Three Greedy Representations of the MRB Constant

Residues, Roots, and Logarithmic Products

Marvin Ray Burns

Research draft for mathematical review • 1 November 2026

Abstract

We compare three constructive greedy representations of the MRB constant. Algorithm A uses reciprocal-kernel residue magnitudes, beginning with the unusually accurate approximation $A(8)$. Algorithm B subtracts the original radical increments $m^{1/m}-1$ from the anchor $B(16)$. Algorithm C employs $\log(m)/m$, seeded by the exact Dirichlet-eta value $\eta'(1)=\gamma\log 2-\frac{1}{2}(\log 2)^2$. We prove convergence for the specified selection rules and obtain a quadratic error bound for the nearest-residue method and a logarithmically improved quadratic bound for both one-sided methods. Algorithm C also gives an exact infinite product for $e^{C_{\mathrm{MRB}}}$. A common Lambert-W backbone generates the integer indices: exact inverse formulas for B and C, and a rigorous, sub-unit-width Lambert bracket for A. Five logarithmic corrections leave an error of approximately 6.85×10^{-105}. These expansions are not irrationality criteria; we do not claim invention of the general greedy principle.

1 Introduction

Put

$$C_{\mathrm{MRB}}=\sum_{n=1}^{\infty}(-1)^n(n^{1/n}-1)=0.187859642462067120248517934054273230\ldots. \tag{1}$$

The sum converges by the alternating-series test after the first few terms. Its arithmetic classification remains open [1, 2]. We compare three MRB-specific instances of a broader idea: approximate a target using entries of an explicitly decreasing positive sequence, choosing indices by the current residual. The A-algorithm uses signed, *nearest* corrections; the B- and C-algorithms use *one-sided* corrections from below. These must not be conflated, since their index-selection rules are different.

2 A common greedy convergence mechanism

Lemma 1 (One-sided decreasing-dictionary expansion)**.** *Let $f(m)>0$ be strictly decreasing to zero on integers $m\ge M$. For $0<r_0\le f(M)$, recursively select*

$$m_k=\min\{m\ge M: f(m)\le r_{k-1}\},\qquad r_k=r_{k-1}-f(m_k), \tag{2}$$

terminating if $r_k=0$. Then $r_k\downarrow 0$, $r_0=\sum_{k\ge1}f(m_k)$, and, whenever $m_k>M$,

$$0\le r_k<f(m_k-1)-f(m_k). \tag{3}$$

Proof. The minimality of m_k gives $f(m_k)\le r_{k-1}<f(m_k-1)$, so (3) holds and $0\le r_k<r_{k-1}$. Let $r_k\to L\ge0$. If $L>0$, choose an integer $J\ge M$ for which $f(J)<L$. At every nonterminating stage $r_{k-1}\ge L$, hence $m_k\le J$, and therefore $r_{k-1}-r_k=f(m_k)\ge f(J)>0$, a contradiction. Thus $L=0$. Telescoping the recurrence yields the asserted exact and absolutely convergent expansion. □

Lemma 2 (Nearest signed corrections)**.** *Let $f(m) > 0$ decrease strictly to zero for $m \geq M$. Given a signed residual E with $0 < |E| < f(M)$, take the member $f(m)$ closest to $|E|$ and set $E' = E - \operatorname{sgn}(E) f(m)$. Repeating this rule gives $|E_k| \to 0$. For $f(j) \leq |E| < f(j-1)$, $j > M$, one has*

$$|E'| \leq \tfrac{1}{2}(f(j-1) - f(j)), \qquad |E'| < |E|. \tag{4}$$

Proof. The closest member is one of the two bracketing values. The distance to its nearer endpoint is at most half the gap and no greater than $|E| - f(j) < |E|$. If the decreasing magnitudes $|E_k|$ tended to $L > 0$, choose J with $f(J) < L$. At all stages the nearest correction would improve the error at least as much as the correction $f(j) \leq |E_k|$, where $j \leq J$, giving $|E_{k+1}| \leq |E_k| - f(J)$, a contradiction. □

