Switcheroo and Lanczos Gravity: MOND done right

Warren D. Smith (warren.wds@gmail.com) + CLAUDE, August-October 2026. This is the flagship of 3½ papers; a short introductory summary is http://viXra.org/abs/2610.0071.

Abstract. Einstein & followers proved GR(Λ) – general relativity with cosmical constant Λ – is the *unique* gravity theory obeying certain postulates. We show a loophole: allowing traceless *Lanczos tensors* L_{abc} permits more theories. That's desirable because GR contradicts observation – with two proposed remedies:

> **DM:** copious invisible "*d*ark *m*atter" (≈5× "normal matter" mass, and distributed differently) exists;
> **MOND:** replace GR by Milgrom's "*mo*dified *N*ewtonian *d*ynamics."

But: most experts claim DM has never been detected; MOND fits many observations better; and allowing *any* undetectable-DM distribution makes DM largely uncheckable/unusable. Meanwhile, MOND is theoretically unacceptable: not "generally covariant," with relativistic-MOND attempts ugly, complicated, highly non-unique, asininely demanding new fields ("dark matter" rematerializes!), and often afflicted with tachyons/acausality/instability.

We propose new simple "Switcheroo" (**SG**) and "Lanczos Gravity" (**LG**). Redone uniqueness arguments suggest LG/SG are the *only* theoretically-acceptable GR(Λ) extensions. LG uses L_{abc} and/or its "superpotential" P_{ab} to *modify* Einstein's field equation. SG obeys the unmodified equation, *except* Λ "switches" among multiple possible values, presumably from different quantum field (QF) vacuum phases in different places. Λ-stairstep phase-transitions ("switcheroos") occur when certain tensorial-invariant scalars $\vec{F}$ cross certain boundaries. The formulas for $\vec{F}$, while not entirely known, importantly involve $L_{abc}L^{abc}$ and/or $L^*_{abc}L^{abc}$ (* means tensor dual). Remarkably, {L_{abc}, P_{ab}} exists *only* in 4 spacetime dimensions. Gravitationally L_{abc} looks analogous to the "4-potential" vector A^u in electromagnetism. In QF theories A^u is central while electric/magnetic fields are not; so analogously L_{abc} might underpin future "quantum gravity" theories, not Weyl's curvature tensor $W^a{}_{bcd}$. GR can be reformulated based on Ricci curvature R_{ab} and {L_{abc} or P_{ab}}, rather than the metric g_{ab}. One should a priori *expect* switcheroos given no known reason for QF vacuum phase uniqueness yielding one Λ value always. For MOND-like SG behavior, $\vec{F}$ *must* involve L_{abc} because invariants involving only the metric tensor and its derivatives cannot work.

LG/SG contain GR, hence fit observations no worse. LG/SG potentially could explain: galaxy rotation curves, early big bang "inflation," recently-alleged nonconstancy of Λ during cosmological (FLRW model) history, "Hubble tension," and solve: galaxy-formation puzzles, "lithium problem," "satellite problem," black hole paradoxes, and perhaps "defend" against singularities. I do not know whether SG/LG really accomplish that, or are garbage. Answering that question could be long and difficult for astronomers. Complications: LG/SG and DM could *coexist*; it's unclear how SG/LG should best-handle Lanczos nonuniqueness; and the full set of possible LG gravity theories remain unclassified.

Signature −+++, MTW curvature signs, indexing-brackets () and [] carrying ½, ε_{abcd} = includes $|g|^{1/2}$ factor.

Table of contents

1. Plan of paper
2. Introduction: Quantum vacuum phase changes and the SG idea
3. MOND (as opposed to Dark Matter)
4. Moffat's MOG and Bekenstein's TeVeS (attempts to make MOND relativistic): both wrong
5. Review of Metric and Curvature tensors
6. Is General Relativity Unique? LG and SG argue "no."
7. "Gravitational Acceleration" in GR: I. spherically symmetric metrics
8. Proof of impossibility of imitating MOND using metric & curvature tensors & derivatives alone
9. Lanczos tensor
10. Big Table of Lanczos
11. Demonstrations traceless Lanczos tensor can overcome those MOND-impossibility proofs' obstacles
12. "Dirty Schwarzschild" (axisymmetric but no longer spherical) black hole
13. Prevalence of horizon-detectors
14. Nonrigorous intuitive reasons to think $L_{abc}L^{abc}\to+\infty$ as approach horizons, and that L_{abc} generically are horizon-detectors
15. Accelerations of "stationary observers" II: the Kerr black hole; and relation to $L^{abc}L_{abc}$
16. Why $L^{abc}L_{abc}=(4/3)A^{u}A_{u}$? The "VAS identity" provides a substantial partial explanation
17. Should (and could) other tensors be involved in $\vec{F}$?
18. GR could be reformulated to be based on Lanczos & Ricci instead of metric tensor: the "Six Foundations Theorem"
19. How should SG handle "Lanczosian gauge freedom"?
20. Do stairsteps violate conservation of energy? Nature & philosophy of the cosmical constant Λ
21. The Superpotential
22. Potential vast power of SG/LG to explain/match observations DM/GR cannot
23. Summary: Future work and Open problems. (Includes "criticisms & responses.")
24. References
25. Appendix: The "Kerr-Newman-de Sitter" metric and associated shorthands, and its Kerr-de Sitter, Kerr-Newman and Kerr special cases
26. Appendix: Kerr pure-horizon Lanczos detector: closed form, invariants, and sparsity patterns

---

1. Plan of Paper

Intro introduces the SG concept that the quantum vacuum might have phase changes triggered by a tuple $\vec{F}$ of curvature-related zero-indexed (scalar) tensors, with different phases having different Einstein cosmical constant Λ values. MOND introduces the DM versus MOND controversy. Wrong argues that the top two attempts to make relativistically acceptable MOND versions (Bekenstein's TeVeS and Moffat's MOG), both are wrong. Review reviews GR tensor formalism.

Unique reviews arguments (begun by Einstein but also investigated by many later authors) that General Relativity GR(Λ) is the *unique* acceptable metric-based theory of gravity, but then points out that by adding

traceless Lanczos tensors L_{abc} to the list of allowed ingredients, a wider class of gravity theories become possible which still obey all Einstein and company's desiderata. Acceleration I initiates the investigation of how to set up a relativistically acceptable version of Milgrom's MOND, by enquiring "what is radial 'gravitational acceleration' in spherically-symmetric metrics?" Impossibility proves it is *impossible* for any metric-based theory of gravity to "imitate MOND" using metric & curvature tensors & their derivatives alone. Lanczos introduces (defines and reviews) the "Lanczos tensor" L_{abc}; here we shall almost wholy be interested in *traceless* L_{abc}, meaning obeying $L_{ab}{}^{a}=0$. This section also shows an important equation in Lanczos theory, the "Dolan-Kim wave equation," was wrong (block of six sign errors), and corrects it. Big Table of Lanczos gives a handy table of Lanczos tensor solutions claimed by many previous papers; then lists a smaller number of such solutions (including new contributions) explicitly. Our explicit list has the advantage that we verified its members by numerical checking. Demonstrations shows that adding traceless Lanczos tensors to the gravity picture can overcome those MOND-imitation impossibility proofs' obstacles. It introduces SG and LG as the prototypical examples of how that can be done. LG uses L_{abc} to modify Einstein field equation, explicitly showing several ways to do that; but we have *not* fully classified *all* possible such modifications. SG keeps the unmodified equation but postulates Λ has multiple possible values, not just a single constant value, with different values in different regions presumably caused by phase change of the quantum vacuum. SG is simpler and easier to think about than LG, at least initially for anybody pre-marinated in the GR literature (which is why this work focuses on SG); but LG seems more likely to be correct. The section shows how SG can produce MOND-like acceleration-thresholded two-gravity-law behavior, as well as black hole "horizon detectors" for the entire Kerr-Newman-deSitter (non-extreme) 4-parameter black-hole family, as well as extreme-Reissner-Nordström. A different interpretation of the same phenomenon is that SG switcheroos "create gravitational effects which make it *appear* as though dark matter exists." Some naive critic might say "Einstein's Λ is too small to affect galaxy-evolution and dynamics." But in fact, Λ-switcheroos for, e.g, the Kottler (Λ-Schwarzschld) metric, force metric parameters, including Kottler's "mass" parameter, to change, as a side effect. Dirty shows that Lanczos-based "horizon detectors" exist for Schwarzschild black holes even if *perturbed* by axially-symmetric static "dirt" — we initially proved this only in first-order perturbation theory, but later found a far simpler proof of a far better (exact, not perturbative) result. Prevalence shows that if at least *one* traceless Lanczos L_{abc} exists that has $L_{abc}L^{abc}$ infinite on almost all points of a hole-horizon, then that "horizon-detector" property is "prevalent" i.e. happens for almost every Lanczos tensor. Nonrigorous offers an intuitive reason to think $L_{abc}L^{abc}\rightarrow+\infty$ as approach black hole horizons from outside, and in view of that plus the results in the prior three sections **conjectures** that *generic* Lanczos tensors in generic physical situations should be "horizon detectors." Accelerations II investigates what "gravitational acceleration" should mean in the Kerr (*rotating* black hole) metric, and shows the "usual relation" $L^{abc}L_{abc}=(4/3)A^{v}A_{v}$ between Lanczos contracted-square and squared gravitational acceleration (appropriately defined) can still hold even for Kerr, except that it is best reformulated in terms of the *complexified* Lanczos tensor (introduced in this work), and an additive error term proportional to the contracted square of the complexified Weyl tensor appears. This error becomes relatively negligible both near and far from Kerr's horizon, and vanishes for nonrotating black holes.

Other tensors investigates what other scalar invariants, besides complexified $\mathring{L}_{abc}\mathring{L}^{abc}$, might be involved inside $\vec{F}$. Surprisingly, almost all the usual suspects, i.e. well known curvature-and-metric-based tensors and algebraic combinations thereof (I'm oversimplifying a bit here), are shown to be ineligible, which suggests a sense in which SG is a remarkably near-*unique* theory.

Reformulated points out (summarizing the first part of the Companion Paper) the theorem that differential geometry, and general relativity, do *not* need to be based on the metric tensor g_{ab} as the fundamental

object that everything else is defined in terms of. There are other possible foundations ("six foundations theorem"). For SG/LG, the alternative foundations that seem most interesting are the "Ricci-Lanczos team" $\{R_{ab}, L_{abc}\}$ and the "Ricci-Superpotential team" (if a superpotential P_{ab} exists) $\{R_{ab}, P_{ab}\}$.

Gauge freedom discusses what should be done about the problem that "Lanczosian gauge freedom" makes traceless Lanczos tensors highly non-unique. Several possible answers are proposed. I do not know which one is right but currently favor the idea everything should be based on the superpotential. Conservation of energy addresses the philosophical worry that SG's "stairsteps" might violate "conservation of energy," arguing they don't.

Superpotential defines and discusses the Lanczos superpotential" P_{ab} which (when one exists) generates $L_{abc}=P_{c[a;b]}$. There has been a widespread myth (which we ourselves initially believed...) that Illge, Andersson, and/or Edgar during 1988-2000 proved that superpotentials can fail to exist. But when we closely examined their papers, we discovered that myth was based on a misinterpetation of their papers. What they really showed was that nonexistence of a *different* object can happen. This myth-debunk does not prove a superpotential exists globally, but by means of closed form constructed solutions and/or existence theorems, we were able to prove that in many cases, plus via computer local examination at random points show "local existence" apparently happens everywhere. The net effect is completely one-sided evidence suggesting the conjecture that superpotentials P_{ab} always exist, and even if we further demand $\nabla^b P_{ab}=0$. And even if they do *not* always exist (a problem occuring on a hypothetical small "bad" subset of metrics), that need not matter physically, because the fact that all of physics and differential geometry can be based on the Ricci-Superpotential team means that with those laws of physics, Einstein's metric would be a *derived* rather than *foundational* quantity, in which case superpotentials would always exist and physics would never enter the "bad" set of metrics.

Observations discusses – briefly – the potentially vast power of SG/LG to explain/match astronomical observations that the prior mainstream GR(Λ) & dark matter picture apparently cannot match. Investigating astronomical datasets to refute, confirm, and/or refine/define SG/LG theory will hopefully be the mission of many future papers by many workers, and perhaps will inspire new observation efforts too. Summary summarizes, and lists open problems for future investigators. Finally we list References and provide two Appendices.

Probabilistic proofs by random checking. Polynomial identities over, e.g. the complex numbers such as $(x+y)^2=x^2+2xy+y^2$ could be proven by simplifying both sides to a canonical form (e.g. fully expanding into monomials) using known properties of the complex field. But this sometimes can be extremely onerous. A classic example is the identity det(AB)=det(A)det(B) where A and B are d×d matrices. The full expansion of det(A) into monomials contains d! terms, which if d=40 would be too much even for all computers on Earth operating for the whole history of the universe. However, if random numbers are substituted into the matrices, my personal computer could check this identity in ≈100 microseconds using machine floating point, and in perhaps 1 millisecond using exact integer arithmetic. If any exact check fails, the identity is disproven. The "Schwartz-Zippel lemma" showed that checks using random integers conducted in arithmetic modulo a prime, would disprove any untrue identity with probability lower bounded by a certain known formula. Therefore we can "probabilistically prove" any such identity simply by repeated random checks, obtaining known confidences exponentially near 100% that the identity is valid. As far as we are concerned, a probabilistic proof with failure chance$\leq 10^{-100}$ is satisfactory for all practical purposes, and often much easier to produce. One of the accomplishments of Smith 2004 was showing how to extend the Schwartz-Zippel lemma to allow "proof by checking" for larger classes of identities: (1) over possibly-noncommutative rings, and (2) *differential-algebraic* identities. All the tensor identities in this paper can be efficiently mechanically proven in the latter manner.

2. Introduction: Quantum vacuum phase changes and the SG idea

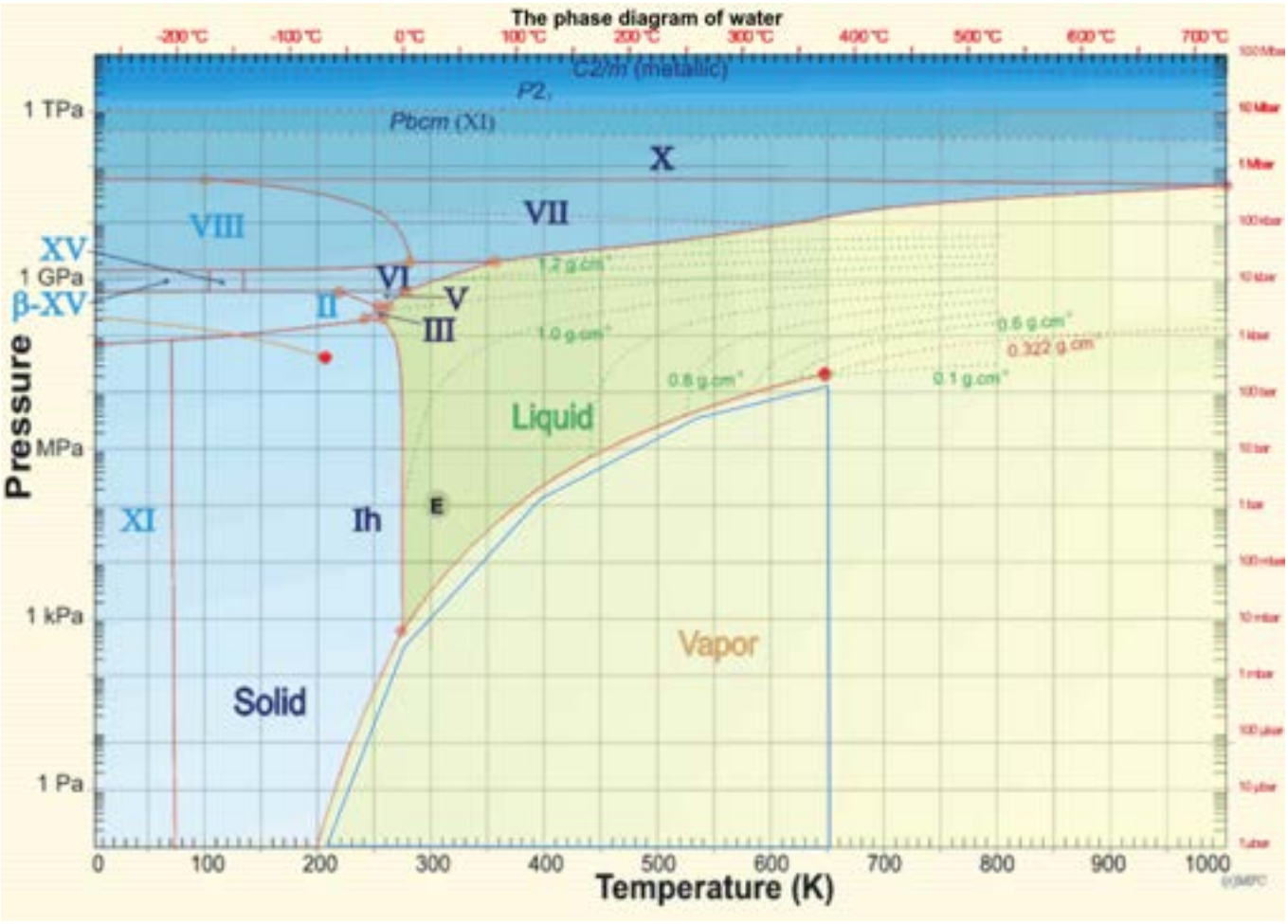
QF-vacuum phases. The QF "standard model" enjoys large experimental support. It claims that the "Higgs scalar field" pervades space with a near-constant value ≈246 GeV everywhere. However, at very high temperatures that would no longer be true; hot Higgs fields instead would be disordered, causing the very concept of "mass" no longer to behave the way we are used to. Gould, Güyer, Rummukainen 2022 estimated from lattice-QF computations that this vacuum transition occurs at temperature 159.5±1,5 GeV/k_B≈1.85×10^{15}Kelvin. When the early universe first cooled below that temperature, it experienced the "electroweak symmetry breaking" phase transition.

Lattice-QCD computational studies also indicate that the standard model has a different phase transition, at the much cooler temperature 155 MeV/k_B≈1.8×10^{12}Kelvin: when the early universe was hotter than that protons and neutrons were "melted" into "quark-gluon plasma," in which "color" was no longer "confined" in the sense we are used to. That allowed color-charge "conduction," unlike today's universe which is a color-insulator.

Some people, despite a complete lack of experimental support for SUSY, have predicted that at very high temperatures (well beyond those attainable by particle accelerators, and probably available nowhere in today's universe except perhaps inside black holes where it cannot be observed) there will be a transition to a "supersymmetric vacuum."

Other people starting with A.H.Guth have advanced the notion that the early big-bang involved exponential "inflation" fueled by a scalar quantum field associated with a new hypothetical "inflaton" particle. Indirect relic evidence suggests inflation happened. The end of the inflationary era could also could be regarded as another vacuum phase transition. Indeed, "inflationary cosmology" actually is the *mainstream* model these days, which can be regarded as *assuming* the existence of a dramatic vacuum phase change; under that view SG should be regarded as very conservative, i.e. not going beyond already-highly-accepted ideas, i.e. *any theorist who disputes SG presumably must also dispute inflation.*

Nitrogen (N) at 2000-2600 Kelvin is a gas, but when subjected to pressures>115 GPa converts into a stiff covalently bonded solid (Eremets et al 2004). Many such phase transitions are thought to be (and in large-atom-count limits of certain oversimplified exactly-solved statistical mechanics models, provably are) *discontinuous* jumps. That also is true to a good approximation *experimentally*: the ratios of the triple-point temperatures for mercury, gallium, and water have been localized to within a few *micro*Kelvins.



Temperature is not the only quantity whose change can cause phase transitions. Two others are *pressure* and *magnetic field*. For example, Martin Chaplin's picture above shows the thermodynamically-stable phases of H_2O (water) as a function of both temperature and log(pressure). It is quite complicated. Although most humans only encounter water's gaseous, liquid, and solid ice-I_h phases, the diagram shows at least 11 different crystal phases of ice exist, plus at ultrahigh pressures (not accessible with today's technology, but likely occurring inside suitable large planets) two more, plus a metallic phase, are predicted.

Suspicious lattice QCD simulations and models reviewed by Alford, Schmitt, et al 2008 predict a complicated quark-matter phase diagram containing many phases when *both* temperature T and "chemical potential μ per unit mass" (i.e. Newtonian gravitational potential, or relatedly hydrostatic pressure) are varied. As a well-confirmed example of the effect of μ, I point out that neutrons are unstable: they beta-decay into a proton, antineutrino, and electron with halflife≈14.6 minutes. *But*, because of μ, neutrons *are* stable inside neutron stars, and also (via a different, nongravitational, mechanism) inside many atomic nuclei.

Switcheroo gravity (SG)'s hypotheses about vacuum phase transitions:

1. Temperature, pressure, and magnetic field are *not* the only quantities whose change can trigger phase transitions. **Tensorial quantities $\vec{\mathbf{F}}$ constructed from the curvature-related traceless *Lanczos* tensor and spacetime *curvature* tensors can, by changing, trigger phase transitions of the quantum vacuum**.
2. Different quantum-vacuum phases can exhibit **different values of the Einstein cosmical**

constant Λ.

3. SG phase transitions are ***not* associated with any "latent heat"** (absorb or release of energy) despite making Λ jump discontinuously.
    - That's unlike: water-ice and water-steam transitions, with latent heats 333.55 & 2260 Joules/gram respectively (at 1 atm and 0 & 100°C) as well as tin's grey↔white solid-solid transition at 285°K with latent heat≈20 kJoule/gram. Superconducting→not transitions caused by applying magnetic fields also exhibit nonzero latent heat, for example for lead at 5.2°K this latent heat is 0.248 Joule/kg.
    - But it is like: the electroweak Higgs transition, and the 1.18°K and 39°K transitions of aluminum and MgB_2 into BCS-superconductive states (causing electrical resistivity and magnetic permeability both to discontinuously drop to zero) in zero external magnetic field, because their latent heats are theorized, and for BCS superconductors also measured, to be zero. Also zero latent heat: iron's ferromagnetic-ordering "Curie point" transition at ≈1044°K, and chromium's antiferromagnetic-ordering "Neel point" transition at ≈311°K, both associated with "corners" or "cusps" in the plot of magnetic-susceptibility (or its reciprocal) versus temperature, e.g. see fig.2 in Mugiraneza & Hallas 2022.
4. Given that the densities of water and ice are not each just one number, but rather vary continuously when we alter temperature and pressure, you might worry that Λ also should vary continuously. However, water's density only varies by ±5% (between 0.95 and 1.05 gram/cm^3) at temperatures 0-100°C and pressures 0-100 atm. The density of ice varies ±1% (between 0.934 and 0.917) at temperatures between -260 and 0°C at 1 atm; and pressurizing to 1000 atm increases its density less than 1%. Therefore, for most human practical purposes, modeling water's density as *constantly* 1 gram/cm^3 and ice's as 0.92 gram/cm^3 are acceptable approximations (for, say, a "physical theory" consisting of using the Navier-Stokes equations to model water and ice), even though under extreme conditions they might not be. Similarly, SG postulates that it is an acceptably-good approximation to regard **Λ as constant for each vacuum phase**, even though under ultra-extreme conditions probably available nowhere in today's universe (except perhaps unobservably inside black holes) that approximation might fail.
5. Relatedly, SG postulates that the switcheroo-surfaces, across which Λ jumps discontinuously – and therefore the Einstein curvature tensor G_{ab} also jumps – are *not* the location of infinitesimally thin, infinite-density 2D layers of "mass." This postulate could be argued not really to be a "postulate" but rather merely part of the "definition of the concept of mass." What *is* "mass"? Well, if you define mass *metrically* via the Einstein field equations then yes: sometimes switcheroo surfaces *must* have such mass-layers, and indeed these supply precisely the "mass change" M_{out}-M_{in} occuring at the spherical switcheroo surface between two Kottler metrics. But if you define mass in terms of particles, then no: there aren't any particles there (at least, not of any particle-type the "standard model" ever heard of, nor any extension of it to include "dark matter" particles) – that switcheroo surface could reside inside pure vacuum. So I would say: switcheroo surfaces can *effectively* act like 2D mass-layers, but do not *actually* have mass.
6. SG also postulates that **vacuum phase transitions are *instantaneous*** or anyhow far quicker than any time scale that matters to people simulating things like galaxy-evolution and colliding black holes.
7. SG demands the metric g_{ab}, and all components of $\vec{F}$ and L_{abc}, be **continuous** functions of spacetime location, including across Λ-switcheroo surfaces. This considerably restricts the class of possible formulas for $\vec{F}$.
8. A considerable further restriction (hopefully not all the way to the empty set!) would come if SG also were to demand that all black-hole **event-horizon** surfaces also be special, i.e. qualify as actual, or at least honorary, Λ-stairstep surfaces. I.e. SG then would also demand that "whether you are on a

black-hole horizon" be *deducible* from $\vec{F}$.

Remark: *if* the quantum-field vacuum exerts a gravitational effect, then it *must* be in the form of Einstein's Λg_{ab} term, because (i) the quantum field vacuum is by assumption the same as viewed by all Poincare-Lorentz observers, and (ii) it by assumption conserves mass-energy, and (iii) the uniqueness proofs by Einstein and successors for the GR(Λ) field equations (to the extent you support them).

To make hypothesis (1) plausible, I will now discuss some simple toy-models of physics-like scenarios in which **curvature definitely *can* trigger a phase change**. The first two toy models involve neither gravitational physics nor quantum fields; they are intended merely to demonstrate my point that curvature can trigger phase changes. The third and fourth models try to be at least a *little* quantum and gravitational.

Model 1: Percolation for unit-density Poisson-random points in a plane. Consider a Poisson point process causing any area=1 subset of the plane to contain, in expectation, one point. [An area=a region contains exactly n points with probability=$a^n e^{-a}/n!$.] Paint the region within an area-A disk centered at each Poisson point, red. Then I claim there will be a critical value A_{crit} of A such that if $A>A_{crit}$ then with probability=1 at least one connected infinite-area red region will exist, but if $A<A_{crit}$ then with probability=1 every connected red component is bounded. If we allow nonEuclidean "hyperbolic planes," then I claim that the value of A_{crit} should be an increasing function of curvature for curvatures<0, see example 3 in Thäles 2014. In the Euclidean plane case (zero curvature) $A_{crit}\approx 1.1281$.

Model 2: "Penny packing." There is unique way to place rigid interior-disjoint unit-radius disks in the Euclidean plane with each disc touching exactly 6 neighbors. This arrangement uniquely achieves the maximum possible packing density $12^{-1/2}\pi\approx 0.9068996821171$. Plainly at sufficiently low temperatures and high pressures, it would be the unique favored phase of hard-disc "matter." The curvature of the Euclidean plane is zero. What happens if we change this constant curvature to make it slightly positive (surface of a sphere) or negative (hyperbolic nonEuclidean plane)? With positive curvature, no matter how slight, it becomes impossible for any disc to touch 6 others of the same radius, and the densest possible packing would in general become unsymmetric. With exactly zero curvature the maximum achievable average number of disc-contacts per disc is 6. But after any slight alteration in curvature (to a generic real of either sign) it discontinuously drops to ≤4 presumably causing dramatically smaller "electrical conductivity" and "stiffness." When curvature decreases to approximately -0.297324642 it becomes possible (in a unique way) to pack the hyperbolic plane with unit-radius pennies each touching exactly 7 others – a quite different "crystal structure." That yields packing density≈0.91429461288746. But if the curvature of the hyperbolic plane is altered slightly away from this magic value (in either direction), then no such packing exists anymore. More generally, the nonEuclidean length of each side of a triangle in the curvature=-1 hyperbolic plane whose internal angles are each $2\pi/n$ radians, is $\ell=\cosh^{-1}(\csc(\pi/n)^2/2-1)$. Its hyperbolic area is $(n-6)\pi/n$. Therefore "pennies" with radius=$\ell/2$ may be packed in that hyperbolic plane in such a way that each penny touches exactly n others. The hyperbolic area of each penny-disk is $4\pi\sinh(\ell/4)^2$, causing the packing density to be $\rho=12\sinh(\cosh^{-1}(\csc(\pi/n)^2/2-1)/4)^2/(n-6)$. This packing density formula yields the correct Euclidean answer $12^{-1/2}\pi$ in the limit $n\to 6$, and is an increasing function of $n\geq 6$, approaching $3/\pi\approx 0.954929658551372$ in the limit $n\to\infty$. The maximum achievable average number of disc-contacts per disc, as a function of the curvature of the geometry, has many jump-discontinuities but can be made arbitrarily large by making the curvature unboundedly negative.

Model 3: gravitational "atoms." Suppose that protons and electrons were electrically *uncharged*, and therefore only attracted each other gravitationally. If we simply replaced Coulomb's intercharge force law

by Newton's law of gravity, the "Bohr radius" would then not be 5.291772×10^{-11} meters, but rather 1.20×10^{29} meters, also known as 1.27×10^{13} light years. Instead of an electron-proton atom, a two-proton "atom" would be considerably smaller, $<10^{7}$ light years wide. The questions of whether such 2-component "atoms" could form and stably persist, evidently then would crucially depend on the value of Einstein's cosmical constant Λ, or equivalently the scalar curvature $R=4\Lambda$, or the characteristic length scale $\Lambda^{-1/2}$. In our universe this length scale is of order 10^{10} light years, which would permit uncharged proton-proton but not electron-proton atoms.

Model 4: supersymmetry. I do not endorse the (experimentally unsupported) concept of supersymmetry in physics. Here I merely want to point out that supersymmetry is mathematically possible in the AdS (anti de Sitter) and Minkowski Λ-vacuum spacetimes, but *impossible* for de Sitter. This possibility→impossibility transition occurs as Λ, or equivalently the scalar curvature $R=4\Lambda$, crosses 0 changing from attractive→repulsive. Scalar curvature R is not the only determinant of supersymmetrizability: The *non*vacuum Einstein 1917 "static universe" $\mathbb{R}\times\mathbb{S}^3$ unstable abandoned cosmological model (in which the attractive effect of "dust" is precisely balanced by the repulsive effect of Λ, causing that universe to be time-invariant), despite having the same-sign R as de Sitter, *can* support supersymmetry in the sense that the $\mathbb{R}\times\mathbb{S}^3$ boundary of AdS_5 can support a supersymmetric conformal field theory.

Non-Apocalypse: S.Coleman, F. De Luccia, A.H.Guth, ond other physicists have at various times offered colorful descriptions of what could happen if we in our universe happen to be living in a "false vacuum," i.e. a quantum-field vacuum state which is not the ground state. According to those accounts, if somewhere a small bubble of "true vacuum" managed to overcome some potential-energy barrier and nucleate, then we would expect the walls of that bubble would then begin expanding at the speed of light leaving a trail of devastation behind them (planet Earth, the Sun, etc, all presumably destroyed, the "ultimate ecological catastrophe") until the whole universe was converted to true vacuum plus a hell of a lot of energy. We reassure the reader that in stairstep gravity **that won't happen.** We are indeed postulating more than one vacuum state, and these states indeed might in general be expected to have different energies, *but* the only switchovers occur when a governing spacetime curvature-related quantity F crosses some threshold at which those two energy curves *cross.* In other words, we only switch when the new and old vacuum states have the *same* energy, and we are always in the ground state. So there will be no devastation, zero released or absorbed energy, and indeed the change (if the two $|\Lambda|$'s were both small) would probably be extremely difficult to detect with local measurements.

Indeed (to illustrate my point): Suppose that inside atomic nuclei, the Einstein cosmical constant Λ has a quite-different-than-usual value. (Which might well be true.) That would be extremely difficult to experimentally prove or disprove.

Unrealistically low "temperatures"? The Λ-switcheroo that SG uses to cause MOND-like effects corresponds (in a vague analogy to a condensed matter system) to a transition happening at very low pseudo-"temperature" compared to the numerical values arising from the allegedly-natural Planck unit system – perhaps so low that some people might worry this is unrealistic. This is a very vague criticism. I have two responses. First, phase transitions do not necessarily occur at the temperatures you might naively guess. The "natural" temperature scale in atomic physics arises from the Rydberg energy $E_{Ryd}\approx13.60569$eV, corresponding to temperature $T_{Ryd}=E_{Ryd}/k_B\approx157887.5$°K. This exceeds the alleged 40 nanoKelvin superconducting transition temperature of gold by 12½ orders of magnitude! Second, great hype has been attached to Planck units based on the speed of light c=299792458 meter/sec, Newton's gravitational constant $G\approx6.6743\times10^{-11}$ $meter^3kg^{-1}sec^{-2}$, Boltzmann constant $k_B\approx1.380649\times10^{-23}$ $meter^2second^{-2}Kelvin^{-1}kg$, and reduced Planck constant $\hbar\approx1.054571817\times10^{-34}$ $meter^2kg/second$, as the

"natural" unit system for quantum gravity. However, I point out that we instead may define a complete unit system ("cosmical units") instead based on c, G, k_B, and Einstein's $\Lambda \approx 1.1056\times10^{-52}$ meter^{-2}. MOND's "switcheroo" happens close to naive expectations based on the *latter* unit system! The following table compares these two systems.

Notion	Planck unit	Cosmical unit
Length	$G^{1/2}\hbar^{1/2}c^{-3/2} \approx 1.6163\times10^{-35}$meter	$\Lambda^{-1/2} \approx 9.510\times10^{25}$meter
Time	$G^{1/2}\hbar^{1/2}c^{-5/2} \approx 5.391\times10^{-44}$second	$\Lambda^{-1/2}c^{-1} \approx 3.172\times10^{17}$second$\approx 1.0053\times10^{10}$years
Mass	$G^{-1/2}\hbar^{1/2}c^{1/2} \approx 21.764\times10^{-8}$kg	$\Lambda^{-1/2}c^3G^{-1} \approx 1.281\times10^{51}$kg
Energy	$G^{-1/2}\hbar^{1/2}c^{5/2} \approx 1.9561\times10^{9}$Joule	$\Lambda^{-1/2}c^3G^{-1} \approx 1.151\times10^{70}$Joule
Temperature	$G^{-1/2}\hbar^{1/2}c^{5/2}k_B^{-1} \approx 1.4168\times10^{32}$Kelvin	$\Lambda^{-1/2}c^3G^{-1}k_B^{-1} \approx 8.34\times10^{92}$Kelvin
Angular momentum	$\hbar \approx 1.054571817\times10^{-34}$meter^{2}kg/second	$\Lambda^{-1}c^2G^{-1} \approx 3.651\times10^{87}$meter^{2}kg/second
Pressure & Energy density	$G^{-2}\hbar^{-1}c^{7} \approx 4.633\times10^{113}$Pascal	$\Lambda c^3G^{-1} \approx 1.338\times10^{-8}$Pascal
Mass density	$G^{-2}\hbar^{-1}c^{5} \approx 5.155\times10^{96}$meter^{-3}kg	$\Lambda c^2G^{-1} \approx 1.489\times10^{-25}$meter^{-3}kg$\approx$89.66 meter^{-3}amu
Acceleration	$G^{-1/2}\hbar^{-1/2}c^{7/2} \approx 5.561\times10^{15}$second^{-2}meter	$\Lambda^{1/2}c^2 \approx 9.4502\times10^{-10}$second^{-2}meter

In the cosmical unit system $c=G=\Lambda_{Ein}=1$, the **natural unit of acceleration** $\Lambda^{1/2}c^2$ happens to equal $8a_0$ where a_0 is Milgrom's MOND critical acceleration!

Another numerological coincidence is as follows. The **deuteron** arguably is the simplest (and certainly smallest) stable particle made of at least two components, at least one of which is stable in isolation. The center-separation between the proton and neutron in a deuteron is believed to be about 4.2×10^{-15} meter. The (Newton law) acceleration caused by one gravitationally attracting each toward the other then is 6.3×10^{-9} meter/sec$^2 \approx 53a_0$.

The **proton** is the smallest stable composite particle. It is made of 2 up-quarks (each charged +2/3) and 1 down-quark (charge=-1/3). If those quarks form an equilateral triangle with sidelength=1.46×10^{-15} meter (roughly true), then the Newtonian gravitational attraction to one of the up-quarks experienced by either of the two others, equals $1a_0$ exactly (to within current experimental errors in the quark masses and Milgrom's a_0)!

Of course, these coincidences are not exactly ironclad evidence for anything. But they have probability-enhancing value. The "cosmical acceleration unit" could have (as far as Einstein knew in the early 1920s) come out to be anything in a very wide numerical range anywhere from 0 to about $+10^{10}a_0$. So the fact that it happens to be within 1 order of magnitude of a_0 presumably is an event of a priori probability ≤0.2. Hence, the a posteriori probability of the correctness of stairstep gravity should in view of this "evidence"

be multiplied by ≥5.

The deuteron & proton coincidences are less impressive since if a_0 had been smaller I likely would instead have provided the (smaller) Newtonian acceleration of the electron in positronium, or Bohr's hydrogen atom [$3\times10^{-7}a_0$], or some such, and declared *that* to be the amazing coincidence. The fact that there are several candidate values makes such a coincidence more likely and hence makes its occurrence less astonishing and hence less-good evidence for anything. Nevertheless, I think this still ought to be considered good enough for a factor≥2 probability gain.

So whatever probability you were going to estimate for the correctness of stairstep gravity – please multiply it by ≥10.

What switcheroo gravity *isn't*: SG does not claim to be, and is not trying to be, "the theory of everything" or even "the final gravity theory." Indeed, I believe the latter would need to be a "quantum gravity" theory that is "microscopic." While SG, unlike GR, tries to maintain diplomatic relations with the quantum world, it is not a quantum theory. Both SG and GR are classical field theories. And both describe gravity sources, i.e. matter and energy, in terms of "density" and "pressure" and "stress" functions implicitly modeling matter as a continuum. Matter actually is made of particles describable using quantum fields, but neither SG nor GR regard it that way.

SG/LG are merely trying to be like GR except hopefully able to better-match observed phenomena, while still making sense relativistically, staying simple, trying to be a unique gravity-theory choice in a way similar to Einstein GR(Λ)'s provable uniqueness as a classical theory of gravity meeting certain desiderata; and trying to remain friendly with possible future attempts to quantize gravity. I used the word "trying" because those attempts cannot be claimed (at least at present) to be wholy successful, and "hopefully" because determining which version of SG or LG best fits observations, and computing how well it fits, is a difficult task that has only barely begun.

Do I really believe SG or LG describe our universe? I'm agnostic on this question. SG/LG might be garbage. Probably even more frighteningly, *both* SG *and* DM might be correct and coexist. At present there are two main camps, the "MOND camp" and the (larger) "DM camp," who each enjoy disparaging the other by pointing out astronomical observations the other supposedly cannot explain (but rarely, if ever, entirely convincingly), plus mocking the other's philosophies as embarrassing/absurd. That has happened for over 40 years, and I see no end in sight. In short, the situation is a mess that probably is hard to resolve. That suggests to me that even if SG or LG is completely wrong or completely right, it will take a lot of time and effort to determine which, become confident of that, and determine the right SG/LG details.

What I will say is: *if* something like MOND is valid, *then* SG/LG are – by far – the nicest, simplest, most mathematically attractive theory that tries to produce MOND-like behavior. It is the first "non-asinine" MOND-imitating relativistically-acceptable theory. And SG/LG also look friendlier than plain-GR to hoped-for future quantization attempts (in fact, SG should take you partway to that goal while providing helpful hints), and is better than GR at "explaining why spacetime is 4-dimensional" and resolving "black hole paradoxes." Beautiful miraculous-looking mathematical things happen. It would be a pity if all that beauty was "wasted" by pertaining to a wrong physics theory. But that sometimes happens.

History of SG/LG idea. I had the main ideas, and wrote the first draft of this paper, by April 2012. The reasons it took so long to get it into a decent state, were mainly

1. my lack of access to GR symbolic-manipulation software tools; and general lack of "horsepower";
2. and the presence of many mistakes in other people's papers;
3. the technical difficulty of trying to compare SG versus observation and trying to decide which MOND

and DM (and anti-MOND and anti-DM) works by prior authors, to believe.
4. Too much stuff. I've cut the paper very substantially versus early versions.

I ultimately solved (1) by finding ways to simplify matters to the point where an unaided human could do it; and/or relying on calculations in GR papers by other authors; and getting access to CLAUDE; and (3) mainly by giving up (i.e. greatly diminishing my ambitions for this paper... hopefully future authors can tackle this) – I will discuss all those questions, but only briefly and with no attempt at a complete discussion or anything near a final evaluation of their status vis-a-vis SG, DM, and MOND. In other words: This is a theory paper, not (which I also want in future) experiment/observation papers. And re my lack of horsepower, let me tell you: when I met CLAUDE, the rubber hit the road. The whole plan I had in 2012... unlike a lot of the time with my plans, where something goes wrong at some point... this time it just progressed pretty smoothly step by step. It just kept on working. The worst two unexpected surprises – our debunking of a widespread myth about Illge, Andersson, and Edgar, and our surprising discovery of complex and bicomplex formulations for 4D Riemannian geometry (see companion paper Smith & Claude 2026) – actually *helped*.

3. MOND (as opposed to dark matter)

> The probability (in the Bayesian sense) of obtaining the observed MONDian phenomenology in galaxies given the prior that we live in a ΛCDM universe is practically zero.
> – Stacy McGaugh.
>
> MOND (or as I prefer to call it, MilgrOmiaNDynamics) is one of the most-verified physical theories: to my knowledge, all predictions made by Mordehai Milgrom in his three historical ground-breaking seminal 1983 papers, Bekenstein & Milgrom 1984) have been verified – no small feat because most of the predictions were done for systems or phenomena that had no data to support or even indication for them in the early 1980s.
> – Pavel Kroupa 2026.
>
> The gold standard: a clear a priori prediction not informed by the data in question. Our conclusion is that MOND is favoured [versus ΛCDM] by a wealth of data across a huge range of astrophysical scales, ranging from the kpc scales of galactic bars to the Gpc scale of the local supervoid and the Hubble tension, which is alleviated in MOND through enhanced cosmic variance.
> – I.Banik & H.Zhao 2022's abstract.

GR-based models of the whole universe, galaxy clusters, and galaxies all disagree with observation. This disagreement is most-simple and most-clear in the case of **rotating near-planar galaxies**. Many "rotation curves" have been measured for many galaxies, starting with Vera Rubin and Kent Ford's 1970 work on the Andromeda galaxy and the 1978 PhD thesis of Albert Bosma at Groningen, then continued by many others.

These curves show the orbital speed of stuff (most commonly hydrogen gas) round the galaxy, as a function of distance-to-axis r. If we assume that the position-dependent mass-density of a galaxy is proportional to its (also position-dependent) measured light-output ("mass to light ratio"), then we can *predict* the rotation curve from the light curve by using Newton's laws. When this is done, we get usually-huge qualitative and quantitative disagreements. In particular, the rotation curves usually stay quite "flat" (near-constant orbital speed) even at the largest distances-to-axis, including distances substantially further than the furthest-from-axis visible star! In contrast, Newton's laws for 2-body systems would predict orbital speeds should decrease toward zero proportionally to $r^{-1/2}$.

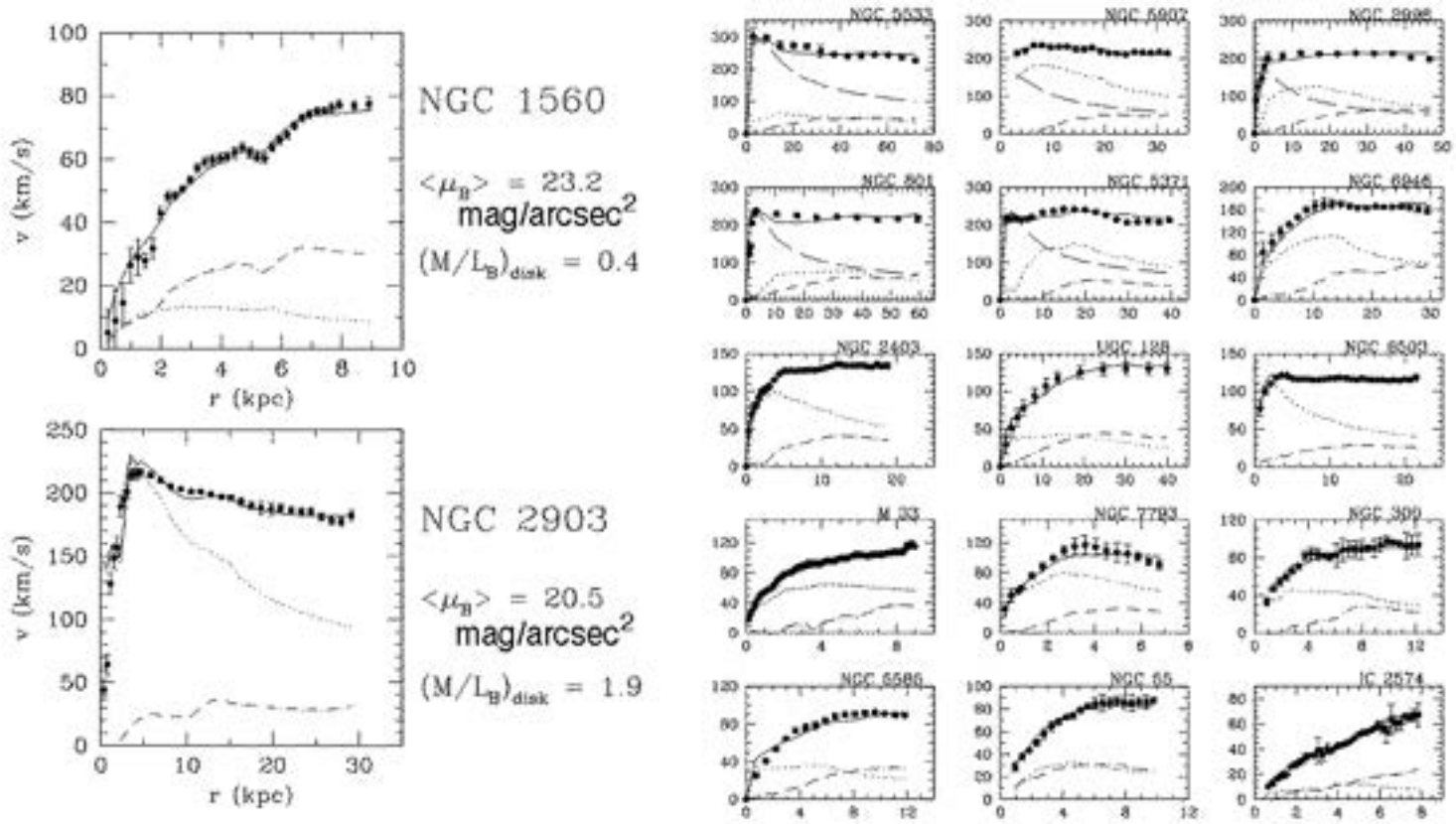


Figure: Galaxy rotation curves (Stolen from figs 3 & 5 of Sanders & McGaugh 2002). At left: 21 cm line rotation curves for NGC1560 (low surface brightness galaxy) and NGC2903 (high surf. bright.) Dotted & dashed curves show Newton-law-based rotation curves from visible & gaseous components of disk; solid curve is MOND fit using $a_0=1.2\times10^{-10}$meter/sec^2 [this number got from 10 nearby galaxies by Begeman et al 1991]; the only free parameter is mass-to-light ratio for each galaxy. At right: Same definitions for dotted, dashed, solid curves now fitting published spiral galaxy data. Obviously, the MOND fits far outperform Newton law.

The review by Sanders & McGaugh 2002 gives many more such pictures and data, e.g. see their figure 4. Still more such data (showing fits for both MOND and the MOG relativistic theory by J.W.Moffat) is in Brownstein & Moffat 2006. Their collection of 101 galaxies with radii 0.02-70 kpc and masses 10^5-10^{11} suns, including low- and high surface brightness and dwarf galaxies, shows that both MOND and MOG deliver good, but not perfect, results: their MOND rotation curve is always a far better fit than the best Newton law fit based on visible matter only, but nevertheless for about 5% of galaxies seems clearly inconsistent with the data, e.g. see galaxies F571-8, NGC2841, NGC2903, NGC4448 in their figure 2. For anyone now objecting "Aha! That bad 5% refutes MOND!" – sorry. The MONDists could simply hypothesize that in those 5% of galaxies, the light-to-mass ratio *varies* in different parts of that galaxy – which undoubtably sometimes is true. For example, consider Hoag's object, a galaxy featuring a ring of young hot blue stars encircling an older yellow nucleus. (It differs from any of Hubble's "galaxy types" because Hubble devised his type system in 1936, before Arthur Hoag discovered this, the first known ringed galaxy, in 1950.) We entirely expect "young hot blue stars" have different light-to-mass ratio than "older yellow" stars. In such situations, a poor fit of MOND to the rotation curve using only 1 fit-parameter – the "single" light-to-mass ratio for that galaxy – might well be a "success," not a "refutation," of MOND. Also, for anybody objecting to MOND's sole fit-parameter, the "mass to light ratios" for each particular galaxy: fitting that parameter is very difficult to avoid. Type-M7V stars have a mass-to-light ratio of about 10, while type-O3V stars have 0.0000013 (both in units of the sun's mass & luminosity), a factor 7.5 million smaller (Massey & Meyer 2001). So it is easy to imagine that two galaxies containing different mixes of

star-types, could easily have *very* different mass-to-light ratios. Hence this fitting parameter at least naively seems *unavoidable*.

Now shift consideration to **ellipsoidal** rather than flat-disk galaxies These have little or no overall rotation and the stars within them move in "random" fashion. This makes it much more difficult to measure their gravitational fields, stellar motions, etc. However, in a few cases, this has been accomplished – and the results seem very damaging to ΛCDM! Specifically, the ellipticals examined by Romanowsky et al 2003 were found to have an amazing *shortage* of dark matter compared with rotating-disk galaxies. Recall that the mainstream ΛCDM estimate is that dark matter outweighs normal matter by a ratio of 4.9:1. But in Romanowsky's elliptical galaxies, *normal* matter outweighed dark matter, by a ratio of over 2:1! To quote their abstract: "[our] data indicates... little if any dark matter in these galaxies' halos." But Milgrom & Sanders 2003 found that Romanowsky's ellipticals behaved as expected according to MOND, with excellent fits as usual.

Then Milgrom 2012 analysed two isolated X-ray ellipticals NGC 720 and NGC 1521, finding:

> [Their] gravitational fields have been measured to very large galactic radii (≈100 and ≈200 kpc), assuming hydrostatic balance of the hot gas enshrouding them. They afford, for the first time to my knowledge, testing MOND in ellipticals with force and quality that, arguably, approach those of rotation-curve tests in disk galaxies. In the context of MOND, it is noteworthy that the measured accelerations span a wide range, from more than $10a_0$ to about $0.1a_0$, unprecedented in individual ellipticals. I find that MOND predicts correctly the measured dynamical mass runs (apart from a possible minor tension in the inner few kpc of NGC 720, which might be due to departure from hydrostatic equilibrium): Meanwhile the predicted mass discrepancy [using plain Newton instead] increases outward from none near the center, to ≈10 at the outermost radii. The implications for the MOND-versus-DM controversy go far beyond the simple fact of two more galaxies conforming to MOND.

In the usual DM picture, one might have expected higher dark/normal matter ratio in more-massive galaxies since the dark matter could escape from a light galaxy. However, larger mass/light ratios occur for dwarfs, precisely the opposite of this intuition! (Such anomalies, however, need not cause any inherent problem with switcheroo gravity.)

"ΛCDM" seems unable to work well for many different galaxy shape and size types without resorting to apparently unrealistically-greatly varying parameters. And other problems for its agreement-with-reality when attempting to model stellar motions in galaxies have been dubbed the "cusp-core controversy," "Tully-Fisher and Faber-Jackson relations," "missing angular momentum problem," "nonconstant phase space density problem," problems with "missing satellite galaxies," wrong distributions of galaxy-types emerging from galaxy-formation simulations, the "globular clusters problem," and discoveries of some galaxies apparently *lacking* dark matter.

Milgrom has also collected many interesting review papers about MOND at http://www.weizmann.ac.il/weizsites/milgrom/mond/scientific-reviews, and written a 2014 review of MOND on "scholarpedia"; there now is a special issue of Canadian J. Physics 93,2 (Feb. 2015) about MOND; and Stacy McGaugh has a website "The MOND pages" about MOND at http://astroweb.case.edu/ssm/mond.

The rotation curves could be explained by postulating enormous amounts of **"dark matter"** (albeit a more-accurate name would have been "invisible matter") – undetectable by every known method *except* via its gravitational effects – distributed quite differently within and around the galaxy than the distribution of everything (such as stars, "MACHO" planets, dust, and gas) that people actually *have* been able to detect

(using many ingenious methods). One can try to infer the dark matter distribution from measured rotation curves and sometimes gravitational lensing, but all such deductions so far have been heavily "underdetermined," causing them to deliver highly non-unique answers (or answers which their authors or somebody else *pretends* are unique). Stunningly, it appears that the dark matter is distributed roughly spherically-symmetrically in 3D, even when that galaxy's visible stuff is distributed 2-dimensionally on or near a plane.

For this reason DM-advocates mainly believe that dark matter is made of some undiscovered **new kind (or kinds) of fundamental particles** (which, in total, considerably outweigh all known particles!). These new particles do not feel electromagnetic forces, i.e. cannot interact with photons, hence are invisible and unable to lose energy by radiating heat. They also do not feel strong/color forces, i.e. cannot interact with gluons (otherwise they would have been detected by now). They *might* be able to feel the weak force, i.e. interact with W and Z bosons, but the fact that many ongoing searches for such "WIMP" dark matter have been widely agreed to have failed, suggests DM cannot do that either. They *might* interact with the Higgs field and thus indirectly interact with other standard model particles, but could avoid that by instead interacting with a hypothetical new "dark Higgs" field which serves the same functions as the usual Higgs field, but only for dark clients. The final and simplest alternative is that DM particles cannot interact with any force at all besides gravity. In that case, direct detection of dark matter presumably will never happen. The stuff passes through you all the time without you noticing. That would be like the previous "phlogiston" and "aether" invisible-substance theories (both ultimately discarded) but this time for real... a disagreeable prospect.

Upping the ante, there seems to be good evidence from astronomical data suggesting that the DM hypothesis would need to be enhanced with a possibly-complicated "dark sector" of unknown dark particles and "dark forces" between them. That would make the DM hypothesis even more disagreeable.

The simplest version of **Mordehai Milgrom's 1983 MOND proposal** is the following. In Newtonian 3-apace with absolute time, let some countable number N of point-masses M_1, M_2, ..., M_N located at $\vec{X}_1(t)$, $\vec{X}_2(t)$, ..., $\vec{X}_N(t)$, at time t interact as follows. The "tentative acceleration" $\vec{A}_J(t)$ experienced by mass number J at time t equals the usual Newtonian formula

$$\vec{A}_J = G \sum_{k\neq J} (\vec{X}_k-\vec{X}_j) M_k |\vec{X}_k-\vec{X}_j|^{-3}$$

where G is Newton's gravitational constant. Its "*actual* acceleration" $d^2\vec{X}_J/dt^2$ equals the tentative acceleration $\vec{A}_J$ if $|\vec{A}_J|\geq a_0$, *but* if $|\vec{A}_J|\leq a_0$ instead equals $(a_0)^{1/2}|\vec{A}_J|^{-1/2}\vec{A}_J$. Here $a_0\approx 1.2\times 10^{-10}$ meter/sec^2 is a proposed new physical constant, "Milgrom's acceleration." This switches Newton's inverse-square force law to an inverse-linear law whenever |acceleration| drops below a_0. Milgrom 2015 emphasizes that dynamics under the inverse-linear law is "spacetime **scale invariant**" meaning that multiplying both space and time coordinates $(\vec{x},t)$ by any common scaling factor, leaves the dynamical predictions unaltered. That causes "circular velocities" (orbital speeds) of test particles to become independent of orbit radius, agreeing with observations that "rotation curves" for disc galaxies become "flat" (constant) outside some radius.

Possible smoother MOND-variants. Bekenstein & Milgrom 1984 (in postulate c on their first page) instead propose that the transition between these two laws of gravity *not* be "sudden" with a "corner" occuring exactly at the acceleration a_0, but rather an unspecified smooth join occuring "within a range [with width] of order a_0 about a_0." Milgrom 1983 his EQ 1 also considered using a possible continuous "joining function" join(x) (for real x≥0), which for "sudden" MOND is join(x)=min(x,1) featuring a corner at x=1, but

Milgrom allows changing it to something smooth such as

$$\text{join}(x) = \tanh(x), \quad 1-\exp(-x), \quad x/(1+x), \quad \text{or} \quad (x^2+1)^{-1/2}x,$$

then proposes that the relation between the true acceleration $\vec{B}$ and tentative acceleration $\vec{A}$ should be $\vec{B}\cdot\text{join}(|\vec{B}|/a_0)=\vec{A}$.

Let us take a little time here to explain **how incredibly stupid and ridiculous** the whole MOND suggestion was. **First** of all, it involves "instantanoeus action at a distance" (not even using "retarded time"), thus transmitting information at infinite speed, violating special relativity's demands that information must travel at speeds≤c and that all Lorentz observers agree what the laws of physics are. Those demands are supported by every experiment ever done, but Milgrom cavalierly dismisses them as old-fashioned garbage. But his Newton-like infinite-speed-transmission approximation arguably would be acceptable in *limits* where all speeds are much slower than c and all gravity weak – a regime galaxies (aside from small regions near neutron stars and black holes) occupy – and provided we only work in *one* special reference frame, namely the center-of-mass frame of that galaxy. (If we were trying to model the interaction of two or more galaxies, that might pose a major problem... which MOND blissfully ignored.) **Second,** Milgrom at all large-enough distances transmutes Newton's inverse-square force law into an inverse-linear force law. That makes it *mathematically impossible* for any body ever to "escape" to unboundedly large distances from any other – they're doomed to return – disagreeing rather seriously with observation. E.g. there are known cases of fast-moving stars that appear to be escaping, or have already escaped, their galaxy; and almost every galaxy appears to be escaping from almost every other, with the universe *not* appearing to be headed for a future "big crunch." **Third,** MOND often violates Newton's law of "equal and opposite reaction" thus causing N-body time-evolution often to violate conservation of momentum, energy, and angular momentum (even when N=2!), again contradicting approximately every experiment ever done in human history.

> Bekenstein & Milgrom 1984 realized this and hence in their EQ 2b presented an inequivalent **Lagrangian-based** MOND-reformulation satisfying momentum conservation, "action=reaction," angular momentum conservation, and a partial notion of energy conservation. But it still is nonrelativistic, still involves "instantaneous action at a distance," and still forbids escapes ("confining"). But B&M's Lagrangian-based MOND seems to have remained largely ignored and unused.

Fourth, MOND has neither "gravitational waves" nor "frame dragging" near rotating bodies; both are predicted by GR and observed.

But: on the plus side, MOND seems to describe the motions of matter inside of galaxies very *well* and very *simply*, with no need for mysterious invisible matter – just use the ordinary matter astronomers can actually detect while choosing the best "mass to light ratio" value (which may change if you change galaxies, but it usually works well to assume it constant within any one particular galaxy). Which suggests that MOND approximates some important correct idea.

The reason the last two paragraphs both are true, is because Milgrom's inspiration for and design of MOND was completely data-driven.

> I didn't know any physical reason why MOND should work. But it worked.
> – Mordehai Milgrom.
> I considered writing a paper "why MOND can't work." But I didn't because it'd be like writing a paper "why humans can't fly."

> – Jeremiah P. Ostriker.
>
> [MOND] cannot be considered a theory, but only a successful phenomenological scheme for which an underlying theory is needed.
>
> – J.D.Bekenstein & M.Milgrom 1984 (p.8).
>
> In my view we are still a far cry from a fully satisfactory MOND theory... The quest for a fundamental MOND theory is by no means over.
>
> – M.Milgrom 2015.

Therefore, it seems reasonable to seek some non-stupid and non-ridiculous theory, that somehow manages to predict behavior similar to MOND if applied to galaxies. Unfortunately, all prior attempts to do that have been ugly and complicated, and many have been shown to suffer from either acausality or instability. But beyond that, they were just **asinine**. To explain what I mean by "asinine," let us consider the two most most widely publicized such theories:

- MOG by John W. Moffat in 2008, and
- TeVeS by Jacob D. Bekenstein in 2004.

"TeVeS" stands for "*Te*nsor-*Ve*ctor-*S*calar" signifying that Bekenstein added new tensor, vector, and scalar fields to Einstein GR. "*Mo*dified *G*ravity" – MOG – was formerly called "Scalar-Tensor-Vector Gravity" but Moffat discarded that name apparently since he considered it too similar to TeVeS, which he thinks MOG obsoletes. Wikipedia's 1-paragraph summary says MOG "is based on an action principle and postulates the existence of a vector field, while elevating the three constants of [GR] theory to scalar fields. In the weak-field approximation, [MOG] produces a Yukawa-like modification of the gravitational force [from] a point source. Intuitively, this result can be described as follows: far from a source gravity is stronger than the Newtonian prediction, but at shorter distances, it is [partially] counteracted by a repulsive fifth force due to the vector field."

Let me review: MOND is absolutely theoretically ridiculous. Its sole advantages are: (1) simplicity, (2) no need for invisible never-detected "dark matter," and (3) fits most galaxy rotation curves well. TeVeS and/or MOG hopefully retain (3) – albeit that is less clear – but sacrifice (1) and then their introductions of *several new never-detected fields which cannot be detected by any presently known experimental method*, also sacrifice (2) by effectively *introducing* their own versions of "dark/invisible matter" – which actually are considerably *more* complicated and arbitrary than merely *one* new "dark particle" type! That is asinine! It would be like "avoiding" the need for phlogiston by instead postulating, as a "brilliant new theory," the existence of three competing kinds of invisible "fairies," "genies," and "unicorns" interacting in complicated ways which sometimes produce fires. If that is your alternative, I'd a priori recommend staying with phlogiston – and I'd also wonder why you even were proposing such an idea at all.

In contrast: **SG** will not introduce *any* new fields not already present in Einstein GR(Λ). SG's only change is that Λ "stairsteps" and those stairstep-locations are based in large part on invariants constructed from the traceless Lanczos tensor L_{abc}. This is *not* insanely speculative. If Einstein were inventing GR(Λ) today instead of 1917 – knowing what is now known about quantum field theory and phase transitions – he would a priori *expect* Λ-altering phase transitions and would be *surprised* if none existed, so that (surprisingly) Λ had only a single universal constant value. And we shall show that the usual invariants cannot work to govern stairstep locations in ways agreeing with Milgrom's observations, but Lanczos-based invariants can. So again, using L_{abc} seems "forced," not "wildly speculative," at least within the tensors known today to exist.

Furthermore, if Einstein in 1915-1918 had been aware of the existence of the Lanczos tensor and/or its superpotential and willing to permit it/them in gravity theory, then he would have *expected* our "**LG**" class of gravity theories, and would be *surprised* if gravity for some unknown reason refused to involve L_{abc}, and

would have demanded some explanation for that surprise.

MOND appears to fail badly when we expand past galactic length scales to **galaxy *cluster*** scales. Specifically the masses of galaxy clusters can be and have been measured (under GR) by gravitational lensing studies; indeed one can try to map their mass *distributions.* (Tyson 1992 and Umetsu 2020 review the lensing mass-mapping technique; Clowe et al. 2006 is the "bullet cluster" analysis, Sanders 2003 and Angus, Famaey & Buote 2008 quantify how much mass MOND still lacks in clusters and groups, and Mahdavi et al. 2007 report a lensing mass peak in Abell 520 largely devoid of galaxies – though Clowe et al. 2012 do not reproduce that peak, so Abell 520 should be cited only with that caveat. I.e, the lensing-observation-based claim is that galaxy-clusters have "mass that MOND cannot supply," distributed differently. I.e. clusters seem to contain a lot of lens-causing mass in places that appear empty. This has been interpreted as "proof" dark matter exists.

(For the MOND-side reading of the same data, see Angus, Shan, Zhao, Famaey 2007, who accept that the bullet cluster is dominated by a collisionless component but deny that this prefers Λ-CDM over MOND plus neutrinos.)

Perhaps MONDists could fight back by proposing a new empirical law (different than Einstein's) of how gravitational lensing works – why should only gravitational *acceleration* be governed by such laws? But ultimately it is not satisfying just to keep postulating new empirical laws ad hoc whenever observations are not fitted by the old laws. We want a *non*-ad-hoc, parsimonious, principled, theoretical framework.

4. Moffat's MOG and Bekenstein's TeVeS (attempts to make MOND relativistically acceptable): both wrong

As of 2012, searching the arXiv finds over 100 papers on MOND, TeVeS and/or Moffat's theory of gravity and their consequences. I only read a small fraction.

Bekenstein and Moffat's gravity theories appear to explain most galaxy rotation curves well without needing to assume dark matter exists, and also enjoy various other successes. But they and MOND (like the mainstream-ΛCDM picture) seem heavily hurt by Karachentsev 2012. Also, MOND has big problems explaining galaxy cluster and supercluster behavior (predicts too-strong galaxy-galaxy attraction; the MOND proponents argue/hope this could be due to normal but non-luminous matter in intergalactic space within the cluster, which is a MONDian version of the "missing baryon problem" afflicting the mainstream ΛCDM model).

I now point out a new and extremely simple yet **devastating theoretical objection to Moffat's theory** and everything like it.

Recall the "no hair theorems" about black holes saying that only observable quantities about a black hole are spin, mass, and charge. Other forces like weak and strong force, which have positive-mass virtual bosons as force carriers, cannot "reach" outside of a black hole because it is effectively an infinite climb and those massive bosons cannot make it without infinitely violating the uncertainty principles which allow them to exist.

Incidentally, I once wrote a paper (Smith 2003) on this, whose point was: the existence of electrically-charged black holes proves that photons must have *exactly* zero mass.

But Moffat stated (I quote from his 2006 paper "Scalar-Tensor-Vector Gravity Theory")

> An important feature of [my] NGT, MSTG,and STVG [gravity] theories is that the modified

acceleration law for weak gravitational fields has a *repulsive* Yukawa force added to the Newtonian acceleration law. This corresponds to the exchange of a massive spin-1 boson.

Note that both at short distances (where the Yukawa potential acts like a Coulomb potential) and at far ones (where it has exponentially vanished) MOG gravity behaves in a Newton-like (inverse square force law) manner, albeit with two different proportionality constants. The latter feature of MOG overcomes the flaw in MOND that escape energies are "infinite" and the related flaw that MOND can mispredict interactions between galaxies and between galaxy-clusters.

For almost any compact mass that is *not* a black hole, Moffat's virtual bosons are so incredibly light (Yukawa falloff distances of many kiloparsecs) that they should be essentially unimpeded by the finite climb they must make... the gravitational contribution to their masses will be at most of the same order as their rest masses.

But we see Moffat's extra kinds of gravity *cannot reach* outside a black hole; only the Einstein component can. In short, I claim Moffat and Einstein gravity ought to *agree* for black holes. Now since the short range Moffat forces/fields are repulsive and cannot reach outside a black hole, we see these highly-disagreeing *predictions*:

- Moffat-gravity: if a neutron star accretes a little more mass hence collapses into a black hole, its gravity viewed by nearby external observer effectively should increase by a factor≈5.9 as a result of this transformation. [Since Moffat gravity explains the 4.9-times-larger mass of "dark matter" in the universe vs normal matter, but without needing the dark matter; and since after the transformation Moffat-gravity is replaced by plain-Einstein gravity.]
- Einstein: gravity should stay about the same. Indeed, due to the Jebsen-Birkhoff theorem the metric outside of a spherically symmetric collapsing star will stay *exactly* the same.

There was also some numerical work by Moffat & Toth 2009 which seemed to confirm some of my thoughts here, albeit without giving my simple intuitive explanation.

This effect should already be enough to kill Moffat's alternative gravity theories as I now explain. Famous analyses determined the neutron star critical mass threshold for gravitational collapse. The simplest forms of these computations were originally done by Chandrasekhar for white dwarfs and by Oppenheimer & Volkoff, for neutron stars. The more recent calculation by Kalogera & Baym 1996 "obtains a secure upper bound on the neutron star mass equal to $2.9M_{sun}$" with "lowest possible upper bound of $2.2M_{sun}$"; Bombaci 1996 claims it lies between 1.5 and 3.0 solar masses. Astronomical observations of 61 pulsars (Zhang et al 2011) found as the heaviest three 4U1700-37 with mass 2.44±0.27, J1748-2012B with mass 2.74±0.2, and PSR B1516+02B with mass 2.08±0.19 M_{sun}. Those calculations should still work fine in Moffat gravity.

In view of Kalogera & Baym's threshold estimates, if a black hole exists with effective mass (judged by its local gravity field) below 13 solar masses, then that *proves* Moffat gravity is wrong, because the minimum threshold size for collapse is ≥2.2 solar masses which would effectively factor-5.9 amplify into ≥13 after collapse because of the cutoff of Moffat's repulsive boson force during the collapse.

However, known stellar-remnant black hole candidates have masses whose central estimates are fairly uniformly distributed between 3 and 16 solar masses, and whose standard errors range between ±3% and ±50% (but usually are about ±10%). See also https://www.ligo.caltech.edu/image/ligo20200902a for more (a picture showing the masses of gravitational-wave emitters detected by LIGO). [Incidentally, the experimentally observed range 3-16 solar masses happens to coincide exactly with the theoretically predicted range gotten from mainstream-GR numerical models of star collapse by Fryer & Kalogera 2001.]

