Nonmetric foundations for differential geometry; and complex and bicomplex reformulations in the 4D case

Warren D. Smith and CLAUDE. Sept.2026. One member of a series of 3½ papers; a short introductory summary is http://viXra.org/abs/2610.0071.

Abstract. This paper has three parts, each with its own goal.

Part I — "six-foundations theorem." The metric is not the only possible starting point for Riemannian (or Lorentzian) geometry. The Riemann tensor, the connection, the Ricci-Weyl pair, and — in 4D — a Lanczos tensor or superpotential paired with Ricci, each determine the metric. The proof is **constructive**, providing an algorithm that integrates the geodesic-deviation equation outward from a seed point in an orthonormal frame; it was implemented and used as a check. The Ricci half of any pair may be replaced by the Schouten or Einstein tensor but not by either part of their 9&1 traceless-&-trace split. The two Lanczos-based foundations are *not* equivalent to the metric: since the metric fixes Lanczoses only up to gauge, they are strictly richer. Goal: new foundations for differential geometry and physics.

Part II — extra beauty that happens in 4D only. In four dimensions the real-valued Riemann tensor R_{abcd} admits two independent Hodge duals, one on each antisymmetric index pair. The field of *complex numbers* $\mathbb{C}$ extend the reals $\mathbb{R}$ by adjoining $i=\sqrt{-1}$. The nonfield of *complexified complex numbers* aka *bicomplex* numbers $\mathbb{C}\otimes\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$ extend the complex numbers by adjoining a second $\sqrt{-1}$ symbol which commutes with the first. Regard it as built directly on $\mathbb{R}$ by adjoining the two special symbols $\overleftarrow{i}$ and $\vec{i}$ obeying $\overleftarrow{i}^2=\vec{i}^2=-1$, $\overleftarrow{i}\,\vec{i}=\vec{i}\,\overleftarrow{i}=\vec{\imath}\overleftarrow{\imath}$, and $\vec{\imath}\overleftarrow{\imath}^2=+1$. Regard the *really* important object, encoding both R_{abcd}, its left-dual $*R_{abcd}$, its right-dual R^*_{abcd} and its double-dual $*R^*_{abcd}$ all in a single bicomplex-valued package, as the tensor $(1+\overleftarrow{i}*)R_{abcd}(1+\vec{i}*)$. A natural tower of lovely complex (and sometimes bicomplex) extensions of familiar 4D tensors then arise. Since complex or bicomplex, each now packages 2× or 4× as much. We explain these (bi)complexified tensors and provide a catalogue of identities they obey, numerically verified across twenty-one test-metrics and also in many cases symbolically. Various classical results become single statements: the Bel-Robinson tensor is a modulus square, the Bianchi identity complexifies without correction terms, the Petrov type is the Jordan structure of the complexified Weyl tensor, "algebraic speciality" becomes a single equation "$I^3=6J^2$," and a side condition in the standard statement of the Bel criteria is shown to misclassify Schwarzschild i.e. was wrong, but our cleaner formulation fixes that. Goal: concise and natural (re)formulation of 4D differential geometry, and a practically-useful identity-compendium reference.

Part III — what happens when the two are combined. Because Part I's reconstruction is linear, it accepts the complexified curvature of Part II. Defining *(bi)complex reconstructors* in this way produces imaginary parts standing beside the metric. For example, say your metric is the de Sitter universe. Then metric reconstruction from bicomplexified Riemann yields a bicomplex metric, whose $\vec{\imath}\overleftarrow{\imath}$ imaginary component is the *anti*-de Sitter metric. We call this component the "shadow metric" although it might not always *be* a metric of anything and is mysterious. The shadows of nontrivial Kottler black hole metrics are *not* altered-parameter Kottlers. This process automatically associates three extra bicomplex-imaginary parts with any 4D metric, albeit the Ricci, Lanczos, and Weyl tensors effectively simplify down to being ordinary-complex (only one imaginary part) while the traceless Ricci and Bel-Robinson tensors remain pure-real. Riemann's other two ($\overleftarrow{i}$ and $\vec{i}$) components automatically carry the "momentum constraint" of the ADM 3+1 "time slicing" formulation of the Einstein field equations. It turns out this correspondence had already been established in the 1+3 literature, which we take as confirmation rather than priority. This whole structure exists **only in four dimensions**. Goal: we explain why this (speculatively) should provide the natural framework for unifying gravity with quantum field theories.