3 Algorithm A: reciprocal-kernel residues

The reciprocal kernel occurring in the analytic study of C_{MRB} is

$$K(u) = u^{u-2} \frac{\sin(\pi u)}{\sin(\pi/u)}.$$

Its simple residues at $u = 1/m$, for positive integers $m \geq 2$, satisfy

$$-2 \operatorname*{Res}_{u=1/m} K(u) = (-1)^m A(m), \qquad \boxed{A(m) = \frac{2}{\pi} m^{-1/m} \sin\!\left(\frac{\pi}{m}\right)}.$$

For $m \geq 4$, $A(m) > 0$ is strictly decreasing to zero and

$$A(m) = \frac{2}{m} + \mathcal{O}\!\left(\frac{\log m}{m^2}\right), \qquad A(m-1) - A(m) \sim \frac{2}{m^2}. \tag{5}$$

For completeness, the logarithmic derivative has the sign of $\log x - 1 - \pi \cot(\pi/x)$. For $x \geq 4$, the elementary estimate $\pi \cot(\pi/x) \geq x - \pi^2/x > \log x - 1$ proves strict decrease.

Set $S_1 = A(8)$, $E_1 = C_{\mathrm{MRB}} - S_1$. For $k \geq 2$, choose $m_k \geq 4$ minimizing $\big||E_{k-1}| - A(m_k)\big|$, and define

$$S_k = S_{k-1} + \operatorname{sgn}(E_{k-1}) A(m_k), \qquad E_k = C_{\mathrm{MRB}} - S_k. \tag{6}$$

Here $E_1 = -1.8918071278800735\ldots \times 10^{-8}$, so Lemma 2 applies. Equations (4) and (5) yield the *worst-case asymptotic bound*

$$|E_k| \leq (\tfrac{1}{4} + o(1)) |E_{k-1}|^2. \tag{7}$$

The conspicuous accuracy of $A(8)$ is an observed feature of the MRB seed; the greedy convergence theorem does not explain that initial near-match.

4 Algorithm B: the original radicals

Put $B(m) = m^{1/m} - 1$. For real $x \geq 3$, $B(x)$ is strictly decreasing because $\log x / x$ is. The useful initial anchor is $B(16) = 2^{1/4} - 1 > C_{\mathrm{MRB}}$. Set

$$r_0 = B(16) - C_{\mathrm{MRB}} > 0, \quad m_k = \min\{m \geq 3 : B(m) \leq r_{k-1}\}, \quad r_k = r_{k-1} - B(m_k). \tag{8}$$

Lemma 1 gives the exact convergent representation

$$\boxed{C_{\mathrm{MRB}} = B(16) - \sum_{k=1}^{\infty} B(m_k)}. \tag{9}$$

Writing $v = \log(1 + r)$, the real inverse of $B(x) = r$ is

$$x = -\frac{W_{-1}(-v)}{v}, \qquad v = \log(1 + r), \tag{10}$$

with W_{-1} the lower real Lambert-W branch; $m = \lceil x \rceil$ except that numerical implementations should guard near integer boundaries. Since $B(x) = \log x/x + \mathcal{O}((\log x)^2/x^2)$, its consecutive spacing is

$$B(m-1) - B(m) \sim \frac{\log m - 1}{m^2}. \tag{11}$$

5 Algorithm C: eta-anchored logarithms

This third construction has a different exact form, even though its local spacing is asymptotically similar to that of B. Let

$$C(m) = \frac{\log m}{m}, \qquad m \geq 3, \qquad H(n) = n^{1/n} - 1 - \frac{\log n}{n}. \tag{12}$$

Write $\eta(s) = (1 - 2^{1-s})\zeta(s)$. Expansion at $s = 1$ gives

$$\eta'(1) = \gamma \log 2 - \tfrac{1}{2}(\log 2)^2 = 0.15986890374243097175694787032 49\ldots. \tag{13}$$

