Magnus quadrature formulas of order 10 on a centered interval

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Abstract

This paper extends the centered Magnus quadrature formulas of orders 2 to 8 in [1] to order 10 (residual error $O(t^{11})$ over the interval $[-t, t]$). The formula uses the centered Legendre moments $g_0, \ldots, g_4$ of the coefficient matrix and nine commutators. No formula with eight commutators was found. Of its 81 nonzero coefficients, 25 are small rationals and 56 are irrational algebraic numbers, defined as the isolated real solution of 70 polynomial equations with rational coefficients; they were refined by Newton's method to more than 190 digits, and the conditioning of the refinement is reported. The moments can be computed with five Gauss-Legendre nodes, with six Gauss-Lobatto nodes (five new samples per step), or with nine nodes that include the order-8 Newton-Cotes nodes. An embedded pair on seven Gauss-Kronrod nodes gives an order-6 estimate with three commutators and an order-10 estimate with seven more, whose difference serves as an error estimate.

Notation and order conditions

The notation follows [1]. The Magnus exponent $\Omega(t)$ propagates the solution from $-t$ to t, and the centered Legendre moments are

$$g_k(t) = \int_{-t}^{t} P_k(\tau/t) \cdot A(\tau)\, d\tau \tag{1}$$

(Eq. 35 of [1]). The moment g_k is $O(t^{k+1})$ and has *weight* $k+1$; g_0, g_2 and g_4 are odd functions of t, g_1 and g_3 are even, and $\Omega(t)$ contains only odd weights. The series of Ω in the moments, derived from the centered Magnus theorem (Eq. 34 of [1]) in exact rational arithmetic, is given through weight 7 in Eq. 41 of [1]. Its weight-9 part, which involves only

$g_0, \ldots, g_4$, has 45 nonzero terms in the Lyndon basis:

$$
\begin{aligned}
\Omega_9 = \; & -\tfrac{1}{2}[g_3,g_4] + \tfrac{3}{28}[g_0,[g_2,g_4]] \\
& + \tfrac{7}{180}[[g_0,g_3],g_3] - \tfrac{1}{28}[[g_0,g_4],g_2] \\
& + \tfrac{9}{140}[g_1,[g_1,g_4]] - \tfrac{1}{4}[g_1,[g_2,g_3]] \\
& + \tfrac{1}{12}[[g_1,g_3],g_2] - \tfrac{3}{280}[g_0,[g_0,[g_1,g_4]]] \\
& - \tfrac{1}{72}[g_0,[[g_0,g_3],g_2]] + \tfrac{3}{280}[g_0,[[g_0,g_4],g_1]] \\
& + \tfrac{1}{280}[[g_0,g_1],[g_0,g_4]] - \tfrac{1}{30}[g_0,[g_1,[g_1,g_3]]] \\
& - \tfrac{5}{168}[g_0,[[g_1,g_2],g_2]] - \tfrac{7}{120}[[g_0,[g_1,g_3]],g_1] \\
& - \tfrac{1}{72}[[g_0,g_2],[g_0,g_3]] + \tfrac{5}{168}[[g_0,g_2],[g_1,g_2]] \\
& - \tfrac{5}{168}[[[g_0,g_2],g_2],g_1] + \tfrac{1}{40}[[[g_0,g_3],g_1],g_1] \\
& - \tfrac{3}{56}[g_1,[g_1,[g_1,g_2]]] + \tfrac{1}{1680}[g_0,[g_0,[g_0,[g_0,g_4]]]] \\
& - \tfrac{1}{720}[g_0,[g_0,[g_0,[g_1,g_3]]]] + \tfrac{1}{240}[g_0,[g_0,[[g_0,g_3],g_1]]] \\
& + \tfrac{1}{240}[g_0,[[g_0,g_1],[g_0,g_3]]] + \tfrac{1}{84}[g_0,[g_0,[g_1,[g_1,g_2]]]] \\
& + \tfrac{1}{84}[g_0,[[g_0,[g_1,g_2]],g_1]] + \tfrac{5}{1008}[[g_0,[g_0,g_2]],[g_0,g_2]] \\
& + \tfrac{1}{112}[g_0,[[[g_0,g_2],g_1],g_1]] - \tfrac{1}{180}[[g_0,[g_0,g_3]],[g_0,g_1]] \\
& + \tfrac{1}{42}[[g_0,g_1],[[g_0,g_2],g_1]] + \tfrac{1}{56}[[[g_0,g_1],g_1],[g_0,g_2]] \\
& + \tfrac{3}{560}[[[[g_0,g_1],g_1],g_1],g_1] + \tfrac{1}{4320}[g_0,[g_0,[g_0,[g_0,[g_0,g_3]]]]] \\
& + \tfrac{1}{5040}[g_0,[g_0,[g_0,[g_0,[g_1,g_2]]]]] + \tfrac{1}{1008}[g_0,[g_0,[g_0,[[g_0,g_2],g_1]]]] \\
& + \tfrac{1}{672}[g_0,[g_0,[[g_0,g_1],[g_0,g_2]]]] + \tfrac{1}{1120}[g_0,[g_0,[[[g_0,g_1],g_1],g_1]]] \\
& - \tfrac{1}{672}[g_0,[[g_0,[g_0,g_2]],[g_0,g_1]]] - \tfrac{1}{504}[[g_0,[g_0,g_1]],[g_0,[g_0,g_2]]] \\
& - \tfrac{1}{1120}[g_0,[[g_0,g_1],[[g_0,g_1],g_1]]] - \tfrac{1}{280}[[g_0,[[g_0,g_1],g_1]],[g_0,g_1]] \\
& + \tfrac{1}{20160}[g_0,[g_0,[g_0,[g_0,[g_0,[g_0,g_2]]]]]] + \tfrac{1}{11200}[g_0,[g_0,[g_0,[g_0,[[g_0,g_1],g_1]]]]] \\
& - \tfrac{1}{3360}[g_0,[g_0,[[g_0,[g_0,g_1]],[g_0,g_1]]]] + \tfrac{1}{6720}[[g_0,[g_0,[g_0,g_1]]],[g_0,[g_0,g_1]]] \\
& + \tfrac{1}{201600}[g_0,[g_0,[g_0,[g_0,[g_0,[g_0,[g_0,g_1]]]]]]]
\end{aligned}
\tag{2}
$$