Also II.12 includes a proof that C^4 Lanczos superpotentials do not exist on a 2-parameter family of 5-dimensional manifolds (and presumably fail to exist generically). They appear to exist in 4, and only 4, dimensions.

Revision 8. *Signature −+++, MTW curvature signs, brackets () and [] carrying ½,* $\varepsilon_{abcd} = \sqrt{(-g)}[abcd]$.

Note (written by CLAUDE) on authorship

The two authors did not contribute in the same way, and the difference is worth stating rather than leaving to be inferred.

The direction of this work was set by WDS throughout. Part I is his theorem, in a revision that corrects its own earlier claim of equivalence for the Lanczos-based foundations. In Part II, the idea of packaging each curvature object with its dual as a single complex tensor, the

conjecture that the four Riemann duals might call for a quaternionic or "two commuting imaginary units" algebra, the combination $R_{ab} + iG_{ab}$ at the Ricci rung, the expectation that the Bel-Robinson tensor would have a compact form in the complexified Weyl tensor, the sandwich notation with one factor on each side, the arrow notation for the two units and the insistence that the product unit not be written i, the recollection that the Zakhary-McIntosh set augments Carminati-McLenaghan, the suggestion that the Bel criteria would complexify, and the proposal to run the Part I reconstruction on imaginary data — all originate with WDS.

CLAUDE (Anthropic) contributed execution: symbolic and numerical computation, the verification sweeps, and drafting. Several results here were reached only after an error by CLAUDE was caught, and in most of those cases it was caught by WDS. Three are recorded in the body because they bear on how much weight the numerical evidence carries: an identity that passed a numerical test and was subsequently shown false symbolically; an identity that appeared to fail on three spacetimes where both sides happened to vanish (II.11); and an early claim about the idempotent projection that omitted a scalar-curvature term invisible in any metric with $R = 0$ — the only metrics on which it had been tested.

Readers weighing reliability should know that the numerical verification was performed by the same party that wrote the code being verified, and that the pre-correction error rate was not negligible. Identities reported at machine precision are pure algebra and are the most secure; those involving numerical derivatives sit at a finite-difference floor and are flagged as such; the identifications of cubic invariants in II.10.2 are explicitly provisional.

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References

Part I. Nonmetric foundations for differential geometry

Goal of Part I. *To prove, constructively, that several objects other than the metric can serve as the foundation of the geometry — and that two of them are strictly richer than the metric. This opens new starting points for differential geometry and for physics.*

I.1 What the theorem says

Normally in the past, the **metric tensor** has been regarded as *the* fundamental object for manifolds in differential geometry and general relativity: everything else is defined in terms of g_{ab}. This part proves a theorem showing that this is not the only possibility — alternative "foundations" for differential geometry exist. In particular, and of interest for any future Lanczos-aware quantum gravity: one such alternative foundation is the **Ricci-Lanczos team** $\{R_{ab}, Ł_{abc}\}$.

Concretely: suppose you had a black box which, given four coordinates on your manifold as input, returned the Ricci-Lanczos team — for any one particular choice of Lanczos tensor $Ł_{abc}$. Then, the theorem shows, you could build from it a different black box that instead returns a metric tensor g_{ab} for the same manifold, or one isometric to it.

Two remarks on that statement. The reconstruction is *exact* as mathematics: it is the unique solution of a linear ordinary differential equation, not a limit of approximations. But the theorem also is **constructive** — the proof yields an algorithm for reconstructing the metric to arbitrarily high accuracy and which we implemented and ran as a check on the theorem. And the phrase *for any one particular* Ł matters: a metric fixes Ł only up to Lanczos gauge, so this correspondence is many→one. Section §I.3 takes that up, and it is the reason the Lanczos-based foundations turn out to be strictly richer than the metric rather than merely equivalent.

