{e}^{-LA} f_\kappa.$$

Then

$$\|u_\kappa(s) - \mathrm{e}^{-sA} f_\kappa\|_{\mathcal{H}} \le \eta_\kappa(s, L)\|b_{\kappa,L}\|_{\mathcal{H}}. \tag{103}$$

Since $A \ge 0$, and hence

$$\|\mathrm{e}^{-LA}\| \le 1,$$

we have

$$\|b_{\kappa,L}\|_{\mathcal{H}} \le \|u_\kappa(L)\|_{\mathcal{H}} + \|f_\kappa\|_{\mathcal{H}} \le C_\varepsilon \mathrm{e}^{\varepsilon L} + \|u(0)\|_{\mathcal{H}}. \tag{104}$$

Consequently,

$$\|u_\kappa(s) - \mathrm{e}^{-sA} f_\kappa\|_{\mathcal{H}} \le \frac{\mathrm{e}^{-\kappa(L-s)}}{1 - \mathrm{e}^{-2\kappa L}} \left( C_\varepsilon \mathrm{e}^{\varepsilon L} + \|u(0)\|_{\mathcal{H}} \right). \tag{105}$$

Choose

$$0 < \varepsilon < \kappa.$$

For fixed $s$ and $L \to \infty$, the term containing $C_\varepsilon$ is bounded by a quantity of order

$$C_\varepsilon \mathrm{e}^{-(\kappa-\varepsilon)L + \kappa s},$$

and the initial-data term by a quantity of order

$$\|u(0)\|_{\mathcal{H}} \mathrm{e}^{-\kappa(L-s)}.$$

Both tend to zero, while

$$1 - \mathrm{e}^{-2\kappa L} \longrightarrow 1.$$

Therefore,

$$P_\kappa u(s) = \mathrm{e}^{-sA} P_\kappa u(0) \qquad \text{for every } s \ge 0. \tag{106}$$

Now let $\kappa \downarrow 0$. By the spectral theorem,

$$E_A([\kappa, \infty)) \xrightarrow[\kappa \downarrow 0]{\mathrm{s}} E_A((0, \infty)) \tag{107}$$

in the strong operator topology. Since $\mathrm{e}^{-sA}$ is bounded and commutes with all spectral projections of $A$, the strong limit in (106) yields (100).

On $\operatorname{Ker} A$, equation (97) holds by the definition of a spectral solution. Hence,

$$u_{\ker A}(s) = a + sb, \qquad a, b \in \operatorname{Ker} A.$$

Every such linear function satisfies (98), so the subexponential condition alone does not eliminate the component $b$. If (102) also holds, the orthogonal projection $u_{\ker A}(s) = E_A(\{0\})u(s)$ is uniformly bounded as well. For the function $a + sb$ on the entire half-line, this is possible only if $b = 0$. □

Thus, the exact decaying sector arises here not by a stronger interpretation of the finite-slab estimate, but as a characterization of an independently specified limiting solution class. On the positive spectrum, condition (98) excludes every component with a positive exponential growth rate and leaves

$$E_A((0, \infty))u(s) = \mathrm{e}^{-sA} E_A((0, \infty))u(0).$$

On $\operatorname{Ker} A$, subexponential growth allows a linear component; the additional condition of uniform forward boundedness removes it.

In particular, if (102) holds and

$$u(0) \in D(A),$$

the resulting representation has a strong right derivative at $s = 0$ and yields

$$u'(0) = -Au(0). \tag{108}$$

On the positive spectrum, this follows from the standard derivative of the semigroup $\mathrm{e}^{-sA}$ on $D(A)$; on $\operatorname{Ker} A$, both sides vanish. Under these additional conditions, the exact graph relation therefore holds on the entire spectrum.

Condition (98) is to be understood as the mathematical definition of an exact limiting class, not as an assertion that an observer or protocol is physically realizable over an unbounded range of the foliation parameter $s \geq 0$. In the reconstructed relativistic description, this geometric range corresponds to an unbounded range of the coordinate

$$x^0_{\mathrm{SR}} = s,$$

or, after the previously established calibration,

$$t = \frac{s}{v_t},$$

but the proof does not assume a physically realizable observer to exist throughout this range.

To apply the Poisson reference description on a finite working interval, one uses the estimates of Theorem 1 and criterion (93), with control of the boundary mismatch and readouts. Convergence of a family of operational regimes to the exact class on the half-line is a separate, stronger result and is not claimed here.

### 4.7 The single-slice law and stationarity as consequences

After the exact selection established in Corollary 4.3, the full solution on the positive spectral subspace

$$\mathcal{H}_+ = E_A((0,\infty))\mathcal{H}$$

is uniquely determined by a single trace:

$$u(s+h) = \mathrm{e}^{-hA}u(s), \qquad h \geq 0. \tag{109}$$

An exact direct Euclidean inter-slice operator thus arises on this sector:

$$P^E_h := \mathrm{e}^{-hA}, \qquad h \geq 0. \tag{110}$$

Here $h = s_2 - s_1$ is the geometric difference of foliation parameters. The operator $P^E_h$ is not identified with observable unitary evolution in reconstructed physical time $t = h/v_t$.

In the Sobolev realization $A = |D|$, the exact single-slice law corresponds to the decaying-branch graph of full data

$$\mathcal{M}^{(r)}_{-,+} := \left\{(f, -Af) : f \in E_A((0,\infty))H^r(\mathbb{R}^3)\right\} \subset \mathcal{X}_r. \tag{111}$$

In other words, for a sufficiently regular representative of the solution,

$$u'(s) = -Au(s) \tag{112}$$

on the positive spectrum, with equality understood in the natural scale of full Cauchy data. In a flat half-space, this relation expresses the classical connection between the Poisson extension and the Dirichlet-to-Neumann operator $|D|$ [8, 9]. The outward normal derivative at $s = 0$ is $-u'(0)$, which determines the sign of the decaying-branch graph.

On the zero spectral subspace, the exact subexponential condition alone allows

$$u_{\ker A}(s) = a + sb, \qquad a, b \in \operatorname{Ker} A,$$

and therefore does not yet specify a single-slice law of the form (110). If the uniform forward-boundedness condition

$$\sup_{s\geq 0} \|u(s)\|_{\mathcal{H}} < \infty,$$

also holds, then $b = 0$. In this stronger class, (110) extends to $\operatorname{Ker} A$, where $P_h^E$ acts as the identity, and the full graph is

$$\mathcal{M}_-^{(r)} = \Big\{(f, -Af) : f \in H^r(\mathbb{R}^3)\Big\} \subset \mathcal{X}_r. \tag{113}$$

As an operator family on $H^r(\mathbb{R}^3)$,

$$\mathrm{e}^{-hA}, \qquad h \geq 0,$$

always forms a strongly continuous contraction semigroup:

$$P_0^E = I, \qquad P_{h_1+h_2}^E = P_{h_1}^E P_{h_2}^E, \qquad \|P_h^E\|_{H^r\to H^r} \leq 1. \tag{114}$$

As a derived law for the full admissible sector, this operator extends to the zero spectral component only after the linear zero mode has been excluded as above.

The semigroup generator is

$$G = -A, \qquad D(G) = H^{r+1}(\mathbb{R}^3), \tag{115}$$

and on $H^{r+2}(\mathbb{R}^3)$,

$$G^2 = A^2 = B. \tag{116}$$

Thus,

$$G = -A, \qquad G^2 = B, \qquad P_{h_1+h_2}^E = P_{h_1}^E P_{h_2}^E \tag{117}$$

are consequences of exact selection already performed, rather than additional premises of single-slice reconstruction.

Stationarity here means homogeneity of direct Euclidean transfer under shifts of the foliation parameter: the operator between two slices depends only on

$$h = s_2 - s_1,$$

and not on the absolute position $s_1$ of the initial slice:

$$P_{s_2,s_1}^E = P_{s_2-s_1}^E. \tag{118}$$

It follows from autonomy of the original equation together with exact selection of the decaying spectral branch. No stationarity of an arbitrary single-slice generator $G_s$ is used before this selection; the counterexample in Section 4.3 shows that stable transfer on a finite slab alone does not ensure it.

This Euclidean stationarity must also be distinguished from subsequent observable quantum dynamics. The family

$$P_h^E = \mathrm{e}^{-hA}$$

is a contraction semigroup of direct inter-slice transfer. The unitary group

$$U(t) = \exp\Big(-\frac{\mathrm{i}}{\hbar} t H_{\mathrm{phys}}\Big)$$

arises only after construction of a reflection-positive Euclidean two-point structure, Osterwalder–Schrader reconstruction, and matching to the already reconstructed relativistic time. Hence the equality

$$t = \frac{h}{v_t}$$

calibrates observable time, but does not directly turn $P_h^E$ into $U(t)$.

For the finite-slab class, the general result is the exact decomposition of Theorem 1,

$$u(s) = S_-(s)f + D_{s,L}b_L$$

together with the quantitative estimate

$$\|u(s) - S_-(s)f\|_{\mathcal{H}} \le \eta_\kappa(s, L)\|b_L\|_{\mathcal{H}}.$$

Given an independent bound on $b_L$ and control of the channels, this estimate yields the operational criterion of Section 4.5. The quantitative approximation and the exact law (110) concern different data classes.

Similarly, closeness of field values does not justify a claim of closeness of unbounded generators without separate control. The present work provides the independent estimate (80) for the normal component; this estimate must be used instead of the formal expression

$$G = -A + O(\varepsilon).$$

# 5 Operational meaning and applicability of the reduction

This section relates the conditions for analytic reduction to accessible readouts, the geometry of the working region, and observable dynamics.