In particular we mention GRS 1009-45 with mass=4.55±0.15 solar masses, GRO J1655-40 with mass=6.3±0.3, and XTE J1118+480 with mass=6.5±0.4. (All three have been confirmed as black holes by at least two independent observing teams. A recent review is ZioŁkowski 2008. Also GRO J0422+32 was estimated to have mass 3.97±0.95, but might not be a black hole and a later mass re-estimate was only 2.1 M_{sun}.) Hence: Moffat gravity is wrong.

This seems to me highly convincing: Because Moffat gravity is supposed to be virtually identical to Einstein gravity at all short (solar system & smaller) length scales, black hole collapse must work the same. But can't.

Moffat in his book did however vaguely raise the possibility on pages 202-204 that there might be a "super-grey star" in his gravity theory which would be a massive compact body much like a black hole, but with no horizon and no singularity. He apparently had in mind huge black holes like the "massive objects in the center of galaxies." The one at the center of the Milky Way is ≈4.1 million solar masses (this hole constitutes about 3×10^{-6} of the entire mass of our galaxy) which according to Einstein GR would have a gravitational radius of 12 million km. The apparent density of very large black holes can be arbitrarily small and in this case would be "only" about 1100 tons per cubic meter (far below the density of white dwarfs). The Milky Way's central black hole is small compared to those at the centers of some other galaxies, which therefore have much lower density. One of the largest known lies in the center of the supergiant elliptical galaxy NGC 4889 (the brightest galaxy in the Coma cluster and ≈336 million light years from Earth) and is estimated to be 2.1×10^{10} solar masses (which perhaps is about 0.3% of the mass of the whole galaxy), which would correspond to a gravitational radius of 6.2×10^{10} km=2.4 light-days (McConnell, Ma, et al 2011) with apparent density only 0.04 kg/meter^3 – slightly less than that of hydrogen gas at room temperature and pressure. But even for a hole this large – about 10× the size of our solar system – it seems doubtful this is large enough for Moffat-gravity to differ from GR enough to permit such an object to be a "grey star" instead of a black hole. And there is considerable observational evidence that the Milky Way center's supermassive black hole as well as numerous small "X-ray novae" holes genuinely do lack surfaces and hence have event horizons, see Broderick et al 2009 and Garcia et al 2001, as well as Narayan et al 2001 who confirmed this for all three of the candidates GRS 1009-45, GRO J1655-40, XTE J1118+480 we mentioned before; and the imaging by the Event Horizon Telescope.

What about Bekenstein's TeVeS theory? Bekenstein informed me by email that my above black-hole collapse argument does not refute his TeVeS theory, for the following reasons:

1. TeVeS has no massive fields (despite having a built in length-scale). Instead its new kinds of field are classical fields apparently with no quantum-field analogue (at least, not one of the usual kinds).
2. Indeed (Bekenstein remarks) all TeVeS propagating fields "show the linear dispersion relation indicative of zero mass (Sagi 2010)." [Although, Sagi finds 6 kinds of gravitational waves exist, which travel at 4 distinct speeds, all different from the speed of light.]
3. And TeVeS's vector field consists (by definition) entirely of unit-Minkowski-"length" vectors, hence cannot vanish outside the black hole or anywhere else (Bekenstein 2004).
4. And finally, explicit TeVeS black hole solutions have been exhibited showing vector and scalar "hair" (Sagi & Bekenstein 2008).

Nevertheless, I believe the following **fact refutes TeVeS:**

> A gamma ray burst was detected by satellites 1.7 seconds after LIGO detected a gravity wave burst from colliding neutron stars on 17 august 2017! (Same direction.) This demonstrated that **gravitational waves travel at the same speed as gamma rays** to within 1 part in 10^{15}. TeVeS, however, predicts *other* wave speeds (not observed) which it seems very hard to

believe could have escaped LIGO observation and not caused much longer time-discrepancies (and the more LIGO observes, the tougher such evidence gets).

I also find TeVeS dubious on theoretical grounds. Why? First of all, in the modern era, just obeying Einsteinian relativity and action principles with some sort of energy-positivity (and with equivalence in some limit to GR), is *not good enough.* It is unacceptable, or at least highly physically dubious, to propose the existence of a new kind of classical field *lacking a quantum analogue.* And when several such new fields are proposed simultaneously, the dubiousness multiplies.

Second, many papers, including some Bekenstein just cited, have found many dangerous problems for TeVeS:

1. Sagi 2010 finds either superluminal wavespeeds violating causality, or if subluminal then probably ultra high energy cosmic rays could not exist (contradicting reality) since they'd rapidly lose energy to new kinds of Cherenkov radiation.
2. Contaldi, Wiseman, Withers 2008 finds that "quite generically, [TeVeS]'s vector field develops caustic singularities on time scales of order the gravitational in-fall time. Having shown singularity formation is generic with or without matter, Bekenstein's original formulation of TeVeS appears dynamically problematic."
3. Seifert 2007 in his section V "rules out" the TeVeS paremeters originally suggested by Bekenstein (and a wide swath of parameter space) because they would "cause the Sun to be unstable on a timescale below 1 month." Other regions of parameter space are ruled out by big bang nucleosynthesis constraints and CMB spatial "power" spectrum (Skordis, Mota, Ferreira, Boehm 2006).
4. Ferreras et al 2012 claims their observations of "strong gravitational lensing over galactic scales" have experimentally refuted both MOND and TeVeS if dark matter does not exist.

Perhaps TeVeS still can successfully navigate through that minefield, i.e. perhaps parameters can be found which still match reality while avoiding acasuality and instabilities, but: (a) it seems unlikely, and (b) even if so, the very fact that the structure of TeVeS makes such problems easily possible makes me worry its fundamentals are wrong. (Incidentally, Moffat on pages 149-150 of his book offers a fairly cursory dismissal of TeVeS saying it was a good initial try showing "tenacity and courage" but which, he feels, is obsoleted by Moffat gravity. Moffat complains TeVeS is "not strictly speaking a fully relativistic theory" – perhaps he has in mind its sub/super-luminal wavespeeds? – and lacks a lower bound on energy.)

Finally, I have aesthetic objections to both TeVeS and Moffat's theories. They both are considerably more complicated than Einstein-GR, and the alterations to GR they made to obtain these theories look unlikely to be unique. (Or at least, nobody has yet offered a uniqueness proof or anything like it.) This is unlike the situation with Einstein GR, where there are uniqueness proofs. Therefore, there probably are many possible comparably "Rube Goldberg" ways to try to modify GR to get Moffat and Bekenstein's kind of effects. If there are N such ways, then the probability they found the right one, is of order 1/N, or less.

5. Review of Metric and Curvature tensors

I first successfully learned GR by reading the two **"old school" books** by Pauli 1958 and Lovelock & Rund 1975. A wonderful online "Catalogue of spacetimes" [mostly exact solutions of GR(Λ) with signature -+++] is Mueller & Grave 2010. A book on GR exact solutions is Stephani et al 2003. Those are excellent for understanding GR's core principles and concepts. But they will not tell you about everything that happened, both theoretically and observationally, after about 1960. There are many more modern books on that (most of which I have not read, such as Peacock 1999 and Carroll 2003), but some that I have

read and can recommend are Misner-Thorne-Wheeler 1973 ("**MTW**"), Weinberg 1972, Hawking & Ellis 1973, Peebles 1993, and Ciufolini & Wheeler 1995.

The introduction of modern tensor notation by Einstein was one of the greatest notational advances in the history of humankind. (Previous ones: the invention of "algebra," and J.W.Gibbs' "vector notation.")

Einstein, in developing GR, wanted to understand "tensors" – indexed-objects and laws *invariant* under, i.e. not caring about, who was observing them, and what coordinate system anybody was using to describe them. Einstein, building on earlier work by Riemann, Beltrami, etc, developed a system of notation and simple rules for how to manipulate that notation, such that a trained monkey can follow the rules, and if he does, then you're safe. Everything magically stays a tensor. They automagically "self-convert" to their versions valid for any other coordinate system. The rules automatically enforce a large number of little lemmas about tensors for you, without you needing to think about them. If you had to think about them every step you took, your life would be a nightmare.

When I see papers written in ad hoc non-Einstein notation – e.g. this means you, Shing-Tung Yau – it *infuriates* me because those jerks have tossed one of the greatest notational innovations in history into the garbage, in return for (in every case I ever saw) little or no benefit.

The fundamental object, which was the foundation upon which all tensor differential geometry was built, is the **metric tensor $g_{ab}=g_{ba}$.** (We shall see later that alternative foundations are possible, but first we'll develop everything in the usual manner starting from g_{ab}.) This is an object with two "downstairs" indices a and b. Hence g_{ab} may be regarded as a position-dependent D×D symmetric matrix if it is the metric tensor for a D-dimensional manifold. Physically D=3+1=4 and our metric tensors have signature −+++. If the manifold has a system of D coordinates x^1, x^2, ..., x^D, then the differential "element of length" (or, more properly "pseudolength" for space*time* pseudometrics) ds is given by

$$ds^2 = g_{ab}\, dx^a\, dx^b.$$

In this expression, and always, for conciseness we employ the "Einstein summation convention" for repeated indices (one upstairs one downstairs):

$$X_a Y^a \text{ means } \sum_a X_a Y^a, \quad Z_a{}^a \text{ means } \sum_a Z_a{}^a, \quad g_{ab}\, x^a\, x^b \text{ means } \sum_a \sum_b g_{ab}\, x^a\, x^b$$

unless we specifically say we do not want Einstein summation to be used on some expression (a rare desire).

Tensor indices can occur in two positions: upstairs and downstairs. Indices are **lowered and raised** with the metric tensor or its matrix-inverse: $M_a=g_{ab}M^b$, $M^a=g^{ab}M_b$. It is useful to employ the unsubscripted letter g to represent tha absolute value of the matrix-determinant of the metric tensor: $\mathbf{g=|det(g_{ab})|}$. This g is useful even though not a tensor. The main reason g is useful is because, if you want to integrate some quantity $Q(x^1,x^2,x^3,x^4)$ over 4-dimensional spacetime, you accomplish that via integral=$\int\int\int Q d^4V$, where the metrical element of spacetime 4-volume is $d^4V=g^{1/2}dx^1dx^2dx^3dx^4$. I will also write $\mathbf{Ⓢ=sign(det(g_{ab}))}$, which in the physical case is Ⓢ=−1.

The **Christoffel symbols** Γ_{ijk} of first kind Γ_{ijk} and second (mixed) kind $\Gamma^i{}_{jk}$ (which each are 3-indexed *non*tensors), expressed in terms of the metric tensor g_{ab} and its matrix inverse g^{ab} are:

$$2\Gamma_{ijk} = 2\Gamma_{ikj} = g_{ij,k} + g_{ik,j} - g_{jk,i} = 2g_{i\ell}\Gamma^{\ell}{}_{jk} \quad \text{and} \quad \Gamma^{i}{}_{jk} = \Gamma^{i}{}_{kj} = g^{i\ell}\Gamma_{\ell jk}.$$

where **comma** before a set of subscripts concisely denotes partial differentiation with respect to those coordinates, for example $g_{ab,k}=\partial_k g_{ab}$ and $g_{ab,jk}=\partial_k\partial_j g_{ab}$, where ∂_k in turn is a shorthand for $(\partial/\partial x^k)$. (Terminology: In differential geometry, the Christoffel symbols sometimes are called "the connection.")

Efficient way to compute **contracted** Christoffel symbols (result still is not a tensor): $2\Gamma^{a}{}_{ab}=2\Gamma^{a}{}_{ba}=\partial_b \ln|g|$.

More generally, taking ordinary partial derivatives of a tensor usually yields *non*tensors. However, "***covariant* derivatives**" of tensors [expressed using **semicolons** before a set of indices, e.g. $X_{ab;k}=\nabla_k X_{ab}$ and $X_{ab;jk}=(X_{ab;j})_{;k}=\nabla_k\nabla_j X_{ab}$] *are* tensors. The covariant derivative of a zero-index tensor, i.e. scalar field Φ, is the same as its ordinary derivative: $\Phi_{;a}=\Phi_{,a}$. Ordinary and covariant derivatives also are the same in the Minkowski flat-spacetime metric $ds^2=dx^2+dy^2+dz^2-dt^2$ because since $g_{ab}=g^{ab}=\mathrm{diag}(1,1,1,-1)$ is constant, all its derivatives are zero, hence all Christoffel symbol components are zero. However, in almost all other metrics, for tensors with one or more indices the two concepts differ:

$$V_{a;b} = V_{a,b} - \Gamma^{c}{}_{ab}V_c$$
$$V^{a}{}_{;b} = V^{a}{}_{,b} + \Gamma^{a}{}_{cb}V^c$$
$$T_{ab;c} = T_{ab,c} - \Gamma^{d}{}_{ac}T_{db} - \Gamma^{d}{}_{bc}T_{ad}$$
$$T^{ab}{}_{;c} = T^{ab}{}_{,c} + \Gamma^{a}{}_{dc}T^{db} + \Gamma^{b}{}_{dc}T^{ad}$$
$$T^{a}{}_{b;c} = T^{a}{}_{b,c} + \Gamma^{a}{}_{dc}T^{d}{}_{b} - \Gamma^{d}{}_{bc}T^{a}{}_{d}$$
$$L_{abc;d} = L_{abc,d} - \Gamma^{e}{}_{ad}L_{ebc} - \Gamma^{e}{}_{bd}L_{aec} - \Gamma^{e}{}_{cd}L_{abe}$$
$$L^{abc}{}_{;d} = L^{abc}{}_{,d} + \Gamma^{a}{}_{ed}L^{ebc} + \Gamma^{b}{}_{ed}L^{aec} + \Gamma^{c}{}_{ed}L^{abe}$$
$$L^{a}{}_{bc;d} = L^{a}{}_{bc,d} + \Gamma^{a}{}_{ed}L^{e}{}_{bc} - \Gamma^{e}{}_{bd}L^{a}{}_{ec} - \Gamma^{e}{}_{cd}L^{a}{}_{be}$$

etc. (hopefully the pattern is clear).

Any covariant derivative of the *metric* tensor automatically equals zero. Covariant derivative indices may be raised and lowered in the usual manner (they carry their semicolon with them). For scalar field Φ one formula more efficient than the straightforward one (Synge 1960, ch.V EQ 1) is $\Phi^{;u}{}_{;u}=g^{-1/2}\partial_u(g^{1/2}g^{uv}\Phi_{,v})$.

The 4-indexed **Riemann curvature tensor $R^{a}{}_{bcd}$** (which *is* a tensor despite being defined in terms of the nontensor Christoffel symbols and with ordinary partial derivatives) then is

$$R_{ijk\ell} = g_{\ell[i,j]k} - g_{k[i,j]\ell} + g^{mn}\left(\Gamma_{mjk}\Gamma_{ni\ell} - \Gamma_{mj\ell}\Gamma_{nik}\right)$$

[The overall sign of this expression was checked numerically in September 2026 against $R^{a}{}_{bcd}$ built from the Christoffel symbols in the MTW convention, on two metrics with no symmetry, at three points, agreeing to relative residual below 10^{-8}. The machinery was itself calibrated on the unit 2-sphere, where it returns $R=+2$ and $R_{\theta\varphi\theta\varphi}=\sin^2\theta$. An earlier draft printed exactly minus the right-hand side above.]

Note: **square brackets** around a set of n subscripts (or superscripts) means total *anti*symmetrization, i.e. $(n!)^{-1}$ times the sum of all versions of the expression with all n! permutations of those n subscripts, with

negative signs for the n!/2 odd permutations, is intended. (Beware: some authors omit my division by n!.) **Round brackets** similarly denote total symmetrization, i.e. the same sum but with all + signs. For example

$$2Q_{[ab]} = Q_{ab}-Q_{ba};\quad 2T_{(ab)} = T_{ab}+T_{ba};\quad 6X_{[abc]} = X_{abc}+X_{bca}+X_{cab}-X_{cba}-X_{bac}-X_{acb}.$$

Although ordinary partial derivatives commute, [$\partial_a\partial_b=\partial_b\partial_a$, rendering it irrelevant whether something's subscripts end ",ab" or ",ba"] covariant derivatives in general do not. The two most important non-commutation examples are (MTW EQ 16.6a p.389; this provides a common and equally valid alternative $R^a{}_{bcd}$ definition) $2V^a{}_{;[bc]}=R^a{}_{dcb}V_d\neq 0$ and $T_{ab;[cd]}=R^s{}_{acd}T_{eb}+R^s{}_{bcd}T_{ae}$. For a scalar they commute: $\Phi_{;[ab]}=0$, due to the fact that $\Gamma^c{}_{[ab]}=0$. Carroll EQ 3.68 gives a general version of these.

Identities obeyed by the Riemann curvature tensor include antisymmetries $R_{(ab)cd}=R_{ab(cd)}=0$; the cyclic identity $R_{a[bcd]}=0$; pair-swap symmetry $R_{abcd}=R_{cdab}$; and Bianchi's differential identity $R_{ab[cd;e]}=0$. (A consequence: $R_{[abcd]}=0$.) Thanks to the antisymmetries, cyclic identity, and pair-swap symmetry, all D^4 components of $R_{ijk\ell}$ are determined by at most $(D^2-1)D^2/12$ of its components, in D dimensions, i.e. in the physical case D=4 by $15\cdot 16/12=$**20** rather than 256.

A metric is **flat** if and only if $R_{abcd}=0$.

The symmetric 2-indexed **Ricci curvature tensor R_{ab}** then (MTW EQ 13.48) is this contraction of the Riemann tensor $R_{(ab)}=R_{ab}=R_{ba}=R^c{}_{acb}=g^{cd}R_{cadb}$. In Einstein "vacuum" metrics, $R_{ab}=0$, while in "Λ-vacuums" $R_{ab}=\Lambda g_{ab}$. *Symmetric* 2-indexed tensors such as R_{ab} and g_{ab} are determined by only (D+1)D/2 of their components, i.e. in the physical case **10** (not $D^2=16$).

Perhaps-faster alternative ways to compute Ricci without first computing the Riemann tensor:

$$R_{uv} = \partial_a\Gamma^a{}_{uv} - (\tfrac{1}{2})\partial_v\partial_u \ln|g| + (\tfrac{1}{2})\Gamma^a{}_{uv}\partial_a \ln|g| - \Gamma^a{}_{ub}\Gamma^b{}_{av} = \partial_c\Gamma^c{}_{uv} - \partial_v\Gamma^c{}_{uc} + \Gamma^c{}_{cd}\Gamma^d{}_{uv} - \Gamma^c{}_{ud}\Gamma^d{}_{vc}.$$

The **scalar curvature tensor R** then is the further contraction $R=R^a{}_a=g^{ab}R_{ab}$; this is also called the "trace" of the Ricci tensor.

The symmetric 2-indexed **traceless Ricci tensor S_{ab}** is $S_{(ab)}=S_{ab}=R_{ab}-g_{ab}R/D$ in D spacetime dimensions (where physically D=3+1=4). The reason I call that "tracefree" is that $S^a{}_a=0$. The symmetric 2-indexed **Einstein tensor G_{ab}** is $G_{(ab)}=G_{ab}=R_{ab}-g_{ab}R/2$. (A better name for it might be "trace-reversed Ricci.") As a consequence of the Bianchi identity it is divergence-free: $G_{ab}{}^{;a}=0$. (The right hand side really should have been 0_b, since it is a 1-indexed tensor with all components zero, but often as a practical abuse of notation people just write 0 for tensors all of whose components are zero, no matter how many indices they really have.)

The **Schouten tensor Q_{ab}** in D dimensions is $Q_{ab}=Q_{ba}=Q_{(ab)}=R_{ab}-Rg_{ab}/(2D-2)$. **Warning on normalization.** The companion paper's Schouten tensor is $(D-2)^{-1}$ times the Q_{ab} defined here - that is, half of it in D=4 - which is the normalization standard in the literature. The two papers' Cotton tensors J_{abc} nevertheless agree exactly, because the companion builds J_{abc} with a compensating factor 2. From the Bianchi-identity-consequence $2R_{ab}{}^{;a}=2R^a{}_{b;a}=R_{;b}$ (Weinberg EQ 6.8.3), we deduce

$Q_{ab}{}^{;b}=Q_b{}^b{}_{;a}=R_{;a}(D-2)(2D-2)^{-1}$; the divergence of the Schouten tensor equals the gradient of its trace, in every dimension.

"Ricci's decomposition" states that

$$R_{abcd} = [(D-1)D]^{-1}(g_{ad}g_{bc}-g_{ac}g_{bd})R + (D-2)^{-1}(S_{ad}g_{bc}-S_{bd}g_{ac}+S_{bc}g_{ad}-S_{ac}g_{bd}) + W_{abcd}$$

where $\mathbf{W_{abcd}}$ is the (and this defines the) **Weyl tensor** (introduced by Weyl in 1918). In the physical case D=4 we have

$$W^{ab}{}_{cd} = R^{ab}{}_{cd} - 2\delta^{[a}{}_{[c}R^{b]}{}_{d]} + \delta^{[a}{}_{[c}\delta^{b]}{}_{d]}R/3 \quad \text{or equivalently} \quad 2W_{abcd} = R_{abcd} - {}^*R^*{}_{abcd} \quad \text{or equivalently}$$
$$W_{abcd} = R_{abcd} - g_{a[c}R_{d]b} - g_{b[d}R_{c]a} + R\, g_{a[c}g_{d]b}/3$$

(MTW eq13.50 p.325; the form involving a "double dual" is from Illge 1993 p.448 and, in its double-dual form, from Lanczos 1938 - the four-dimensional identity now usually called the "Ruse-Lanczos identity," after Lanczos 1938 and Ruse 1944. Lanczos's own 1962 presentation of the Weyl-Lanczos relation was likewise written with dual operators rather than in the modern index form.) A D≥4 dimensional metric is "**conformally flat**" (equivalently, "Petrov type O") if and only if its Weyl tensor vanishes everywhere: $W_{abcd}=0$. In ≤3 dimensions, the Weyl tensor turns out to be identically zero, which would cause Einstein's GR in spacetimes with 3 or fewer dimensions, to be incapable of transmitting gravity through vacuum. In other words "gravity cannot exist unless spacetime is at least 4 dimensional." The Weyl tensor is traceless: $W^a{}_{bad}=0$. In that Weyl definition we have used the **"Kronecker delta" tensor $\delta^a_b=\delta^b_a=g_{ac}g^{cb}$** which equals 1 if a=b, otherwise 0. More generally, $\delta^{abc...d}_{wxy...z}$ equals 1 if all superscripts abc...d are distinct integers and an *even* permutation of wxy...z, but -1 if all superscripts abc...d are distinct integers and an *odd* permutation of wxy...z, and 0 otherwise.

Related are the **Levi-Civita D-indexed tensors**, which I'll define in the physical case D=4:

$$\epsilon_{abcd} = g^{1/2}\delta^{1234}_{abcd}, \qquad \epsilon^{abcd} = \text{Ⓢ}g^{-1/2}\delta^{abcd}_{1234}.$$

Covariant derivatives of Levi-Civita tensors are zero. One thing 4-dimensional Levi-Civita tensors are useful for is defining the **tensor dual $\mathbf{X^*_{ab}}$** of a 2-indexed antisymmetric tensor $X_{ab}=X_{[ab]}$ via

$$X^*{}_{ab} = \epsilon_{abcd}X^{cd}/2.$$

Dualizing twice results in multiplication by Ⓢ, i.e. in the physical case *negation*: $X^{**}{}_{ab}=\text{Ⓢ}X_{ab}$. So for example since $W_{abcd}=W_{bacd}$ we may define the **left dual of the Weyl tensor**

$$({}^*W)_{abcd} = \epsilon_{abpq}W^{pq}{}_{cd}/2$$

and since $W_{abcd}=W_{abdc}$ we may similarly define the right dual: $(W^*)_{abcd}=\epsilon_{cdpq}W_{ab}{}^{pq}/2$; but these are *equal* in view of the pair-swap identity $W_{abcd}=W_{cdab}$. Therefore, one may just speak of *the* "**Weyl dual**" $W^*{}_{abcd}$.

Complexified tensors. Once duals are in our arsenal, it turns out to be natural to use them to crate complex-valued tensors such as the **complexified Weyl tensor**

$$\mathring{W}_{abcd} = W_{abcd} + iW^*_{abcd}.$$

Here the circle underneath the W, as in $\mathring{W}$ means "complexified." This object exists only in 4 dimensions. The Companion Paper introduces these in greater detail, also devising *bi*complexified tensors such as $\underset{\circ\circ}{R}_{abcd}$, with *two* circles underneath to denote that this is the bicomplexified Riemann tensor. Notice that dualizing $\mathring{W}_{abcd}$ yields $i\mathring{W}_{abcd}$.

Several additional important tensors in D dimensions known to be useful for thinking about GR:

The **Bel-Robinson** tensor (Senovilla 2000) is defined (in two equivalent ways, albeit the second only is defined when D=4 because it employs the Weyl dual) by

$$T_{abcd} = T_{(ab)(cd)} = T_{cdab} = W_{aecf}W_b{}^e{}_d{}^f - (3/2)\, g_{a[b}W_{jk]cf}W^{jk}{}_d{}^f = W_{eabf}W^e{}_{cd}{}^f + W^*_{eabf}W^{*e}{}_{cd}{}^f.$$

But actually there is a nicer definition, given here for the first(?) time, in terms of the *complexified* Weyl tensor:

$$T_{abcd} = \mathring{W}_{aecf}\, \overline{\mathring{W}}_b{}^e{}_d{}^f.$$

where the overbar means "complex conjugate." If D=4 or D=5 it is "totally symmetric": $T_{abcd}=T_{(abcd)}$. If D=4 it is traceless: $B^a{}_{acd}=0$ (Senovilla 2000 after EQ 13). In "Λ-vacuum" regions (where $R_{ab}=\Lambda g_{ab}$) it also is divergenceless: $T_{abcd}{}^{;a}=0$, hence "conserved." (Theorem 6.1 in Senovilla 2000, which claims this divergence-freeness happens, more generally, whenever $R_{c[b;a]}=0$.) Balfagon & Jaen 2000 claim there is a *unique* (up to constant multiples) 4-indexed totally-symmetric tensor quadratic-in-Riemann that is divergence-free in Λ=0 vacuum on arbitrary-dimensional manifolds (and give a formula for it). And if the dimension is D=4, it is the Bel-Robinson tensor. A remarkable **positivity property** of the Bel-Robinson tensor in (3+1)-dimensional spacetimes is that its contraction with any four future-timelike vectors is always positive: $T_{abcd}W^aX^bY^cZ^d>0$.

Cotton tensor: The 3-indexed tensor in D dimensions

$$J_{abc} = R_{c[a;b]} - g_{c[a}R_{;b]}/(2D-2) = Q_{c[a;b]}$$

In D≥4 dimensions this is

$$J_{abc} = R_{c[a;b]} + g_{c[b}R_{;a]}/6 = W_{abcd}{}^{;d}$$

where the equality of the leftmost and rightmost sides has been called "Schouten's Bianchi identity." Properties: Antisymmetry $J_{(ab)c}=0$, cyclic identity $J_{abc}+J_{bca}+J_{cab}=0$ or equivalently $J_{[abc]}=0$, divergence-freeness $J_{abc}{}^{;c}=0$ and tracelessness $J_{ab}{}^a=J_{ab}{}^b=J^a{}_{ab}=0$. If D≤2 then $J_{abc}=0$. If D≤3 then conformal flatness is equivalent to $J_{abc}=0$.

In an Einstein Λ-*vacuum* (i.e. when $R_{ab}=\Lambda g_{ab}$, constituting the Einstein field equations in Λ-vacuum) Schouten's Bianchi identity implies

$$W_{abcd}{}^{;d} = W^*_{abcd}{}^{;d} = 0,$$

because in Λ-vacuum the Schouten tensor Q_{ab} is merely a fixed multiple of the metric tensor g_{ab}, whose covariant derivatives all are zero, causing the Cotton tensor to be zero (J_{abc}=0). Hence in Λ-vacuum both the Weyl tensor and its dual are divergence-free with respect to every index, hence "conserved."

Bach tensor: $B_{ab} = B_{(ab)} = W_{abcd}{}^{;db} - R^{bd}W_{abcd}/2$.

It is tracefree: $B^a{}_a$=0. And if D=4 it is divergence-free: $B_{ab;b}$=0.

When D=4, according to Kozameh, Newman, Tod 1985, the metrics conformally equivalent to Λ-vacuums are precisely those obeying B_{ij}=0.

According to Bergman, Edgar, Herberthson 2005 if D=4 the Bach tensor is the unique (up to constant multiples) conformally-invariant divergence-free symmetric 2-indexed tensor expressible as a polynomial function of the components of the metric tensor g_{ab}, its matrix-inverse g^{ab}, and a finite set of their derivatives, whose Thomas-replaced version is *quadratic* in Riemann curvature components; but if D≥5 no such tensor exists.

All the tensors defined in this section are expressible as **polynomials** in the metric tensor g_{ab}, its matrix-inverse g^{ab}, and a finite set of their derivatives. Bel-Robinson, Schouten, and the curvature tensors R_{abcd}, W_{abcd}, R_{ab}, S_{ab}, G_{ab}, and R require at most second-differentiation of the metric, but Cotton includes third and Bach fourth derivatives. Therefore, if anybody tells you formulas for the components of some metric g_{ab} in terms of 4 named coordinates, you can, purely mechanically, compute formulas for all components of any of those tensors. Unfortunately the amount of symbolic manipulation work required to do that can be enormous. So it really helps to have computers do it (MacCallum 2018), not humans; some systems for that purpose include "MathTensor" by Steven M. Christensen & Leonard Parker, "GRtensorII and III" by Peter Musgrave, Denis Pollney and Kayll Lake for MAPLE, "TTC" by X.Jaen & E.Llanta for Mathematica, "Xact" by a large number of developers for Mathematica, and – the only one available at no cost, but it is very weak – the itensor and ctensor packages in Maxima.

Killing vectors and Symmetries. Many spacetime metrics enjoy parameterized familes of symmetries. For example, axisymmetric spaceitmes enjoy a 1-parameter group of rotation symmetries, while static spacetimes have a 1-parameter group of time-translation symmetries. In such spacetimes, there exist a vector field, which is a tensor, that tells you the infinitesimal generators of the symmetry. For example, in axisymmetric metrics where adding a constant angle to the coordinate φ implements a rotation by that angle, a Killing vector is the $\hat{\varphi}_a$ vector field, always pointing in the φ-coordinate direction and obeying $\hat{\varphi}^a\hat{\varphi}_a$=1. The condition that *defines* K_a *being* a Killing vector field is "Killing's equation" $\mathbf{K_{(a;b)}=0}$. Generally speaking, no Killing field exists. They only exist on atypically *nice*, i.e. symmetric, spacetimes.

"Weak-field gravity" in de Donder (more precisely, exact-harmonic) gauge. In "weak field gravity," the metric tensor $g_{ab}=\eta_{ab}+h_{ab}$ where η_{ab}=diag(-1,1,1,1) is the flat spacetime metric tensor, while h_{ab} is a small perturbation with trace $h=h^a{}_a$. The box (□) symbol will denote the flat-spacetime d'Alembertian operator $\Box\equiv\eta^{cd}\partial_c\partial_d$. Our convention in weak-field gravity is that all indices are raised and lowered (trivially) with the flatspace metric η_{ab}, and we will express everything in terms of plain "comma" derivatives. The (exact) harmonic gauge condition is $g^{ab}\Gamma_{cab}$=0.

Two abbreviations: The (exact, not perturbative) Christoffel symbols are

$$\Gamma_{abc} = h_{a(b,c)} - h_{bc,a}/2 = (h_{ab,c} + h_{ac,b} - h_{bc,a})/2, \quad \Gamma^a{}_{bc} = \eta^{ad}\Gamma_{dbc} \quad \text{(exact)}$$

and we shall also make use of the "exact harmonic gauge defect"

$$W_a = h^{bc}\Gamma_{abc} = h^{bc}h_{ab,c} - h^{bc}h_{bc,a}/2.$$

The 2nd order inverse metric tensor is

$$g^{ab} = \eta^{ab} - h^{ab} + h^{ac}h_c{}^b + O(h^3).$$

The second order Riemann tensor (use just the first two terms on the right for first order) is

$$R_{abcd} = h_{a[d,c]b} + h_{b[c,d]a} + \Gamma^f{}_{ad}\Gamma_{fbc} - \Gamma^f{}_{ac}\Gamma_{fbd} + O(h^3).$$

To first (using only leftmost term on right) and second perturbative order, the Ricci tensor is:

$$R_{ab} = -\Box h_{ab}/2 + W_{(a,b)} + \Gamma^{ec}{}_b\Gamma_{eac} - h^{cd}(h_{c[b,d]a} + h_{a[d,b]c}) + O(h^3)$$

while the curvature scalar is

$$R = -\Box h/2 + O(h^2) = -\Box h/2 + (h^{bc}\Gamma_{abc})^{,a} + \Gamma_{abc}\Gamma^{abc} + h^{ab}\Box h_{ab} + O(h^3).$$

At first order the Weyl tensor is

$$W_{abcd} = h_{a[d,c]b} + h_{b[c,d]a} + (\eta_{a[c}\Box h_{d]b}/2 + \eta_{b[d}\Box h_{c]a}) - (\Box h)\,\eta_{a[c}\eta_{d]b}/6.$$

These weak-field formulas have been checked numerically on flatspace perturbed with quadratic terms.

Sign-convention hell. In an unfortunate recaptitulation of the "tower of Babel" story in the Bible, different general relativity authors have used different sign conventions. The three top issues are

1. The sign of the **metric** tensor: this work, MTW 1973, Stephani et al 2003, Wald 1984, S.Carroll's *Spacetime and Geometry*, Mueller & Grave 2010, and everybody from Euclid onward doing no-time 3-dimensional differential geometry all use the "mostly plus" signature -+++, but unfortunately many others use +---.
2. The overall sign used to define the ***Riemann*** *curvature tensor*,
3. The sign of the ***Einstein*** *tensor*; MTW and myself agree G_{ab} is the same sign as T_{ab}, while Synge 1960 (e.g. chIV§5, EQ 108) disagrees.

(Related games that can be played: reordering the indices of the Riemann tensor, and contracting different index-pairs to define the Ricci tensor, which can change sign in a more "stealthy" way.) These choices have no physical significance by themselves, but carelessly mixing different authors' sign conventions is a prolific source of errors. In terms of these three sign choices MTW classify themselves, me, Bergman-Edgar-Herbertson 2005, and Hawking & Ellis 1973, as **(Metric, Riemann, Einstein)=(+++)**; Weinberg 1972 and Pauli 1958 are (+−−); Stephani et al 2003 and MacCallum 2018 are (+−+); Synge 1960 is (++−); Landau & Lifshitz 4th English Edition 1980 [but earlier editions may differ], Muratori's 1997 PhD, Peebles 1993, Dolan & Kim 1994a, and Penrose & Rindler 1984 all use (−++); Peacock 1999 is (−+−). When discussing the Einstein's cosmical Λ constant I prefer to say it has "repulsive" (de Sitter space) or "attractive" sign (anti de Sitter) instead of actually saying + or − since the latter depends on the author but

the former does not. MTW claim all authors agree that mass-density is positive: $T_{tt}>0$.

Quantity	Reverse metric $g_{ab} \to -g_{ab}$	Reverse Riemann sign $R_{abcd} \to -R_{abcd}$	Reverse Einstein sign $G_{ab} \to -G_{ab}$
g_{ab}, g^{ab}	negates	unchanged	unchanged
For each index-raising or index-lowering, you get the same sign-effect as multiplication by g_{ab}.			
$\Gamma^a{}_{bc}$, Kronecker δs	unchanged	unchanged	unchanged
$R^a{}_{bcd}$	unchanged	negates	unchanged
R_{abcd}, W_{abcd}, L_{abc}	negates	negates	unchanged
R_{ab}, G_{ab}, Q_{ab}, Λg_{ab}, J_{abc}	unchanged	negates	negates
T_{abcd}	unchanged	unchanged	unchanged
B_{ab}	negates	unchanged†	unchanged
R	negates	negates	negates
covariant derivatives ∇_a or ";" in subscript	unchanged	unchanged	unchanged
tensor duals *	unchanged	unchanged	unchanged

† The Bach tensor B_{ab} is the odd one out in the middle column, and deserves a word. As a *geometric object* it is unchanged, because it is the metric variation of $\int W_{abcd}W^{abcd}(-g)^{1/2}d^4x$, and that integrand is quadratic in W_{abcd} and so cannot feel a reversal of the Riemann sign. But unlike every other row of this table, the usual *formula* $B_{ab} = W_{acbd}{}^{;cd} + R^{cd}W_{acbd}/2$ is not sign-covariant: under $R_{abcd}\to -R_{abcd}$ its first term negates while its second, being bilinear in R_{cd} and W_{acbd}, does not. An author using the opposite Riemann sign must therefore flip the relative sign between the two terms in order to name the same tensor. This is a live trap, since the same formula is printed in sources using opposite conventions.

With MTW's signs, the standard sphere using all-plus metric signature has $R>0$ and all eigenvalues of R_{ab} also >0.

6. Is General Relativity Unique? LG and SG argue "no."

The first claims general relativity is "the unique acceptable theory of gravity" date to Einstein himself. Later works trying to give stronger, broader, alternative, and/or simpler versions of Einstein's uniqueness argument include: Vermeil 1917; pp.315-318 of Weyl 1921; Lovelock 1969, 1970, 1971, 1972, and 1974,

and in the final chapter of the book by Lovelock & Rund; Anderson & Lovelock 1976; Aldersley 1977; Anderson 1978 & 1984; McKellar 1981 & 1987; Navarro & Sancho 2008; Navarro & Navarro 2011; and Curiel 2019.

The stress-energy tensor T_{ab} of mass-energy is a type (0,2) divergence-free (since locally conserved) tensor, meaning $T_{ab}{}^{;a}=0$. It is symmetric ($T_{jk}=T_{kj}$) either because

a. Angular momentum is locally conserved and there should not be positive torques on infinitesimal chunks of matter. Wikipedia calls this "Cauchy's second law of motion."
b. Of the definition of T_{jk} in terms of a "variational derivative" of the matter action with respect to the metric g_{jk}, given that the latter is symmetric.

So any field equation in any generally covariant metric-based gravity theory that wants to locally-conserve mass-energy and either locally-conserve angular-momentum or assume "matter" has a lagrangian description, *must* be expressible in the form $X_{ab}=T_{ab}$ where X_{ab} is a symmetric 2-downstairs-indexed divergence-free tensor expressible in terms of the metric tensor g_{ab}, its matrix-inverse g^{ab}, and a (hopefully finite) set of their derivatives.

Lovelock showed that in 4 spacetime dimensions the *only* (0,2) tensors divergence-free in at least one index, that are functions of the metric tensor g_{ab} and its first two derivatives, are the Einstein tensor G_{ab} and the metric g_{ab} itself (and linear combinations). Consequently **Einstein's GR(Λ) field equations (MTW ch.17)**

$$\mathbf{G_{ab} + \Lambda g_{ab} = 8\pi G c^{-4} T_{ab}}$$

constitute the *unique* possible 4D locally-energy-conserving generally-covariant theory of gravity based solely on the metric tensor, its first two derivatives, and the stress-energy tensor of mass-energy (well, "unique" up to changing the numerical values of the constants Λ and $8\pi G c^{-4}$). Note: Lovelock did not even need to assume the symmetry $T_{jk}=T_{kj}$, and thus did not need to use arguments (a) or (b) for that; this symmetry instead arose as an *output* from his arguments.

What if instead of "tensors" we more generally (given that spacetime dimensionality is 3+1) permit "spinors"? Anderson & Lovelock 1976 showed nothing extra comes. More precisely: The only 2-downstairs-indexed tensors which are divergence-free on at least one index, and "concomitants of a spinor field together with its first two partial derivatives and scalars under spin transformations" are linear combinations of the Einstein and metric tensors.

What if instead of basing things on the "metric tensor" we base it on a "frame field" aka "tetrad"? McKellar 1987 showed, assuming Lorentz invariance, nothing extra comes. Curiel in some respects generalized Lovelock to all dimensions≥4 and also pointed out that another consequence is that "gravitational energy" *cannot* be described by any tensor density, i.e. is nonlocal, because its stress-energy tensor would have to be the Einstein tensor G_{ab}, but cannot be because $G_{ab}=0$ in vacuum, i.e. even though vacuum can and does transmit energy in the form of gravitational waves.

What if you permit *more* than the first two derivatives of the metric? Then other classical gravity theories [e.g. "f(R) gravity" and "quadratic gravity"] become permissible. Troubles with that: (1) the metric might not be smooth enough to *have* third or fourth derivatives, (2) in any such higher-derivative gravity theory, the initial conditions and boundary conditions would need to be much more complicated, e.g. to predict the future you would need to know not only the initial metric, but also its initial derivatives some number of

levels deep, everyplace. That's a big reason why no accepted fundamental physical theory involves partial differential equations (PDEs) with more than two derivatives.

Aldersley 1977 assumed (3+1)-dimensional spacetime. He claimed In "natural units" with c=G=1 that Newton's law acceleration=mass/distance^2 plus "dimensional analysis" forces length, time and mass all to have same units. Hence any relativistic laws of gravity should enjoy the "scaling theorem" that scaling all lengths, timespans, and masses by any particular common scaling factor s, should leave the theory unaltered. That indeed is a true statement about GR with $\Lambda=0$. But if $\Lambda\neq 0$, or if we adjoin Planck's $\hbar$ to our list of involved physical constants, that inputs a "natural length scale" into the theory, ruining the scaling theorem. [Aldersley would have refused to accept the Λg_{ab} term in Einstein's field equation because it scales like s^0. I.e. Aldersley's demand contradict Nobel-winning observational evidence that $\Lambda\neq 0$.] Aldersley blissfully ignored that quibble, then claimed that the *only* possible Lagrangian densities of "class C^2" were those proportional to Hilbert's $g^{1/2}R$. That leads uniquely to the Einstein $\Lambda=0$ field equations.

> There are more things in heaven and earth, Horatio, Than are dreamt of in your philosophy.
> – William Shakespeare: *Hamlet*.

All those "provers of Einstein GR(Λ) uniqueness" **ignored** the *traceless Lanczos tensor* L_{abc}, which is a tensor that always exists on 4-dimensional manifolds but is *not* a pointwise function of g_{ab}, g^{ab}, and any finite set of derivatives thereof. What if we do not ignore it?

LG Theorem: Lanczosian gravity theories. Einstein GR(Λ) no longer is the unique theory of gravity if we stop ignoring the Lanczos tensor.

First Proof: Consider the new tensor $Z_{ab}=Z_{ba}\equiv L_p{}^q{}_a L_q{}^p{}_b=-L_p{}^q{}_a L^p{}_{qb}$. This tensor has 2 downstairs indices, is symmetric, and Aldersley-scales the same way G_{ab} does, namely proportionally to s^{-2}. In general it fails to be divergence-free, i.e. $Z_{ab}{}^{;b}\neq 0$. However, as we shall see, the traceless Lanczos tensor L_{abc} is a highly *non*unique object, with 6-arbitrary-functions-of-spacetime worth of "gauge freedom." We therefore are free to demand that L_{abc} satisfy up to 6 equations of our choice ("gauge-fixing," or if we impose fewer than 6 equations, "partial gauge fixing"). Since $Z_{ab}{}^{;b}=0_a$ is only 4 equations, we could even demand two more if we wanted. So any tracefree Lanczos tensor L_{abc} obeying the $Z_{ab}{}^{;b}=0$ demand can be used to produce new terms κZ_{ab} that you can add to the Einstein field equations to get a wider class of theories of gravity still obeying all the demands of Einstein, Lovelock, and Aldersley. **Q.E.D.**

Second Proof: Instead let $Z_{ab}=Z_{ba}\equiv L_{ijk}L^{ijk}g_{ab}$. Again this tensor has 2 downstairs indices, is symmetric, and Aldersley-scales the same way G_{ab} does, namely proportionally to s^{-2}. Again in general it fails to be divergence-free, i.e. $Z_{ab}{}^{;b}\neq 0$. However again we could insist $(L_{ijk}L^{ijk})_{;a}=0_a=0$ as four gauge-fixing demands, which would force $Z_{ab}{}^{;b}=0$. This also would yield a possible new term κZ_{ab} you could add to the Einstein field equations. **Q.E.D.**

Third Proof (sort of): Suppose the traceless Lanczos tensor L_{abc} arises from a symmetric "*superpotential*" $P_{ab}=P_{(ab)}$ via $L_{abc}=P_{c[a;b]}$. Also suppose P_{ab} is divergence-free: $P_{ab}{}^{;a}=0$. In that case, would P_{ab} also be eligible for adding to the Einstein field equation? Aldersley would contend "no" because P_{ab} should scale like ln(s), not like s^{-2}. However, if you *disregarded* Aldersley, it would seem acceptable.

Of course, you might here object that I have not proven that a Lanczos-superpotential always exists, or (further) that a "nicer" divergence-free superpotential exists. But we shall see later that there was a widely circulating misconception that it had been proven that superpotentials P_{ab} need not exist. Our investigation has revealed that *was* a misconception. The truth: nobody ever found any proof of P_{ab}-nonexistence for any spacetime metric. And by computerized local power-series investigations at many points tried in many metrics (selection includes every Petrov type) tried, we accumulated considerable computational evidence suggesting the conjecture that P_{ab}, and indeed divergence-free P_{ab}, always (at least locally) exist – exactly the opposite of the former widespread misconception-based belief! This suggests to me that even if the P_{ab} existence conjecture is false, that doesn't matter for physics. That is: Hypothesize that ***not* all metrics g_{ab} can arise physically. The only ones that can arise are those such that divergence-free superpotentials exist.** This is a very interesting hypothesis. And we have shown in our "six foundations" theorem that {P_{ab}, R_{ab}} is a "foundation" for differential geometry in 4 spacetime dimensions, *whenever P_{ab} exists*. Hence the "physical" g_{ab} are *producable* from P_{ab} and R_{ab}, now serving as the foundation upon which all physics rests *instead* of g_{ab}. And if the physically-achievable g_{ab} truly are merely a subset of all g_{ab}, and the g_{ab} for which a superpotential exists truly are merely a subset of all 4-dimensional g_{ab}, then it would be a *better* foundation, in the sense it gets closer to what is physical. Physics based on that alternative foundation automatically would only generate metrics in the "good" subset which possess a superpotential. **Q.E.D.**

Interlude: what if GR(Λ)'s Hilbert action had included other terms?

One way to derive Einstein's field equations was to begin with the "Hilbert action" $\int ((16\pi G c^{-4} L_{matter} + R) g^{1/2} d^4x$. One could consider altering the scalar curvature "R" inside this action, e.g. to add other possible terms to it, to explore possible other gravity theories. For example, adding a constant -2Λ, i.e. replacing "R" by "R-2Λ," leads to the GR(Λ) field equations upen demanding that the variational derivative of the action integral with respect to the metric tensor, be zero. The table below summarizes the effects of several possible extra such terms.

#	Action term to be added to R	Effect E_{ab} on field equation	Comment
1	R	$G_{ab} = R_{ab} - ½\, g_{ab}R$	Einstein; 2nd order
2	-2Λ	Λg_{ab}	Einstein cosmical constant term
3	R^2	$2RR_{ab} - ½\, g_{ab}R^2 + 2g_{ab}\Box R - 2R_{;ab}$	4th order; local
4	$R_{cd}R^{cd}$	$-½\, g_{ab}R_{cd}R^{cd} + \Box R_{ab} + ½\, g_{ab}\Box R - R_{;ab} + 2R_{acbd}R^{cd}$	4th order; local
5	$R_{abcd}R^{abcd}$	$4E^{[4]} - E^{[3]}$ (4D Gauss-Bonnet)	4th order; local
6	f(R)	$f'(R)R_{ab} - ½\, g_{ab}f(R) + g_{ab}\Box f'(R) - f'(R)_{;ab}$	up to 4th order; local; generalizes R and R^2
		$2L_{acd}L_b{}^{cd} + L_{cda}L^{cd}{}_b - ½$	algebraic mass term;

7	$L_{abc}L^{abc}$ ("free" L)	$g_{ab}L_{cde}{}^{cde}$	L=0 on-shell
8	$L_{abc}L^{abc}$ (demanding $L_{ab}{}^{b}=0$, $L_{abc}{}^{;c}=0$, and that L_{abc} be a Lanczos tensor)	$E^{[7]}_{ab} + \underline{B}_{ab}[\lambda] - D_{ab}[\lambda,L]$, with $W^{\dagger}(\lambda)=2L$, $W(L)=C[g]$	coupled; nonlocal / higher-derivative

Rows 3-5 of the table consider the effect of "quadratic gravity," while row 6 is "f(R) gravity" and we use the notation $\Box Z \equiv Z_{;u}{}^{;u}$.

Row 8 is the one of interest for considering possibly adding $L_{abc}L^{abc}$ to the R inside the Hilbert action, where L_{abc} is a traceless Lanczos tensor, here assumed to obey the Lanczos differential gauge.

To make "L is a Lanczos potential of *this* metric" part of the theory, we impose the Lanczos→Weyl relation as the constraint. that the Weyl $W_{abcd}(L)$ arising from our Lanczos L_{abc} – the standard first-order map built from $L_{abc;d}$ and its traces – must satisfy $W_{abcd}(L)=W_{abcd}$, so that L is a Lanczos potential of the metric whose Weyl tensor is W_{abcd}. Introduce a new tensor field λ^{abcd} as a Lagrange multiplier, carrying the same algebraic symmetries as the Weyl tensor. Because it multiplies the traceless combination C−W, this multiplier may be taken **traceless**.

Variation with respect to λ – the constraint: $W_{abcd}(L)=W_{abcd}[g]$ (with $g^{bc}L_{abc}=0$, $L_{abc}{}^{;c}=0$).

Since W is first order in L, integrating $\lambda^{abcd}W_{abcd}(L)$ by parts moves the derivative onto λ, producing the formal adjoint $W^{\dagger}$. Hence $W^{\dagger}(\lambda)^{abc}=2L^{abc}$+gauge-multiplier terms. This is a first-order differential equation determining λ from L: the multiplier is essentially an anti-derivative of the Lanczos potential, hence nonlocal with respect to the metric.

The three contributions to the field equation in row 8 would be as follows:

- **$E^{[7]}_{ab}$:** the "bare" stress tensor from row 7 of the table.
- **$\underline{B}_{ab}[\lambda]$:** the metric variation of $\lambda^{cdef}W_{cdef}[g]$. Since λ is traceless, $\lambda^{cdef}W_{cdef}=\lambda^{cdef}R_{cdef}$, so this is the variation of $\lambda^{cdef}R_{cdef}$ with λ held fixed – a second-order operator on the multiplier,
$$\underline{B}_{ab}[\lambda] = 2\,\lambda_{cabd}{}^{;cd} + \lambda\cdot(\text{Riemann algebraic terms}) - \tfrac{1}{2}\,g_{ab}\lambda^{cdef}R_{cdef}.$$
This is a "Bach-type operator" acting on λ, in the sense that setting λ=W would reduce $\underline{B}$ to the Bach tensor $W_{acbd}{}^{;cd}+R^{cd}W_{acbd}/2$ (up to normalization; that expression is indeed the standard Bach tensor), i.e. the same operator that generates the $R_{abcd}R^{abcd}$ or Weyl-squared field equations. So the "extra terms" that encode "L generates Weyl" are literally the Bach operator, but evaluated on the multiplier rather than the Weyl tensor.
- **$D_{ab}[\lambda,L]$:** the metric variation of $\lambda^{cdef}W_{cdef}(L)$. The Weyl-Lanczos operator carries the metric through its covariant derivative (Christoffel symbols) and its trace contractions, so this term is local, bilinear in (λ,L), and first order in derivatives. It has no shorter name; it is assembled directly from the derivative and trace structure of C.

The field content is a **coupled system**: the constraint fixes L from g (up to residual gauge), the adjoint equation fixes λ from L, and the metric equation feeds back. Eliminating λ and L would render it

(considered as a theory about the metric g_{ab}) an effectively **nonlocal, higher-derivative theory**, reflecting the fact that $\int\sqrt{|g|}\,L^2$ is a nonlocal functional of the metric.

On a vacuum background ($R_{ab}=0$, $W_{abcd}=R_{abcd}$) the Bach operator loses its Ricci pieces, $\underline{B}_{ab}[\lambda]=2\lambda_{cabd}{}^{;cd} - \frac{1}{2}\, g_{ab}\lambda^{cdef}R_{cdef}$, so that the multiplier equation $C^{\dagger}(\lambda)=2L$ is driven purely by the Lanczos potential. The residual Lanczos gauge freedom – with no canonical fixing – enters through the gauge multipliers v, ρ. The field equations therefore inherit a gauge dependence unless that freedom is pinned. (End of interlude; this paper shall not consider this kind of alteration of the Hilbert action any more.)

Returning to main discussion: The **4-way classification of the scalar quantity $L_{abc}L^{abc}$ as either "zero," "positive," "negative," or "infinite"** is something all observers agree about (since L_{abc} is a tensor), and which is *unaltered* by Aldersley rescalings. Therefore we could, for example, change GR's two "constants" Λ and $8\pi G c^{-4}$ depending on that classification, whereupon Aldersley, Lovelock, etc would not object.

Switcheroo gravity (SG). This paper mainly investigates SG, with LG largely just coming along for the ride as an "innocent bystander." The reason I'm concentrating on SG in this work is *not* because I feel it is more likely than LG to be correct. (Indeed, I'd guess the opposite.) It is simply because I have *better intuition* about SG vis-a-vis MOND, because SG is very similar to GR(Λ), an area that has benefitted from a large amount of prior work, sourcing a large amount of intuition. [If you lose your keys on a dark street, it is best to first search for them under the streetlamps.] In contrast, I have almost no intuition about LG, aside from: (i) realizing that at least several LG schemes exist, and (ii) believing (or hoping) that some LG flavor ought to be able to emulate SG (and MOND-like effects), at least qualitatively.

SG is a very simple add-on to GR(Λ) gravity: SG postulates that Λ switches among some discrete set of values depending on the values of a tuple $\vec{\mathrm{F}}$ of scalar invariants, one member of that tuple being $L_{abc}L^{abc}$.

Complex invariant. Actually: although throughout this paper I have focused on $L_{abc}L^{abc}$ for simplicity, there is a second quadratic scalar invariant $L^*_{abc}L^{abc}$ where L* means tensor dual (which exists since L_{abc} is antisymmetric in its first two indices). There naively are 18 possible quadratic scalar invariants obtainable by permuting the indices in 3!=6 ways and by using the dual or primal, but there are many linear relationships among these 18 caused by the algebraic symmetries of the Lanczos tensor. Those cause these *two* to be a full basis for the full set of 18. That suggests packaging them both as one ***complex-valued*** **invariant $\mathring{L}_{abc}=L_{abc}L^{abc}+iL^*_{abc}L^{abc}$.** Then (it turns out) the real and complex parts of $\mathring{L}^{abc}\mathring{L}_{abc}$ are precisely (except for a factor of 2) those two invariants! This "complex package" idea suggested a whole pattern of natural complex- and bicomplex-valued generalizations of standard tensors, which is the topic of part II of the Companion Paper (Smith & Claude 2026) but will not be discussed further here.

Motivating arguments. Given that supernova observations show Λ≠0 in our universe, we should dismiss Aldersley's scaling theorem (since not true). In that case we could change the Einstein field equation's two constants depending on where $L_{abc}L^{abc}$ lies on the extended-real number line (which now could be partitioned into an *arbitrary* number of regions, not just 4). Then the metric g_{ab} and Einstein G_{ab} tensors still both would be symmetric and divergence-free ($G_{ab}=G_{ba}$ and $G_{ab}{}^{;b}=0$) on both sides of switcheroo-transitions. Some quantities would discontinuously jump, but general relativists historically have happily accepted that. For example, the Schwarzschild interior+exterior glued-metric planet model involves matter-

density (and hence components of the Einstein tensor) which discontinuously jump whenever we cross the planet's surface.

GR, in order to work, does not need everything to be continuous. It only needs some things to be continuous. Specifically the metric g_{ab} needs to be continuous funcrtion of the coordinates. But the second derivative of the metric need not be continuous, and in the Schwarzschild planet model, is not.

Now thinking quantum-field-theoretically, it might well **make sense that Λ should have different values** depending on the "phase of the quantum vacuum," which quite possibly might have multiple phases in different locations in a way dependent on $L_{abc}L^{abc}$ and/or certain other curvature-tensor-based invariant scalars. These phase transitions would need to absorb/release *zero* heat (unlike water↔ice), but many known phase transitions do that.

By contrast **I doubt that the values of G and/or c could change**. Newton's G is required by quantum field theoretic arguments to be a *universal* coupling constant between mass and gravity, the same for *every* kind of mass; while c is built into the fundamental geometric structure of everything, and not supposed to change ever.

Experimental evidence ⟪Λ⟫ changes while c and G do not. Lunar laser ranging and historical planet observations have estimated G is changing $\lessapprox 10^{-13}$ parts per year (amounting to $\lessapprox 10^{-5}$ over the age of the Solar System, assuming constant drift); big bang nucleosynthesis models estimate G was within 15% of today's value during the big bang (otherwise isotope ratios would be different). That all obsoletes the simple, but nevertheless quite strong, older argument by H-bomb expert Edward Teller that the sun's luminosity ought to depend on G like a high power (G^7 or worse) indicating G could not have changed much during the history of life on Earth (otherwise that life would have frozen or fried). And the fine structure constant (containing c^{-1} inside its defining formula) appears to be changing $\lessapprox 10^{-17}$/year based on atomic clock measurements, and changed $\lessapprox 10^{-7}$ or 10^{-8} fractionally versus 2 Gyr ago based on analyses of remnants of the ancient natural fission reactor at Oklo.

In contrast, it is quite possible that the "universe-wide average" Einstein cosmical "constant" ⟨Λ⟩ (and therefore also local Λ values) *have* noticeably changed with time. There currently is no consensus on that question. There have been published papers, e.g. from DESI, contending Λ has indeed time-evolved, but their claimed significances 2.8-4.2 "sigma" have not yet overwhelmed everybody. The "Hubble tension" where two different classes of measurement methods obtain two different values for Hubble expansion rate, and consequantly two different Λ values, keeps getting more worrying (>5σ as of year 2026) as both camps narrow their claimed error bounds while their disagreement remains about the same. Let me just say that if we *combine* the DESI results, with significance≥2.8σ (confidence≥0.99744=1-0.00256), with the Hubble tension significance>5σ (confidence>$1-2.87\times10^{-7}$) by regarding them as *independent*, then the net confidence>$1-7.4\times10^{-10}$ that ⟨Λ⟩ time-evolved is already equivalent to over 6σ, the level the physics community accepts as "experimental proof" of important discoveries.

Cai & Wang's 2026 review begins "This Hubble-constant tension has likely become a real crisis for modern cosmology."

Important questions about existence, uniqueness, and theory-incompleteness: We are getting ahead of ourselves, since we have not yet come to the section explaining the Lanczos tensor. But for now let me pose three crucial questions:

1. Some gauge-fixing demands are ok in the sense proofs are known for global L_{abc}-existence under

those demands. But others might not always be permissible! For example the demand "$(L_{abc}L^{abc})^2=-1$" obviously would not be permissible because squares of real numbers are never negative. Historically, the attitude of practitioners of both GR and Maxwell electromagnetism has been that "*any* reasonable-sounding set of equations can be freely imposed as a gauge-fixing demand," so you should use whichever set of such equations are most convenient for solving your particular problem. Call that the **Physicist's heuristic.** That attitude has been supported by experience – it always seems to work in practice – but rarely by any rigorous proof. If LG is to be based on certain carefully-chosen gauge-fixing demands, then this lack-of-existence-proof is not good, and future authors need to supply suitable existence proofs. Since my above two LG proofs each only demand 4 gauge-fixing equations, not the full 6, it seems likely to me they do not impair existence; and with *some* choice of the other two demands we could attain uniqueness. But I have not proven either.

2. The other side of the same coin: The *non-uniqueness* of the traceless Lanczos tensor L_{abc} needs to be overcome in order to make SG or LG become unique, and hence well-defined and explicit, theories. The attitude of theoretical physicists during the last 40 years has been bipolar:

- In *quantum* field theories gauge invariance plays a crucial, in fact *the* fundamental role, and it is a wonderfully powerful thing, capable of telling you the *unique* laws of various parts of the "standard model.".
- In *classical*, i.e. non-quantum, field theories (GR and Maxwell EM), gauge freedom is an annoyance that you need to get rid of by arbitrarily imposing gauge-fixing demands.

The traceless Lanczos tensor, if involved in physics, brings with it a new kind of "gauge freedom." And we need to know what should be done about it. This paper is not about quantum gravity, it is about classical field theories, so from the latter perspective we want to dream up gauge-fixing demands to get rid of Lanczosian gauge freedom. Lanczos himself suggested uniquifying L_{abc} by demanding divergence-freeness in its rightmost index: $L_{abc}{}^{;c}=0_{ab}$. Those 6 demands, in combination with boundary conditions in view of the Dolan-Kim "wave equation," are known to generically force L_{abc} to be unique; and L_{abc} has been shown always to exist even under this "Lanczos differential gauge." But for LG to work, at least for the first two possible LG Einstein-field-equation modifications I pointed out, the "Lanczos differential gauge" would need to be abandoned in favor of certain alternative gauge-demands.

If Lanczos tensor really plays a role in physics, it will influence the development of "quantum gravity" in important, probably crucial, ways; and that will pose additional gauge-freedom-related puzzles.

3. Exactly *what* formulas do we want for LG's revised-Einstein field equation? Or in SG, *what* formulas do we want for $\vec{\mathrm{F}}$, what are the mathematical descriptions of the quantum-phase-diagram regions, and what specific values of Λ should be associated with each region? These questions need to be answered by a long interplay between observation and theoretical computer-aided modeling/fitting, which will take many years of work by many people at a minimum. But it is certainly possible to say some things immediately (and we will).

7. "Gravitational Acceleration" in GR

What acceleration will be measured (by accelerometer) by somebody who, by firing his rocket to "counteract gravity," arranges to keep staying in the "same place"? That whole concept of "gravitational acceleration" depends on having a notion of what "same" place *means*. There is no difficulty defining this for "static" spacetimes containing a "timelike Killing vector field" corresponding to "t-translation" where t is the time-coordinate. All spherically-symmetric black hole metrics, such as Schwarzschild, Reissner-

Nordström, and Kottler's Λ-Schwarzschild metric, are static, so for them "staying in the same place" means "traveling along a worldline curve, which is everywhere tangent to that Killing field." But for *generic* spacetimes, which have *no* Killing vectors, there is no obvious way to define this notion. (Restricting to "static" spacetimes is the general-relativity analogue of, in electromagnetism, restricting the study to "electrostatics.")

Even for spacetimes which *do* have Killing vector fields, if they are nonstatic it remains unclear what "gravitational acceleration" should mean. This includes the Kerr-Newman spinning, hence nonstatic, axisymmetric black hole family. Kerr-Newman, and more simply plain Kerr, features a region called the "ergosphere" whose inhabitants cannot prevent themselves from rotating round the hole (no matter how hard they blast their rockets), although they can escape the ergosphere to locations far away, because the ergosphere lies outside the (inescapable) "event horizon." Should we therefore consider the "gravitational acceleration" to be infinite everywhere on the boundary of the ergosphere? Everywhere inside it? I would argue "no"; I think it would be best for MOND-like purposes if Kerr's "gravitational acceleration" were finite everywhere outside the event horizon but infinite on it. (I'm not saying what the Kerr "gravitational acceleration" should be; I'm merely saying I want it to obey this horizon-detection property.) Call that the **Kerr/MOND horizon-detection demand**.

What *is* "inward acceleration"? For a particle in Kottler's Λ-Schwarzschild metric which moves purely *radially*, we can ignore the angular coordinates and focus only on t and r. In that case our metric effectively is only two-dimensional:

$$ds^2 = -[1-2M(r)/r]\,dt^2 + [1-2M(r)/r]^{-1}\,dr^2.$$

Here M(r)=M for the Schwarzschild mass=M black hole (working in units with G=c=1 for simplicity) But for Kottler's generalization permitting Λ≠0, it instead is $M(r)=M+\Lambda r^3/6$. We now consider the "geodesic equations"

$$(x'')^a + \Gamma^a{}_{bc}\,(x')^b(x')^c = 0^a$$

obeyed by a geodesic trajectory $x^a(\zeta)$. Here $\Gamma^a{}_{bc}$ is the Christoffel symbol nontensor. We have (cf. Mueller & Grave 2010)

$$\Gamma^r{}_{tt}=(r-2M)(rM'-M)/r^3, \quad \Gamma^t{}_{rt}=\Gamma^t{}_{tr}=(rM'-M)r^{-1}/(r-2M), \quad \Gamma^r{}_{rr}=(rM'-M)r^{-1}/(r-2M).$$

The geodesic equation at *apogee* where x' is pointing in the t-direction is

$$(r'') + \Gamma^r{}_{tt}\,(r')^2 = 0.$$

If we choose $r'=[1-2M/r]^{-1/2}$ at apogee that makes the x' vector have unit |length| so that the geodesic parameter ζ corresponds at apogee to proper time for that particle. This causes $r''(\zeta)=(r-2M)(rM'-M)r^{-3}(1-2M/r)^{-1}=(rM'-M)r^{-2}$. However, note that r''(ζ) is *not* the "acceleration" experienced by somebody staying at fixed r. That acceleration instead is s''(ζ) where s is the *true* radial distance (which is not r).

The new answer is the same as the old except we need to multiply it by $(g_{rr})^{1/2}=[1-2M(r)/r]^{-1/2}$ to convert the second derivative of r to the second derivative of radial *length.* The final result (where the negative sign here means "inward") is

$$\text{acceleration} = s''(\zeta) = [rM'(r)-M(r)]\, r^{-2}\, [1-2M(r)/r]^{-1/2}.$$

Using Kottler's explicit formula for M(r), and hence $M'(r)=\Lambda r^2/2$, we can rewrite this explicitly in terms of the mass-parameter M of the black hole and the Einstein cosmical constant Λ, and then restore the Newton gravitational constant G and speed of light c (instead of using units with G=c=1):

$$\text{acceleration} = (c^2\Lambda r^3/3-GM)\, r^{-2}\, [1-2GMc^{-2}/r-\Lambda r^2/3]^{-1/2}.$$

8. Proofs of impossibility of imitating MOND using metric & curvature tensors & derivatives alone

The central feature of MOND is Milgrom's magic constant acceleration threshold $a_0 \approx 1.2\times 10^{-10}$ meter/sec^2. So any physical theory that wants to imitiate MOND must somehow be able to *distinguish* **Milgrom's region**, in which "gravitational |acceleration|<a_0," from the complement-set. And any physical theory that wants to be relativistically acceptable, i.e. Einstein "general covariant," must accomplish this distinguishment in the following way:

> There exists an integer k≥1 (simplest: k=1) and a k-dimensional vector $\vec{\mathrm{F}}$ whose components each are zero-indexed *tensors*, such that "Milgrom's region" corresponds to "$\vec{\mathrm{F}}\in$M" where M is a certain (specified by your physical theory) subset of $\mathbb{R}^k$.

In SG, $\mathbb{R}^k$ will be partitioned into various regions, each associated with a particular value of Einstein's cosmical constant Λ. This partitioning is called the "quantum vacuum **phase diagram**," and if k≤2 will be drawable on a sheet of paper just like water's phase diagram.

To allow that to work, the "gravitational |acceleration|" must be a *function* of $\vec{\mathrm{F}}$. (We decided in the previous section what "gravitational acceleration" should mean.)

Initial Impossibility Theorem. The gravitational acceleration in the Schwarzschild metric is not deducible from any $\vec{\mathrm{F}}$ whose components are zero-indexed tensors that are *algebraic functions of the metric- and curvature-tensors*.

Proof: The proof consists in evaluating (1) the gravitational acceleration and (2) an algebraically-complete (or larger) set of "curvature invariants," both for the Schwarzschild metric; then demonstrating the impossibility of expressing (1) as any function of (2).

Here are the details: The Schwarzschild length-element (written for simplicity in units with c=G=1) is

$$ds^2 = -\,(1-H/r)\, dt^2 + (1 - H/r)^{-1}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

where H is the "Schwarzschild radius," i.e. r-value on the horizon-sphere, which (with c and G temporarily restored and M denoting the mass of the black hole) is $H=2GMc^{-2}$. The Jebsen-Birkhoff uniqueness theorem says this ds^2 is the unique spherically-symmetric asymptotically-flat solution of Einstein's vacuum field equations. It is static, i.e. unchanged by translations of the time-coordinate t.