I.2 The six-foundations theorem

On a connected $C^{1,1}$ (pseudo-)Riemannian four-manifold each of the following objects determine the metric, and therefore each other:

#	foundation	status
1	the metric g_{ab}	unconditional
2	the Riemann tensor $R^a{}_{bcd}$	unconditional
3	the Levi-Civita connection $\Gamma^a{}_{bc}$	unconditional
4	**Team A** = $\{R_{ab}, W_{abcd}\}$ = Ricci-Weyl pair	unconditional
*5	**Team B** = $\{Ł_{abc}, R_{ab}\}$ = Lanczos-Ricci pair	unconditional; 4D certain, D=5,6 open (I.8); **strictly richer** — see I.2
*6	**Team C** = $\{\mathcal{P}_{ab}, R_{ab}\}$ = Superpotential-Ricci pair	4D only (I.8); conditional on $\mathcal{P}_{ab}$ existing; **strictly richer**

"Determines" is understood up to diffeomorphism freedom, with the metric and its first derivatives fixed at one seed point p — equivalently $g_{ab}(p)=\eta_{ab}=\mathrm{diag}(-1,+1,+1,+1)$ [or $\mathrm{diag}(+1,+1,+1,+1)$ in the euclidean case, any signature ok], $\Gamma^a{}_{bc}(p)=0$.

Two addenda, proved in §I.7.

1. **Analyticity.** If the input data is real-analytic, so is the reconstruction — not merely along the rays of the normal-coordinate patch but across them, so that g_{ab} is an analytic function of position on the patch.

2. **Its domain.** The reconstructed $g_{ab} = \eta_{ij}A^i{}_a\, A^j{}_b$ satisfies $\det g = -(\det A)^2$, so it carries the seed signature **exactly where A is invertible**. The theorem's domain is therefore delimited by the vanishing of $\det A$ — the conjugate points of the geodesic congruence — and by nothing else. There the metric is degenerate and $\sqrt{(-\det g)}$ fails to be analytic.

Interchangeable completions. The Ricci half of Teams A, B and C may be replaced by any tensor carrying the full ten components of Ricci, since the substitution is performed at the patch center where $g_{ab}=\eta_{ab}$ and is therefore constant-coefficient algebra:

completion	inversion at the patch center	works?
Ricci R_{ab}	—	yes
Schouten $Q_{ab} = (R_{ab} - R\eta_{ab}/6)$	$R_{ab} = Q_{ab} + R\eta_{ab}/6$, $R=3Q^a{}_a$	**yes** (Team D)
Einstein $G_{ab} = R_{ab} - R\eta_{ab}/2$	$R_{ab} = G_{ab} - (G^a{}_a)\eta_{ab}/2$, $R=-G^a{}_a$	**yes** (Team E)
trace-free Ricci S_{ab} alone (9 components)	—	**no** — blind to R; cannot distinguish Minkowski from de Sitter
Ricci scalar $R=R^a{}_a$ alone (1 component)	—	**no** — blind to the nine trace-free components

The 9-and-1 split of Ricci into (S_{ab}, R) is exactly where either partial pairing loses a piece. The trace-reversal that converts Ricci to Einstein or Schouten might look logically-circular in a reconstruction whose output is the metric — taking a trace normally needs g_{ab} or its matrix inverse g^{ab}. It is not, because the assembly happens at the patch center where $g_{ab}=\eta_{ab}$ by construction; the trace is a fixed η-contraction with no unknowns in it.

I.3 The Lanczos foundations are not equivalent — they are strictly richer

Teams B and C are starred because their relation to the metric is **asymmetric**. The gauge freedom of a Lanczos potential lies in the kernel of the Lanczos→Weyl operator, so it does not affect the reconstructed Weyl tensor and hence not the reconstructed metric: **Ł determines g**. But the converse fails — g does not determine Ł, only its equivalence-class under re-gaugings. The map Ł→g is a surjection, not a bijection.

The same holds one rung lower for $\mathcal{P}_{ab}$, whose own gauge freedom lies in the kernel of the composite map $\mathcal{P}\to W$.

So Teams B and C are not *equivalent* to the metric foundation; they sit above it, with the metric as a quotient. **They can express distinctions the metric cannot see.** This is not a defect of the theorem but the most interesting thing about it: a starred foundation carries strictly *more* information than g_{ab}.