## 5.1 Conditions for application to a reconstruction class

Theorem 1 establishes an exact decomposition of an existing solution and a bound on its deviation from the Poisson extension. Applying this bound to a specific reconstruction class requires independent control of the boundary mismatch on the spectral sector under consideration.

For fixed $\mathfrak{C}$, $O$, and $\Pi$, let $\mathscr{U}_{\mathfrak{C},\Pi}$ denote the set of admissible full Cauchy-data variations with a prescribed initial uncertainty budget; dependence on the fixed observer $O$ is suppressed in the notation. For each $z \in \mathscr{U}_{\mathfrak{C},\Pi}$, a spectral solution $u_z$ with traces in $H^r$ on $[0, L]$ is assumed to exist. This set need not be linear and does not represent an ensemble of fundamental worlds. It is specified relative to a single actual configuration by conditions preserving the observer and protocol, by operational accuracy, and by independently established data constraints, not by requiring closeness to a preselected branch.

Set

$$e_z(s) := u_z(s) - S_-(s)u_z(0), \qquad b_L(z) := e_z(L). \tag{119}$$

For fixed $\kappa > 0$, control sufficient for the estimates below takes the form

$$\sup_{z \in \mathscr{U}_{\mathfrak{C},\Pi}} \|P_\kappa b_L(z)\|_{H^r} \le M_{\mathfrak{C},\Pi,\kappa}(L) < \infty, \qquad P_\kappa = E_A([\kappa, \infty)). \tag{120}$$

Dependence of the bound on $\kappa$ is explicit: uniformity as $\kappa \downarrow 0$ is not assumed. For a specific class, (120) is verified using independent constraints on admissible field data; the mere existence of an observer and finiteness of the protocol do not suffice. Operational smallness is determined by the product of the boundary bound, the transfer factor, and channel sensitivity, not merely by finiteness of $M_{\mathfrak{C},\Pi,\kappa}(L)$.

In particular, an arbitrary finite set of readouts cannot control the full norm. An observability estimate of the form

$$\|v\|_{H^r} \le C \max_{1\le j\le N_\Pi} |\ell_{j,L}(v)| \tag{121}$$

is impossible on all of $H^r(\mathbb{R}^3)$: the map $v \mapsto (\ell_{1,L}(v), \dots, \ell_{N_\Pi,L}(v))$ has a finite-dimensional range and an infinite-dimensional kernel. For a nonzero element of this kernel, the right-hand side of (121) vanishes.

On an independently specified finite-dimensional model subspace, such an estimate follows from injectivity of the readout and equivalence of norms; its constant is determined by the conditioning of the readout. On the full slice $\mathbb{R}^3$, however, restriction to a finite spectral band alone does not make the data space finite-dimensional. Another route to (120) is through independent quantitative a priori bounds on admissible field data. Local regularity alone, without uniform bounds, does not provide this control.

Thus, condition (120) gives a sufficient norm-based method of control for the chosen class. An alternative is to estimate the remainder's action on the channels directly, without recovering its full norm. In both cases, constraints on admissible variations are specified independently of the conclusion about the Poisson approximation.

## 5.2 Distances, boundaries, and infrared modes

In the finite-slab theorem, $L$ is the extent along the normal $n$, and $L - l$ is the distance from the working interval $0 \le s \le l < L$ to the second slab boundary. The quantity $\operatorname{dist}(K, \partial\Omega)$ is measured within the slice and has a different meaning. A relation between these distances does not follow from $\Omega_O \Subset K \Subset \Omega$ and requires a separate estimate for the specific reconstruction geometry.

Condition (120) and Theorem 1 imply

$$\|P_\kappa e_z(s)\|_{H^r} \le \eta_\kappa(l, L) M_{\mathfrak{C},\Pi,\kappa}(L), \qquad 0 \le s \le l, \tag{122}$$

uniformly in $z \in \mathscr{U}_{\mathfrak{C},\Pi}$. This estimate concerns only the sector $A \ge \kappa I$. It does not automatically control the full signal if admissible variations also contain components with $0 \le A < \kappa I$.

To estimate the full signal, denote its low-frequency contribution by the following quantity, where the subscript IR indicates the infrared, that is, low-frequency, component:

$$\mathsf{R}^{\mathrm{IR}}_{j,\kappa,l} := \sup_{\substack{z\in\mathscr{U}_{\mathfrak{C},\Pi}\\ 0\le s\le l}} \left|\ell_{j,s}((I - P_\kappa)e_z(s))\right|. \tag{123}$$

Using the decomposition $e_z = P_\kappa e_z + (I - P_\kappa)e_z$, the triangle inequality, and the notation $\Lambda_{j,l}$, $\varepsilon^{\min}_{j,l}$ from Section 4.5, we obtain a sufficient criterion for indistinguishability of pre-threshold signals:

$$\Lambda_{j,l}\eta_\kappa(l, L) M_{\mathfrak{C},\Pi,\kappa}(L) + \mathsf{R}^{\mathrm{IR}}_{j,\kappa,l} \le \varepsilon^{\min}_{j,l}, \qquad j = 1, \dots, N_\Pi. \tag{124}$$

If the solutions under consideration lie entirely in the positive sector $P_\kappa H^r$, the low-frequency remainder is zero and (124) reduces to (93). In general, independent control of (123) is needed. Preserving discrete registration labels additionally requires control of the threshold margin, as stated in Section 4.5. The spectral decomposition here is an analytic device; it does not assume that the observer directly separates signal components according to this projection.

Neither $\kappa$ nor the spatial region $K$ introduces a fundamental spectral cutoff. When $\kappa(L-l) \lesssim 1$, the exponential factor may not provide the required suppression. High frequency also does not guarantee that an individual mode is resolvable: this depends on its amplitude, channel sensitivity, and tolerance. Thus, (124) is a sufficient approximation criterion, not a classification of physically admissible modes.

The analytic problem is still posed on the full slice $\Sigma \simeq \mathbb{R}^3$; no boundary conditions are imposed here on $\partial K$ or $\partial\Omega$. The boundary $\partial\Omega$ belongs to the large working domain of reconstruction. Its remoteness alone does not control the effect of unknown external data: the corresponding corrections must be estimated in the same channels and included in the overall error budget. In particular, (122) does not simultaneously bound all spatial boundary corrections.

Thus, $L-l$, $\operatorname{dist}(K,\partial\Omega)$, and $\kappa^{-1}$ characterize, respectively, slab depth, spatial localization, and a spectral scale. Their interrelation, as well as control of low-frequency and external corrections, is not tacitly included in the first main theorem.

### 5.3 An exact operational quotient requires invariance

Consider the exact decaying sector with trace space $X_{\mathfrak{C}} = H^r(\mathbb{R}^3)$ and transfer $P_h^E = \mathrm{e}^{-hA}$, $A = |D|$. On the slice $\Sigma_s$, exact indistinguishability under the protocol is defined by $N_{\Pi,s} = \bigcap_j \ker \ell_{j,s}$. Closure of the full field sector alone does not suffice for transfer to act on the quotient.

For $s_2 \geq s_1$, the induced map

$$\overline{P}^{\Pi}_{s_2,s_1} : X_{\mathfrak{C}}/N_{\Pi,s_1} \longrightarrow X_{\mathfrak{C}}/N_{\Pi,s_2}, \qquad [f]_{s_1} \longmapsto [P^E_{s_2-s_1} f]_{s_2}, \tag{125}$$

is well defined if and only if

$$P^E_{s_2-s_1} N_{\Pi,s_1} \subseteq N_{\Pi,s_2}. \tag{126}$$

Indeed, two representatives of the same initial class differ by an element of $N_{\Pi,s_1}$, and their images represent the same final class precisely when (126) holds. In the quotient norms, the resulting map is bounded with norm at most $\|P^E_{s_2-s_1}\| \leq 1$. If the condition holds for all pairs of slices, the induced maps compose consistently.

If $N_{\Pi,s} = N_\Pi$ is independent of $s$, in particular for a stationary channel family, the criterion takes its usual form

$$P_h^E N_\Pi \subseteq N_\Pi. \tag{127}$$

However, stationarity of field transfer ensures neither stationarity of the channels nor this invariance. For $X = \mathbb{R}^2$, $A = \operatorname{diag}(1,2)$, and $\ell(f) = f_1 + f_2$, the vector $v = (1,-1)$ belongs to $\ker \ell$, but

$$\ell(P_h^E v) = \mathrm{e}^{-h} - \mathrm{e}^{-2h} \neq 0, \qquad h > 0. \tag{128}$$

Thus, exact indistinguishability on one slice may fail to be preserved even under exact stable field transfer.

For a channel family defined on a finite interval $0 \leq s \leq s_*$, one may separately define the subspace of indistinguishability on all subsequent slices of this interval:

$$N^{\Pi}_{\mathrm{fut,s;s_*}} := \{ f \in X_{\mathfrak{C}} : \ P_h^E f \in N_{\Pi,s+h} \text{ for all } 0 \leq h \leq s_* - s \}. \tag{129}$$

Equivalently,

$$N^{\Pi}_{\mathrm{fut,s;s_*}} = \bigcap_{0 \leq h \leq s_* - s} \ \bigcap_{j=1}^{N_\Pi} \ker(\ell_{j,s+h} \circ P_h^E). \tag{130}$$

These subspaces are closed, and the semigroup property gives

$$P_a^E N^{\Pi}_{\mathrm{fut,s;s_*}} \subseteq N^{\Pi}_{\mathrm{fut,s+a;s_*}}, \qquad 0 \leq a \leq s_* - s. \tag{131}$$

Indeed, for $0 \leq h \leq s_* - s - a$ we have $P_h^E P_a^E f = P^E_{h+a} f \in N_{\Pi,s+a+h}$. If the channel family is independently extended to the entire half-line, one may similarly intersect over all $h \geq 0$; such an extension does not follow from finiteness of the physical protocol.