The "acceleration" experienced by anybody firing rockets in order to keep (r,θ,φ) fixed while time t advances (i.e. traveling along the time-translating "Killing vector") is

$$\text{acceleration} = -G\,M\,r^{-2}\,[1-H/r]^{-1/2}.$$

As our "algebraically complete set of curvative invariants" we use the Carminati-McLenaghan-Zachary set. The only C-McL-Z invariants that are not identically zero for the Schwarzschild metric are:

$$W_{abcd}W^{abcd} = W^*_{abcd}W^{*abcd} = 48M^2r^{-6}, \quad \text{and} \quad W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab} = 96M^3r^{-9}.$$

Therefore, anybody inside the Schwarzschild spacetime, by measuring the C-McL-Z invariants at his location, can deduce his value of Mr^{-3}, but nothing more. **Q.E.D.**

Horizon Remark. The horizons of black holes have this property: if your spaceship is sitting some small *fixed* distance $\delta>0$ above the horizon, then in order to keep δ constant (or increase it) you must fire your rockets to accelerate yourself, with minimum possible |acceleration| approaching ∞ in the limit $\delta\to 0+$. Therefore, any $\vec{\mathrm{F}}$ capable of computing acceleration, must also be capable of detecting horizons by outputting "infinite acceleration."

Tensor Classification Remark. The "Thomas replacement theorem" (so-called because versions of it appeared in 1934 and 1961 books by Tracy Y. Thomas) states that (1) *every* tensor expressible in terms of the metric tensor g_{ab} and its first and second derivatives is obtainable by: tensor producting, contracting indices, and symmetrization and antisymmetrization, starting from g_{ab}, g^{ab}, and R_{abcd}; and more generally (2), every tensor expressible in terms of the metric tensor g_{ab} and finitely many derivatives is similarly obtainable instead starting from g_{ab}, g^{ab}, and R_{abcd} and finitely many of its *covariant* derivatives.

Therefore (in view of Thomas case 1), the above Impossibility Theorem rules out MOND-imitation for all theories of gravity based solely on the metric tensor and its first two derivatives. We now strengthen that to *all*-order derivatives:

Improved Impossibility Theorem. It is not possible to determine whether you are on a horizon if your only source of information is an $\vec{\mathrm{F}}$ whose components are zero-indexed tensors that are *functions solely of the metric tensor and finitely many derivatives*.

Proof (A): The proof uses both the Horizon remark and the full Thomas replacement theorem, in the context of the Oppenheimer-Snyder "collapsing spherically-symmetric dust-ball" metrics. These metrics consist of two pieces glued together on a common spherical surface (which shrinks with time)

1. Inner piece: a ball-shaped chunk [$r\le A(t)$] of the FLRW homogeneous and isotropic cosmological model metric, which is (using units with c=G=1 for simplicity)

   $$ds^2 = -\,dt^2 + (1-k\tilde{r}^2)^{-1}\,A(t)^2\,d\tilde{r}^2 + A(t)^2\,\tilde{r}^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

   or rewritten in an equivalent form with "circumferential" radial coordinate $r=\tilde{r}A(t)$

   $$ds^2 = -\,dt^2 + (1-kr^2A^{-2})^{-1}\,[dr - (A'/A)\,dt]^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

   where $A=A(t)$ is the time-dependent "scale factor" of the FLRW universe and $k\in\{-1, 0, +1\}$ is a constant, which for our present purposes we shall demand be k=0 ("flat cosmology"). One may deduce A(t) from the "equation of state" obeyed by whatever forms of mass-energy FLRW assume fill their universe, plus the demand that the FLRW metric solve the Einstein field equations, but I won't discuss that. The only fact we'll need about A(t) is that for an "expanding universe" A(t)

increases with t starting from A(0)=0 ("big bang"). But notice that the time-reversal of an expanding universe, obtained by negating t and assuming A(t)=A(-t), is a contracting ("big crunch") universe which also is an exact solution.

2. Outer piece: forced by the Jebsen-Birkhoff uniqueness theorem to be the only spherically-symmetric asymptotically-flat solution of Einstein's vacuum field equations, i.e. the static Schwarzschild metric stated in the first impossibility proof, when r̃>A(t).

Both these pieces are spherically symmetric, and have "circumferential" radial coordinate r. The two pieces "match" across the ball's outer boundary provided the mass-energy inside the FLRW universe is assumed to be *pressureless* "dust." (Actually, all we really need is zero pressure at the ball *surface*, but since the FLRW universe is *homogeneous* that forces the surface pressure to equal the pressure throughout its interior.) How to handle this kind of collapsing 2-piece glued metric was first understood in work by a young Indian student, B.Datt, at Presidency College in Calcutta (now renamed Kolkata) in 1938. Tragically, Datt died in 1940. The dust ball collapse then was independently analyzed (much more famously) by Oppenheimer & Snyder 1939.

As the dust-ball collapses, ultimately to a point, it remains homogeneous at all times (i.e. density independent of position but increasing with time). There initially is no event horizon. However, at some moment the centerpoint of the dust ball becomes unable to emit a lightspeed signal that can escape to reach locations with unboundedly large r. A horizon-sphere then expands outward from that point at the speed of light, enclosing more and more of the dust-ball, until it hits the outer boundary of the ball. Then the horizon-circumference stays constant (equal to the Schwarzschild horizon-circumference of the final black hole) forever after. (One can deduce the location of the outer ball-boundary as a function of time, but we shall not do so here.)

Now for the crux of the argument. Because the FLRW universe is *homogeneous* (i.e. invariant under spatial translations and rotations at any fixed time t), all invariants constructed from the metric and curvature tensors and their covariant derivatives depend *only* on t and *not* on spatial location (r,θ,φ) within the dust-ball. Therefore those invariants **cannot** locate the expanding horizon-sphere and **cannot** output any sort of "radial acceleration" function that depends nontrivially on r at any given time t. **Q.E.D.**

Additional Proof (B): Instead of the Oppenheimer-Snyder "collapsing dust ball," use a collapsing "thin spherical shell" of pressureless dust. Outside the shell we have a Schwarzschild vacuum metric. Inside it, we have Minkowski "flat spacetime" vacuum:

$$ds^2 = - dt^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2).$$

Both these pieces are spherically symmetric, enjoy t-translation invariance, and have "circumferential" radial coordinate r.

As the shell collapses, ultimately to a point, it remains spherically symmetric at all times (i.e. with areal density independent of angular position but increasing with time). There initially is no event horizon. However, at some moment the sphere's centerpoint becomes unable to emit a lightspeed signal that can escape to reach locations with unboundedly large r. A horizon-sphere then expands outward from that point at the speed of light, until it coincides with the dust-shell. Then forever after, the horizon-circumference stays constant (equal to the Schwarzschild horizon-circumference of the final black hole).

Because Minkowski flat spacetime is invariant under spacetime translations and rotations at any fixed time t, invariants constructed from the metric and curvature tensors and their covariant derivatives **cannot** determine whether you (if located inside the shell) are inside or outside the expanding horizon-sphere.

(Indeed, for Minkowski spacetime, all the usual invariants equal zero everywhere and always.) **Q.E.D.**

Further obstacle. If SG wants to discontinuously change Λ whenever you cross certain surfaces in spacetime, then many curvature invariants are **impermissible** as $\vec{F}$-components because they would (or at least sometimes *oould*) change discontinuously. For example (most obviously) the scalar curvature R is forbidden as an $\vec{F}$-component; and so is essentially any formula that depends on R.

It now might appear that we are **dead**: no relativistically-acceptable MOND-like theory is possible! However, there is a loophole in that death-certificate: what if $\vec{F}$ involves a new tensor *not* expressible in terms of the metric tensor and any finite set of its derivatives? If spacetime is 4-dimensional then – remarkably – another tensor, called the Lanczos tensor, happens always to exist. And we shall see that "traceless Lanczos tensors" are capable of evading both the above impossibility-proofs and the further obstacle.

9. Lanczos tensor

The Lanczos tensor on a 4-dimensional spacetime is a 3-indexed tensor L_{abc} obeying these 40 antisymmetry demands and the 4 "cyclic identities":

$$L_{(ab)c} = 0 \quad \text{and} \quad L_{[abc]} = 0.$$

I will call it **trace-free** if it also obeys the 4 equalities $\mathbf{L_{ab}{}^{b}=0_{a}}$. (Equivalent conditions: $L_a{}^b{}_b=0$ and $L^b{}_{ab}=0$. Prior authors unfortunately have mostly called this tracefreeness demand the "Lanczos algebraic gauge" condition.) Note that due to the antisymmetry, *all* traces of traceless Lanczos tensors are zero: $L_{ab}{}^{b}=L_{ab}{}^{a}=L^{a}{}_{ab}=0$. SG only uses tracefree Lanczos tensors. It is possible (albeit with a more complicated definition!) and *occasionally* convenient to allow Lanczos tensors with *non*zero trace. Those will be denoted with a the alternative "slashed L" symbol: $Ł_{abc}$. These always can be converted to a tracefree form via the **standard projection**

$$L_{abc} = Ł_{abc}+(2/3)g_{c[a}Ł_{b]k}{}^{k}$$

which still is a valid Lanczos tensor if the original $Ł_{abc}$ was. Conversely, if you have some tracefree L_{abc}, then you can obtain infinitely many non-tracefree $Ł_{abc}$ by $Ł_{abc}=L_{abc}+\Phi_{[a}g_{b]c}$ where Φ_a is an *arbitrary* 4-vector.

Effect of trace-removal on contracted square . Writing $T_a=Ł_{ab}{}^{b}$ for the trace vector of $Ł_{abc}$, and L_{abc} for its trace-free representative:

$$Ł^{abc}Ł_{abc} = L^{abc}L_{abc} + (2/3)T^{a}T_{a}.$$

Because the Lanczos tensor is antisymmetric in its first two indices [$L_{(ab)c}=0$], one may define its **dual**

$$(L^*)_{abc} = (½)\epsilon_{abpq}L^{pq}{}_{c}$$

We call L_{abc} "**dual-orthogonal**" if it obeys $L^{abc}L^*{}_{abc}=0$.

Also, if we define the **auxiliary 4-indexed tensor $K_{abcd}=L_{ab[c;d]}$** via an antisymmetrized covariant differentiation, then because of the antisymmetries $K_{abcd}=K_{[ab]cd}=K_{ab[cd]}$ we similarly may in D=4 dimensions define its left dual $^*K_{abcd}$, right dual K^*_{abcd}, and **double dual $^*K^*_{abcd}=\epsilon_{abrs}\epsilon_{pqcd}K^{rspq}/4$**.

Why do I insist on tracefree Lanczos? I regard $Ł_{ab}{}^c$ as containing useless stupid extra degrees of freedom. (Or as Edgar & Höglund 1997 put it, "there seems nothing to be gained" by not demanding tracefreeness.) Sure, some idiot could formulate GR in terms of *un*symmetric-matrix metric tensors $\tilde{g}_{ab}$, but the resulting "more general" gravity-theory would really only depend on the symmetrized-matrix metric tensor $g_{ab}=\tilde{g}_{(ab)}$, so those 6 "extra" degrees of freedom would just be stupid. And some other idiot could allow Lanczos tensors not obeying the antisymmetry $L_{(ab)}{}^c=0$ but again the "extra" degrees of freedom thus obtained would just be stupid because really, all Lanczos-tensor physics would involve only the antisymmetrized version $L_{[ab]}{}^c$. As far as I'm concerned, permitting Lanczos tensors with nonzero trace similarly is stupid, because the 4-vector Φ_a representing the "extra" degrees of freedom thus obtained is completely arbitrary, and therefore those extra degrees of freedom are meaningless. Also:

- Andersson & Edgar 2001 (e.g. their EQ 7) note that Lanczos-tracefreeness seems necessary in order to make L_{abc} naturally correspond to a spinor; and Illge 1988's spinorial existence proof assumes it.
- Vishwakarma 2021 near his EQ 33d points out that tracefreeness is a necessary condition to allow $L^a{}_{bc}$ to hope to be conformally invariant.
- The condition $L^*_{ak}{}^k=0$ of tracefreeness for the Lanczos *dual*, is equivalent to the cyclic symmetry condition $L_{[abc]}=0$.
- And tracefreeness for Lanczos is equivalent to cyclicity for the dual.
- With tracefreeness, the upcoming definition of "Lanczos" simplifies.

If tracelessness is demanded, then the Lanczos tensor does not have 64 degrees of freedom at each point (which one might have naively thought since it has 3 indices each 4-valued in 4 spacetime dimensions, and $64=4^3$) but rather, in view of both tracelessness, the above-listed symmetries, and trivially-satisfied linear equations, only **16**=64-40-4-4.

The adjective "Lanczos" denotes the defining demand that L_{abc} obey

$$W_{abcd} = K_{abcd} + K_{cdab} - {}^*K^*_{abcd} - {}^*K^*_{cdab}.$$

(This was given as EQ 1.16+1.17 in Dolan & Kim 1994a. It also arises in Takeno 1962. It also is is the traceless-L case of EQ15 from Roberts 1992, which Dolan & Gerber 2003 cite as the source for their EQ 5, *except* Roberts unfortunately changed the final two − signs to +.] A 2004 Linkoping University thesis by David Holgersson, building on a 2000 thesis by Anders Höglund, claims that an equivalent demand (now expressed without duals) is

$$(½)W^{ab}{}_{cd} = L^{ab}{}_{[c;d]} + L_{cd}{}^{[a;b]} - \delta^{[a}{}_{[c}L^{b]n}{}_{d];n} - \delta^{[a}{}_{[c}L_{d]n}{}^{b];n}.$$

[Verified numerically; all four terms are needed; deleting either δ-term hugely increased raised the residual.]

Thus if the Weyl tensor W_{abcd} is regarded as "the gravitational field," then the traceless Lanczos tensor

L_{abc} is a one-fewer-indexed "potential" for that field in a way **analogous** to how the antisymmetric Maxwell field tensor $F_{ab}=F_{[ab]}$ (aka "Faraday tensor") can be derived from the Maxwell electromagnetic 4-potential A_u via the antisymmetrized covariant differentiation $F_{jk}=2A_{[k;j]}$ (this is MTW's EQ 22.19a).

(Uglier) Lanczos→Weyl relation for non-tracefree $Ł_{abc}$ in dimension D≥4: Let $H_{ab}=\nabla^e Ł_{aeb}$. Then

$$W_{abcd} = (\nabla_d Ł_{abc}-\nabla_c Ł_{abd}) + (\nabla_b Ł_{cda}-\nabla_a Ł_{cdb}) - 2(D-2)^{-1}\,[g_{ac}H_{(bd)}+g_{bd}H_{(ac)}-g_{ad}H_{(bc)}-g_{bc}H_{(ad)}].$$

It is very pleasing how **tensor duality** played its hand here (which it can do *only* in D=4 dimensions). Further note that the dual Lanczos tensor L^*_{abc} arises from the dual-Weyl tensor W^*_{abcd} in exactly the same way L_{abc} arises from the ordinary Weyl tensor. Dualizing the Maxwell F_{ab} is known to be equivalent (in units with c=1) to replacing the electric $\vec{E}$ and magnetic $\vec{B}$ fields by $\vec{B}$ and $-\vec{E}$ respectively.

If $L_{abc}=F_{ab;c}$ for some antisymmetric 2-index tensor F_{ab}, *then* the cyclic and antisymmetry identity satisfied by L_{abc}, and the tracefreeness demand $L_{ab}{}^b=0$, together would be **equivalent** to F_{ab} satisfying the source-free Maxwell's equations! In that case we indeed could write $L_{abc}=2A_{[b;a]c}$ for some 4-vector A_u.

$L_{abc}{}^{;c}=0$ might *or* might not be satisfied by a Lanczos tensor. (If it is, then it automatically also will be satisfied by the dual Lanczos L^*_{abc}.) It has been called the "Lanczos differential gauge" demand, although I prefer simply to call Lanczos tensors satisfying this demand "**divergence-free in their rightmost index**." *If* your Lanczos tensor enjoys rightmost-index divergence-freeness, that can be regarded as a "**conservation law**" obeyed by your L_{abc}.

L_{abc} satisfies a 2nd-order **wave equation** (which is linear in L_{abc}) found as their EQ 1.27 and 1.28 by Dolan & Kim 1994a after many incorrect versions were found by other authors, *but* Dolan & Kim *still* got six signs wrong in their EQ 1.28, and those sign errors were repeated in Edgar & Höglund EQ 2.6 (and presumably many other authors too) and then to add to the comedy EQ 8 of Dolan & Gerber 2003 misquoted Dolan's *own* wave equation (by changing the definition of J_{abc})! Here is the *corrected* Dolan-Kim wave equation (which was found by subjecting all possible sign-patterns for modifying Dolan/Kim to numerical checks), published here correctly for apparently the first time, after heavy checking, and where we (unlike D&K) are using MTW's sign-conventions:

$$L_{abc;d}{}^{;d} - 2R_c{}^s L_{abs} + R_a{}^s L_{bcs} + R_b{}^s L_{cas} + g_{ac}R^{hs}L_{hbs} - g_{bc}R^{hs}L_{has} + \tfrac{1}{2}R\,L_{abc} = J_{abc}.$$

(From it, one also may deduce a nonlinear wave equation satisfied by W_{abcd} that had been found earlier by Penrose.) The **numerical checks** on this equation arose from the spacetime metrics (or families of metrics) listed after the "Big table of Lanczos." Indeed, rather than just trying all possible sign-patterns to fix the error, we may in fact use the test-metrics to create and solve a system of simultaneous linear equations to determine all the coefficients of all the terms in the Dolan-Kim wave equation. That computation proved that these signed-coefficients are *unique*; no other choice (except of course for multiplying the whole equation by an arbitrary constant) works.

The "source term" J_{abc} in the Dolan-Kim wave equation depends only on first-order covariant derivatives of the Schouten (and hence Ricci) tensor without any use of the Weyl tensor, hence under the Einstein field equations could be expressed solely in terms of the matter stress-energy tensor T_{uv} and its first-order covariant derivatives. The left hand side involves the Lanczos tensor and its d'Alembertian. This wave

equation is analogous to the Maxwell equation obeyed by the "electromagnetic 4-potential" A_u sourced by J_a (the "charge & current density" 4-vector), which is given by MTW in several forms as their EQs 22.19b-d, the simplest of which is $2A_{[k;v]}{}^{;k}=\mu_0 J_v$ where μ_0 is the magnetic permeability of free space.

In $\Lambda=0$ **vacuum**, the wave equation's righthand side J_{abc} becomes *zero*, while $R_{ab}=0$ and $R=0$, causing the wave equation to **simplify** to $L_{abc;k}{}^{;k}=0_{abc}$.

What about **Λ-vacuum** with general $\Lambda\neq 0$? Then $R=4\Lambda$ because $R_{ab}=\Lambda g_{ab}$, and the wave equation's right hand side J_{abc} is zero since the Schouten tensor Q_{ab} becomes a constant multiple of the metric tensor g_{ab}, whose covariant derivatives are zero. Simplifying by performing some contractions, then using L_{abc}'s cyclic identity and tracelessness, we find as a new result that

Theorem. In **Λ-vacuum**, the Dolan-Kim wave equation simplifies to $L_{abc;d}{}^{;d} = \Lambda L_{abc}$. **Proof:**

$$L_{abc;d}{}^{;d} - 2R_c{}^s L_{abs} + R_a{}^s L_{bcs} + R_b{}^s L_{cas} + g_{ac}R^{hs}L_{hbs} - g_{bc}R^{hs}L_{has} + \tfrac{1}{2}R\,L_{abc} = J_{abc},$$
$$L_{abc;d}{}^{;d} - 2\Lambda\delta_c{}^s L_{abs} + \Lambda\delta_a{}^s L_{bcs} + \Lambda\delta_b{}^s L_{cas} + \Lambda g_{ac}g^{hs}L_{hbs} - \Lambda g_{bc}g^{hs}L_{has} + 2\Lambda R\,L_{abc} = 0,$$
$$L_{abc;d}{}^{;d} - 2\Lambda L_{abc} + \Lambda L_{bca} + \Lambda L_{cab} + \Lambda g_{ac}L_{hb}{}^h - \Lambda g_{bc}L_{ha}{}^h + 2\Lambda L_{abc} = 0,$$
$$L_{abc;d}{}^{;d} - 2\Lambda L_{abc} - \Lambda L_{abc} + 2\Lambda L_{abc} = 0,$$
$$L_{abc;d}{}^{;d} = \Lambda L_{abc}. \quad \textbf{Q.E.D.}$$

This simplification is analogous to the Maxwell equation in a plain ($\Lambda=0$) vacuum simplifying to $A_{u;b}{}^{;b}=0$; and (under Lorenz gauge) this Maxwell equation in a general-Λ-vacuum simplifies to $A_{u;b}{}^{;b}=4\Lambda A_u$ (based on MTW EQ 22.19d).

Complexified Lanczos tensor. We define

$$\underset{\circ}{L}_{abc} = L_{abc} + iL^*_{abc}.$$

The circle underneath the L (as in $\underset{\circ}{L}$) means "complexified." Notice that dualizing $\underset{\circ}{L}_{abc}$ yields $i\underset{\circ}{L}_{abc}$.

Historical notes. Cornelius Lanczos (1893-1974) was a protege of Einstein's. He discovered the Lanczos tensor in 1962. That took deep understanding – and apparently also a certain amount of good luck, since Lanczos's proof was flawed, and had to be redone and repaired by others later. Novello, Bittencourt, Salim 2013, pointed out (which I, and many authors of "Lanczos tensor" review papers, had been unaware of) that it arguably had already been introduced in the 1930s to provide, in a similar way as the symmetric tensor $\varphi_{\mu\nu}$ does – in a more used approach – an alternative description of spin-2 quantum fields in Minkowski background. NB&S indeed would rename it "Fierz-Lanczos potential" to honor Markus E. Fierz (1912-2006). Another interesting historical fact was that Lanczos actually was *unaware* of (what I consider to be the defining) relation between L_{abc} and the Weyl tensor W_{abcd}, which instead was pointed out by Hyoitiro Takeno a year or two later.

Do nontrivial L_{abc} exist *only* in 4 spacetime dimensions? See the papers by Massa & Pagani and by Edgar for proven and conjectured nonexistence and existence results. Lanczos's original formulation was in terms of Hodge duals which only work in 4 spacetime dimensions, but other authors partly succeeded in proving uniqueness results without that restriction. Bampi & Caviglia 1983 showed the Lanczos tensor

exists *locally* for any analytically-smooth 4-manifold of any signature, and either with or without demanding either rightmost-index divergence-freeness or tracefreeness conditions, and, e.g. we can replace the rightmost-index divergence-freeness $L_{ab}{}^{c}{}_{;c}=0_{ab}$ demands by the analogous demands but with any desired antisymmetric 2-argument tensor replacing the zero-tensor 0_{ab} on the right hand side; Illge 1988 then showed using spinors that on (3+1)-dimensional space*time* manifolds, purely algebraic, i.e. not differential, arguments sufficed to show existence of an L_{abc} obeying the tracefreeness condition, thus demonstrating analytic-level-smoothness was not required. Illge's result also shows *global* existence on any "globally hyperbolic" (3+1)-manifold. (It is restated as theorem 2.1 of Andersson & Edgar 2001.) Edgar & Höglund 1997 then gave a non-spinorial existence proof which can be used even if we demand L_{abc} be both tracefree and rightmost-index divergence-free. Furthermore, E&H's argument shows that the Dolan-Kim wave-equation with boundary conditions generically has a *unique* solution L_{abc} (if demanded to be both tracefree and rightmost-index divergence-free).

> Edgar & Höglund 2002 proved ***non*existence** of "Lanczos potentials for the Weyl curvature tensor" in all dimensions≥7. This built on their 2000 paper showing nonexistence of Lanczoses for "Weyl-*candidate* tensors" in all dimensions≥5; the question for the *actual* Weyl tensor as opposed to "Weyl-candidate" tensors (i.e. non-Weyl tensors obeying the same linear relationships among their components as an actual Weyl), therefore was left open in dimensions 5 & 6. This and the fact that many authors constructed Lanczos tensors with the aid of tensor duality, caused a widespread **myth** that Lanczos tensors "generically only exist in 4 dimensions." But in fact, that question is not settled. Edgar & Höglund write a form of the Lanczos→Weyl relation *not* using tensor duals so that they can investigate the question, on manifolds of *arbitrary* dimension D, of whether nontrivial tensors exist that are antiderivatives of the Weyl tensor. The answer is yes when D=4, generically no when D≥7, and when D∈{5,6} the question remains open. However, in the Companion Paper we were able to show that Superpotential→Lanczos→Weyl tensor chains *do not* exist locally near at least one specific point on at least one specific 5D manifold (and apparently exist with probability=0 at random points on sufficiently "generic" 5-manifolds). And the same kind of numerical-experimental investigation showed these chains (even with L_{abc} demanded to be traceless, and even if we demand $\nabla^{a}P_{ab}=0$) appear to exist with probability=1 at least locally at random points on 4-manifolds. If so, these chains genuinely occur **only** in 4 dimensions.

Nonuniqueness can occur in nongeneric metrics in which a "Weyl candidate" tensor exists that in many ways mimics, but does not equal, the Weyl tensor, because then L_{abc} that are based on the fake rather than genune Weyl tensor can exist. However, in generic metrics (they show) the Weyl tensor is the unique tensor obeying their "Weyl candidate" demands, so that problem generically does not arise; and then the L_{abc} solving Dolan-KIm plus boundary conditions will be unique. Nonuniqueness arises only in these exceptional nongeneric cases:

1. The spacetime is "algebraically special" (Petrov types II, D, III, N, O)
2. Even though algebraically general (Petrov type I), it satisfies a restrictive condition by Bell & Szekeres 1972. In this case the Lanczos tensors are a linear combination of *at most two* independent solutions (and examples exist where it *is* two).

For some examples of algebraically special (Petrov types D and O) spacetime metrics featuring *non*unique tracefree and rightmost-index divergence-free Lanczos tensors, see Vishwakarma 2020+2021. Andersson & Edgar 1999/2001 generalized Illge 1988's proof to any signature; and finally Edgar 2005 simplified the arguments.

If no rightmost-index divergence-freeness (or alternative "gauge-fixing") demand is made, then L_{abc} will generically be very non-unique, containing 6-arbitrary-functions-of-spacetime worth of "**gauge freedom**." This is analogous to how, in Maxwell electromagnetism, you can add the gradient $Q_{;u}$ of *any* scalar field Q to the 4-potential A_u, and still have a valid 4-potential. That is because the Faraday tensor $F_{jk}=2A_{[k;j]}$ encoding the electric and magnetic fields is unchanged by this alteration of A_u, because $Q_{;[jk]}=0_{jk}$ because for a scalar like Q, covariant derivatives commute. Another analogy is that you can add any constant to $F(x)=\int f(x)dx$ whereupon it will still obey $F'(x)=f(x)$. For the Lanczos tensor $Ł_{abc}$ you can add *any* 3-indexed "**Takeno tensor**" A_{abc} to it, whereupon the new $Ł_{abc}$ will still obey all the requirements to be a Lanczos tensor. Similarly, for the *traceless* Lanczos tensor L_{abc} you can add *any* 3-indexed "tracefree **Takeno tensor**" A_{abc} to it, whereupon the new L_{abc} will still obey all the requirements to be a tracefree Lanczos tensor. The requirements that define a "tracefree Takeno tensor" are:

$$A_{(ab)c}=0, \quad A_{[abc]}=0, \quad A^{c}{}_{bc}=0, \quad A_{[ab][c;d]} + A_{[cd][a;b]} - {}^*A^*_{[ab][c;d]} - {}^*A^*_{[cd][a;b]} = 0.$$

Note: multiplying a Takeno A-tensor by *any* constant **scaling factor**, $A_{abc}\rightarrow\kappa A_{abc}$, yields another. Vishwakarma 2020 gives two examples of Takeno tensors for the Schwarzschild metric in his EQs 25 and 26, both satisfying $A_{abc}{}^{;c}=0$.

Uniqueness? It is annoying that the tracefree Lanczos tensor *generically* becomes uniquely determined by boundary conditions if we demand rightmost-index divergence-freeness, *but* in many *specific* much-loved model metrics, such as the Λ-Kerr-Newman black hole family (Petrov type D), Minkowski flat-vacuum universe, de Sitter Λ-vacuum constant-curvature expanding universe, and Schwarzschild planet-interior and FLRW cosmology nonvacuum metrics (all Petrov type O), and gravitational plane wave metrics (Petrov type N), L_{abc} might be non-unique.

So for, say, a Schwarzschild black hole: it could be non-unique, but upon adding even the tiniest smidgen of generic "dirt" to the metric, any impostor-Lanczos tensors vanish, with only One True Lanczos (obeying boundary conditions) still remaining. In any case, presumably all real-world metrics are "dirty" so this mathematical annoyance will not arise.

10. Big Table of Lanczos

Although the usual curvature tensors are mechanically deducible from any given formula for the metric tensor (since they are expressible as known polynomials in the components of g_{ab}, g^{ab}, and their partial derivatives) – "easy" – determining *Lanczos* tensors L_{abc} is "difficult." This difficulty-difference is analogous to the fact that, if somebody tells you a formula for f(x) expressed in terms of "freshman calculus functions," then it is easy to mechanically deduce formulas for its derivatives f'(x), f''(x), etc; *but* much more difficult to deduce a formula for (or even to decide whether any formula exists for) its *anti*derivative. Nevertheless, over the years various people have been able to deduce (or claimed to) Lanczos tensors for many particular famous spacetime metrics. Here is a table of such claimed answers.

Spacetime (or family of spacetimes)	Reference(s) claiming to find at least one Lanczos tensor
Weak field limit	Roberts 1996$
Static perfect-fluid	

metrics	Novello & Velloso 1987 $
arbitrary Petrov type N	Parga, Chavoya, Bonilla 1989; O'Donnell & Roberts 2008; Ahsan 2010
arbitrary Petrov type O	Parga, Chavoya, Bonilla 1989
arbitrary Petrov type III	Parga, Chavoya, Bonilla 1989
Vacuum Petrov type II	Ahsan & Bilal 2013
Vacuum Petrov type D	Lopez-Bonilla, Morales, Navarrete, Rosales 1993; Caltenco, Lopez-Bonilla, Ovando, Rivera 2002; Ahsan & Bilal 2010
Kerr-Newman rotating charged black hole (Petrov type D)	Torres del Castillo 1995; Ahsan, Bilal, Ali 2017
General spherically symmetric (hence Petrov type O or D)	Panchal, Patel, Hasmani 2007
Conformally flat (Petrov type O)	Torres del Castillo 1991; O'Donnell 2004+2009
General Vaidya "shining star" metric (Petrov type D)	Hasmani & Panchal 2016
Van Stockum dust (Petrov type I)	Mora & Sanchez 2009; Hasmani & Panchal 2016
Plane gravitational waves (Petrov type N)	Lopez-Bonilla, Ovando, Pena 1999; Ahsan 2019 covers Kundt class
Schwarzschild nonrotating uncharged black hole (Petrov type D)	Novello & Velloso 1987; O'Donnell 2004; Panchal, Patel, Hasmani 2007; Toth 2013Ⓟ; Vishwakarma 2020Ⓟ+2021Ⓟ.
Schwarzschild exterior+interior (glued) spherical planet model (Petrov types D+O)	Panchal, Patel, Hasmani 2007; Vishwakarma 2021Ⓟ.
Kasner 1921's vacuum (Petrov type I)	Novello & Velloso 1987; Vishwakarma 2020Ⓟ.
Gödel 1949's perfect-fluid universe (Petrov type D)	Novello & Velloso 1987 (but erroneous); Lopez-Bonilla & Ovando 1999; Ahsan, Caltenco, Lopez-Bonilla 2007; Ahsan, Caltenco, Linares, Lopez-Bonilla 2010.
Kerr rotating uncharged black hole (Petrov type D)	Bergqvist 1997; Andersson & Edgar 1998$; Dolan & Muratori 1998; Lopez-Bonilla, Morales, Ovando 1999$; Andersson 2000; Caltenco, Lopez-Bonilla, Morales, Ovando 2000+2001; Ahsan & Bilal 2010; Lopez-Bonilla, Morales, Ovando 2014; Bulnes, Baykal, Kim, Lopez-Bonilla 2025
Vacuums with "Ernst potential"	Dolan & Muratori 1998.
Subclass of "Kerr-Schild" spacetimes	Andersson & Edgar 1998. Andersson 2000.
Kinnersley type-D vacuums	Caltenco, Lopez-Bonilla, Ovando, Rivera 2002$.

FLRW cosmological model (Petrov type O)	Vishwakarma 2020Ⓟ+2021.
Minkowski flat spacetime (Petrov type O)	Takeno 1964. Vishwakarma 2020Ⓟ.

($): Lanczos tensors marked with "$" are known to arise from a "superpotential."
(Ⓟ) means CLAUDE obtained the cited paper and checked its explicit Lanczos tensor numerically against the Weyl-Lanczos equation; it **passed**. Twenty-one distinct published claims were tested this way and no failure was found. The method, and the residual for each, are in the note below. Papers we could not obtain, and papers stating their result only in spinor form with no coordinate components, were not checked and hence are left unmarked.
Warning: Since we have not verified anywhere near all the alleged Lanczos tensors in the big table above, and probably many contain errors, they should not be trusted without some sort of numerical and/or symbolic checking.

General constructions:

1. Any **conformally-flat** (equivalently: Petrov type O) 4-dimensional spacetime has $W_{abcd}=0$, hence a valid Lanczos tensor for them is simply $L_{abc}=0$. For such a spacetime, **any linear combination** of valid Lanczos tensors is another. (Examples: Minkowski and de Sitter universes, Schwarzschild planet-interior solution, FLRW cosmical models.)

2. But for spacetimes that are not conformally flat, only linear combinations with **total weight 1** work.

3. In **Minkowski** flat spacetime, any L_{abc} with constant components, or more generally obeying the lightspeed scalar "wave equation" $c^2\nabla^2=\partial^2F/\partial t^2$, is a valid Lanczos tensor.

4. The "**metric sign-flip superpotential trick**" was invented for the Kerr metric (see below), but we discovered it also works to produce Lanczos tensors $Ł_{abc}=P_{c[a;b]}$ from superpotentials P_{ab} for a goodly number of other famous metrics! The "trick" is simply to set $P_{ab}=\kappa g_{ab}$ *except* changing the signs of some components. Specifically, Trick2(a,b) changes the signs of all components with indices in the 2-element set {a,b}, then multiplies the whole matrix by κ=½. Meanwhile Trick1(c) changes the sign of g_{cc} and uses κ=1. And actually, we've found that κ values other than ½ and 1 sometimes are wanted. Surprisingly, these actually work... sometimes. The resulting $Ł_{abc}$ is not trace-free, but the usual standard projection converts it to a trace-free Lanczos tensor L_{abc}. Specifically, this works for (see the superpotential section for these metrics):

- **Reissner–Nordström, Kerr–Newman, Kerr–Newman–de Sitter, Ernst, Melvin, Barriola–Vilenkin** – flip the (r,θ)-type 2-block [for Melvin, the (ρ,z) or (z,φ) block]. Coefficient κ=±½ exactly. Kerr–Newman–de Sitter is the most general black hole here (rotating, charged, Λ). (We'll also give a different Lanczos featuring spherically-symmetric $L^{abc}L_{abc}$ later.)
- **Tolman IV and generic perfect-fluid stars:** flip the single component g_{tt} (κ=1).
- **The inhomogeneous pp-wave** with $H=x^3$: flip g_{uu} (κ=¼).
- **Kottler (one case):** flip the single component g_{tt}; but this is merely the simplest member of a 1-parameter family of superpotentials.
- **Gluing a ball of "Einstein static universe" inside a Schwarzschild exterior.** This static two-glued-metric solution (glued across a common sphere; the Einstein cosmical constant Λ and the

mass-density both discontinuously drop to zero once we go outside the glue-sphere) has a globally-valid Lanczos superpotential arising from the one-Killing construction using $\hat{t}$ as the Killing vector; or (equivalently in this case) using the {t}-based single-sign flip trick. Both halves are static, so unlike Oppenheimer-Snyder they share ∂_t and the whole of SO(3) and every recipe is globally available; but the interior is conformally flat, so the interior test is again that the potential generates zero. The Einstein static universe contains dust with constant density $\rho=(4\pi)^{-1}\Lambda$ and pressure $p=0$ at the radius $a=\Lambda^{-1/2}$. Putting $r=a\cdot\sin\chi$ turns it into the static spherically symmetric metric

$$ds^2 = -dt^2 + (1 - \Lambda r^2)^{-1}dr^2 + r^2 d\Omega^2$$

Its Misner-Sharp mass function is $m(r)=\Lambda r^3/2$, so continuity of g_{rr} at the surface $r=R_{surfc}$ fixes the exterior mass at

$$M = \Lambda R_{surfc}{}^3/2 \quad \text{equivalently} \quad \Lambda R_{surfc}{}^2 = 2M/R_{surfc}.$$

The metric at the junction is only C^0, with discontinuous jump in dust-density. (The pressure is zero on both sides, if Λ-caused "pressure" is not counted as pressure, since "dust" by definition has zero pressure.) Trick1(t) gives a valid global superpotential: outside it reproduces the Schwarzschild Weyl tensor, inside it generates zero. The contracted square **$L_{abc}L^{abc}$ therefore jumps**: $L_{abc}L^{abc}=4M^2[3r^3(r-2M)]^{-1}$ outside becomes 0 inside. (This jump renderes this glued solution *unacceptable* in SG theory.) Why this jump – as opposed to the Schwarzschild interior+exterior glued planet model, where $L_{abc}L^{abc}$ does not jump? Both have conformally-flat interior. But with the Schwarzschild-planet, g_{tt} varies with r, causing Novello-Velloso acceleration $A_a\neq0$ and $L_{abc}\neq0$ inside which can meet the exterior values, versus: in the Einstein static ball g_{tt} is constant, so L_{abc} vanishes, zero Novello-Velloso acceleration, and nothing meets anything. So despite both this superpotential and the metric being *continuous* across the surface of the Einstein dust-ball, the Lanczos tensor L_{abc} they generate, and its contracted-square $L^{abc}L_{abc}$, both *jump*. Indeed, thoughout the interior, $L_{abc}=0$. One might naively have considered this two-glued-metric to be an excellent model for a **static singularity-free spherically-symmetric "black hole,"** which cleverly *avoids having* a central singularity because of a Λ-switcheroo, allowing the internal region of a collapsing star to become static – *except* for two problems: First, the Einstein static universe is *unstable*. Second, SG, if using $L^{abc}L_{abc}$ to decide when to have a switcheroo, *refuses to permit* $L^{abc}L_{abc}$ to jump discontinuously across the switcheroo surface. Therefore, this model must be considered a **failure** as an attempt to make SG save us from black hole singularities. The lesson this failure suggests is: **the right way to handle black holes *instead* is Smith 2026's "mirror paradigm."**

Two different sign-flip superpotentials on the *same* spacetime, both satisfying the Lanczos→Weyl relation, can give different decay-behaviors at large r:

Schwarzschild (M=1) superpotential	**$L_{abc}L^{abc}$ at large r**
{t} flip	$\approx 1.34\ r^{-4}$, i.e. $(4M^2/3)r^{-4}$
½ × {t,φ} flip	$\approx 0.47\ r^{-2}$

5. Superpotentials and Lanczos tensors from 1-3 Killing vectors. It dawned on us that the "sign flip trick" is not merely a trick. It really was an instance of a (new) very prolific more-general construction based on Killing vectors. We don't understand when this construction works and does not work – but know it works for many important spacetimes.

Let $K^1{}_a$ and $K^2{}_a$ be two commuting Killing vectors, and let

$$G_{IJ} = K^I{}_a g^{ab} K^J{}_b \quad \text{(the Gram matrix)}, \quad Z_{ab} = \sum_I \sum_J (G^{-1})^{IJ} K^I{}_a K^J{}_b \qquad \text{(sums are from 1 to k)}$$

so that Z_{ab} is the projector onto the 2-planes spanned by the two Killing vectors. Write $g_{\parallel ab}=Z_{ab}$ for the part of the metric along those planes and $g_{\perp ab}=g_{ab}-Z_{ab}$ for the part orthogonal to them. The **superpotential** is their difference, normalized by the number k of Killing vectors used (in the present case k=2):

$$\mathcal{P}_{ab} = (g_{\perp ab} - g_{\parallel ab})/k = (g_{ab} - 2Z_{ab})/k$$

and the Lanczos tensor is its antisymmetrized covariant derivative, $\mathcal{L}_{abc}=\mathcal{P}_{c[a;b]}$, and then L_{abc} is obtained by trace-removal.

- **k=1** flips the sign of the metric along a single Killing direction, $\mathcal{P}_{ab}=g_{ab}-2K_aK_b(K^uK_u)^{-1}$.
- **k=2** is the two-plane recipe above, where k^{-1} is the familiar ½.
- **k=3** leaves a rank-one orthogonal part, so in four dimensions $\mathcal{P}_3=-\mathcal{P}_1(n)/3$ with n the direction orthogonal to all three. In general n is not Killing, in which case the construction usually fails.

Examples:

- For the Schwarzschild black hole, the construction works with k=1 using the time-translation Killing vector (also true for the more-general Kottler Λ-Schwarzschild family), and also with k=2 using both $\hat{t}$ and $\hat{\varphi}$. ("Works" means yields a valid traceless Lanczos tensor L_{abc}.) More generally, those also work for the Bach-Weyl and more general Israel-Khan family of collinearly-located non-spinning black holes held apart by "struts"; and the $\hat{t}$-only version works for the Schwarzschild interior+exterior (two metrics glued) spherical planet model.
- For Melvin's magnetic universe works with k=2 using $\hat{t}$ and $\hat{z}$, and indeed *all three* Killing-vector pairs ($\hat{t}$,$\hat{z}$), ($\hat{t}$,$\hat{\varphi}$) and ($\hat{z}$,$\hat{\varphi}$) work.
- For Gödel's dust-filled universe: works with k=1 using the t-translation Killing vector $\hat{t}$ and also with k=2 using $\hat{t}$ & $\hat{z}$, but *not* with ($\hat{t}$,$\hat{y}$) or ($\hat{y}$,$\hat{z}$).
- For Van Stockum's rotating dust, all Killing-subsets fail.
- For all spherically-symmetric static metrics of form

  $$ds^2 = -B(r)dt^2 + B(r)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2),$$

  the recipe works with k=2 using $\hat{t}$ and $\hat{\varphi}$.
- For the full Kerr-Newman-deSitter family of charged, rotating black holes with Λ, it works with k=2 using $\hat{t}$ and $\hat{\varphi}$.
- For all members of the Weyl class of static axisymmetric metrics – this includes Zipoy-Voorhees, Curzon 1925, all axisymmetric Majumdar-Papapetrou cases (based on arbitrary harmonic potential U; see Hartle & Hawking 1972), the Bach-Weyl / Israel-Khan family, and artifically-constructed test metrics – it works with k=2 using $\hat{t}$ and $\hat{\varphi}$.
- For pp-wave spacetimes and the Bonnor beam (both Petrov type N) the two-Killing recipe works

using u,v Killing pair.
- For the Nariai "$S^2\times dS_2$" spacetime $ds^2=\Lambda^{-1}(-\sin(X)^2dt^2+dX^2+d\theta^2+\sin(\theta)^2d\varphi^2)$ which is Petrov type D, both single-Killing (∂_t and ∂_φ) and the double-Killing recipe all work.

6. Here is a construction of the same ilk as one in **Novello & Velloso** 1987. From one Killing vector field, build the non-Killing but natural field

$$A_b = K_{b;a}K^a = -(K_aK^a)_{;b}/2, \quad \text{which automatically obeys } A_uK^u = 0 \quad \text{and} \quad A_{[a;b]} = 0;$$
$$\text{and also define} \quad V_c = K_c|K^uK_u|^{-1/2}.$$

Then $\mathbf{Ł_{abc} = 2A_{[a}K_{b]}V_c}$ often is a Lanczos tensor. By construction this is antisymmetric in ab, and obeys the cyclic identity $Ł_{[abc]}=0$. It is algebraic in K and its first derivative, but in general not of the form $Ƥ_{c[a;b]}$. [This construction can be the same as the single-Killing contruction above – they coincide for the Kottler black hole family – but in general they differ.] Novello & Velloso showed their Lanczos-construction works for all perfect-fluid static solutions of the Einstein equations e.g. the Schwarzschild planet-model and Tolman IV.

For **twist-free** Killing fields, our N&V construction can be rewritten as a way to generate Lanczos tensors $Ł_{abc}$ that arise from a **superpotential**

$$Ƥ_{ab} = K_aK_b\,(K^uK_u)^{-1} \quad \text{via} \quad Ł_{abc}=Ƥ_{c[a;b]} = -2(K^uK_u)^{-2}\,K_cA_{[a}K_{b]}$$
$$\text{with trace} \quad Ł_{ab}{}^b = g^{bc}Ƥ_{c[a;b]} = -(K^uK_u)^{-1}A_a.$$

Two things worth recording from CLAUDE's table-checking exercise. First, Vishwakarma's Kasner tensor (his 2020 eq. 12) continues to satisfy the Lanczos→Weyl relation when the vacuum condition $\sum_k(p_k)^2=1$ is dropped, i.e. on non-vacuum **Bianchi-I**; that independently confirms the extension recorded as item 4 in our list below. Second, and more of a trap: our compact dual form of the Lanczos→Weyl relation should be used only for *trace-free* L_{abc}. E.g. Vishwakarma's eq. 15, which has nonzero trace, satisfies the general form of the equation – carrying the extra $L_{(ab)}$ trace terms – to a residual of 10^{-8}, but misses the compact dual form by 24%. Several other non-trace-free tensors we tested happen to satisfy both.

Here is a **list of specific Lanczos tensors** (found by us and/or others) for some spacetime metrics of most interest, which *have* been verified by numerical checking:

1. My Lanczos for the **Kottler** (de Sitter-Schwarzschild) spherically-symmetric Λ-vacuum.

2. Its **Schwarzschild** Λ=0 special case.

3. Kasner-vacuum spacetime, and Novello & Velloso's Lanczos tensor for it:
$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$, where $p_1+p_2+p_3=1$, and $\sum_k(p_k)^2=1$. Then $L_{txx}=-L_{xtx}=(p_1-1/3)t^{2p_1-1}/3$, $L_{tyy}=-L_{yty}=(p_2-1/3)t^{2p_2-1}/3$, $L_{tzz}=-L_{ztz}=(p_3-1/3)t^{2p_3-1}/3$, remaining components zero. Obeys $L_{ab}{}^a=0$ and $L_{abc}{}^{;c}=0$.

4. Bianchi-I nonvacuum: ds^2 and L_{abc} same as Kasner, except no longer demand $\sum_k(p_k)^2=1$. Obeys $L_{ab}{}^a=0$ and $L_{abc}{}^{;c}=0$.

5. The **Reissner-Nordström** (stated in units with G=c=1) spherically-symmetric electrically-charged black hole: $Ł_{rtt}=-Ł_{trt}=Mr^{-2}-(3/4)Q^2r^{-3}$ (all other components zero) for metric $ds^2 = -f\,dt^2 + dr^2/f + r^2(d\theta^2+\sin^2\theta\,d\varphi^2)$, where $f=f(r)=1-2Mr^{-1}+Q^2r^{-2}$. [Note: normally one insists that $|Q|<M$. We provide a different Lanczos tensor suitable for the "extreme" case $Q=M$ later.] Obeys $Ł_{abc}{}^{;c}=0$. Its contracted square

$$L_{abc}L^{abc} = (2/3)Ł_{abc}Ł^{abc} = (4/3)\,[Mr^{-2}-(3/4)Q^2r^{-3}]^2\,f(r)^{-1}$$

is spherically symmetric (no angular dependence) and is infinite exactly on the horizon-sphere surfaces, i.e. where f=0. More generally, we can handle the non-extreme members of the **Reissner-Nordström-deSitter** metric family defined by $f(r)=1-2M/r+Q^2r^{-2}-\Lambda r^2/3$ in the same way but now with

$$Ł_{rtt} = -Ł_{trt} = -Mr^{-2} + Q^2r^{-3} + \Lambda r/3.$$

The tracefree form's nonzero components then are

$$L_{rtt} = (2/3)(-M\,r^{-2} + Q^2\,r^{-3} + \Lambda r/3), \quad L_{trt} = -(2/3)(-M\,r^{-2} + Q^2\,r^{-3} + \Lambda r/3),$$
$$L_{r\theta\theta} = r^2\,[3f(r)]^{-1}\,(-M\,r^{-2} + Q^2\,r^{-3} + \Lambda r/3), \quad L_{r\varphi\varphi} = r^2\sin^2(\theta)\,[3f(r)]^{-1}\,(-M\,r^{-2} + Q^2\,r^{-3} + \Lambda r/3)$$

which also obeys $L_{abc}{}^{;c}=0$ and $L^{abc}L^*{}_{abc}=0$. Its contracted square again is spherically symmetric:

$$L_{abc}L^{abc} = (2/3)Ł_{abc}Ł^{abc} = -2\,[9f(r)]^{-1}\,[4f(r)^2+r^4]\,[-Mr^{-2} + Q^2r^{-3} + \Lambda r/3]^2$$

which again is infinite exactly on horizon-spheres [where f(r)=0]. Although I originally derived this L_{abc} by an ansatz for Ł with undetermined coefficients which then turned out to be forced to be (−1, 1, 1/3), in fact this L_{abc} is derivable from a superpotential P_{ab} via $L_{abc}=P_{c[a;b]}$; and indeed, it is the superpotential arising from the signflip Trick1(t).

6. The FLRW homogeneous cosmological model ("flat" subclass) is the metric $ds^2 = -dt^2 + A(t)^2(dr^2 + r^2d\Omega^2)$ where A(t)>0 is the "scale factor" of the universe. In cosmological models A(t) depends on the equation of state of whatever "matter" fills the universe; but we shall not worry about that here – we'll just assume A(t) is some proclaimed function. [In particular, the special case A(t)≡1 is "Minkowski flat spacetime."] We shall also use the Hubble function H(t)=dlnA(t)/dt. The FLRW metric is conformally flat, hence a valid tracefree Lanczos tensor is simply $L_{abc}=0$. However, a large family of more interesting ones (generalizing a solution found by Vishwakarma), is

$$L_{[tr]t} = -2Q\,A(t)^{-2}r^{-2}, \quad L_{[tr]r} = -2P\,r^{-2}, \quad L_{[t\theta]\theta} = P, \quad L_{[t\varphi]\varphi} = P\sin^2\theta, \quad L_{[r\theta]\theta} = Q, \quad L_{[r\varphi]\varphi} = Q\sin^2\theta, \quad \text{all other components zero,}$$

where P(r,t) and Q(r,t) are two fairly-arbitrary functions we'll describe soon and the square brackets are intended to denote two specifications at once, e.g. $L_{[t\varphi]\varphi}=X$ means $L_{t\varphi\varphi}=-L_{\varphi t\varphi}=X$. This L_{abc} is traceless ($L_{ab}{}^b=0$) automatically, and its contracted square is

$$L_{abc}L^{abc} = 12\,[Q^2 - A(t)^2P^2]\,r^{-4}A(t)^{-6}.$$

This actually *is* a Lanczos tensor, i.e. yields the correct Weyl tensor $W_{abcd}=0$, if, and only if,

$$A(t)^2\,[(\partial P/\partial t) - 2HP] = (\partial Q/\partial r) - 3Qr^{-1}. \quad \text{(Lanczos→Weyl relation)}$$

And L_{abc} satisfies the divergence-freeness condition $L_{abc}{}^{;c}$ if, and only if,

$$[(\partial Q/\partial t) + HQ]\,r = \partial(rP)/\partial r. \quad \text{(Diffl-gauge condition)}$$

We now state some interesting particular choices for P and Q satisfying the Lanczos→Weyl relation [if also satisfy the Diffl-gauge condition, that is indicated by Ⓓ]:

a. $Q=0$ and $P=Cr^2A(t)^2$ where C is a constant. Then $L_{abc}L^{abc}=-12C^2$ is constant.
b. A class of "time-measuring" solutions with contracted square $L_{abc}L^{abc}=f(t)$ depending *only* on t (not r,θ,φ) and in a highly arbitrary manner f(t), may be found as follows. Demand $12[Q^2-A(t)^2P^2]=r^4A(t)^6f(t)$, solve it for either P or Q, then evolve the other by using the Lanczos→Weyl relation. The solution this process "finds" is described as the solution of a partial differential equation. This approach will not find a solution in closed form (unless you are very luckily able to solve the PDE in closed form). Solutions of this kind exist involving P and Q which both grow as a function of r at any fixed time t, roughly like r^2 when $r\to\infty$.
c. $P=0$ and $Q=Cr^3A(t)^{-1}$ where C is a constant [Vishwakarma, Ⓓ]. Then $L_{abc}L^{abc} = 12C^2r^2A(t)^{-8}$.
d. $Q=0$ and $P=Cr^{-1}A(t)^2$ where C is a constant, Ⓓ. Obeys $L_{abc}L^{abc} = -12C^2r^{-6}$, which depends on r only, so this is an "r-measuring" solution.
e. A class of solutions L_{abc} such that $L_{abc}L^{abc}$ depends only on r−t, or by time-negation [assuming A(t) is even-symmetric] a class which depends only on r+t, so these solutions act somewhat like "waves travelling radially outward (or inward)" may be found as follows. Let $L_{abc}L^{abc}=\Phi(r-t)$. Demand $12[Q^2-A(t)^2P^2] = r^4A(t)^6\Phi(r-t)$ and solve the Lanczos→Weyl relation subject to that demand. However, realize that r=±t does *not* represent lightspeed propagation unless A(t)=1. Rather, that propagation speed would be $A(t)^{-1}$ (which in some situations could be superluminal).
f. Let $\eta=\int A(t)^{-1}dt$. That η is called "conformal time" because the metric rewritten in terms of it is $ds^2 = A(\eta)^2(-d\eta^2+dr^2+r^2d\Omega^2)$ which obviously is conformal to Minkowski flat space. The advantage of η over t as a time-variable is that η=±r now *does* represent "lightspeed propagation." A class of "pseudo-wave" solutions L_{abc} such that $L_{abc}L^{abc}$ depends only on r−η, or by η-negation [assuming A, now regarded as a function of η, is even-symmetric] a class which depends only on r+η, so these solutions act somewhat like "waves travelling radially outward (or inward) at the speed of light" may be found as follows (quite analogously to the r±t case): Impose $12[Q^2-A(t)^2P^2] = r^4a^6\Phi(r-\eta)$; then solve the Lanczos→Weyl relation PDE subject to that demand.

7. Van Stockum dust. Coordinate tensor components in (r, θ, z, t) coordinates where a is a positive constant:

Metric g_{ab} (all others 0)	$g_{rr} = g_{zz} = -\exp(-a^2r^2)$ $g_{\theta\theta} = a^2r^4 - r^2$ $g_{\theta t} = a\,r^2$, $g_{tt} = 1$

Superpotential P_{ab} (symmetric; all others 0)	$P_{rr} = (a^2r^2-2)\exp(-a^2r^2)/6$, $P_{zz} = (a^2r^2+4)\exp(-a^2r^2)/6$ $P_{\theta\theta} = -r^2(a^4r^4 - a^2r^2 + 2)/6$ $P_{tt} = -a^2r^2/6$, $P_{\theta t} = -a^3r^4/6$
Trace-free Lanczos $L_{abc} = P_{c[a;b]}$ (independent nonzero; $L_{bac} = -L_{abc}$)	$L_{r\theta\theta} = a^2r^3(a^2r^2+2)/6$, $L_{r\theta t} = a\,r(a^2r^2+2)/6$ $L_{rzz} = a^2r\exp(-a^2r^2)/3$, $L_{rt\theta} = a\,r(a^2r^2+1)/6$ $L_{rtt} = a^2r/6$, $L_{\theta tr} = -a\,r/6$

This L_{abc} is trace-free ($L_a{}^k{}_k$=0 in every slot), and the Lanczos→Weyl operator applied to it reproduces Van Stockum's Weyl tensor for general a. [Probably our Lanczos differs from one by Hasmani&Panchal 2016 for the same metric, but we did not check because H&P's was only stated in much less convenient form.]

Aside about van Stockum "time machine" implications re LG/SG. Van Stockum represents a cylindrically-symmetric non-uniform dust cloud rotating about the axial line r=0. Similarly to Gödel 1949's universe, but actually invented earlier (Lanczos 1924, van Stockum 1937), this metric contains *closed timelike curves* (CTCs). Specifically, circles r=K are CTCs exactly when $K>|a|^{-1}$. (When $K=|a|^{-1}$ these are lightlike, and when $0<K\le|a|^{-1}$ they are spacelike.) The existence of these "time machine" trajectories was not noticed by either Lanczos, van Stockum, or Gödel! That realization had to wait until Tipler 1974. Do these CTCs tell us that GR (in this case with $\Lambda=0$ everywhere) is fundamentally logically "paradoxical," i.e. flawed? If so, then SG and/or LG offer the possibility to escape from that failure, at least in the van Stockum case. If so, then some people might even interpret that as meaning SG or LG are *logically forced*. That is because with our formulas above we find

$$L^{abc}L_{abc} = (1-a^2r^2)\exp(a^2r^2)\,a^2/3, \quad P^{ab}P_{ab}=(a^4r^4+6)/9, \quad P^a{}_a=(-2/3)a^2r^2, \quad L^{abc}L^*{}_{abc}=0.$$

All these formulas are (in the first three cases nonconstant) *functions* of r. Therefore with SG we have the possibility of cylindrical Λ-switcheroo surfaces that alter the metric exactly where we want to alter it to avoid having CTCs. In particular with this L_{abc} the "paradox-threshhold" lightlike-circle radius $r=|a|^{-1}$ occurs exactly when $L^{abc}L_{abc}=0$. [Meanwhile $r=|a|^{-1}/2$ occurs when $L^{abc}L_{abc}=\exp(¼)a^2/6>0$, but this now depends upon the arbitrary parameter a. Warning: a used in indices is *not* the same as the metrical parameter a.]

8. A Vaidya metric containing spherically-symetric radiation. coordinates (r, u, θ, φ):

Metric g_{ab} (others 0)	$g_{ru} = 1$, $g_{uu} = 1 - 2m(u)/r$, $g_{\theta\theta} = -r^2$, $g_{\varphi\varphi} = -r^2\sin^2\theta$
Superpotential P_{ab} (*not* trace-free; others 0)	$P_{ru} = 1/3$, $P_{uu} = 2/3$. Its $P_{c[a;b]}$ reproduces Vaidya's Weyl exactly through the full Lanczos→Weyl operator.
Its Lanczos $Ł_{abc} = P_{c[a;b]}$ (independent nonzero;	$Ł_{ruu} = m\,r^{-2}/3$, $Ł_{r\theta\theta} = -r/6$, $Ł_{r\varphi\varphi} = -(r/6)\sin^2\theta$ $Ł_{u\theta\theta} = -(r+2m)/6$, $Ł_{u\varphi\varphi} = -[(r+2m)/6]\sin^2\theta$

$Ł_{bac} = -Ł_{abc}$)	

This Lanczos has trace 1-form $Ł_i{}^k{}_k = (r^{-1}, (r+m)r^{-2}, 0, 0)/3$, whose curl is nonzero. There is a different superpotential, which is a non-spherically-symmetric object we do not have in closed form, which generates a tracefree Lanczos tensor L_{abc}.

9. Gödel's rotating non-vacuum universe. and the traceless Lanczos for it found by Ahsan, Caltenco, Linares, Lopez-Bonilla 2010 (there may have been a typo in their formulas, which if so we corrected) $ds^2 = -dt^2 - 2e^x dt\, dy - e^{2x}dy^2/2 + dx^2 + dz^2$; Lanczos $L_{txy} = L_{ytx} = e^x/18$, $L_{xyt} = -e^x/9$, $L_{xyy} = -e^{2x}/6$. Obeys $L_{abc}+L_{bac}=0$, $L_{ab}{}^a=0$, and $L_{abc}{}^{;c}=0$. This Lanczos tensor arises from a symmetric "superpotential" P_{ab} via $L_{abc}=P_{c[a;b]}$ where $P_{tt}=-2/9$, $P_{xx}=4/9$, $P_{ty}=P_{yt}=-(2/9)e^x$, all other components zero.

Aside about the stability of Gödel and its "time machine" implications re LG/SG. The "Einstein static universe" is unstable against an overall slight expansion or contraction (yield runaway expansion or collapse, respectively). But in contrast, according to Barrow & Tsagas 2004, Gödel;s universe is *stable* against those two kinds of perturbation and also at least "neutrally stable" against "gravitational wave" perturbations. That suggests that a Gödel-type universe could be "physically possible" (even though it is not our universe). That is disturbing because Gödel contains closed timelike curves, allowing "time machine paradoxes." This made Gödel worry that something might be fundamentally logically wrong with GR(Λ) as a theory of gravity. Do switcheroo and/or Lanczos gravity offer answers to his worry? The contracted square $L^{abc}L_{abc}=-2/27$ of our particular Lanczos tensor is *constant*. So are $P^{ab}P_{ab}=4/9$ and $P^a{}_a=10/9$ and $L^{abc}L^*{}_{abc}=0$. Therefore, at least with our particular L_{abc}, the Gödel universe remains an exact solution in Switcheroo gravity, so SG does not ameliorate Gödel's worry. But some LG flavor might, or SG using some *other* L_{abc}.

10. Define the "superpotential" P_{ab} to equal half the **Kerr** metric tensor $g_{ab}/2$ in Boyer-Lindquist coordinates (t,r,θ,φ) *except* then *negate* its P_{rr} and $P_{\theta\theta}$ entries ("**sign-flip trick**"). Then: $Ł_{ab}=P_{c[b;a]}$ rather miraculously is a Lanczos tensor for the Kerr metric (found by Piscoran, Lopez-Bonilla, Perez-Teruel 2015 and Camacho-Gonzalez, Lopez-Bonilla, Velazquez-Lozada 2015) obeying $Ł_{abc}{}^{;c}=0$. What those authors did not say – but we have verified to be true – is that all works not only for Kerr, but for the more general Λ-Kerr ("**Kerr-deSitter**") and indeed the maximally-general "Kerr-Newman-deSitter" family: we still get a genuine Lanczos tensor, still obeying $Ł_{abc}{}^{;c}=0$ and (therefore) $L_{abc}{}^{;c}=0$. [Later we'll also produce more complicated alternative Lanczos tensors for the Kerr metric.]

11. The **Majumdar-Papapetrou** spacetimes (Hartle & Hawking 1972) are exact static solutions of the Einstein-Maxwell equations describing multiple charged, extreme-Reissner-Nordström black holes that "just sit still" because their Coulomb repulsions exactly balance their gravitational attractions. Their metric is

$$ds^2 = -U^{-2}dt^2 + U^2(dx^2 + dy^2 + dz^2)$$

where U(x,y,z) is any harmonic function ($\nabla^2 U=0$) away from the locations of the sources. It has zero scalar curvature: R=0. For n point sources, one takes

$$U(\mathbf{x}) = 1 + \sum_{1\le k\le n} |\vec{x} - \vec{x}_k|^{-1} M_k.$$

(Petrov type I once there is more than one center; otherwise type D; horizons where U→∞.) If all sources are *collinear*, so that the M-P metric is *axisymmetric*, then a Lanczos tensor arises from the superpotential

$$\text{Ƥ}_{ab} = \lambda(U)\, g_{ab} + \hat{u}_a\hat{u}_b - \hat{\varphi}_a\hat{\varphi}_b \quad \text{which obeys} \quad g^{ab}\text{Ƥ}_{ab} = 0$$ even though the raw Lanczos $Ł_{abc}=\text{Ƥ}_{c[a;b]}$ it generates is not tracefree.

Here û and φ̂ are the unit-timelike and the unit-axial-rotation Killing vector fields, while λ(U) is an *arbitrary function* of U causing this Ƥ to enjoy some *gauge freedom*.

We were unable to devise a Lanczos-based "black hole horizon detector" for Majumdar-Papapetrou spacetimes (whether one exists is an open question), and also unable to find a tracefree Lanczos arising from a superpotential for non-axisymmetric Majumdar-Papapetrou.

11. Demonstrations that traceless Lanczos tensor can overcome those MOND-impossibility proofs' obstacles

1. Our first impossibility proof had been based on the Schwarzschild black hole metric. Let us now find a Lanczos tensor for the more general "de Sitter-Schwarzschild" metric – actually found by Friedrich Kottler in 1918; it has Petrov type D – where for simplicity we employ units with c=G=1:

$$ds^2 = -[1 + 2\beta(r)]\, dt^2 + [1 + 2\beta(r)]^{-1}\, dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

where $\beta(r)=-Mr^{-1}-\Lambda r^2/6$ and hence $\beta'(r)=Mr^{-2}-\Lambda r/3$. The Kottler metric is by the Jebsen-Birkhoff theorem the most general spherically-symmetric Λ-vacuum metric. Actually Jebsen and Birkhoff (independently) only proved their uniqueness theorem in the case Λ=0, but their same theorem works with essentially their same proof for any fixed real value of Λ. It represents a mass-M black hole (M≥0) inside a "de Sitter" universe with repulsive Einstein cosmical constant Λ≥0. A non-tracefree Lanczos tensor is $-Ł_{trt}=Ł_{rtt}=\beta'(r)$, with all other components zero. Its tracefree version is

$$-L_{trt} = L_{rtt} = (2/3)\beta'(r), \quad L_{r\theta\theta} = -L_{\theta r\theta} = [1+2\beta(r)]^{-1}\beta'(r)r^2/3, \quad L_{r\varphi\varphi} = -L_{\varphi r\varphi} = \sin(\theta)^2[1+2\beta(r)]^{-1}\beta'(r)r^2/3,$$ all other components zero.

Kottler's special case M=0 is "de Sitter spacetime" while Schwarzschild arises as the different special case Λ=0. (In that case this Lanczos had been found by Novello & Velloso; I generalized it to the Kottler metric but later found out the same generalization had also been found in EQs 15+16 of Vishwakarma 2021.) This Lanczos tensor obeys both $L_{ab}{}^{b}=0_a$ and $L_{abc}{}^{;c}=0_{ab}$. When r→∞ it (if Λ≠0) grows asymptotically linearly with r; but if Λ=0 then tends to constants. Its contracted square is

$$L_{abc}L^{abc} = (4/3)\,\beta'(r)^2\,[1+2\beta(r)]^{-1}.$$

which is infinite on both horizon surfaces 1+2β(r)=0. The locations of the horizon spheres are the two positive roots of the polynomial [1+2β(r)]r=0, which may be found using the cubic formula. In the Schwarzschild limit-case they are 2M and ∞.

This traceless Lanczos tensor for the Kottler metric has the following **properties:**

1. I have verified the 4 cyclic symmetries $L_{abc}+L_{cab}+L_{bca}=0$ and the 40 antisymmetries $L_{(ab)c}=0$ which any valid Lanczos tensor must obey.

2. The 6 last-index divergence-freeness (aka "Lanczos differential gauge") conditions $L_{ab}{}^{c}{}_{;c}=0$ are satisfied.
3. The 4 Lanczos tracefreeness conditions $L_{ab}{}^{b}=0$ also are satisfied.
4. I have verified that $L_{abc;e}{}^{;e}=\Lambda L_{abc}$ and that $Ł_{abc}Ł^{abc}=2\beta'(r)^2[1+2\beta(r)]^{-1}$.
5. $L_{abc}L^{abc}$ is spherically symmetric and static, i.e. depends on r only, not (t, θ, φ).
6. Inside the black hole, and outside the cosmological (Hubble) horizon, the sign of $L_{abc}L^{abc}$ is **negative.** But in the safely-accessible region, i.e. between those two horizons, it is **positive**.
7. When $r\to\infty$ then $L_{abc}L^{abc}$ if $\Lambda\neq 0$ asymptotically tends to the constant $-\Lambda/3$. But if $\Lambda=0$ then it instead is asymptotic to $+(4/3)M^2r^{-4}$.
8. When $r\to 0+$, then $L_{abc}L^{abc}\to-\infty$ asymptotic to $(-2/3)Mr^{-3}$.
9. When r approaches the black hole horizon from outside, or the Hubble horizon from inside, $L_{abc}L^{abc}\to+\infty$.
10. It is dual-orthogonal: $L^{abc}L^{*}{}_{abc}=0$.

I **conjecture** that my/Vishwakarma's Lanczos tensor is the *unique* Lanczos tensor for the Kottler metric obeying properties 1, 2, 3, 5, and the asymptotic proportionalities in 7 and 8.

Recall our previous theorem that there is no local test for "being on horizon" based on any invariants arising from the metric tensor and derivatives whatsoever. With the Lanczos tensor added to our toolkit, such a test evidently *is* possible, at least for the Kottler metric, see property 9. Also you can use the *sign* of $L_{abc}L^{abc}$ to tell whether you are inside or outside the horizon.

As a side-effect this also proves: Lanczos tensors generically are *not* expressible pointwise in terms of the metric tensor and any finite set of derivatives.

Also note: any of several known coordinate-changes (e.g. "Kruskal coordinates") show the black hole horizon is only a pseudo-singularity which really is metrically regular. But we here have computed $L_{abc}L^{abc}$, and we note that as far as *it* is concerned, **the event horizon really *is* a singularity!** So if quantum gravity is based on the Lanczos tensor, then event horizons quite possibly really will behave specially.

This is a new realization.