A formula of order 10 must reproduce the series at weights 3, 5, 7 and 9. The free Lie algebra generated by $g_0, \ldots, g_4$ has 1, 5, 16 and 48 basis elements at these weights (not counting the single letters g_2 and g_4, which occur in neither the series nor the formulas), so there are 70 *order conditions* (Table 1, which extends Table 2 of [1]), compared with 0, 1, 6 and 20 at orders 2, 4, 6 and 8.

Table 1: Order conditions and commutator count at order 10.

Order	Residual	Moments	Conditions at weights 3, 5, 7, 9	Total	Commutators
10	$O(t^{11})$	$g_0, \ldots, g_4$	1, 5, 16, 48	70	9

These formulas were first derived for the conventional interval $[0, h]$ with A expanded about the midpoint. The Legendre moments over $[0, h]$, with $P_k(2\tau/h - 1)$, are the centered moments of Eq. 1 with $h = 2t$, and the two series agree term by term (as verified in [1] through weight 9), so the formulas carry over to the centered interval unchanged. All results below were re-verified against the centered series.

Number of commutators

As in [1], each formula is a sequence of nested commutators $C_i = [X_i, Y_i]$, where X_i and Y_i are linear combinations of the moments and earlier commutators and each has definite parity, followed by $\Omega = g_0 + \sum_i c_i \cdot C_i$. The orders 4, 6 and 8 require 1, 3 and 6 commutators, the minimum numbers stated by Blanes, Casas and Ros [2]; no order-10 formula has been published, as far as we know. The search at order 10 computed, for each pattern of operand parities, the maximum generic rank of the Jacobian of the 70 order conditions with respect to the free coefficients, and then searched for exact solutions with Levenberg-Marquardt iterations from random starting points (Table 2):

Table 2: Search for the minimum number of commutators at order 10.

Commutators	Patterns	Max. rank	Restarts	Result
8	all 2,187	64	322	residual 9×10^{-4}
9	all 6,561	69	about 100	exact, 3 families
10	3,000 sampled	70 (26 patterns)	11	exact

A generic rank below 70 does not by itself exclude eight commutators, because the nine-commutator solutions lie on a special set where the Jacobian rank drops to 55-56; the failure at eight commutators is strong numerical evidence, not a proof.

The nine-commutator formula

The operand parities of the chosen formula are, for C_1 to C_9, OE, OE, OO, OE, OO, OE, OO, OE and OE (O = odd, E = even); the commutators C_3, C_5 and C_7 are even and enter only as operands. With $C_i = [X_i, Y_i]$ and coefficients $x_1, \ldots, x_{81}$ (Appendix A), of which 25

are rational and 56 irrational,

$$
\begin{aligned}
X_1 &= x_1 g_0 + x_2 g_2 \\
Y_1 &= x_3 g_1 + x_4 g_3 \\
X_2 &= x_5 g_0 + x_6 g_2 + x_7 C_1 \\
Y_2 &= x_8 g_1 + x_9 g_3 \\
X_3 &= x_{10} g_0 + x_{11} g_2 \\
Y_3 &= x_{12} g_0 + x_{13} g_2 + x_{14} g_4 + x_{15} C_1 \\
X_4 &= x_{16} g_0 + x_{17} g_2 + x_{18} g_4 + x_{19} C_2 \\
Y_4 &= x_{20} g_1 + x_{21} g_3 + x_{22} C_3 \\
X_5 &= x_{23} g_0 + x_{24} g_2 + x_{25} g_4 \\
Y_5 &= x_{26} g_0 + x_{27} g_2 + x_{28} g_4 + x_{29} C_4 \\
X_6 &= x_{30} g_0 + x_{31} g_2 + x_{32} g_4 + x_{33} C_2 + x_{34} C_4 \\
Y_6 &= x_{35} g_1 + x_{36} g_3 + x_{37} C_3 + x_{38} C_5 \\
X_7 &= x_{39} g_0 + x_{40} g_2 + x_{41} g_4 + x_{42} C_2 + x_{43} C_4 + x_{44} C_6 \\
Y_7 &= x_{45} g_0 + x_{46} g_2 + x_{47} g_4 + x_{48} C_1 + x_{49} C_4 + x_{50} C_6 \\
X_8 &= x_{51} g_0 + x_{52} g_2 + x_{53} g_4 + x_{54} C_1 + x_{55} C_2 + x_{56} C_4 + x_{57} C_6 \\
Y_8 &= x_{58} g_1 + x_{59} g_3 + x_{60} C_3 + x_{61} C_5 + x_{62} C_7 \\
X_9 &= x_{63} g_0 + x_{64} g_2 + x_{65} g_4 + x_{66} C_1 + x_{67} C_2 + x_{68} C_4 + x_{69} C_6 + x_{70} C_8 \\
Y_9 &= x_{71} g_1 + x_{72} g_3 + x_{73} C_3 + x_{74} C_5 + x_{75} C_7 \\
\Omega(t) &= g_0 + x_{76} C_1 + x_{77} C_2 + x_{78} C_4 + x_{79} C_6 + x_{80} C_8 + x_{81} C_9 + O(t^{11})
\end{aligned}
\tag{3}
$$

The solution families have 16-20 free parameters beyond 24 rescaling freedoms. The member of Eq. 3 was chosen by minimizing the leading (t^{11}) error coefficients, with each moment weighted by its leading size, and then fixing as many coefficients as possible to simple rationals while keeping that weighted error within 25% of the minimum; rescalings of the operands were then used to simplify the coefficients. Its weighted error is about 150 times smaller than for the member with the simplest coefficients, and the best points of the other families are within 6% of it on the test problems. Attempts to make all coefficients rational failed: in every order of fixing that was tried, some coefficients became determined at irrational values.