Neither the current quotient $X_{\mathfrak{C}}/N_{\Pi,s}$ nor the quotient by (129) is automatically identified with the internal observer's state, the event network, or the quantum state space. The second quotient includes subsequent readouts, and its representability by current registers requires separate justification. Here "subsequent" refers to the ordering of slices under the chosen direct transfer.

Finally, exact equality of linear readouts must be distinguished from $d_{\Pi,s}(f,g) \le 1$, which is generally nontransitive at finite precision. As shown in Section 2, this condition does not define a linear quotient. Error estimates are used in this regime; preservation of a discrete registration label also requires control of the nonlinear response and the threshold margin.

### 5.4 Euclidean transfer, observable time, and physical causality

For $A = |D|$ and $h > 0$, the operator $P_h^E$ has the Poisson kernel

$$p_h(y) = \frac{1}{\pi^2} \frac{h}{(h^2 + |y|^2)^2} > 0, \qquad y \in \mathbb{R}^3. \tag{132}$$

It therefore does not have finite propagation of spatial support. It describes elliptic extension of analytic data in the chosen harmonic sector, not retarded signal transmission. In particular, a mathematical variation of data is not by itself a physically realizable intervention by the internal observer.

At the fundamental Euclidean level there are no events or causal cone. In the inherited relativistic regime, they arise operationally relative to the observer [1]. Thus, the support of the elliptic extension kernel and causal accessibility of a signal concern different objects. Conclusions about signal transmission are determined by admissible operations and registrations consistent with the reconstructed causal structure, not by the spatial support of (132) alone.

In the corresponding IFR, interval calibration takes the form

$$h = s_2 - s_1 = \Delta x^0_{\mathrm{SR}} = v_t \Delta t. \tag{133}$$

It does not identify the operators

$$P_h^E = \mathrm{e}^{-hA}, \qquad U(\Delta t) = \exp\!\left(-\frac{\mathrm{i}}{\hbar} \Delta t H_{\mathrm{phys}}\right). \tag{134}$$

The first acts on analytic field traces; the second acts in the reconstructed quantum space.

For quantum reconstruction, a reference two-point form is constructed and characterized below, the associated Gaussian measure is constructed, and the hypotheses of Osterwalder–Schrader reconstruction are verified. The operational interpretation of the resulting Lorentzian group uses the same observable time as the relativistic sector. Matching the temporal calibration gives

$$\Delta x^0_{\mathrm{OS}} = \Delta x^0_{\mathrm{SR}} = v_t \Delta t. \tag{135}$$

Matching to a registration protocol includes both this temporal calibration and identification of the relevant observables and their statistics. The second condition is verified separately: agreement of parameters does not establish a correspondence between operators and register readings.

The spectral weight of the free reference form is characterized under two conditions: a consistent action of observable Lorentz transformations on the algebra of effective quantum observables, and invariance of the chosen reference state. This action concerns the quantum description, not direct $O(4)$ rotations of the fundamental configuration. In the scalar positive-frequency sector, these conditions fix the spectral weight independently of the canonical Green's-function definition.

**Lemma 5.1** (Conditional Lorentzian selection of the reference spectral weight)**.** *Suppose a free translation-invariant, spatially isotropic scalar reference form in the positive massless sector has, for $r = |k| > 0$, the form*

$$\widehat{C}_E(\tau_E; k) = q(r)\mathrm{e}^{-|\tau_E|r}, \qquad q(r) \geq 0, \tag{136}$$

*where $q$ is measurable, locally integrable away from zero, and defines a tempered form. Suppose its positive-frequency continuation*

$$W(x^0, \mathbf{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{(2\pi)^3}\, q(|k|)\mathrm{e}^{-\mathrm{i}|k|x^0 + \mathrm{i}k\cdot\mathbf{x}}, \qquad x^0 = v_t t, \tag{137}$$

*is the two-point correlation of one fixed reference state $\omega_0$ for an effective scalar field $\varphi$: $W(x) = \omega_0(\varphi(x)\varphi(0))$. Suppose further that the observable Lorentz transformations $M_\Lambda$ of the reconstructed sector are associated with a consistent action $\alpha_\Lambda$ of $SO^+(1,3)$ by $*$-automorphisms of the algebra of effective observables, acting on the scalar field as*

$$\alpha_\Lambda(\varphi(x)) = \varphi(\Lambda x). \tag{138}$$

*The equality for the field is understood distributionally, after smearing with test functions. This action concerns effective observables, not direct $O(4)$ rotations of the fundamental configuration $\Phi_0$. Finally, assume* invariance of the reference state itself*:*

$$\omega_0 \circ \alpha_\Lambda = \omega_0 \qquad (\Lambda \in SO^+(1,3)). \tag{139}$$

*In particular, these conditions imply invariance of its correlation as a distribution:*

$$W(\Lambda x) = W(x) \qquad \text{for all } \Lambda \in SO^+(1,3), \tag{140}$$

*Then, almost everywhere for $r > 0$,*

$$q(r) = \frac{c}{2r}, \quad c \geq 0, \qquad \widetilde{C}_E(p_0, k) = \frac{c}{p_0^2 + |k|^2}. \tag{141}$$

*For a nontrivial nonzero sector, $c > 0$. Possible distributional terms supported at zero momentum are not fixed by this statement.*

*Proof.* By (138) and (139),

$$W(\Lambda x) = \omega_0(\alpha_\Lambda(\varphi(x)\varphi(0))) = \omega_0(\varphi(x)\varphi(0)) = W(x).$$

Here $\Lambda(0) = 0$ is used; no transformation of the fundamental field is performed. The spectral measure in (137) is supported on the positive massless shell $(p^0, k) = (r, k)$, $r = |k| > 0$. Relation (140) and uniqueness of the Fourier transform imply invariance of this measure under proper orthochronous Lorentz transformations. For a boost taking $(r, k)$ to $(r', k')$, the standard Jacobian relation is

$$\mathrm{d}^3k' = \frac{r'}{r}\, \mathrm{d}^3k,$$

since $\mathrm{d}^3k/(2r)$ is the invariant measure on the positive null shell. Invariance of $q(r)\,\mathrm{d}^3k$ therefore gives $r'q(r') = rq(r)$ almost everywhere. Rotations and boosts act transitively on the nonzero positive shell, so $rq(r) = c/2$ almost everywhere; nonnegativity gives $c \geq 0$. The inverse Fourier transform in $\tau_E$ in (136) has multiplier $2rq(r)/(p_0^2 + r^2)$, proving (141). □

If $q \in C^1(0,\infty)$, the same conclusion is seen directly from an infinitesimal boost in the induced action on effective momenta along the $z$-axis, with small parameter $\epsilon_{\mathrm{b}}$: $r' = r + \epsilon_{\mathrm{b}} k_z + O(\epsilon_{\mathrm{b}}^2)$ and $\mathrm{d}^3k' = (1 + \epsilon_{\mathrm{b}} k_z/r)\,\mathrm{d}^3k + O(\epsilon_{\mathrm{b}}^2)$. Invariance of the spectral measure then gives

$$q'(r) + \frac{q(r)}{r} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}r}(rq(r)) = 0.$$

This is an additional way to verify the lemma, not a smoothness requirement in its main formulation.

The consistent action (138) on the quantum algebra and invariance (139) of one reference state are additional hypotheses beyond the Lorentz form of observable transformations established in [1]. Covariance of a family allows joint transformation of the state and observables; the lemma uses invariance of $\omega_0$ itself. This property is not required of the state of a localized observer and its apparatus. When the lemma is applied to registration statistics, one verifies the correspondence between the effective action and operational re-descriptions, and the invariance of the reference preparation. The independent Euclidean construction through the Green's function does not use this route.

### 5.5 The growing inverse operator and the cosmological context

On the exact decaying sector, $P_L^E = \mathrm{e}^{-LA}$ is injective, and the spectral functional calculus gives

$$(P_L^E)^{-1} = \mathrm{e}^{LA}, \qquad D\Big((P_L^E)^{-1}\Big) = \operatorname{Ran} P_L^E = D(\mathrm{e}^{LA}). \tag{142}$$

Indeed, if $g = \mathrm{e}^{-LA} f$, then $g \in D(\mathrm{e}^{LA})$ and $\mathrm{e}^{LA} g = f$; conversely, for $g \in D(\mathrm{e}^{LA})$, the vector $f = \mathrm{e}^{LA} g$ satisfies $\mathrm{e}^{-LA} f = g$. For unbounded nonnegative $A$, the inverse operator is unbounded in the original norm.

Here $\mathrm{e}^{LA}$ recovers data from their Poisson image within the already selected sector. In the full Cauchy problem, the same spectral multiplier described the other branch of direct extension. Equality of the multipliers does not identify these problems: existence of an unbounded inverse does not require the full growing branch to be included in a stable direct law.