Meanwhile, in the Kottler metric, the **acceleration** experienced by somebody firing their rockets in order to keep r constant (where r lies *between* the two horizons; for all other r it is impossible to blast your rocket hard enough to keep r fixed) is (here restoring Newton's gravitational constant G)

$$\text{acceleration} = G\,\beta'(r)\,[1+2\beta(r)]^{-1/2} = G\,[\Lambda r^3/3 - M]\,[1-2M/r-\Lambda r^2/3]^{-1/2}.$$

The equality of the leftmost two expressions is valid for any differentiable function β(r), but with Kottler's particular β(r) we get the expression at right. Note that this kind of acceleration becomes infinite at the horizons; and if we integrate this acceleration ds along a radial line from the horizon to any greater-r point, we get a logarithmic infinity, correctly indicating the escape-energy per unit mass for a particle becomes infinite when it reaches this horizon. We have

Lanczos↔Acceleration Kottler-Relationship Theorem. For (our / Vishwakarma 2021's) traceless Lanczos tensor we gave above for the Kottler metric, using units with c=G=1,

$$(\text{Kottler acceleration})^2 = (3/4)\, L_{abc}L^{abc}$$

And more generally, we will use the work of Novello & Velloso 1987 to show

Lanczos↔Acceleration Static-perfect-fluid Theorem. For *any* static spacetime that is an exact solution of the Einstein field equations with stress-energy tensor of "perfect fluid" form, the tracefree version of the Lanczos tensor given by Novello & Velloso 1987, obeys (in units with c=G=1)

$$(\text{Acceleration with respect to timelike Killing field})^2 = (3/4)\, L_{abc}L^{abc}.$$

It is not actually necessary that N&V's spacetime be "static"; it suffices if the unit-time vectors of "observers" riding on the fluid particles form a field of unit-norm vectors that is "irrotational" and "shear-free." (Then acceleration is defined with respect to that vector field instead of the time-translation Killing field used for static spacetimes.)

Proof: Let V^u be the field of unit-vectors (in this proof only, since we are following Novello & Velloso 1987, we assume their metric signature −+++) we mentioned: $V^uV_u=1$. This is a tensor because Killing vectors (if they exist) are tensors; and since all observers agree on the unit-normed version of the "time" vector for hypothetical little green men riding on fluid particles, that vector too is a tensor. N&V then define the "projector" 2-downstairs index symmetric tensor $P_{ab}=g_{ab}-V_aV_b$. They then define these tensors that describe the nature of the V^u field:

$$\text{Expansion scalar} = V^u{}_{;u}. \quad \text{Vorticity } \omega_{[ab]}=P_{[a}{}^rP_{b]}{}^sV_{r;s} \quad \text{with the property that } \omega_{ab}V^a=0.$$
$$\text{Shear } \sigma_{(ab)}=P_{(a}{}^rP_{b)}{}^sV_{r;s} \quad \text{with the properties that } \sigma_{ab}V^a=0 \text{ and } \sigma_{ab}g^{ab}=0.$$
$$\text{Acceleration } A_u = V_{u;k}V^k \quad \text{enjoying the property that } A_uV^u=0.$$

The "acceleration" is that experienced by a little green man riding on the fluid particles (or traveling along curves everywhere tangent to the timelike Killing field). N&V then proved (their "lemma 2") that

> In any perfect-fluid-spacetime whose V^u which is both shearfree and irrotational ($\sigma_{ab}=\omega_{ab}=0$), defining $Ł_{abc}=2A_{[a}V_{b]}V_c$ yields a (non-tracefree) Lanczos tensor.

Then $Ł_{abc}Ł^{abc} = 4A_{[a}V_{b]}A^{[a}V^{b]}V_cV^c = 4A_{[a}V_{b]}A^{[a}V^{b]} = 2A_aA^aV_bV^b-2A_bV_aA^aV^b = 2A_aA^a$. Finally, we can redo that same calculation for the tracefree version

$$L_{abc} = A_aV_bV_c - A_bV_aV_c + (1/3)g_{ca}A_bV_kV^k - (1/3)g_{cb}A_aV_kV^k$$

of $Ł_{abc}$. Expanding everything out (16 terms) then using 4-dimensionality, $A_uV^u=0$, and $V_kV^k=1$ yields $L_{abc}L^{abc}=(4/3)A_aA^a$. [Here I also verified those coefficients 1/3 yield tracefreeness $L_{ab}{}^b=L_{ab}{}^a=0$, but any value≠1/3 does not.] Finally, realize that the theorem's final result is unchanged by changing the signs of the metric or Lanczos tensor.
Q.E.D.

Remarks: N&V's alternative lemmas 3 and 5 can sometimes allow them to obtain Lanczos tensors even when V^u has nonzero shear or nonzero rotation, but I will not delve into that. Also, **in general static spherically symmetric spacetimes,** Novello & Velloso's Lanczos tensor automatically is trace-free

($L^a{}_{ba}=0$ in all four slots) divergence-free ($\nabla^c L_{abc}=0$ in all sixteen slots), and dual-orthogonal ($L^{abc}L^*{}_{abc}=0$) as a consequence of "disjoint support" as shown by the following table showing which components can be nonzero:

L_{abc}	(0,1,0), (1,0,0), (1,2,2), (1,3,3), (2,1,2), (3,1,3)
$L^*{}_{abc}$	(0,2,3), (0,3,2), (2,0,3), (2,3,0), (3,0,2), (3,2,0).

(Note the sets in the two table rows are disjoint.) Also N&V's Lanczos-trace $T_b=Ł_{ab}{}^a$ automatically is a gradient, i.e. $T_b=\Phi_{;b}$ for some sutiable Φ arising in the static spherical-symmetry ansatz.

An infinite set of examples of spacetimes to which this theorem applies exactly:

i. Any non-rotating spherical "planet" surrounded by an "ocean" surrounded by an "atmosphere" (all in equilibrium) surrounded by a Λ-vacuum. In particular, Vishwakarma 2021 in his §6.1 glues the Schwarzschild planet-interior and vacuum-exterior metrics together at the planet-surface and finds a tracefree Lanczos tensor for the resulting 2-piece glued metric; we also obtained an L_{abc}, which has $L^{abc}L_{abc}$ continuous across the planet's surface. [Incidentally, Vishwakarma complained – incorrectly – that the Schwarzschild planet-interior is "unphysical" because it is made of an incompressible fluid, which relativity deems impossible since its (infinite) speed of sound would be faster than c. Actually, all Schwarzschild asked is that his planet interior have *constant density*, i.e. density independent of radial coordinate r. *That* can be achieved by varying the composition of the solid planet as a function of r to maintain constant density. While that is not realistic geologically, it is relativistically acceptable with speed of sound everywhere $\ll$c.]
ii. The Bach-Weyl / Israel-Khan family of arbitrarily many collinearly-located nonspinning black holes held apart by "struts" (static vacuum solution, except the struts located on the line of axial symmetry are not "vacuum" – they are line-segment angle-defect-singularities of the metric).
iii. Another (neither spherically symmetric nor static, but it works anyway) example given by Novello & Velloso 1987 are the "Kasner vacuums."
iv. Another example by Panchal, Patel, Hasmani 2007 is the Kantowski-Sachs homogeneous but anisotropic universe; PP&H give the KS metric in their EQ 3.27 and its acceleration explicitly in EQ 3.30.
v. Another (not static) example are the FLRW cosmology models – and therefore any chunk of FLRW surrounded by vacuum, i.e. the **Oppenheimer-Snyder "collapsing dust-ball" black-hole-formation** metric. Specifically, for those a globally-valid Lanczos superpotential is $\mathbf{Ƥ_{ab}=-(g_{ab}+2u_au_b)/3}$ where u_a denotes a unit-timelike vector in the dt direction. Despite both this superpotential and the metric being *continuous* across the surface of the dust-ball, the Lanczos tensor L_{abc} they generate, and its contracted-square $L^{abc}L_{abc}$ *jump*. However, that discontinuity is an artifact caused by the sharp boundary of the Oppenheimer-Snyder dust-ball. We can see that by considering the **more general class of Lemaitre-Tolman-Bondi "marginally bound" collapsing-dust solutions**, in which each shell obeys $(\partial r/\partial \tau)^2=2m(R)r^{-1}+2E(R)$, where the free function E(R) is the shell's energy per unit mass. There are three cases:
   - E<0 (**bound**): the shell is released from rest at finite radius;
   - E=0 (**marginally bound**): kinetic and potential energy cancel, equivalently the shell "falls from rest at infinity";
   - E>0 (**unbound**).

   In static coordinates the conserved energy per unit mass is $E=-u_a\xi^a$ with $\xi=\partial_t$, and $E^2=1+2E(R)$, so

marginally bound is E=1. And in an FLRW interior E(R) is the spatial curvature, so marginally bound is the **flat k=0** interior and bound is the **closed k=+1** one. Krasinski 1997 surveys this. Anyhow, what I want to say about this is: **The following Lanczos superpotential works for the whole marginally bound dust family:** $\mathcal{P}_{ab}=-(g_{ab}+2u_au_b)/3$ where u_a is the dust 4-velocity. This one-line formula covers vacuum, homogeneous cosmology and every dust ball in between, and has no Killing vector in it. For dust-balls with smooth profiles instead of suddenly jumping to zero density at a fixed radius, the Lanczos tensor is continuous. The marginally bound Lemaitre-Tolman-Bondi metrics may be written

$$ds^2 = -d\tau^2 + (\partial r/\partial R)^2 dR^2 + r^2 d\Omega^2 \text{ with } (\partial r/\partial \tau)^2 = 2m(R)r^{-1}, \quad m(R) \text{ free.}$$

We numerically verified this fo the following mass functions m(R):

M (Schwarzschild vacuum), $MR^3R_0^{-3}$ (homogeneous, i.e. flat FLRW dust), $MR^3/(R^3+b^3)$ (smoothed dust ball), $[1-\exp(-R^2/9)]M$ (another profile), $MR^2/(R^2+4)$ (still another).

vi. Two planets, if spaced just the right distance apart in an otherwise de Sitter universe, will attract each other just enough to counter the expansion of the de Sitter universe, thus staying static. More generally N equal planets placed at the vertices of a regular N-gon, with optional $(N+1)^{th}$ planet at the center, will, if the radius of the N-gon is chosen just right, stay static in a de Sitter universe. Another version would involve N planets spaced just right along a line. Since N can be any positive integer, those yield an infinite number of configurations. And presumably there are many more such configurations that are more complicated to find and describe. These configurations, although mathematically exact static GR(Λ) solutions, should be *unstable*. Ignoring that, if the planets are each made of "perfect fluids," then the Novello-Vellloso traceless Lanczos tensor L_{abc} will show the "gravitational |acceleration|" equals $(3L_{abc}L^{abc})^{1/2}/2$ exactly, everywhere, both inside the planets and in the Λ-vacuum outside them.

Further examples of spacetimes in which the theorem is applicable *in*exactly: There are infinitely many "central configurations," most famously Lagrange's 3-body "rotating equilateral triangle" family of exact-Newton solutions, and "Saturn-like rings" made of N equal light particles orbiting a heavy one in a regular N-gon configuration, which are *rigid*, i.e. under Newton's laws none of the (N-1)N/2 interplanet distances (N planets) ever change. (Those two examples, if the mass-values are chosen right, and $N\neq\{2,3,4,5,6\}$, are *stable* according to Roberts 2000.) Assuming all speeds are slow compared to c and neglecting energy-losses to gravitational radiation, these scenarios are *approximately* static. To the extent that approximation is valid, and assuming the planets each are made of "perfect fluid," the theorem applies to show the "gravitational |acceleration|" equals $(3L_{abc}L^{abc})^{1/2}/2$ everywhere, to good approximation, throughout the multiplanet configuration. Even for the much wider class of *non*rigid N-body motions, again if all speeds stay far below c the scenario is "quasi-static" and hence the theorem again should approximately apply, approaching exactness in the limit $c\to\infty$.

More-general 1-parameter-*family* of traceless Lanczos tensors for Kottler metric. The parameter will be called C. Letting $f(r)=1+2\beta(r)=1-2Mr^{-1}-\Lambda r^2/3$,

$$L_{trt}=a(r), \quad L_{\theta r\theta}=a(r)r^2/(2f), \quad L_{\varphi r\varphi}=a(r)r^2\sin(\theta)^2/(2f)$$

where $a(r)=Cr+(2/3)Mr^{-2}$, and with C an arbitrary constant.

($L_{abc}=-L_{bac}$; all other L_{abc} components zero.)

This L_{abc} is tracefree and satisfies $L_{abc}L^{abc}=3a(r)^2/f(r)$.

Can it be derived from a superpotential $P_{ab}=P_{(ab)}$ via $L_{abc}=P_{c[a;b]}$? The answer is yes. See the table below for several – entries are diag($\mathcal{P}_{tt}$, $\mathcal{P}_{rr}$, $\mathcal{P}_{\theta\theta}$, $\mathcal{P}_{\varphi\varphi}$) arising from the coordinate-ordering (t, r, θ, φ). Let $k=C+2\Lambda/9$.

$\mathcal{P}_{ab}$	generates a(r) =	valid for
diag(2f + 3kr^2, 3kr^{2}f^{-2}, 0, 0)	$Cr + (2/3)Mr^{-2}$	**all M, Λ, C** — the general form, k=C+2Λ/9
diag(3r^2, 3r^{2}f^{-2}, 0, 0)	r	all M, Λ — the homogeneous piece
diag(2f, 0, 0, 0)	$-(2\Lambda/9)r + (2/3)Mr^{-2}$	all M, Λ; the k=0 member
diag(f, f^{-1}, r^2, r^{2}sin^{2}θ)	$-(2\Lambda/9)r + (2/3)Mr^{-2}$	all M, Λ; the metric with g_{tt} sign-flipped
diag(2/3 − 2Mr^{-1}, −2M(3rf^2)$^{-1}$, 0, 0)	$(2/3)Mr^{-2}$	**Λ=0 and C=0 only** — a third gauge

The last three rows are all superpotentials for the *same* Lanczos tensor when their conditions coincide, despite looking quite different. Any two rows provide an independent check on the others, which is a practical value of having several. Rows 3 and 4 differ by exactly g_{ab}: subtracting the metric from diag(f, f^{-1}, r^2, r^{2}sin^{2}θ) leaves diag(2f, 0, 0, 0), which changes nothing.

Reissner-Nordström electrically-charged black hole metric (charge=Q, mass=M). The squared acceleration $A^2=(1/12)(Q^2-Mr)^2r^{-4}[Q^2-2Mr+r^2]^{-1}$ is *not* proportional to $L_{abc}L^{abc}=(1/12)(3Q^2-4Mr)^2r^{-4}[Q^2-2Mr+r^2]^{-1}$ unless Q=0, although both are infinite at exactly the same places: horizon and r=0.

***Extreme* Reissner–Nordström: a degenerate-horizon detecting Lanczos.** The electrical charge Q of mass-M Reissner-Nordström black holes is normally only allowed to obey |Q|<M. Then RN if Q≠0 features two concentric horizons. (If |Q|>M the metric is not a "black hole" at all, but rather a "naked singularity," and seems clearly non-astrophysical.) But in the limit case Q→M we get the so-called "extreme" Reissner–Nordström black hole, which has only a single horizon, which is of a weaker kind ("degenerate horizon") at which "surface gravity" vanishes. Our usual RN Lanczos tensor above, while normally a "horizon detector," in the Q→M- limit cannot detect this weaker kind of horizon because the residue associated to the pole of $L^{abc}L_{abc}$ vanishes in that limit.

Extreme RN also is believed to be unphysical, but let us investigate it anyway.

The metric: $ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2d\Omega^2$; $f = (1 - Mr^{-1})^2$; degenerate horizon at r=M.

1-parameter family of superpotentials (one nonzero component; parameterized by C):

$$\mathcal{P}_{rr}(r) = r^2(Cr^4 - 2Mr + 3M^2/2)\,[(r - M)^3(r - 2M)]^{-1},\quad \text{all other components zero.}$$

The trace-free Lanczos tensors arising from those superpotentials: With $N(r) = 2Cr^4 - 4Mr + 3M^2$:

$$L_{trt} = L_{-rtt} = -(6r^3)^{-1}N, \quad L_{r\theta\theta} = -L_{\theta r\theta} = [12(r-M)^2]^{-1}rN, \quad L_{r\varphi\varphi} = -L_{\varphi r\varphi} = \sin^2\theta\cdot[12(r-M)^2]^{-1}rN.$$

The contracted-square scalar:

$$L^{abc}L_{abc} = N(r)^2[12r^4(r-M)^2]^{-1}, \quad \lim_{r\to M} (r-M)^2\, L^{abc}L_{abc} = (2CM^2-1)^2 / 12.$$

So for every $C\neq(2M^2)^{-1}$ this Lanczos is a "horizon detector," now with a **second-order pole** — stronger than the simple pole in the previous RN-Lanczos at non-degenerate horizons. At the single exceptional $C=(2M^2)^{-1}$ value, N(r) acquires a double root at r=M, cancelling out the pole. The residue is θ-independent, so there is no axis singularity, and since $L^{abc}L_{abc}$ is a perfect square, it never changes sign across the horizon.

Invariants for that extreme-RN superpotential & Lanczos:

quantity	value
$\mathcal{P}_{rr}$	$r^2\,[2(r-M)^3(r-2M)]^{-1}\,N$
$g^{ab}\mathcal{P}_{ab}$	$[2(r-2M)(r-M)]^{-1}\,N$
$\mathcal{P}_{ab}\mathcal{P}^{ab}$	$[4(r-2M)^2(r-M)^2]^{-1}\,N^2$ = (trace)^2, since $\mathcal{P}$ has a single component
$\mathcal{P}_{rb}{}^{;b}$	$(4Cr^5 - 10CMr^4 - 2Mr^2 + 8M^2r - 3M^3)\,[r(r-2M)^2(r-M)]^{-1}$
T_r	$-(2r-M)\,[4r(r-2M)(r-M)^2]^{-1}\,N$
$Ł^{abc}Ł_{abc}$	$(2r^2 - 4Mr + 3M^2)\,[8r^4(r-2M)^2(r-M)^2]^{-1}\,N^2$
$L^{abc}L_{abc}$	$[12r^4(r-M)^2]^{-1}N^2$
$L_{abc}{}^{;c}$	0
$L^{abc}L^*{}_{abc}$	0

$Ł^{abc}Ł_{abc}$ carries a spurious $(r-2M)^{-2}$ pole at an ordinary point of the geometry but the trace removal cancels it exactly. $\mathcal{P}_{rr}$ itself is still singular there, albeit a better representative should exist using the unused freedom in A and E.

Weak-field gravity. The "weak field limit" of general relativity involves a metric tensor $g_{ab}=\eta_{ab}+h_{ab}$ where η_{ab}=diag(-1;1,1,1) is the Minkowski flat space metric and h_{ab} is "small." In practically every astronomical scenario ever observed and simulated, except near black holes and neutron stars, we are, to good approximation, in the weak field limit. MTW's exercise 18.3 p.438 and EQ 18.15 gives this formula (derived originally by Einstein in 1916), in the de Donder, aka harmonic, gauge $2h^{ab}{}_{,b}=h^{,a}$ for the GR weak-fields created by K stationary mass-balls M_k centered at (x_k,y_k,z_k) for each k=1,2,3,...,K – we follow MTW by calling such scenarios "**near-Newtonian**":

$$h_{tt} = h_{xx} = h_{yy} = h_{zz} = 2c^{-2}G \sum_{1\le k\le K} M_k / [(x_k)^2+(y_k)^2+(z_k)^2]^{1/2} = -2c^{-2}\Phi$$

with all other h_{ab}=0. (MTW call that "Lorentz gauge" in a highly dubious naming decision. Note that in this sort of linearized gravity, if we only ask for first-order accuracy it is permissible to raise & lower indices using η_{ab} rather than the full metric; also covariant and ordinary derivatives are the same to first-order accuracy.) In other words, all entries of h_{ab}, and hence also of $h_a{}^b$ and h^{ab}, and its trace $h\equiv h^a{}_a$, are proportional to the **Newtonian gravitational potential Φ per unit test-mass**: $h=-4c^{-2}\Phi$. [Also confirmed by EQ 31 & 42 of chapter V of Synge 1960. Also is true even for moving sources, provided we employ "retarded distances" and are allowed to neglect mass-alteration effects of relative order $(v/c)^2$, where v is the velocity of a source.] That prepares us to state the

Weak-field Theorem. In weak-field gravity (to leading-order accuracy, stated in units with c=G=1), there exists a tracefree Lanczos tensor L_{abc} obeying $L_{abc}{}^{;c}=0$, such that

$$(\text{Acceleration})^2 = (3/4)\, L_{abc}L^{abc} = (\vec{\nabla}\Phi)^2;$$

(where the rightmost equality is only claimed valid in near-Newtonian situations) and this Lanczos is constructible from a superpotential P_{ab} via $L_{abc}=P_{c[a;b]}$.

Proof: We shall begin by showing that a **valid Lanczos tensor L_{abc} in any weak field**, obeying both Lanczos gauges and the de Donder gauge simultaneously, to first-order accuracy (i.e. neglecting terms of quadratic or higher order in the h_{ab}) – and note this also yields a valid Lanczos superpotential P_{ab} – obeys

$$96\, L_{abc} = 96\, P_{c[a;b]} = 24\, h_{ab,c}h^{b[a,c]} - h_{,a}h^{,a}. \quad \text{where} \quad 12P_{ab} = \eta_{ab}h - 6h_{ab}. \quad \text{and} \quad h=h^a{}_a.$$

[Incidentally, we've also shown by examining enough specific example-metric-perturbations that the coefficients (12,1,−6) in the last equation are *unique* up to overall rescaling under the demand that the Lanczos tensor $L_{abc}=P_{c[a;b]}$ must be tracefree.] In this equation we insist that the background metric η_{ab} be *flat* Minkowski spacetime. Similar L_{abc} were already stated by Lanczos 1962 and Roberts 1996, but I am going to give my own derivation different from theirs. Note: while semicolons ";" before indices denote covariant differentiation, in our flat background case that is the same thing as ordinary partial differentiation (normally denoted using ",") to our level of accuracy.

We use our de Donder gauge weak field expressions for R_{ab} and R. We substitute them into the corrected Dolan-Kim wave equation, and use Edgar & Höglund 1997's theorem that obeying it generically *suffices* for L_{abc} to be a Lanczos tensor obeying both Lanczos gauge conditions in (3+1)-spacetime. Letting "≈" mean "ignoring quadratically-small additive errors," our Dolan-Kim wave equation, in weak-field gravity at leading order, becomes

$$12\Box L_{abc} \approx 6\Box h_{c[b,a]} + \Box\eta_{c[a}h_{,b]}.$$

If we get rid of the □ on both sides, the result is that

$$12\, L_{abc} \approx 6\, h_{c[b,a]} + \eta_{c[a}h_{,b]}$$

is 12 times a valid (automatically traceless) Lanczos tensor to first order accuracy. (I tried to consider second-order accuracy too, but aborted that effort as too messy, and there is no analogous cute "get rid of

the boxes" trick for it.) We may now compute (taking advantage of de Donder gauge to enjoy more simplification)

$$96\, L_{abc}L^{abc} = 24\, h_{ab,c}h^{b[a,c]} - h_{,a}h^{,a}.$$

It now is immediate from the Einstein result that $L_{abc}L^{abc}$ necessarily is proportional to Acceleration^2. The proportionality constant must be the same as in our prior Schwarzschild metric calculation (by considering the latter's weak-field limit).

Note that I have structured this argument in a quite "robust" way, by which I mean that there are a large number of ways to err about the Dolan-Kim wave equation and de Donder weak-field calculations (since that in all was a horribly large amount of symbolic manipulation), and in fact I might have erred, and/or previous authors might have erred. But I do not believe any of those errors matter, in the sense that via our argument, the same final conclusion will be reached. **Q.E.D.**

Remark 1: divergence-freedom. This Lanczos tensor automatically obeys $L_{abc}{}^{;c}=0$ at leading order. Three-line proof: Contract the defining formula with ∂^c. Since the background is flat, covariant and partial derivatives agree, so

$$\partial^c L_{abc} = (1/4)[\, \partial_a(\partial^c h_{cb}) - \partial_b(\partial^c h_{ca})\,] + (1/24)[\, h_{,ba} - h_{,ab}\,].$$

The second bracket vanishes because partial derivatives commute. For the first, linearised de Donder gauge says $\partial^c h_{cb} = \tfrac{1}{2}h_{,b}$, so that bracket is $(1/8)(h_{,ba}-h_{,ab})$, which vanishes for the same reason. **Q.E.D.**

Remark 2: Dual-orthogonality? the parity-odd invariant, corrected. An earlier draft asserted that this Lanczos tensor also obeys $L^{abc}L^*_{abc}=0$ automatically. **That is false in general**, and a corrected statement is that it holds precisely when the weak field is *static*. On random de Donder-gauge weak fields the invariant comes out the same order as $L_{abc}L^{abc}$ itself — three trials gave $L^{abc}L^*_{abc}$ = +4.376, −16.25, +0.664 against $L_{abc}L^{abc}$ = +0.430, −11.48, −2.291. Restricting the field:

weak field	**$L^{abc}L^*_{abc}$**
general (de Donder imposed)	nonzero, and of the same size as $L_{abc}L^{abc}$
stationary: $\partial_0 h_{ab}=0$ but $h_{0i}\neq 0$	nonzero
static: $\partial_0 h_{ab}=0$ *and* $h_{0i}=0$	**zero**, exactly
Newtonian: $h_{ab} = 2\Phi\, \delta_{ab}$	**zero**, exactly

The static case has a one-line proof, and explains the pattern. Under time reversal $x^0 \to -x^0$ a static h_{ab} is unchanged, hence so is L_{abc}, which is built from its first derivatives; but ε_{abcd} is a pseudo-tensor and changes sign, so $L^{abc}L^*_{abc}$ changes sign. A quantity equal to minus itself is zero. **Q.E.D.**

What the invariant measures is the gravitomagnetic content of the field, which is exactly what h_{0i} and

$\partial_0 h_{ab}$ carry. Since the acceleration argument above is applied to static and Newtonian configurations, nothing in it depends on the false general claim.

All that suggests that adding Lanczos tensors to GR's toolkit will enable creating a relativistically-acceptable MOND-like theory, because our former impossibility theorem, claiming GR invariants "cannot compute acceleration," is now overcome. And note that merely the weak-field and Kottler metric theorems *alone* suffice to show that "galaxy simulations" should obey the Lanczos↔acceleration relationship.

2. Let us now consider our **second impossibility "proof B"**, which involved a collapsing *thin spherical shell* of dust. The region inside the shell is Minkowski flat spacetime $ds^2=dx^2+dy^2+dz^2-dt^2$. Because Minkowski's Weyl tensor $W_{abcd}=0$, the simplest valid Lanczos tensor for Minkowski is just $L_{abc}=0$. However, there are a vast number of nonzero Lanczos tensors also-valid for Minkowski spacetime. Specifically, by solving the Lanczos wave equation, which in $\Lambda=0$ vacuum is the ordinary wave equation, and using the fact that covariant and ordinary derivatives are the same in Minkowski spacetime, we see that filling in the components of L_{abc} in *any* way (compatible with the symmetries Lanczos tensors must obey) with solutions of the "wave equation" $\partial^2U/\partial t^2=c^2\nabla^2U$ in $\mathbb{R}^3$ will yield a valid Lanczos tensor.

As many textbooks (Evans 2010, Landau & Lifshitz 1971, Courant & Hilbert 1962) will tell you, the general *spherically symmetric* solution of the 3D wave equation is $U(r,t)=[F_-(r-ct)+F_+(r+ct)]/r$. If you insist this solution remain finite at r=0, then the two "arbitrary" functions F_- and F_+ cannot be independent but rather must obey $F_-(s)+F_+(-s)=0$.

So for an *arbitrary even-symmetric* function F(s)=F(-s), we can write the *general finite-at-origin spherically-symmetric wave-equation solution* as $U(r,t)=[F(ct-r)-F(ct+r)]/r$. An example of a solution U(r,t) which goes infinite at ct=r ("expanding wave of death") but is finite everywhere else, would be to use $F(x)=x^{-2}$, whereupon

$$U(0,t)=0 \text{ for all } t\neq 0, \quad U(r,t) \text{ is finite (and same sign as } ct-r) \text{ if } 0<r<ct \text{ or } ct<r, \quad U(r,t)=\infty \text{ if } r=ct.$$

This might be criticizable for having a "cone point" at r=0, in which case this U would not be differentiable and the laplacian ∇^2U not finite at the origin. But it solves the wave equation everywhere r≠0. The point of this solution is that it demonstrates that, *if the Lanczos tensor is added to GR's toolkit*, a Lanczos tensor L_{abc} exists which is infinite on a sphere expanding at the speed of light from the center of our spherical shell of dust, causing $L_{abc}L^{abc}$ to be infinite exactly on the expanding "horizon," *despite* our proof that no invariants arising pointwise from the metric and its derivatives, can distinguish any point of Minkowski spacetime from any other.

Also, any function of the form U(r,t)=A+Bt (where A and B are arbitrary constants) is a spherically-symmetric solution of the wave equation since $\nabla^2U=0$ and $\partial^2U/\partial t^2=0$. Therefore Lanczos tensors in flat spacetime would be capable of "measuring time" and hence could cause physics to act different starting at some threshold time.

By using the Schwarzschild-metric Lanczos tensor we gave before (in the outside-the-shrinking-dust-shell region) as a "boundary" or "matching" condition for the spherically-symmetric wave-equation inside the shell, we readily determine the unique F's that work, thus obtaining a Lanczos tensor for the full inside-glued-to-outside metric. Because this outer Lanczos tensor has components infinite on the horizon, at the moment the dust-shell and black-hole horizon coincide, i.e. the moment the horizon "swallows" the dust, we get an infinity, which the wave equation (going backward in time) propagates inward spherically

symmetrically at the speed of light. Therefore, this matched Lanczos tensor will indeed have $L_{abc}L^{abc}$ infinite on the horizon *both* during black hole formation and horizon-sphere growth, *and* after we reach a static state with a fixed-size horizon-sphere.

But I must criticize this Lanczos tensor (and/or the associated story) as *non-causal*. That is: mathematically we found it by an outside→inside propagation going backward in time. But if the Lanczos tensor is part of the laws of physics and makes horizons special, then presumably propagation occurs from the dust-shell centerpoint outward, going forward in time. How did that centerpoint "know when" to emit that Lanczos wave? This is mysterious.

3. Let us now consider our **second impossibility "proof A"**, which involved a homogeneous collapsing *ball* of dust. The dust-ball-interior's metric was a special case of the more-general FLRW "big crunch" homogeneous cosmology model class of metrics. Its exterior is the Schwarzschild vacuum. A Lanczos tensor for this 2-piece glued metric was actually explicitly worked out by Vishwakarma 2021 in his §6.2, as well as by us. At the instant the horizon coincides with the boundary of the dust-ball (i.e. the dust ball gets completely "swallowed") it gives infinite acceleration at that horizon. Prior to that time it gives the exactly-right fluid-particle-measured radial acceleration everywhere inside the dustball, while outside it gives the Schwarzschild radial acceleration relative to its timelike Killing field.

Also I remark that we've given Lanczos tensors L_{abc} for FLRW which can exhibit $L_{abc}L^{abc}$ depending only on r-η or only on r+η and thus are lightspeed-expanding "waves"; so evidently that can happen with Lanczos tensors despite its impossibility with invariants from old-style GR given the homogeneity of the FLRW metric. Such things could hope to duplicate the "lightspeed-expanding horizon" in the collapsing dust-shell example. And speaking of "horizon detecting":

Theorem: Kerr/MOND horizon-detector I. The Kerr-de Sitter metric in Boyer-Lindquist coordinates (see Appendix for that metric and certain shorthands associated with it) Lanczos tensor $Ł_{abc}$ found in the plain-Kerr case by Piscoran, Lopez-Bonilla, Perez-Teruel 2015 and Camacho-Gonzalez, Lopez-Bonilla, Velazquez-Lozada 2015 has the property that $Ł_{abc}Ł^{abc}$ is **infinite** both on the Kerr-dS "**horizon**" surfaces $\Delta=0$, the **rotation axis** $\theta=0$ and $\theta=\pi$, and the **ring singularity** $\Sigma=0$, but finite everywhere else. Here $\mathbf{\Delta}=r^2-2Mr+a^2$ is the standard Boyer-Lindquist "horizon function" ($\Delta=0$ precisely on the horizon surfaces; M is the mass and ac the specific angular momentum of the hole), while $\mathbf{\Sigma}=r^2+a^2\cos(\theta)^2$ is another standard Boyer-Lindquist quantity which vanishes only on the "ring singularity" deep inside Kerr's inner horizon. But $Ł_{abc}Ł^{abc}$ is finite everywhere else, including on the "ergosphere" boundary; $Ł_{abc}Ł^{abc}\sin(\theta)^2\Delta$ is finite everywhere outside (and on) the inner horizon. Its trace vector $T_a=Ł_{ab}{}^b$ is usually not zero. The **tracefree version** of this is $L_{abc}=Ł_{abc}-(T_{[a}g_{b]c})/6$. This still is a Lanczos tensor and still obeys $L_{abc}{}^{;c}=0$, and again is infinite on the horizons and spin-axis but finite everywhere else in the "accessible region" between the hole- and Hubble-horizons. When $\theta\to 0$ approaches the spin-axis, it obeys $3L^{abc}L_{abc}=[r^2+a^2]^{-1}\csc(\theta)^2+O(\csc\theta)$.

Proof (for L_{abc}, the tracefree version, only): The Kerr-de Sitter $L_{abc}L^{abc}$ is a rational function with **denominator**$=3\Delta_r\Sigma^3\sin(\theta)^2$ and numerator which is a rather complicated polynomial in r and $v=\cos(\theta)^2$ (and a, M and Λ). This denominator yields a *simple* pole at each horizon $\Delta_r=0$, a *cubic* pole at the ring-singularity $\Sigma=0$, and a *double* pole on the spin-axis $\sin\theta=0$ (note the *squared* $\sin\theta$ in the denominator). In the **plain-Kerr** case ($\Lambda=0$) the **numerator** polynomial is

$$\text{Numerator}_{\text{Kerr}} = r^6 + M\,r^5\,(3v - 5) + r^4\,[\,7M^2(1 - v) + 3a^2\,v\,] + M\,a^2\,r^3\,(\,-9v^2 + 8v - 3\,) + a^2\,r^2\,v\,[\,(3a^2 + 16M^2)\,v - 16M^2\,] + M\,a^4\,r\,v\,(9 - 11v) + a^4\,v^2\,[\,a^2\,v + M^2(1 - v)\,].$$

I will not provide the numerator in the full Kerr-deSitter case, since it appears to be contain well over 1000 terms. But I will say that it simplifies in the case where we are on the equatorial plane ($\theta=\pi/2$) to

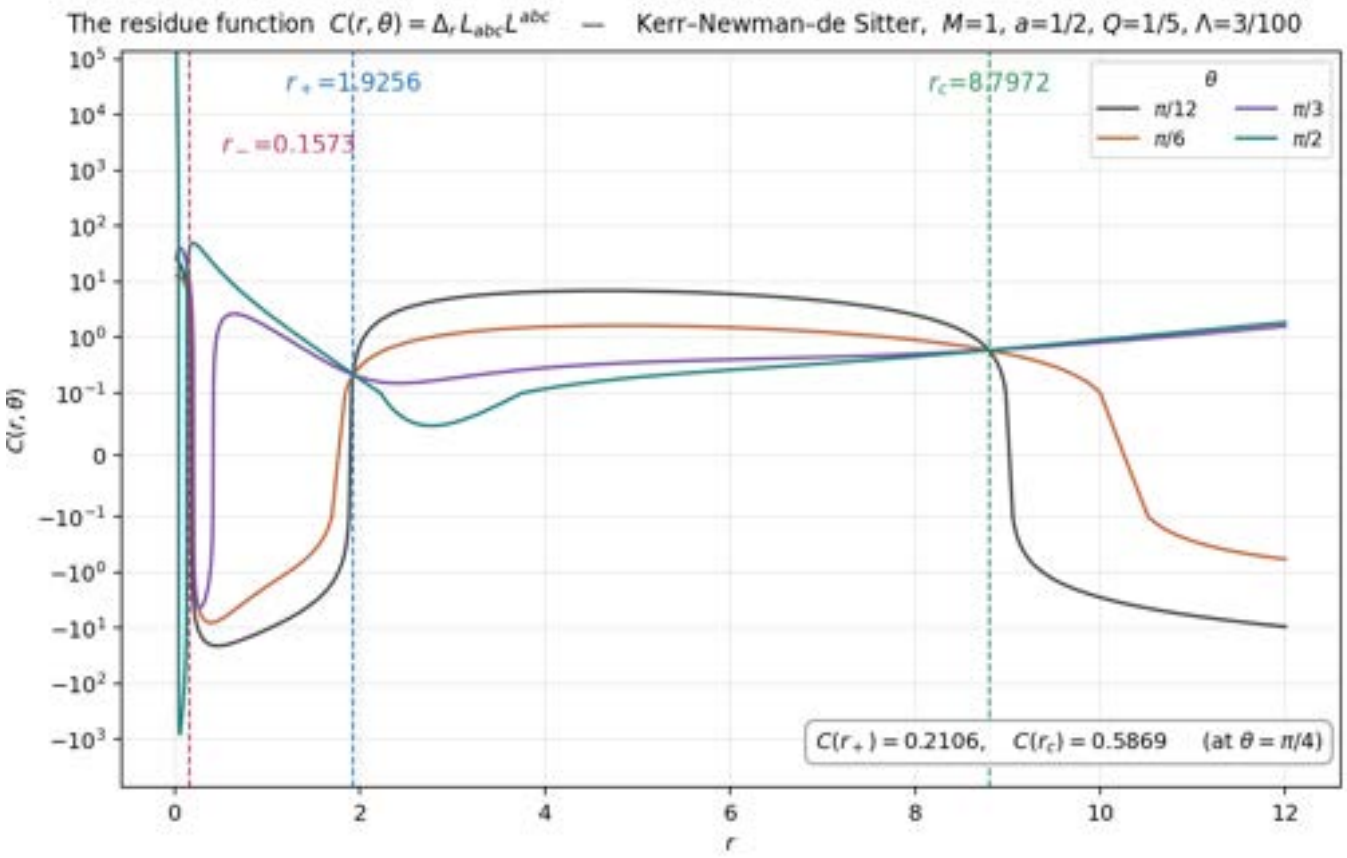
$$L_{abc}L^{abc} = (\,r^3 - 5Mr^2 + 7M^2r - 3Ma^2\,) / (\,3\,r^3\,\Delta\,).$$

Q.E.D.

Extension of {horizon+axis}-detector theorem to full Kerr-Newman-deSitter (charged, rotating, Λ) black hole family. Very heavy symbolic manipulation reveals that the trace-free Lanczos $L_{abc}=Ł_{abc}-g_{c[a}T_{b]}/6$ (where $T_b=L_{ab}{}^a$) built from the sign-flip-trick superpotential obeys $L_{abc}{}^{;c}=0$ and has the property that its contracted square $L_{abc}L^{abc}$ is finite and smooth throughout the accessible region **goes infinite on axis** and also

diverges as a simple pole on every Killing horizon: $L_{abc}L^{abc} \approx C(r,\theta) / \Delta_r$,

where Δ_r is the horizon function ($\Delta_r=0$ defines the horizons) and $C(r,\theta)=L_{abc}L^{abc}\Delta_r$, as you can see from the plot, is constant on each horizon surface, although non-constant when we are not on horizons. (The constants for the cosmological and hole horizons usually differ, and also differ in the picture.) For Kerr-Newman-deSitter *both* horizons bounding the static region (event r_+ and cosmological r_c) show the pole, while inside the accessible region $r_+<r<r_c$ the scalar stays finite ($|L\cdot L|\lesssim 1$) with no other divergence: $\Sigma=r^2+a^2\cos^2\theta$ and Δ_θ never vanish there, so neither the ring singularity nor the axis *on* the horizon produce a blow-up. However, there is an on-axis singularity *outside* the horizon. Since Kerr and Kerr-Newman are the $\Lambda\to 0$ (and $Q\to 0$) limits of Kerr-Newman-deSitter, the same $1/\Delta_r$ pole structure carries over to them. In effect this particular Lanczos invariant is a **horizon+axis detector**: the trace-free superpotential Lanczos "knows" where the Killing horizons are, but has the flaw of also detecting the spin-axis when *off* the horizons.



Remark. The raw-sign-flip-trick superpotential obeys $P^a{}_a=0$, but $P_{ab}{}^{;b}\neq 0$, and $P_{ab}P^{ab}=4$. For the "cooked" superpotential (appropriate for the tracefree L_{abc}), it is the opposite: $P_{ab}{}^{;b}=0$, but $P^a{}_a\neq 0$; and $P_{ab}P^{ab}=4+(4/9)\ln(\Delta_r\Delta_\theta \sin^2\theta)^2$.

Theorem: Kerr/MOND horizon-detector II. (This theorem was produced with help from SymPy and superintelligence CLAUDE.) For the Kerr metric in Boyer-Lindquist coordinates (t,r,θ,φ), except use y=cosθ for brevity, so the coordinates really are (t,r,y,φ). Here are the eight nonzero components of a Lanczos tensor:

$$Ł_{trt} = M(r^2-a^2y^2) / (2\Sigma^2),$$
$$Ł_{tyt} = a^2M\, r\, y / \Sigma^2,$$
$$Ł_{ty\varphi} = -a\, M\, r\, (r^2+a^2)\, y / \Sigma^2,$$
$$Ł_{tr\varphi} = -a(1-y^2)\, Ł_{trt},$$
$$Ł_{r\varphi t} = +a(1-y^2)\, Ł_{trt},$$
$$Ł_{y\varphi t} = +a\, M\, r\, (r^2+a^2)\, y / \Sigma^2,$$
$$Ł_{r\varphi\varphi} = r(1-y^2)/2 - a^2(1-y^2)^2\, Ł_{trt},$$
$$Ł_{y\varphi\varphi} = -(r^2+a^2)\, y/2 - a^2M\, r\, y\, (1-y^2)[a^2(1+y^2)+2r^2] / \Sigma^2.$$

Its trace vector $T_a=Ł_{ab}{}^b$ is usually not zero. Let the tracefree form of this be L_{abc}. Both this Lanczos and its tracefree form obey $L_{abc}{}^{;c}=0$, i.e. are divergence-free in their rightmost index; and their contracted squares

$Ł_{abc}Ł^{abc}$ and $L_{abc}L^{abc}$ both are expressible as exact rational functions with denominator=$3\sin(\theta)^2\Sigma^3\Delta$ and numerators that are complicated polynomials in r,y,M,a. When *on-axis*, the $L_{abc}L^{abc}$ numerator simplifies to $(r^2+a^2)^2\Delta$. Even though all *components* of both $Ł_{abc}$ and L_{abc} are finite everywhere outside the horizon, their *contracted squares* are infinite on the spin-axis. That can be avoided by the following modification: $Ł_{abc}=L_{abc}+(\Phi_a g_{bc}-\Phi_b g_{ac})$ where Φ has all components zero except that $6g^{tt}(\Phi_t)^2=-1/[3(r^2+a^2)\sin^2\theta]$. Then Φ_t is real everywhere outside the horizon, since $g^{tt}<0$ there. And $Ł_{abc}$ still is a genuine Lanczos tensor because this kind of gauge-move preserves Lanczos-validity. However, after this modification, the *components* of $Ł_{abc}$ no longer are finite on the rotation axis and $Ł_{abc}$ no longer is either tracefree or divergence-free. But it does have *contracted square* $Ł_{abc}Ł^{abc}$ enjoying finiteness everywhere between Kerr's inner and outer horizons (infinite on those horizons). Indeed, it is expressible exactly as a rational function with denominator=$3(r^2+a^2)\Sigma^3\Delta$ and

$$\text{numerator} = (a^6M\,y^4 - 2a^6r\,y^4 + 9a^6r\,y^2 + a^4M\,r^2y^4 - 16a^4M\,r^2y^2 + 14a^4r^3y^2 - 3a^4r^3 - 16a^2M\,r^4y^2 + 7a^2M\,r^4 + 9a^2r^5y^2 - 8a^2r^5 + 7M\,r^6 - 3r^7\ )\ M.$$

Q.E.D.

Corollary: non-uniqueness of L for the Kerr metric. The L_{abc} in Kerr/MOND horizon-detector I and the different L_{abc} in Kerr/MOND horizon-detector II, both are traceless Lanczos tensors with $L_{abc}L^{abc}$ infinite everywhere on the Kerr horizon, and obeying $L_{abc}{}^{;c}=0$. So evidently those demands are insufficient to force uniqueness.

Two types i and ii of Lanczos. Numerical work then also indicated (non-rigorously) that

i. A *tracefree* Lanczos L_{abc} exists whose contracted square $L_{abc}L^{abc}$ is finite and bounded both on the horizon and everywhere outside it, including on the axis. Actually the existence of such a Lanczos tensor is guaranteed by known existence theorems because the horizon is metrically regular in suitable coordinatizations (e.g. Kruskal coordinates) and indeed the whole Kerr metric is analytically smooth except at the ring singularity.
ii. It also indicated a different tracefree Lanczos exists, whose contracted square $L_{abc}L^{abc}$ is infinite on the horizon, but finite everywhere outside it, including on the axis. CLAUDE was able to find numerically a 1-parameter *family* of $L_{abc}=Ł_{abc}+\lambda A_{abc}$ of tracefree Lanczos tensors (where λ is an *arbitrary* nonzero real parameter and A_{abc} is a traceless Takeno tensor). Finally on 4 Sept. 2026 CLAUDE was able to produce an exact closed form pure-horizon detector traceless Lanczos tensor for the Kerr metric. It is described in the last appendix and also as a C program, and is a remarkable object.

Both (i) and (ii) *dis*obey $L_{abc}{}^{;c}=0$. **Remark:** It should be **impossible** for a Lanczos L_{abc} to exist with $L_{abc}L^{abc}$ that is finite and bounded *everywhere* because near the ring singularity, $W_{abcd}W^{abcd}$ goes infinite like Σ^{-6}, which should be severe enough to force $|L_{abc}L^{abc}|\to\infty$ also.

Continuity across "switcheroo" surfaces. If Λ stairsteps to a new value when we cross some surface in spacetime, then certain things will jump discontinuously. Most obviously, the Einstein tensor G_{ab} will do that. However, SG demands that the *metric* g_{ab} remain continuous. The discontinuity in G_{ab} forces a

discontinuity in *second derivatives* of the metric, but not the metric itself. SG further demands that anything in $\vec{F}$, the tuple the universe uses to decide whether to *have* a switcheroo, must be continuous. Because L_{abc} is an *antiderivative* of the Weyl tensor W_{abcd}, it is capable of remaining continuous (and should be expected to) even if W_{abcd} itself jumps discontinuously. If so, then $L_{abc}L^{abc}$ also will remain continuous. Therefore, using $L_{abc}L^{abc}$ as a component of the phase-diagram coordinate-vector $\vec{F}$ seems, at least a priori, acceptable.

Two-Kottler *glued* metric. To explore continuity, we now consider the "two-Kottler metric." This consists of a Kottler metric inside a sphere, say when $0\le r\le r_{switch}$, glued at that common sphere (serving as a "switcheroo surface") to a second Kottler metric when $r\ge r_{switch}$. Both Kottler metrics are exact solutions of the Einstein GR(Λ) vacuum field equations, but with two different Einstein cosmical constants $\Lambda_{in}\neq\Lambda_{out}$.

The tracefree Lanczos tensor we shall use in both regions has nonzero components

$$-L_{trt} = L_{rtt} = h(r), \quad L_{r\theta\theta} = -L_{\theta r\theta} = -[1+2\beta(r)]^{-1}r^2/2, \quad L_{r\varphi\varphi} = -L_{\varphi r\varphi} = -\sin(\theta)^2-[1+2\beta(r)]^{-1}r^2/2.$$

where $\beta(r)=-Mr^{-1}-\Lambda r^2/6$, and automatically obeys $L_{abc}{}^{;c}=0$. The Lanczos→Weyl condition that this *be* a Lanczos tensor turns out to adopt the simple form

$$r^{-1}h(r) - h'(r) = 2Mr^{-3}.$$

The general solution of this first-order ordinary differential equation then is

$$h(r) = (2/3)Mr^{-2} + Cr, \quad \text{where C is an arbitrary constant.}$$

Continuous gluing of the metric at $r=r_{switch}$ requires $\beta(r)$ to be continuous, which forces [and now we put in Newton's gravitational constant G and the speed of light c to restore correct units]

$$\mathbf{(M_{out}-M_{in})\, Gc^{-2} = [\Lambda_{in}-\Lambda_{out}]\, (r_{glue})^3 / 6}$$

Note: That's a change in Kottler's mass-parameter M.

Meanwhile, the demand that L_{abc} be continuous forces $h(r_{switch}-)=h(r_{switch}+)$. Finally, I want $|L_{rtt}|$ to fall toward 0 when $r\to\infty$. The reason I want that is because only Lanczos tensors that fall toward zero at far distances allow "easy compatibility" so one can think about many different black holes, such that when all the holes are mutually far-separated, each will behave pretty independently of the others. (Since physics obviously seems to possess that far-separated ⇒ behave independently "locality" property.) This demand forces $C_{out}=0$. (Of course, you more generally could glue further Kottler metrics at an "onion" of additional concentric switcheroo-spheres, in the same way.) So the final glued-metric Lanczos tensor arises from

$$h(r) = (2/3)Mr^{-2} + C_{in}r, \quad \text{when } 0<r\le r_{switch}$$
$$h(r) = (2/3)Mr^{-2} + 0, \quad \text{when } r\ge r_{switch}.$$

where $C_{in}=(\Lambda_{in}-\Lambda_{out})/9$. Although L_{abc} is continuous across the stairstep sphere, its radial *derivative* is not.

Lesson: the **apparent mass of the central black hole *changes* (M_{in} versus M_{out}) as we cross the switcheroo-sphere**, i.e. the "law of gravity" changes. Therefore although the "gravitational *acceleration*" stays continuous as a function of r, its radial *derivative* jumps discontinuously. That produces a somewhat "MOND-like" effect where we have "two laws of gravity" glued together. That also is MOG-like. Alternatively, you can regard it as simulating the existence of "dark matter" because the $M_{in}\rightarrow M_{out}$ apparent-mass change across the switcheroo makes it seem as though there were "extra" mass, which is "dark." However, unlike Milgrom's MOND, SG's outer law will *not* be an inverse-linear force law, so that SG (unlike Milgrom MOND) will permit two bodies to escape to unboundedly large distances from each other. (But one could, if desired, make SG *emulate* a 1/r force law – or many other laws – arbitrarily well within any finite-width r-interval by using an "onion" with enough layers and the right Λ_k values in layer k. Then the 1/r function could be approximated piecewise analogously to a smooth curve getting approximated by a polygon. What we actually want is merely to make SG fit galaxy rotation-curve data well, which is not the same thing as fitting Milgrom's 1/r function well, and it would be better to aim directly at that more-correct goal.] In Milgrom's MOND, the gluing surface was the surface where the radial acceleration equals a_0. Exactly Milgrom's surface could be forced to happen in SG by making $\vec{F}$ just be $L_{abc}L^{abc}$, then demanding a switcheroo when F crosses $(4/3)(a_0)^2$.

Stability? I now would like to argue, intuitively (although I make no pretense this is a rigorous proof), that under SG the 2-Kottler glued metric (static exact solution) is **stable**. More generally, we really want to investigate the stability of vacuums with SG or LG field equations. But mostly, nobody (including me) has investigated that – this will be my sole attempt, to start the ball rolling.

First of all, under old-fashioned plain GR(Λ), I think everybody agrees the Kottler metric is stable, so let's assume that. The question is whether the *two*-Kottler glued metric is stable – or can its switcheroo surface somehow develop some kind of exponentially-growing oscillating infinitesimal perturbation? If it did, it should emit gravitational waves. The reason I think this cannot happen is: I do not see where the energy to power that, is coming from. Since the wave emission loses energy, the mass of the Kottler black hole must decrease. But it seems to me SG must forbid that.

Major issue for SG: *dis*continuous acceleration. We deduced that M-jump formula by demanding the metric be continuous, while also demanding that Λ jump. But this also would force the radial acceleration to be discontinuous, and hence, for test particles circularly-orbiting a two-glued-Kottler metric, would force their orbital velocity (as a function of r) to exhibit a discontinuity across $r=r_{glue}$.

Do "rotation curves" of galaxies actually exhibit such discontinuities? If so, that would be dramatic confirmatory evidence for SG. If not, that would refute it, and would suggest that instead we want (if we insist Lanczos tensors be involved in gravity) some flavor of **LG**. But three issues confuse this matter:

a. orbits are not exactly circular, but rather (e.g.) elliptical;
b. gas velocities will not exhibit discontinuities, even if they "want to," because of (i) diffusion of gas atoms across r_{glue} and (ii) viscosity;
c. SG could employ many small Λ-jumps rather than one big one, making such discontinuities much less obvious.

Real-world numbers for Milky Way. Consider the Milky Way galaxy. Its mass (including "dark matter" halo) is commonly claimed to be $M_{out}\approx(10\text{-}20)\times10^{11}M_{sun}\approx(2\text{-}4)\times10^{42}$kg. However, according to Jiao et al 2023, $M_{out}\approx(2.13\pm0.15)\times10^{11}M_{sun}\approx(4.2\pm0.3)\times10^{41}$kg within distance 123.8±2.8 kPc≈404000 lightyears≈38.2×10^{20} meters from the center! I am astounded by this huge disagreement even about the

mass of our *own* galaxy. Jiao et al 2023 claims based on "Gaia data release 3" that the rotation curve of the Milky Way is atypical: "unlike those determined for other large spiral galaxies, it is not flat" and instead declines in Kepler/Newton fashion so that "the ratio of DM to baryonic mass is only a factor of about 3 [for the Milky Way], instead of a factor of approximately 6 from ΛCDM" cosmology-based estimates based on year-2020 Planck data. According to Kafle, Sharma et al 2012, the mass located within 25kPc≈81540 lightyears≈7.7×10^{20} meters of the center is $M_{in}\approx2.1\times10^{11}M_{sun}\approx4.2\times10^{41}$kg, while Watkins et al 2019 based on Gaia data release 2 say that is the mass located within 21.1kPc≈68820 lightyears≈6.5×10^{20} meters of the center. Watkins et al say this corresponds to a circular velocity at radius 21.1kPc of V=206±13 km/sec, while Jiao et al say 200 km/sec. Milgrom's acceleration $a_0\approx12\times10^{-11}$ meter/sec^2 is achieved when $V^2/r=a_0$. This happens at radius $r_{glue}\approx14$ kiloparsec≈45700 lightyears≈4.3×10^{20}meters according to Jiao et al's rotation curve in their fig.1, which gives V≈225 km/sec and $V^2/r\approx12\times10^{-11}$ meter/sec^2. In contrast, the Earth is located about 8.2 kPc≈26700 lightyears from galactic center. At this r_{glue}, Jiao et al's fig.8 says the enclosed mass is about $1.65\times10^{11}M_{sun}$. That figure exhibits no sign of any *discontinuity* in enclosed mass at any r with 0<r<22.5 kiloparsec. The immediatele naive conclusion from this is that SG is false – and needs to be replaced by some LG flavor – or: the discontinuity occurs, but at a greater r; or that multiple smaller, not just one big, Λ-jump happens. With multiple small-jump switcheroos, it would not be impressive for SG to match one particular galaxy vary well; but matching *many* different galaxies of many different kinds, with no failures, *would* be impressive and offers at least the potential to be very convincing. In any case, if we use that r_{glue} and $M_{out}-M_{in}\approx0.48\times10^{11}M_{sun}\approx9.5\times10^{40}$kg, we would deduce

$$\Lambda_{in}-\Lambda_{out}\approx5.3\times10^{-48}\text{ meter}^{-2}$$

which is about 50000 times our universe-wide single-number Einstein cosmical constant Λ estimate. In other words, if Λ adopts only two values Λ_{in} and Λ_{out}, with $\Lambda_{out}\approx1.1\times10^{-52}$ meter^{-2}, and the Milky Way's switcheroo surface is roughly spherical at radius $r_{glue}\approx14$ kiloparsec, then to explain the gross features of the Milky Way's rotation curve, Λ_{in} would need to be about 50000 times greater than Λ_{out} – a very dramatic predicted contrast.

A 50000× greater Λ might be (barely) detectable by observing the most distant Oort cloud objects in our solar system (or, probably better, purpose-built spacecraft sent to that area); should noticeably affect the Milky Way; and would have huge effects on galaxy clusters (although under SG presumably the large-Λ would switch down to a much smaller value at cluster length scales, hence the latter effects would not happen). Over a 100-year timespan, a 50000× greater &Lambda would perturb the motion of the Voyager spacecraft by only around 5 meters, i.e. 17 nanoseconds delay at lightspeed. Although in principle that could be detectable for suitable spacecraft, unfortunately, the Voyagers lacked atomic clocks, and the so-called "Pioneer anomaly" (essentially the same kind of effect, but observed for the Pioneer 10 & 11 spacecraft) was ultimately blamed (Turyshev & Toth 2010), most probably, on acceleration caused by thermal radiation the Pioneers emitted anisotropically. It might be too difficult to eliminate such confusor-effects.

What if we instead used the Alpha Centauri 3-star system as our laboratory? This consists of two sunlike stars A & B elliptically orbiting each other with period 79.76 Earth years at mutual separations 11.2-35.6 AU, plus a third dwarf star C (with two known planets) orbiting AB at a distance of about 13000 AU≈0.21 lightyears with period≈550 kyr. The masses of A and B are each known to within about ±1 parts in 400, while C's mass is known to ±2%. Since the postulated large Λ would alter C's period by about 2 parts per million, i.e. about 1 Earth year, its effect could only be detectable if Alpha Centauri ABC were

characterized 10^5 times more precisely, and indeed considerably more precisely than our own solar system.

Two-Reissner-Nordström-deSitter glued metric. We now redo that two-Kottler exploration, but instead based on the more general Reissner-Nordström-deSitter family of *electrically-charged* spherical black holes (mass=M, charge=Q), with metric

$$ds^2 = -f\,dt^2 + dr^2/f + r^2(d\theta^2+\sin^2\theta\,d\varphi^2) \quad \text{where} \quad f=f(r)=1-2Mr^{-1}+Q^2r^{-2}-\Lambda r^2/3,$$

representing a spherically-symmetric electrically-charged black hole in a de Sitter background. This is an exact solution of the combined Einstein-Maxwell equations in Λ-vacuum. (Plain Reissner-Nordström is its Λ=0 special case.) A tracefree Lanczos tensor obeying $L_{abc}{}^{;c}=0$ has the following nonzero components:

$$-L_{rtt} = L_{trt} = h(r), \quad -L_{r\theta\theta} = L_{\theta r\theta} = h(r)\,r^2/(2f), \quad -L_{r\varphi\varphi} = L_{\varphi r\varphi} = h(r)\,r^2\sin^2\theta/(2f).$$

Its contracted square is $L_{abc}L^{abc}=3h(r)^2/f(r)$, which (note) goes infinite exactly on the horizon-surfaces f(r)=0, and since it depends on r *only* is spherically symmetric. The demand that this L_{abc} *be* a Lanczos tensor is equivalent to

$$r^{-1}h(r) - h'(r) = 2Mr^{-3} - 2Q^2r^{-4}.$$

with solution

$$h(r) = (2/3)Mr^{-2} - Q^2r^{-3}/2 + C\,r, \quad \text{where C is an arbitrary constant.}$$

We now glue *two* RNdS metrics at the common switcheroo-sphere $r=r_{switch}$, using two different Einstein cosmical constants $\Lambda_{in}\neq\Lambda_{out}$ inside their f-definitions. The conditions that the metric and electric field be continuous across the switcheroo-sphere $r=r_{switch}$ then amount to $Q_{out}=Q_{in}$ and $M_{in}-M_{out}=[\Lambda_{out}-\Lambda_{in}](r_{glue})^3/6$. (And as before, you should multiply the mass-difference on the left hand side by Gc^{-2} to restore correct units.] The glued solution then has

$$h(r) = 2M_{in}/(3r^2) - Q^2/(2r^3) + [(\Lambda_{in}-\Lambda_{out})/9]\,r, \quad 0<r\le r_{switch}$$
$$h(r) = 2M_{out}/(3r^2) - Q^2/(2r^3), \quad \text{when} \quad r\ge r_{switch}.$$

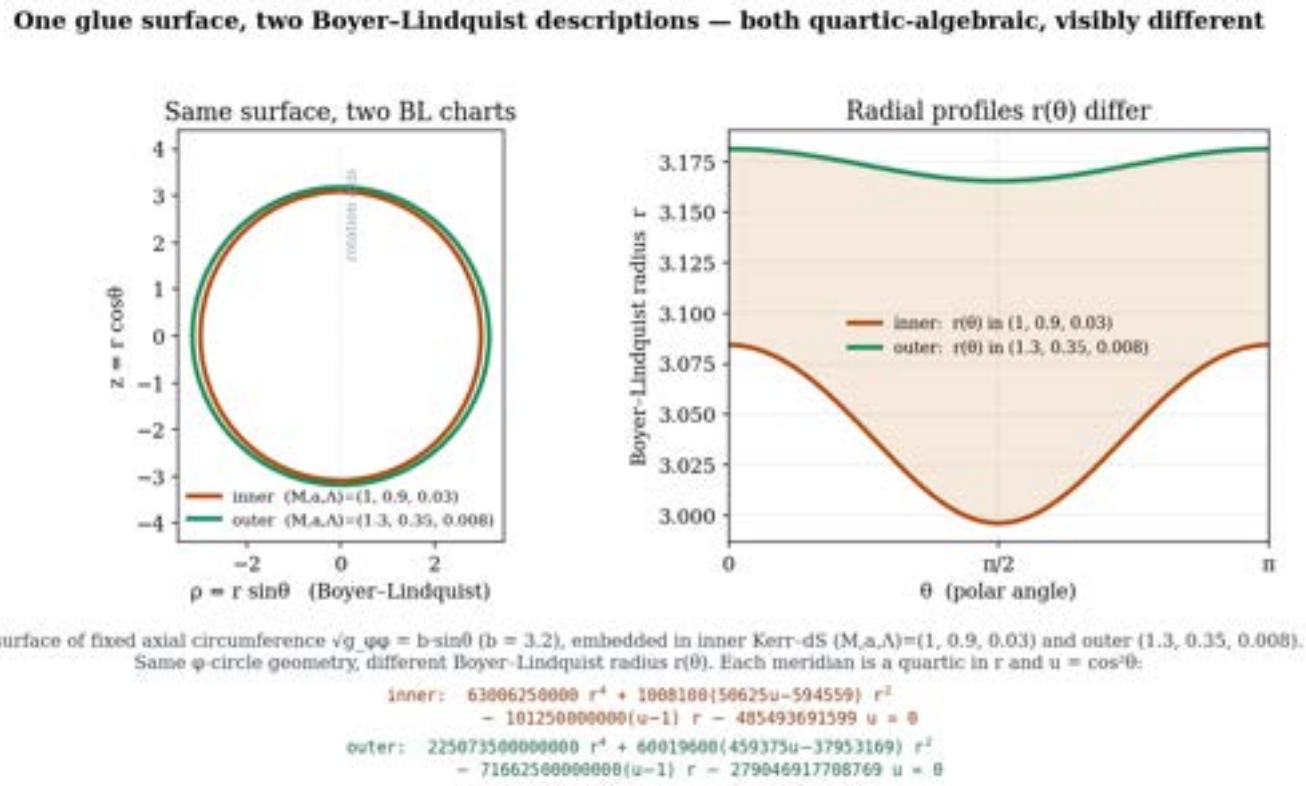
Important **lesson:** Introducing electric field does *not* cause a disaster for SG-theory where (say) the radial electric field had to be discontinuous.

***Non*spherical switcheroo-surfaces are possible.** Some reader might now worry that although switcheroos worked nicely for spherical black holes, that might have been merely a lucky consequence of exact spherical symmetry, making something possible that in less-symmetrical (i.e. generic) situations would have been impossible. I can largely quell that fear. It *is* possible to have a topologically-spherelike axisymmetric switcheroo-surface Ω, which in general is not a literal round sphere (in fact it is a rotated quartic-algebraic curve), at which one "inner" Λ-Kerr metric (whose inner ring singularity is enclosed by Ω) is glued to a different, outer, Λ-Kerr metric, both of them using Boyer-Lindquist coordinate systems. The inner and outer Kerr metrics are described by *different* parameter 3-tuples (M,a,Λ), all three of whose components, in general, change discontinuously as we cross Ω. But the 2-piece glued metric is continuous. Since the gluing-surface, aka switcheroo surface, is axisymmetric, it may be described by stating a formula relating r and θ on its meridian-curve. There actually are two *different* meridian curves,

described by two different quartics, for describing Ω's shape; one is for use in the inner, the other for use in the outer, Boyer-Lindquist coordinate system. However, despite those two quartic curve formulas differing, both actually describe *isometric* Ω reckoned by their surface-metrics, i.e. the two Λ-Kerr metrics *match* at Ω, provided the two curves are just right, and provided their (M,a,Λ) tuples are "compatible" – M_{in} and a_{in} and the two meridian-describing curves are deducible from Λ_{in} and $(M,a,\Lambda)_{out}$. The formula for the quartics (in which $\Xi=1+\Lambda a^2/3$ is a standard Kerr-dS Boyer-Lindquist abbreviated quantity, see appendix):

$$\Xi\, r^4 + \Xi(a^2(1+\cos^2\theta) - b^2\Xi)\, r^2 + 2Ma^2\sin^2\theta\, r + \Xi\, a^2\cos^2\theta\,(a^2 - b^2\Xi) = 0.$$

A picture depicting one example match-up:



Now although such a quartic surface gluing is *mathematically* possible, the reader may now be wondering whether thresholding $L_{abc}L^{abc}$ as a test to decide where SG should "switcheroo," can actually *physically* produce that exact surface. If no traceless Lanczos tensor L_{abc} and constant K can exist such that our quartic glue-surface is exactly described by $L_{abc}L^{abc}$=K, then that would refute $L_{abc}L^{abc}$-thresholded SG right there. Well, I believe such Lanczos tensors *do* exist. The first (intuitive) reason to suspect that is: traceless L_{abc} have exactly 6 degrees of "Lanczosian gauge freedom" per point, while quartic gluing surfaces of the kind shown in the figure, are described by 5 parameters. So existence would follow (*if* the "physicist's heuristic" works) from the fact that 6>5. Second, if we, even merely for some *single* generic two-Kerr-dS gluing, *find* a suitable K and L_{abc}, then that *proves* (well, more precisely, "demonstrates with probability=1") that L_{abc}-existence for *all* (except perhaps a zero-measure set) of two-Kerr-dS gluings. (Because: Mityagin 2020 showed that the zero-set of any analytic function of any finite number of variables, has measure=0. Our scenario is analytic. Therefore, if you pick *random* parameters defining your Kerr-dS gluing, and then prove L_{abc} and K exist with $L_{abc}L^{abc}$=K everywhere on its gluing-surface, then with probability=1 that implies such L_{abc} and K exist in 100% of all Kerr-dS-gluing scenarios.) Well,

we in fact *did* (actually 4 times!) generate random-parameter two-Kerr-dS glued metrics, then found L_{abc} and K for them to high accuracy. One specific example had

$$(a/M, \Lambda M^2, M)_{in}=(0.154, 0.010, 1.48), \quad (a/M, \Lambda M^2, M)_{out}=(0.407, 0.026, 0.97), \quad K=0.0338$$

and enjoyed tracelessness accurate to 16 decimal places, $L_{abc}L^{abc}\approx K$ accurate to 1 part per million everywhere on the gluing surface, and the Lanczos→Weyl transform applied to the numerically-found L_{abc} matched the exact W_{abcd} accurate to 4 significant figures at all sample points tried.

Summary. All the feats we'd proven old-style GR(Λ) could *not* do, we've seen SG by adding a suitable traceless Lanczos tensor L_{abc} to the picture, *can* do:

1. We can compute "gravitational acceleration" exactly for all spherically-symmetric black holes and perfect-fluid "planets" and in all weak-field vacuum scenarios (which together seem adequate for all galaxy-simulating purposes) from $L_{abc}L^{abc}$. If the black hole is Reissner-Nordström, i.e. electrically charged, then our usual method of computing acceleration delivers wrong answers; but that is irrelevant to observation because all large astronomical objects are believed to have charge negligible by comparison with their mass.
2. SG then can use $L_{abc}L^{abc}$ to trigger both Λ and law-of-gravity "switcheroos" in an acceleration-thresholded way, just like Milgrom wanted for MOND. We've also demonstrated these cause gravitational effects of "dark matter" even *without* any dark matter.
3. For spherically-symmetric black holes (including if they are electrically charged), we've exhibited traceless Lanczos tensors L_{abc} such that $L_{abc}L^{abc}$ can serve as a "horizon detector" since it goes infinite precisely on horizons. Further, this horizon detector works for the full Kerr-Newman-deSitter (charged, rotating, Λ) black hole family, and the L_{abc} may be taken to be the traceless version of the Ł arising from the "sign-flip-trick" superpotential P_{ab}, which happens to obey $L_{abc}{}^{;c}=0$. With spin, this horizon detector unfortunately also has $L_{abc}L^{abc}=\infty$ *on axis*, which might not matter for SG purposes since the axis is a 1-dimensional line, not a 2-dimensional sphere-like surface, hence cannot yield a "switcheroo."
4. Although all that concerned spherically-symmetric metrics, one may show by example that there does not appear to be any obstacle to having *non*spherical switcheroo-surfaces gluing two metrics that are not spherically symmetric, e.g. Kerr-deSitter metrics. Indeed we exhibited a quartic formula specifying the switcheroo-surface gluing two Kerr-deSitter metrics together.
5. Furthermore, we found (high-accuracy approximate) tracefree Lanczos tensors L_{abc} and constants K such that $L_{abc}L^{abc}=K$ exactly on that quartic surface.
6. 1,2,3, and 5 all can be accomplished, if desired, only with L_{abc} that obey the differential gauge condition $L_{abc}{}^{;c}=0$ and with $L_{abc}L^{abc}$ which go to zero as $r\to\infty$ along any radial ray other than the spin axis.
7. That allows SG to cause special behavior on horizons, such as Smith 2026's "mirror paradigm."
8. For Kerr there also is another tracefree L_{abc} with $L_{abc}L^{abc}$ finite everywhere outside the horizon (including on the spin axis) but infinite on (a full measure subset of) the horizon. It does not arise from a superpotential and usually does not obey $L_{abc}{}^{;c}=0$. When $r\to\infty$ with θ fixed, empirically $L_{abc}L^{*}{}_{abc}$ appears to fall toward zero proportionally to r^{-2} (and equals 0 in the equatorial plane), while $L_{abc}L_{abc}$ appears to approach a θ-independent negative constant, with the difference between

it and that constant falling ultimately proportionally to r^{-1}.