The defining equations and root polishing

With the 25 rational coefficients substituted, the order conditions become a system of polynomial equations in the vector x of the 56 irrational coefficients. For each Lyndon word ℓ of odd weight 3 to 9 in the letters $g_0, \ldots, g_4$, the equation $E_\ell(x) = 0$ states that the coefficient of the word ℓ in the expansion of the formula equals its coefficient in the series. Because a Lie polynomial is determined by its coefficients on the Lyndon words, the 70 equations are equivalent to the order conditions. They have rational coefficients, total degree at most 13

and 4951 terms in all; the weight-3 equation, for $\ell = g_0 g_1$ and with $x_1 = 5/3$ substituted, is

$$\tfrac{1}{2} + \tfrac{5}{3} x_3 x_{76} + x_5 x_8 x_{77} + x_{16} x_{20} x_{78} + x_{30} x_{35} x_{79} + x_{51} x_{58} x_{80} + x_{63} x_{71} x_{81} = 0 \qquad (4)$$

Root polishing. The coefficients were refined by Gauss-Newton iteration (Newton's method for the overdetermined, consistent system), $x \leftarrow x - (J^T J)^{-1} J^T E(x)$, with E and its Jacobian J evaluated from the exact rational coefficients in 320-digit decimal arithmetic. Convergence is quadratic. The largest $|E_\ell|$ at each iteration, starting from the coefficients rounded to 17 and to 30 significant digits, is shown in Table 3.

Table 3: Largest residual $|E_\ell|$ in Gauss-Newton polishing.

Start	Iteration 0	1	2	3	4	5
17 digits	10^{-15}	10^{-31}	10^{-53}	10^{-107}	10^{-206}	10^{-318}
30 digits	10^{-28}	10^{-57}	10^{-105}	10^{-210}	10^{-318}	

The final residual is at the level of the arithmetic precision, and the iterates agree with an earlier 150-digit solution to all of its digits. The values in Appendix A are rounded from this solution.

Conditioning. The Jacobian $J = \partial E / \partial x$ (70×56) has full column rank. Its singular values range from 11.5 to 5.9×10^{-8}, so its condition number is $\kappa = 2.0 \times 10^8$; with each column scaled to unit norm, $\kappa = 2.3 \times 10^7$. The two smallest singular values, 5.9×10^{-8} and 3.6×10^{-7}, lie well below the rest (the next is 1.1×10^{-5}). Full rank means that the solution is isolated, so every x_k is a real algebraic number. It also bounds the error of the polished coefficients by the final residual divided by the smallest singular value, about 10^{-310} (the solution was stored to 200 digits). In the opposite direction, rounding the coefficients to double precision changes the residuals by at most about 10^{-15}, far below the t^{11} truncation error at any useful step size, so the condition number does not limit the accuracy of the formula in double-precision use. Integer-relation tests (LLL on 140-digit values of x_2 and x_{76}) excluded minimal polynomials of degree 12 or less with integer coefficients below about 10^{10}, so closed forms, if they exist, are large.

Moment quadrature

By the criterion of [1] (moment-quadrature section), any rule of polynomial degree 9 gives moments accurate enough for order 10: g_0, which occurs alone, is then accurate to $O(t^{11})$, and g_k for $k \geq 1$ occurs only in terms of weight at least $2k + 1$. Three such rules are given here. Each was verified by substituting its moments, expressed in the exact moments $g_0, \ldots, g_9$, into Eq. 3 and comparing every word through weight 10 with the series.

Gauss-Legendre, five nodes. $g_k \approx 2t \cdot \sum_i w_i \cdot P_k(u_i) \cdot A(u_i \cdot t)$ (Eq. 42 of [1], with $\sum_i w_i = 1$), with the nodes and weights of Table 4, which extends Table 3 of [1]. The nodes are approximately ± 0.5384693101 and ± 0.9061798459.

Table 4: Gauss-Legendre nodes and weights for order 10.

Nodes n	Order	Nodes u_i	Weights w_i
5	10	0	$64/225$
		$\pm\frac{1}{3}\sqrt{5-2\sqrt{10/7}}$	$(322+13\sqrt{70})/1800$
		$\pm\frac{1}{3}\sqrt{5+2\sqrt{10/7}}$	$(322-13\sqrt{70})/1800$

Gauss-Lobatto, six nodes. The same form, with the nodes and weights of Table 5, which extends Table 5 of [1]. The interior nodes are approximately ± 0.7650553239 and ± 0.2852315165. The rule is exact for polynomials of degree 9. Because it includes both endpoints, the last sample of one step is the first of the next, so it needs five new samples of A per step, like the Gauss-Legendre rule, and it also provides A at the step boundaries. After a step is split, however, only the two endpoint samples can be reused, so the two half-steps need nine new samples. The weights must be applied to the products $P_k \cdot A$ as written; the alternative of integrating the interpolating polynomial through the six samples is exact only for A of degree 5, which is not enough for $g_1, \ldots, g_4$.

Table 5: Gauss-Lobatto nodes and weights for order 10.