Perturbed data may fail to belong to $\operatorname{Ran} P_L^E$. To estimate error propagation, we therefore restrict inverse recovery to the spectrum $0 \le A \le \mu$. For $P_{\le\mu} := E_A([0,\mu])$ and an error $\xi \in H^r$, we obtain

$$\|\mathrm{e}^{LA} P_{\le\mu} \xi\|_{H^r} \le \mathrm{e}^{L\mu} \|\xi\|_{H^r}. \tag{143}$$

This operator is bounded on all of $H^r$; the estimate follows from $\sup_{0\le\lambda\le\mu} \mathrm{e}^{L\lambda} = \mathrm{e}^{L\mu}$. For predetermined $0 < \delta < \delta_*$, the condition

$$\delta \mathrm{e}^{L\mu} \le \delta_* \tag{144}$$

suffices to bound the amplified data error by $\delta_*$ whenever $\|\xi\|_{H^r} \le \delta$. It is equivalent to

$$\mu \le \lambda_{\max}(L) := \frac{1}{L} \log \frac{\delta_*}{\delta}. \tag{145}$$

For fixed $\delta$ and $\delta_*$, this upper bound decreases as $L$ increases. It characterizes the stability of recovery within the chosen range, not the total reconstruction error: discarding high frequencies adds the remainder $(I - P_{\le\mu})f$ to the data error, and this requires a separate estimate.

Unlike the lower bound $\kappa$ in the finite-slab theorem, $\lambda_{\max}(L)$ bounds the spectrum from above in the inverse problem. These parameters belong to different estimates and do not by themselves specify fundamental spectral bounds. Likewise, the Euclidean extension parameter $L$ is not automatically identified with an observable cosmological distance.

The relation of this instability to an operational horizon and a cosmological contribution is studied in [6] under additional conditions on remote reconstruction. Here we use only estimate (145), which characterizes the stability of inverse continuation.

Finally, replacing $n \mapsto -n$ reindexes the slices by $s \mapsto -s$ and exchanges the roles of the extension directions. It does not turn the same solution with a nonzero positive spectral component into one that decays on both half-lines. Choosing the orientation of direct transfer does not by itself introduce a fundamental arrow of time; its relation to the observable ordering of registrations belongs to operational reconstruction.

# 6 The canonical free two-point structure

## 6.1 The reference form as the Green's function of the Laplacian

The quadratic form in a real variational function $\vartheta$,

$$\mathcal{E}(\vartheta) = \int_{\mathbb{E}^4} |\nabla\vartheta|^2 \, \mathrm{d}^4x, \tag{146}$$

is canonically associated with the fundamental equation (1); its Euler–Lagrange equation is $\Delta_{\mathbb{E}^4}\vartheta = 0$. The quadratic form (146) is canonically associated with the fundamental Laplace operator; using it does not add a fundamental field.

Define the reference two-point form using the Green's function of the Laplacian associated with this Dirichlet form:

$$(-\Delta_{\mathbb{E}^4})C = c\,\delta, \qquad \widetilde{C}_E(p) = \frac{c}{|p|^2}, \qquad c > 0. \tag{147}$$

Here and below, $\widetilde{C}_E(p)$ denotes the Fourier multiplier of the two-point-form operator. The relation between the inverse Laplacian and the covariance of the Gaussian free field is standard [21]. In the present setting it is used to define the reference structure; this structure is then characterized within the class of forms compatible with inter-slice transfer.

With this definition, equal-slice regularity, positivity and spatial isotropy of the equal-slice form, the one-argument transfer law, and $O(4)$ invariance—that is, all the hypotheses of Theorem 2 below—become consequences. Form (146) is translation-invariant and $O(4)$-invariant and contains no normal $n$; the independent status of this condition is discussed in Section 6.3. Its positivity is immediate. The equal-slice restriction and inter-slice transfer are computed as

$$\int_{\mathbb{R}} \frac{\mathrm{d}p_0}{2\pi} \, \frac{c\,\mathrm{e}^{\mathrm{i}p_0\tau_E}}{p_0^2 + |k|^2} = \frac{c}{2|k|}\,\mathrm{e}^{-|\tau_E||k|}, \tag{148}$$

which gives $Q = c(2|D|)^{-1}$; the one-argument transfer law formulated below in (150) holds identically for $P_h^E = \mathrm{e}^{-h|D|}$. Local integrability of $|p|^{-2}$ in four dimensions, the absence of contact terms, and the status of the normalization $c$ are discussed separately in Section 6.5.

The Green's function is specified at the level of the quadratic form and inverse operator. The associated Gaussian measure is constructed next, in Section 7. Canonicity here refers to full Euclidean space; on a bounded domain, the Green's function also depends on the chosen boundary conditions (Appendix B).

Theorem 2 characterizes the canonical representative within the stated class. The essential roles of transfer compatibility and $O(4)$ invariance are demonstrated in Sections 6.2 and 6.3. Conditions for applying the constructed reference form to registration statistics are formulated in Section 9.2.

### 6.2 Freedom in the two-point form at fixed transfer

We examine the freedom to choose a two-point form for a given inter-slice transfer before fixing the canonical representative (147). The next subsection considers consistency of a family of such forms under changes of foliation.

Consider the exact flat reference model $P_h^E = \mathrm{e}^{-h|D|}$. Using it after finite-slab reduction means choosing a mathematical reference model with a separately estimated application error. The local approximate theorem does not by itself specify a global four-dimensional two-point form.

Before probability is introduced, $C(s, s')$ is understood as a bilinear kernel, not as an already known covariance of a random fundamental field. Having $P_h^E$ does not yet determine which two-point structure is physically required. For example, for any positive equal-slice form $Q$, the expression

$$C_{\mathrm{det}}(s, s') = P_s^E Q (P_{s'}^E)^*, \qquad s, s' \geq 0, \tag{149}$$

is consistent with transferring both arguments from one reference slice. It is generally nonstationary and differs from $Q\mathrm{e}^{-|s-s'|A}$.

Passing from the two-argument notation $C(s, s')$ to notation in terms of a single difference $C(\tau_E)$ itself assumes stationarity of the reference form, and is used below in precisely this sense. The one-argument transfer condition for a stationary reference form must therefore be stated separately:

$$C(\tau_E + h) = P_h^E C(\tau_E), \qquad \tau_E, h \geq 0. \tag{150}$$

This is a compatibility condition between transfer and the two-point structure. The canonical form (147) satisfies it identically; in other reconstructions the law must be verified.

### 6.3 Symmetries of the law and selection of the reference form

The canonical reference form (147) is $O(4)$-invariant and singles out no foliation normal. In characterizing reference forms generally, this property must be distinguished from consistency of their family under changes of foliation. Euclidean symmetry of the fundamental equation preserves its solution class but does not force every solution to be invariant. Similarly, a consistent transformation of a family of reference forms,

$$\widetilde{C}_{E,Rn}(Rp) = \widetilde{C}_{E,n}(p) \tag{151}$$

is not equivalent to independence of $\widetilde{C}_{E,n}$ from $n$.

For example, set $p_\perp^{(n)} = p - (n \cdot p)n$. For any admissible nonnegative function $q$, the family

$$\widetilde{C}_{E,n}(p) = \frac{2|p_\perp^{(n)}|\, q(|p_\perp^{(n)}|)}{|p|^2} \tag{152}$$

satisfies (151), although for general $q$ it depends on the chosen normal. Thus, passing from consistency under changes of foliation to full $O(4)$ invariance of a single reference form requires an additional condition: the absence of data that single out $n$.

Without the canonical definition (147), this requirement must be introduced as a specification of a *free reference sector with Euclidean invariance*. It is not asserted for the actual configuration, for every observer, or for all reconstruction classes. A fixed spatial spectral band is likewise not invariant under rotations mixing normal and tangential coordinates. The exact symmetry theorem is therefore formulated in an ideal four-dimensional reference sector, not for an arbitrary local spectral truncation.

Condition (151) has the same logical form as the foliation-independence condition on the transfer law in the operational reconstruction of special relativity [1]: there it is written as a

similarity transformation of matrix representations of transfer under changes of foliation, and the resulting equivalence of inertial frames is explicitly attributed to the *form of the law*, not to the data. The family (152) satisfies that condition fully: the reference form in each frame is obtained by rotating the one in a neighboring frame. Operational equivalence of frames alone therefore does not select the reference form.

Comparison with the derivation of invariance of $v_{\max}$ in the same paper clarifies the limits of this argument. There, invariance follows from physical equivalence of all directions in $\mathbb{E}^4$, and the argument is valid because $v_{\max}$ is a scalar: an equivariant assignment of a number to each normal is constant when $SO(4)$ acts transitively on unit vectors. The argument does not carry over to a two-point form: equivariance leaves dependence on $n$ through $p_\perp^{(n)}$, exactly as realized by (152).

Reference [1] itself notes that the foliation-independence condition is consistent with $O(4)$ symmetry of the equation but does not imply it, and that constructing a full theory may require further conditions. Item (iv) of Theorem 2 is such a condition. Under definition (147), it holds exactly. Lemma 5.1 provides an alternative characterization: a consistent action of observable Lorentz transformations on the effective quantum algebra, together with invariance of *one* free reference state under that action, yields the same spectral weight. Covariance of a family of states or the time normalization (135) alone does not establish this invariance.

### 6.4 Uniqueness of the free two-point form

All Fourier transforms of test functions below use unitary normalization. As above, $\widetilde{C}_E(p)$ is the Fourier multiplier of the two-point-form operator; the inverse transform of the scalar kernel in the slice difference uses the measure $\mathrm{d}p_0/(2\pi)$. Here $\tau_E = s - s'$ is the Euclidean difference of slice parameters introduced in Section 1; see (4).