Presumably some LG flavors also could engender SG-like behavior, although I have not investigated that.

However, to turn SG into a unique physical theory (or at least near enough to unique to be useful), we need to do something about the annoying fact that traceless Lanczos tensors have 6-arbitrary-functions worth of "Lanczosian gauge freedom." And as we saw in the above corollary, even with Lanczos differential gauge imposed, which naively would entirely remove L_{abc}'s Lanczosian gauge freedom, plus imposing certain boundary-condition-like demands, L_{abc} *still* can be non-unique.

Horizon-detector conjecture. In any (3+1)-dimensional spacetime, there exists a tracefree Lanczos tensor L_{abc} such that $L^{abc}L_{abc}$ is infinite on a full-measure subset of horizons.

Furthermore (which supports Smith 2026's "mirror paradigm" idea), there is **an argument that, in some sense, a "full measure subset" of L_{abc} automatically have the "horizon detector" property:** If you have two traceless Lanczos tensors, say L^{i} and L^{ii}, then any linear combination $aL^{i}+bL^{ii}$ with a+b=1 will still be a traceless Lanczos tensor (and will still be divergence-free in its rightnmost index, if the original two both were). But – and this is the key – if these are of types i and ii respectively in terms of the two numerically-validated classes above, then *all* members of that one-parameter family of Lanczos tensors will be of the horizon-detecting type ii, i.e. infinite almost everywhere on horizons, *except* for (a,b)=(1,0). Therefore, **if gravitational physics truly is Lanczos-aware, then it probably will be *automatic* that physics knows how to detect horizons; horizons truly are singular surfaces as far as $L_{abc}L^{abc}$ is concerned; and prior notions based on old-style GR(Λ) that horizon surfaces are *not* special, and could *not* be detected locally, are overthrown.** We'll later give a theorem backing up this intuition using "prevalence theory."

12. Lanczos horizon detector for distorted ("dirty") Schwarzschild black hole

Dirty Schwarzschild Theorem. For Schwarzschild distorted by an external multipole field of order ℓ = 1, 2, 3, 4 — and for arbitrary linear combinations of these distortions — a trace-free Lanczos tensor exists whose contracted square $L_{abc}L^{abc}$ has a simple pole exactly on the event horizon and is finite off it *except* on the rotational-symmetry axis. That flaw can be corrected by instead using $(L_{abc}L^{abc})/(H_{abc}H^{abc})$ as the detector, where H_{abc} is a certain *other* tracefree Lanczos tensor we'll describe, whose contracted square is infinite on the symmetry axis. The Lanczos tensor L_{abc} is obtained from a single covariant recipe applied to the distorted metric, and is *verified* to satisfy the Lanczos→Weyl equation at **first order** in the distortion amplitude. All results here are first order in that amplitude. (In the *un*perturbed-Schwarzschild case, L_{abc} is given by exact formula, satisfies Lanczos→Weyl relation exactly, and yields an exact horizon detector.)

Later Nonperturbative version. However, we later realized how to that a Lanczos superpotential could be constructed using the two-Killing trick for the full Weyl class of static axisymmetric spacetimes, again enjoying the properties that the trace-free version L_{abc} of the Lanczos tensor $Ł_{abc}=Ᵽ_{c[a;b]}$ arising from that superpotential Ᵽ(ab) has contracted square $L_{abc}L^{abc}$ that goes infinite exactly on the event horizon (for non-degenerate horizons) and is finite off it *except* on the rotational-symmetry axis. That axis-flaw again can be corrected by instead using the *ratio* $(L_{abc}L^{abc})/(H_{abc}H^{abc})$ as the detector, where H_{abc} is the

tracefree Lanczos tensor arising in the same way from the *one*-Killing-based superpotential (using $\hat{\varphi}$ as the Killing vector). This improved result is exact, *not* merely perturbative and merely for a finite number of allowed multipoles. It can be proved using the "verification = probabilistic proof" paradigm for computer-proving differential-rational identities in Smith 2004.

The rest of this section will prove the original theorem.

Why distorted Schwarzschild is the right test case. A horizon detector is only interesting if it works on a spacetime that is not artificially symmetric. The "clean" black holes — Kerr-Newman-deSitter family, and also Vaidya — are all Petrov type D everywhere, so no test based on Petrov type can detect their horizons. A "dirty" black hole is one which is Petrov type I almost everywhere, and for those the Tanatarov-Zaslavskii theorem says the *horizon* is algebraically special (i.e, not type I), which enables a horizon-detector to be built. We have verified T&Z's theorem by numeric testing on the dirty class here; it can indeed detect horizons. However that T&Z has the flaws of also detecting the rotational-symmetry axis and the "sphere of radius infinity"; and stops seeing the horizon when the dirt is cleaned off.

The Weyl class supplies dirty black holes in closed form. Every static axisymmetric vacuum spacetime can be written in (t,ρ,z,φ) coordinates as

$$ds^2 = -e^{2U}dt^2 + e^{-2U}[\, e^{2k}(d\rho^2+dz^2) + \rho^2 d\varphi^2 \,]$$

in terms of two functions U(ρ,φ,z) and k(ρ,φ,z). On this class of metrics, Einstein's vacuum field equations reduce to two conditions: **U is harmonic** in a fictitious flat three-space defined by cylindrical coordinates (ρ, φ, z); and k arises from the following partial differential equations with right hand sides *quadratic* in ∇U:

$$k_{,\rho} = \rho(U_{,\rho}{}^2 - U_{,z}{}^2), \quad k_{,z} = 2\rho U_{,\rho}U_{,z}.$$

Because U is harmonic, external "multipole fields" can be **added** to the Schwarzschild seed. Or arbitrary linear combinations thereof. The resulting spacetime is Schwarzschild immersed in an external gravitational field — the standard distorted black hole by Geroch & Hartle 1982. It is an exact Einstein-vacuum solution. It is Petrov type I almost everywhere. And it retains a genuine Killing horizon. That precisely captures, in closed form, the "dirty Schwarzschild" concept.

Our computation is done in prolate spheroidal coordinates (x,y,z):

$$\rho = M\sqrt{(x^2-1)(1-y^2)},\ z = Mxy, \text{ with } x \geq 1 \text{ and } |y| \leq 1.$$

In these coordinates the Schwarzschild seed is defined by

$$U_0 = (\tfrac{1}{2})\ln[(x-1)(x+1)^{-1}], \quad k_0 = (\tfrac{1}{2})\ln[(x^2-1)(x^2-y^2)^{-1}],$$

where the subscript "0" denotes "zero perturbation." The Schwarzschild **horizon is the surface x=1**. The **symmetry axis is y=±1**. Exponentiating, every metric component is a *rational*(x,y), for example $e^{2U_0} = (x-1)(x+1)^{-1}$ and so on — which is what makes the symbolic computation tractable.

The multipoles. The external harmonics (index by an integer ℓ≥0) are $u_\ell = P_\ell(x)P_\ell(y)$, where $P_\ell(x)$ denotes Legendre polynomials, each verified harmonic with respect to the prolate Laplacian $\partial_x[(x^2-1)\partial_x] + \partial_y[(1-y^2)\partial_y]$. Write $U = U_0 + Au_\ell$ and $k = k_0 + Ak_1 + A^2k_2 + \ldots$ for small A governing the perturbation size. Actually we only used terms here up to first order A^1. However in some calculations

mentioned below we also computed terms at order A^2. The PDEs for k were solved in closed form and each solution checked against then both:

ℓ	$u_\ell = P_\ell(x)P_\ell(y)$	k_1	axis y = ±1 regular?
1	xy	$2y$	**no** — conical defect (strut)
2	$(3x^2-1)(3y^2-1)/4$	$3x(y^2-1)$	yes
3	$P_3(x)P_3(y)$	$y[15x^2(y^2-1) - 5y^2 + 9]/2$	yes
4	$P_4(x)P_4(y)$	$5x[x^2(35y^4-42y^2+7) - 21y^4 + 30y^2 - 9]/8$	yes

The ℓ=1 dipole has $k_1=2y\neq 0$ on the axis, so it carries a conical singularity — physically the strut holding the hole in an external uniform field. It is included for completeness but is the least-clean member of the family.

Lanczos superpotential. A trace-free Lanczos tensor L_{abc} is generated from a symmetric superpotential $\mathcal{P}_{ab}$ by $\mathcal{L}_{abc} = \mathcal{P}_{c[a;b]}$ followed by trace removal. The superpotential used here is the **reflection form**

$$\mathcal{P}_{ab} = g_{ab}/2 + u_a u_b - \hat{\varphi}_a \hat{\varphi}_b$$

with u^a the unit timelike Killing direction and $\hat{\varphi}^a$ the unit azimuthal direction. Every Lanczos tensor below was verified by applying the Lanczos→Weyl operator then comparing with the exact Weyl tensor.

Results. For each ℓ, first order in A throughout: the metric is checked to be vacuum, the Lanczos tensor is checked to be trace-free, the Lanczos→Weyl equation is checked componentwise, and $L_{abc}L^{abc}$ is computed.

ℓ	vacuum at O(A)	trace-free	LW[L] = W	denominator of $L_{abc}L^{abc}$
0 (undistorted)	yes	yes	**0 of 6 fail**	$3(x-1)(x+1)^3(y-1)(y+1)$
1	yes	yes	**0 of 6 fail**	$3(x-1)(x+1)^3(y-1)(y+1)$
2	yes	yes	**0 of 6 fail**	$6(x-1)(x+1)^3(y-1)(y+1)$
3	yes	yes	**0 of 6 fail**	$6(x-1)(x+1)^3(y-1)(y+1)$
4	yes	yes	**0 of 6 fail**	$96(x-1)(x+1)^3(y-1)(y+1)$

The denominator is the same in every case up to a numerical factor; the distortion enters only in the numerator. The factor (x−1) is never cancelled, so we have a simple pole sitting exactly on the horizon x=1 for every multipole tested. Is x=1 really the horizon of the *distorted* metric, as opposed to merely the background? Yes: g_{tt} vanishes at x=1 at **both** orders in A, so x=1 remains the Killing horizon after distortion.

Linear superposition. Since U is harmonic the multipoles superpose exactly in U; and although k comes

from quadratures quadratic in ∇U, its *first-order* part is linear in the perturbing harmonic. The Lanczos→Weyl map is linear in L_{abc}. Hence the Lanczos tensor depends linearly on the perturbation, and a detector working for each multipole should work for every linear combination of multipoles (to first order in perturbation theory). This is a theorem. However, we also checked all these assertions numerically inside the computer program.

The axis "wire" flaw. The factor (y−1)(y+1) survives in the denominator after full cancellation, so $L_{abc}L^{abc}$ also diverges on the symmetry axis y=±1.

Invariants of these Lanczos tensors and superpotentials. Several natural invariants turn out to be constant or zero, which is worth recording both as a check and because it identifies a distinguished subclass of Lanczos tensors.

1. The superpotential is both "unit" and traceless:

invariant	value	holds for
$\mathcal{P}^a{}_a$	**0**	every ℓ, both orders
$\mathcal{P}_{ab}\mathcal{P}^{ab}$	**1**, exactly	every ℓ, both orders

That $\mathcal{P}_{ab}\mathcal{P}^{ab} = 1$ exactly, undisturbed by the distortion, follows from the reflection structure: with u^a and $\hat{\varphi}^a$ unit and orthogonal, $g/2 + uu - \hat{\varphi}\hat{\varphi}$ contracts to a pure number independent of the geometry. The superpotential is a unit-norm object in a precise sense, and the perturbation does not touch it.

2. Three Lanczos quantities vanish identically:

quantity	value	significance
$L^a{}_{ba}$	**0**	trace vector — confirms the trace removal
$L_{abc}{}^{;c}$	**0** in every slot	the tensor is **automatically in the "Lanczos differential gauge"**, which normally does not happen without intentional design
$L_{abc}L^{*abc}$	**0**	see below; a genuine condition, valid for ours, but *not* all, Lanczos tensors

The two quadratic contractions are proportional:

$2L_{abc}L^{acb} = L_{abc}L^{abc}$, at both orders: for ℓ=2 the two share an identical numerator over denominators differing by a factor of two. The alternative contraction carries no independent information here.

"Dual-orthogonal" Lanczos subclass: $L_{abc}L^{*abc}=0$. Writing out L_{abc} and its tensor-dual L^*_{abc}, one observes they have **exactly disjoint supports,** meaning that each has twelve nonzero components but share none: L occupies (0,1,0), (0,2,0), (1,2,1), (1,3,3), … while L* occupies only slots carrying a φ index — (0,1,3), (0,3,1), (1,0,3), (3,0,1), …. So $L^*_{abc}L^{abc}$ vanishes not by cancellation but because the two never meet. **The support pattern is identical at ℓ=0 and ℓ=2**: the distortion changes values, not which slots are occupied.

Two dual relations hold, and it matters that they have quite different status. Testing six random tensors with Lanczos symmetries (built from no metric at all):

relation	random tensors	status
$L^*_{abc}L^{*abc} = -L_{abc}L^{abc}$	ratio −1.0000 in all six	**an identity** — automatic for anything antisymmetric in its first pair, since $F^*_{ab}F^{*ab} = -F_{ab}F^{ab}$ for any 2-form in Lorentzian signature
$L^{abc}L^*_{abc} = 0$	0.034, 8.69, 2.90, −2.77, −3.92, −3.18	**a genuine condition** — since generically violated; one scalar equation

So the Lanczos tensors found here lie in a distinguished subclass, characterised by the single equation $\mathbf{L^{abc}L^*_{abc}=0}$ — the Lanczos analogue of a vanishing Pontryagin invariant, i.e. no electric-magnetic overlap. The tensor is purely electric with respect to the static Killing observer and its dual is purely magnetic; the −1 in $L^{*abc}L^*_{abc} = -L_{abc}L^{abc}$ is a result of the norm reversal between the two sectors.

A conjecture worth testing. If this subclass is where well-behaved Lanczos tensors live, then *searching within it* should be both more likely to succeed and cheaper than searching the full space — and the disjoint-support structure suggests the condition can be imposed **structurally** (restrict the ansatz to slots carrying no φ index) rather than as an equation, which shrinks the search space instead of adding a constraint to it.

But: for the metrics here dual-orthogonality follows from **staticity** — there is no magnetic Weyl part relative to the Killing observer — so it may be automatic for any reasonable superpotential on a static metric and therefore does no selective work there. Its potential value could lie in the **non-static** cases, e.g. Kerr and Vaidya, where magnetic parts genuinely exist. **We do not know whether the Kerr black hole admits a dual-orthogonal Lanczos tensor**; the simple "sign flip trick" is not dual-orthogonal.

Wire-free horizon detector I: The "wire" can be removed here. The denominator of $L_{abc}L^{abc}$ is $(x-1)(x+1)^3(y-1)(y+1)$ and the norm of the axial Killing vector equals $g_{\varphi\varphi} = M^2(x+1)^2(1-y^2)(1-2Au)$, which supplies exactly one factor of $(1-y^2)$. So the product

$$g_{\varphi\varphi} \cdot L_{abc}L^{abc}$$

cancels the axis divergence while leaving the $(x-1)$ pole intact. This is a true horizon-detecting scalar-invariant without the "wire" flaw. It uses the axial Killing vector, which is a tensor available in the Weyl metric due to its exact axial symmetry, but unaviable in general spacetimes. But on the other hand, in the general case there is no coordinate axis for a wire to sit on, so such a repair presumably is not needed.

Wire-free horizon detector II: built entirely from Lanczos objects. Wire removal I works, but imports a factor $g_{\varphi\varphi}$ from outside the Lanczos framework. There is an alternative route which uses Lanczos tensors only. Novello & Velloso 1987 gave a way to construct Lanczos tensors in Einstein-vacuum spacetimes from a Killing field. Weyl family metircs have two Killing fields, one corresponding to time-translation symemtry, the other to axial-rotation. Their construction from aa unit timelike irrotational and shear-free vector field V^a is

$$Ł_{abc} = V_{a;\lambda}V^{\lambda}V_bV_c - V_{b;\lambda}V^{\lambda}V_aV_c$$

which becomes a traceless Lanczos tensor L_{abc} after trace removal. For the static observer in the Schwarzschild-de Sitter (aka Kottler) metric this gives the single raw component $Ł_{rtt} = \Lambda r/3 - m/r^2$; our computation reproduces $-M/r^2$ at $\Lambda=0$ exactly.

Sign convention note. The resulting tensor passes our Lanczos→Weyl check only after multiplication by −1; with that sign and α=−½ it matches on all six independent components. This is a sign-convention difference between us and Novello-Velloso.

If instead of using the unit *timelike* Killing field, we instead feed the unit *spacelike* azimuthal direction $\hat{\varphi}^a = (0,0,0,(r\sin\theta)^{-1})$ into Novello-Velloso's construction, the result, at least in our case, still is a genuine Lanczos potential: cyclic identity satisfied, trace-free after removal, and **0 of 6** Lanczos→Weyl components failing with the same c=−1 convention.

The two traceless Lanczos tensors thus-obtained are

From timelike Killing:

$L_{trt} = 2M(3r^2)^{-1}$, $L_{r\theta\theta} = Mr[3(2M-r)]^{-1}$, $L_{r\varphi\varphi} = Mr\sin^2\theta[3(2M-r)]^{-1}$

From azimuthal Killing:

$L_{trt} = (2M-r)(3r^2)^{-1}$, $L_{tt\theta t} = (2M-r)(3r\tan\theta)^{-1}$, $L_{r\theta r} = r[3(r-2M)\tan\theta]^{-1}$,
$L_{r\theta\theta} = -r/3$, $L_{r\varphi\varphi} = 2r\sin^2\theta/3$, $L_{\theta\varphi\varphi} = r^2\sin(2\theta)/3$

Their invariants:

invariant	from timelike Killing	from azimuthal Killing
$L^a{}_{ba}$	0	0
$L_{abc}{}^{;c}$	**0** (divergence-free)	**0** (divergence-free)
$L_{abc}L^{*abc}$	**0** (dual-orthogonal)	**0** (dual-orthogonal)
$L_{abc}L^{acb}$ / $L_{abc}L^{abc}$	½	½
$L_{abc}L^{abc}$	**$4M^2[3r^3(r-2M)]^{-1}$**	**$4(r\sin^{-2}\theta - 2M)(3r^3)^{-1}$**
behaviour at r = 2M	**diverges**	finite: $(3M^2\tan^2\theta)^{-1}$
behaviour on axis θ → 0, π	finite	**diverges**

Both are dual-orthogonal, tracefree, and divergence-free in their rightmost index (Lanczos differential gauge).

Quotient detector. Rename the azimuthal-based Lanczos tensor H_{abc}. The two contracted squares

diverge on **complementary** sets, so their quotient

$$(L_{abc}L^{abc}) / (H_{abc}H^{abc})$$

diverges at the horizon, vanishes on the axis, and is finite elsewhere. This is a wire-free horizon detector built entirely from Lanczos tensors of the spacetime, with no Killing-vector norm or other non-Lanczos factor imported. But "underneath the hood" it used two Killing vectors, so like the $g_{\varphi\varphi}$ trick it is available in the Weyl family but not (or, at least, not obviously) for general metrics – but every ingredient is a Lanczos object.

Wire caveat: This Lanczos pair is verified on **undistorted Schwarzschild only**. Since the azimuthal Killing vector survives the Weyl distortion, the construction must carry over. But we did not explicitly verify that by computer. **Second order caveat:** Everything here is O(A). We never attempted to go to order A^2. **ℓ=1,2,3,...?** Because it works for ℓ=1,2,3,4, wee presume it keeps working for all ℓ≥5 too. But our brute-force proof method only works for one particular ℓ and keeps getting harder, so we stopped at ℓ=4.

13. Prevalence of horizon-detectors

Definition. A trace-free Lanczos tensor L_{abc} is a **detector** for a horizon Ω if the scalar $L_{abc}L^{abc}$ is infinite at every point of Ω except possibly for a measure-zero subset.

Detectors are known to exist for several classes of stationary black holes. The question is whether they are *typical* among admissible Lanczos tensors, or exceptional.

If we have two traceless Lanczos tensors L_{abc} and H_{abc}, one of which is a detector, the other not, then the linear combination $(1-q)L_{abc}+qH_{abc}$, if q is sampled randomly from any probability density on the real line, will, with probability=1, be a detector.

That makes it sound plausible that a "random Lanczos tensor" will be a horizon detector "with probability=1." But can this be made rigorous? It turned out that measure-theorists have already addressed this kind of scenario. The right notion is

Definition of "prevalence" (Christensen 1972; Hunt-Sauer-Yorke 1992): A Borel set S in a completely metrizable topological vector space is *shy* if for some compactly supported Borel probability measure μ we have μ(S+w)=0 for every w. And S is *prevalent* if its complement is shy.

Remarks about prevalence. A measure μ witnessing shyness is called a **probe measure**; when μ is normalised Lebesgue measure on a bounded subset of a finite-dimensional subspace E ⊆ V, then E is called a **probe subspace**.

Prevalence is translation invariant, agrees with "Lebesgue-almost every" in finite dimensions, is preserved under countable intersection, and a prevalent set is dense. See Ott & York 2005 for a survey.

Remarks about our prevalence theorems below.

i. The definition does *not* require finite-dimensional probes, but they are convenient since the easiest witnesses to exhibit.
ii. Prevalence is a *measure-theoretic*, not topological, notion. It is logically independent of Baire category: shy sets can be residual and prevalent sets can be meager.

iii. Our prevalence results will bound the singular locus (e.g. black hole horizon) from below only. I.e. we conclude that the singular set *contains* the locus, never that it *equals* it. So "pure" horizon-detection – infinite $L^{abc}L_{abc}$ on the horizon but nowhere else – can never follow from an argument of this kind. But if we *only permit* L_{abc} such that $L^{abc}L_{abc}$ is bounded in regions external to horizons, then purity is automatic.
iv. Prevalence statements are about *one* named space. It does not descend to subspaces.
v. Our Abstract Prevalence Theorem below gives its conclusion at almost$_\mu$everywhere p for a fixed μ; upgrading from a *countable* family of loci to a continuum would need a separate uniformity argument.

Two useful tools:

Theorem of Mityagin 2020: Let d≥1. The zero-set of a nontrivial real-*analytic* function in d-dimensional space has zero measure. (And Hausdorff dimension≤d-1.)

But even when d=1, for any integer k≥0 (and including k=∞) there exist nontrivial C^k-smooth functions f(x) whose zero-set has positive measure, e.g. f(x)=0 for x≤0, but f(x)=exp(-x^{-1}) when x>0.

Tonelli's Lemma: Let (X, μ) and (Y, ν) be σ-finite measure spaces and let f : X × Y → [0, ∞] be measurable with respect to the product σ-algebra. Then the two iterated integrals agree with each other and with the product integral,

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_X \left[\int_Y f(x,y)\, d\nu(y) \right] d\mu(x) = \int_Y \left[\int_X f(x,y)\, d\mu(x) \right] d\nu(y),$$

all three being equal as elements of [0, ∞] and possibly infinite.

Useful for: when you know the answer along one family of slices and want it along the transverse family.

Armed with those tools, our mission now is to prove an "abstract prevalanece theorem" that assumes certain hypotheses, and then to apply it to the concrete Lanczos horizon-detector question.

The structure to be abstracted is this. *Admissible fields* (for our application, Lanczos tensors) form a linear (or in our case affine) space; each field has a *leading singular part* near the *locus* of interest (which for our application is "black hole horizons"), depending linearly on the field; and whether the field "detects" is decided by a quadratic expression in that leading part, evaluated pointwise on the locus.

Setting. Let

- $\mathbb{A}$ be an *affine space* modelled on a completely metrizable topological vector space V (the *admissible fields*): V acts freely and transitively on $\mathbb{A}$ by translation, so fixing any L_* identifies $\mathbb{A}$ with L_* + V. Shyness and prevalence in $\mathbb{A}$ are defined by translating by elements of V, exactly as in Definition 1. (In the Lanczos application $\mathbb{A}$ is the set of potentials generating a *given* Weyl tensor and V=**K** is the vector space of those generating *zero* Weyl; the distinction is used below.);
- (M, μ) be a measure space with μ(M) > 0 (the *locus*, e.g. a horizon);
- π : $\mathbb{A}$→W be an *affine* map into a real vector space W, with linear part π_0 : V→W (the *leading-part map*);
- for each p ∈ M, Q_p : W → $\mathbb{R}$ be a quadratic form, with polar bilinear form B_p, so that $Q_p(w)=B_p(w, w)$.

Define R : V × M → $\mathbb{R}$ by $R(L, p)=Q_p(\pi(L))$, then put

$$\mathbb{D} = \{ L \in \mathbb{A} : \mu^+(Z_L) = 0 \}.$$

where μ^+ denotes the outer measure induced by μ, and $Z_L = \{p \in M : R(L,p)=0\}$. Elements of $\mathbb{D}$ are called **detectors**.

Some possible hypotheses.

> **(Admissible probe hypothesis)** The direction q of (Detector exists) lies in the **translation space**, $q \in V$ &mdash *not* merely $q \in \mathbb{A}$ — so that $L + aq \in \mathbb{A}$ for every $L \in \mathbb{A}$ and every $a \in \mathbb{R}$. (In the Lanczos application this says the witness must generate *zero* Weyl, not the Weyl tensor of the spacetime.)
>
> **(Detector exists)** There is $q \in V$ with $Q_p(\pi_0(q)) \neq 0$ for almost$_\mu$everywhere $p \in M$.
>
> **(0-1 hypothesis)** For every $L \in V$ and $a \in \mathbb{R}$, the function $p \mapsto R(L + aq, p)$ either vanishes identically on M or is nonzero almost$_\mu$everywhere. [This hypothesis would follow on a connected real-analytic manifold M with μ equivalent to Lebesgue measure in a coordinate-chart, from supposing $p \mapsto R(L + aq, p)$ is real-*analytic* on M for each fixed a.]

We know two hypothesis-sets that both yield the same prevalence conclusion: with (0-1 hypothesis) we do not need measurability, while with σ-finiteness and joint measurability we do not need (0-1). These two routes differ only in how sharply they bound the exceptional set E_L.

Abstract Prevalence Theorem. Assume (Admissible probe hypothesis) and (Detector exists). Let $L \in V$ be arbitrary. Write $E_L = \{ a \in \mathbb{R} : L + aq \notin \mathbb{D} \}$. Then:

(a) If (0-1 hypothesis) holds	then	E_L contains at most two points;
(b) If μ is σ-finite and $(a, p) \mapsto R(L+aq, p)$ is jointly measurable	then	E_L has Lebesgue measure zero.

Either way, $\mathbb{A} \setminus \mathbb{D}$ is contained in a shy set, hence $\mathbb{D}$ is prevalent in $\mathbb{A}$, with one-dimensional probe subspace $\mathrm{span}\{q\} \subset V$. (Shyness in the Christensen / Hunt-Sauer-Yorke sense is defined for universally measurable sets and extended to arbitrary sets by containment; the slice estimate is what that definition requires, and no measurability of $\mathbb{A} \setminus \mathbb{D}$ itself is asserted.)

Proof. *The pencil.* Fix $L \in \mathbb{A}$. By (Admissible probe hypothesis) the whole line L+aq lies in $\mathbb{A}$, so the statement is about V and not a larger space. By linearity of π, $\pi(L+aq) = \pi(L) + a\pi_0(q)$, and hence for each p

$$R(L+aq, p) = Q_p(\pi(L) + a\pi_0(q)) = Q_p(\pi(L)) + 2a\, B_p(\pi(L), \pi_0(q)) + a^2\, Q_p(\pi_0(q)). \quad (*)$$

By (Detector exists) the set $M_0 = \{p : Q_p(\pi_0(q)) \neq 0\}$ has outer-measure-null complement, hence is nonempty because μ(M)>0. For $p \in M_0$ the right-hand side of (∗) is a real polynomial in a of degree exactly two, its leading coefficient being nonzero, so it has at most two real roots.

(a) Under (0-1 hypothesis). Let $a \in E_L$, that is, $R(L + aq, \cdot)$ fails to be nonzero almost everywhere. By (0-1 hypothesis) it then vanishes identically on M, in particular at any single point we care to choose. Since M_0

has full measure it is nonempty; fix $p_0 \in M_0$. Then every $a \in E_L$ is a root of the degree-two polynomial $(*)$ at $p = p_0$, so E_L has at most two elements, and a fortiori Lebesgue measure zero.

(b) Under σ-finiteness and measurability. Put

$$E = \{ (a, p) \in \mathbb{R} \times M : R(L + aq, p) = 0 \}.$$

By $(*)$ the coefficients of the polynomial are measurable functions of p, so E is measurable. *Slice in p*: for $p \in M_0$ the root set $E^p = \{ a : (a,p) \in E \}$ has at most two elements, hence $\mathrm{Leb}(E^p) = 0$ for almost$_\mu$everywhere p. *Transpose*: apply Tonelli's Lemma on $\mathbb{R} \times M$ with $f = \mathbf{1}_E$, nonnegative and product-measurable, both measures being σ-finite. The two iterated integrals are the two ways of slicing E, so

$$\int_{\mathbb{R}} \mu(E_a)\, da = \int_M \mathrm{Leb}(E^p)\, d\mu(p) = 0,$$

where $E_a = \{ p : (a,p) \in E \}$. A nonnegative function with zero integral vanishes almost everywhere, so $\mu(E_a) = 0$ for Lebesgue-almost-every a - which says exactly that $R(L + aq, \cdot) \neq 0$ almost$_\mu$everywhere, i.e. $L + aq \in \mathbb{D}$, for almost-every a. Hence E_L is Lebesgue-null. Note that (0-1 hypothesis) is nowhere used.

Prevalence, common to both. Either clause gives $\mathrm{Leb}(E_L) = 0$ for every $L \in V$. Let μ_q be the push-forward of normalised Lebesgue measure on [0, 1] under $t \mapsto tq$; by (Admissible probe hypothesis) this is a compactly supported Borel probability measure on V. For any $w \in V$,

$$\mu_q((V \setminus \mathbb{D}) + w) = \mathrm{Leb}\{ t \in [0,1] : -w + tq \notin \mathbb{D} \} \leq \mathrm{Leb}(E_{-w}) = 0.$$

Hence $V \setminus \mathbb{D}$ is shy and $\mathbb{D}$ is prevalent, with probe subspace span{q}. **Q.E.D.**

The two slicings we used Tonelli on are: (i) Fixing p and integrating over a measures, for that point of the locus, the set of coefficients a at which the residue happens to vanish; (ii) fixing a and integrating over p measures, for that member of the pencil, the set of points at which it fails to detect. Tonelli's Lemma says the two have the same total; the first is zero because a quadratic has at most two roots.

How the two arguments differ:

a. fixes a single $p_0 \in M_0$ then uses (0-1 hypothesis) to propagate the conclusion from that one point to all of M, yielding the finite bound.
b. never fixes a point: it integrates the pointwise nullity in a over p. That is enough for prevalence and needs no regularity (neither analyticity nor continuity) of the residue along M whatever, only measurability.

Remark on sharpness. The bound "at most two" cannot be improved to "at most one": One root always comes from the possibility $Q_p(\pi(L) + a\pi_0(q)) = 0$ with w+av=0; the second is a genuine cancellation between the cross term $2aB_p(\pi(L), \pi_0(q))$ and the diagonal terms, and it occurs whenever Q_p restricted to span{$\pi(L)$, $\pi_0(q)$} is indefinite.

Application to Lanczos horizon detectors. The *admissible fields* for us shall mean the Lanczos tensors obeying certain regularity demands on- and off-horizon, for example bounded $L^{abc}L_{abc}$ throughout the "accessible region," and having at worst simple poles on horizons.

Lanczos tensors form an *affine space*: any linear combination of them with weight=1 is another. This remains true under various common restrictions such as demanding tracelessness $L^a{}_{ba}=0$, rightmost index divergence freeness $L^{abc}{}_{;c}=0$, or being derivable from a superpotential $L^{abc}=P_{c[a;b]}$. But the theorem does *not* permit you to define your subclass of Lanczos tensors using *quadratic* demands, such as dual-orthogonality $L^{abc}L^*{}_{abc}=0$, although the "disjoint support" common-*reason* for dual-orthogonality *is* a linear demand.

The *leading-part* notion we shall use is "leading Laurent coefficient" for simple poles of $L^{abc}L_{abc}$ on nondegenerate horizons (or in the case of an application to extreme Reissner-Nordstorm black holes, dounle poles). I.e. we let Q_p be the quadratic form induced on those leading parts by the contraction $L_{abc}L^{abc}$. Then

$$Q_p(\pi(L)) = \lim_{\Delta\to 0} \Delta \cdot L_{abc}L^{abc} \;=\; \text{the residue at p.}$$

Corollary: one horizon-detector implies prevalence. Suppose some $q \in K$ has nonzero residue almost everywhere on H. Then the horizon detectors form a prevalent subset of $\mathbb{A}$, and for any admissible L the horizon-subset of a at which L+aq fails to detect is Lebesgue-null. If in addition the residue is real-*analytic* along H, then that failure-set consists of at most two points.

Corollaries about specific black hole families:

1. Horizon-detectors are prevalent among the Lanczos tensors $Ł_{abc}$ for the Kottler, and more generally Reissner-Nordström-deSitter black hole, family even if we demand divergence-freeness in the rightmost index $L_{abc}{}^{;c}=0$, dual-orthogonality $L^{abc}L^*{}_{abc}=0$, and tracefreeness $L^a{}_{ba}=0$.

2. Horizon-detectors are prevalent among the Lanczos tensors $Ł_{abc}$ for extreme Reissner-Nordström black holes, even if we demand divergence-freeness in the rightmost index $L_{abc}{}^{;c}=0$, dual-orthogonality $L^{abc}L^*{}_{abc}=0$, and tracefreeness $L^a{}_{ba}=0$.

3. Horizon-*and*-axis detectors are prevalent among the Lanczos tensors $Ł_{abc}$ for the Kerr black hole family, even if we demand tracefreeness $L^a{}_{ba}=0$ and divergence-freeness in the rightmost index $L_{abc}{}^{;c}=0$.

Kerr axis annoyance. The fact that for Kerr metric the axis *also* gets detected, is very annoying because it cannot be physical – in the real "dirty" physical world, Kerr will *not* be exactly axisymmetric and Petrov type D. Therefore, I expect that for real-world "dirty Kerr" black holes, the horizons will be detected ($L^{abc}L_{abc}\to\infty$) by almost all tracefree Lanczos tensors; *but* the symmetry axis will *not* have $L^{abc}L_{abc}\to\infty$, since there is no symmetry axis. So to understand realistic physical behavior, we really want to find some traceless L_{abc} whose $L^{abc}L_{abc}$ goes infinite almost everywhere on the Kerr horizon, but nowhere outside the horizon (including: stays finite on the spin axis). But the shyness of such objects in the "clean" Kerr metric makes them difficult to find, despite their presumed prevalence under real-world "dirty" Kerr conditions. It took us a long time to overcome that but we finally did, as described in the final appendix. Consequently,

4. Horizon detectors are prevalent among those traceless Lanczos tensors L_{abc} for the Kerr black hole family which have contracted square $L^{abc}L_{abc}$ *finite* everywhere *outside* the horizon (which the appendix

finally showed is a nonempty set).

14. Nonrigorous intuitive reasons to think $L_{abc}L^{abc}\to+\infty$ as one approaches horizons, and that L_{abc} generically are horizon-detectors

Simpler & more familiar analogous problem. If photons had mass, then the potential between two charges Q_1, Q_2 separated by distance r would behave like $Q_1Q_2r^{-1}\exp(-\mu r)$ with μ proportional to photon mass. This is the "Yukawa potential" as opposed to the "Coulomb potential" which is its μ=0 special case. The reason the Yukawa potential arises is that the "harmonic equation" $\nabla^2\Phi=0$ the electrostatic potential Φ obeys in vacuum, is replaced by $\nabla^2\Phi=\mu^2\Phi$. The Coulomb potential solves the former equation, the Yukawa potential the latter.

Smith 2003 argued that with positive photon mass, it would be impossible to feel electrostatic forces exerted by a charged black hole, essentially because photons carrying that force would be unable to climb out of the gravitational field of the hole. Essentially, an infinite number of Yukawa exponential e-factor losses would occur during any such attempt. Others have called such arguments "black hole **no-hair** theorems": Black holes are characterized by their mass, charge, and angular momentum *only*. They necessarily exert *zero* weak force and Higgs force on the outside world, because the carriers of those forces (W, Z, and Higgs bosons) have mass. Only with *exactly zero* photon mass could black holes exert electrostatic or magnetic forces on external objects. Therefore, Smith 2003 concluded, any experimental evidence of such forces constitutes experimental proof of the exact masslessness of the photon.

Nonrigorous argument that $L^{abc}L_{abc}\to\infty$ on full-measure subset of black hole horizons. What does all that have to do with Lanczos tensors? We showed that in Λ-vacuum, the Dolan-Kim wave equation they obey simplifies to $L_{abc;d}{}^{;d}=\Lambda L_{abc}$. Consequently, the positive quantity $X=L^{abc}L_{abc}$ must obey the PDE

$$X_{;u}{}^{;u} = 2\Lambda X + 2L^{abc;d}L_{abc;d}.$$

With the MTW sign conventions used in this paper, Λ>0 in de Sitter spacetime, i.e. our universe. If we work in the rest frame of the black hole, and regard that hole as approximately static, and regard the Lanczos tensor near the hole as also approximately static (i.e. time-translation invariant) then the left hand side of the equation becomes just ∇^2X, while both terms on the right hand side are *positive* (or if Λ=0 then only one is). By the "maximum principle" for harmonic and sub-harmonic functions the maximum of X must lie on the *boundary* of any region that PDE applies in. The boundaries of the "accessible region" of the universe are black hole and cosmological horizons. So X should be expected to attain large, perhaps infinite, values on those boundaries. That all would be true even if the right hand side of the PDE were 0. However, as we said, it is positive — which makes it more true. This PDE is analogous to the one that produced the Yukawa potential, except ours includes an additional *non*linear term on the right hand side. This should yield a stronger effect than Yukawa's. As a result, the same hairless conclusion about black holes should be expected:

The ratio $X(\omega)/X(\text{in accessible region})$ should be infinite for ω on a full-measure subset of the black hole horizon Ω.

This argument predicts this for the definition of "horizon" being the "surface that light cannot escape from

into the outside world" and from which escape requires infinite rocket fueled/unfueled mass ratio. That argument assumed we were in Λ-vacuum. However, if we are not (i.e. matter is present), then that matter outside horizons presumably has *bounded* density. Therefore the (non-Λvacuum) terms in the Dolan-Kim wave equation will, as we approach the horizon , become negligible by comparison with $2L^{abc;d}L_{abc;d}$. Therefore the leading order asymptotics of X, as we approach the horizon, ultimately should not care whether it is Λ-vacuum or not. "**Q.E.D.**"

Nonrigorous argument that traceless Lanczos tensors L_{abc} should generically detect black hole horizons (in Λ-vacuum). Consider a collection of black holes in a de Sitter universe. For simplicity and to avoid technical issues, I'll assume no two holes ever merge (which could be ensured by making all pairs of them be escaping from each other to unboundedly-far distances), and every hole has a *non-degenerate* horizon. Consider a 3-dimensional spacelike surface "t=0" of time-reflection symmetry. We'll regard that surface as the "initial conditions." Make the initial L_{abc} be finite throughout the "accessible region" but with $L^{abc}L_{abc}$ approaching ∞ as we approach any black hole horizon from outside it. We don't care what the initial data is inside the holes since that cannot affect anything in the accessible region. By the existence theorems of Illge 1988 and Andersson & Edgar 2001, this initial data can be extended to t>0 to yield L_{abc} thoughout the future. The preceding nonrigorous argument suggests that the infinity initally imposed on the black hole horizons, will stay infinite forever after. So we now have nonrigorously established that at least one L_{abc} exists that is a horizon detector. Finally by the "prevalence theorem" the existence of that one, implies that *generic* L_{abc} are horizon detectors. "**Q.E.D.**"

Between this intuitive reasoning and all the rigorous results we established in the three preceding sections, I regard the case as now being strong for the

Generic Horizon-detector Conjecture: Generic traceless Lanczos tensors L_{abc} in the physical universe have the property that $L^{abc}L_{abc}$ approaches +∞ as we approach generic black hole event horizons from the outside (except perhaps on a measure-zero subset of the horizon). This also is true if restricted to the subset of L_{abc} that arise from superpotentials and/or obey $L_{abc}{}^{;c}$ (if those subsets are nonempty).

Anybody who wants to attack this conjecture perhaps should look first at Vaidya's family of radial-radiation nonvacuum metrics, since we tried, but failed, to find a closed-form horizon-detecting L_{abc} for Vaidya. (We also failed for the Majumdar-Papapetrou metrics, but that failure bothers me less because the M-P metrics' charged black holes all are "extreme," i.e. nongeneric and unphysical. And the methods in the Companion Paper do enable detecting M-P horizons *without* using Lanczos by using the fact that they are algebraically special, unlike the rest of the metric which generically is Petrov type I. In nongeneric cases such as extreme RN, or axisymmetric M-P, though, algebraic-speciality detectors fail.)

15. Accelerations of "stationary observers" II: the Kerr black hole; and relation to $L^{abc}L_{abc}$

MTW conventions, signature (−+++), Boyer-Lindquist coordinates, M=1 in all numbers. Throughout, $\Sigma=r^2+a^2\cos^2\theta$ and $\Delta=r^2-2Mr+a^2$. Latin indices u,v,w run over spacetime; but i,j run over the two directions (r, θ) transverse to the Killing orbits, with $g^{rr}=\Delta\Sigma^{-1}$ and $g^{\theta\theta}=\Sigma^{-1}$. Every formula below was checked numerically.

For spherically symmetric static metrics (e.g. Schwarzschild), it was fairly clear what "gravitational acceleration" ought to mean. For Kerr, which is rotating, *axi*symmetric, and *non*static, it was *not* immediately clear what it should mean, and investigating this is considerably more complicated. Nevertheless, we were able to do such an investigation, and found there are four (or five) natural acceleration notions, some of which do interesting things. All this will be of little interest for Milgromian observational-astronomy because accelerations small enough to trigger Milgrom-MOND's postulated gravity-law-switchover will only happen very far outside Kerr-hole horizons, so far that Kerr effectively becomes Schwarzschild. But this investigation feels centrally important for purposes of SG-theory *understanding*.

A 1-parameter class of vector fields $V^u(p)$ and with acceleration $A^u(p)$. The Kerr metric has two Killing vectors, in Boyer-Lindquist coordinates the two fields of unit-length vectors $\hat{t}^u$ and $\hat{\varphi}^u$. For a **real parameter p** set $V^u(p)=\cosh(p)\hat{t}^u+\sinh(p)\hat{\varphi}^u$, so that $V^uV_u=-1$ for every p.

This V^u is *not* a Killing vector field. Why? Although constant-coefficient linear combinations of two Killing fields are Killing, V^u, due to the scaling factors required to make norm=±1, is a *position-dependent* linear combination. Despite this non-Killingness, it turns out that the *acceleration* of somebody who travels along the integral curves of V^u is still given by the formula

$$A^u = V^v\nabla_v V^u = \Gamma^u{}_{vw}V^vV^w$$

which is related to the fact that r and θ are constant along the integral curves, so $V^v\partial_v V^u=0$, causing A^u to depend only on the *value* of V^u at the point, not on how it is extended. [We verified this numerically.] Then also $A^uV_u=0$ exactly, with A^u spacelike so $A^uA_u\geq 0$.

The four (or five) natural "gravitational acceleration" notions we shall consider:

1. If p=0, we get an observer who **refuses to rotate** the hole and insists on staying at constant r (by firing his rockets); what acceleration does he experience?
2. We can choose p to *minimize* the observer's experienced-acceleration. And actually it will turn out there can be *two* local minima.
3. Between those two local minima is a local *max*imum; what is the experienced-acceleration for *that* observer?
4. Finally, there is another special choice of p yielding what we call "the **distinguished observer**."

Domain. $\hat{t}^u$ is timelike only **outside the ergosphere**. Therefore the p-parametrisation using sinh and cosh only makes sense in this outside-ergosphere region. We can overcome that limitation with an **alternative variable q**, which will be well-defined all the way down to the horizon.

2. Reformulation: the vectors P, Q, R. Writing $\Omega = V^\varphi(V^t)^{-1}$ and $W(\Omega) = -(g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi})$, the Killing result gives $A_i = \partial_i \ln(W)/2$ with Ω **held fixed**, hence

$$A^uA_u = [\Delta(\partial_r W)^2 + (\partial_\theta W)^2][4W^2\Sigma]^{-1}.$$

Using the Kerr identity $g_{t\varphi}{}^2-g_{tt}g_{\varphi\varphi}=\Delta\sin^2\theta$, the two **null** angular velocities are $\Omega_\pm = (-g_{t\varphi} \pm \Delta^{1/2}\sin\theta)g_{\varphi\varphi}{}^{-1}$ and $W = g_{\varphi\varphi}(\Omega_+ - \Omega)(\Omega - \Omega_-)$. Setting

$$q = (\Omega - \Omega_-)(\Omega_+ - \Omega)^{-1} \in (0, \infty),$$

the acceleration becomes linear in q and q^{-1}:

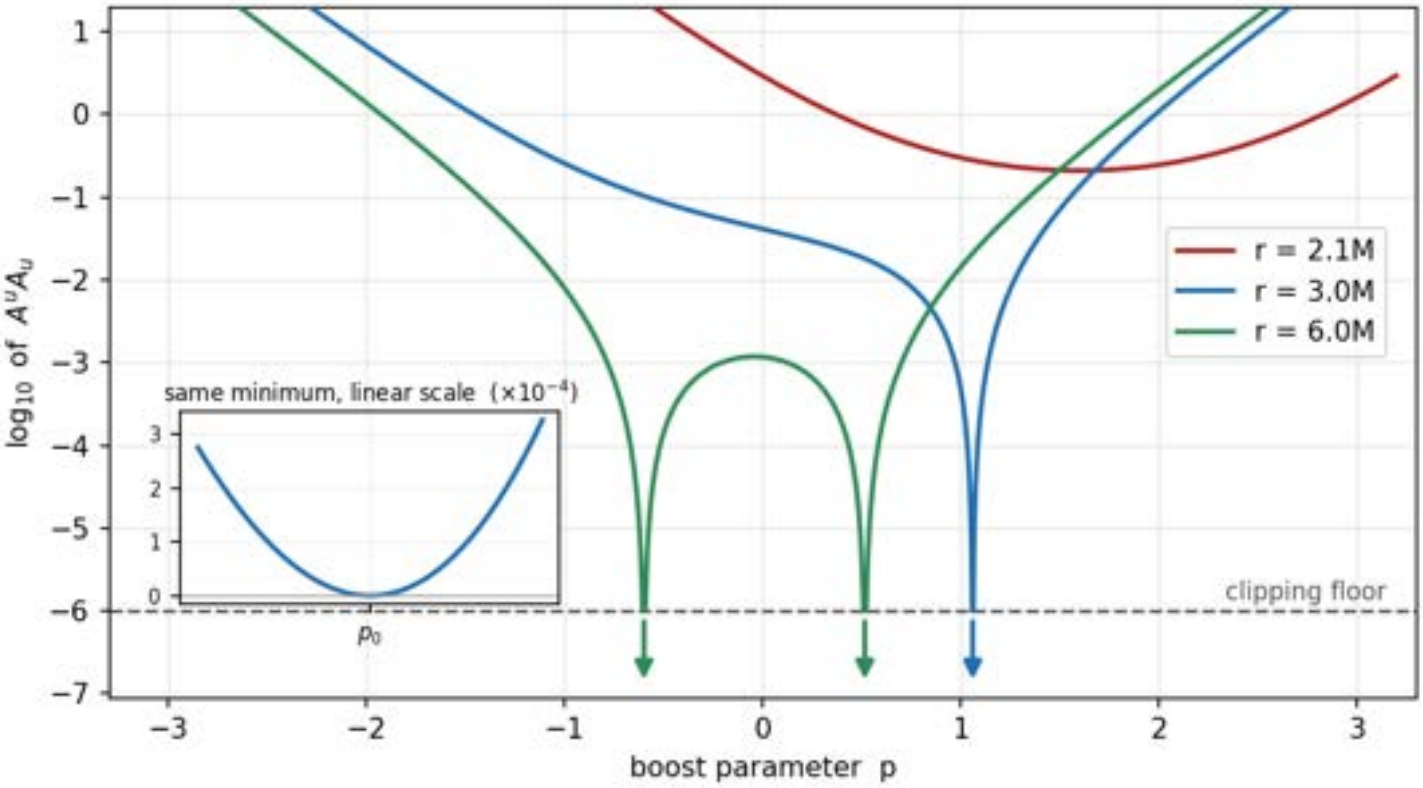
$$\mathbf{A_i = P_i + Q_i q - R_i q^{-1}}, \quad A^u A_u = (P_i + Q_i q - R_i q^{-1})(P^i + Q^i q - R^i q^{-1})$$

and, because $g_{\varphi\varphi}(\Omega_+ - \Omega_-) = 2\Delta^{1/2}\sin\theta$ exactly,

$$P_i = \partial_i \ln(\Delta^{1/2}\sin\theta)/2 \qquad [\, P_r = (r-M)(2\Delta)^{-1},\ \ P_\theta = \tfrac{1}{2}\cot\theta \,]$$
$$Q_i = \partial_i \Omega_+ (2D)^{-1}, \quad R_i = \partial_i \Omega_- (2D)^{-1}, \quad D = \Omega_+ - \Omega_-$$

P_i is metric-explicit and refers to neither null direction: it is the gradient of the log **area density of the Killing orbits**, since $\Delta^{1/2}\sin\theta = \sqrt{(g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi})}$. The relation to p is $q = e^{2(p + p_0)}$ with $p_0 = \operatorname{arctanh}[g_{t\varphi}(\Delta^{1/2}\sin\theta)^{-1}]$.

Structure of $A^u A_u$ as a function of p: minima, maxima, and large-|p| behavior. As $p \to \pm\infty$ the orbit becomes null, $W \to 0$ and $A^u A_u \to +\infty$. So the **supremum over p is $+\infty$ at every point of the spacetime**, and the interesting quantities are the interior stationary points. Since the function diverges at both ends, minima and maxima alternate beginning and ending with a minimum, so the count is always **odd**. In practice it is 3 or 1.



The three regimes, at the equator with a = 0.6M. The vertical axis is $\log_{10}(A^u A_u)$, and the plot is **clipped at 10^{-6}**, marked by the dashed floor. **The downward arrows below that floor mark minima that are exactly zero**, where the logarithm descends without bound: those observers are circular *geodesics* hence feel no acceleration at all. Blue (r=3M) has one such zero, the prograde geodesic. Green (r=6M) has two, prograde and retrograde, with a genuine local maximum between them at 1.187×10^{-3}. The red curve (r=2.1M) is different in kind: it has a single minimum that is **strictly positive**, 2.098×10^{-1}, so it levels off at $\log_{10}=-0.678$, and never approaches the floor hence carries no arrow. Away from the minima all three curves rise unboundedly as $p\to\pm\infty$.

The apparent cusps at the arrowed minima are not cusps. They are merely an artifact of our plot's use of a *logarithmic* vertical axis: near a simple zero of f(q) the acceleration behaves as $A^u A_u \propto (p - p_0)^2$. The plot-inset confirms this for the blue minimum by replotting it on a **linear** vertical scale.

What controls the minima/maxima structure (and their count) is the **sign of Q_r and R_r** on the equator, and each flips at a photon orbit – Q_r at the prograde one, R_r at the retrograde one, because $\partial_r\Omega_\pm = 0$ is exactly the circular photon orbit condition. At a = 0.6M these sit at 2.18891M and 3.62985M:

region (equator)	sgn Q_r	sgn R_r	stationary points	minimum $A^u A_u$
$r > r_c$ (retrograde photon orbit)	−	+	min, max, min	0 (two circular geodesics)
$r_{photon,prograde} < r < r_c$	−	−	one min	0 (prograde geodesic only)
$r < r_{photon,prograde}$ (prograde photon orbit)	+	−	one min	strictly positive

As we approach the horizon with every observer forced to co-rotate, even the minimum acceleration approaches infinity:

$$(A^u A_u)_{min} \to (M^2-a^2)^{1/2}\,[2\Sigma_+(r-r_+)]^{-1}, \qquad \Sigma_+ = r_+^2 + a^2\cos^2\theta$$

so the proper acceleration itself grows like $(r - r_+)^{-1/2}$, which $-\frac{1}{2}$ is the same power as occurs for the Schwarzschild static observer.

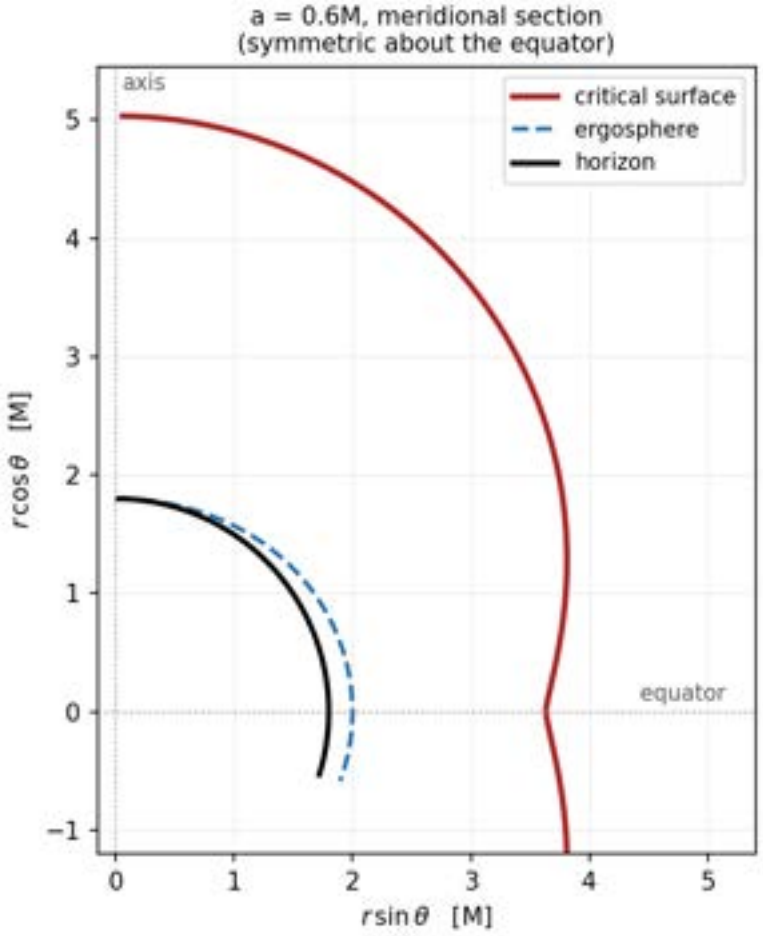
The critical surface There is a *transition* from three stationary points to one. This transition happens when we go inside the "critical surface." That surface corresponds to the acquisition of a positive *double root* by the quartic

$$Q^h Q_h q^4 + P^h Q_h q^3 + P^h R_h q - R^h R_h = 0,$$

i.e. two of its roots merge into one, then depart from the real axis as a complex-conjugate pair, when its discriminant vanishes. With $a=Q^i Q_i$, $b=P^i Q_i$, $d=P^i R_i$, $e=-R^h R_h$ and no q^2 term the discriminant collapses to six terms:

$$\text{discrim} = 256a^3e^3 - 192a^2bde^2 - 27a^2d^4 - 6ab^2d^2e - 27b^4e^2 - 4b^3d^3 = 0.$$

The **figure** shows a meridional section of the critical surface for a Kerr hole with spin parameter a=0.6M. Only a thin strip below the equator is drawn, since all three surfaces are symmetric under $\theta \to \pi-\theta$. The critical surface is narrowest at the equator and widens on either side of it, because r_c grows quadratically away from the equator faster than $\sin\theta$ falls. At zero spin (a=0) the critical surface is a **sphere**. It becomes **prolate**, i.e. elongated along the rotation axis when we spin up the hole (0<a<M). This critical surface lies entirely outside both the ergosphere and the horizon, and its θ-dependence runs *opposite* to the ergosphere's: the ergosphere is fattest at the equator and pinches onto the horizon at the poles, while the critical surface is smallest at the equator and widest on axis:



θ	π/2	1.30	1.00	0.70	0.50	0.30	axis
r_c (a = 0.6M)	3.62985	3.94682	4.34393	4.68001	4.84950	4.96641	5.03354
ergosphere	2.00000	1.98704	1.94599	1.88849	1.85014	1.81941	1.80000

On the equator, $\partial_\theta W \equiv 0$ by reflection symmetry, $R_r \to 0$, and the quartic factors as $q^3(Q^iQ_iq+P^iQ_i)$, whereupon the critical surface reduces to $\partial_r\Omega_- = 0$, meaning the critical surface, in the equatorial plane, is the **retrograde circular photon orbit**:

$$r_c(\pi/2) = 2M\left\{1 + \cos[(2/3)\arccos(a/M)]\right\}.$$

In the Schwarzschild case a=0 this is just r_c=3M ("photon-orbit sphere"). At extremality (a=M) we get r_c=4M. At a=0.6M (pictured) r_c=3.629850M.

On the spin-axis the location of r_c increases starting from $r_c(0)$=3M, and rising toward $r_c(M)\approx$6.158629990160713M in the extreme-Kerr (maximum attainable spin) case a=M. The magic value 6.158629990160713M arises as the positive root of the irreducible quintic

$$4r^5 - 12Mr^4 - 56M^2r^3 - 115M^3r^2 - 114M^4r - 31M^5 = 0.$$

Closed forms for A^uA_u on the equator. On the equator P_i, Q_i, R_i are purely radial, so $A^uA_u = g^{rr}f(q)^2$ with $f(q) = P_r + Q_r q - R_r q^{-1}$, and we get elementary closed forms for everything. Two special values of q occur. The zeros of f are the roots of the quadratic $Q_r q^2 + P_r q - R_r = 0$, and the critical point of f is at

$$\hat{q} = (-R_r Q_r^{-1})^{1/2}$$

which is a *different* point from the q_* in the "migration limit" discussion to follow: that one carries the opposite sign under the root. Exactly one of these two is real at any radius, since $-R_rQ_r^{-1} > 0$ precisely when $R_rQ_r^{-1} < 0$. They are complementary, and between them account for every stationary point:

stationary point	region	located at q =	value of A^uA_u there
the two minima	$r > r_c$	both roots of $Q_rq^2 + P_rq - R_r = 0$	**0 exactly** (prograde and retrograde circular geodesics)
the local maximum	$r > r_c$	$\hat{q}$	$\mathbf{g^{rr}[P_r - 2(-Q_rR_r)^{1/2}]^2}$
the single minimum	$r_{photon,pro} < r < r_c$	the one positive root of that quadratic	**0 exactly** (the prograde circular geodesic)
the single minimum	$r < r_{photon,prograde}$	$\hat{q}$	$\mathbf{g^{rr}[P_r + 2(-Q_rR_r)^{1/2}]^2}$

The last column is the value of the squared acceleration at the stationary point named in the first, with $g^{rr} = \Delta\Sigma^{-1} = \Delta r^{-2}$ on the equator. The maximum and the nonzero minimum are the same expression with the two signs, the sign being forced by that of Q_r. The minimum is zero precisely when a timelike circular geodesic exists, and the minimising Ω then is the Kepler frequency $\pm M^{1/2}(r^{3/2} \pm aM^{1/2})^{-1}$.

P^aP_a: "migration limit" and "distinguished observer." The single quantity

$$\mathbf{P^aP_a = [(r-M)^2\Delta^{-1} + \cot^2\theta](4\Sigma)^{-1}}, \quad \text{which simplifies on the equator to} \quad (r-M)^2(4r^2\Delta)^{-1}$$

is the acceleration with *both* null-direction contributions switched off. It plays three different roles depending on where you are, and the switch is again the sign of $R_rQ_r^{-1}$:

region	$R_rQ_r^{-1}$	status of P^iP_i
$r_{photon,prograde} < r < r_c$	> 0	**attained** at $q_* = (R_rQ_r^{-1})^{1/2}$ by a real observer – the **"distinguished observer"** – for whom the two null-gradient terms cancel

on r_c	$= 0$	attained only as $q\to 0$ (or equivalently, outside the ergosphere where our simpler paramterization based on p is usable, when $p\to -\infty$): the **migration limit**. Normally this limit would diverge, but $R_r=0$ removes the q^{-1} pole, so $A_i \to P_i$ is finite.
outside r_c, *or* inside $r_{photon,prograde}$	< 0	not attained by any observer; only an algebraic remainder

Therefore P^a is an attained acceleration for some observer precisely in **the radius interval between the two photon orbits**, 2.18891M<r<3.62985M when a=0.6M – the same interval in which the minimum acceleration is zero. (The sign of $R_rQ_r^{-1}$ flips at *both* ends of this interval: at r_c because R_r changes sign, and at $r_{photon,prograde}$ because Q_r does.) The reason for our word "**migration**" is: outside the critical surface there are two minima with one local maximum between them. Once we cross inside that surface, suddenly the maximum and retrograde minimum both *migrate* to p=−∞. However, the squared-acceleration A^uA_u they provide remains finite.

Relations connecting Lanczos contracted square $L^{abc}L_{abc}$ to squared-accelerations. Consider the trace-free Lanczos tensor $L_{abc}=\mathcal{P}_{c[a;b]}$ generated by the normalised sign-flip superpotential

$$\mathcal{P}_{ab} = ½ \times \text{(metric with its t-φ block sign-reversed)},$$

which satisfies the Lanczos→Weyl relation. Then, **exactly and at every (r, θ)**,

$$\mathbf{L_{abc}L^{abc} = (4/3)P^hP_h - (r^2 - 3a^2\cos^2\theta)\, Mr\, \Sigma^{-3}}.$$

This identity has been verified to $\le 8\times 10^{-16}$ at a/M = 0, 0.3, 0.6, 0.9 and many (r, θ), inside and outside the critical surface alike. The **first** term is $\nabla^h\ln(\Delta^{1/2}\sin\theta)\nabla_h\ln(\Delta^{1/2}\sin\theta)/3$; the **second** is the "Coulomb" (mass-monopole) part $\mathrm{Re}\psi_2$ of the Weyl tensor, where

$$\bar{\psi}_2{}^2 = \mathring{W}^{abcd}\mathring{W}_{abcd} / 96 = (W^{abcd}W_{abcd}+iW^{abcd}W^*_{abcd}) / 48$$

which for a type-D vacuum is determined by the metric alone. Here the overbar on $\bar{\psi}$ denotes *complex oonjugation*, and $\mathring{W}_{abcd}=W_{abcd}+iW^*_{abcd}$ is the **complexified Weyl tensor**, discussed much more in the Companion Paper. In the Schwarzschild case this is $\psi_2=-Mr^{-3}$ (with zero imaginary part), which is the transverse tidal eigenvalue felt by a static observer.

To explain our rather odd-looking notation "ψ_2" and our mysterious word "Coulomb": ψ_2 denotes one of the five complex Newman-Penrose Weyl scalars, ψ_0 through ψ_4, obtained by projecting the Weyl tensor W_{abcd} onto a null tetrad (two real null vectors and a complex pair). These five repackage Weyl's ten independent components as five complex numbers. Since Kerr is Petrov type D, with the right tetrad only ψ_2 is nonzero — a single complex number carries the entire Weyl tensor. "Coulomb" is just a traditional name for it.

We can interpret this identity in an acceleration-based manner in each region:

Where	Equation	Interpretation/comment
Inside r_c	$L_{abc}L^{abc} = (4/3)(A^uA_u)_{q_*} + re(\psi_2)$,	The "distinguished observer's" value
On r_c	$L_{abc}L^{abc} = (4/3)(A^uA_u)_{migration} + re(\psi_2)$	At a=0.6M and r_c=3.629850: $L_{abc}L^{abc} = (4/3)(0.02090904) + re(\psi_2) = 0.00696968$.
Outside r_c		The identity still holds, but P^hP_h is no longer any observer's acceleration. But...:

Asymptotically at large r, the maximum is recovered. The leading term of $L_{abc}L^{abc}$ is

$$(L_{abc}L^{abc})_{Flat} = (3r^2)^{-1}\csc^2\theta$$

a pure flat-space artifact of our superpotential which survives at M=a=0 despite there being no mass. Removing it leaves exactly the curvature term, and since the maximum sits at the static (refusing to rotate) observer,

$$(A^uA_u)_{no\text{-}rotate},\ (A^uA_u)_{max} \to r^2\ \ [L_{abc}L^{abc} - (1/3)\nabla^i\ln(\Delta^{1/2}\sin\theta)\nabla_i\ln(\Delta^{1/2}\sin\theta)]^2\ \ [\ 1 \pm O(r^{-1})\ ] = r^2$$
$$[(L_{abc}L^{abc})_{Kerr} - (L_{abc}L^{abc})_{Flat}\ ]^2\ \ [\ 1 \pm O(r^{-1})\ ]$$

[Note the r^2 factor in the right hand sides, and the squaring on the other factor – you might not have been expecting either. The two right hand sides are not exactly identical but are both valid at leading order.] or equivalently **the proper acceleration tends to $-r\,re(\psi_2)\to Mr^{-2}$.** [The $1\pm O(r^{-1})$ term in the displayed equation was, numerically, 1.112, 1.020, 1.005 at r = 20M, 100M, 400M, identical for a = 0.3M and 0.6M and for θ = π/2 and 1.2.] For contrast, Schwarzschild under the simpler {t}-flip-based superpotential has no flat-space artifact that must be removed, and obeys $L_{abc}L^{abc} \to (4/3)(A^uA_u)_{max}$ with no need to subtract a flattened version – but the {t}-only flip stops generating a Lanczos tensor once a≠0, so for Kerr the subtracted form is the only one available.

"Improved" Lanczos tensor L_{abc}? It was annoying, in the large-r limit above, that we needed to subtract $(L_{abc}L^{abc})_{Flat}$ from $L_{abc}L^{abc})_{Kerr}$. This leads us to ask: is there some "improved" traceless Lanczos tensor $\mathfrak{L}_{abc}$ for Kerr, which effectively already has its "flattened version" subtracted off? The answer is yes: using both the complexified Weyl and complexif Lanczos tensor we find the god-given **master identity**

$$\mathbf{[\mathfrak{L}_{abc}\mathfrak{L}^{abc}/2 - (4/3)P^iP_i]^2 = \mathfrak{W}_{abcd}\mathfrak{W}^{abcd}\ /\ 96}.$$

In our Kerr metric, we have

$$|\psi_2|^2 = [(W_{abcd}W^{abcd})^2 + (W_{abcd}{}^*W^{abcd})^2]^{1/2}\ /\ 48 = M^2\Sigma^{-3},$$
$$|\mathfrak{W}_{abcd}\mathfrak{W}^{abcd}| = 96M^2\Sigma^{-3},\quad |\mathfrak{L}_{abc}\mathfrak{L}^{abc}/2 - (4/3)P^iP_i| = M\Sigma^{-3/2}.$$

So $W_{abcd}W^{abcd}$ equals $48|\psi_2|^2$ **only when $W^{abcd}W^*_{abcd}$ vanishes**, i.e. only in the static nonrotating cases. The master identity for Kerr may be rewritten

$$|\mathring{L}_{abc}\mathring{L}^{abc}/2 - (4/3)P^iP_i| = M\Sigma^{-3/2}$$

recovering the "usual" Lanczos-accelration relation, *but* with a rotation-correction curvature-based term: the right hand side.