The alternating logarithmic series is $\sum_{n\geq 1}(-1)^n \log(n)/n = \eta'(1)$. Moreover $H(n) = \mathcal{O}((\log n)^2/n^2)$, so its series is absolutely convergent. Consequently,

$$R := C_{\mathrm{MRB}} - \eta'(1) = \sum_{n=2}^{\infty}(-1)^n H(n). \tag{14}$$

For $n \geq 3$, $H(n)$ decreases to zero (it is an increasing function of $\log n/n$). Direct bounds for the first four terms give

$$H(2) - H(3) + H(4) - H(5) > 0.00139.$$

The remaining paired tail $\sum_{j\geq 3}(H(2j) - H(2j + 1))$ is nonnegative. Therefore $R > 0$. Also $R < H(2) - H(3) + H(4) < 0.060 < C(3)$. Numerically,

$$R = 0.02799073871963614849157006372935665955968107114368\ldots. \tag{15}$$

Theorem 3 (Eta-anchored logarithmic greedy expansion)**.** *Starting with $r_0 = R$, choose recursively*

$$m_k = \min\left\{m \geq 3 : \frac{\log m}{m} \leq r_{k-1}\right\}, \qquad r_k = r_{k-1} - \frac{\log m_k}{m_k}. \tag{16}$$

Then $r_k \downarrow 0$ and

$$\boxed{C_{\mathrm{MRB}} = \gamma \log 2 - \tfrac{1}{2}(\log 2)^2 + \sum_{k=1}^{\infty}\frac{\log m_k}{m_k}}. \tag{17}$$

Equivalently, the following product converges to a positive finite value:

$$\boxed{e^{C_{\mathrm{MRB}}} = 2^{\gamma} e^{-(\log 2)^2/2}\prod_{k=1}^{\infty} m_k^{1/m_k}}. \tag{18}$$

If r is a sufficiently small current residual, the next one satisfies

$$0 \leq r_{\mathrm{next}} \leq (1 + o(1))\frac{r^2}{\log(1/r)} \qquad (r \downarrow 0). \tag{19}$$

The selected index is

$$\boxed{m = \left\lceil -\frac{W_{-1}(-r)}{r} \right\rceil}, \tag{20}$$

with boundary verification required in finite-precision computation.

Proof. We established $0 < R < C(3)$, so Lemma 1 applies to the strictly decreasing $C(m)$. It yields (17). Exponentiation is legitimate because the series of nonnegative logarithmic terms sums to the finite value R, proving (18). Solving $\log x/x = r$ for real $x > e$ gives (20). Finally, the mean-value theorem and $C'(x) = (1 - \log x)/x^2$ give

$$C(m-1) - C(m) \sim \frac{\log m - 1}{m^2}.$$

Since $C(m) \le r < C(m-1)$, one has $r \sim \log m/m$ and $\log(1/r) \sim \log m$. Inserting these relations into (3) gives (19). □

Remark 4 (Shared asymptotic rate of B and C)**.** *As $m \to \infty$, $B(m) \sim C(m)$ and both have the spacing $(\log m - 1)/m^2$. Thus the same logarithmically improved bound $r_{\text{next}} \le (1 + o(1))r^2/\log(1/r)$ also holds for the one-sided B-algorithm. The novelty of C is the precise eta anchor and multiplicative identity,* not *a fundamentally faster asymptotic order.*

6 The Lambert-W backbone of all three greedy algorithms

An additional unifying feature is a *Lambert backbone*: all three integer orbits can be generated from the current residual by inverting a continuous approximation or the exact underlying dictionary, using real branches of Lambert W [3]. The distinction between an exact inverse and an exact *index bracket* matters.

For the B-dictionary, $v = \log(1 + r)$ gives the exact real solution $x_B = -W_{-1}(-v)/v$, as in (10). For the C-dictionary the exact solution is $x_C = -W_{-1}(-r)/r$, as in (20). Both one-sided algorithms choose $m = \lceil x \rceil$, with a finite-precision boundary check. The residue dictionary A, however, has a sine factor. Its real inverse does not follow from a single ordinary Lambert-W call. Nevertheless, the same function gives an unusually sharp, provable bracket.