**Theorem 2** (Uniqueness of the Euclidean-invariant free form)**.** *Consider a real symmetric translation-invariant two-point form $C$ in flat four-dimensional space. Suppose its reference sector has the following properties.*

*(i) As a function of the slice difference $\tau_E$, the form has a weakly continuous equal-slice representation on spatial test functions. Its value at $\tau_E = 0$ is denoted by $Q$.*

*(ii) $Q$ is a positive, spatially homogeneous and isotropic form. Away from $k = 0$, it has a Fourier multiplier $q(|k|) \geq 0$ that is locally integrable and admits the required tempered extension.*

*(iii) For $\tau_E, h \geq 0$, law (150) holds with $P_h^E = \mathrm{e}^{-h|D|}$, initially on tests with Fourier support away from zero. Its independence from the mere existence of the semigroup is shown in Section 6.2.*

*(iv) The four-dimensional form is $O(4)$-invariant. Section 6.3 shows that this condition does not follow from consistency of a family under changes of foliation; its relation to the matching condition* (135) *is discussed in Section 5.4.*

*Then, away from zero momentum,*

$$\widetilde{C}_E(p_0, k) = \frac{c}{p_0^2 + |k|^2}, \qquad c \geq 0, \tag{153}$$

*and*

$$C(\tau_E) = \frac{c}{2|D|}\mathrm{e}^{-|\tau_E||D|}. \tag{154}$$

*A nontrivial form on the nonzero sector has $c > 0$. Terms supported entirely at zero momentum are not fixed by this statement.*

*Proof.* The weak equal-slice limit and transfer law imply $C(\tau_E) = P_{\tau_E}^E Q$ for $\tau_E \geq 0$. Spatial

isotropy makes $Q$ a function of $|D|$; symmetry gives, for all $\tau_E$,

$$\widehat{C}_E(\tau_E; k) = q(|k|)\mathrm{e}^{-|\tau_E||k|}. \tag{155}$$

The transform in $\tau_E$ for $r = |k| > 0$ is

$$\widetilde{C}_E(p_0, k) = \frac{2rq(r)}{p_0^2 + r^2}. \tag{156}$$

Let $r_1, r_2 > 0$. Choose $R > \max(r_1, r_2)$ and two four-dimensional momenta of length $R$ whose tangential lengths are $r_1$ and $r_2$. They are related by a Euclidean rotation. Applying $O(4)$ invariance of the form to (156) yields $2r_1q(r_1) = 2r_2q(r_2)$, with the usual almost-everywhere interpretation for measurable multipliers. Hence $2rq(r) = c$, giving (153) and (154). Nonnegativity of $c$ follows from positivity of $Q$. □

The canonical definition (147) satisfies all the hypotheses of the theorem. The constructed free reference form is thus characterized uniquely within the stated class, up to overall normalization and the separately considered zero components. The examples in Sections 6.2 and 6.3 establish the essential roles of compatibility and invariance. Applying the reference form to a specific registration protocol requires relating its correlations to the statistics of accessible registrations; the conditions for this identification are distinct from the present analytic theorem and are discussed in Section 9.

Zero-momentum additions to the distributional two-point kernel at $p = 0$ must be distinguished from the zero eigenspace $\ker A$ in the Cauchy problem. For $A = |D|$ on $H^r(\mathbb{R}^3)$, this eigenspace is trivial, whereas distributional additions supported at zero four-momentum belong to a different space of objects and are considered separately in Appendix B.

### 6.5 Existence, contact terms, the Green identity, and normalization

The form (153) with $c > 0$ indeed exists on $\mathscr{S}(\mathbb{R}^4)$ and has the properties stated in the theorem. In this dimension, $|p|^{-2}$ is locally integrable:

$$\int_{|p|<1} \frac{\mathrm{d}^4 p}{|p|^2} < \infty. \tag{157}$$

Likewise, $|k|^{-1}$ is locally integrable on $\mathbb{R}^3$. Thus, for smooth rapidly decreasing tests, there is no automatic need to remove the point $k = 0$. The situation is different on a compact domain with a discrete zero eigenmode, or for other test spaces; this is considered in Appendix B.

Weak continuity of $C(\tau_E; f, g)$ in $\tau_E$ after spatial smearing excludes terms proportional to $\delta(\tau_E)$ and its derivatives. It does not imply finiteness of the pointwise value $C(x, x)$. The absence of contact terms follows from the stated class of reference forms independently of the Green identity considered next.

For the resulting form, direct calculation gives

$$(-\partial_{\tau_E}^2 + A^2)C(\tau_E) = c\,\delta(\tau_E)I. \tag{158}$$

The derivative of $C$ has a jump of $-cI$ at zero. Thus, the two-point kernel is a Green's function, not a solution of the homogeneous equation at coincident arguments. Since $\Delta_{\mathbb{E}^4} = \partial_s^2 - A^2$ and $\partial_s$ acts as $\partial_{\tau_E}$ on a translation-invariant kernel, identity (158) is the defining relation (147) written in terms of the slice difference. It therefore agrees with the definition of the reference form in Section 6.1.

The number $c > 0$ specifies the overall normalization of the quadratic form. The theorem determines its spectral dependence up to this factor. Field normalization and energy calibration are distinguished in Section 7.3.

# 7 Reduction to standard free quantum reconstruction

## 7.1 Positivity and reflection positivity

For the chosen nontrivial reference kernel,

$$C(F,F) = c\int_{\mathbb{R}^4} \frac{|\widehat{F}(p)|^2}{|p|^2}\,\mathrm{d}^4p \geq 0, \qquad F \in \mathscr{S}(\mathbb{R}^4). \tag{159}$$

Integrability at zero follows from (157); at infinity, rapid decay of the tests suffices. Splitting the integral into $|p| \leq 1$ and $|p| > 1$ also directly establishes continuity of the form in the Schwartz topology.

The reflection $\theta$ sends $s$ to $-s$. For tests supported in $s > 0$, the reflected quadratic form is

$$C(\theta F, F) = \int_{\mathbb{R}^3} \frac{c}{2|k|}\left|\int_0^\infty \mathrm{e}^{-s|k|}\widehat{F}(s,k)\,\mathrm{d}s\right|^2 \mathrm{d}^3k \geq 0. \tag{160}$$

The same statement in operator notation on finite sets of spatial tests, with $Q$ the equal-slice form of Theorem 2, is

$$\sum_{i,j}\langle f_i, Q\mathrm{e}^{-(s_i+s_j)A}f_j\rangle = \left\|\sum_i Q^{1/2}\mathrm{e}^{-s_iA}f_i\right\|^2 \geq 0. \tag{161}$$

Reflection positivity of the constructed kernel follows from (161) and is not introduced as an additional hypothesis.

The scope of this calculation should be noted. Identity (161) uses only nonnegativity of $Q$ and its commutation with $A$; the specific form of the multiplier $q$ does not enter. Reflection positivity therefore holds throughout the family of reference forms satisfying hypotheses (i)–(iii) of Theorem 2 and does not by itself select the reference form. The implications for that choice are discussed in Section 5.4.

If physical correlators are known only approximately, their reflection positivity does not follow automatically from closeness to (160): a nonnegative form may have null directions in which a small correction changes its sign. Exact positivity is established for the reference kernel; comparison with an actual protocol requires separate error control.

## 7.2 Construction of the Gaussian measure

On the real nuclear space $\mathscr{S}(\mathbb{R}^4)$, define

$$\mathcal{Z}_C(F) = \exp[-\tfrac{1}{2}C(F,F)]. \tag{162}$$

Continuity and positivity of $C$ ensure positive definiteness of $\mathcal{Z}_C$ and $\mathcal{Z}_C(0) = 1$. The Bochner–Minlos theorem yields a unique centered Gaussian probability measure $\mu_C$ on $\mathscr{S}'(\mathbb{R}^4)$ with characteristic functional (162) [20, 17].

Uniqueness concerns the centered *Gaussian* measure with the prescribed characteristic functional. Its higher correlations are specified by Wick's rule; a two-point form alone does not determine them for an arbitrary state. For example, multiplying an auxiliary Gaussian field by an independent scalar random variable $\zeta$ with $\mathsf{E}\zeta^2 = 1$ preserves the covariance but generally changes the fourth moments. The Gaussian measure construction is therefore a precisely specified free construction, not a proof that all admissible states of the model are Gaussian.

The measure $\mu_C$ is not interpreted as a collection of actually existing fundamental fields $\Phi_0$. Its generalized samples $\varphi_{\mathrm{G}}$ are not required to solve $\Delta\varphi_{\mathrm{G}} = 0$ either: the covariance satisfies the inhomogeneous identity (158). This is a mathematical representation of the free reference structure. Conditions for relating it to registration statistics are formulated in Section 9.2.

### 7.3 OS reconstruction and the free state space

The Gaussian measure determines the Schwinger hierarchy by Wick's rule. For the exact kernel $c/p^2$, Euclidean invariance, permutation symmetry, temperedness, reflection positivity, and the standard free-field growth conditions hold. In the absence of a separate constant zero component (Appendix B), the cluster property also holds. Hence the Osterwalder–Schrader theorems and standard free-field reconstruction [10, 11, 12, 13] apply. Euclidean invariance of the hierarchy is verified for the chosen kernel; the basis for choosing it is explained in Sections 6.3 and 5.4.

The one-particle space is obtained by completing spatial tests in the form

$$\|f\|_1^2 = c \int_{\mathbb{R}^3} \frac{|\widehat{f}(k)|^2}{2|k|}\, \mathrm{d}^3k. \tag{163}$$

The full free space is the symmetric Fock space over its complex completion. The one-particle generator is represented by $A = |D|$, whereas the full free Hamiltonian is the second-quantized operator $H = \mathrm{d}\Gamma(A)$. These two generators are not identified. The unitary group is $\mathrm{e}^{-\mathrm{i}x^0 H}$, where $x^0$ is the length coordinate along the reconstructed time direction. Normalization with respect to observable time $t = x^0/v_t$ is specified in (166), namely $H_{\mathrm{phys}} = \hbar v_t H$. The complex structure, canonical commutation relations (CCR), and free microcausal Lorentzian realization are standard consequences of reconstruction here, not structures postulated anew.