Approaching the horizon. The horizon limit is cleaner than the r→∞ limit, because there the curvature term drops out. As r → r_+ the factor Δ → 0 and $P_r = (r-M)(2\Delta)^{-1}$ diverges, while re(ψ_2) stays finite. Writing $\Sigma_+ = r_+^2 + a^2\cos^2\theta$ and using $(r_+ - M)^2 = M^2 - a^2$ together with $\Delta \to 2(M^2-a^2)^{1/2}(r - r_+)$, all three quantities diverge like $(r-r_+)^{-1}$, and with elementary coefficients:

$$(A^uA_u)_{\text{Migration-limit}} = P^iP_i \to (M^2-a^2)^{1/2}\,[8\Sigma_+(r-r_+)]^{-1}$$

$$\mathbf{(A^uA_u)_{min} \to (M^2-a^2)^{1/2}\,[2\Sigma_+(r-r_+)]^{-1} = 4P^iP_i}$$

$$\mathbf{L_{abc}L^{abc} \to (M^2-a^2)^{1/2}\,[6\Sigma_+(r-r_+)]^{-1} = (4/3)P^iP_i}$$

Since re(ψ_2) is finite it is swamped, so the exact identity degenerates to its first term alone and the Lanczos scalar locks onto both the **minimum** and **migration limit** accelerations:

$$\mathbf{L_{abc}L^{abc} \to (4/3)(A^uA_u)_{migration\ limit}} \quad \text{and} \quad \mathbf{L_{abc}L^{abc} \to (1/3)(A^uA_u)_{min}} \quad \text{as} \quad r \to r_+$$

The first of these shows that

> For Kerr, the "usual" acceleration↔Lanczos relation $L^{abc}L_{abc}=(4/3)A^vA_v$ becomes asymptotically exact when one approaches the horizon;
> and that also is true everyplace inside and on the critical surface where "tides" are neglectibly small – i.e. wherever $|re\psi_2| \ll |A^uA_u|$.

So the two ends of the spacetime give opposite pairings. **At large r** the Lanczos scalar tracks the **maximum** squared-accelertion, but only after Lanczos's flat-space "gauge" term (arising from M=a=0) is subtracted, since that term dominates in this near-flat region. **At the horizon** it tracks the **minimum** and **migrated-limit** accelerations, with no subtraction needed, since there the gauge term is what dominates and the curvature term is negligible. Note also that P^iP_i is *not* attained by any observer this close in – the horizon lies well inside $r_{photon,prograde}$ – so near the horizon it is again a purely algebraic quantity, related to the minimum by the factor 4 but, since we are well inside the ergosphere, not equal to any observer's acceleration, but nevertheless regardable as a *limit* of accelerations experienced by actual observers.

?

16. Why $L^{abc}L_{abc}=(4/3)A^uA_u$? The "VAS identity" provides a substantial partial explanation

Although we have seen that for all the classes of spacetimes most important for simulating stellar systems, galaxies and galaxy-clusters the relationship $3L_{abc}L^{abc}=4A_aA^a$ between a natural Lanczos tensor L_{abc} and

"gravitational acceleration" A^a holds either exactly or approximately – it has remained somewhat mysterious why that is, and for which spacetimes this is true. I expect it only is true for a fairly small set of spacetimes – since, for one thing, I see no obvious way even to assign a *meaning* to "gravitational acceleration" for fully general spacetime metrics. In this section, we exhibit some simple constructions, similar to our previous Killing-vector-based constructions, but instead now based on *arbitrary* "unit line fields" u_a, for "Lanczos-candidate" tensors L_{abc}. I warn you that these usually are *not* Lanczos tensors. But they are tensors whenever the input unit-line-fields are, and obey all the algebraic restrictions that Lanczos tensors must obey. The construction's output L_{abc} is bilinear in $(u_a, u_{a;b})$. The main purpose of this section is to show the **"VAS identity"**

$$\mathbf{L_{abc}L^{abc} = A_aA^a\ 4/3 - 6\ \omega_{ab}\omega^{ab} - 2\ \sigma_{ab}\sigma^{ab}}$$

in the case of a single unit-line-field, where ω_{ab} describes the unit-line-field's "*v*orticity," A_a describes the *a*cceleration of curves tangent to the field, and σ_{ab} is its "*s*hear." This identity does not depend on curvature, symmetry, or assumed field equations. This is only the k=1 case of a more general rank-k version, valid for any number k of commuting unit line fields. This identity goes a long way toward explaining why the Lanczos↔acceleration relation works (to the extent that it does).

The construction, and that its output is a genuine tensor. Given any vector field ξ^a with $\xi_k\xi^k\neq 0$, put

$$\mathcal{P}_{ab} = g_{ab} - 2\xi_a\xi_b / (\xi_k\xi^k), \quad \text{Ł}_{abc} = \mathcal{P}_{c[a;b]}, \quad L_{abc} = \text{Ł}_{abc} + (g_{ca}T_b - g_{cb}T_a) / 3$$

with $T_b=g^{kc}\text{Ł}_{bkc}$ being the removed trace. ($\mathcal{P}_{ab}$ is the "sign flip" of the metric along the ξ direction.)

If ξ^a is tensorial (and everything is smooth enough to take all the derivatives asked for), then so are all these derived quantities. Moreover L_{abc} carries *every* Lanczos algebraic symmetry exactly: antisymmetry in the first index-pair, trace-freeness, and the cyclic identity – including for wildly non-Killing ξ. The resulting **"Lanczos-candidate"** tensor L_{abc} thus *resembles* a true traceless Lanczos tensor, except that it usually *fails* to generate that metric's W_{abcd}. **It depends on ξ only through the "unit line field"** in the sense that $\xi_a\xi_b/(\xi_k\xi^k)$ is *unchanged* by $\xi\to f\xi$; only ξ's *direction* matters. Writing $u_a=\xi_a|\xi_k\xi^k|^{-1/2}$ and $\varepsilon=u_au^a=\text{sgn}(u_au^a)=\pm 1$, the construction is simply $\mathcal{P}_{ab}=g_{ab}-2\varepsilon u_au_b$.

Why the VAS identity must hold - the closed form. Differentiating $\mathcal{P}_{ab}$ gives

$$\text{Ł}_{abc} = [\ u_{c;a}u_b - u_{c;b}u_a + u_c(u_{b;a} - u_{a;b})\ ]\ \varepsilon$$

Then Ł_{abc}, and hence L_{abc} (since trace removal only adds terms in g_{ab} and traces), is *bilinear* in u_a and $u_{a;b}$. (No curvature, second derivative, or Killing condition involved.) So $L_{abc}L^{abc}$ is a quadratic form in $u_{a;b}$ with coefficients built from u_a and g_{ab}. But $u_{a;b}$ decomposes irreducibly as

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + h_{ab}\theta/3 + u_bA_a\varepsilon$$

into shear, vorticity, expansion and acceleration (soon to be defined) and those four are all there is. Hence $L_{abc}L^{abc}$ **cannot** be anything other than a fixed linear combination of A_aA^a, $\omega_{ab}\omega^{ab}$, $\sigma_{ab}\sigma^{ab}$ and θ^2. Then the only work left is to find four numbers.

The four coefficients.

$$L_{abc}L^{abc} = A_aA^a\,4/3 + (6\,\omega_{ab}\omega^{ab} + 2\,\sigma_{ab}\sigma^{ab})\,\varepsilon$$

with $\varepsilon=u_au^a=\pm1$, so that the timelike case has −6 and −2 and the spacelike case +6 and +2, while the acceleration coefficient 4/3 and the vanishing of the θ^2 coefficient are the same in both. We'll derive a k-field version of this later.

Definitions of the four kinematic quantities. The coefficients depend on these, so they are given in full. Write $u_a = \xi_a|\xi_k\xi^k|^{-1/2}$ for the unit line field, $\varepsilon=u_au^a=\pm1$, and

$$h_{ab} = g_{ab} - \varepsilon u_au_b \quad (\text{ so } h_{ab}u^b = 0 \text{ and } h^a{}_a = 3\ )$$

for the projector into the 3-space orthogonal to u_a. Then

expansion	$\theta = u^a{}_{;a}$
acceleration	$A_b = u^au_{b;a}$
vorticity	$\omega_{ab} = h_a{}^ch_b{}^du_{[c;d]}$
shear	$\sigma_{ab} = h_a{}^ch_b{}^du_{(c;d)} - h_{ab}\theta/3$

so that $A_bu^b = 0$, $\omega_{ab}u^b = 0$, $\sigma_{ab}u^b = 0$, $\sigma_{ab}g^{ab} = 0$, and

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + h_{ab}\theta/3 + u_bA_a\varepsilon$$

The invariants $\omega_{ab}\omega^{ab}$ and $\sigma_{ab}\sigma^{ab}$ are taken with no factor of ½.

Relation to Novello and Velloso. These are, deliberately, the definitions of Novello & Velloso 1987 – the paper whose observer-based construction supplies several of the Lanczos tensors in the Companion Paper's big table – as set out in their EQ2.5a-d. They write vorticity by antisymmetrizing the two projectors, $\omega_{ab} = h_{[a}{}^ch_{b]}{}^du_{c;d}$, and shear by symmetrizing them; because h_{ab} is symmetric those are identically the same as the table's forms, which move the bracket onto the derivative instead. Our expansion θ and acceleration agree with theirs.

The one real difference is metric signature. N&V use +−−− with a timelike unit field normalized to $V_aV^a=+1$, so their timelike case is $\varepsilon=-1$ in the convention here, and their projector $g_{ab} - V_aV_b$ is the h_{ab} above. Nothing else differs, and no coefficient in this section depends on the choice.

How this affects/explains the "law" that $3L_{abc}L^{abc}=4A_aA^a$. A hypersurface-orthogonal Killing field has $\omega_{ab}=0$, and any Killing field is rigid so $\sigma_{ab}=\theta=0$. Only A_aA^a survives, so that "law" is recovered exactly. But more generally, the VAS identity now covers the entire Kerr observer family rather than merely one distinguished observer, and covers arbitrary "congruences" inside any spacetime. Where we previously had to pick a preferred observer to make the statement true, we can now state the VAS identity and let the observer be whatever the reader wants. We'ss soon extend this further, to base it on any number k of

mutually-commuting line fields.

The **purpose** of $3L_{abc}L^{abc}=4A_aA^a$ in Switcheroo Gravity is to allow acceleration A^a to come out of the theory so that that can be used to produce MOND-like effects. That all is bolstered by the VAS identity. But since the VAS identity holds for any unit line field: Could SG be rebuilt on an *arbitrary* line field rather than on L_{abc}, causing it to become far more arbitrary and non-unique? No. The trouble with trying to create such a large generalization of SG is physics does not normally hand you any observer-independent line-fields. To build a gravity theory on one you must postulate that it arises as some extra dynamical field, which turns the theory into gravity-plus-vector-field reminiscent of the TeVeS / Einstein-aether type we've attacked. By contrast, the Lanczos tensor just always exists. So the VAS identity is *not* a route to a wider class of MOND-capable physics theories. Rather, its value is because it explains *why* the Lanczos tensor was able to do that job.

General-k VAS identity based on k commuting line fields. In our previous discussion (re the "Gram matrix") given k mutually commuting, mutually independent Killing vectors $K^I{}_a$, we had formed the Gram matrix $G^{IJ} = K^I{}_a g^{ab} K^J{}_b$ and the projector

$$Z_{ab} = \sum_I \sum_J (G^{-1})_{IJ}\, K^I{}_a K^J{}_b, \qquad \mathcal{P}_{ab} = (g_{ab} - 2Z_{ab})\, k^{-1}$$

onto the k-plane. That Z_{ab} is the Z_{ab} used throughout this section – same object, same letter. Nothing here needs the fields to be Killing; we only need that they commute, so that Z_{ab} projects onto an integrable k-plane distribution. The k=1 case is $Z_{ab}=\varepsilon u_a u_b$.

The identity. Everything below is written in terms of Z_{ab} and its covariant derivative $Z_{ab;c}$. The result is

$$6\, k^2\, L_{abc}L^{abc} = 24\, Z_{c[a;b]}Z^{c[a;b]} - 4\, Z^a{}_{b;a}Z^{cb}{}_{;c}$$

Four-line Proof.

1. Since $g_{ab;c} = 0$, $Ł_{abc} = \mathcal{P}_{c[a;b]} = -2k^{-1}Z_{c[a;b]}$. So $Ł_{abc}$ is *linear* in $Z_{ab;c}$.
2. Hence $Ł_{abc}Ł^{abc} = Z_{c[a;b]}Z^{c[a;b]}\, 4k^{-2}$.
3. The trace vector is $T_b = g^{kc}Ł_{bkc} = -k^{-1}(Z^a{}_{b;a} - Z^a{}_{a;b}) = -k^{-1}Z^a{}_{b;a}$, because $Z^a{}_a = k$ is constant.
4. Trace removal $L_{abc} = Ł_{abc} + (g_{ca}T_b - g_{cb}T_a) / 3$ gives $L_{abc}L^{abc} = Ł_{abc}Ł^{abc} - T_bT^b\, 2\cdot 3^{-1}$. Substituting steps 2 and 3 gives the identity. **Q.E.D.**

A lemma worth recording. Because $Z^a{}_cZ^c{}_b=Z^a{}_b$, differentiating gives $Z^a{}_cZ^c{}_{b;d} + Z^a{}_{c;d}Z^c{}_b = Z^a{}_{b;d}$. Contracting that with $Z_{ab;c}$ and using the symmetry of Z_{ab} yields

$$2\, Z^{ad}Z_{ab;c}Z_d{}^{b;c} = Z_{ab;c}Z^{ab;c}.$$

This is what collapses the apparently five-term quadratic form one first meets to the two terms above, and it is the only place the projector property – as opposed to mere symmetry – is used.

Recovering the k=1 kinematics. Put $Z_{ab} = \varepsilon u_a u_b$ with $\varepsilon=u_au^a=\pm1$. Then $Z_{ab;c}=(u_{a;c}u_b+u_au_{b;c})\varepsilon$ and $Z^a{}_{b;a}=(\theta u_b+A_b)\varepsilon$, since $u_au^{a;c}=0$. Feeding those into the general-k VAS identity, and then substituting

$u_{a;b}=\sigma_{ab}+\omega_{ab}+h_{ab}\theta/3+u_bA_a\varepsilon$, every cross term dies due to $u_aA^a=0$ whereupon the four coefficients drop out. The θ^2 terms cancel between the three contributions – which is why the expansion coefficient is exactly zero rather than merely small.

When is the candidate L_{abc} a *genuine* Lanczos tensor?

We numerically tried the Lanczos-candidate built from ∂_t (or the analogous natural field), for some specific metrics we'd tabulated, asking whether it yielded a multiple λW_{abcd} of the Weyl tensor W_{abcd} for that metric. The answers were:

spacetime	Petrov type	image = λW_{abcd}	apparent verdict
Schwarzschild	D	$\lambda = 1.00000$, residual 2×10^{-16}	genuine Lanczos tensor
Kottler (Schwarzschild-de Sitter)	D	$\lambda = 1.00000$, residual 2×10^{-16}	genuine
Reissner-Nordström	D	$\lambda = 0.98246$	genuine
Janis-Newman-Winicour	D	$\lambda = 1.05986$	genuine
Kasner vacuum	I	$\lambda = -3.00000$	genuine
Gödel's universe	D	$\lambda = 9.00000$	genuine, despite $\omega_{ab}\omega^{ab} = 1.0$
Melvin magnetic universe	D	$\lambda = 0.20635$	genuine
arbitrary static, non-Einstein	-	$\lambda = 0.65436$	genuine
Morris-Thorne wormhole	D	$\lambda = 0.00000$	degenerate: generates nothing while $W \neq 0$
Kerr, a = 0.6	D	residual 0.19	candidate only
Kerr, a = 0.9	D	residual 0.62	candidate only
pp-wave, circular polarization	N	residual 1.0	candidate only
FRW, Bertotti-Robinson	O	$W = 0$	vacuous

Two observations. **Rotation is not the obstruction**: Gödel's ∂_t has $\omega_{ab}\omega^{ab}=1.0$, an order of magnitude more vorticity than Kerr at a=0.9, and the construction works there and fails in Kerr. **Petrov type is not the obstruction either**: type D appears on both sides of the table, and Kasner is type I and works. The two clear failures – Kerr and the pp-wave – are the two spacetimes here with twisting or radiative structure, but that is a description rather than a criterion (we have no criterion); and we were able to obtain a genuine Lanczos tensor for the circular-polarized pp-wave by using a different ("helical") Killing vector.

Perturbing ξ away from the natural field destroys genuineness immediately in every case: for

Schwarzschild, $\partial_t+0.05\partial_\varphi$ already gives residual 0.72. So genuineness happens only for specific line fields, not generic ones – while the kinematic identity of §3 holds for all of them indifferently.

Is the image a linear combination of W_{abcd} and W^*_{abcd}? The answer usually is "no."

17. Should (and could) other tensors be involved in $\vec{F}$?

I computed the formulas for many **curvature-invariants** (including Carminati & McLenaghan's algebraically-complete set) in the Kottler, aka Λ-Schwarzschild, metric:

$$R = 4\Lambda,\quad G_{ab}G^{ab} = 4\Lambda^2,\quad W_{abcd}W^{abcd} = {}^*W_{abcd}{}^*W^{abcd} = 48M^2r^{-6},\quad W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab} = 96M^3r^{-9},$$
$${}^*W_{abcd}W^{abcd} = {}^*W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab} = 0,\quad T_{abcd}T^{abcd} = 576M^4r^{-12}$$

and $S_{ab}=0$ everywhere $r\neq0$ so that all the Carminati-McLenaghan-Zakhary invariants involving it are 0. (Indeed, $S_{ab}=0$ in any Λ-vacuum.) All of these computations are simply a matter of plugging the Λ-Schwarzschild metric into the definitions of these tensors – highly laborious, but straightforward and amenable to computer aid. Note that all the invariants so far happen to be functions of *just two* of them, e.g. R and $W_{abcd}W^{abcd}$. I also computed the Bach tensor (all components are zero) and the "Karlhede invariant"

$$W_{abcd;e}W^{abcd;e} = R_{abcd;e}R^{abcd;e} = 240M^2r^{-9}\,(3r-\Lambda r^3-6M).$$

Question: which of these invariants (or linear combinations, or simple polynomial combinations) are *continuous* across the gluing-surface in the two-Kottler glued metric?

Answer: The following invariants (or certain "szygzy" combinations thereof) obviously are continuous because *identically zero*:

Invariants: ${}^*W_{abcd}W^{abcd}$, ${}^*W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab}$, the Bach invariants, and every S_{ab}–based invariant.
Szygzies: $R^2 - 4G_{ab}G^{ab}$, $(W_{abcd}W^{abcd})^2 - 4T_{abcd}T^{abcd}$, $(W_{abcd}W^{abcd})^3 - 12(W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab})^2$.

Of the six *non*vanishing invariants, *none* are continuous on their own: R and $G_{ab}G^{ab}$ jump with Λ; $W_{abcd}W^{abcd}$, the cubic Weyl scalar, and $T_{abcd}T^{abcd}$ jump with M; the Karlhede scalar jumps with both. Furthermore, the *only* linear and indeed polynomial combinations (with constant coefficients) of those invariants which stay continuous are the obvious ones that are identically zero. To prove that in the linear case is easy (simultaneous linear equation solve).

Proof that no non-trivial *polynomial* of those invariants is continuous: the continuous datum $f=1-2Mr^{-1}-\Lambda r^2/3$, whose continuity is *equivalent* to the continuity of the metric across the gluing surface, contains a term linear in M, whereas every invariant contains M only as M^2, M^3, M^4 or $M^2\Lambda$, i.e. no "lone-M" invariant exists. Therefore f cannot be built polynomially. **Q.E.D.**

So anything continuous (if nontrivial) intrinsically must be a *non*polynomial, for example rational, function of those invariants. And indeed, the following – each arising from a ***quotient* of two *separately jumping* invariants** – are nontrivial continuous zero-indexed tensors:

$$I_1 \equiv (W_{abcd;e}W^{abcd;e}) / (W_{abcd}W^{abcd}) = 15\, f\, r^{-2},$$
$$I_2 \equiv (R/12) + (W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab}) / (W_{abcd}W^{abcd}) = (1-f)\, r^{-2}.$$

Ineligibility Theorem: Under the conjecture that the meridian curve defining the shape of the gluing surface for the two-Kerr-dS glued metric-pair is **non-algebraic** (aka transcendental): *None* of the invariants listed so far this section, nor any linear, or polynomial, or rational, or algebraic, *combination* of them, is continuous across the switcheroo surfaces in all three of the two-Kottler, two-RNdS, and two-Kerr-dS glued metric-pairs, *except* for "trivial" invariants, meaning ones that are identically constant throughout all three metrics. Furthermore, no nontrivial function $F(I_1, I_2)$ is thus-continuous.

WARNING: This proof was devised at a time when I incorrectly thought the two-Kerr gluing surface was defined by a transcendental curve. But later, we found a closed formula describing its shape using a quartic algebraic curve. Therefore, the part of the present proof depending on curve transcendence, is no good. I state it nevertheless because

a. I feel it is educational,
b. quartic-algebraic is complicated enough that for practical purposes, it still excludes a vast number of candidates, probably including all "nice" candidates,
c. The proof-idea here probably *does* work if we consider more-generic switcheroo-surfaces which really are transcendental. Plenty of such surfaces presumably exist, the problem is the great difficulty of finding explicit examples and proving them transcendental. Indeed: applying the Christensen-Hunt-Sauer-Yorke theory of **prevalence/shyness**, the known fact that $f(x)+e^x a$ is an algebraic function(x) for at most one real value of a already suffices to imply that algebraic functions are shy within both $C^\infty[0,1]$ and the real-analytic functions on [0,1]. We'll discuss that later.

Proof: If we test all those invariants, and the two-Kottler-continuous scalars I_1, I_2, for continuity across the gluing sphere in the two-Reissner-Nordström-deSitter glued metric (not a Λ-vacuum due to the presence of an electric field; three metric-defining parameters M, Q, Λ all jump as we cross the switcheroo-sphere), we find that the following are continuous because identically zero on both sides:

$$W^*_{abcd}W^{abcd}, \quad W^*_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab}, \quad (W_{abcd}W^{abcd})^2 - 4\, T_{abcd}T^{abcd},$$
$$(W_{abcd}W^{abcd})^3 - 12(W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab})^2.$$

while all S_{ab}-based invariants, Bach-based invariants, $R^2 - 4G_{ab}G^{ab}$, and I_1 and I_2 jump.

Finally, if we (numerically) test the Kottler-continuous scalars for continuity across the non-spherical switcheroo surface we found earlier for gluing two Λ-Kerr metrics, we find that the *only* continuous ones are those that are identically zero in every Λ-vacuum, namely:

All the CMZ-invariants based on S_{ab} (since under the Einstein field equations $S_{ab}=0$ in any Λ-vacuum), the Bach-based invariants (since $B_{ab}=0$ in any Λ-vacuum), and R^2-$4G_{ab}G^{ab}$.

The **set-intersection** of the sets of (our above) invariants continuous for two-RNdS *and* continuous for two-Λ-Kerr is **the empty set!**

Furthermore, two-Kottler gluing by itself already proved that no polynomial combination of the above invariants is continuous across all three switcheroo-surfaces, unless it is "trivial," i.e. identically zero or constant, for all three metrics.

Basically the reason for the empty set-intersection is that the Kottler-survivors are governed by which symmetry each of the two Kottler-metric-generalizations preserves: Kerr preserves the Λ-vacuum property, while RNdS preserves spherical symmetry, and each keeps exactly the invariants protected by the property it retains. Furthermore, under our **conjecture** that the meridian-curve defining the shape of the switcheroo surface for the two-Λ-Kerr glued metric is **transcendental**, i.e. not an algebraic curve (for example if it were described by a formula involving "elliptic integrals") we can prove that **no rational (indeed, more generally, no algebraic) combination** of the above invariants is continuous across all three switcheroo-surfaces (unless trivial).

Proof: Across the Kerr glue-surface numerical checking shows *every* non-constant invariant jumps, e.g. R=4Λ jumps since the two regions have different Λ; and every Weyl-sector invariant jumps. You might hope that some nontrivial rational combination of these would "cancel out" all the jumps to enable continuity – but under our transcendence conjecture, that cannot happen. (Proof: if it did happen, then the equation saying "jump cancels to zero" would be an algebraic description of the gluing meridian-curve, contradicting its transcendence.) So the only rational-combinations continuous across the Λ-Kerr gluing are

$$\text{The } S_{ab}\text{-based and Bach invariants, } R^2 - 4G_{ab}G^{ab}\text{, and combinations thereof –}$$

which all happen to vanish identically for all Λ-vacuum spacetimes. But all of those jump across the RNdS gluing. Now consider the RNdS gluing using electric charge Q. We want to prove this **lemma**: The only combinations, polynomial, rational, or algebraic, that are continuous across the Reissner-Nordström-de Sitter glue-sphere are those that vanish identically on the RNdS family.

Proof of lemma: The rational and algebraic problems can be reduced to the polynomial-combination problem. Namely: a rational function is zero if and only if its numerator (polynomial) is zero. An algebraic function is zero if and only if the product of all its algebraic conjugates (another polynomial) is zero. So now all we need to do is to prove that no nontrivial *polynomial* combination Ψ of the still-eligible invariants can be continuous across the RNdS glue-sphere surface. By considering varying RNdS's charge-parameter Q, we see that the only way that could happen is for Ψ to depend on (M, Λ, Q) *only* through f. So Ψ must be a function of f and r only. Fix (f, r) and vary the charge Q, adjusting M or Λ to hold f fixed. Along this path Ψ remains constant. A constant equal to its own Q→0 limit (namely 0 in the Kottler Λ-vacuum special case of RNdS) must be 0. So Ψ=0 at every (f, r), i.e. Ψ vanishes identically on the RNdS family. So the postulated polynomial-combination of invariants must be identically zero for all Λ-vacuum spacetimes, and for every member of the RNdS family of spacetimes containing electric fields too. So it is one of the universal identities already 0 for all three metric classes, not a new survivor.

Functions $F(I_1, I_2)$.

Let $(A,B)=(I_1, I_2)$ for brevity. On Kottler (A,B) is an *invertible chart for the continuous data itself*:

$$(A, B) = (15\, f\, r^{-2}, [1-f]\, r^{-2}),$$

so the Kottler-gluing junction does not move (A,B) at all and *every* F(A,B) is continuous. That is precisely the property that adding charge destroys.

Both A and B have irreducible charge terms inside them: $B = (1-f)\, r^{-2} - Q^2\, r^{-4}$, and likewise $(W_{abcd;e}W^{abcd;e}) / (W_{abcd}W^{abcd}) - 15\, f\, r^{-2}$ are proportional to Q^2. The specific linear combination A+15B that equals $15r^{-2}$ becomes $r^{-2}+Q^2(\ldots)$ in RNdS, hence vanishes only when Q=0. Since Q jumps, all are discontinuous across the RNdS gluing. and each also jumps across the Kerr gluing.

On the RNdS gluing-sphere, at fixed continuous data (f, r), the pair sweeps a full **two-dimensional** region as the junction parameters vary. B collapses to depend on the charge alone,

$$B = (1-f)\, r^{-2} - Q^2\, r^{-4} \quad \text{(independent of } \Lambda\text{)}, \quad \partial_\Lambda A \neq 0,$$
$$\partial(A, B) / \partial(Q, \Lambda) = -72\, Q^3\, f\, (\Lambda\, r^4 + 5Q^2 + 3f\, r^2 - 3r^2) \cdot [r^2(\Lambda\, r^4 + 3Q^2 + 3f\, r^2 - 3r^2)^3]^{-1} \neq 0.$$

A function F(A, B) is continuous across a glue-surface only if constant on the set of (A, B)-values the junction connects. On Kottler that set is a single point; on RNdS it is a 2-D region — so only a *constant* F can be continuous. The obstruction is *dimensional*, not a matter of function class: no choice of F, from any subclass, survives once the image is two-dimensional. The pair has gone from a chart for the continuous (f, r) to two independent jumping directions — B carrying a bare $-Q^2\, r^{-4}$, and A an *independent* Λ-dependence — so no combination of them can live on the continuous data alone.

Q.E.D.

Conclusion. For SG purposes, *only* the Lanczos invariant-square $L^{abc}L_{abc}$ and $L^{abc}L^*_{abc}$ (among the invariant scalars we've examined) are allowed as $\vec{\mathrm{F}}$-components. *None* of the curvature-based invariants and their combinations that we've considered, is allowed.

Critisism. Unfortunately, as we warned, we later refuted our conjecture that the Kerr-gluing surface is transcendental. Also, all three metrics used in the proof above were in the Kerr-Newman-deSitter black hole family, all of which are Petrov type D (algebraically special) and exact Einstein-Maxwell "Λ-electrovac" solutions. So any curvature-invariant that is a "Petrov-D detector" i.e. is constant on Petrov-D metrics, but nonconstant on metrics with some other Petrov type, would still be permitted. (Also any that is a "Λ-electrovac detector" would similarly be permitted, although that prospect bothers me less.) However, our proof would be resuscitated and immunized against those criticisms if we had any switcheroo surface that really was transcendental, for some metric family of fully general Petrov type I. So...

Presumably, generic switcheroo surfaces are *not* describable as algebraic surfaces. Here is a candidate metric for possibly proving that: Consider the Bianchi-I anisotropic-cosmology model which is an exact Λ-vacuum Einstein solution

$$ds^2 = -dt^2 + a_1(t)^2dx^2 + a_2(t)^2dy^2 + a_3(t)^2dz^2.$$

where

$$a_j(t) = \sinh([3\Lambda]^{1/2}t)^{1/3} \tanh(\, [3\Lambda]^{1/2}\, t/2\,)^{(p_j - ⅓)}.$$

and where the three **Kasner constants** p_1, p_2, p_3 obey

$$p_1+p_2+p_3 = (p_1)^2+(p_2)^2+(p_3)^2 = 1.$$

This metric was given by Pereira, Pitrou, Uzan 2007 then again by Furuya, Niiyama, Sendouda 2019. It reduces to the ordinary "Kasner vacuum"

$$ds^2 = -dt^2 + t^{(2p_1)}dx^2 + t^{(2p_2)}dy^2 + t^{(2p_3)}dz^2$$

in a rescaled-time limit when $\Lambda \to 0$ (both metrics have fully-general Petrov type I), and in the far future becomes de Sitter spacetime.

Note the presence of transcendental functions such as sinh and tanh, and also the fact that the Kasner constants p_j can be transcendental irrational numbers. So if two such metrics with different Λ were glued across some switcheroo surface, it would be surprising if that surface were not transcendental. Assuming transcendence, we would conclude that **the *only* scalar invariants algebraically constructible from the metric g_{ab}, its inverse g^{ab}, and their first- and second-order partial derivatives, permissible as components of $\vec{\mathrm{F}}$ in switcheroo gravity, are "szygzy" combinations which are identically constant for all Λ-vacuum metrics – i.e. useless.**

And one presumably could, by also considering suitably trancendental nonvacuums, also rule out "vacuum detector" invariants. Another point is that the number of transcendental analytic functions has the cardinality of the continuum, while algebraic function *forms* (with undetermined coefficients) are countable. This contrast strongly suggests that switcheroo surfaces should generically be transcendental.

If so, curvature-based tensors then could only play a role in $\vec{\mathrm{F}}$ if they only occur *combined* with the Lanczos tensor.

Switcheroo-surface transcendence conjecture. Switcheroo surfaces exist which are not describable algebraically, i.e. are transcendental. Indeed I would conjecture that generically happens.

Summary. Our investigation has not been fully satisfactory since it needed to rest on unproven transcendence conjectures. But those conjectures are highly plausible. Under them, the result of this continuity investigation has been remarkably simple: *no* nontrivial curvature-invariant simply-constructible from among those investigated here (which includes the full Carminati-McLenaghan-Zakhary set) exists that is continuous across all switcheroo-surfaces. More precisely, the only algebraic combinations made from them, that are continuous across all switcheroo surfaces, are the trivial ones:

- constants
- universal algebraic identities that are identically zero for every metric.

And while there were two clever nontrivial rational functions that managed to be continuous for two-Kottler, they do not survive the other two glued-metrics we investigated.

This indicates that **SG is a remarkably *unique* theory** – much more than I initially expected.

18. GR could be reformulated to be based on Lanczos (or Superpotential) & Ricci instead of metric tensor

[This section contains the statement of the six-foundations theorem extracted almost verbatim from the Companion Paper – Smith & CLAUDE 2026.]

Normally in the past, the **metric tensor** has been regarded as *the* fundamental object for manifolds in differential geometry and general relativity: everything else is defined in terms of g_{ab}. This part proves a theorem showing that this is not the only possibility — alternative “foundations” for differential geometry exist. Of particular interest for us is the fact that the **Ricci-Lanczos team** $\{R_{ab}, Ł_{abc}\}$ provides one such alternative foundation.

Concretely: suppose you had a black box which, given four coordinates on your manifold as input, returned the Ricci-Lanczos team — for any one particular choice of Lanczos tensor $Ł_{abc}$. Then, the

theorem shows, you could build from it a different black box that instead returns a metric tensor g_{ab} for the same manifold, or one isometric to it.

Two remarks on that statement. The reconstruction is *exact* as mathematics: it is the unique solution of a linear ordinary differential equation, not a limit of approximations. But the theorem also is **constructive** — the proof yields an algorithm for reconstructing the metric to arbitrarily high accuracy and which we implemented and ran as a check on the theorem. And the phrase *for any one particular* Ł matters: a metric fixes Ł only up to Lanczos gauge, so this correspondence is many→one, which is why the Lanczos-based foundations turn out to be strictly richer than the metric rather than merely equivalent.

The six-foundations theorem: On a connected $C^{1,1}$ (pseudo-)Riemannian four-manifold each of the following objects determine the metric, and therefore each other:

#	foundation	status
1	the metric g_{ab}	unconditional
2	the Riemann tensor $R^a{}_{bcd}$	unconditional
3	the Levi-Civita connection $\Gamma^\lambda{}_{\mu\nu}$	unconditional
4	**Team A** = $\{R_{ab}, W_{abcd}\}$ = Ricci-Weyl pair	unconditional
*5	**Team B** = $\{Ł_{abc}, R_{ab}\}$ = Lanczos-Ricci pair	unconditional; 4D only; **strictly richer**
*6	**Team C** = $\{Ƥ_{ab}, R_{ab}\}$ = Superpotential-Ricci pair	4D only; conditional on $Ƥ_{ab}$ existing; **strictly richer**

“Determines” is understood up to diffeomorphism freedom, with the metric and its first derivatives fixed at one seed point p — equivalently $g_{ab}(p)=\eta_{ab}=\mathrm{diag}(-1,+1,+1,+1)$ [or diag(+1,+1,+1,+1) in the euclidean case; any signature ok], $\Gamma^a{}_{bc}(p)=0$.

Two addenda:

1. **Analyticity.** If the input data is real-analytic, so is the reconstruction – not merely along the rays of the normal-coordinate patch but across them, so that g_{ab} is an analytic function of position on the patch.
2. **Its domain.** The reconstructed $g_{ab}=\eta_{ij}A^i{}_a\,A^j{}_b$ satisfies $\det(g_{ab})=-\det(A)^2$, so it carries the seed signature **exactly where A is invertible**. The theorem's domain is therefore delimited by the vanishing of det A – the conjugate points of the geodesic congruence – and by nothing else. There the metric is degenerate and $[-\det(g_{ab})]^{1/2}$ fails to be analytic.

Interchangeable completions. The Ricci half of Teams A, B and C may be replaced by any tensor carrying the full ten components of Ricci, such as the Einstein and Schouten tensors G_{ab} and Q_{ab}, but *not* by the traceless Ricci tensor S_{ab}.

The Lanczos foundations are not equivalent – they are strictly richer.

The tables stars (*) teams B and C because their relation to the metric is **asymmetric**. The gauge freedom of a Lanczos potential lies in the kernel of the Lanczos→Weyl operator, so it does not affect the reconstructed Weyl tensor and hence not the reconstructed metric: **Ł determines g**. But the converse fails — g does not determine Ł, only its equivalence-class under re-gaugings. The map Ł→g is a surjection, not a bijection.

The same holds for $Ƥ_{ab}$, whose own gauge freedom lies in the kernel of the composite map Ƥ→W.

So Teams B and C are not *equivalent* to the metric foundation; they are richer. **They can express distinctions the metric cannot see.** A starred foundation carries strictly *more* information than g_{ab}.

19. How should SG handle "Lanczosian gauge freedom"?

Six possibilities:

1. One possible approach to uniquify the Lanczos tensor would be **variational**: among all the too-many L_{abc} permitted by Lanczosian gauge freedom, hypothesize that physics uses one that (presumably generically uniquely) minimizes some sort of "cost" functional. Admittedly, I do not know what that functional ought to be... perhaps $\int|Ł^{abc}Ł_{abc}|^2d^4V$... but it might not actually matter very much.

2. Another possible approach would be to actually impose some **specific gauge-fixing demand**, such as (perhaps) Lanczos's suggestion $L_{abc}{}^{;c}=0$, which combined with suitable boundary conditions ought to make L_{abc} unique.

3. Although SG is not trying to be a quantum-gravity theory, there presumably is some unknown quantum-gravity theory underlying everything. So now is a good time to ask: "what would quantum field theory *want* SG to do about Lanczosian gauge freedom?" One thing QFT might want, is for that gauge freedom *not* to be completely eliminated as in suggestions 1 & 2. QFT-theorists *like* gauge freedom. It historically has been extremely powerful and helpful for them. That is why, for example, in Peskin & Schroeder's quantum field theory book, they claim that if A^u is the Maxwell 4-vector field, then A^uA_u is *not permitted* to arise as part of the laws of physics, even though it is relativistically acceptable, i.e. a tensor. That's because A^uA_u is gauge-dependent, but Peskin & Schroeder insist the laws of quantum field theory be invariant under a large gauge group. Motivated by that analogy, one could consider, in suggestion (2), merely imposing demands that only *partially* fix the Lanczos gauge. Then a subgroup of the original gauge-freedom-group will remain present, permitting QFT-theorists to be happier. I.e:

"Quadratic-constrained gauge freedom" compromise suggestion: Consider as a starting "base point," the L_{abc} that obeys the Lanczos algebraic and differential gauge conditions (or some other starting point could be considered, we just choose this one for concreteness). Suppose the full Lanczos gauge group is that adding any infinitesimal multiple of Φ_{abc} to L_{abc} preserves its validity as a Lanczos tensor, for some large set of allowed Φ_{abc}. Now if Φ_{abc} obeys the scalar identity-constraint $\Phi_{abc}L^{abc}=0$, then we still get a valid Lanczos tensor which still has the *same* $L_{abc}L^{abc}$ everywhere as it did before. This is a 1-parameter linear restriction at each point, i.e. decrementing the dimensionality d of the gauge group, at each point, by one. Repeatedly doing such infinitesimal perturbations still defines an enormous gauge group – everywhere there are d-1 infinitesimal functions worth of gauge freedom instead of d. Since tracefree Lanczos with differential-gauge left free would have d=6, this still with d reduced to 5 leaves a lot of gauge freedom. [Note: in the case of Maxwell electrodynamics, there is only *one* function worth of gauge

freedom, so that if we had tried the analogous constraint-trick to try to allow A^uA_u to occur in electrodynamics, we would have eliminated gauge freedom completely. I.e. this trick can work for SG, but would have been unacceptable, or anyhow not different from plain gauge-fixing, in the context of Maxwell electrodynamics.]

We similarly have proposed two possible LG schemes involving partially fixing the gauge of L_{abc}, removing 4 of its 6 degrees of gauge freedom per spacetime point.

4. Another possible partial-gauge-fix demand that might be interesting is to demand that L_{abc} be **dual-orthogonal**: $L^{abc}L^*_{abc}=0$. Since this and $L_{abc}{}^{;c}=0$ actually are *tensorial* demands, I'm not sure it is right to call them "gauge fixing." We could just regard these demands as part of the theory, or part of the definition of what a permitted Lanczos tensor *is*. But whatever you call it, demanding dual-orthogonality would remove 1 degree of gauge-freedom.

5. Multiple Lanczos tensors. If Lanczosian gauge freedom makes Lanczos tensors non-unique, maybe the right thing to do is to accept that: spacetimes can simultaneously have more than one Lanczos tensor. We could have, for example, two, *both* of which are involved in the laws of physics (as in our prior quotient detector). Or possibly even the *full set* of *all possible* Lanczos tensors could be involved, for example via some sort of integration over all of them.

6. This is the option I currently prefer as a priori "mathematically nicest": Demand that $L_{abc}=P_{c[a;b]}$ arise from a symmetric **superpotential**. This should generically eliminate *all* Lanczosian gauge freedom from L_{abc}, leaving only the Einsteinian (i.e. change-of-coordinates) gauge freedom that GR already had. The superpotential-based option has several nice features:

a. It eliminates the undesired gauge freedom *without* actually making any gauge-fixing demand – it just springs naturally from the structure of the theory. It seems the most "parsimonious" option, in the sense that if we demand some artificial Lanczos-gauge-fixing, or variational minimization principle, then that is proposing a new law of physics, which might have considerable arbitrariness to it. If all we ask is thet Lanczos arise from a superpotential, that seems non-arbitrary and not really an additional law of physics, or not much of one, anyhow.
b. The Einsteinian gauge freedom that remains, which has long seemed the "natural" gauge group for gravity, would not be expanded.
c. The superpotential seems nice for possible future gravity+quantum unification attempts. Indeed it can behave in ways similar to the way the metric and Einstein tensor behave.
d. On 4-dimensional manifolds, Superpotential→Lanczos→Weyl chains apparently (although we have not proven it) *always exist* for traceless Lanczos tensors, even if we additionally demand $P_{ab}{}^{;a}=0$. But the Companion Paper proves these chains can fail to exist if dimension≥5. So superpotential-based gravity "explains" why spacetime is 4-dimensional.
e. And even if they do not always exist – for rare exceptional "bad" metrics – then still due to the "six foundations" approach, one could base everything on the superpotential+Ricci "team" foundation, whereupon the metric will be *derived* so that physics would only employ the "good" metrics. This would be very interesting.
f. Indeed, we've seen that superpotentials exist for all weak-field scenarios, all Novello-Velloso static perfect fluid scenarios, and the full family of (non-extreme) Kerr-Newman-deSitter black hole metrics, plus extreme-Reissner-Nordström – and presumably plenty of them, given that we found 1-parameter family of superpotentials for each Kottler spacetime.
g. If a superpotential P_{ab} really exists for traceless Lanczos tensors *and* with $P_{ab}{}^{;a}=0$, that means it can be used by LG.

The table below summarizes the superpotentials and corresponding Lanczos tensors for our most important metrics. For every one of the tabulated metrics, a tracefree Lanczos L_{abc} exists obeying both $L^a{}_{ba}=0$ and $L_{abc}{}^{;c}=0$ that arises from a superpotential $P_{(ab)}$ [which is *not* necessarily the same as $\mathcal{P}_{(ab)}$, but in all cases tabulated may be derived from it] via $L_{abc}=P_{c[a;b]}$, and such that $L^{abc}L_{abc}=\infty$ on horizons and in $\Lambda=0$ vacuums decays like r^{-2} or faster when $r\to\infty$.

#	superpotential or construction	spacetime	trace-free	div-free	dual-orthog	genuine Lanczos	horizon detector?	$L_{abc}L^{abc}$ when $r\to\infty$
1	{t} sign flip for g_{ab}	Schwarzschild; generalizes to Reissner-Nordström-deSitter	yes	yes	**yes**	**yes**	**yes** - simple pole at r = 2M; $L_{abc}L^{abc}\Delta_r \to \frac{1}{3}$ exactly, independent of θ and of M	$(4M^2/3)r^{-4}$, from the closed form $4M^2[3r^3(r-2M)]^{-1}$
1b	½ × {t,φ} flip	Schwarzschild	yes	yes	**yes**	**yes**	**yes**	$[3\sin^2\theta]^{-1}r^{-2}$ exactly – same spacetime as row 1, different Lanczos & exponent
2	½ × {t,φ} flip	**Reissner-Nordstrom**	yes	yes	**yes**	**yes**	**yes** - simple pole; $L_{abc}L^{abc}\Delta_r \to 0.0718$ at r_+	$[3\sin^2\theta]^{-1}r^{-2}$ exactly; independent of M and Q
3	½ × {t,φ} flip	**Kerr**	yes	yes	**NO**	**yes**	**yes but also detects spin-axis**	$[3\sin^2\theta]^{-1}r^{-2}$; independent of M and a
4	½ × {t,φ} flip	**Kerr-Newman**	yes	yes	**NO**	**yes**	**yes but also detects spin-axis**	$[3\sin^2\theta]^{-1}r^{-2}$; independent of M, Q and a
5	½ × {t,φ} flip	**Kerr-Newman-de**	yes	yes	**NO**	**yes**	**yes but also detects spin-axis**, for both hole and	No decay: $L_{abc}L^{abc} \to -\Lambda/9$,

		Sitter					Hubble horizons bounding accessible static region	independent of θ, M, Q and a
6	$\mathcal{P}_{ab}$ = diag(2f+3kr^2, 3kr^{2}f^{-2}, 0, 0), every C	Kottler	yes	yes	yes	yes	yes - poles at the roots of f	If C≠0: grows as 3C^{2}r^2; If C=0: −4M^{2}Λ^{-1}r^{-6}
7	horizon detector, every C	extremal Reissner-Nordstrom	yes	yes	yes	yes	yes - *second-order* pole at r = M	If C≠0: grows as (C^2/3)r^2; If C=0: (4M^2/3)r^{-4}
8	reflection $\mathcal{P}_{ab}$ = ½g_{ab} + $\hat{u}_a\hat{u}_b$ − $\hat{\varphi}_a\hat{\varphi}_b$, ℓ = 0 to 4	Weyl-distorted ("dirty") Schwarzschild	yes	yes	yes	yes (to 1st order in pertrbtn theory)	yes - pole on x=1; but also an axis wire at y=±1	~ x^{-4} times multipole-dependent numerator
9	½ × {t,φ} flip	**Weyl** static axisymmetric-spacetime class	yes	yes	yes	yes (for non-degenerate horizons) but also detects sym-axis	yes	
10	Novello-Velloso based on time-translation Killing vector	Schwarzschild	yes	yes	yes	yes(−)	yes - simple pole at r = 2M	(4M^2/3)r^{-4}, same spacetime & Lanczos as row 1
11	Novello-Velloso based on *azimuthal* Killing vector	Schwarzschild	yes	yes	yes	yes(−)	**no** - finite at r = 2M; detects the *axis* instead	4[3sin^{2}θ]$^{-1}$r^{-2}
12	Novello-Velloso, time-based Killing	**static perfect fluid**, general	yes	yes	yes	yes(−)	*inapplicable* in the sense fluid interiors have no horizon, but may regard Λ-vacuum as a perfect fluid, in	

	vector						which case always yields horizon detector for non-degenerate horizons	
13	$12\mathcal{P}_{ab} = \eta_{ab}h - 6h_{ab}$	general weak field, de Donder gauge	yes*	yes	yes if static	yes	*inapplicable* - the linearised regime excludes horizons	$\sim M^2r^{-4}$ for a Newtonian source
14	*control*: arbitrary symmetric $\mathcal{P}_{ab}$	Schwarzschild	yes*	**NO**	**NO**	**NO**	*inapplicable*	*inapplicable*

* trace-freedom in these rows is by construction, since the standard $\frac{1}{3}(g_{ca}T_b - g_{cb}T_a)$ trace-removal is applied in every case; it is a check on the arithmetic, not a discovery. And in the "yes(−)" cases, the minus sign is to indicate that Novello-Velloso's Lanczos tensor formula actually generates a *negated* Lanczos tensor according to our MTW sign conventions.

Dual-orthogonality fails for every *rotating* metric tabulated, but occurs for all the nonrotating metrics, i.e. all the static metrics, covered by this table, including the full static axisymmetric Weyl class. (Interesting question: Does a dual-orthogonal Lanczos tensor exist for every static spacetime?] This dual-orthogonality cause $L_{\circ abc}L_{\circ}^{\ abc}$ merely to equal $2L_{abc}L^{abc}$, which is pure-real, for the nonrotating tabulated metrics. But for the rotating cases these two expressions usually differ since the former one has a nonzero imaginary part.
Tabulated superpotentials are never divergence-free: Neither $\mathcal{P}_{ab}{}^{;b}$ nor $P_{ab}{}^{;b}$ vanish in any table row. Arbitrary symmetric superpotentials on the Schwarzschild background produce Lanczos-symmetric tensors with $\nabla^c L_{abc}$ nonzero in several of its sixteen slots. So the $L_{abc}=P_{c[a;b]}$ construction by itself does **not** force divergence-freedom. *Yet* every case in the table, including the general Weyl class, have $\nabla^c L_{abc}=0$. So the table at least makes it *look* as though the **Lanczos differential gauge** is not a gauge choice at all, but rather a consequence.

property	test
trace-free	$g^{ac}L_{abc} = 0$ in all four slots
divergence-free	$\nabla^c L_{abc} = 0$ in all sixteen slots
dual-orthogonal	$L^{abc}L^*_{abc} = 0$
genuine	the Lanczos→Weyl relation is satisfied: the operator applied to L_{abc} returns the actual

	Weyl tensor W_{abcd}

Superpotentials for L versus Ł. $\mathcal{P}_{ab}$ yields Lanczos tensor $Ł_{abc}=\mathcal{P}_{c[a;b]}$ generally with nonzero trace $T_b=Ł_{ab}{}^a$. Its trace-free version L_{abc} has a different superpotential P_{ab}, which might not exist.

The relation, and when P_{ab} exists. Shifting $\mathcal{P}_{ab} \to \mathcal{P}_{ab} + K\,S\,g_{ab}$ for a scalar S changes the Lanczos tensor by $(g_{c[a}\nabla_{b]}S)K$. Matching the standard trace removal $-(g_{ca}T_b - g_{cb}T_a)/3$ requires $K=-2/3$ and $\nabla_b S=T_b$. Hence

$$P_{ab} = \mathcal{P}_{ab} - (2/3)\,S\,g_{ab}, \quad \text{where} \quad \nabla_b S = T_b,$$

which exists ***only* if T_b is an exact covariant gradient**. That is a real condition, not a mere formality. The very existence of P_{ab} is therefore a discriminating property in its own right, and for all the examples tabulated above it tracks "genuine Lanczos" perfectly.

The trace-free Lanczos tensor L_{abc} is divergence-free in every "genuine" case tabulated. But its superpotential is divergence-free in none of them. So evidently **the Lanczos differential gauge does not lift to the superpotential;** whatever causes $\nabla^c L_{abc}=0$ acts on the Lanczos tensor, not on the superpotential it is built from.

---

20. Do stairsteps violate conservation of energy? Nature & philosophy of the cosmical constant Λ

In Einstein's field equations, the cosmical constant term acts just like a gas of matter would act – if it were a very peculiar sort of matter. The "mass-energy density" of this matter is $\Lambda/(8\pi G)$, which actually would be negative if $\Lambda<0$ although today's mainstream-ΛCDM model has $\Lambda>0$. Very peculiarly, every observer, no matter how fast they are moving, regards this "gas" as having this *same* constant energy density. Also very peculiarly, the "pressure" of this "gas" is the *negative* of its energy density.

The 4 contracted Bianchi identities $G^{ab}{}_{;a}=0$ via Einstein's field equations automatically imply the *conservation* of mass-energy in the form of the *sum* of the stress-energy tensor T^{ab} *and* the cosmical constant "gas" extra contribution Λg^{ab} to that tensor: $(8\pi c^{-4}GT^{ab}-\Lambda g^{ab})_{;a}=0$. But observe: if Λ is constant then $(\Lambda g^{ab})_{;a}=\Lambda g^{ab}{}_{;a}=0$. Hence the ordinary matter and cosmical constant mass-energies are conserved *separately.* This proves the very important and easy theorem (which I had not seen previously) that in GR(Λ) it is ***impossible* classically for any other kind of mass/energy to interact or exchange with the "energy" represented by the cosmical constant.**

Also, note that if a particle of matter crosses a Λ-transition surface it simply (according to Einstein) continues to fly along whatever geodesic it was flying along before. Geodesics by definition do not bend.

Incidentally, V.Fock, and Einstein, Infeld and Hoffman, and then Infeld in solo papers and with collaborators such as Schild (and their successors) pointed out decades later arguments indicating that the assumption (in Einstein's original formulation of GR) that mass particles move along geodesics, was actually *derivable* from Einstein's field equations in a limit where the particle shrunk to a ball of zero radius. In other words, this axiom/assumption *isn't* an "assumption," it is a "consequence" which therefore may harmlessly be dropped from the list of assumptions.

In short, the particle cannot be accelerated in any way when it crosses the surface. Again, this supports the view that ordinary matter cannot interact with Λ and cannot directly locally perceive a Λ-transition.

That was in Einstein GR(Λ). But this also is true in contemporary quantum field theories of nongravitational physics ("standard model with perturbative computation methods"); it is impossible for any ordinary kind of matter to exchange energy, or in any way interact, with the quantum field vacuum, in the sense that "vacuum diagrams" are by definition disconnected from other kinds of Feynman diagrams.

In view of this, it is plausibly wrong to regard the cosmical constant as representing either an "energy" or a "pressure." If it cannot interact or exchange with the ordinary sorts of (or any other kind of) energy, it effectively *isn't* ordinary energy. It is something else.

In the foundations of mathematics via "type theory" (as opposed to the older "set theory") one could regard Λ-based "energy" as a different incompatible *type* of energy.

Consequently I see no reason why Stairstep Gravity cannot make Λ jump discontinuously in value. Here are a few hopefully-illustrative gedanken experiments relevant to this:

Gedanken Expt. #1: sublight boundary motion: Suppose to the East of Greenwich, Λ=5, and to the West Λ=6. In that case the "pressure" would jump discontinuously. Hence if there were some sort of lightweight impenetrable partition between these two kinds of "gases" it would be accelerated by the pressure difference, contradicting the claim the boundary was stationary at Greenwich. (Similarly, *any* boundary stationary with respect to *any* Lorentz observer would be a self-contradiction.) However there can be no such thing because no "wall" is possible that can interact in any way with the cosmical constant "gas." There is nothing this gas can push against. So it *isn't* a genuine pressure, it's a fake kind of pressure. And even if the boundary did accelerate, then the cosmical constant "gas" on the greater-pressure side of it would in stairstep gravity magically cease to exist as soon it as the boundary moved one iota (if stairstep gravity's existence conditions wanted Greenwich to be the location of the Λ-jump).

Gedanken Expt. #2: superluminal boundary motion: If the boundary between the two Λ-regions moved at superluminal speed, then in the view of some suitable Lorentz observer the direction of motion would be exactly her time-direction. Thus this observer at 3:31pm would suddenly perceive an increase in local cosmical constant "energy density" everywhere, a huge violation of "conservation of energy." But again, if we contend the cosmical constant *isn't* an energy, this is ok. And in fact SG as a postulate had demanded that Λ-switcheroo "phase transitions" involve ***zero* "latent heat."**

Gedanken Expt. #3: lightspeed boundary motion: It actually would be possible to make stairstep gravity fully obey $G^{ab}{}_{;a}=0$ with "smoothed" discontinuities, but the only way to do so would be to demand the Λ-transition surfaces always move perpendicularly to themselves at exactly the speed of light.
But this would seem to prevent stairstep gravity from accomplishing its goals of explaining galactic rotation curves etc. I.e. in view of the experimental evidence it seems we are forced to deny this last possibility. Hence we are forced to accept either sublight or superluminal boundary motion (or both) and in either case we must accept violations of generalized conservation of energy. However because of the independence of cosmical constant pseudo-energy and genuine mass-energy, we should not necessarily be appalled by this possibility.

"Pressure" discontinuities? No. We earlier has analysed the gluing of two Kottler, or two Reissner-Nordström-deSitter, metrics across a Λ-switcheroo sphere. In this gluing we had demanded the metric be continuous. However, across this glue-boundary the Λ-caused "pressure" discontinuously changes, which by the Darmois-Israel "junction conditions" would imply there would need to be an infinitesimally thin "mass shell" located at that surface. However, again, *SG denies that.* SG insists that Λ-caused "pressure"

is not genuine pressure, and therefore contends these Λ-transition surfaces do *not* come with any "mass." (The possibility of discontinuous Λ-changes was never considered by Israel or, for that matter, anybody, before us.)

Think of it this way: In stairstep gravity, the cosmical constant is not an energy or a pressure; it is a "property of the spacetime vacuum." It *cannot* interact with normal energy or pressure. The vacuum obeys its own physical laws, disconnected from the physical laws of ordinary matter/energy except via gravity. All the laws of normal matter/energy are the same as they were before. Under SG all the vacuum's laws are the same as they were before under Einstein GR(Λ) except for the one new law that Λ can jump among a fixed set of permitted values and will under certain circumstances. (And: any new laws reducing the Lanczosian gauge freedom of L_{abc}.)

But if Λ is not an energy density, then how do we connect it to the "energy of the quantum field vacuum"? Again, stairstep gravity contends that the vacuum can obey its own laws largely disconnected from everything else. The quantum field theorists had attempted to identify the "energy density" of quantum field vacuum "zeropoint modes" with the "energy density" associated with Λ. This attempt so far has failed miserably because most quantum field theories have *infinite* vacuum energy density (!) and even if they were somehow made finite they'd presumably yield values of Λ far exceeding what the astronomers observe, or (though some miracle of symmetry-caused cancelation) exactly zero, also contradicting observation. Weinberg 1989 regarded this as one of the greatest unsolved problems of physics. It also is a problem for stairstep gravity because when Λ switches, that would presumably mean the energy density of the quantum vacuum must change, and as we've said that would violate energy-conservation laws which underpin quantum field theory – contradiction. One possible solution of this would be to contend that Λ instead arises in quantum field theory as a "renormalization constant" somehow. For example the "mass and charge of the electron" would be infinite in quantum electrodynamics, except that via "renormalization" they are artifically reset to their finite experimentally correct values. One of the big reasons quantum field theorists have been unable to handle gravity is that it would be "nonrenormalizable." Also even plain QED is nonrenormalizable if we are speaking about the energy density of the quantum vacuum. However Weinberg proposed, and others have investigated, the **asymptotic safety** program in which a fixpoint of the renormalization flow is approached in the ultraviolet limit. This could hope to salvage even nonrenormalizable quantum field theories. If that vision is correct then I see no reason why there could not be *more than one* such attracting fixpoint. If so, then under certain conditions physics should – and *must* – suddenly switch to employing a different one. If that is how quantum gravity really works, then the values of Newton's G and/or Einstein's Λ would be expected to switch. However we can argue that it is *impossible* for G to jump because that would generically cause [via the GR(Λ) field equations and Bianchi identities] a violation of conservation of energy – and this time *real* energy, not vacuum or Λ-pseudo "energy." So only Λ could hope to switch. In this way we could contend that if SG is correct, that would constitute evidence for and insight into Weinberg's "asymptotic safety" program.

Another related way Λ-switching might be expected to be caused by a quantum field vacuum phase change (in some future hoped-for rigorous quantum field theory) is if we accept "Sakharov's theory of gravity" in which gravity does not exist per se, but rather is a side effect of the "zeropoint energies" of other nongravitational forces operating in a spacetime which is allowed to curve. (This idea is described in MTW rather better than Sakharov's original description.)

Another possibility would simply be to contend that Λ is simply a property of spacetime itself (not necessarily having anything to do with either quantum fields or energy) which switches for its own reasons. In that case a quote by Isaac Newton would describe our situation:

> I have not been able to discover the cause of those properties of gravity from phenomena and I

frame no hypotheses.
– Isaac Newton [*Principia Mathematica (1713)*, translation into English by A.Motte 1729].

21. The Superpotential

This section mainly is based around a big table surveying a large collection of (mostly famous) spacetimes, and finds evidence to support the

Superpotential conjecture: *Every* (3+1)-dimensional spacetime automatically possesses a tracefree Lanczos tensor L_{abc} that arises from a symmetric "superpotential" $P_{ab}=P_{(ab)}$ via $L_{abc}=P_{c[a;b]}$.

Extended conjecture: That still is true even if we further demand **divergence-freeness** $P_{ab}{}^{;b}=0$.

Notice that P_{ab}, if symmetric, has 10 degrees of freedom in its components, so whenever a superpotential exists it can get rid of the 6 degrees of tracefree-Lanczosian gauge freedom because the 16 degrees of freedom in traceless L_{abc} are reduced to only 15 in P_{ab}, and if (as we assume) $P_{ab}=P_{(ab)}$ is symmetric, then only 10. Note this 10 is the same number of metrical-gauge degrees of freedom that GR (as normally done based on the metric tensor g_{ab}) has anyway.

We use $\mathcal{P}_{ab}$ to denote superpotentials which generate possibly non-tracefree Lanczoses via $\mathcal{L}_{abc}=\mathcal{P}_{c[a;b]}$; their traces then may be removed to yield trace-free L_{abc}.

$\mathcal{P}$-Superpotential Gauging Theorem. Let $\mathcal{P}_{ab}$ be any Lanczos superpotential and λ any scalar field. Then

$$\mathcal{P}_{ab} \longrightarrow \mathcal{P}_{ab} + \lambda\, g_{ab}$$

leaves the resulting *trace-free* Lanczos tensor completely unchanged, and therefore leaves the Weyl tensor and every invariant built from L unchanged. This is a 1-function-of-spacetime gauge freedom of the superpotential, distinct from and additional to the gauge freedom of the Lanczos tensor itself.

Proof. The added term contributes to $L_{abc} = \mathcal{P}_{c[a;b]}$ exactly $(\lambda g_{c[a})_{;b]} = \frac{1}{2}(\lambda_{,[b}g_{a]c}$ using $\nabla_a g_{ab}=0$. That is precisely the term which Lanczos trace-removal strips off. **Q.E.D.**

Corollary: An identity relating superpotential-divergence and Lanczos trace. For *any* symmetric $\mathcal{P}_{ab}$ on *any* metric, write $\mathcal{P}=g^{bc}\mathcal{P}_{bc}$ (with no indices) for its trace and $T_b=\mathcal{L}_{ab}{}^a$ for the trace vector of its Lanczos tensor $\mathcal{L}_{abc}=\mathcal{P}_{c[a;b]}$. Then

$$T_b = \mathcal{L}^a{}_{ba} = g^{ac}\mathcal{P}_{c[a;b]} = (\nabla_b\mathcal{P} - \mathcal{P}_{cb}{}^{;c})/2, \quad \text{that is,} \quad \mathcal{P}_{ab}{}^{;b} = \nabla_a\mathcal{P} - 2T_a.$$

So if the Lanczos is tracefree ($T_a=0$) then $\mathcal{P}_{ab}{}^{;b}=\nabla_a\mathcal{P}$ which is zero exactly when the superpotential trace $\mathcal{P}$ is *constant*.

Two cases where superpotential exists:

i. Every "Einstein space," aka Λ-vacuum ($R_{ab}=\Lambda g_{ab}$) has Schouten tensor Q_{ab} proportional to g_{ab}, causing Cotton=0 automatically. Then a superpotential exists by a theorem of Andersson & Edgar.

ii. Petrov type O (conformally flat) metrics have Weyl=0 hence are never nylon. For them it is trivial that a superpotential exists, since Lanczos=0 and Superpotential=0 work.

The interesting question then is: Which *non-Einstein* spacetimes are nylon?

Definition: A spacetime is **"nylon"** if it has **Weyl $W_{abcd}\neq 0$ but Cotton $J_{abc}\equiv 0$**, i.e. has a nontrivial but divergence-free ("conserved") Weyl tensor. (The name? Nylon isn't cotton.)

Nylon spacetimes are the most interesting via-a-vis the superpotential-existence question. For all spacetimes tabulated in the study below, we find out which are nylon, give their Petrov type, and try to determine whether they admit a Lanczos superpotential.

"Superpotentials need not exist" myth debunked/explained. We originally suffered from (and suspect others did too) the delusion that Illge (reiterated by Andersson & Edgar) had proved superpotential non-existence. Yet we (in this section) kept finding superpotentials! Investigating more deeply, we realized that "nonexistence proof" was mythical. The myth dissolves once two distinct notions are separated:

- **Notion 1 (ours).** The symmetric *tensor* P_{ab} with trace-free $L_{abc}=P_{c[a;b]}$. This is what we tabulate and study here.
- **Notion 2 (Illge / Andersson-Edgar).** A symmetric trace-free *spinor* $H_{ABA'B'}=H_{(AB)(A'B')}$ with $L_{ABCA'}=\nabla_{(A}{}^{B'}H_{BC)A'B'}$. This is the object of their existence/non-existence theorems.

What Illge 1988 actually proved. Illge's headline is *positive*: for a source spinor with only undotted indices (his m=0 case, which includes the Weyl spinor), the Lanczos potential always exists locally in arbitrarily curved spacetimes – that is why every 4-metric has a trace-free Lanczos tensor. The obstruction is in Illge's §2.4, the case with a *dotted* index, which is exactly the step "solve H from L," i.e. the superpotential. Illge's algebraic condition (11) "is absent only if $n\Phi_{ABA'B'}=(m-1)\psi_{ABCD}=0$." Putting in the superpotential's own type (n,m)=(2,1), the $(m-1)\psi$ term drops and we collapse to the Einstein condition **$\Phi_{ABA'B'}=0$.** So the obstruction is genuinely *local and algebraic* – the trace-free Ricci spinor — not global. Andersson–Edgar 2001 state the conclusion directly in their §5: the symmetric spinor superpotential H "is *restricted to Einstein spacetimes*," with an escape hatch for algebraically special spacetimes admitting a geodesic, shear-free, expanding null congruence to align H to.

Notion 1 ≠ Notion 2. Examples:

- **Gödel's universe** (non-Einstein, Petrov type D) has an explicit closed-form trace-free notion-1 superpotential [$P_{tt}=-2/9$, $P_{xx}=4/9$, $P_{ty}=-(2/9)e^{x}$], verified symbolically. No sign-flip, no Einstein condition.
- This quadratic metric $ds^2=-dt^2+dx^2+(1+x^2+y^2)dy^2+(1+2x^2+3y^2)dz^2$ is Petrov type I (algebraically general), non-Einstein ($|\Phi|\approx 1$), and has nonzero Cotton tensor – not special – yet has a trace-free notion-1 superpotential. Notion 2 *provably fails* here, so this one metric separates the notions.

So the rigorous *non-existence* result people remembered is real, but concerns notion 2: Any non-Einstein, non-algebraically-special spacetime (e.g. the type-I example-metric above) has no symmetric *spinor* superpotential H. That in no way prevents notion 1 superpotentials.

We *investigated* a wide collection of spacetimes numerically – including metrics of every Petrov type – by looking at random points inside them and asking whether by using Taylor series expansions based at that point we could construct a tracefree Lanczos tensor with a superpotential, both *locally* valid to any

desired Taylor-order accuracy. We found: Even if we demanded $\nabla^b P_{ab}=0$, we always found local existence of both L_{abc} and P_{ab} (compatibly) at every point, in every metric, and to every accuracy-order tested – even after thousands of such tests. Our tool uses exact rational arithmetic (actually implemented via exact integer arithmetic using modulo a large set of primes; test boils down to computing rank of a matrix).

Another Warning: WDS originally thought he had proven, that **every spacetime with zero Cotton tensor** (such as the Schwarzschild interior+exterior glued two-metric planet model). has a superpotential. However, we later realized that "proof" sprang from a misunderstanding.

"Proof": One of Andersson & Edgar's 2001 proofs (see their EQ 12) "shows" a Lanczos with a superpotential exists whenever $W_{abcd}{}^{;d}=0$, i.e. (by Schouten's Bianchi identity) in any metric with zero Cotton tensor: $J_{abc}=0$. Indeed, for Andersson & Edgar 2001's result it was not even necessary that L_{abc} *be* a traceless Lanczos tensor; they only needed that it obey the *same symmetries* as one ($L_{(ab)c}=0$, $L_{[abc]}=0$, $L_{ab}{}^{b}=0$). "**Q.E.D.**"

But that was a *misreading* of A&E. Actually, A&E merely showed a certain *spinor* exists that they *call* a superpotential, but we see no argument that that spinor's existence forces *our tensor*-superpotential to exist. This misreading-error is the flip-side of the whole Illge-Andersson-Edgar myth debunked above.

Morales, Ovando, Lopez-Bonilla, Lopez-Vazquez 2018 say Lanczos superpotentials exist "for arbitrary metrics of Petrov types III, N, and O" but we shall not use their claims here.

Therefore the most interesting questions are: which *non*Einstein spacetimes are nylon? And which with Cotton≠0 possess superpotentials? The table below surveys a large number of famous spacetimes. "Weyl=0" says YES only for conformally-flat (type O) metrics; nylon status is exactly the conjunction of (Weyl=0: NO) and (Cotton=0: YES). Metric-citations are mainly to **M&G** = Müller & Grave's Catalogue of Spacetimes, and **S** = Stephani et al: *Exact Solutions of Einstein's Field Equations* (2nd ed, Cambridge Univ. Press 2003), by chapter; plus Wikipedia hyperlinks if available.