Theorem 5 (Lambert localization for the residue dictionary)**.** *Let $0 < r \le A(8)$ and let $x \ge 8$ be the unique real solution of $A(x) = r$. Set $y = 2/r$. Then*

$$\boxed{W_0\left(e^{y-1}\right) < x < W_0(e^y)\,.} \tag{21}$$

The bracket has width strictly less than 1. Consequently, the nearest residue index is among the at most three integers in

$$\left[\left\lfloor W_0(e^{y-1}) \right\rfloor, \lceil W_0(e^y) \rceil\right]. \tag{22}$$

More precisely, it is the integer minimizing $|A(m) - r|$ over that finite set. The W_0-based bracket thus generates an exact *greedy integer index after only a constant-sized local comparison.*

Proof. Write $L = \log x$, $u = L/x$, and $z = \pi^2/(6x^2)$. For $x \geq 8$, $0 < z < 1$. The elementary inequalities $1 - z \leq \sin(\pi/x)/(\pi/x) < 1$ and $0 < e^u - 1 - u \leq \frac{1}{2}u^2 e^u$ yield

$$\Delta(x) := \frac{2}{A(x)} - x - \log x \tag{23}$$

$$= \frac{\pi e^{(\log x)/x}}{\sin(\pi/x)} - x - \log x > 0, \tag{24}$$

$$\Delta(x) \leq \frac{\dfrac{(\log x)^2}{2x} e^{(\log x)/x} + \dfrac{\pi^2(x + \log x)}{6x^2}}{1 - \dfrac{\pi^2}{6x^2}}. \tag{25}$$

For $x \geq 8$, the functions $(\log x)^2/x$, $(x+\log x)/x^2$, and $(\log x)/x$ are decreasing, while the denominator in the last fraction increases. Hence the upper bound is maximized at $x = 8$, where it is $0.62561964\ldots < 1$. Thus $0 < \Delta(x) < 1$ and

$$y - 1 < x + \log x < y.$$

The inverse of $t \mapsto t + \log t$ on $t > 0$ is $W_0(e^y)$, proving (21). Since $d(t + \log t)/dt = 1 + 1/t > 1$, its inverse increases by less than one when its argument increases by one; therefore the bracket has width < 1. A nearest member of a strictly decreasing sequence is attained at either $\lfloor x \rfloor$ or $\lceil x \rceil$. These lie in the integer range (22), which contains at most three integers. □

There is also an exact *implicit* Lambert representation of the residue inverse:

$$\boxed{x = W_0\left(\exp\left[\frac{2}{r} - \Delta(x)\right]\right), \qquad \Delta(x) = \frac{\pi x^{1/x}}{\sin(\pi/x)} - x - \log x.} \tag{26}$$

Here

$$\Delta(x) = \frac{(\log x)^2/2 + \pi^2/6}{x} + \mathcal{O}\left(\frac{(\log x)^3}{x^2}\right), \qquad x \to \infty. \tag{27}$$

This formula explains why the uncorrected Lambert predictor $W_0(e^{2/r})$ works especially well as $r \to 0$, without falsely claiming that A has an elementary closed-form Lambert inverse.

Remark 6 (Stable evaluation: the Wright omega function)**.** *Although $W_0(e^{2/r})$ is symbolically compact, direct construction of $e^{2/r}$ is computationally undesirable for small r. Define $\omega(y) = W_0(e^y)$, the real Wright omega function [3]. It satisfies $\omega + \log \omega = y$ and can be evaluated stably by Newton iteration*

$$w \longleftarrow w - \frac{w + \log w - y}{1 + 1/w}, \qquad w_0 = y - \log y. \tag{28}$$

The A-bracket becomes $\omega(2/r-1) < x < \omega(2/r)$. The companion Wolfram Language program implements this bounded search, whereas B and C use the negative real branch W_{-1} directly.

Table 1: The common Lambert backbone, with different exact selection rules.

Family	Continuous inversion or bracket	Integer selection
A	$\omega(2/r - 1) < x < \omega(2/r)$	Nearest $A(m)$ (signed)
B	$x = -W_{-1}[-\log(1 + r)]/\log(1 + r)$	$m = \lceil x \rceil$ (one-sided)
C	$x = -W_{-1}(-r)/r$	$m = \lceil x \rceil$ (one-sided)

7 Numerical comparison

The calculations below use an independently evaluated high-precision value of C_{MRB}. We report errors after incorporating each displayed term or correction. The values are numerical demonstrations rather than certified interval bounds.

Table 2: Algorithm A: nearest signed residue corrections; first term is $A(8)$.

Terms	Last index	Absolute residual
1	8	1.89181×10^{-8}
2	105719004	2.58394×10^{-17}
3	77401040908104410	8.04971×10^{-35}
4	248456304020606417026698184311215 05	1.39211×10^{-69}
5	143666925298660404624304474518299 3 372702370706363156398992231559707684	2.55097×10^{-140}

Table 3: Algorithm B: one-sided radical subtraction from $B(16)$.