For a positive Euclidean increment $\tau_E$, the direct and observable levels must be distinguished:

$$\boxed{P^E_{\tau_E} = \mathrm{e}^{-\tau_E A}} \qquad \text{and} \qquad \boxed{U(t) = \exp\!\left(-\frac{\mathrm{i}}{\hbar} t H_{\mathrm{phys}}\right)}. \tag{164}$$

The first operator describes direct Euclidean transfer between slices; the second describes observable unitary dynamics in the already reconstructed Lorentzian quantum sector. They are not the same operator with a differently named parameter.

Osterwalder–Schrader reconstruction relates the Euclidean correlation structure to a Lorentzian quantum representation. However, the physical interpretation of the resulting unitary group's parameter must be matched to the time already reconstructed in the relativistic sector. In length coordinates, this condition is

$$x^0_{\mathrm{OS}} = x^0_{\mathrm{SR}} = v_t t. \tag{165}$$

Thus, OS reconstruction introduces no independent second physical time. It must yield quantum dynamics with respect to the same observable Lorentzian temporal structure that previously emerged from operational reconstruction of special relativity.

Fix a reconstruction class $\mathfrak{C}$ and introduce a positive coefficient $\hbar_{\mathfrak{C}}$ with the dimension of action, relating the reconstructed time-translation generator to the physical energy normalization of this class. For brevity, write $\hbar = \hbar_{\mathfrak{C}}$ below. For the massless free sector with

$$A = |D|$$

and a common energy normalization, its physical generator is

$$H_{\mathrm{phys}} = \hbar v_t\ \mathrm{d}\Gamma(|D|), \tag{166}$$

when $s$ and $x^0$ are measured in units of length. In particular, the one-particle dispersion relation takes the form

$$E(\mathbf{k}) = \hbar v_t |\mathbf{k}| = \hbar v_{\max} |\mathbf{k}|. \tag{167}$$

Condition (165) must be distinguished from identifying direct Euclidean transfer with observable time evolution. It concerns matching two reconstructed Lorentzian descriptions—the

relativistic and quantum descriptions. The full operational bridge from the observer's internal registration protocols to observable operators and POVMs on the reconstructed Hilbert space is treated separately and is not tacitly included in OS reconstruction.

Normalization of the reference two-point form and energy calibration play different roles. The factor $c$ in (163) normalizes the quadratic form and can be absorbed by rescaling the free field. The operator $v_t\,\mathrm{d}\Gamma(A)$ generates time translations and has the dimension of inverse time, while $\hbar_{\mathfrak{C}}$ converts this generator into energy units. In the free sector considered here, the coefficient is the same for all modes and is not fixed by $c$. Its value is a reconstruction-class parameter to be identified operationally. For the class describing our observable physics, this parameter is identified with the reduced Planck constant. Universality of this calibration beyond the present free scalar sector, in particular for interacting fields, is not established here. This status of the normalization is consistent with the preceding studies of the quantum core and reconstruction classes [3, 4].

The general GNS construction for a positive functional must not be identified with the entirety of OS reconstruction: the latter requires a complete Euclidean hierarchy and its associated conditions. Here, these are verified for the free hierarchy of the Gaussian measure constructed here.

### 7.4 Markov realization after construction of the covariance

The Markov property is derived after construction of the Gaussian measure and is used neither in Theorem 2 nor in constructing the measure. Let $X_s(f)$ be the generalized Gaussian field on a slice, and let $\mathcal{F}_s$ be generated by all its smearings on slices $s' \le s$. All expressions are initially understood on tests with Fourier support away from zero, and are then extended in the corresponding covariance space.

For $h \ge 0$, set

$$\xi_{s,h}(f) = X_{s+h}(f) - X_s(P_h^E f). \tag{168}$$

Write $Q = c(2A)^{-1}$, so that $C(\tau_E) = Q\mathrm{e}^{-|\tau_E|A}$, with $Q$ commuting with functions of $A$. For $s' \le s$ and $h \ge 0$, one has $s + h - s' \ge 0$, so

$$\mathrm{Cov}(X_{s+h}(f), X_{s'}(g)) = \langle f, Q\mathrm{e}^{-(s+h-s')A} g\rangle, \tag{169}$$

while self-adjointness of $\mathrm{e}^{-hA}$ and its commutation with $Q$ give

$$\mathrm{Cov}(X_s(P_h^E f), X_{s'}(g)) = \langle \mathrm{e}^{-hA} f, Q\mathrm{e}^{-(s-s')A} g\rangle = \langle f, Q\mathrm{e}^{-(s+h-s')A} g\rangle. \tag{170}$$

Consequently, $\mathrm{Cov}(\xi_{s,h}(f), X_{s'}(g)) = 0$ for all $s' \le s$. Only three properties have been used: nonnegativity of $s + h - s'$, self-adjointness of $\mathrm{e}^{-hA}$, and commutation of $Q$ with functions of $A$. The first is precisely where the restriction $s' \le s$ enters: for $s' > s + h$, the absolute value in $\mathrm{e}^{-|s+h-s'|A}$ cannot be removed, and equality (169) does not have the stated form. Joint Gaussianity turns orthogonality into independence from $\mathcal{F}_s$. Therefore,

$$\mathsf{E}[X_{s+h}(f) \mid \mathcal{F}_s] = X_s(P_h^E f). \tag{171}$$

The conditional covariance is

$$N_h = Q - P_h^E Q (P_h^E)^* = \frac{c}{2A}(I - \mathrm{e}^{-2hA}) \ge 0. \tag{172}$$

This gives a stationary Gaussian Markov realization [18, 21]. It does not imply $X_{s+h} = P_h^E X_s$ for every sample: the nonzero term (168) is necessary for a stationary nonzero variance.

This proof concerns the full Gaussian slice field. Projection onto a finite set of signals or registers generally destroys the Markov property. Thus, (171) must not be transferred automatically to an arbitrary operational state of the observer or derived from consistency of causal ordering alone. A related remark on nonstationarity of effective variables upon passing to an observer-dependent representation is given in Appendix C.

# 8 Operational measurements and affine statistics

Passing from the free Gaussian structure to measurements requires a relation between the quantum state and probabilities of register readings. The conditions under which such a relation admits a POVM representation are formulated below.

## 8.1 Registers and probabilities

For a fixed protocol $\Pi$, distinguishable stable register states form the outcome space. On a finite partition with $m_\Pi$ distinguishable outcomes, it defines a commutative algebra $\mathcal{C}_\Pi \simeq \mathbb{C}^{m_\Pi}$. The number $m_\Pi$ refers to protocol outcomes and is distinct from the number $N_\Pi$ of linear channels in Section 2. This is not a global observer-independent event space. A general measurable structure is used only under the corresponding mathematical idealization of the protocol.

A positive normalized functional on such an algebra gives outcome probabilities. For a commutative unital $C^*$-algebra, the Riesz–Markov representation on its compact spectrum applies; on a measurable algebra, countable additivity requires the corresponding normality condition [19]. On a finite algebra, this reduces to finite additivity.

Relating the register algebra to the reconstructed quantum algebra requires a suitable embedding or positive map. A measurable function of the Euclidean Gaussian field defines a classical distribution; representing it by an operator in the OS space requires additional justification. Likewise, the Markov property of the full slice field is generally lost under projection onto finite registers (Section 7.4).

Applying Proposition 8.1 to an apparatus presupposes stable, distinguishable register states and the hierarchy (31). Such a construction belongs to an effective interacting regime: retaining a record over a finite interval requires control of the harmonic-representation error relative to readout margins. The absence of strictly $s$-independent global free $H^r$ profiles does not exclude this regime (Section 9.3). Here registers are taken as operational data of the protocol; their physical construction is not part of the result on representation of affine statistics.

## 8.2 Standard representation of affine statistics

The following proposition provides a POVM representation of probabilities that depend affinely on a normal quantum state.

**Proposition 8.1** (POVM representation of statistics)**.** *Let $\mathcal{H}_\mathrm{q}$ be the reconstructed quantum Hilbert space, and let $(Y_\Pi, \mathscr{B}_\Pi)$ be the protocol's outcome space with its $\sigma$-algebra. Suppose each normal state $\rho$ on $\mathcal{H}_\mathrm{q}$ is assigned a probability measure $p^\Pi_\rho$, and, for each $\mathsf{B} \in \mathscr{B}_\Pi$, the map $\rho \mapsto p^\Pi_\rho(\mathsf{B})$ is affine. Then there is a unique POVM $E_\Pi$ such that*

$$p^\Pi_\rho(\mathsf{B}) = \mathrm{Tr}(\rho E_\Pi(\mathsf{B})), \qquad 0 \le E_\Pi(\mathsf{B}) \le I, \qquad E_\Pi(Y_\Pi) = I. \tag{173}$$

*Proof.* For a positive trace-class operator $T$, $T \neq 0$, set $F_\mathsf{B}(T) = \mathrm{Tr}(T)\, p^\Pi_{T/\mathrm{Tr}(T)}(\mathsf{B})$ and $F_\mathsf{B}(0) = 0$. Affinity gives additivity on the positive cone; $0 \le F_\mathsf{B}(T) \le \mathrm{Tr}(T)$ ensures a bounded linear extension to the whole trace class by decomposing a self-adjoint trace-class operator into its positive and negative parts. Trace-class duality, $\mathcal{T}_1(\mathcal{H}_\mathrm{q})^* = \mathcal{B}(\mathcal{H}_\mathrm{q})$, yields the effect $E_\Pi(\mathsf{B})$. Normalization and countable additivity of all the scalar measures yield normalization and weak-operator countable additivity of $E_\Pi$ [14, 19]. □

For a pure state, one obtains the usual expression $\langle \psi, E_\Pi(\mathsf{B})\psi\rangle$. Applying the proposition to an actual protocol requires its statistics to depend only on $\rho$ and to be defined on all normal states in a way that preserves convex mixing.