Nodes n	Order	Nodes u_i	Weights w_i
6	10	± 1	$1/30$
		$\pm\sqrt{1/3+2\sqrt{7}/21}$	$(14-\sqrt{7})/60$
		$\pm\sqrt{1/3-2\sqrt{7}/21}$	$(14+\sqrt{7})/60$

Nine nodes (modified Newton-Cotes). With $u_i = -1, -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}, 1$, samples $a_i = A(u_i \cdot t)$, and weights obtained by integrating the interpolating polynomial against P_k,

$$
\begin{aligned}
g_0 &= \tfrac{t}{115500} \cdot (6235a_0 + 24576a_1 + 22440a_2 + 47025a_3 + 30448a_4 \\
&\quad + 47025a_5 + 22440a_6 + 24576a_7 + 6235a_8) \\
g_1 &= \tfrac{t}{23100} \cdot (-1247a_0 - 4096a_1 - 2992a_2 - 3135a_3 + 3135a_5 \\
&\quad + 2992a_6 + 4096a_7 + 1247a_8) \\
g_2 &= \tfrac{t}{1270500} \cdot (58295a_0 + 196608a_1 - 34320a_2 - 91575a_3 - 258016a_4 \\
&\quad - 91575a_5 - 34320a_6 + 196608a_7 + 58295a_8) \\
g_3 &= \tfrac{t}{84700} \cdot (-3429a_0 - 8192a_1 + 9856a_2 + 11055a_3 - 11055a_5 \\
&\quad - 9856a_6 + 8192a_7 + 3429a_8) \\
g_4 &= \tfrac{2t}{4129125} \cdot (69210a_0 + 80896a_1 - 282535a_2 + 1650a_3 + 261558a_4 \\
&\quad + 1650a_5 - 282535a_6 + 80896a_7 + 69210a_8)
\end{aligned}
\tag{5}
$$

The rule is exact for polynomials of degree 9 (degree 8 by interpolation, and degree 9 by symmetry). It needs eight new samples per step. Uniformly spaced nine-point Newton-Cotes nodes are not used, because of Runge's phenomenon: interpolation through equally spaced samples becomes unstable at high degree, and the nine-point Newton-Cotes weights alternate in sign, which amplifies rounding and data errors. (The g_0 weights above are all

positive.) The nodes used here nevertheless lie on a uniform grid of spacing 1/6 ($6u_i = 0, \pm2, \pm4, \pm5, \pm6$), and when a step is split into two half-steps, the nodes of the two halves include all nine nodes of the original step, as with conventional Newton-Cotes rules. The split then needs only eight new samples instead of sixteen. The nodes also contain the seven equally spaced nodes of the order-8 Newton-Cotes rule of [1], so an order-8 result is available from the same samples.

The leading quadrature error of g_0, $\int_{-t}^{t} A(\tau)\,d\tau - (\text{rule}) = \kappa \cdot t^{11} \cdot A^{(10)}(0)$, has the coefficients in Table 6, which extends Table 4 of [1]; the Newton-Cotes column is for the nine-node rule. For the six-node Gauss-Lobatto rule $\kappa = -1/1031443875$, and the seven-node Gauss-Kronrod rule below is exact to degree 11, so its error enters only at $O(t^{13})$.

Table 6: Leading quadrature-error coefficients κ of g_0 at order 10.

Order	Gauss-Legendre κ	Newton-Cotes κ (nine nodes)
10	$+1/1237732650$	$-7/15588936000$

Embedded pair on seven Gauss-Kronrod nodes

This variant gives an order-6 estimate M_6 and an order-10 estimate M_{10} from the same samples, and the difference $\|M_{10} - M_6\|$ estimates the error of M_6 and is a conservative bound for that of M_{10}. The seven Gauss-Kronrod nodes [3] contain the three Gauss-Legendre nodes (Table 7).

Table 7: Gauss-Kronrod and Gauss-Legendre nodes and weights.

Node u_i	Kronrod weight w_i^K	Gauss weight w_i^G
0	11008/48825	4/9
$\pm\sqrt{3/5}$	6250/46557	5/18
$\pm\frac{1}{3}\sqrt{5-6\sqrt{10/33}} \approx \pm0.4342437493$	$\frac{2028807}{16036300} + \frac{1852389}{13745400}\sqrt{10/33}$	
$\pm\frac{1}{3}\sqrt{5+6\sqrt{10/33}} \approx \pm0.9604912687$	$\frac{2028807}{16036300} - \frac{1852389}{13745400}\sqrt{10/33}$	

The Kronrod moments $g_k = 2t \cdot \sum_{i=1}^{7} w_i^K \cdot P_k(u_i) \cdot A(u_i \cdot t)$ $(k = 0, \ldots, 4)$ are exact for polynomials of degree 11 and behave like exact moments here. The Gauss moments $h_k = 2t \cdot \sum w_i^G \cdot P_k(u_i) \cdot A(u_i \cdot t)$ ($k = 0, 1, 2$, over the three Gauss nodes) are exact only to degree 5; in terms of exact moments,

$$\begin{aligned} h_0 &= g_0 - \tfrac{429}{100} g_6 + \tfrac{2431}{800} g_8 + \ldots \\ h_1 &= g_1 - \tfrac{99}{50} g_5 - \tfrac{22}{25} g_7 + \tfrac{2223}{1000} g_9 + \ldots \\ h_2 &= g_2 - \tfrac{27}{20} g_4 - \tfrac{91}{1000} g_6 - \tfrac{5151}{8000} g_8 + \ldots \end{aligned} \tag{6}$$

Stage 1 (order 6). From the Gauss samples alone, with three commutators,

$$\begin{aligned} D_1 &= \left[\tfrac{1}{2}h_0 + h_2,\ h_1\right] \\ D_2 &= \left[h_2 + \tfrac{1}{5}D_1,\ h_0 + D_1\right] \\ D_3 &= \left[3h_0 + 11h_2 + D_1,\ h_1 - \tfrac{5}{54}D_2\right] \\ M_6 &= h_0 - \tfrac{14}{5}D_1 + \tfrac{3}{10}D_3 + O(t^7) \end{aligned} \tag{7}$$

which is an order-6 formula in the Gauss moments, equivalent to Eq. 47 of [1].