Terms	Last index	Absolute residual
1	16	1.34747×10^{-3}
2	6523	8.55325×10^{-8}
3	224837282	1.93623×10^{-16}
4	205895605429798299	2.45194×10^{-34}
5	33359379752022502970352634406641 0006	5.60969×10^{-70}

Table 4: Algorithm C: one-sided logarithmic corrections from $\eta'(1)$.

Corrections	Last index	Positive residual
1	187	1.68958×10^{-5}
2	804837	1.13983×10^{-11}
3	2504677193278	1.36579×10^{-24}
4	43219281590481694906765145	3.44675×10^{-51}
5	35102694747288081860097103451656600300718619565955788	6.84728×10^{-105}

The initial anchor accuracies differ substantially, so the table does not constitute a fair speed benchmark. In particular $A(8)$ is already an unexpectedly accurate approximation before greedy corrections begin.

8 Interpretation, originality, and limits

The selection mechanism is an instance of the classical greedy-expansion idea, familiar from Sylvester's Egyptian-fraction expansions [4]. Its use with A, B, and C is not a proof of a new universal theorem, nor is the mere existence of such expansions unique to C_{MRB}. The claims developed here are instead:

1. an explicit comparison of signed residue corrections with two distinct one-sided algorithms;
2. the exact $\eta'(1)$ anchoring and positive logarithmic expansion (17), and the product (18);
3. a shared Lambert-W backbone—with exact inverses for B and C and a rigorously narrow index bracket for A—together with asymptotic residual bounds and reproducible MRB-specific orbits.

All three expansions presuppose access to C_{MRB} (or a sufficiently accurate independent enclosure) in their index-selection rules. Consequently, they do not by themselves give a new, oracle-free algorithm

for evaluating the constant. In particular, rapid convergence and exact products do not establish irrationality or transcendence.

9 Conclusion

The three families connect the MRB problem with complex-analytic residues, algebraic radicals, and logarithmic/eta-function structure. Their Lambert-W backbone supplies a common method for generating indices without a search over enormous integer ranges, while preserving the necessary distinction between exact and bracketed inversions. The logarithmic method is an appealing third member because it leads to an exact infinite product while retaining a provable and rapidly decreasing greedy residual. The distinction between these identities and genuine new arithmetic information should be preserved in any publication or computational discussion.