Classical mixing of preparations ensures affinity if the map from preparations to $\rho$ preserves mixtures and procedures with the same $\rho$ are indistinguishable under the protocol. For statistics defined only on a subset of states, existence and uniqueness of an operator extension require separate verification. Identification of preparations with quantum states is among the application conditions in Section 9.2.

# 9 Structure of the proofs and conditions of applicability

The model-specific part of the argument has two main branches. The first establishes reduced Euclidean transfer:

$$\begin{gathered}\text{full Cauchy data and the harmonic equation}\\ \Downarrow\\ \text{spectral decomposition and quantitative control of the second branch}\\ \Downarrow\\ \text{approximate reduction on a finite slab}\\ \Downarrow\\ \text{stronger growth condition: an exact stable graph}\\ \Downarrow\\ P_h^E = \mathrm{e}^{-hA}.\end{gathered} \tag{174}$$

For the two-point structure, construction must be distinguished from characterization. The canonical representative is specified by the Green's function of the fundamental Laplacian, after which its equal-slice, positivity, and symmetry properties are verified (Section 6.1). Theorem 2 establishes its uniqueness within the stated class of reference forms. The following scheme concerns this characterization part of the argument.

$$\left.\begin{gathered}P_h^E = \mathrm{e}^{-hA},\\ \text{positivity},\\ \text{homogeneity and isotropy},\\ \text{compatibility with inter-slice transfer},\\ O(4)\text{ invariance}\end{gathered}\right\} \Longrightarrow C \Longrightarrow \text{reflection positivity}. \tag{175}$$

Standard results on Gaussian measure construction and Osterwalder–Schrader reconstruction are then applied, yielding a Hilbert space, a positive generator, and a Lorentzian unitary group.

For exact Poisson transfer, one also has

$$\|P_h^E\| \le 1$$

for $h \ge 0$. This contractivity concerns Euclidean extension, not observable dynamics.

## 9.1 Hypotheses and established results

The initial conditions include equation (1) and the operational framework for the internal observer. The working spaces, spectral projections, and free reference form specify the analytic classes to which the results apply. The table lists the hypotheses and scope of each application.

| Statement | Mathematical status |
|---|---|
| Full data $(f, \pi)$ | The initial second-order analytic formulation; single-slice completeness is not assumed. |
| Two spectral branches | The solution is exact on bounded spectral bands; domains are controlled on the full spectrum. |
| Approximate graph | Estimate (80) and Corollary 4.2 on the thickness of the admissible layer of normal data are proved. Criterion (93) relates the bound $M_L$, channel sensitivity, and readout tolerances. |
| Exact reduction | Corollary 4.3: exact transfer on the positive spectral subspace under subexponential growth. The zero component is treated separately; the graph is understood in the specified Cauchy-data scale, with the domain of the derivative taken into account. |
| Linearity and stationarity | Established for transfer on the exactly selected sector. On a finite slab, an exact decomposition with a quantitatively bounded remainder is obtained. |
| Operational quotient | The necessary and sufficient criterion (126) for well-defined transfer on the exact quotient is established. It must be verified for the chosen channels. |
| Canonical reference form | Constructed as the Green's function (147); it satisfies all the hypotheses of Theorem 2. Matching to registration statistics is a condition of application (Section 9.2). |
| The form $c/p^2$ | Theorem 2: uniqueness up to normalization and zero components within the stated class. The canonical representative satisfies all its conditions. |
| $O(4)$ invariance of the reference form | Proved for the canonical Green's function. Lemma 5.1 yields the same weight under a consistent Lorentz action on the effective algebra and invariance of one reference state; these hypotheses differ from covariance of re-descriptions alone. |
| Reflection positivity | Proved for the canonical kernel using (161). The same calculation applies to the whole family satisfying hypotheses (i)–(iii), so positivity does not replace the symmetry condition in characterizing the form. |
| Gaussian measure and OS | The unique centered Gaussian measure associated with the reference form is constructed; the conditions for free OS reconstruction are verified for its hierarchy. A Fock space, a positive generator, and a unitary group are obtained (Section 7.3). |
| Energy-normalization parameter | $\hbar_{\mathfrak{C}} > 0$ is the energy-normalization parameter of class $\mathfrak{C}$ in (166), common to the modes of the sector considered and distinct from the two-point-form normalization $c$. The scope of this calibration is specified in Section 7.3. |
| Markov property | Derived from the Gaussian covariance for the full slice field (Section 7.4), not for an arbitrary finite readout. |

| Statement | Mathematical status |
|---|---|
| POVM and the Born form | A representation of positive affine probabilities defined on all normal states is proved. Application to a protocol requires verifying the identification of preparations and statistics with this state space. |

## 9.2 Conditions for application to registration protocols

The results specify analytic conditions for applying the free quantum representation. For a specific reconstruction class and fixed protocol, the following aspects of verification are distinguished.

Control of admissible full data may be expressed, for example, by (120). It may be based on independently specified constraints of the chosen class, a priori field estimates, or properties of a specific realization of the observer and protocol. These constraints are specified independently of the conclusion about the Poisson approximation. Direct control of the remainder in readout channels is also possible, as noted in Section 5.1.

For a finite working range, errors are compared with operational tolerances using (93) and (124). Depending on the setting, one accounts for the low-frequency remainder, spatial boundary corrections, and registration threshold margins. If the working slab must be related to the boundary of a large reconstruction domain, a separate geometric estimate specifies that relation.

When the canonical form (147) is used as a reference for observable statistics, its correlations must be related to accessible preparations and registrations. The alternative characterization in Lemma 5.1 applies under its stated conditions on the effective-algebra action and invariance of the reference state. These conditions differ from covariance of a family of states under re-description.

For Proposition 8.1 to apply, preparations are identified with quantum states in a way that preserves convex mixing, and the protocol's statistics depend only on that state. Probabilities are defined on all normal states. Register properties and finite-interval stability of the record are verified for the chosen realization. Construction of such a measurement protocol and its joint harmonic realization are outside the scope of this paper; the analytic theorems do not use them as established results.

## 9.3 Technical conditions and scope of the results

Analytic domains, weak limiting procedures, and the zero mode are specified in the main text and appendices. In the Sobolev setting, the topology is specified before the continuous functionals. Continuity of individual outputs does not replace strong continuity of the semigroup; the latter is proved spectrally here after exact selection [15]. Appendix D summarizes the standard theorems used and the hypotheses to be verified before applying them.

The constructed canonical free kernel is massless: the operator $A = |D|$ is obtained from $\Delta_{\mathbb{E}^4}\Phi_0 = 0$ through (35), and the multiplier $c/|p|^2$ of the reference form (147) contains no mass scale. Direct replacement by $A = \sqrt{-\Delta_\Sigma + m^2}$ and $c/(|p|^2+m^2)$ within the same identification with the fundamental operator would lead to $\Delta_{\mathbb{E}^4}\Phi = m^2\Phi$ instead of the harmonic equation. This is a limitation of the particular canonical construction, not a prohibition of effective masses in all reconstruction sectors. Their origin is not investigated here.

The global Sobolev class considered here contains no nonzero strictly $s$-independent free harmonic profiles. Indeed, if $u(s) \equiv f$, then $u'' = 0$, and (36) gives $\Delta_\Sigma f = 0$, that is, $|k|^2 \widehat{f} = 0$. Such a profile therefore vanishes in $H^r(\mathbb{R}^3)$. This conclusion concerns strict stationarity of

the global free sector. Finite-interval stability of a localized register is a different property. When an effective interacting hyperbolic regime is used on the intermediate scale $\Omega_O \Subset K \Subset \Omega$, transferring that stability to a harmonic representation requires error control over the entire storage interval in norms governing the readout, and errors small relative to threshold margins. Geometric scale separation is supplemented by these estimates; constructing the registers is outside the paper's scope. Neither the actual configuration $\Phi_0$ nor the internal observer is identified here with a free mode.

The origins of masses, bound configurations, and long-lived memory carriers belong to the further construction of effective sectors. The present results characterize the specified analytic classes of the free sector and impose no global prohibition on growing harmonic solutions.

## 10 Discussion and conclusions

This work establishes analytic results for two elements of the free quantum core used in the preceding operational reconstruction [3]. Starting from the full elliptic data pair $(u, u')$, we derive an exact finite-slab decomposition of the solution and a quantitative bound on its deviation from the Poisson extension. In the spectral sector $A \geq \kappa I > 0$, this deviation is suppressed with distance from the second slab boundary when the boundary mismatch $b_L = u(L) - \mathrm{e}^{-LA}u(0)$ is controlled. Subexponential growth on the half-line yields the exact decaying-branch graph away from the zero mode. Approximate reduction and exact selection thus have distinct, explicitly stated conditions.

For the free massless scalar sector, a canonical mathematical representative is constructed: the Green's function of the Laplacian $-\Delta_{\mathbb{E}^4}$. Theorem 2 establishes uniqueness of the multiplier $\widetilde{C}_E(p) = c/|p|^2$ under a regular equal-slice limit, a positive, spatially homogeneous and isotropic equal-slice form, compatible transfer of one argument, and $O(4)$ invariance. The freedom of choice is restricted to overall normalization and the separately specified zero components.

Lemma 5.1 establishes the same spectral weight under a consistent action of observable Lorentz transformations on the effective scalar quantum algebra and invariance of one fixed positive-frequency reference state. Under these hypotheses, the invariant measure on the positive massless shell determines $q(r) = c/(2r)$.