Stage 2 (order 10). Seven further commutators $C_i = [X_i, Y_i]$, $i = 4, \ldots, 10$, may use $g_0, \ldots, g_4$, h_0, h_1, h_2, D_1, D_2, D_3 and each other. Besides the 70 order conditions, they must cancel the terms containing $g_5, \ldots, g_8$ through weight 9 that enter through the h_k (Eq. 6), which gives 78 conditions. With coefficients $y_1, \ldots, y_{96}$ (Appendix B), of which 35 are rational and 61 irrational,

$$\begin{aligned}
X_4 &= y_1 h_0 + y_2 h_2 \\
Y_4 &= y_3 g_1 + y_4 g_3 + y_5 h_1 + y_6 D_2 \\
X_5 &= y_7 g_4 + y_8 h_0 + y_9 h_2 + y_{10} D_1 + y_{11} D_3 + y_{12} C_4 \\
Y_5 &= y_{13} g_1 + y_{14} g_3 + y_{15} h_1 \\
X_6 &= y_{16} g_2 + y_{17} g_4 + y_{18} h_0 + y_{19} h_2 + y_{20} D_1 + y_{21} D_3 + y_{22} C_4 \\
&\quad + y_{23} C_5 \\
Y_6 &= y_{24} g_0 + y_{25} g_2 + y_{26} g_4 + y_{27} h_0 + y_{28} h_2 + y_{29} D_1 + y_{30} D_3 \\
&\quad + y_{31} C_4 \\
X_7 &= y_{32} g_0 + y_{33} g_4 + y_{34} h_0 + y_{35} h_2 + y_{36} D_1 + y_{37} C_4 \\
Y_7 &= y_{38} g_1 + y_{39} g_3 + y_{40} h_1 + y_{41} D_2 + y_{42} C_6
\end{aligned} \tag{8}$$

$$\begin{aligned}
X_8 &= y_{43} g_0 + y_{44} g_2 + y_{45} g_4 + y_{46} h_0 + y_{47} h_2 + y_{48} C_4 + y_{49} C_7 \\
Y_8 &= y_{50} g_0 + y_{51} g_2 + y_{52} g_4 + y_{53} h_0 + y_{54} h_2 + y_{55} D_3 + y_{56} C_5 \\
&\quad + y_{57} C_7 \\
X_9 &= y_{58} g_2 + y_{59} g_4 + y_{60} h_0 + y_{61} h_2 + y_{62} D_1 + y_{63} D_3 + y_{64} C_4 \\
&\quad + y_{65} C_5 + y_{66} C_7 \\
Y_9 &= y_{67} g_1 + y_{68} g_3 + y_{69} h_1 + y_{70} D_2 + y_{71} C_6 + y_{72} C_8 \\
X_{10} &= y_{73} g_0 + y_{74} g_2 + y_{75} g_4 + y_{76} h_0 + y_{77} h_2 + y_{78} D_1 + y_{79} D_3 \\
&\quad + y_{80} C_4 + y_{81} C_5 + y_{82} C_7 + y_{83} C_9 \\
Y_{10} &= y_{84} g_1 + y_{85} g_3 + y_{86} h_1 + y_{87} D_2 + y_{88} C_6 + y_{89} C_8 \\
M_{10} &= g_0 + y_{90} D_1 + y_{91} D_3 + y_{92} C_4 + y_{93} C_5 + y_{94} C_7 + y_{95} C_9 + y_{96} C_{10} + O(t^{11})
\end{aligned} \tag{9}$$

The large coefficients in C_8 multiply small combinations ($h_k - g_k$ and g_4); this is how stage 2 cancels the quadrature errors of the Gauss moments. With six new commutators (nine in all) the generic Jacobian rank reaches only 76, and 67 Levenberg-Marquardt restarts stalled at residual 1.1×10^{-2}; seven new commutators give exact solutions. The nodes must be the Kronrod nodes: a seven-node rule containing the Gauss nodes and the endpoints, $u = 0, \pm\frac{1}{3}, \pm\sqrt{3/5}, \pm 1$, is exact to degree 9, which suffices for Eq. 3, but not for this pair, because stage 2 uses g_4 in weight-7 terms to cancel the g_4 component of h_2.

Root polishing. The 61 irrational coefficients were polished as above, with the 78 conditions taken on the Lyndon words in $g_0, \ldots, g_8$. The Jacobian (78×61) has full rank 61, so the solution is again isolated; its condition number is 3.2×10^{10} (1.9×10^8 with column scaling). Starting from the coefficients rounded to 20 significant digits, the largest residual fell from 10^{-18} to 10^{-36}, 10^{-70}, 10^{-139} and 10^{-208} in successive iterations, and the polished values differ from the 20-digit values by less than 4×10^{-20} relative.

Cost per step

Table 8, which extends Table 6 of [1], gives the cost per step of Eq. 3; the Newton-Cotes column is for the nine-node rule. The embedded pair uses 10 commutators (20 matrix products) and 7 new samples per step.

Table 8: Commutators, matrix products and samples of A per step at order 10.