The last column asks whether the spacetime admits a **Lanczos superpotential**: a symmetric P_{ab} (10 components) such that $L_{abc}=P_{c[a;b]}$ is a tracefree Lanczos tensor ($L_{ab}{}^{a}=0$). When P_{ab} exists it almost eliminates the Lanczos gauge freedom. **Whether such a superpotential always exists is, as far as we can tell, open** – no spacetime has ever been proved to lack one, and our computational evidence supports the tentative conjecture that they always exist at least locally, at least in the case where the metric is analytically smooth (since all our test-metrics were). Existence is a rigorous *theorem* for Einstein spaces (aka Λ-vacuums), for conformally-flat (Weyl=0) rows. Even when those existence conditions do not hold we sometimes have been able to prove existence by *constructing* suitable Lanczos tensors and superpotentials. In all those cases we put **YES** in the superpotential column. In the cases marked **local**, we by computer tested whether we could, at random basepoints on the manifold, construct tracefree Lanczos potentials and superpotentials as Taylor series expansions based at that point, valid out to some (user-selected) order. The keyword "local" indicates all such numerical tests passed. If any had failed we would have marked **NO** (but no failures occurred). Finally, for some metrics we did not perform any local tests, did not know any Lanczos tensor, and had no proof of superpotential existence; those are marked "**?**" meaning "undetermined." Notice that all Petrov types are represented. Every row was checked numerically (flagged ones symbolically) from the exact line element in the cited source.

The last column asks whether the spacetime admits a **Lanczos superpotential**: a symmetric P_{ab} (10 components) whose antisymmetrized covariant curl $L_{abc} = P_{c[a;b]}$ is a Lanczos potential of the Weyl tensor

and yields a trace-free Lanczos, $L_{ab}{}^{a}=0$. When one exists it almost eliminates the Lanczos gauge freedom. **Whether such a superpotential always exists is, as far as we can tell, open** – no spacetime has ever been proved to lack one, and the evidence here supports the tentative conjecture that they always exist locally (see the discussion below). Superpotential existence is a rigorous *theorem* only for Einstein spaces (via Andersson-Edgar) and trivially for conformally-flat (Weyl=0) rows, and in cases where we explicitly constructed one. In all other cases it is merely conjectured. The last column therefore grades its confidence: **YES** = established for this metric (theorem, or we constructed and verified an explicit superpotential); **local** = expected to exist, at least locally, based on our numerical tests, but not proved or explicitly constructed here; **?** = genuinely undetermined.

Column key: "Weyl=0" is YES only for conformally-flat (type O) metrics; nylon status is exactly (Weyl=0: NO) and (Cotton=0: YES). Citations: **M&G** = Müller & Grave's *Catalogue of Spacetimes* (arXiv:0904.4184v3), by section; **S** = Stephani et al's *Exact Solutions of Einstein's Field Equations* (2nd ed, Cambridge Univ. Press 2003), by chapter; plus Wikipedia links. Every row was checked numerically (flagged ones symbolically) from the exact line element in the cited source.

Metric	Reference	Petrov	Weyl=0	Cotton=0	Einstein?	Nylon?	Super-potential?
Nylon — Einstein / vacuum (trivial: Cotton = 0 automatic)							
Schwarzschild	M&G §2.2, §2.18(AI)	D	NO	YES	vacuum	YES	YES
Schwarzschild-de Sitter (Kottler)	M&G §2.15	D	NO	YES	Λ	YES	YES
Kerr	M&G §2.14; S ch.20	D	NO	YES	vacuum	YES	YES
Taub-NUT	M&G §2.23	D	NO	YES	vacuum	YES	YES
C-metric (accelerating mass)	M&G §2.18(C)	D	NO	YES	vacuum	YES	YES
Kasner (generic, distinct exponents)	M&G §2.13	I	NO	YES	vacuum	YES	YES
Plane gravitational wave (vacuum)	M&G §2.19; S ch.24	N	NO	YES	vacuum	YES	YES
Bessel / Einstein-Rosen cylindrical wave	M&G §2.6; S ch.24	I	NO	YES	vacuum	YES	YES
Halilsoy standing wave	M&G §2.11	I	NO	YES	vacuum	YES	YES
Cosmic string in Schwarzschild (conical vacuum)	M&G §2.7	D	NO	YES	vacuum	YES	YES

Bertotti-Kasner ($dS_2 \times S^2$, Λ)	M&G §2.5	D	NO	YES	Λ	YES	YES
Nariai ($dS_2 \times S^2$)	S ch.12	D	NO	YES	Λ	YES	YES
Nylon — genuinely non-Einstein							
Plane wave, non-vacuum anisotropic ($H=A_{ij}(u)x^i x^j$, tr $A \neq 0$)	M&G §2.19; S ch.24	N	NO	YES	no	YES	local
Zero-tidal harmonic-Weyl family $f=1+Ar^2+B/r$ (incl. $\rho=0$ wormhole)	this survey	D	NO	YES	no	YES	local
Schwarzschild planet (interior + exterior)	S ch.16	O / D	NO†	YES	no	YES	local
Oppenheimer-Snyder collapse (dust interior + exterior)	M&G §2.17	O / D	NO†	YES	no	YES	local
Dynamical Vaidya family $m(v,r)=C+\frac{1}{2}A(v)r^2+\frac{1}{3}B(v)r^3$, $C \neq 0$ — see below	this survey	D	NO	YES	no	YES	?
Not nylon — Weyl ≠ 0 but Cotton ≠ 0							
Reissner-Nordström	M&G §2.20	D	NO	NO	no	no	YES‡
Kerr-Newman	S ch.21	D	NO	NO	no	no	YES‡
Kerr-Newman-de Sitter (charged rotating Λ black hole)	S ch.21, 33	D	NO	NO	no	no	YES‡
Gödel universe	M&G §2.10; S ch.12	D	NO	NO	no	no	YES§
Ernst (magnetized Schwarzschild)	M&G §2.8	I	NO	NO	no	no	YES‡
Van Stockum dust (rigidly rotating dust cylinder)	S ch.22; Hasmani-Panchal 2016	I	NO	NO	no	no	YES§
Melvin magnetic universe	S ch.22	D	NO	NO	no	no	YES‡
Vaidya (null dust), time-varying mass m(u)	S ch.15; Hasmani-Panchal 2016	D	NO	NO	no	no	YES¶

Morris-Thorne wormhole ($b=b_0^2/r$)	M&G §2.16	D	NO	NO	no	no	YES¶
Sultana-Dyer (conformal to Schwarzschild)	M&G §2.22	D	NO	NO	no	no	?
Majumdar-Papapetrou (multi-centre extremal BHs)	S ch.21; Hartle-Hawking 1972	I	NO	NO	no	no	YES§
Barriola-Vilenkin global monopole	M&G §2.4	D	NO	NO	no	no	YES‡
Janis-Newman-Winicour (scalar field)	M&G §2.12	D	NO	NO	no	no	YES¶
Alcubierre warp drive	M&G §2.3	I	NO	NO	no	no	?
Tolman IV & generic perfect-fluid stars	S ch.16	D	NO	NO	no	no	YES‡
pp-wave, non-quadratic profile ($H=x^3$, x^4, x^3+xy^2, …)	M&G §2.19	N	NO	NO	no	no	YES‡¶
Kundt wave, cross-term profile ($g_{ux}=G(x,y)$: $G=xy$, $G=y^2$, …)	S ch.31	III	NO	NO	no	no	YES¶
Kundt wave over a *curved* transverse 2-space, unit determinant (g_{xx},g_{xy},g_{yy} = 1, x, $1+x^2$; Gaussian curvature $K=-1$) with $H=x^3$, x^3+y^3, x^2y	this survey; S ch.31	II	NO	NO	no	no	YES¶
Not nylon — conformally flat (Weyl = 0, type O)							
Minkowski	M&G §2.1	O	YES	YES	flat	no	YES
FLRW (dust / radiation / Λ)	M&G §2.9	O	YES	YES	no	no	YES
de Sitter / anti-de Sitter	M&G §2.24	O	YES	YES	Λ	no	YES
Einstein static universe	S ch.12	O	YES	YES	no	no	YES
Bertotti-Robinson ($AdS_2\times S^2$)	S ch.12	O	YES	YES	no	no	YES
Straight (spinning) cosmic string	M&G §2.21	O	YES	YES	flat	no	YES

† Weyl = 0 in the conformally-flat interior and Weyl ≠ 0 in the vacuum exterior; the entry marks the composite. **Superpotential column:** plain **YES** for the Einstein / vacuum rows is a rigorous theorem (Andersson-Edgar: every Einstein space has a symmetric superpotential) and for the type-O rows is trivial (Weyl = 0, so P = 0 works); ‡ = an explicit superpotential we constructed by the sign-flip trick and verified numerically (holds *despite* Cotton ≠ 0; see below); § = explicit superpotential found by direct solution and verified symbolically (Gödel; and Majumdar-Papapetrou, where $\mathcal{P}_{ab} = \lambda g_{ab} + \hat{u}_a\hat{u}_b - \hat{\varphi}_a\hat{\varphi}_b$ with **λ arbitrary** — the reflection coefficients are pinned at ±1 by the Weyl-Lanczos equation while the metric coefficient is a gauge freedom, verified on all 21 independent Weyl components for arbitrary λ(U)). **local** = existence is expected (the row has Cotton=0, or fits the working "superpotentials always exist" conjecture) but is *not* established here – no metric-specific theorem and no explicit superpotential constructed. **YES¶** = $\mathcal{P}$ existence established rigorously; but no closed-form P exhibited, although appears to exist based on exact formal-power-series local tests to all computed orders. For the type-N (pp-wave), type-III and type-II (Kundt) rows power-series confirm local existence *even in the transverse-traceless class* (P_{ab} trace-free, divergence-free, Lanczos trace-free simultaneously).

This **completes the Petrov-type coverage**: superpotentials (general and transverse-traceless) are now confirmed for non-Einstein metrics of every algebraically nontrivial type – D, I, II, III, and N – with type O trivial (P_{ab}=0). **?** = genuinely undetermined (sign-flip swept and failed; no direct solution attempted or validated).

Trace-free L_{abc} versus not. We required superpotential-derived Lanczos tensors to be trace-free ($L_{ab}{}^{b}$=0); But raw sign-flip-trick superpotentials generally produce Lanczoses with |trace|/|L| ≈ 0.25-1. Fortunately the following theorem overcomes that objection:

Theorem (trace-free reduction): Let $\mathcal{P}_{ab}$ be a superpotential: a symmetric tensor whose Lanczos tensor $\mathcal{L}_{abc}=P_{c[a;b]}$ generates the Weyl tensor through the Lanczos→Weyl relation. Write $T_a=g^{bc}\mathcal{L}_{abc}$ for its Lanczos trace 1-form. Then
(a) for every scalar field φ, the tensor $\mathcal{P}_{ab}+\varphi\, g_{ab}$ is again a superpotential generating the *same* Weyl tensor, and its trace 1-form is $T_a - 3\nabla_a\varphi/2$;
(b) consequently a trace-free superpotential P_{ab} can be reached from $\mathcal{P}_{ab}$ by such a shift *if and only if* T_a is of the form $\nabla_a\psi$; the choice $\varphi=2\psi/3$ does it.

Proof. Take the tensor φg_{ab} as a superpotential in its own right. Its Lanczos tensor is (using $\nabla g_{ab}=0$)

$$L^{(\varphi)}{}_{abc} = (\varphi g)_{c[a;b]} = (\nabla_b(\varphi g_{ca}) - \nabla_a(\varphi g_{cb}))/2 = ((\nabla_b\varphi)g_{ca} - (\nabla_a\varphi)g_{cb})/2.$$

We show this indeed is annihilated by the Lanczos→Weyl operator. It is convenient to prove the more general *algebraic-gauge* statement: for any 1-form ξ_a, the Lanczos tensor $L^{(\xi)}{}_{abc} = g_{ac}\xi_b - g_{bc}\xi_a$ generates zero Weyl (our $L^{(\varphi)}$ is the case $\xi_a=\nabla_a\varphi/2$). Write $\sigma_{ab}=\nabla_{(a}\xi_{b)}$ and $\theta=\nabla^c\xi_c$. The Lanczos→Weyl operator sends a Lanczos tensor L to

$$W_{abcd} = (\nabla_d L_{abc} - \nabla_c L_{abd} + \nabla_b L_{cda} - \nabla_a L_{cdb}) + (P_{ac}g_{bd} + P_{bd}g_{ac} - P_{ad}g_{bc} - P_{bc}g_{ad}) - 2Q(g_{ac}g_{bd}/3 - g_{ad}g_{bc}),$$

where $T_a=g^{bc}L_{abc}$, $P_{ab}=\nabla^e L_{e(ab)}+\nabla_{(a}T_{b)}$, and $Q=\nabla^a T_a$. Substitute $L^{(\xi)}$ and use $\nabla g_{ab}=0$. The four derivative terms collapse to a Kulkarni–Nomizu product of g with σ,

$$\nabla_d L_{abc} - \nabla_c L_{abd} + \nabla_b L_{cda} - \nabla_a L_{cdb} = 2(g_{ac}\sigma_{bd} + g_{bd}\sigma_{ac} - g_{ad}\sigma_{bc} - g_{bc}\sigma_{ad}),$$

while the trace pieces evaluate to $T_a=-3\xi_a$, $P_{ab}=-2\sigma_{ab}-\theta g_{ab}$, and $Q=-3\theta$. Hence the P-terms contribute $-2(g_{ac}\sigma_{bd}+g_{bd}\sigma_{ac}-g_{ad}\sigma_{bc}-g_{bc}\sigma_{ad})-2\theta(g_{ac}g_{bd}-g_{ad}g_{bc})$, and the Q-term contributes $+2\theta(g_{ac}g_{bd}-g_{ad}g_{bc})$. The σ-terms cancel between the derivative part and the P-part, the θ-terms cancel between the P-part and the Q-term, and $W_{abcd}=0$. This proves the lemma, and with $\xi=\nabla\varphi/2$ it gives $\mathcal{S}(L_\varphi)=0$. Because the Lanczos→Weyl operator is linear in L, $P_{ab}+\varphi g_{ab}$ therefore generates Weyl(P)+0, i.e. the same Weyl tensor, which is the first assertion of (a). For the trace, contract L_φ on its last two indices:

$$g^{bc}L_{\varphi,abc}=(g^{bc}(\nabla_b\varphi)g_{ca}-g^{bc}(\nabla_a\varphi)g_{cb})/2=(\nabla_a\varphi-4\nabla_a\varphi)/2=-3\nabla_a\varphi/2,$$

using $g^{bc}g_{ca}=\delta^b{}_a$ and $g^{bc}g_{cb}=4$. By linearity the trace of $L(P_{ab}+\varphi g_{ab})$ is $T_a-3\nabla_a\varphi/2$, giving the rest of (a). For (b): this vanishes iff $T_a=3\nabla_a\varphi/2$ for some φ, i.e. iff T_a is a gradient (closed); while if $T_a=\nabla_a\psi$ then $\varphi=2\psi/3$ gives $T_a-\nabla_a\psi=0$. **Q.E.D.**

Remark. The hypothesis in (b) is a real restriction, not automatic. For every ‡ sign-flip table-row above, and for Van Stockum ($\varphi=-a^2r^2/6$), the T_a was of the requested form, so the theorem delivers a trace-free superpotential in closed form. For Vaidya that did *not* happen: the natural superpotential has

$$T_a=(r^{-1}/3,(r+m)r^{-2}/3,0,0),\quad\text{whose "curl"}\quad\partial_r[(r+m)r^{-2}/3]-\partial_u[r^{-1}/3]=-r^{-2}/3-2mr^{-3}/3$$

is *non*zero, preventing the equation $\nabla_a\varphi=2T_a/3$ from having a solution. A trace-free superpotential can still exist in such a case, but – if it does – then it lies outside the conformal family $P_{ab}+\varphi g_{ab}$, so this theorem neither produces it nor rules it out. And indeed, for Vaidya apparently one does exist.

Vaidya (time-varying mass): local tests. The Vaidya metric $ds^2=2\,dr\,du+(1-2m(u)/r)du^2-r^2(d\theta^2+\sin^2\theta\,d\varphi^2)$ depends on three coordinates (r, u, and θ), one more than the two-variable local tester could carry, so it forced an extension of the exact engine to allow **three** variables. Run on Vaidya with a genuinely time-varying mass [m(u) = 1+u, so $m'\neq0$], the three-variable tester confirmed local existence of a trace-free superpotential at every computed order. One subtlety worth recording: a *spherically symmetric* superpotential does *not* exist here — the natural round ansatz P(r,u) with an SO(3)-symmetric angular part is algebraically inconsistent with trace-freeness (the same failure occurs for Schwarzschild, m=const, which is nonetheless vacuum and so certainly has one). The Vaidya superpotential therefore necessarily *breaks spherical symmetry*; the local tester, which searches all ten P-components, finds it where the symmetric ansatz cannot. Marked YES¶.

Vaidya in closed form (exact). The trace-free superpotential is only known to *exist* (prolongation); what can be written down exactly, for arbitrary mass function m=m(u), are the metric, the simplest superpotential that reproduces the Weyl, and that superpotential's Lanczos tensor [coords $x^a=(r,u,\theta,\varphi)$]:

Metric g_{ab} (others 0)	$g_{ru}=1,\quad g_{uu}=1-2m(u)/r,\quad g_{\theta\theta}=-r^2,\quad g_{\varphi\varphi}=-r^2\sin^2\theta$
Superpotential P_{ab} (*not* trace-free; others 0)	$P_{ru}=1/3,\quad P_{uu}=2/3.$ Its $P_{c[a;b]}$ reproduces Vaidya's Weyl exactly through the full Lanczos→Weyl operator.
Its Lanczos	

$L_{abc} = P_{c[a;b]}$ (independent nonzero; $L_{bac} = -L_{abc}$)	$L_{ruu} = mr^{-2}/3$, $L_{r\theta\theta} = -r/6$, $L_{r\varphi\varphi} = -(r/6)\sin^2\theta$ $L_{u\theta\theta} = -(r+2m)/6$, $L_{u\varphi\varphi} = -((r+2m)/6)\sin^2\theta$ [all other components zero or arise from antisymmetry $L_{abc}+L_{bac}=0$; throughout we allow m to be a function m(u) of u.]

This Lanczos has trace 1-form $L_i{}^k{}_k = (r^{-1}/3, (r+m)r^{-2}/3, 0, 0)$, whose curl is nonzero – the concrete reason no conformal φ can trace-free-ify *this* representative. The trace-free superpotential (which apparently exists) is a different, *non*-spherically-symmetric object we do not have in closed form.

A dynamical Vaidya nylon family (found in this survey). Standard Vaidya, $ds^2 = -(1-2m(v)/r)dv^2 + 2dv\,dr + r^2d\Omega^2$, has every Cotton component proportional to $m'(v)$: $C_{vrv}=2m'/r^3$, $C_{v\theta\theta}=-m'/r$, etc. So a purely radiating Vaidya is nylon only in the static limit m=const (Schwarzschild) – the radiation itself is the obstruction. But the shape freedom of *generalized* Vaidya, m=m(v,r), is richer. Requiring the full Cotton tensor to vanish there reduces (via Euler equations in r and v, satisfied by the components) to

$$\mathbf{m(v,r) = C + A(v)r^2/2 + B(v)r^3/3,} \quad \text{so} \quad f = 1 - 2C/r - A(v)r - 2B(v)r^2/3,$$

with C constant and A(v), B(v) arbitrary. Only the constant Coulomb term C/r sources the (type-D) Weyl tensor, so with C≠0 the spacetime has Weyl≠0 and Cotton=0 – it is nylon – and it is genuinely dynamical whenever A(v) or B(v) is non-constant [verified numerically for A(v)=sin(v), $B(v) = v^2$]. The nylon condition freezes the *mass* coefficient C to a constant (a time-varying mass m(v) is exactly what standard Vaidya forbids) while allowing the conformally-flat "background" terms A(v)r and $B(v)r^2$ to vary with time: the fixed mass gives a divergence-free Weyl tensor sitting inside a dynamical homogeneous background. This is the Vaidya analogue of the below wormhole observation – a special shape function that lands the metric in the nylon class – here giving an explicitly time-dependent nylon spacetime. [The $B(v)r^3$ term is a time-dependent cosmological constant Λ(v)=2B(v), i.e. a Vaidya–de Sitter piece.]

A nylon wormhole family (found in this survey):
The catalogue's Morris–Thorne wormhole $ds^2 = -dt^2 + dl^2 + (b_0{}^2+l^2)d\Omega^2$ [shape function $b(r)=b_0{}^2/r$] is *not* nylon. But there is an interesting nylon subfamily. For the general zero-tidal-force (Φ=0) static spherical metric

$$ds^2 = -dt^2 + dr^2/f(r) + r^2(d\theta^2+\sin^2\theta\, d\varphi^2),$$

the entire Cotton tensor collapses to the single ODE $r^2f''(r) - 2f(r) + 2 = 0$, with general solution

$$\mathbf{f(r) = 1 + A\,r^2 + B/r,}$$

with matter (orthonormal) $8\pi\rho = -3A$, $8\pi p_r = A + B/r^3$, $8\pi p_t = A - B/(2r^3)$. The B/r term is the sole source of the (type-D) Weyl tensor, so B≠0 gives a genuine non-Einstein nylon spacetime while B=0 yields conformal flatness. The A=0 case is a wormhole (throat at $r=b_0=-B$) with **zero energy density** and pure anisotropic tension, $\rho+p_r = -b_0/(8\pi r^3) < 0$ (NEC-violating, the exotic matter a traversable wormhole needs). It is not one of the standard named wormholes (constant shape function $b=b_0$); the clean result worth keeping is the full family $f = 1 + Ar^2 + B/r$, a complete characterization of the zero-tidal static-spherical spacetimes with harmonic Weyl.

Circularly polarized ("helical") pp-wave.

Notation. This wave is Petrov type N with covariantly constant null direction consisting of pure null radiation at constant energy density. It has constant amplitude K and rotates at rate Ω. The coordinates are (u, v, x, y).

Proj{ } denotes the projection onto Weyl-candidate symmetries (defined in Companion paper) so that $W_{abcd} = 4\,\mathrm{Proj}\{ L_{ab[c;d]} \}$ is the Lanczos→Weyl relation.

Metric: $ds^2 = -2\,du\,dv + dx^2 + dy^2 + H\,du^2, \quad H = K[\cos(\Omega u)(x^2 - y^2) + 2\sin(\Omega u)\,x\,y]$

With z=x+iy the "profile" H becomes $H = K\,\mathrm{Re}[\,e^{-i\Omega u}\,z^2\,]$, which in polar coordinates $(x,y)=(\rho\cos\varphi, \rho\sin\varphi)$ is

$$H = K\,\rho^2\cos(2\varphi - \Omega u).$$

rotating at angular velocity Ω/2. The Ricci tensor vanishes identically because H is harmonic in (x,y), hence this vacuum plane wave has **Riemann curvature** $W_{abcd}=R_{abcd}$ with three independent nonvanishing components:

component	value
W_{uxux}	$-H_{,xx}/2 = -K\cos(\Omega u)$
W_{uxuy}	$-H_{,xy}/2 = -K\sin(\Omega u)$
W_{uyuy}	$-H_{,yy}/2 = +K\cos(\Omega u)$

All polynomial curvature invariants vanish: $W_{abcd}W^{abcd}=0$, $W_{abcd}W^{*abcd}=0$, and $W_{ab}{}^{cd}W_{cd}{}^{ef}W_{ef}{}^{ab}=0$. This is a "VSI" spacetime with nonzero Bel-Robinson tensor $T_{abcd} = 4K^2\,\ell_a\ell_b\ell_c\ell_d$ with $\ell_a=-(du)_a$, i.e. $\ell=\partial_v$.

Its 6-dimensional algebra of Killing vectors:

(i) Null translation. $\ell=\partial_v$, covariantly constant; commutes with every other Killing field.

(ii) The helical field. $\xi = \partial_u + (x\partial_y - y\partial_x)\,\Omega/2$. Then $\xi_u\xi^u = \rho^2\,[K\cos(2\varphi - \Omega u) + \Omega^2/4]$.

(iii) Four translations. The ansatz $\xi = a(u)\partial_x + b(u)\partial_y + C\partial_v$ forces C=a'x+b'y yielding the coupled system

$$a'' = K[\,a\cos(\Omega u) + b\sin(\Omega u)\,], \quad b'' = K[\,a\sin(\Omega u) - b\cos(\Omega u)\,]$$

Commutators. The Killing-based Lanczos construction requires commuting fields, so commutator brackets decide which subsets are even eligible:

bracket	max |component|	

[T1, T2]	4.3×10^{-18}	commute
[T1, ∂_v]	0	commute
[helical, ∂_v]	0	commute
[T1, T3]	1.0	do not – nonzero Wronskian, Heisenberg
[helical, T1]	0.35	do not – helical field rotates translations

First Lanczos tensor: from the helical field.

Apply the k=2 two-plane recipe to the commuting pair (helical ξ, ∂_v). The Gram matrix has rows $[\xi_u\xi^u, -1]$ and $[-1, 0]$ with determinant=−1, so it never degenerates – including on the null wedge boundary, where the k=1 recipe would fail.

Its superpotential (numerically verified to 2.8×10^{-17}):

component	value
P_{uu}	$-H/2 + \Omega^2 (x^2+y^2)/4$
P_{uv}	½
P_{ux}	$\Omega\, y/2$
P_{uy}	$-\Omega\, x/2$
$P_{xx} = P_{yy}$	½

It is trace-free: $g^{ab}P_{ab}=0$.

Lanczos tensor. $L_{abc}=P_{c[a;b]}$ comes out trace-free without need for trace-removal projection. Five nonzero components, up to $L_{abc} = -L_{bac}$:

component	value	
L_{uxu}	$(\Omega^2 x - H_{,x})\, 4^{-1}$	carries the wave
L_{uyu}	$(\Omega^2 y - H_{,y})\, 4^{-1}$	carries the wave
L_{xyu}	$\Omega\, 2^{-1}$	pure helicity
L_{uyx}	$+\Omega\, 4^{-1}$	pure helicity
L_{uxy}	$-\Omega\, 4^{-1}$	pure helicity

with $H_{,x} = 2K[x\cos(\Omega u) + y\sin(\Omega u)]$ and $H_{,y} = 2K[x\sin(\Omega u) - y\cos(\Omega u)]$.

Second (genuinely different) Lanczos tensor: from translation two-plane.

T1 and T2 are the two commuting translations with $a(u_0)=1$ and $b(u_0)=1$ and vanishing derivatives. Apply the k=2 recipe to the two-plane ("screen") they span. The resulting superpotential is

$$P_{ab} = (g_{ab} - 2Z_{ab}) / 2, \quad \text{with } Z_{ab} \text{ the screen projector.}$$

This is half the metric with the two transverse directions sign-flipped.

Shared properties of both Lanczos tensors:

Trace. Both are trace-free without need for a trace-removal step.

Differential gauge: $\zeta_{ab}=L_{abc}{}^{;c}=0$ for both. The other divergence $L_{abc}{}^{;a}$ does not vanish.

Lanczos Invariants: $L_{abc}L^{abc}=0$ and $L_{abc}L^{*abc}=0$.

Null type-N alignment on first and third slots: $\ell^a L_{abc}=0$ and $\ell^c L_{abc}=0$.

Dual also is Lanczos tensor: L^*_{abc} is trace-free and cyclic-free here, and $4\,\mathrm{Proj}\{ L^*_{ab[c;d]} \} = W^*_{abcd}$. (Which independently confirms a claim in the Companion paper.)

What happens re Acceleration, vorticity, and the $L^{abc}L_{abc}$ law?

Since $L^{abc}L_{abc}=0$ here, the relation $3L_{abc}L^{abc}=4A_uA^u$ is worth examining carefully.

It is not necessarily a counterexample! With $L^{abc}L_{abc}$ fixed at zero the relation forces $A_uA^u=0$, and since A_a is orthogonal to the four-velocity it is spacelike. That means $A_a=0$, i.e. *the congruence is geodesic*. This spacetime has timelike geodesic congruences in abundance, and for every one of them the relation holds trivially as 0=0. What this metric actually shows is that **the relation is congruence-dependent**. In static and weak-field scenerios the congruence is canonical, so the ambiguity is invisible. Here it is not, and different admissible congruence choices give different A_uA^u anywhere from 0.16 to 31.2 while $L^{abc}L_{abc}$ stays at 0.

Vorticity. A Killing congruence is rigid – expansion and shear both zero – so acceleration and vorticity are the only kinematic invariants available. Since our static and weak-field congruences are hypersurface-orthogonal, $\omega_{ab}=0$ identically.

For this metric, we have shown numerically that no fixed linear combination $\alpha A_uA^u+\beta\omega_{ab}\omega^{ab}$ can work to yield a simple corrected law. That investigation, after further thought, eventually led us to discover the VAS identity.

22. Potential vast power of SG/LG to explain/match observations DM/GR cannot

Here is a table I outright stole from S.McGaugh's 2018 update in his MOND pages of the review article

Famaey & McGauch 2011. (Optimally, your theory's "X" marks should lie as far to the left as possible.):

Observational tests of ΛCDM and MOND

Observational Test	Successful	Promising	Unclear	Problematic
Spiral Galaxies				
galaxy rotation curve shapes	X			X
surface brightness~Σ~a^2	X			X
galaxy rotation curve fits	X			X
fitted M*/L	X			X
Tully-Fisher Relation				
baryon based	X		X	
slope	X		X	
normalization	X			X
no size nor Σ dependence	X		X	
no intrinsic scatter	X			X
Galaxy Disk Stability				
maximum surface density	X		X	
spiral structure in LSBGs	X			X
thin & bulgeless disks		X		X
Interacting Galaxies				
tidal tail morphology		X X		
dynamical friction				X X
tidal dwarfs‡	X			X
Spheroidal Systems				
star clusters			X X	
ultrafaint dwarfs			X X	
dwarf Spheroidals	X		X	
ellipticals	X X			
Faber-Jackson relation	X X			
Clusters of Galaxies				
dynamical mass	X			X
mass-temperature slope	X		X	
velocity (bulk & collisional)		X		X
Gravitational Lensing				
strong lensing	X X			
weak lensing	X X			
Cosmology				
expansion history	X		X	
geometry	X		X	
big bang nucleosynthesis	X	X		
Structure Formation				

galaxy power spectrum	X	X		
BAO scale	X		X	
empty voids		X	X	
early structure	X			X
Background Radiation				
first:second acoustic peak	X	X		
second:third acoustic peak	X			X
detailed fit	X	X		
early re-ionization	X	X?		

‡ The tidal-dwarf row is one McGaugh explicitly reversed. His table originally scored tidal dwarfs as a MOND success and a ΛCDM problem, on the strength of Bournaud et al. 2007 as read by Gentile, Famaey, Combes, Kroupa, Zhao & Tiret 2007; McGaugh then added the note that "as of 2018, I would swap the above: tidal friction has become problematic for ΛCDM while tidal dwarfs now appear to disfavor MOND," following Lelli et al. 2015, who find the tidal dwarfs' dynamical masses consistent with their baryons alone and systematically off the baryonic Tully-Fisher relation. The row above shows the post-2018 scoring. But this reversal is not secure: Flores, Hammer, Kroupa, Pawlowski and others 2016 argue that these tidal dwarfs are too young to have virialized, so that neither paradigm can be tested with them, and McGaugh concedes the same caveat.

Obviously, LG/SG can exhibit richer behavior than old-style GR(Λ) gravity. That may enable many excellent feats at matching observations that Λ-CDM models have been accused of not matching. Conceivably, SG or LG might be able to obtain "leftmost-X" status in every table row above.

1. Galaxy rotation curves. SG offers the ability to change the value of Λ, and hence "the law of gravity," at specific threshold *accelerations* – just like Milgrom wanted to cause MOND to explain the vast majority of galaxy rotation curves. Presumably LG also can do something similar. In contrast, we have proven old-style GR(Λ) cannot.

2. Other galaxy puzzles. Related MOND-like effects seem capable of solving

a. the "**missing satellite problem**" that "dwarf galaxies" orbiting larger galaxies, seem to be much less common than DM-based computer simulations predict, and
b. puzzles about how galaxies managed to **form** as quickly as they did starting from small initial perturbations away from uniformity of the original "hot soup" – models indicate it could not have happened that quickly. Also possibly related is the question
c. of how "supermassive black holes" managed to form so quickly.

E.g. computer simulation work led by Pavel Kroupa has indicated MOND can solve the missing satellite problem.

3. Time-evolution of ⟪Λ⟫. As the universe evolved, it got "clumpier" and "less uniform." Those clumps in turn, in SG, could trigger local Λ-switcheroos, which in turn could amplify clumpiness. Therefore the evolving clumpiness of the universe could alter the universe-wide average ⟨Λ⟩, causing it *not* to be constant, even if the two particular values of Λ on either side of all those switcheroos, remained fixed constants. SG can do that. GR(Λ) cannot.

4. "Hubble tension." One approach to assessing the time-evolution of ⟨Λ⟩ as well as the whole history of the FLRW-model universe-scale function(t) since the big bang, relies on a "distance ladder" deduced from

various kinds of "standard candle" astronomical objects. A different approach is based directly on the CMB (cosmic microwave background) sky "acoustics." Under the FLRW cosmological model family based on GR(Λ) and "cold dark matter," these two approaches obtain significantly *different* "Hubble" redshift↔distance relationships from observations plus theory-based number crunching, namely 73 km/sec/Mpc and 67.5 km/sec/Mpc respectively. (Over "5 sigma" discrepancy as of year 2026, with some recent re-analyses, e.g Riess, Anand et al 2024 claim ing 8σ.) So something is wrong.

> We shouldn't call [the "Hubble tension"] a tension or a problem, but rather a *crisis*.
> – (2004 Nobel laureate) David J. Gross.

Possible rather obvious SG-based solution: All the standard candles were based on objects comparatively *close* to Earth, hence coming fairly *late* in the history of the universe (post-galaxy formation era). In contrast, the CMB came early. Between those two eras, the universe got "clumpier," see (3).

5. "Inflation." Inflation can be regarded as a giant change in the effective, or actual, value of Λ (i.e. during inflation it was vastly repulsive), perhaps related to a phase-change in the quantum vacuum. That phase change could have been triggered (at least in part) by high temperatures, although a "Switcheroo" based (at least in part) on Lanczos is also conceivable, given our arguments that Lanczos tensors exist that are capable of "measuring time" in homogeneous cosmological models.

6. "Lithium problem." Big bang nucleosynthesis models are often claimed to have been been highly successful at predicting isotopic abundances for the lightest elements. The "lithium problem" refers to the less-publicized fact that for lithium isotopes, those models failed. E.g. the most widely accepted models overpredict lithium-7 by a factor of 3-4, while Lithium-6 is >200 times (maybe even 1000×) *more* abundant than big bang nucleosynthesis models predict! (Two reviews of this are Fields 2011 and Spite, Spite, Bonifacio 2012.) Anders et al 2014 performed accelerator experiments to replicate big bang conditions re lithium synthesis; they confirmed the "problem." Several authors have gone so far as to proclaim the lithium problem must be solved with "new physics" such as new kinds of fundamental particles. Could SG or LG cure that disease, e.g. by interaction of a "switcheroo" and some nuclear "resonance"? Maybe.

7. Singularities ruining everything. GR can self-generate singularities of the metric, which it might be unable to then handle. SG/LG and the "mirror paradigm" offer the potential ability to defend against them.

8. Inability to quantize. If you try to regard GR as a quantum field theory of spin-2 massless "gravitons" which couple to mass-energy, after some initial success you lose because GR is "nonrenormalizable" hence cannot be married into the "standard model" framework. SG/LG offer potential possibly to alter that situation.

9. What do SG and/or LG say about the "bullet cluster" and its ilk? I have not attempted any detailed fit. However, I will say *intuitively* that stairstep, even without any dark matter, might be able to produce effects at least *qualitatively* similar to the bullet cluster observations that some claimed "proved" the existence of dark matter. That is because a moving blob of "heavy vacuum" (whose surface is a stairstep Λ-transition surface; and within which |Λ| is postulated to be considerably larger than in space faraway from any galaxy) might intuitively be expected to "keep going" after a galaxy-cluster collision, due to "momentum conservation," even though the visible gas cloud would be expected to slow due to colliding with other cloud. (Which is exactly analogous to the alleged dark matter blob "keeping going" after the collision.)

> A common refrain is that MOND works well in galaxies, but not in clusters of galaxies... the residual missing baryon problem MOND suffers in clusters is a serious one. I do not see a reasonable solution.

– Stacy McGaugh 2024.

MOND fails to explain why galaxy clusters have masses nearly as large as their gravitational-lensing-measured values.

10. "Antimatter problem." The universe consists of >99.9% matter, and <0.01% antimatter, despite the laws of quantum field theory exactly obeying "CPT symmetry." It is possible in principle (as was realized, e.g. by A.Sakharov) to generate more matter than anitmatter during the early big bang by exploiting the non-symmetry of quantum field theories under any nonempty strict subset of {C,P,T}, and assuming the early universe was *not* in thermodynamic equilibrium due to rapid expansion. However, known laws and models fail to predict matter-favortism numerically anywhere near large enough to agree with observation. (Reviewed in Dine & Kusenko 2003.) By inserting new "stairsteps" triggered in the right moments during the early big bang, one obtains tremendous control over the thermodynamic "arrow of time" (one of Sakharov's three crucial ingredients). The same remarks, for the same reason, also can be made about stairstep gravity's possible curative effects on the lithium problem, as well as its possible explanation/forcing of early-universe "inflation" without needing to have any "inflatons." Also, as I remarked in Smith 2026, the "mirror paradigm" could permit the "big bang" to be a sheet-ending boundary of spacetime, at which new physical laws necessarily must be postulated, and we could then simply postulate such an asymmetry.

11. Missing dynamical friction problem: a moving body (e.g. galaxy) passing through a sea of dark matter (with which it interacts gravitationally) ought to be slowed by the well understood mechanism of Chandrasekhar "dynamical friction." Their putative dark matter halos should cause galaxies, rather than passing each other at high speed on hyperbolic orbits, instead to sink towards each other and merge. But this friction is not observed. The data on this is allegedly highly inconsistent with ΛCDM theory, cf. Kroupa "Galaxies as simple dynamical systems" 2015, Asencio, Banik, Kroupa 2020, Hernandez & Kroupa 2025.

12. External field problem for free-falling galaxies: Chae, Lelli, et al 2020 test the fundamental GR hypothesis that "the internal dynamics of a self-gravitating system (e.g. galaxy) under freefall in an external gravitational field should not depend on the external field strength." They find it is strongly *falsified* by their large galaxy dataset! MOND however predicts such "external field effects."

13. Apparent Newton law symmetry-violations. Kroupa et al 2022 & 2024 argue that Newton's law of "equal and opposite reaction" for gravitational forces is observationally falsified. MOND can do that. SG can effectively do that.

14. Black hole "paradoxes." The theoretical prediction (by GR enhanced with the simplest possible quantum additions) of "Hawking radiation" and black hole "Bekenstein-Hawking entropy" have led to contradictions between various well-accepted laws of physics, indicating something needs to change. These include the "information loss puzzle" where you throw something containing a lot of entropy into a black hole, making it vanish, making entropy decrease, contradicting the 2nd law of thermodynamics. That is repaired by ascribing Bekenstein-Hawking entropy to black holes, but then when that black hole later "Hawking evaporates" into radiation, that radiation apparently must be *unrelated* to whatever you threw in, contradicting the "unitarity" (reversibility) of all quantum field theories. Also: after a star made of >99.9% baryons collapses into a black hole, the Hawking radiation consists of >99.9% *non*baryons, contradicting "baryon number" and "lepton number" conservation laws. Also: the entropy equals a known constant times hole-horizon surface area, contradicting GR's assertion that horizons are not metrically special and that all infalling matter ends up far inside them.

Smith 2026 proposed one idea, the "**mirror paradigm**," that can solve those black hole paradoxes. The mirror paradigm proposes that the (3+1)-dimensional universe is like a 2-dimensional sheet of paper in

that both have *two sides*. Our side is the "universe," containing mainly matter particles. The other side is the "anti-universe" containing mainly antimatter particles. The two kinds of particles move in sync – an electron in the universe simultaneously *is* a positron in the (CPT reversed) anti-universe at the same location on the other side of the sheet. This matter and antimatter normally cannot annihilate because the sheet normally is impenetrable. However, black hole event horizons are postulated to be sheet-*boundaries*. An ant walking on a sheet of paper, when it reaches the end of the sheet, in order to continue walking forward would need to proceed onto the other side of the sheet. At black hole horizons, the mirror paradigm postulates the sheet ends – so there are no black hole "interiors." Matter-antimatter annihilations can therefore occur; but we cannot observe them because, classically, no signal is capable of being emitted from a black hole horizon to the outside world. Any mass-energy falling into a black hole therefore gets "flattened" onto the horizon surface and "trapped" there, classically forever, since with no black-hole interior there is nowhere for it to go. This classically causes infinite density of mass-energy on that surface. That in turn causes (via the GR field equations) horizons to be very metrically special and singular, causing the sheet to be *damaged*, explaining (admittedly via somewhat "circular" logic) why it ends. A paper by Einstein & Rosen proposed mathematics equivalent to this picture of black hole horizons as sheet-boundaries in 1935, but my re-interpretation of their mathematics as "mirror paradigm" came in 2026. At almost the same time (1 year before me) Tzanavaris, Boyle, Turok also independently invented almost the same idea, under the name "black mirror." Because there is no black hole interior, only a surface, the areal-entropy and information-loss paradoxes no longer are paradoxes. The baryon and lepton number nonconservations no longer are paradoxes either, since the pure energy (from matter-antimatter annihilations) trapped on that surface can Hawking-radiate out, via quantum tunneling, in any form. And regarding Hawking radiation as quantum tunneling actually now makes *sense*.

Also the "big bang" singularity also might be regardable as a sheet-boundary and we could postulate new laws of physics at sheet boundaries that could, e.g, solve the antimatter problem.

All that sounds very nice; but as I said, there was a worrying bit of "circular logic" within that story. SG/LG may offer a way to eliminate that worry. What is needed is some relativistically-acceptable way for spacetime points on black hole horizons to *know* they are on horizons, so that those points can act specially (sheet-damage, sheet-ending). As we have proven, under old-style GR(Λ) no local "on-horizon test" is possible, even in principle. But with the addition of the traceless Lanczos tensor to that picture, it looks like it *is* possible: I have produced a traceless Lanczos tensor L_{abc}, divergence-free in its rightmost index, such that $L_{abc}L^{abc}$ is infinite exactly on the horizons of the Reissner-Nordström de Sitter metric; and we have found a Lanczos for the Kerr metric with that same property.

23. Summary: Future work and Open problems

Dark matter has been attacked as a "theory of anything." The potential power of the SG/LG ideas is so great that it might ultimately end up subject to the same criticism! But, I think, not yet. I contend SG/LG are "more conservative" or "more parsimonious" changes to physics than either dark matter or MOND, because their key "new" ingredient, the traceless Lanczos tensor L_{abc}, was *already there* inside Einstein general relativity – it was simply a mathematical miracle that has been sitting around unused. And likely even more so for the Lanczos superpotential P_{ab}. In contrast, DM postulated new never-detected, particles that quite likely never will be detected. Meanwhile MOND postulated theoretically unacceptable laws in order to fit observation. SG/LG stays within Einstein's brilliant framework defining what is theoretically acceptable versus what is not, while offering MOND-like ability to fit observations old style GR(Λ), with or without DM, apparently cannot.

So for now, we need to investigate to see what SG and LG really can do and how they should be

formulated. (Open problem: Classify all permitted Lanczos-involving modification of Einstein's tensor equation.) They open up a whole new world. One of the most productive mathematicians ever, Pal Erdös – who had a drug-fueled quasi-religious approach to math – imagined that God possessed a big book, in fact *The* Big Book, containing all the best theorems and all their best proofs. And sometimes, if you worked very hard, you might be able to glimpse one of the pages of that Book. On this paper, I feel that both CLAUDE and I have worked hard, pushing our limits. And I think we may have revealed a few pages from The Physics Book.

Future efforts will need to be heavily driven by observational data to try to fill in the remaining details of SG or LG theory to both devise a nice theory and that fits data. We will need: computer-aided numerical work to explore different SG/LG versions and contrast them versus, and fit them to, observation datasets. And: A serious investigation of how to use statistics *well* to compare different theory-versions, and the DM+GR(Λ) picture, *fairly*, versus observational datasets, producing valid confidence numbers.

Key theoretical questions: I direct your attention to the horizon-detector conjecture, the superpotential conjecture (and extension), the switcheroo-surface-transcendence conjecture, my three crucial theoretical questions, and also this: Prove or disprove: traceless Lanczos tensors L_{abc} always exist such that $L_{abc}L^{abc}$ is infinite everywhere on every black hole horizon, but finite at every metrically-regular point in the "accessible region" outside those horizons. (Warning: May depend on precise notion of what a "horizon" is.)

Some criticisms and responses.

> "They are entitled to their stupid opinions."
> – Magnus Carlsen, 28 Nov. 2018, responding to criticism of the manner in which he'd just won the World Chess Championship versus Fabiano Caruana.

Criticism 1. Switcheroo is not a complete fundamental physical theory.

Response 1. I agree. SG's "vacuum phase changes" must be the result of some unknown more fundamental gravito-quantum field theory which would predict the whole phase diagram. And until SG's phase diagram and the rules for handling its Lanczosian gauge freedom are both specified, it will remain only a partly-specified theory of physics. I suggest: Determine SG's phase diagram and Lanczos gauge-handling rules by a long process featuring interaction between observation, modeling, theory design/refinement, development of numerical techniques to enable SG simulation, development of further SG understanding (e.g: Stability proof? Superpotential conjecture proof?), and development of statistical techniques capable of honestly assessing observation-based confidences for theory-comparisons in the light of all that. That all will take many years of work by many people. The situation may be similar to semi-empirical nuclear physics models for, e.g. inter-nucleon forces, which have been refined for decades and probably will continue to be refined forever. Something resembling that is probably the best SG could hope for.

After all that happens, if a phase diagram & etc completing SG is found to provide excellent agreement with observations, *then* those who want to develop future more-fundamental "final theory" will have something experiment-based to guide their quest. Without such guidance, they're dead in the water with nothing.

Criticism 2. Switcheroo is too hard to refute. (Unlike, say MOG and TeVeS, which I've refuted here.)

Response 2. That may not have been a "criticism" exactly, but yes: SG and LG are tough to refute. Probably they cannot be refuted purely theoretically, although you're welcome to try. To disprove SG, you'd

probably need to argue that among *all* possible vacuum phase diagrams, combined with *all* possible ways of handling Lanczosian gauge freedom, *none* are capable of matching observation. That sounds like a difficult task. It might not be obvious how to accomplish it. Let's compare that with the current-mainstream ΛCDM picture. How can you refute *it*? The invisible undetectable dark matter could be anywhere, distributed any way, doing anything. Somehow you'd need to argue that *none* of those possible dark matter scenarios could match observation. Or none of the "reasonable" ones (whatever that even means). Difficult. Maybe even impossible. SG offers this advantage for refutors/confirmers: While SG's phase diagram & gauge-handling presciption is not currently known, whatever it is, it is *fixed*. If you manage to determine that, you might enjoy centuries of excellent agreement with data, whereupon it'd be hard to argue with all that success. The point of "fixed" is: SG is the same for galaxy N+1 as for galaxy N, so the number of degrees of freedom in the theory does not grow with N. In contrast, with Λ-CDM, every galaxy could have different dark matter – far *more* degrees of freedom (growing proportional to N), a far *worse* "moving target."

Since SG and LG include GR(Λ) as a special case (and dark matter could, optionally, be added to both) any agreement to data GR(Λ) can manage, SG and LG can match or outperform. Since SG permits acceleration-thresholded MOND-like effects (similarly to MOG) but is far richer than MOND, it presumably should be able to match observations at least as well as MOG and MOND – and also presumably far better, because MOND's and MOG's problems at the galaxy-cluster scale presumably could easily be avoided by just adding one or more additional switcheroos to SG's phase diagram.

But I want to make it clear: SG is *not* merely refined GR(Λ). We have impossibility theorems saying GR(Λ) cannot do MOND-like feats. SG can. So SG is a *new and richer theoretical framework that overcomes proven fundamental limitations.*

Criticism 3. For some of their tensor identities, Smith & CLAUDE did not prove them using full symbolic simplification; they used "proof by numerical verification."

Response 3. If you can supply enough compute-resources, you will be able to do a full symbolic proof. We consider "probabilistic proofs" *more* reliable than deterministic proofs in situations where the probabilistic proof is simpler. Smith 2004 supplied the underlying theory (generalizing "Schwartz-Zippel lemma") of probabilistic proofs-by-random-checking for differential-algebraic identities, and if you want you could use that to determine how vast the number of "nines" are in the confidence the proofs-by-checking are right (99.999...%).

Criticism 4. Smith & CLAUDE do not fully understand their "sign flip" and "Killing vector" tricks for finding a lot of Lanczos tensors in closed form.

Response 4. So what? What matters is, Smith proved the same Lanczos↔Acceleration connection, with the same constants, for Kottler, general weak field scenarios, and static (and some nonstatic) perfect fluid scenarios, plus we've got further proven understanding for Kerr. That should more than suffice to compare versus observations.

Criticism 5. Smith & CLAUDE do not know the right way to handle Lanczosian "gauge freedom."

Response 5. This is a "stage of development," not a "defect." We have plenty of options to answer this. It is the future observation/refinement process's job to determine the right one.

Criticism 6. MOND failed to produce any consensus in 43 years. Therefore Smith & CLAUDE's stuff is just more lame quantum-gravity nuttiness that'll never yield consensus.

Response 6. *Of course* MOND failed to produce consensus, because it was obvious from day one, on

principle, that it was wrong. That made consensus impossible. It also meant those who worked a lot on MOND got marginalized and shunned, causing the total manpower, and quality of that manpower, working on it, to be small. With SG and LG, that changes. SG and LG are *not* "wrong on principle"; and SG and/or LG can (and presumably will) outperform MOND on matching observation. In §III.7 of the Companion Paper Smith numerically estimated 99.9996% as the chance this whole picture is the right path to travel now toward progress in fundamental physics. (And that estimate did not even use several other facts pointed out in *this* paper.) You can look at that estimate and decide whether to believe it and what exactly to have high confidence in... which e.g. might perhaps be only part, not the whole, of our picture... but seems to pertain to *something*. That combined with LG & SG's accessibility to observation should make SG/LG the pre-eminent top topic to investigate right now. (The comparison, versus, say, superstring theory is so vastly on SG/LG's side that it is absurd.)

The picture advanced in the present 3-paper team, including "superpotential," "complexified geometry," "magic generation of the ADM timeslicing paradigm," and "six foundations," offers a huge number of rich new avenues for investigation that could hook on to work of others. E.g. $\mathrm{im}(T_{ab})$ as a connection to quantum matter; and the consequence that would follow that the imaginary part of the stress tensor must separately be divergence-free. CLAUDE remarks "The self-dual complexified sector is already how some of the most productive gravity-reformulations work. Ashtekar's variables are a self-dual complex connection and the loop programme descends from them; the Plebanski action is built on self-dual two-forms; Penrose's nonlinear graviton is the anti-self-dual sector. These are the $W_{\circ abcd}=W_{abcd}+iW^*_{abcd}$ half of the Companion Paper, but arrived at independently and repeatedly by people trying to quantize gravity."

Criticism 7. Oh my God, Smith 2026 assassinated my sacred cow, "ER=EPR"!

Response 7. That is what happens when theoretical physicists adopt a lifestyle of treating rigorous mathematics as an irrelevant joke. And I've got news for you from the land of rigorous math: Even a single contradiction within your physical theory means *it is wrong.* This can exceed even the power of experiment. The "black hole paradoxes" should not be treated as a joke. They are fundamental contradictions in present physical law. If you believe the right answer is to paper them over with a giant pile of bullshit like "ER=EPR" and "AdS-CFT," then you should not be doing physics. You need a serious solution to those very real problems such as Smith 2026's Gordian-knot-cutting "mirror paradigm." That is enabled by SG and the Lanczos↔horizon-detector connections here.

24. References

A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z.

Zafar Ahsan: The Potential of Fields in Einstein's Theory of Gravitation, Springer, July 2019.

Zafar Ahsan, J.H.Caltenco, J.Lopez-Bonilla: Lanczos potential for the Gödel spacetime, Annalen der Physik 16,4 (April 2007) 311-313.

J.Lopez-Bonilla, V.Gaftoi, G.Ovando: Lanczos potential for the Gödel cosmological model, Czechoslovak J. Physics 52,6 (2002) 811-813.

Zafar Ahsan & Mohd Bilal: A Solution of Weyl-Lanczos Equations for Arbitrary Petrov Type D Vacuum Spacetimes, Int'l J. Theoretical Physics 49,11 (2010) 2713-2722.

Zafar Ahsan & Mohd Bilal: Lanczos Potential and Perfect Fluid Spacetimes, International J. of Theoretical Physics 50,6 (June 2011) 1752-1768.

Zafar Ahsan & Mohd Bilal: Lanczos Potential for Arbitrary Petrov Type II Spacetimes, International J. Theoretical Physics 52,12 (July 2013) 4275-4282,

Zafar Ahsan, Mohd Bilal, Musavvir Ali: Petrov Type D Spacetimes and Lanczos Potential, J. Tensor Society 11,1 (Sep.2017) 97-105.

Z.Ahsan, Jose Hector Caltenco, R.Linares, J.Lopez-Bonilla: Lanczos generator in Gödel Geometry, Communications in Physics 20,1 (2010) 9-14.

S.J.Aldersley: Dimensional analysis in relativistic gravitational theories, Physical Review D 15,2 (Jan.1977) 370-377.

S.J.Aldersley: Comments on certain divergence-free tensor densities in a 3-space, J.Math'l Physics 20,9 (Sep.1979) 1905-1907.

Mark G. Alford, Andreas Schmitt, Krishna Rajagopal, Thomas Schäfer: Color superconductivity in dense quark matter, Reviews of Modern Physics 80,4 (2008) 1455-1515.

M.Anders & 28 others (LUNA Collaboration): First Direct Measurement of the $^{2}H(\alpha,\gamma)^{6}Li$ Cross Section at Big Bang Energies and the Primordial Lithium Problem, Phys. Rev. Lett. 113,4 (July 2014) #042501.

Ian M. Anderson: Natural variational principles on Riemannian manifolds, Annals of Maths. 120,2 (Sep.1984) 329-370.

Ian M. Anderson & David Lovelock: A characterization of the Einstein tensor in terms of spinors, J. Mathematical Physics 17,6 (June 1976) 1001-1003.

Fredrik Andersson: Lanczos potentials and curvature-free connections aligned to a geodesic shear-free expanding null congruence, arXiv:gr-qc/0002016 (4 Feb. 2000). No journal version appears to exist.

F.Andersson & S. Brian Edgar: The wave equations for the Weyl tensor/spinor and dimensionally dependent tensor identities, International Journal of Modern Physics D 5,3 (June 1996) 217-225.

F.Andersson & S. Brian Edgar: Curvature-free asymmetric metric connections and Lanczos potentials in Kerr-Schild spacetimes, J. Math'l. Physics 39,5 (May 1998) 2859-2861.

Fredrik Andersson & S.Brian Edgar: Local existence of spinor and tensor potentials (1999); Existence of Lanczos potentials and superpotentials for the Weyl spinor/tensor, Classical & quantum gravity 18,12 (June 2001) 2297-2304.

G.W.Angus, H.Y.Shan, H.S.Zhao, B.Famaey: On the Proof of Dark Matter, the Law of Gravity, and the Mass of Neutrinos, Astrophys. J. 654,1 (2007) L13-L16. [The direct reply to Clowe et al. 2006: the bullet cluster is dominated by a collisionless component in MOND as well as in GR.]

G.W.Angus, B.Famaey, D.A.Buote: X-ray group and cluster mass profiles in MOND: unexplained mass on the group scale, Monthly Notices Royal Astron. Soc. 387,4 (2008) 1470-1480.

Elena Asencio, Indranil Banik, Pavel Kroupa: A massive blow for ΛCDM - the high redshift, mass, and collision velocity of the interacting galaxy cluster El Gordo contradicts concordance cosmology, Monthly Notices Royal Astron. Soc. 500,4 (2021) 5249-5267.

R.Bach & H.Weyl: Neue Losungen der Einsteinschen Gravitationsgleichungen. B. Explizite Aufstellung

statischer axialsymmetrischer Felder. Mit einem Zusatz uber das statische Zweikorperproblem von H. Weyl, Math. Zeitschrift 13 (1922) 134-145; English translation Gen. Relat. Gravit. 44 (2012) 817-832. W.Israel & K.A.Khan: Collinear particles and Bondi dipoles in general relativity, Nuovo Cimento 33 (1964) 331-344. These are two- and N-hole axisymmetric static Weyl solutions with struts. There is a rotating generalisation called "double-Kerr" by D.Kramer & G.Neugebauer: The superposition of two Kerr solutions, Phys. Lett. A 75 (1980) 259-261, obtained by Backlund transformation; see also W.Dietz & C.Hoenselaers: Stationary system of two masses kept apart by their gravitational spin-spin interaction, Phys. Rev. Lett. 48 (1982) 778, with the full treatment in W.D.&C.H: Two mass solutions of Einstein's vacuum equations: the double Kerr solution, Ann. Phys. (NY) 165 (1985) 319-383.

A.Balfagon & X.Jaen: Review of some classical superenergy tensors, Classical & Quantum Gravity 17 (2000) 2491-2497.

Franco Bampi & Giacomo Caviglia: Third-order tensor potentials for the Riemann and Weyl tensors, part I: General Relativity and Gravitation 15,4 (1983) 375-386; part II singular solutions: 16,5 (1984) 423-433.

Indranil Banik & Hongsheng Zhao: From galactic bars to the Hubble tension: weighing up the astrophysical evidence for Milgromian gravity, Symmetry 14 (2022) 1331.

Manuel Barriola & Alexander Vilenkin: Gravitational field of a global monopole, Phys. Rev. Lett. 63,4 (1989) 341-343.

John D. Barrow & Christos G. Tsagas: Dynamics and stability of the Gödel universe, Classical & Quantum Gravity 21,7 (2004) 1773-1790.

K.G.Begeman, A.H.Broeils, R.H.Sanders: Extended rotation curves of spiral galaxies: dark haloes and modified dynamics, Monthly Notices Royal Astron. Soc. 249,3 (1991) 523-537. [The source of the MOND acceleration constant $a_0 \approx 1.2 \times 10^{-10}$ meter/sec^2, from ten nearby galaxies.]

Jacob D. Bekenstein & Mordehai Milgrom: Does the missing mass problem signal the breakdown of Newtonian gravity?, Astrophysical J. 286,1 (Nov.1984) 7-14.

Jacob D. Bekenstein: Relativistic gravitation theory for the MOND paradigm, Phys. Rev. D 70 (2004) #083509; erratum Phys. Rev. D 71 (2005) #069901. [This is TeVeS, and is what the text cites; the 2005 date belongs to the erratum only. A separate review exists - "Modified gravity vs dark matter: relativistic theory for MOND", Proc. of Science JHW2004 (2005) 012, astro-ph/0412652 - but the unit-timelike vector field the text appeals to is defined in the 2004 paper itself.]

P.Bell & P.Szekeres: Some properties of higher spin rest-mass-zero fields in general relativity, Intl J Theor Phys 6,2 (Aug.1972) 111-121.

Jonas Bergman, S.Brian Edgar, Magnus Herberthson: The Bach tensor and other divergence-free tensors, Int'l. J. Geometric Methods in Modern Physics 02,01 (2005) 13-21.

Göran Bergqvist: A Lanczos potential in Kerr geometry, J. Math'l. Phys, 38,6 (June 1997) 3142-3154.

Ignazio Bombaci: The maximum mass of a neutron star, Astron. & Astrophys. 305 (1996) 871-877.

W.B.Bonnor: The gravitational field of light, Commun. Math. Phys. 13 (1969) 163-174; see also V.Faraoni & R.M.Dumse: The gravitational interaction of light: from weak to strong fields, Gen. Relat. Gravit. 31 (1999) 91-105.

Albert Bosma: THE DISTRIBUTION AND KINEMATICS OF NEUTRAL HYDROGEN IN SPIRAL GALAXIES OF VARIOUS MORPHOLOGICAL TYPES, PhD Thesis, Groningen University 1978.

Frederic Bournaud, Pierre-Alain Duc, Elias Brinks, 6 others: Missing Mass in Collisional Debris from Galaxies, Science 316,5828 (2007) 1166-1169.

Avery E. Broderick, Abraham Loeb, Ramesh Narayan: The event horizon of Sagittarius A*, Astrophysical J. 701,2 (2009) 1357-1366.

J.R.Brownstein & J.W.Moffat: Galaxy rotation curves without nonbaryonic dark matter, Astrophysical J. 636,2 (2006) 721-741. [Fits 101 galaxies with both MOND and Moffat's MOG.]

J.D.Bulnes, A.Baykal, T.Kim, J.Lopez-Bonilla: A generator of the Lanczos potential in Kerr spacetime via complexification of real coordinates, Advanced Studies in Contemporary Maths 35,3 (July 2025) 255-258.

Rong-Gen Cai & Shao-Jiang Wang: The Hubble tension: A decade review, Research in Astronomy and Astrophysics 26 (2026) #084011.

J.H.Caltenco, J.Lopez-Bonilla, G.Ovando, J.M.Rivera: Lanczos Superpotential for Kinnersley Spacetimes, Apeiron 9,1 (Jan.2002) 38-42.

G.M.Camacho-Gonzalez, J.L.Lopez-Bonilla, E.Velazquez-Lozada: Lanczos Generators for the Weyl Tensor in Kerr Geometry Prespacetime Journal 6,12 (Dec.2015) 1388-1391. J.L.Lopez-Bonilla, J.Morales, G.Ovando: General Relat. Gravit. 31,3 (1999) 413-415.

John Carminati & Raymond G. McLenaghan: Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space, J. Math'l. Phys. 32 (1991) 3135-3140. Emil Zakhary & C.B.G.Mcintosh: A Complete Set of Riemann Invariants, Gen.Relat.Gravit. 29,5 (1997) 539-581 Emil Zakhary & John Carminati: On the problem of algebraic completeness for the invariants of the Riemann Tensor I, J.Math'l Physics 42,3 (2001) 1474-1485. J.Carminati, E.Zakhary, R.G.McLenaghan: On the problem of algebraic completeness for the invariants of the Riemann tensor II, J.Math'l. Phys. 43,1 (2002) 492-507. John Carminati & Emil Zakhary: On the problem of algebraic completeness for the invariants of the Riemann tensor III. J.Mathematical Physics 43,8 (2002) 4020-4034.

Sean Michael Carroll: Spacetime and Geometry: An Introduction to General Relativity, Addison-Wesley 2003, reprinted 2019. QC173.6.C377.

Kyu-Hyun Chae, Federico Lelli, Harry Desmond, Stacy S. McGaugh, Pengfei Li, James M. Schombert: Testing the Strong Equivalence Principle: Detection of the External Field Effect in Rotationally Supported Galaxies, Astrophysical Journal 904,1 (Nov. 2020) #51, 20 pp.

Subrahmanyan Chandrasekhar: The Mathematical Theory of Black Holes, Oxford Univ. Press 1983 (Int'l Series of Monographs on Physics 69; Oxford Classic Texts reissue 1998). QB843.B55C48.

Jens P.R. Christensen: On sets of Haar measure zero in abelian Polish groups, Israel J. Maths. 13,3-4 (Sep.1972) 255-260.

Steven M. Christensen & Leonard Parker: MathTensor: A System for Performing Tensor Analysis by Computer, The Mathematica Journal 1,1 (1990) 51-61. MathTensor, Inc., PO Box 16175, Chapel Hill, NC 27516, USA, United States.

Steven M. Christensen: Large scale tensor analysis by computer, Computer Physics Communications

115,2â€“3 (Dec.1998) 245-263

Ignazio Ciufolini & John A. Wheeler: Gravitation and inertia, Princeton Univ. Press 1995. QC173.59.G44C58

Douglas Clowe, Marusa Bradac, Anthony H. Gonzalez, Maxim Markevitch, Scott W. Randall, Christine Jones, Dennis Zaritsky: A Direct Empirical Proof of the Existence of Dark Matter, Astrophysical J. 648,2 (2006) L109-L113. [The "bullet cluster." For the more recent lensing analyses see S.Cha & 7 others, Astrophys. J. Letters 987,1 (2025) L15, arXiv:2503.21870; and G.Rihtaršič & 27 others, Astron. & Astrophys. 710 (2026) A207. For the MOND-side reading of the same data see Angus, Shan, Zhao & Famaey 2007 above, and B.Famaey, arXiv:2605.10022 (2026).]

Douglas Clowe & 5 others: On Dark Peaks and Missing Mass: A Weak Lensing Mass Reconstruction of the Merging Cluster System Abell 520, Astrophysical J. 758 (2012) 128. [Reports no detection of the "dark core" that Mahdavi et al. 2007 and Jee et al. 2012 report in the same cluster. The disagreement is unresolved; so Abell 520 should be cited only with this caveat.]

C.D. Collinson: A Class of Conserved Tensors in an Arbitrary Gravitational Field, Mathematical Proceedings of the Cambridge Philosophical Society 58,2 (1962) 346-362

C.R.Contaldi, T.Wiseman, B.Withers: TeVeS gets caught on caustics, Phys. Rev. D 78,4 (2008) #044034.

Emile C. Cotton: Sur les varietes a trois dimensions, Annales de la faculte des sciences de Toulouse (II) 1,4 (1899) 385-438.

Richard Courant & David Hilbert: Methods of Mathematical Physics vol. II: Partial Differential Equations, Interscience/Wiley 1962 (Wiley Classics Library reissue 1989). QA401.C6813.

Erik Curiel: On Tensorial Concomitants and the Non-Existence of a Gravitational Stress-Energy Tensor, Studies in History & Philosophy of Modern Physics 66 (2019) 90-102. Also a 2010 version here: https://arxiv.org/abs/0908.3322.

H.E.J.Curzon: Cylindrical solutions of Einstein's gravitation equations, Proc. London Math. Soc. s2-23,1 (1925) 477-480.

W.J.G. de Blok: The Core-Cusp Problem, Advances in Astronomy 2010 (2010), Article #789293, 14 pages.

Michael Dine & Alexander Kusenko: Origin of the matter-antimatter asymmetry, Reviews in Modern Physics 76,1 (2003) 1-30.

P.Dolan & Annelies Gerber: The Weyl-Lanczos Equations and the Lanczos Wave Equation in 4 Dimensions as Systems in Involution, J.Math'l.Phys. 44 (2003) 3035-3045.

Patrick Dolan & Chul-Won Kim: The wave equation for the Lanczos potential: I. Proc. Royal Society London A447,1931 (1994a) 557-575.

Patrick Dolan & Chul-Won Kim: Some solutions of the Lanczos vacuum wave equation, Proc. Royal Soc. London A447,1931 (Dec.1994b) 577-585.

Patrick Dolan & Bruno D. Muratori: The Lanczos potential for vacuum spacetimes with an Ernst potential, J. Math'l. Phys. 39,10 (Oct.1998) 5406-5420.

P.Dolan & B.D.Muratori: The Lanczos potential for vacuum spaceâ€“times with an Ernst potential, J. Math'l. Phys. 39 (Oct.1998) 5406â€“5420 https://doi.org/10.1063/1.532580

S.Brian Edgar: The wave equations for the Lanczos tensor/spinor, and a new tensor identity, Modern Phys. Letters A 9,6 (1994) 479-482.

S.Brian Edgar: Proofs of existence of local potentials for trace-free symmetric 2-forms using dimensionally dependent identities, Journal of Geometry and Physics 54,3 (July 2005) 251-261.

S.Brian Edgar: On Effective Constraints for the Riemann-Lanczos System of Equations, J.Math'l.Phys. 44 (2003) 5375-5385.

S.Brian Edgar: Nonexistence of the Lanczos potential for the Riemann tensor in higher dimensions, General Relativity and Gravitation 26,3 (1994) 329-332.

S.Brian Edgar & Anders Höglund: The Lanczos potential for the Weyl curvature tensor: existence, wave equation and algorithms, Proc. Royal Soc. London A 453,1959 (Apr.1997) 835-851.

S.Brian Edgar & Anders Höglund: The Lanczos potential for Weyl-candidate tensors exists only in four dimensions, Gen.Relat.Gravit. 32,12 (2000) 2307-2318.

S.Brian Edgar & Anders Höglund: Dimensionally dependent tensor identities by double antisymmetrization, J. Math'l Physics 43 (2002) 659-677.

S.Brian Edgar & Anders Höglund: The non-existence of a Lanczos potential for the Weyl curvature tensor in dimensions n≥7, Gen.Rel.Grav. 34,12 (2002) 2149-2153.

M.I.Eremets, A.G.Gavriliuk, I.A.Trojan, D.A.Dzivenko, R.Boehler: Single-bonded cubic form of nitrogen, Nature Materials 3,8 (2004) 558-563.

Lawrence C. Evans: Partial Differential Equations, 2nd edition, American Math. Society (Graduate Studies in Mathematics 19) 2010. QA377.E95.

Sandra M. Faber & Robert E. Jackson: Velocity dispersions and mass-to-light ratios for elliptical galaxies, Astrophysical J. 204,1 (1976) 668-683.

Benoit Famaey & Stacy McGaugh: Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions, 2011-2012. This is a 164-page review article. Also available as *Living reviews in relativity* 15 (2012) 10 at http://relativity.livingreviews.org/Articles/lrr-2012-10/.

Joan Josep Ferrando & Juan Antonio Saez: Obtaining the Weyl tensor from the Bel-Robinson tensor, General Relativity & Gravitation 42,6 (2010) 1469-1490.