A Wolfram Language verification

The following standalone code computes an independent high-precision reference through the Chebyshev–Euler alternating-series transform, then generates the greedy indices rather than inserting them. The first five logarithmic indices and residuals were additionally checked in a Wolfram Language kernel.

```
(* ::Title:: *)
(* Three Greedy Representations of the MRB Constant *)
(* Marvin Ray Burns -- Research companion, 1 November 2026 *)
(* Mathematica/Wolfram Language standalone reproducibility script. *)
(* No precomputed greedy indices are hard-coded. *)

ClearAll["Global`*"];
wp = 180;
guard = 45;
prec = wp + guard;

(* High-precision MRB reference via the Chebyshev-Euler / CVZ
   alternating-series transform. The recurrence follows the publicly
   documented OEIS A037077 Maple algorithm. *)
nCVZ = Ceiling[1.31 wp] + 70;
z = N[3 + 2 Sqrt[2], prec];
d = (z^nCVZ + z^(-nCVZ))/2;
b = SetPrecision[-1, prec];
c = -d;
s = SetPrecision[0, prec];
Do[
  c = b - c;
  b = 2 b (k^2 - nCVZ^2)/((2 k + 1) (k + 1));
  a = If[k == 0, SetPrecision[1, prec],
    Exp[N[Log[k + 1], prec]/(k + 1)]];
  s = s + c a;
, {k, 0, nCVZ - 1}];
cmrb = N[1/2 - s/d, wp];
Print["CMRB = ", N[cmrb, 60]];

(* Algorithm A: signed nearest residue, after the m=8 anchor. *)
aRes[m_Integer?Positive] :=
   N[(2/Pi) Exp[-N[Log[m], prec]/m] Sin[N[Pi, prec]/m], prec];

(* Lambert backbone: omega[y] = W0[Exp[y]] computed stably via
   omega[y] + Log[omega[y]] = y. Avoid Exp[2/r] for small r. *)
omega[t_?NumericQ] := Module[{y = N[t, prec], w},
  w = y - Log[y];
  Do[w = w - (w + Log[w] - y)/(1 + 1/w), {9}];
  w
```

```
];

(* For 0 < r <= A(8), the real A-inverse is strictly between
   omega[2/r - 1] and omega[2/r]. Width < 1: at most 3
   integer candidates suffice. Compare their A-values to r. *)
aLambertBracket[r_?NumericQ] :=
  {omega[2/r - 1], omega[2/r]};

nearestA[err_?NumericQ] := Module[
  {r = Abs[err], bounds, candidates},
  bounds = aLambertBracket[r];
  candidates = Range[Max[8, Floor[bounds[[1]]]],
    Ceiling[bounds[[2]]]];
  First[MinimalBy[candidates, Abs[r - aRes[#]] &]]
];

approxA = aRes[8];
errA = cmrb - approxA;
indicesA = {8};
errorsA = {Abs[errA]};
Do[
  mA = nearestA[errA];
  approxA = approxA + Sign[errA] aRes[mA];
  errA = cmrb - approxA;
  AppendTo[indicesA, mA];
  AppendTo[errorsA, Abs[errA]];
, {4}];

(* Algorithm B: one-sided nth-root terms from B(16). *)
bRoot[m_Integer?Positive] := Exp[N[Log[m], prec]/m] - 1;

errB = bRoot[16] - cmrb;
indicesB = {16};
errorsB = {errB};
Do[
  v = Log[1 + errB];
  mB = Ceiling[-ProductLog[-1, -v]/v];
  While[bRoot[mB] > errB, mB++];
  While[mB > 3 && bRoot[mB - 1] <= errB, mB--];
  errB = errB - bRoot[mB];
  AppendTo[indicesB, mB];
  AppendTo[errorsB, errB];
, {4}];

(* Algorithm C: logarithmic eta-anchored greedy terms. *)
cLog[m_Integer?Positive] := N[Log[m], prec]/m;
eta1 = N[EulerGamma Log[2] - Log[2]^2/2, prec];
errC = cmrb - eta1;
indicesC = {};
errorsC = {};
Do[
  mC = Ceiling[-ProductLog[-1, -errC]/errC];
  While[cLog[mC] > errC, mC++];
  While[mC > 3 && cLog[mC - 1] <= errC, mC--];
  errC = errC - cLog[mC];
  AppendTo[indicesC, mC];
  AppendTo[errorsC, errC];
, {5}];

formatErrors[x_] := ScientificForm[#, 10] & /@ x;
Print["A indices: ", indicesA];
Print["A errors: ", formatErrors[errorsA]];
Print["B indices: ", indicesB];
Print["B errors: ", formatErrors[errorsB]];
Print["C indices: ", indicesC];
Print["C errors: ", formatErrors[errorsC]];

(* Test the exponential product at five correction terms. *)
prod5 = Exp[eta1] Times @@ (Exp[cLog[#]] & /@ indicesC);
Print["C 5-factor product error: ",
```

```
  ScientificForm[Abs[Exp[cmrb] - prod5], 12]];

(* Expected beginning: A {8, 105719004, 77401040908104410, ...};
   B {16, 6523, 224837282, ...};
   C {187, 804837, 2504677193278, ...}.
   C error after five corrections: 6.84727717028875...*10^-105. *)
```

References

[1] N. J. A. Sloane et al., *OEIS A037077: Decimal expansion of the MRB constant*, `https://oeis.org/A037077` (accessed November 2026).

[2] E. W. Weisstein, *MRB Constant*, MathWorld—A Wolfram Resource, `https://mathworld.wolfram.com/MRBConstant.html`.

[3] NIST Digital Library of Mathematical Functions, *Section 4.13: Lambert W-Function*, `https://dlmf.nist.gov/4.13`.

[4] N. J. A. Sloane et al., *OEIS A006525: Denominators of greedy Egyptian fraction for* $e - 2$, `https://oeis.org/A006525`; see also the historical Sylvester expansion.