Order	Commutators	Matrix products	Samples per step Gauss-Legendre	Newton-Cotes (new)	Gauss-Lobatto (new)
10	9	18	5	8	5

Results

The formulas were tested on the three test problems of [1] (random, commutator-heavy, and rapidly varying relative to its non-commutativity), with the same reference solution. Table 9 lists the error in $\Omega(t)$ (Frobenius norm) for half-steps $t = 0.1$ and $t = 0.05$, and the error ratio between $t = 0.1$ and $t = 0.05$ for the random test. The table has the format of Table 7 of [1]. The order-10 rows give Eq. 3 with each moment rule (Newton-Cotes, 9 is the nine-node rule), followed by the embedded pair: M_{10}, M_6 and the error estimate $\|M_{10} - M_6\|$. The last row is the order-8 formula of [1] with four Gauss-Legendre nodes (Table 7 of [1]). The order-10 coefficients were read from the MATLAB file described below and used in double precision.

Table 9: Errors in $\Omega(t)$ (Frobenius norm) for the three test problems.

Order	Moments	Random		Comm.-heavy		Rapid		Ratio
		$t = 0.1$	$t = 0.05$	$t = 0.1$	$t = 0.05$	$t = 0.1$	$t = 0.05$	
10	Gauss-Legendre, 5	1.1e-10	5.6e-14	9.2e-08	4.6e-11	2.3e-12	2.8e-15	2041
	Newton-Cotes, 9	1.1e-10	5.6e-14	9.2e-08	4.6e-11	1.1e-12	1.9e-15	2041
	Gauss-Lobatto, 6	1.1e-10	5.6e-14	9.2e-08	4.6e-11	2.5e-12	2.0e-15	2041
	Gauss-Kronrod, 7	1.1e-10	5.6e-14	9.2e-08	4.6e-11	3.2e-13	2.2e-15	2041
	exact moments	1.1e-10	5.6e-14	9.2e-08	4.6e-11	3.2e-13	2.2e-15	2041
10 (pair)	Gauss-Kronrod, 7	8.7e-11	4.3e-14	8.9e-08	4.5e-11	2.9e-13	2.1e-15	2033
6 (pair)	Gauss-Legendre, 3	5.7e-07	4.5e-09	3.8e-05	3.1e-07	7.1e-08	5.5e-10	127
estimate	$\|M_{10} - M_6\|$	5.7e-07	4.5e-09	3.8e-05	3.1e-07	7.1e-08	5.5e-10	127
8	Gauss-Legendre, 4	6.7e-09	1.3e-11	1.9e-06	3.8e-09	3.9e-10	7.7e-13	508

Halving t reduces the order-10 errors by factors approaching $2^{11} = 2048$ and the M_6 errors by factors approaching $2^7 = 128$. (In the rapidly varying test at $t = 0.05$ the order-10 errors reach the rounding level of about 2×10^{-15}.) In the first two tests all moment rules give the same errors to within 1%. In the rapidly varying test the quadrature error of g_0 dominates: at $t = 0.1$, exact or Kronrod moments are about 7 times more accurate than Gauss-Legendre moments, the nine-node rule is about 2 times more accurate, and the Lobatto rule is about 10% less accurate. The embedded M_{10} is as accurate as Eq. 3, and $\|M_{10} - M_6\|$ tracks the error of M_6 to two or three digits. Compared with order 8, the order-10 formula costs three

more commutators and one more sample per step, and at $t = 0.1$ it is 20-170 times more accurate in these tests.

MATLAB coefficients

The accompanying function magnus10coef.m, called as [x,y] = magnus10coef(), returns the coefficients $x_1, \ldots, x_{81}$ of Eq. 3 and $y_1, \ldots, y_{96}$ of Eqs. 8 and 9 as column vectors, in the order of Appendices A and B. Rational coefficients are written as exact fractions and irrational ones to 30 significant digits.

Code availability

MATLAB implementations of the formulas (explde.m, magnus_exp.m and magnus10coef.m), these documents and their LyX and LaTeX sources are available at `https://github.com/kjinnovation/explde` and on MATLAB File Exchange, `https://www.mathworks.com/matlabcentral/fileexchange/183695-explde`.

Acknowledgment

This work was carried out with the assistance of Claude Opus 5.5 (Anthropic), a large language model. Under the author's direction, Claude derived the Magnus series in Legendre moments, searched for minimal-commutator formulas and optimized their leading error, and drafted portions of the text.

Appendix A: coefficients of the nine-commutator formula

Tables 10-12 list the coefficients x_k of Eq. 3; rationals are exact and irrational values are given to 30 significant digits. The position X_i, g_j means the coefficient of g_j in X_i.

Table 10: Coefficients $x_1, \ldots, x_{27}$.

	Position	Value
x_1	X_1, g_0	$5/3$
x_2	X_1, g_2	$0.090121751105428617920000 6432152$
x_3	Y_1, g_1	$0.132144147153367034682119532902$
x_4	Y_1, g_3	$-1/5$
x_5	X_2, g_0	$-0.472685390399902712406002183602$
x_6	X_2, g_2	$3.234017832472208244199078 49435$
x_7	X_2, C_1	1
x_8	Y_2, g_1	$0.161670971415242270251705225531$
x_9	Y_2, g_3	$-0.525919202398768195923304595384$
x_{10}	X_3, g_0	$-1.530795899667269237752891 56382$
x_{11}	X_3, g_2	$-0.910924396469683553350121373571$
x_{12}	Y_3, g_0	$-19/20$
x_{13}	Y_3, g_2	$-0.455192455511641898707317211627$
x_{14}	Y_3, g_4	$-7/160$
x_{15}	Y_3, C_1	$1/20$
x_{16}	X_4, g_0	$0.999065729598492021595447046029$
x_{17}	X_4, g_2	$-0.169685143102619456375901958438$
x_{18}	X_4, g_4	$3.920519298895263967344208834 85$
x_{19}	X_4, C_2	1
x_{20}	Y_4, g_1	$0.555796983434714384253066760376$
x_{21}	Y_4, g_3	$-0.607334116992801989897737814155$
x_{22}	Y_4, C_3	1
x_{23}	X_5, g_0	$-8.074209177861560751839231 02445$
x_{24}	X_5, g_2	$-10.049506946162107588048978 7299$
x_{25}	X_5, g_4	$80/3$
x_{26}	Y_5, g_0	$19/2$
x_{27}	Y_5, g_2	$12.101746197386900083920468 6308$

Table 11: Coefficients $x_{28}, \ldots, x_{54}$.