Reflection positivity is verified for the canonical kernel. The associated Gaussian measure specifies a free hierarchy of Schwinger functions to which Osterwalder–Schrader reconstruction applies. It yields a free quantum state space, a positive generator, and unitary dynamics. The Markov property of the full Gaussian slice field is derived from the constructed covariance. The reference measure represents this correlation structure, not an ensemble of fundamental configurations.

Thus, within the specified analytic classes, we establish quantitative reduction of full Cauchy data, exact selection of Poisson transfer under a growth condition, and a characterization of the canonical free two-point form. The conditions for standard quantum reconstruction are verified for the hierarchy of the associated Gaussian measure, and the Markov property of the full slice field is derived. Application of this representation to a specific internal observer is governed by the conditions for matching to registration protocols stated in Section 9.2. The main contribution is the establishment of sufficient conditions for passing from the full harmonic problem to a reduced free sector, the characterization of its canonical reference form, and explicit accuracy estimates for the single-slice description on a finite slab. Standard semigroup and free-field constructions serve as the mathematical apparatus, with their applicability conditions verified.

# A A detail of the local-instability proof

Suppose the hypothesized continuous extension of the functional in Proposition 4.1 exists. The Fourier transform of a compactly supported distribution is a smooth function of real $k$ with polynomial growth. On tests with compact Fourier support away from zero, the representative of the dual Sobolev functional must agree with $\mathrm{e}^{h|k|}\widehat{\ell}(k)$. Thus, (58) holds away from zero; near zero, the required local integrability follows from smoothness of $\widehat{\ell}$.

For $0 < a < h$ and any integer $m \geq 0$, the Cauchy–Schwarz inequality gives

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathrm{e}^{a|k|}(1+|k|)^m|\widehat{\ell}(k)|\,\mathrm{d}^3k \\ &\quad\leq \left\|(1+|k|^2)^{-r/2}\mathrm{e}^{h|k|}\widehat{\ell}\right\|_2 \left\|\mathrm{e}^{-(h-a)|k|}(1+|k|)^m(1+|k|^2)^{r/2}\right\|_2 < \infty.\end{aligned}$$

Hence the inverse transform defines a holomorphic function for $|\operatorname{Im} x| < a$; differentiation under the integral is valid. On the real slice, this function represents the distribution $\ell$. It vanishes on a nonempty open region outside its compact support. The identity theorem for real-analytic functions gives $\ell = 0$.

The proof does not require a physical channel to be a smooth $C_c^\infty$ profile. It uses compactness of support, continuity in the chosen Sobolev topology, and the exponential multiplier on an unbounded spectrum. On a bounded spectral sector, continuity does not imply exponential integrability of the entire transform, and no contradiction arises.

# B The zero mode and a finite spatial domain

For $A = 0$, the second-order equation has solution $u(s) = a + sb$. Formulas involving $A^{-1}$ do not apply on this subspace. Boundedness on the half-line forces $b = 0$, whereas subexponential growth allows the linear term. The zero mode is therefore separated in Corollary 4.3.

On $L^2(\mathbb{R}^3)$, the operator $|D|$ has no nonzero eigenvector with zero eigenvalue, but its inverse is nevertheless unbounded near zero. Absence of a nonzero zero-eigenvalue vector must thus be distinguished from boundedness of the inverse. The forms $c/p^2$ on $\mathscr{S}(\mathbb{R}^4)$ and $c/(2|k|)$ on $\mathscr{S}(\mathbb{R}^3)$ are well defined without removing the zeroth moment of the test function, by local integrability.

On a finite domain, one may use a positive self-adjoint realization $B_K$ and set $A_K = \sqrt{B_K}$. Under Dirichlet conditions on a bounded regular domain, there is a spectral gap, and Theorem 1 applies with that gap. Under Neumann conditions, the constant mode requires separate treatment. Choosing boundary conditions is part of the analytic formulation, not a consequence of merely designating a region $K$.

Local independence from the choice of boundary conditions is checked by estimating differences of solutions, resolvents, or kernels on tests inside $K$. Comparison with a specific large reconstruction domain requires bounds on the corresponding boundary corrections. Finite $K$ specifies a working region for the analysis, not a physical boundary of the observable world.

# C Changing the observer-dependent representation

Even an exact stationary analytic operator $G = -A$ does not ensure stationarity in every system of effective variables. If, in a given sector, $\psi(s) = W_s u(s)$ and $W_s$ is invertible on that sector, then, under sufficient regularity,

$$\psi'(s) = \left(\dot{W}_s W_s^{-1} - W_s A W_s^{-1}\right)\psi(s). \tag{176}$$

Explicit dependence of $W_s$ on $s$ generally persists. For reducing, noninvertible $W_s$, the formula involving $W_s^{-1}$ is not valid at all: a separate intertwining or image-closure condition must be checked.

Likewise, an error bound for the full trace does not guarantee the same bound after arbitrary nonlinear threshold processing. Each protocol requires either a stability margin for its registration label or an established modulus of continuity of the primary processing. The nonlinearity of event formation is neither automatically transferred to the linear data space nor removed by construction of a free reference kernel.

# D Map of standard results and hypotheses to be verified

| Framework | What must be verified before application |
|---|---|
| Spectral calculus | Self-adjointness of $A$, the chosen space, and operator domains; growing exponentials are first treated on bounded spectral projections. |
| Poisson semigroup | Selection of the decaying-branch graph; strong continuity and the generator then follow from spectral theory. |
| Bochner–Minlos | A nuclear space, a continuous positive-definite characteristic functional, and normalization at zero. |
| OS reconstruction | The complete Euclidean hierarchy, its regularity, symmetries, reflection positivity, and the required growth conditions. |
| Gaussian Markov realization | A joint Gaussian hierarchy with kernel $Q\mathrm{e}^{-\lvert s-s'\rvert A}$, not merely deterministic transfer of a trace. |
| Riesz–Markov | A positive normalized state on a specific commutative algebra; countable additivity/normality is verified for the chosen measurable realization. |
| POVM representation | A probability measure for each normal state, affinity, and operational identification of this state space. |

For the constructed free sector, the hypotheses of the spectral and semigroup results are verified in Section 4, and those of Gaussian measure construction and OS reconstruction in Section 7. Proposition 8.1 uses prescribed affine probabilities; conditions for matching them to a registration protocol are stated in Section 9.2.

# References

[1] A. N. Smirnov, “Special Relativity as an Emergent Structure in a Timeless Euclidean Model,” *International Journal of Quantum Foundations* **12** (2026), no. 2, 272–312.

[2] A. N. Smirnov, “Einstein Equations from a Timeless Euclidean Model: Operational Reconstruction and the Compensation Principle,” *International Journal of Quantum Foundations* **12** (2026), no. 2, 372–402.

[3] A. N. Smirnov, “The Quantum Core of the Effective Theory as an Operational Reconstruction in a Timeless Euclidean Model,” *International Journal of Quantum Foundations* **12** (2026), no. 2, 531–583.

[4] A. N. Smirnov, "Reconstruction Classes and the Fine-Tuning Problem in a Timeless Euclidean Model," *International Journal of Quantum Foundations* **12** (2026), no. 2, 738–765.

[5] A. N. Smirnov, "The Standard Model and Dark Matter in a Timeless Euclidean Model," *International Journal of Quantum Foundations* **12** (2026), no. 2, 805–836.

[6] A. N. Smirnov, "The Reconstruction Horizon and the Emergence of the Cosmological Constant in a Timeless Euclidean Model," *International Journal of Quantum Foundations* **12** (2026), no. 2, 867–917.

[7] U. Tautenhahn, "Optimal Stable Solution of Cauchy Problems for Elliptic Equations," *Zeitschrift für Analysis und ihre Anwendungen* **15** (1996), no. 4, 961–984. doi:10.4171/ZAA/740.

[8] L. Caffarelli and L. Silvestre, "An Extension Problem Related to the Fractional Laplacian," *Communications in Partial Differential Equations* **32** (2007), nos. 7–9, 1245–1260. doi:10.1080/03605300600987306; arXiv:math/0608640.

[9] P. R. Stinga and J. L. Torrea, "Extension Problem and Harnack's Inequality for Some Fractional Operators," *Communications in Partial Differential Equations* **35** (2010), no. 11, 2092–2122. doi:10.1080/03605301003735680; arXiv:0910.2569.

[10] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's Functions," *Communications in Mathematical Physics* **31** (1973), 83–112. doi:10.1007/BF01645738.

[11] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's Functions. II," *Communications in Mathematical Physics* **42** (1975), 281–305. doi:10.1007/BF01608978.

[12] B. Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*, Princeton University Press, Princeton, 1974.

[13] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, New York, 1987.

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*, revised and enlarged ed., Academic Press, San Diego, 1980.

[15] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.

[16] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, 2nd ed., Springer, Berlin, 1990.

[17] V. I. Bogachev, *Gaussian Measures*, American Mathematical Society, Providence, 1998.

[18] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2nd ed., Cambridge University Press, Cambridge, 2014.

[19] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol. 1*, 2nd ed., Springer, Berlin, 1987.

[20] R. A. Minlos, "Generalized Random Processes and Their Extension to a Measure," *Trudy Moskovskogo Matematicheskogo Obshchestva* **8** (1959), 497–518 (in Russian). MathNet.Ru: mmo94.

[21] S. Sheffield, "Gaussian free fields for mathematicians," *Probability Theory and Related Fields* **139** (2007), 521–541. doi:10.1007/s00440-006-0050-1; arXiv:math/0312099.