Ignacio Ferreras, Nick E. Mavromatos, Mairi Sakellariadou, Muhammad Furqaan Yusaf: Confronting MOND and TeVeS with strong gravitational lensing over galactic scales: an extended survey, Phys. Rev. D 86,8 (2012) #083507.

Brian D. Fields: The Primordial Lithium Problem, Annual Review Nuclear & Particle Science 61 (Nov. 2011) 47-68. Update: arXiv:1412.1408.

E.A.Flanagan & S.A.Hughes: The basics of gravitational wave theory, New J. Physics 7 (2005) 204, 51pp.

Hector Flores, Francois Hammer, 5 others (incl. P.Kroupa): Young tidal dwarf galaxies cannot be used to probe dark matter in galaxies, Monthly Notices Royal Astron. Soc. Letters 457,1 (2016) L14-L18.

Chris L. Fryer & Vassiliki Kalogera: Theoretical black hole mass distributions, Astrophysical J. 554,1 (2001) 548-560.

Alberto A. Garcia, Friedrich W. Hehl, Christian Heinicke, Alfredo Macias: The Cotton tensor in Riemannian spacetimes, Classical & Quantum Gravity 21,4 (2004) 1099-1118.

Janusz Garecki: Some remarks on the Bel-Robinson Tensor, Annalen d. Physik 10 (2001) 911-919.

Gianfranco Gentile, Benoit Famaey, Francoise Combes, Pavel Kroupa, HongSheng Zhao, Olivier Tiret: Tidal dwarf galaxies as a test of fundamental physics, Astron. & Astrophys. 472,2 (2007) L25-L28.

Robert Geroch & James B. Hartle: Distorted black holes, J. Math'l. Physics 23 (Apr.1982) 680-692.

Kurt Gödel: An example of a new type of cosmological solution of Einstein's field equations of gravitation, Reviews in Modern Physics 21,3 (July 1949) 447-450.

Oliver Gould, Sinan Güyer, Kari Rummukainen: First-order electroweak phase transitions: A nonperturbative update, Physical Review D 106 (2022) #114507.

Alan H. Guth: Eternal inflation and its implications, J.Physics A 40 (2007) 6811-6826. [Review article.]

J.B.Hartle & S.W.Hawking: Solutions of the Einstein-Maxwell Equations with Many Black Holes, Communications in Mathematical Physics 26,2 (Jun.1972) 87-101.

Moritz Haslbauer, Indranil Banik, 3 others: The Magellanic Clouds are very rare in the IllustrisTNG simulations, Universe 10,10 (Oct.2024) #385.

A.H.Hasmani & Ravi Panchal: Lanczos potential for some non-vacuum spacetimes, European Physical Journal Plus 131,9 (Sep.2016) 1-6.

Stephen W. Hawking: Chronology protection conjecture, Physical Review D. 46,2 (July 1992) 603-611.

S.W.Hawking & G.F.R.Ellis: The large scale structure of space-time, Cambridge Univ. Press 1973. QC173.59.S65H38.

Sean A. Hayward: General laws of black-hole dynamics, Phys. Rev. D 49,12 (June 1994) 6467-6474.

Xavier Hernandez & Pavel Kroupa: A recent confirmation of the wide binary gravitational anomaly, Monthly Notices Royal Astron. Soc. 537,3 (2025) 2925-2930.

Gregory W. Horndeski: Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space, International J. of Theoretical Physics 10,6 (Sept.1974) 363-384.

Brian R. Hunt, Tim Sauer, James A. Yorke: Prevalence: a translation-invariant “almost every” on infinite-dimensional spaces, Bull. Amer. Math. Soc. 27,2 (1992) 217-238; addendum 28,2 (1993) 306-307.

Reinhard Illge: On potentials for several classes of spinor and tensor fields in curved space-times, General Relativity & Gravitation 20,6 (June 1988) 551-564. [Illge's EQ 34 was later shown to be missing a term, although supposedly the spinor equation it came from was correct; there also is an error related to an argument about superpotential supposedly corrected by Andersson & Edgar.]

Yongjun Jiao & 6 others: Detection of the Keplerian decline in the Milky Way rotation curve, Astronomy & Astrophysics 678 (27 sept. 2023) A208.

Prajwal R. Kafle, Sanjib Sharma, Geraint F. Lewis, Joss Bland-Hawthorn: Kinematics of the stellar halo and the mass distribution of the Milky Way using BHB stars, The Astrophysical Journal 761,2 (2012) #17.

Vassiliki Kalogera & Gordon Baym: The maximum mass of a neutron star, Astrophysical J. 470 (1996) L61-L64.

I.D. Karachentsev: Missing Dark Matter in the Local Universe, Astrophysical Bulletin 67,2 (2012) 123-134.

I.D.Karachentsev & K.N.Telikova: Stellar and dark matter density in the Local Universe, Astronomische Nachrichten 339,615 (Aug.2018) 615-622.

Willem H. Keesom & Johannes A. Kok: On the change of the specific heat of tin when becoming superconductive, Akademie der Wetenschappen Amsterdam Proc. 35 (1932) 744-748.

Willen H. Keesom & Johannes A. Kok: Measurements of the specific heat of thallium at liquid helium temperatures, Physica 1,1-6 (1934) 175-181.

Roy P. Kerr: Gravitational field of a spinning mass as an example of algebraically special metrics, Phys. Rev. Lett. 11,5 (1963) 237-238.

William M. Kinnersley: Type D Vacuum Metrics, J. Math'l. Phys. 10,7 (1969) 1195-1203.

Friedrich Kottler: Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie, Annalen der Physik 361,14 (1918) 401-462. [The Schwarzschild-de Sitter metric.]

Carlos N. Kozameh, Ezra T. Newman, K.P. Tod: Conformal Einstein Spaces, Gen. Relat. & Gravit. 17 (April 1985) 343-352.

Andrzej Krasinski: Inhomogeneous Cosmological Models, (Cambridge University Press, 1997). QB981.K76. Includes survey of dust solutions and of the "bound," "marginally bound," and "unbound" classification.

Pavel Kroupa: Dwarf spheroidal satellite galaxies without dark matter, New Astronomy 2,2 (July 1997) 139-164.

Pavel Kroupa: The dark matter crisis: falsification of the current standard model of cosmology, Publications of the Astronomical Society of Australia (CSIRO Publishing) 29,4 (2012) 395-433.

Pavel Kroupa: Lessons from the Local Group (and beyond) on dark matter, Seychelles conference on galaxy evolution, "Lessons from the Local Group," ed. K.C. Freeman, B.G. Elmegreen, D.L. Block, and M.Woolway (Dordrecht: Springer) 2015.

Pavel Kroupa: Galaxies as simple dynamical systems: observational data disfavor dark matter and stochastic star formation, Canadian J. Physics 93,2 (2015) 169-202. Part of a special issue on MOND: modified Newtonian dynamics.

Pavel Kroupa, Marcel Pawlowski, Mordehai Milgrom: The failures of the standard model of cosmology require a new paradigm, Int'l J. Modern Physics D 21,14 (2012) 13 pages.

Pavel Kroupa, Christian Theis, Christian M. Boily: The great disk of Milky-Way satellites and cosmological sub-structures, Astron.Astrophys. 431 (2005) 517-521.

Pavel Kroupa, Tereza Jerabkova, & 15 others: Asymmetrical tidal tails of open star clusters: stars crossing their cluster's prah challenge Newtonian gravitation, Monthly Notices of the Royal Astronomical Society 517,3 (Dec.2022) 3613-3639.

Pavel Kroupa, Jan Pflamm-Altenburg, 5 others: Open Star Clusters and Their Asymmetrical Tidal Tails, The Astrophysical Journal 970,1 (July 2024) #94, 17 pp.

Kornel Lanczos: Über eine stationäre Kosmologie im Sinne der Einsteinschen Gravitationstheorie, Zeitschrift für Physik 21,1 (1924) 73-110. Reprinted as On a Stationary Cosmology in the Sense of Einstein's Theory of Gravitation, General Relativity & Gravitation 29,3 (Mar.1997) 363-399.

Cornelius Lanczos: A remarkable property of the Riemann-Christoffel tensor in four dimensions, Annals of Mathematics 39 (1938) 842-850. [The source of both the Lanczos/Gauss-Bonnet identity and, with Ruse 1944, of the four-dimensional double-dual "Ruse-Lanczos identity."]

Cornelius Lanczos: The splitting of the Riemann tensor, Reviews Modern Physics 34,3 (July 1962) 379-389. [The historically first paper on this topic was C.Lanczos: Lagrangian Multiplier and Riemannian Spaces, Reviews of Modern Physics 21,3 (1949) 497-502.]

Lev D. Landau & Evgeny M. Lifshitz: The Classical Theory of Fields (Course of Theoretical Physics vol.2), 4th revised English edition, transl. M.Hamermesh, Pergamon 1975, reprinted with corrections 1980. QC665.E4L2313. [The 3rd revised English edition is 1971; the text cites both years, and they should be reconciled to one.]

J.Lee & E.Komatsu: Bullet Cluster: A Challenge to ΛCDM Cosmology, Astrophys. J. 718 (2010) 60-65.

Federico Lelli, Stacy S. McGaugh, James M. Schombert: SPARC: Mass Models for 175 Disk Galaxies with Spitzer Photometry and Accurate Rotation Curves, Astronomical J. 152,6 (2016) #157.

G.Lemaitre: L'univers en expansion, Annal. Soc. Sci. Bruxelles A53 (1933) 51-85, English translation by M.A.H.MacCallum *The expanding universe* in Gen.Relat.Gravit. 29,5 (1997) 641-680; R.C.Tolman: Effect of inhomogeneity on cosmological models, Proc. Natl. Acad. Sci. USA 20 (1934) 169-176; H.Bondi: Spherically symmetrical models in general relativity, Monthly Notices Royal Astron. Soc. 107 (1947) 410-425.

J.L.Lopez-Bonilla, Jean Pierre Morales, D.Navarrete, M.Rosales: Lanczos spin tensor for empty type D spacetimes, Classical & Quantum Grav. 10,10 (1993) 2153-2156.

Jose L. Lopez-Bonilla & Gerardo Ovando: A Lanczos Spintensor for the Gödel Metric, Gen. Relat. & Gravit. 31,7 (1999) 1071-1074.

Jose L. Lopez-Bonilla, G. Ovando, J.J. Pena: A Lanczos potential for plane gravitational waves, Foundation of Physics Letters 12,4 (1999) 401-405.

David Lovelock: The uniqueness of the Einstein-Maxwell field equations, General Relativity & Gravitation 5 (1974) 399-408.

David Lovelock & Hanno Rund: Tensors, Differential Forms, and Variational Principles, Wiley 1975; reprinted by Dover 1989. QA433.L67.

David Lovelock: The uniqueness of the Einstein field equations in a four-dimensional space, Archive Rational Mech. Anal. 33 (1969) 54-70.

David Lovelock: Divergence-free tensorial concomitants, Aequationes Math. 2 (June 1969) 394-395 (preview) and (main article) 4 (Feb. 1970) 127â€“138.

David Lovelock: The Einstein tensor and its generalizations, J. Math'l. Phys. 12 (1971) 498-501.

David Lovelock: The four-dimensionality of space and the Einstein tensor, J. Math'l Physics 13,6 (1972) 874-876.

Malcolm A.H. MacCallum: Computer algebra in gravity research (Review Article), Living Reviews in Relativity 21 (Aug.2018) #6.

Andisheh Mahdavi, Henk Hoekstra, Arif Babul, David D. Balam, Peter L. Capak: A Dark Core in Abell 520, Astrophysical J. 668,2 (2007) 806-814. [A weak-lensing mass peak "largely devoid of galaxies." See also M.J.Jee & 6 others, Astrophys. J. 747,2 (2012) 96, arXiv:1202.6368; and the contrary reconstruction of Clowe et al. 2012 above.]

Sudhansu Datta Majumdar: A Class of Exact Solutions of Einstein's Field Equations, Physical Review 72,5 (1947) 390-398.

Enrico Massa & Enrico Pagani: Is the Riemann tensor derivable from a tensor potential? Gen. Rel. Gravit. 16,9 (Sept.1984) 805-816.

Philip Massey & Michael R. Meyer: Stellar Masses, article #1882 in *Encyclopedia of Astronomy and Astrophysics*, ed. P.Murdin, IOP Publishing / Nature Publishing Group 2001. (These articles are numbered, not paginated.)

Stacy McGaugh: Distinguishing between CDM and MOND: Predictions for the Microwave Background, Astrophys.J. 523,2 (1999) L99-L102.

Stacy McGaugh: MOND pages http://astroweb.case.edu/ssm/mond/. Includes large bibliography of papers about MOND: http://astroweb.case.edu/ssm/mond/litsub.html, Tabular comparison of MOND versus ΛCDM: http://astroweb.case.edu/ssm/mond/LCDMmondtesttable.html, lecture slides, and humorous flowchart giving simple algorithm to evaluate observations/theories: http://astroweb.case.edu/ssm/mond/LCDMMONDflowchart.png

Stacy McGaugh: A Tale of Two Paradigms: the Mutual Incommensurability of ΛCDM and MOND, Canadian J. Physics 93,2 (2015) 250-259. Part of a special issue on MOND: modified Newtonian dynamics.

Stacy McGaugh, Federico Lelli, Jim Schombert: The Radial Acceleration Relation in Rotationally Supported Galaxies, Phys. Rev. Lett. 117,20 (2016) #201101.

Stacy McGaugh & Moti Milgrom: Andromeda Dwarfs in Light of MOND. II. Testing Prior Predictions http://arxiv.org/abs/1308.5894.

Stacy McGaugh & Mordehai Milgrom: Andromeda Dwarfs in Light of MOND, Astrophysical Journal 766,1 (2013) 22 and see also part II: Testing Prior Predictions (2013).

R.J. McKellar: The uniqueness of gravity as a Poincare or Lorentz gauge theory, J. Mathematical Physics

22,12 (Dec. 1981) 2934-2942.

R.J.McKellar: Lorentz invariance and the equivalence of metric, spinor, and frame field concomitants, J. Mathematical Physics 28,8 (Aug.1987) 1840-1843.

Martin A. Melvin: Pure magnetic and electric geons, Physics Letters 8,1 (1964) 65-68.

M.Milgrom: A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis (part I), Astrophysical Journal 270,2 (July 1983) 365-370; see also part II, implications for galaxies: 371-383 and III: galaxy systems: 384-389.

Mordehai Milgrom: MOND mass-to-light ratios for galaxy groups, Astrophys.J. 577,2 (2002) L75-L77.

Mordehai Milgrom: MOND and the unique void galaxy KK246, http://arxiv.org/abs/1104.1118 (2011).

Mordehai Milgrom: Testing MOND Over a Wide Acceleration Range in X-Ray Ellipticals, Phys. Rev. Lett. 109,13 (2012) #131101.

Mordehai Milgrom & Robert H. Sanders: Rings and shells of "dark matter" as MOND artifacts, Astrophys. J. 678,1 (2008) 131-143.

Mordehai Milgrom & Robert H. Sanders: Modified Newtonian Dynamics and the "Dearth of Dark Matter in Ordinary Elliptical Galaxies", Astrophysical Journal 599,1 (2003) L25-L28.

Mordehai Milgrom: MOND theory, Canadian J. Physics 93,2 (2015) 107-118, special issue devoted to MOND; The a_0-cosmology connection in MOND, arXiv:2001.09729 (2020), contribution to BonnGravity2019; no journal version.

C.W.Misner & D.H.Sharp: Relativistic equations for adiabatic, spherically symmetric gravitational collapse, Phys. Rev. 136 (1964) B571-B576.

Charles Misner, Kip Thorne, John Wheeler: Gravitation, W.H. Freeman & Co. 1973. QC178.M57.

Boris S. Mityagin: The zero set of a real analytic function, Mathematical Notes 107,3 (2020) 529-530; Mat. Zametki 107,3 (2020) 473-475.

John W. Moffat: Reinventing Gravity: A Physicist Goes Beyond Einstein, Smithsonian books 2008. QC178.M64.

Cesar Mora & Ruben Sanchez: A heuristic review of Lanczos potential, Latin American J. Physic Education 1,1, (Sept.2007) 78-82.

Cesar Mora & Ruben Sanchez: Lanczos Potential for the van Stockung Space-Time, Int'l J. Theor. Phys. 48,5 (2009) 1357-1368.

J.Morales, G.Ovando, J.Lopez-Bonilla, R.Lopez-Vaì□zquez: Lanczos Potential for The Weyl Tensor, To Physics Journal 1,3 (2018) 189-195. [This is indeed the paper the text refers to. Its §3, headed "Andersson-Edgar's potential for the Lanczos spinor", gives that generator for arbitrary Petrov types O, N and III in its eq.(20) and for Petrov type D vacuums in its eq.(22), leaving types I and II open. Note for the two-notions discussion that what it constructs is a *spinor* $T_{ABC}{}^{E}$ with $L_{AB\dot{C}D} = \nabla_{\dot{E}D}T_{ABC}{}^{E}$, not a symmetric two-index tensor P_{ab}; the word "superpotential" does not occur in its body text.] Also R.Sivaraman, J.Morales, G.Ovando, J.Lopez-Bonilla: Lanczos Spintensor via the Andersson-Edgar's Generator Europ.

J. Theoretical and Applied Sciences 1,5 (Sep.2023) 1383-1387.

Michael S. Morris & Kip S. Thorne: Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, American J. Physics. 56,5 (1988) 395-412.

Thomas Mueller & Frank Grave: Catalogue of Spacetimes (arXiv 2010).

Uwe Mueller, Christian Schubert, Anton van de Ven: A Closed Formula for the Riemann Normal Coordinate Expansion, Gen.Relat.Gravit. 31 (1999) 1759-1768.

Sam Mugiraneza & Alannah M. Hallas: Tutorial: a beginner's guide to interpreting magnetic susceptibility data with the Curie-Weiss law, Communications Physics 5 (2022) #95.

Ramesh Narayan, Michael R. Garcia, Jeffrey E. McClintock: X-ray novae and the evidence for black hole event horizons, pp. 405-425 in Proc. IX Marcel Grossmann Meeting, World Scientific 2001.

Alberto & Jose Navarro: Lovelock's theorem revisited, J.Geometry & Physics 61 (2011) 1950-1956.

Jose Navarro & Juan B. Sancho: On the naturalness of Einstein's equation, J.Geometry & Physics 58,8 (2008) 1007-1014.

Gunnar Nordström: On the Energy of the Gravitational Field in Einstein's Theory, Verhandelingen Koninkl. Ned. Akad. Wetenschap., Afd. Natuurk., Amsterdam 26 (1918) 1201-1208.

M.Novello, E.Bittencourt, J.M.Salim: The Quasi-Maxwellian Equations of General Relativity: Applications to Perturbation Theory, Brazilian J. Physics 44,6 (2014) 832-894; arXiv:1302.7248, February 2013. [Cited in the text as "Novello, Bittencourt, Salim 2013" from the arXiv date. This is the quasi-Maxwellian paper, not the six-author "Geometric scalar theory of gravity", JCAP 06 (2013) 014: it is the one with exactly these three authors, and the one whose abstract credits the formalism to Jordan, Ehlers and Kundt and contrasts it with Lifshitz - matching the historical remark in the text.]

Mario Novello & A.L.Velloso: The connection between the general observers and Lanczos potential, General Relativity and Gravitation 19,12 (Dec.1987) 1251-1265.

Peter O'Donnell: Letter: A solution of the Weyl-Lanczos equations for the Schwarzschild space-time, General Relativity & Gravitation 36,6 (June 2004) 1415-1422.

Peter J. O'Donnell: A method for finding Lanczos coefficients with the aid of symmetries, Czechoslovak J. Phys. 54,9 (2004) 889-896.

Peter J. O'Donnell: Lanczos tensor potential for conformally flat space-times, Il Nuovo Cimento B 119,4 (April 2004) 341-345. Conformal transformations of the Lanczos potential, Relativity, Gravitation, & Cosmology 2,2 (2005) 195-200. Lanczos potential for conformally flat space-times II, Relativity, Gravitation, & Cosmology 3,3 (2009).

Peter O'Donnell: On a new approach of deriving the Weyl-Lanczos equations giving rise to a physical interpretation of the Lagrange multiplier q, European Physical J. Plus 126,9 (Sep.2011) #87 (6 pages).

Peter O'Donnell & H.Pye: A Brief Historical Review of the Important Developments in Lanczos Potential Theory, Electronic J. Theoretical Physics 7,24 (2010) 327-350.

P.J.O'Donnell & A.Roberts: A Solution of the Weyl-Lanczos Equations for a Type N Space-Time, ICFAI

University Journal of Physics 1,3 (2008) 7-16.

J. Robert Oppenheimer & Hartland Snyder: On Continued Gravitational Contraction, Physical Review 56,5 (1939) 455-459. O&S's scenario was investigated earlier by B.Datt: Über eine Klasse von Lösungen der Gravitationsgleichungen der Relativitat, Zeitschrift der Physik 108 (May 1938) 314-321, reprinted in translation, Gen.Relat.Gravit. 31 (1999) 1619.

William Ott & James A. Yorke: Prevalence, Bull. Amer. Math. Soc. 42,3 (2005) 263-291. (Another.)

Ravi Panchal, Ashok C. Patel, Abdulvahid H. Hasmani: On Lanczos Potential for Spherically Symmetric Spacetimes, J. of the Tensor Society 13,1 (June 2007) 31-43.

Achille Papapetrou: A static solution of the equations of the gravitational field for an arbitrary charge-distribution, Proc. Royal Irish Academy A 51 (1945) 191-204. [With Majumdar 1947: the Majumdar-Papapetrou multi-black-hole solutions.]

Gonzalo Ares de Parga, O. Chavoya, Jose L. Lopez Bonilla, Jesus Morales R.: The Lanczos superpotential (paper is in Spanish), Revista Mexicana de Fisica 36,1 (January 1990) 85-93. Gonzalo Ares de Parga, Oscar Chavoya A, Jose L. Lopez Bonilla: Lanczos potential, J. Mathematical Physics 30,6 (1989) 1294-1295.

Wolfgang Pauli: Theory of Relativity, Dover reprint of 1958 English-language edition (updates 1921 book in German). QC6.P323.

Adam G. Riess, Gagandeep S. Anand & 8 others: JWST Observations Reject Unrecognized Crowding of Cepheid Photometry as Explanation for the Hubble Tension at 8σ Confidence, Astrophysical Journal Letters 962,1 (Feb.2024) L17.

Ji Rigelesaiyin, Adrian Diaz, Weixuan Li, Liming Xiong, Youping Chen: Asymmetry of the atomic-level stress tensor in homogeneous and inhomogeneous materials, Proc. Math. Phys. Eng. Sci. 474,2217 (Sep.2018) #20180155.

John A. Peacock: Cosmological Physics, Cambridge University Press 1999. QB981.P37.

P.J.E.Peebles: Principles of physical cosmology, Priceton Univ. Press 1993. QB981.P424.

Roger Penrose & Wolfgang Rindler: Spinors and space-time, Cambridge University Press 1984. QC20.7.S65P46

Thiago S. Pereira, Cyril Pitrou, Jean-Philippe Uzan: Theory of cosmological perturbations in an anisotropic universe, J. Cosmol. Astropart. Phys. 09 (2007) 006. See also Yu Furuya, Yuki Niiyama, Yuuiti Sendouda: A quantisation procedure in the presence of an initial Kasner singularity: primordial gravitational waves from triaxially anisotropic pre-inflation, Classical and Quantum Gravity 36,8 (2019) #085007.

Michael E. Peskin & Daniel V. Schroeder: An introduction to quantum field theory, Perseus/Westview Press 1995.

Aleksei Zinov'evic Petrov: Einstein spaces, Pergamon Press, Oxford, 1969. QA689.P4713

Laurian-Ioan Piscoran, Jose L. Lopez-Bonilla, G.Perez-Teruel: Rotating black hole and a potential for its Weyl tensor, Global J. Advanced Research on Classical and Modern Geometries 4,1 (2015) 62-64.

Hans Reissner: Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie, Annalen der Physik 355,9 (1916) 106-120.

Gareth E. Roberts: Linear Stability of the 1+n-gon Relative Equilibrium, Hamiltonian Systems and Celestial Mechanics (HAMSYS-98), Proceedings of the III International Symposium, Patzcuaro, Mexico. World Scientific Monograph Series in Mathematics 6 (2000) 303-330.

Mark D. Roberts: The double duals and gauges of the Lanczos tensor, Czechoslovak Journal of Physics 42,2 (Feb.1992) 151-158.

Mark D. Roberts: The physical interpretation of the Lanczos tensor, Il Nuovo Cimento B110,10 (1996) 1165-1176.

Aaron J. Romanowsky, Nigel G. Douglas & 6 others: A dearth of dark matter in ordinary elliptical galaxies, Science 301,5640 (2003) 1696-1698. 15-page extended version: http://arxiv.org/abs/astro-ph/0308518.

Vera C. Rubin & W.Kent Ford Jr: Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions, Astrophysical J. 159 (feb.1970) 379-403. [The rotation-curve measurements that opened the subject.] Headline result: The rotation curve does not fall; i.e. the mass enclosed within radius r is still increasing at the outermost object measured, r=24 kpc. Their later paper V.C.Rubin, W.K.Ford Jr, N.Thonnard: Rotational properties of 21 Sc galaxies with a large range of luminosities and radii, from NGC 4605 (R=4 kpc) to UGC 2885 (R=122 kpc), The Astrophysical Journal 238 (June 1980) 471-487 extended that to 21 Sc galaxies with same result in all of them.

> "The conclusion is inescapable: mass, unlike luminosity, is not concentrated near the center of spiral galaxies."
> – Vera C. Rubin, Scientific American 248,6 (June 1983) 96-106.

More recent review: Yoshiaki Sofue & Vera Rubin: Rotation Curves of Spiral Galaxies, Ann.Rev.Astron.Astrophys. 39 (2001) 137-174.

Harold S. Ruse: On the Line-Geometry of the Riemann Tensor, Proc. Royal Society of Edinburgh A 62,1 (1944) 64-73. [With Lanczos 1938: the "Ruse-Lanczos identity," the four-dimensional double-dual identity.]

Andrei D. Sakharov: Violation of CP invariance, C asymmetry, and baryon asymmetry of the universe, Journal of Experimental and Theoretical Physics 5 (1967) 24-27 republished in Soviet Physics Uspekhi 34,5 (1991) 392-393.

Robert H. Sanders, Eva Sagi & Jacob D. Bekenstein: Black holes in the tensor-vector-scalar theory of gravity and their thermodynamics, Phys. Rev. D 77,2 (2008) #024010.

Eva Sagi: Propagation of gravitational waves in the generalized tensor-vector-scalar theory, Phys. Rev. D 81,6 (2010) #064031.

Robert H. Sanders: Clusters of galaxies with modified Newtonian dynamics, Monthly Notices Royal Astron. Soc. 342,3 (2003) 901-908. [MOND reduces but does not remove the dark component in clusters.]

Robert H. Sanders: NGC 2419 does not challenge MOND and part 2 (`http://arxiv.org/abs/1107.4953` and `http://arxiv.org/abs/1111.4334`) 2011. Shorter version published later in Monthly Notices Royal Astronomical Soc. 419 (2012) L6-L8.

Robert H. Sanders & Stacy S. McGaugh: Modified Newtonian Dynamics as an Alternative to Dark Matter,

Ann. Rev. Astron. Astrophys. 40 (2002) 263-317.

Jan Arnoldus Schouten: Ueber die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Massbestimmung auf eine Mannigfaltigkeit mit Euklidischer Massbestimmung, Math. Zeitschrift 11 (March 1921) 58-88.

Michael D. Seifert: Stability of spherically symmetric solutions in modified theories of gravity, Phys.Rev. D 76 (2007) 064002.

Jose M.M. Senovilla: Super-energy tensors, Classical and Quantum Gravity 17,14 (2000) 2799â€“2841. Applications of Super-energy tensors (arXiv 1999).

Constantinos Skordis, David F. Mota, Pedro G. Ferreira, Celine Boehm: Large scale structure in Bekenstein's theory of relativistic modified Newtonian dynamics, Phys. Rev. Lett. 96 (2006) #011301.

Warren D. Smith: Reasons the Photon Mass Should be Exactly Zero (2003).

Warren D. Smith: Quaternions, Octonions, and Now, 16-Ons and 2^n-Ons: New Kinds of Numbers (2004).

The other SG/LG-related papers in this series (in chronological order with approximate page counts):

I. Warren D. Smith: Turing-Unsimulability, and instantaneous & sustained Cosmic-Censorship-falsity, for Einstein-Vacuum General Relativity (2025). 80 pages. Not really part of the Switcheroo/Lanczos theoretical framework, but motivated II.
II. Warren D. Smith: Mostly-Destructive Consequences of Non- and Nowhere-Analyticity of the Spacetime Metric; and new "2-sided Mirror Paradigm" hypothetical grandiose universe picture (2026), 39 pages. (v1 15 July 2026; v3 of 30 Aug. 2026 added an appendix giving no-go theorem against "ER=EPR" and Kruskal wormhole).
III. Warren D. Smith & CLAUDE: Switcheroo and Lanczos Gravity: MOND done right (2026). The present paper and flagship of the series.
IV. Warren D. Smith & CLAUDE: Nonmetric foundations for differential geometry; and complex and bicomplex reformulations in the 4D case (2026). 47 pages. The "COMPANION PAPER," cited throughout as "the companion."

There also is a *short* summary/introduction to the entire series: Warren D. Smith: 14-page guide to Switcheroo Gravity & its companion papers (Oct.2026).

Monique & Francois Spite, Piercarlo Bonifacio: The cosmic lithium problem: an observer's perspective, Memorie della Societa Astronomica Italiana Supplementi 22 (2012) 9.

Hans.Stephani & 4 others: Exact Solutions of Einstein's Field Equations, Cambridge University Press 2003. QC173.6.E96.

Willem J. van Stockum: The gravitational field of a distribution of particles rotating around an axis of symmetry, Proc. Royal Soc. Edinburgh A 57 (1937) 135.

John Lighton Synge: Relativity: the General Theory, North-Holland 1960. QC6.S92.

P.Szekeres: Conformal tensors, Proc. Royal Soc. London A 304 (1968) 113-122.

Hyoitiro Takeno: On the spintensor of Lanczos, Tensor N.S. 15 (1964) 103-119.

I.V.Tanatarov & O.B.Zaslavskii: What happens to Petrov classification on horizons of axisymmetric dirty black holes, J. Math'l. Phys. 55,2 (2014) 022502.

Christoph Thäle: Hausdorff dimension of visibility sets for well-behaved continuum percolation in the hyperbolic plane, Brazilian J. Probab. Statist. 28 (2014) 73-82.

Tracy Yerkes Thomas: Concepts from Tensor Analysis and Differential Geometry, Academic Press 1961. QA433.T48. T.Y.Thomas: The differential invariants of generalized spaces, Cambridge Univ. Press 1934, reprinted Chelsea 1991. QA689.T5. Also, appendix B of Igor Khavkine, Alberto Melati, Valter Moretti: http://arxiv.org/abs/1710.01937 On Wick polynomials of boson fields in locally covariant algebraic QFT, Annales Henri Poincare 26 (2019) 929-1002 provides "a mostly self-contained proof of the Thomas Replacement theorem," also described in §2 of Anderson 1984.

Frank J. Tipler: Rotating Cylinders and the Possibility of Global Causality Violation, Physical Review D 9,8 (Apr.1974) 2203-2206. Tipler conjectured that some *finite* rotating cylinder (rather than the infinitely-long cylinder he analysed) could act as a time machine. However, in 1992 S.W.Hawking conjectured the opposite: "It can't be done with positive energy density everywhere! To build a finite time machine, you need ["exotic matter" with] negative energy."

Gerardo F. Torres del Castillo: Lanczos potential for algebraically special space-times, Revista Mexicana de Fisica 37,2 (Jan.1991) 246-254.

G.F.Torres del Castillo: Lanczos potential for some type D spacetimes, J. Math'l Physics 36,1 (Jan. 1995) 195-200.

Viktor Toth: Lanczos tensor at Schwarzschild horizon (9 Jan. 2013) http://www.vttoth.com/CMS/physics-notes/256-the-lanczos-tensor-at-the-schwarzschild-horizon.

Andrzej Trautman: Boundary conditions at infinity for physical theories, Bull. Acad. Polon. Sci. 6 (1958) 403-406. Radiation and boundary conditions in the theory of gravitation, Bull. Acad. Polon. Sci. 6 (1958) 407-412.

R.Brent Tully & J.Richard Fisher: A new method of determining distances to galaxies, Astronomy and Astrophysics 54,3 (Feb. 1977) 661-673.

Slava G. Turyshev & Viktor T. Toth: The Pioneer Anomaly, Living Reviews in Relativity 13,1 (Sep.2010) 4.

Anthony Tyson: Mapping dark matter with gravitational lenses, Physics Today 45,6 (June 1992) 24-33. Tyson's CL0024 density map is also briefly discussed in "Mapping dark matter", Discover Magazine (June 1997).

Kostas Tzanavaris, Latham Boyle, Neil Turok: Black Mirrors: CPT-Symmetric Alternatives to Black Holes, arXiv ["Submitted on 12 Dec 2024 (v1), last revised 20 May 2025 (this version, v2)."]

Keiichi Umetsu: Cluster-galaxy weak lensing, Astronomy & Astrophysics Review 28 (2020) #7. [Review of cluster lensing mass maps and profiles.]

P.C.Vaidya: The external field of a radiating star in general relativity, Current Science 12 (1943) 183-189; reprinted Gen. Rel. Grav. 31 (1999) 119-120.

Hermann Vermeil: Notiz über das mittlere Krümmungsmass einer n-fach ausgedehnten Riemannschen Mannigfaltigkeit, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-

Physikalische Klasse. 21 (1917) 334-344. (See also wikipedia "Vermeil's theorem.")

Ram Gopal Vishwakarma: Relativistic potential of Weyl: a gateway to quantum physics in the presence of gravitation? Classical & Quantum Gravity 37,6 (2020) #065020.

Ram Gopal Vishwakarma: Lanczos potential of Weyl field: interpretations and applications, Europ. Physical J. C 81 (Feb.2021) #194, 14pp.

Robert M. Wald: General Relativity, Univ. of Chicago Press 1984. QC173.6.W35.

Laura L. Watkins, Roeland P. van der Marel, Sangmo Tony Sohn, N. Wyn Evans: Evidence for an Intermediate-Mass Milky Way from Gaia DR2 Halo Globular Cluster Motions, The Astrophysical Journal 873,2 (2019) #118.

Steven Weinberg: Gravitation and Cosmology, Wiley 1972. QC6.W47.

Steven Weinberg: The cosmological constant problem, Reviews in Modern Physics 61,1 (1989) 1-23.

Hermann Weyl: Space-time-matter (4th ed.). Dover 1952 reprint of English translation by H.Borse of 1921 German edition (first ed. was 1918). QC6.W55.

C.M.Zhang, J.Wang, Y.H.Zhao, 6 others: Study of measured pulsar masses and their possible conclusions, Astron. & Astrophys. 527 (2011) A83. [61 pulsar masses.]

Janusz Ziołkowski: Masses of black holes in the Universe, Chinese J. Astron. Astrophys. Supplement 8 (2008) 273-280.

D.M.Zipoy: Topology of some spheroidal metrics, J. Math. Phys. 7 (1966) 1137-1143; B.H.Voorhees: Static axially symmetric gravitational fields, Phys. Rev. D 2 (1970) 2119-2122.

Joseph D. Zund: Algebraic invariants and the projective geometry of spinors, Annali di Matematica Pura ed Applicata Serie IV, 82 (1969) 381-412. That was part 1 of a 2-part paper; the second is J.D.Zund: The Theory of the Lanczos Spinor, Annali di Matematica Pura ed Applicata, 104,1 (1975) 239-268.

25. Appendix: The "Kerr-Newman-de Sitter" metric and associated shorthands, and its Kerr-de Sitter, Kerr-Newman and Kerr special cases

In coordinates (t, r, θ, φ), with mass M, spin a, electric charge Q, and cosmological constant Λ, the metric used throughout is (all other components zero):

- $g_{tt} = (a^2 \Delta_\theta \sin^2\theta - \Delta_r) / \Sigma$
- $g_{t\varphi} = g_{\varphi t} = a \sin^2\theta (\Delta_r - (r^2+a^2)\Delta_\theta) / (\Xi \Sigma)$
- $g_{rr} = \Sigma / \Delta_r$
- $g_{\theta\theta} = \Sigma / \Delta_\theta$
- $g_{\varphi\varphi} = \sin^2\theta [(r^2+a^2)^2 \Delta_\theta - a^2 \Delta_r \sin^2\theta] / (\Xi^2 \Sigma)$

Equivalently, in line-element form (this reproduces the components above):

$$ds^2 = -(\Delta_r/\Sigma)(dt - (a \sin^2\theta/\Xi)\, d\varphi)^2 + (\Sigma/\Delta_r)\, dr^2 + (\Sigma/\Delta_\theta)\, d\theta^2 + (\Delta_\theta \sin^2\theta / \Sigma)(a\, dt - ((r^2+a^2)/\Xi)\, d\varphi)^2$$

The charge enters in exactly one place, through the single extra term $+Q^2$ in Δ_r; Σ, Δ_θ, Ξ and ρ are unchanged. The metric is therefore identical in form to the Kerr-de Sitter metric, which is recovered by setting Q=0.

The special cases. Three sub-families of this metric are used elsewhere in this work. Each is obtained simply by switching off parameters; no other change to the shorthands below is needed.

- **Kerr-de Sitter** (Q=0). The $+Q^2$ term leaves Δ_r, giving $\Delta_r = (r^2+a^2)(1 - \Lambda r^2/3) - 2Mr$. Σ, Δ_θ, Ξ and ρ are untouched. The vector potential A_a vanishes, so the solution becomes a Λ-vacuum, and the Weyl scalar reverts to $\Psi_2 = -M\rho^{-3}$.
- **Kerr-Newman** (Λ=0). Then Δ_θ=1 and Ξ=1, so the Ξ factors disappear from $g_{t\varphi}$ and $g_{\varphi\varphi}$, and $\Delta_r = r^2 - 2Mr + a^2 + Q^2$, which is the usual Boyer-Lindquist Δ. Σ and ρ are untouched.
- **Kerr** (Q=0 and Λ=0). Both of the above at once: Δ_θ=Ξ=1, A_a=0, and $\Delta_r = \Delta = r^2 - 2Mr + a^2$, whose event-horizon root is $r_h = M + (M^2 - a^2)^{1/2}$.

Unlike Kerr-de Sitter this is *not* an electrovacuum-free solution: it is sourced by an electromagnetic field, with vector potential

$$A_a\, dx^a = -(Q\, r / \Sigma)\, (\, dt - (a \sin^2\theta/\Xi)\, d\varphi\,)$$

and it satisfies $G_{ab} + \Lambda g_{ab} = 2T_{ab}$ with T_{ab} the Maxwell stress tensor of that potential (checked numerically; coefficient recovered as 1.9989 against an expected 2, with a finite-difference residual of 2.7e−4). Because the Maxwell stress is trace-free, the Ricci scalar is unaffected by the charge:

$$R = 4\Lambda$$ (verified: 0.120001 against 4Λ = 0.120000 at M=1, a=0.4, Q=0.3, Λ=0.03).

The shorthands are:

symbol	definition	meaning
Σ	$r^2 + a^2 \cos^2\theta$	Kerr radial-polar combination; Σ=0 is the ring singularity. $\Sigma = \lvert\rho\rvert^2$ with $\rho = r - i\, a \cos\theta$. *Unchanged by Q.*
Δ_r	$(r^2 + a^2)(1 - \Lambda r^2/3) - 2Mr + \mathbf{Q^2}$	radial horizon function; its real roots are the (inner, event, cosmological) horizons. **The only place the charge appears.** Reduces to the Kerr-Newman $\Delta = r^2 - 2Mr + a^2 + Q^2$ when Λ=0, and to the Kerr-de Sitter Δ_r when Q=0.
Δ_θ	$1 + (\Lambda a^2/3) \cos^2\theta$	polar horizon function; a de Sitter deformation of the sphere; becomes 1 when Λ=0 or a=0. *Unchanged by Q.*
Ξ	$1 + \Lambda a^2/3$	becomes 1 when Λ=0 or a=0. *Unchanged by Q.*

ρ	$r - i\,a\cos\theta$	the "complex radius". For Kerr-Newman-de Sitter the Weyl scalar acquires a charge term: $\Psi_2 = -M\rho^{-3} + Q^2(\rho^3\bar{\rho})^{-1}$, with $\bar{\rho} = r + i\,a\cos\theta$ the conjugate. Setting $Q=0$ restores $\Psi_2 = -M\rho^{-3}$.
A_a	$-(Qr/\Sigma)(dt - (a\sin^2\theta/\Xi)d\varphi)$	the electromagnetic potential sourcing the solution; vanishes when $Q=0$.
r_h	event-horizon root of Δ_r	for Kerr-Newman, $r_h = M + (M^2 - a^2 - Q^2)^{1/2}$. Extremality is $a^2 + Q^2 = M^2$, so charge and spin both work to shrink the horizon; the Kerr condition $a^2 = M^2$ is the $Q=0$ case.

26. Appendix: Kerr pure-horizon Lanczos detector: closed form, invariants, and sparsity patterns

This appendix is a companion to the C implementation `kerr_detector2.c`, which computes everything stated here.

Contents of this appendix. 25.1 Summary · 25.2 Definitions and notation · 25.3 The closed form · 25.4 The pencil · 25.5 Algebraic invariants · 25.6 Sparsity patterns · 25.7 The contracted square · 25.8 Horizon and axis limits · 25.9 The parity-odd invariant $L^{abc}L^*_{abc}$ · 25.10 The divergence · 25.11 Empirical large-r asymptotics · 25.12 Verification · 25.13 The C program

26.1 Summary

There is a trace-free Lanczos potential L_{abc} for the Kerr metric, given in closed form below as a combination of eight algebraically defined basis tensors, such that

- The Lanczos→Weyl transform maps L_{abc} to W_{abcd} exactly, i.e. $WL[L]_{abcd}=W_{abcd}$;
- $L_{abc}L^{abc}$ is finite everywhere on the open exterior, the symmetry axis included;
- $L_{abc}L^{abc}$ diverges like Δ^{-1} at the outer horizon $\Delta=0$, with a residue we'll give in closed form.

The parity-odd invariant $L^{abc}L^*_{abc}$ also has a closed form, and the two residues combine into the square of a single complex number, whose modulus never vanishes. So the *pair* of quadratic Lanczos invariants diverges at every point of the Kerr horizon, with no exception.

26.2 Definitions and notation

$G = c_{light} = 1$; signature $-+++$; MTW; brackets on indices include the 1/2. Boyer-Lindquist coordinates (t, r, θ, φ), indices 0,1,2,3.

Abbreviations used throughout, to keep the formulae short:

$$\mathbf{c = \cos\theta}, \quad \mathbf{s = \sin\theta}, \quad \Sigma = r^2 + a^2c^2, \quad \Delta = r^2 - 2Mr + a^2, \quad \Delta^- = r^2 + 2Mr + a^2, \quad r_+ = M + (M^2 - a^2)^{1/2}.$$

Δ^- is Δ with the sign of the mass term reversed; it happens to occur twice below. Σ and Δ are always ordinary scalar functions of position, never operators; where one of them multiplies a bracketed expression it is written *after* the bracket so that this is unmistakable.

The dual is taken on the antisymmetric pair of indices, and is written with the star after the kernel letter rather than before it, to avoid any suggestion of an operator acting from the left:

$$L^*_{abc} = \varepsilon_{ab}{}^{pq} L_{pqc} / 2, \quad \varepsilon_{abcd} = (-g)^{1/2}[abcd], \quad (-g)^{1/2} = s\,\Sigma.$$

26.2.1 The Carter frame

Four covectors; all components not listed are zero, in coordinate order (t, r, θ, φ):

T_a	$T_t = -(\Delta\Sigma^{-1})^{1/2}, \quad T_\varphi = (\Delta\Sigma^{-1})^{1/2}\, a\, s^2$
R_a	$R_r = (\Sigma\Delta^{-1})^{1/2}$
P_a	$P_\theta = \Sigma^{1/2}$
Z_a	$Z_t = a\, s\, \Sigma^{-1/2}, \quad Z_\varphi = -(r^2+a^2)\, s\, \Sigma^{-1/2}$

Orthonormal: $T_aT^a = -1$, $R_aR^a = P_aP^a = Z_aZ^a = +1$, all cross-contractions zero. T and R span the 2-plane containing the two principal null directions; P and Z span its orthogonal complement, tangent to the surfaces of constant t and r.

26.2.2 The projection Π

For Y_{abc} antisymmetric in ab, first remove the cyclic part,

$$\hat{Y}_{abc} = Y_{abc} - (Y_{abc} + Y_{bca} + Y_{cab})\, 3^{-1},$$

then, with $v_a = g^{pq}\hat{Y}_{apq}$, remove the trace,

$$\Pi[Y]_{abc} = \hat{Y}_{abc} + (g_{ac}v_b - g_{bc}v_a)\, 3^{-1}.$$

26.2.3 The eight basis tensors, and what "weight" means

Each basis tensor is a bivector built from two frame legs, tensored with a third leg, projected by Π, and multiplied by an explicit power s^w.

The weight w. Consider a rotation by angle ψ in the 2-plane spanned by P_a and Z_a, the plane tangent to the constant-(t,r) surfaces. Under it P and Z turn into each other while T and R are fixed. A tensor built with a net of *n* legs drawn from {P, Z} then oscillates n times as ψ runs once round; equivalently it carries

azimuthal quantum number ±n. That n is the weight (elsewhere called the spin weight, or the helicity), and **w = n is exactly the power of s attached to the tensor**.

The reason the power must be there: the axis is where the P-Z plane degenerates, and a quantity with nonzero azimuthal number that does not vanish like s^n there is not a smooth tensor field. Attaching s^w by definition therefore makes axis regularity a condition on the *coefficients* instead of on the tensors, which is what allows it to be read off by inspection in §26.3.

	definition	net legs from {P,Z}	weight w
$B^{(1)}_{abc}$	$\Pi[\, T_{[a}R_{b]}T_{c}\,]$	0	0
$B^{(2)}_{abc}$	$\Pi[\, P_{[a}Z_{b]}T_{c}\,]$	0 ($P\wedge Z$ is rotation invariant)	0
$B^{(3)}_{abc}$	$\Pi[\, T_{[a}R_{b]}Z_{c}\,]\ s$	1	1
$B^{(4)}_{abc}$	$\Pi[\, P_{[a}Z_{b]}Z_{c}\,]\ s$	1	1
$B^{(5)}_{abc}$	$\Pi[\, T_{[a}P_{b]}T_{c}\,]\ s$	1	1
$B^{(6)}_{abc}$	$\Pi[\, T_{[a}Z_{b]}R_{c}\,]\ s$	1	1
$B^{(7)}_{abc}$	$\Pi[\, T_{[a}P_{b]}Z_{c} + T_{[a}Z_{b]}P_{c}\,]\ s^2$	2	2
$B^{(8)}_{abc}$	$\Pi[\, R_{[a}P_{b]}P_{c} - R_{[a}Z_{b]}Z_{c}\,]\ s^2$	2	2

The two terms of $B^{(7)}$ and of $B^{(8)}$ are combined precisely so that each has a definite weight; a single term would not.

These eight are drawn from a set of sixteen spanning the whole trace-free Lanczos space (6×4 − 4 cyclic − 4 trace = 16). The other eight are unused (§26.6.1).

26.3 The closed form

$$\mathbf{L_{abc} = \Sigma_{k=1}^{8}\, f_k\, B^{(k)}_{abc}}$$

The mass scales out. The coefficients below are stated at M = 1, so r and a are in units of M. For general M, $f_k(r, \theta, M, a) = f_k(r\,M^{-1}, \theta, 1, a\,M^{-1})\,M^{-1}$.

26.3.1 The four coefficients carrying $\Delta^{-1/2}$

$$P_1 = -(3/8)(r^3 + 5a^2 r)\,s^2 + [\ 13/4 - a^2 - (3/4)c^2\ ]\,r^2 - a^2(½ + a^2)\,c^2$$

$$P_2 = [\ ¼\,a^{-1}r^4 + (a - a^{-1})\,r^3 + ⅛\,a\,(13 - c^2)\,r^2 - (\ (15/4)a + (¼a - a^3)\,c^2\ )\,r + (5/8)\,a^3(3 - c^2)\ ]\,c$$

k	f_k

1	$P_1\ \Sigma^{-3/2}\Delta^{-1/2}$
2	$P_2\ \Sigma^{-3/2}\Delta^{-1/2}$
7	$-\frac{1}{8}\ a\ c\ \Delta^{1/2}\Sigma^{-3/2}$
8	$-\frac{1}{8}\ r\ \Delta^{1/2}\Sigma^{-3/2}$

f_7 and f_8 belong to this group because they arise as (something)Δ divided by $\Delta^{1/2}$; the surviving factor $\Delta^{1/2}$ means they *vanish* on the horizon rather than diverging. Only f_1 and f_2 blow up there.

26.3.2 The four coefficients without $\Delta^{-1/2}$

k	f_k
3	$[\ -\frac{1}{4}\ a^{-1}r^3 + \frac{1}{4}\ a^{-1}(3 - 4a^2)\ r^2 + \frac{1}{4}\ a\ (c^2 - 8)\ r + \frac{1}{4}\ a\ (1 - 4a^2)\ c^2\ ]\ \Sigma^{-3/2}$
4	$\frac{1}{4}\ (r^2 - 4r + 7a^2)\ c\ \Sigma^{-3/2}$
5	$-\frac{1}{4}\ \Delta^{-}\ c\ \Sigma^{-3/2}$
6	$\frac{1}{4}\ a^{-1}\ r\ \Delta^{-}\ \Sigma^{-3/2}$

No negative power of s occurs anywhere. That is the whole content of axis regularity, and it is visible by inspection. The known sign-flip potential fails exactly here: one of its coefficients carries an uncancelled s^{-2}.

26.4 The pencil

The Killing-2-plane sign-flip potential - generated by the superpotential $\mathcal{P}_{ab} = -\frac{1}{2}(2k_{ab} - g_{ab})$ with k_{ab} the projector onto the span of ∂_t and ∂_φ - uses the **same eight basis tensors**. Its coefficients, also at M = 1:

k	f_k^{flip}
1	$[\ -\frac{1}{2}r^3 + 2r^2 - (3/2)a^2r + a^2c^2(r - 1)\ ]\ \Sigma^{-3/2}\Delta^{-1/2}$
2	$(3/2)\ a\ c\ \Delta^{1/2}\Sigma^{-3/2}$
3	$-2\ a\ r\ \Sigma^{-3/2}$
4	$(r^2+a^2)\ c\ \Sigma^{-3/2}\ s^{-2}$
5	$-a^2\ c\ \Sigma^{-3/2}$
6	$a\ r\ \Sigma^{-3/2}$

7	$-\tfrac{1}{2}\, a\, c\, \Delta^{1/2}\Sigma^{-3/2}s^{-2}$
8	$-\tfrac{1}{2}\, r\, \Delta^{1/2}\Sigma^{-3/2}s^{-2}$

The s^{-2} in f_4^{flip} is the axis wire. Those in f_7^{flip} and f_8^{flip} are harmless, because those basis tensors carry weight 2, and s^4 in the Gram matrix absorbs them.

Both potentials generate the same Weyl tensor, so their difference generates zero, and

$$\mathbf{L_{abc}(\lambda) = (1 - \lambda)\, L_{abc}^{\,det} + \lambda\, L_{abc}^{\,flip}}$$

is a one-parameter family of exact Kerr Lanczos potentials, verified to satisfy the field equation at $\lambda = -3$, -1, 0, 0.25, 0.37, 1, 2.5 and 10. Every member is trace-free and generates W_{abcd}; every member diverges on the horizon; but **only $\lambda = 0$ is axis-regular**. The parameter is therefore no help in improving the detector; it is retained because $\lambda = 1$ is an independently known object against which the machinery can be checked.

26.5 Algebraic invariants

quantity	value	status
$L_{abc} + L_{bac}$	0	identically, by construction
$L_{abc} + L_{bca} + L_{cab}$	0	identically (step 1 of Π); measured 3×10^{-16}
trace one-form $L_{ab}{}^{b}$	0	identically (step 2 of Π); measured 4×10^{-15}
$\mathrm{WL}[L]_{abcd} - W_{abcd}$	0	exact; see §26.12

26.5.1 The Gram matrix

$G_{jk} = B^{(j)}{}_{abc}B^{(k)abc}$ **does not depend on r or a at all**, only on θ. Every entry not listed is exactly zero.

entry	value
G_{11}	+1/3
G_{22}	−1/3
G_{33}, G_{66}	$-s^2\, 3^{-1}$
G_{44}, G_{55}	$+s^2\, 3^{-1}$
$G_{36} = G_{63}$, $G_{45} = G_{54}$	$-s^2\, 6^{-1}$
G_{77}	$-s^4$

G_{88}	$+s^4$

The form is **indefinite**, which is why $L_{abc}L^{abc}$ can and does change sign, and why the horizon residue can vanish on a ring.

26.5.2 It is NOT superpotential-derived

A Lanczos potential is called *superpotential-derived* if there is a symmetric $\mathcal{P}_{ab}$ with $L_{abc} = \mathcal{P}_{c[a;b]}$, the trace then being removed. ($\mathcal{P} \to \mathcal{P} + \lambda\, g$ leaves the trace-free result unchanged, so one of the ten functions is pure gauge.) The detector is **not** of this form.

The map $(\mathcal{P}_{ab}, \partial_r\mathcal{P}_{ab}, \partial_\theta\mathcal{P}_{ab}) \to f_k$ is linear, 30 numbers in and 16 out. It is *onto* at every point tested, so there is no pointwise algebraic obstruction; the question is global. Tying the derivatives together on a spectral grid gives 10 unknowns and 16 equations per node - overdetermined by 6 per node - and the residual behaves as follows, with the sign-flip potential as a control, since that one *is* superpotential-derived by construction:

grid	sign-flip potential (control)	the detector
6×6	6.5×10^{-5}	0.187
8×8	8.1×10^{-7}	0.215
10×10	5.9×10^{-9}	0.234
12×12	1.6×10^{-10}	0.209

The control converges spectrally, about two orders per refinement. The detector is flat at roughly 0.2 and does not converge at all. The map itself is verified correct independently: fed $\mathcal{P}_{ab} = \lambda\, g_{ab}$ it returns zero to 10^{-10}, so the known pure-trace gauge freedom lies in its kernel as it must.

Why this matters. The prevalence paper defines its admissible space **A** to consist of superpotential-derived Lanczos tensors. The detector is therefore not an element of **A**, and any prevalence statement about horizon detection would have to be made in a wider space. This is the outcome anticipated in §26.7 of the lessons document, which suspected on numerical evidence - but could not establish - that superpotential-derivation and the horizon pole were in tension for Kerr.

The sharper statement. The tension is not between superpotential-derivation and the pole. The sign-flip potential is superpotential-derived and *does* have the Δ^{-1} pole; what it lacks is axis regularity. The detector is axis-regular and has the pole, and is not superpotential-derived. On this evidence a possible conjecture – a better word would be "guess" – would be:

for Kerr, superpotential-derived + horizon pole ⇒ axis wire.

That is not proved.

26.5.3 Scale covariance

Kerr of mass M is the image of Kerr of mass 1 under $r \to Mr$, $t \to Mt$ with $g \to M^2 g$, and the coefficients inherit $f_k \to f_k M^{-1}$. Hence **$L_{abc}L^{abc}M^2$ is a function of $r M^{-1}$, c and $a M^{-1}$ alone**, and the horizon residue is dimensionless. Verified across $M \in \{0.25, 0.5, 1, 2, 5, 17.3\}$ at fixed $r M^{-1} = 3$, $a M^{-1} = 0.6$: $L_{abc}L^{abc}M^2 = -0.056032113995$ and residue = 0.325925925926 at every mass, to all twelve digits. A cheap sharp test for any reimplementation.

26.6 Sparsity patterns

26.6.1 Basis sparsity: eight of sixteen

Exactly eight of the sixteen basis tensors are used. Writing XY×W for $\Pi[X_{[a}Y_{b]}W_{c]}$ with its weight, the unused eight are TR×R, PZ×R, TR×P, PZ×P, TZ×T, TP×R, TP×P−TZ×Z, RP×Z+RZ×P. This is not a free choice: each of the eight that *are* used is necessary - deleting any one and re-solving degrades the field equations by at least two orders of magnitude.

26.6.2 Branch sparsity: four and four

Four coefficients carry $\Delta^{-1/2}$ and no regular part (k = 1, 2, 7, 8); four carry a regular part and no $\Delta^{-1/2}$ (k = 3, 4, 5, 6). None contains both. **The sign-flip potential has exactly the same split**, tensor for tensor - the detector inherits the branch structure of the object it repairs.

26.6.3 Coefficient degrees

k	1	2	3	4	5	6	7	8
degree in r	3	4	3	2	2	3	2	3
degree in c	2	3	2	1	1	0	1	0

Degree 4 in r is the maximum anywhere, and occurs once. The distinct rational constants appearing are few: ±1/4, ±1/8, ±3/8, ±5/8, ±15/4, ±15/8, 13/4, 13/8, 1/2, 3, 5, 7, 8.

26.6.4 Component sparsity, and a discrete symmetry

In Boyer-Lindquist coordinates L_{abc} has 24 independent components (antisymmetry in the first pair). Exactly **12 of the 24 vanish identically**:

$L_{abc} = 0$ unless an *even* number of the indices a, b, c lies in {t, φ}.

That is invariance under the discrete isometry of Kerr $(t, \varphi) \to (-t, -\varphi)$: each index in {t, φ} contributes a factor −1, so only the even components survive. The twelve that do are

$$L_{tr\theta},\ L_{tr\varphi},\ L_{t\theta t},\ L_{t\theta\varphi},\ L_{t\varphi r},\ L_{t\varphi\theta},$$
$$L_{r\theta r},\ L_{r\theta\theta},\ L_{r\varphi t},\ L_{r\varphi\varphi},\ L_{\theta\varphi t},\ L_{\theta\varphi\varphi}.$$

Equivalently: for the four "mixed" bivector pairs {tr, tθ, rφ, θφ} the third index runs over the Killing directions {t, φ}, and for the two "pure" pairs {tφ, rθ} it runs over {r, θ}.

26.7 The contracted square

Because the Gram matrix of §26.5.1 is explicit, so is this scalar:

$$L_{abc}L^{abc} = 3^{-1}\,(f_1^2 - f_2^2) + (3^{-1}\,f_4^2 - f_3^2 + f_5^2 - f_6^2 - f_3f_6 - f_4f_5)\,s^2 + (f_8^2 - f_7^2)\,s^4$$

Checked against direct contraction of the full rank-3 tensor: agreement to 3×10^{-17}. It holds for every member of the pencil, and is manifestly quadratic in λ - which the C program tests directly, by second differences.

26.8 Horizon and axis limits

26.8.1 The horizon residue

As $\Delta \to 0$, $f_3 \ldots f_6$ stay finite, f_7 and f_8 vanish, and only f_1, f_2 diverge, each like $\Delta^{-1/2}$. Since $B^{(1)}$ and $B^{(2)}$ are Gram-orthogonal to everything, the first line of §26.7 is all that survives:

$$\lim_{r\to r_+} (\,L_{abc}L^{abc}\,)\,\Delta = [\,P_1(r_+,c)^2 - P_2(r_+,c)^2\,]\,[\,3\,\Sigma(r_+)^3\,]^{-1}$$

where $\Sigma(r_+) = r_+^2 + a^2c^2$. The minus sign between the squares is $G_{22} = -1/3$. Values:

a/M	c = 0	c = 0.25	c = 0.50	c = 0.75	c = 1 (axis)
0.2	+0.5020431	+0.3866944	+0.0415290	−0.5308282	−1.3260571
0.3	+0.4784380	+0.4343916	+0.3030268	+0.0866323	−0.2110872
0.6	+0.3483128	+0.3426003	+0.3259259	+0.2995885	+0.2654815
0.9	+0.1151202	+0.1123416	+0.1047515	+0.0941365	+0.0824192

Spin threshold. The axis value vanishes at a/M = 0.347810. Above that the residue is nonzero over the whole horizon; below it there is one ring where it vanishes (at a/M = 0.3, c = 0.830; at a/M = 0.2, c = 0.522). The ring has measure zero, so the divergence holds on a full-measure subset of H at every spin – and §26.9.3 shows the exceptional ring disappears entirely once the parity-odd invariant is included.

a^{-1} occurs in f_2, f_3, f_6, so this family is singular as $a \to 0$ and says nothing about Schwarzschild.

26.8.2 The axis value

On the axis every s-weighted term drops out of §26.7, leaving

$$L_{abc}L^{abc}|_{axis} = [\,P_1(r,1)^2 - P_2(r,1)^2\,]\,[\,3\,(r^2+a^2)^3\,\Delta\,]^{-1}$$

with $P_1(r,1) = (5/2 - a^2)\,r^2 - a^2(½ + a^2)$, and $P_2(r,1) = ¼\,a^{-1}r^4 + (a - a^{-1})\,r^3 + (3/2)\,a\,r^2 - (4a - a^3)\,r + (5/4)\,a^3$. At a = 0.6, r = 3 this gives 0.03790432, matching the numerical limit to eight figures.

26.9 The parity-odd invariant $L^{abc}L^*_{abc}$

26.9.1 The dual Gram matrix

Write $G^*_{jk} = B^{(j)}_{abc}B^{(k)*abc}$. Like G_{jk} it is symmetric and depends only on θ, and it is **purely off-diagonal** - complementary to G, which is concentrated on the diagonal:

entry	value
G^*_{12}	+1/3
G^*_{34}	$-s^2\,3^{-1}$
G^*_{35}	$+s^2\,6^{-1}$
G^*_{46}, G^*_{56}	$-s^2\,6^{-1}$
G^*_{78}	$-s^4$
all diagonal entries, all others	0

Within the weight-1 quartet {3,4,5,6}, G supplies the pairings (3,3), (4,4), (5,5), (6,6), (3,6), (4,5) and G* supplies exactly the complementary ones (3,4), (3,5), (4,6), (5,6). Between them they fill the 4×4 block. Hence

$$\mathbf{L^{abc}L^*_{abc} = (2/3)\,f_1f_2 - 2\,f_3f_4\,s^2\,3^{-1} + f_3f_5\,s^2\,3^{-1} - f_4f_6\,s^2\,3^{-1} - f_5f_6\,s^2\,3^{-1} - 2\,f_7f_8\,s^4}$$

$L^{abc}L^*_{abc} = 0$ when θ=π/2 (equator).

Checked against direct contraction of the dual: agreement to 10^{-17}.

26.9.2 The complex horizon residue

On the horizon only f_1 and f_2 survive. G contributes $f_1^2 - f_2^2$, and G* contributes $2f_1f_2$; these are the real and imaginary parts of one square, so the two residues combine into

$$\mathbf{\lim_{r\to r_+}\,(\,L_{abc}L^{abc} + i\,L^{abc}L^*_{abc}\,)\,\Delta = [\,P_1(r_+,c) + i\,P_2(r_+,c)\,]^2\,[\,3\,\Sigma(r_+)^3\,]^{-1}}$$

with the *same* two polynomials of §26.3.1. At a/M = 0.6, θ = π/3: Re = 0.3259259259, Im = −0.1326126126, and the direct numerical limits agree to six figures.

26.9.3 Consequence: the exceptional ring disappears

The modulus is a sum of squares,

$$|\,\lim\,(\,L_{abc}L^{abc} + i\,L^{abc}L^*_{abc}\,)\,\Delta\,| = [\,P_1^2 + P_2^2\,]\,[\,3\,\Sigma(r_+)^3\,]^{-1}.$$

The ring on which the real residue vanishes is exactly where $P_1 = \pm P_2$, and there the imaginary part is $\pm 2P_1^2 \neq 0$. A scan over $0.02 \le a/M \le 0.98$ across the whole range of c finds no zero of the modulus; its global minimum over that range is 0.0166 at a/M = 0.98, tending to zero only in the extremal limit. Therefore

the pair ($L_{abc}L^{abc}$, $L^{abc}L^*_{abc}$) diverges like Δ^{-1} at *every* point of the outer horizon, for every 0<a<M, with no exceptional ring.

The two zero sets are complementary, as parity requires. P_2 carries an overall factor c, so the parity-odd residue vanishes identically on the equatorial plane - exactly where the residue of $L_{abc}L^{abc}$ is largest. Conversely the residue of $L_{abc}L^{abc}$ vanishes on a ring at low spin, where $L^{abc}L^*_{abc}$ is nonzero.

26.10 The divergence

$d_{ab} = L_{abc}{}^{;c}$ is antisymmetric, so has six components. For every member of the pencil, **four of the six vanish identically**:

$$d_{tr} = d_{t\theta} = d_{r\varphi} = d_{\theta\varphi} = 0,$$

leaving only $d_{t\varphi}$ (on the Killing 2-plane) and $d_{r\theta}$ (on its orthogonal complement). This is a property of the whole eight-tensor family, not of the particular solution, and it explains why imposing the full Lanczos differential gauge $d_{ab} = 0$ is so badly over-determined: four of the six conditions are automatic, and each of the remaining two is incompatible with axis regularity - imposing either one degrades the field equations by three to four orders of magnitude.

26.11 Empirical large-r asymptotics

When $r \to \infty$ with θ fixed, empirically $L_{abc}L^*_{abc}$ appears to fall toward zero proportionally to r^{-2}, while $L^{abc}L_{abc}$ appears to approach a θ-independent negative constant, with the difference between it and that constant falling ultimately proportionally to r^{-1}.

26.12 Verification

The Weyl-Lanczos equation used is

$$WL[L]_{abcd} = L_{abc;d} - L_{abd;c} + L_{cda;b} - L_{cdb;a}$$
$$- (g_{ac}H_{bd} + g_{bd}H_{ac} - g_{ad}H_{bc} - g_{bc}H_{ad}), \quad H_{ab} = \tfrac{1}{2}(L_a{}^e{}_{b;e} + L_b{}^e{}_{a;e}).$$

H_{ab} is the symmetrised divergence of L. It is *not* the traceless Ricci tensor, and it is given a letter of its own here for exactly that reason. The coefficient of the second line was not assumed: writing it with an unknown α and requiring $g^{ac}WL[L]_{abcd} = 0$, which the Weyl tensor satisfies identically, gives α = −1 to 2×10^{-16}.

The right-hand side is built algebraically rather than from the Riemann tensor: Kerr is type D and the Carter frame is aligned with it, so the frame components of W_{abcd} are constant integers (only ±1 and ±2,

24 nonzero of 256) times $\mathrm{Re}\psi_2$ and $\mathrm{Im}\psi_2$, where

$$\mathrm{Re}\psi_2 = M\,r\,(3a^2c^2 - r^2)\,\Sigma^{-3}, \quad \mathrm{Im}\psi_2 = M\,a\,c\,(a^2c^2 - 3r^2)\,\Sigma^{-3}.$$

Arbitrary-precision check. Metric, Christoffels, the eight basis tensors and their derivatives, and the eight coefficients and their derivatives were built symbolically, then evaluated in arbitrary precision and assembled:

working precision	max|WL[L] − W| / max|W|
30 digits	1.4×10^{-30}
60 digits	1.2×10^{-61}
100 digits	2.2×10^{-101}

The residual tracks the working precision, which only an exact identity does. Repeated at 40 random points with $a/M \in [0.05, 0.97]$, r from just outside r_+ to $r \approx 13$, $c \in [-0.98, 0.98]$, at 60 digits: worst relative residual 1.0×10^{-59}.

Not done: a symbolic proof. sympy's expression swell defeated a direct simplification; it would need the computation organised sector by sector.

26.13 The C program

kerr_detector2.c is self-contained (libm only). Entry points:

routine	what it does
kerr_metric(r, th, M, a, g, gi)	metric and inverse
kerr_carter_frame(r, th, M, a, legs)	the four covectors of §26.2.1
kerr_basis_tensor(k, r, th, M, a, B)	$B^{(k+1)}{}_{abc}$, k = 0..7
kerr_coefficients(r, th, M, a, lambda, f)	the eight f_k, including the M-scaling and the pencil blend
kerr_lanczos(r, th, M, a, lambda, L)	L_{abc}, indices down, Boyer-Lindquist
kerr_dual_lanczos(r, th, M, a, lambda, dL)	$L^*{}_{abc}$
kerr_LL(r, th, M, a, lambda)	$L_{abc}L^{abc}$, direct contraction
kerr_LL_gram(r, th, M, a, lambda)	the same from §26.7 - independent route
kerr_LdualL(r, th, M, a, lambda)	$L^{abc}L^*{}_{abc}$, direct
kerr_LdualL_gram(r, th, M, a, lambda)	the same from §26.9.1

kerr_horizon_residue(th, M, a)	§26.8.1, for the λ = 0 member
kerr_horizon_residue_complex(th, M, a, &re, &im, &mod)	§26.9.2 and its modulus
kerr_weyl(r, th, M, a, W)	the Kerr Weyl tensor, algebraically
kerr_wl_residual(r, th, M, a, lambda)	|WL[L] − W|/|W| by finite differences - the self-test

Build with `gcc -O2 kerr_detector2.c -lm`; λ is argv[1]. The header of that file documents usage, the expected magnitude of each self-test, and the caveats.