	Position	Value
x_{28}	Y_5, g_4	$-31.453154734991168509416348784 9$
x_{29}	Y_5, C_4	$1/20$
x_{30}	X_6, g_0	$0.27518845713816718187563006449 1$
x_{31}	X_6, g_2	$-0.95432684704216793410595406832 6$
x_{32}	X_6, g_4	$-0.78472726514475881257882377888 9$
x_{33}	X_6, C_2	1
x_{34}	X_6, C_4	$-1/4$
x_{35}	Y_6, g_1	$1.610457643768733072583672568 28$
x_{36}	Y_6, g_3	$2.948077284203842349930297106 40$
x_{37}	Y_6, C_3	-11
x_{38}	Y_6, C_5	1
x_{39}	X_7, g_0	$-0.24651749649211567203312757977 6$
x_{40}	X_7, g_2	$-2.90152105321322178449420125787$
x_{41}	X_7, g_4	$-0.72023195103780940629794777066 4$
x_{42}	X_7, C_2	1
x_{43}	X_7, C_4	$-1/8$
x_{44}	X_7, C_6	$-0.08348503464461511275951956460 48$
x_{45}	Y_7, g_0	$0.59664885877857046430944189247 1$
x_{46}	Y_7, g_2	$-0.08487262988818287782438284768 24$
x_{47}	Y_7, g_4	$0.18943929364205089599055087924 5$
x_{48}	Y_7, C_1	-1
x_{49}	Y_7, C_4	$1/11$
x_{50}	Y_7, C_6	$-2/9$
x_{51}	X_8, g_0	$-1.22504203400172707389155574478$
x_{52}	X_8, g_2	$2.93982839425824894217792481282$
x_{53}	X_8, g_4	$1.89056056479215080737398254598$
x_{54}	X_8, C_1	1

Table 12: Coefficients $x_{55}, \ldots, x_{81}$.

	Position	Value
x_{55}	X_8, C_2	$-9/4$
x_{56}	X_8, C_4	0.49372686399541858591636763951 3
x_{57}	X_8, C_6	0.14337154598526183963458903272 2
x_{58}	Y_8, g_1	$-1.10171588444014784576499503191$
x_{59}	Y_8, g_3	$-1.33521791511463352749639989683$
x_{60}	Y_8, C_3	3
x_{61}	Y_8, C_5	$-0.30027747599016455175300348526 1$
x_{62}	Y_8, C_7	0.04222263211698891530421004613 07
x_{63}	X_9, g_0	$-0.70731516563903405332737410465 4$
x_{64}	X_9, g_2	$-1.00852021385304266703414054617$
x_{65}	X_9, g_4	0.52231946246038520354754494644 2
x_{66}	X_9, C_1	$-0.03952502855118174506466119194 90$
x_{67}	X_9, C_2	1/2
x_{68}	X_9, C_4	0.12488418098607260497559711828 8
x_{69}	X_9, C_6	$-0.50143255551050646163555395341 6$
x_{70}	X_9, C_8	0.33534623821201253951659685903 0
x_{71}	Y_9, g_1	0.35907766379494446284474174786 2
x_{72}	Y_9, g_3	0.83784788218820374663773074501 1
x_{73}	Y_9, C_3	1/5
x_{74}	Y_9, C_5	$-0.02775154319653633604068644852 63$
x_{75}	Y_9, C_7	$-0.07515364995428795181286842778 51$
x_{76}	Ω, C_1	$-0.59473938159601729898776317705 3$
x_{77}	Ω, C_2	0.34540085021790638657804466361 1
x_{78}	Ω, C_4	$-0.11904162944066179896268903994 5$
x_{79}	Ω, C_6	0.19587905021773140087939791596 0
x_{80}	Ω, C_8	$-0.21969901114168458614611992313 4$
x_{81}	Ω, C_9	0.26305537297102741920339524027 0

Appendix B: coefficients of the embedded pair

Tables 13-15 list the coefficients y_k of Eqs. 8 and 9; rationals are exact and irrational values are given to 30 significant digits.

Table 13: Coefficients $y_1, \ldots, y_{32}$.

	Position	Value
y_1	X_4, h_0	0.41042202944903352379476037445 7
y_2	X_4, h_2	$-0.76718432679069514474857457384 9$
y_3	Y_4, g_1	$1/2$
y_4	Y_4, g_3	$-1.31232865221517550984653286316$
y_5	Y_4, h_1	$-0.06392671365308394402180740347 69$
y_6	Y_4, D_2	$3/32$
y_7	X_5, g_4	0.91162034425928592725629901078 3
y_8	X_5, h_0	$-0.12573510519491326294732844770 7$
y_9	X_5, h_2	0.44404634496789828993102076369 4
y_{10}	X_5, D_1	$1/6$
y_{11}	X_5, D_3	$-27/1760$
y_{12}	X_5, C_4	$1/12$
y_{13}	Y_5, g_1	0.75009726256748316045749537094 2
y_{14}	Y_5, g_3	$-0.89337268795514027245028122440 6$
y_{15}	Y_5, h_1	0.08343378421721751868455778502 21
y_{16}	X_6, g_2	-1
y_{17}	X_6, g_4	-1
y_{18}	X_6, h_0	0.83338475975721104940270366883 7
y_{19}	X_6, h_2	0.25982816228515838966914864286 5
y_{20}	X_6, D_1	$-1/3$
y_{21}	X_6, D_3	$81/6160$
y_{22}	X_6, C_4	$-0.17081364907504811429749003400 8$
y_{23}	X_6, C_5	1
y_{24}	Y_6, g_0	$-1/3$
y_{25}	Y_6, g_2	$25/2$
y_{26}	Y_6, g_4	$-18.68668399034902112742736671 67$
y_{27}	Y_6, h_0	0.65273783056228439496376137073 4
y_{28}	Y_6, h_2	$-13.49315587416866032272328378 85$
y_{29}	Y_6, D_1	$-1/2$
y_{30}	Y_6, D_3	$81/1760$
y_{31}	Y_6, C_4	$-1/3$
y_{32}	X_7, g_0	$-1/3$

Table 14: Coefficients $y_{33}, \ldots, y_{64}$.

	Position	Value
y_{33}	X_7, g_4	$-0.595796530427459543546087044697$
y_{34}	X_7, h_0	$0.544039246895247658107386805015$
y_{35}	X_7, h_2	$0.084049259577342630112910100425 3$
y_{36}	X_7, D_1	$-1/8$
y_{37}	X_7, C_4	$1/6$
y_{38}	Y_7, g_1	$0.552499056229169796055542507726$
y_{39}	Y_7, g_3	$0.033543624240028260800033040849 2$
y_{40}	Y_7, h_1	$-1.405542658789070875598302749 08$
y_{41}	Y_7, D_2	$-1/8$
y_{42}	Y_7, C_6	$1/3$
y_{43}	X_8, g_0	$-119/2$
y_{44}	X_8, g_2	$-1387/2$
y_{45}	X_8, g_4	$928.988353643688265914130123001$
y_{46}	X_8, h_0	$60.059420805047975262584109298 0$
y_{47}	X_8, h_2	$693.692381734424602201762733442$
y_{48}	X_8, C_4	1
y_{49}	X_8, C_7	$3/2$
y_{50}	Y_8, g_0	-92
y_{51}	Y_8, g_2	$-914.900054948477505605246355089$
y_{52}	Y_8, g_4	$1225.472334762136795897003468 68$
y_{53}	Y_8, h_0	$92.836306930719425723720982950 3$
y_{54}	Y_8, h_2	$915.140420340519614481104711094$
y_{55}	Y_8, D_3	$243/1760$
y_{56}	Y_8, C_5	-5
y_{57}	Y_8, C_7	6
y_{58}	X_9, g_2	6
y_{59}	X_9, g_4	$-8.159699309212123232846164286 82$
y_{60}	X_9, h_0	$0.074800449757430869782317096742 2$
y_{61}	X_9, h_2	$-6.048138787784094449037469038704$
y_{62}	X_9, D_1	$-1/20$
y_{63}	X_9, D_3	$27/8800$
y_{64}	X_9, C_4	$-0.049461934310637546152303509417 8$

Table 15: Coefficients $y_{65}, \ldots, y_{96}$.

	Position	Value
y_{65}	X_9, C_5	0.10022359004354862347599331759 6
y_{66}	X_9, C_7	$-0.08005196468721350546087089886 14$
y_{67}	Y_9, g_1	7.26294248076323644126729654658
y_{68}	Y_9, g_3	0.78985068023217125163717437860 5
y_{69}	Y_9, h_1	0.41701391065865161097463307389 5
y_{70}	Y_9, D_2	$-0.34513714361129632515113932922 6$
y_{71}	Y_9, C_6	10/3
y_{72}	Y_9, C_8	$-0.54450135961105547036158366956 8$
y_{73}	X_{10}, g_0	$-1/5$
y_{74}	X_{10}, g_2	9.51291317660040269267562552719
y_{75}	X_{10}, g_4	$-12.80105381569001968309354211 32$
y_{76}	X_{10}, h_0	0.15897496826812543017576691503 5
y_{77}	X_{10}, h_2	$-9.50378384438196258445870705920$
y_{78}	X_{10}, D_1	$-1/10$
y_{79}	X_{10}, D_3	0.00709141050666411711074437892981
y_{80}	X_{10}, C_4	0.00693770793595351213892558834982
y_{81}	X_{10}, C_5	0.06190896248426303430805421312 62
y_{82}	X_{10}, C_7	$-0.11783925113404158817571367995 9$
y_{83}	X_{10}, C_9	0.05956381295160887295145776675 42
y_{84}	Y_{10}, g_1	10
y_{85}	Y_{10}, g_3	20.98593651445981031692613136 93
y_{86}	Y_{10}, h_1	$-1.00602720808865272131737227031$
y_{87}	Y_{10}, D_2	0.46629344140130846335207534782 0
y_{88}	Y_{10}, C_6	0.59843516756383301733693476484 8
y_{89}	Y_{10}, C_8	0.10270499190060546929413557328 3
y_{90}	M_{10}, D_1	$-0.07450518628375078286451725689 42$
y_{91}	M_{10}, D_3	0.00514753783338198345546041267870
y_{92}	M_{10}, C_4	$-0.35880884571145524907498222040 3$
y_{93}	M_{10}, C_5	0.52042851337788882123213667180 3
y_{94}	M_{10}, C_7	$-0.07513321661315413792891928021 31$
y_{95}	M_{10}, C_9	$-0.48308516397863698101887145746 1$
y_{96}	M_{10}, C_{10}	0.25860321739725094932312872386 3

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