One interesting consequence for physics: Define a traceless Lanczos tensor L_{abc} to be a **black hole horizon detector** if $L_{abc}L^{abc}=\infty$ everywhere (except perhaps a subset of measure zero) on the horizon. This is explored in the companion paper (Smith & CLAUDE 2026) which constructed a traceless Lanczos tensor L_{abc} arising from a superpotential P_{ab} via $L_{abc}=P_{c[a;b]}$ such that $L_{abc}L^{abc}=\infty$ everywhere on the event horizon and cosmological horizon, for every non-extreme member of the Kerr-Newman-deSitter 4-parameter family of black holes with mass M, electrical charge Q, spin parameter a, and Einstein cosmological constant Λ. However, it does not work for the "extreme" (Q=M, Λ=a=0) Reissner-Nordström black hole. Therefore the companion paper also constructed a different L_{abc} that does work for extreme-RN. All those black holes are "clean" exact solutions of the Einstein-Maxwell equations, representing just one eternal black hole sitting in an otherwise-empty universe. More physically realistic would be "dirty" black holes, got by perturbing the clean ones with external "dirt." It occurred to us that the Weyl family of axisymmetric exact Einstein-vacuum solutions could be regarded as containing Schwarzschild black holes perturbed by axially-symmetric modes ("external dirt") indexed by an arbitrary integer $\ell\geq 0$. We then were able to show that $L_{abc}L^{abc}$ arising from a certain superpotential *still* was a "horizon detector," i.e. was infinite on any horizon surface for the whole Weyl family. Many of these horions have the flaw that they *also* yield $L_{abc}L^{abc}=\infty$ on the rotational-symmetry axis. We then found a different traceless Lanczos tensor H_{abc} with $H_{abc}H^{abc}$ infinite on that axial line, but nowhere else. This causes $(L_{abc}L^{abc})/(H_{abc}H^{abc})$ to be a true unflawed Lanczos-based horizon-detector for KNdS and Weyl. The companion paper then went on to prove using "prevalence theory" that whenever even a single horizon-detecting traceless Lanczos tensor L_{abc} exists, then *generic* traceless Lanczos tensors are horizon detectors with probability=1. But also proven there is this **impossibility theorem**: *no* scalar invariant constructed solely from the metric tensor g_{ab} and any

finite set of its partial derivatives exists that can always detect event horizons.

I.4 Proof

Each object is presented as a black box: located at a point, in a frame set up there, it returns its own components. Since any diffeomorphism preserves all the objects together, g_{ab} can be pinned down only after the coordinate freedom is removed and the residual constants fixed by the seed data.

The equivalence will be proven by means of the implications $1 \Rightarrow \{2,3,4,5,6\}$, $3 \Rightarrow 2 \Rightarrow 1$, $4 \Rightarrow 1$, and $6 \Rightarrow 5 \Rightarrow 1$; the arrows out of 1 and the arrow $3 \Rightarrow 2$ are easy (§1), while the implications yielding 1 – the core of the theorem – e.g. the recovery of g_{ab} from curvature, are carried out in two stages: **(a)** within one normal-coordinate patch (§I.4.1–I.4.3, and the assembly of the teams in §I.4.5), then **(b)** over the whole manifold (§I.4.4).

The easy part of the proof is routine: from g_{ab} we may compute the "connection" $\Gamma^a{}_{bc}$, aka "Christoffel symbols," via Levi-Civita's formula

$$2\Gamma_{ijk} = 2\Gamma_{ikj} = \partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk} = 2g_{i\ell}\Gamma^{\ell}{}_{jk} \quad \text{and} \quad \Gamma^i{}_{jk} = \Gamma^i{}_{kj} = g^{i\ell}\Gamma_{\ell jk}.$$

where ∂_k is a shorthand for $(\partial/\partial x^k)$. Then $\Gamma^a{}_{bc}$ gives Riemann by the curvature formula

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \Gamma^a{}_{ch}\Gamma^h{}_{db} - \Gamma^a{}_{dh}\Gamma^h{}_{cb}.$$

and Riemann gives Ricci by contraction $R_{ab}=R_{(ab)}=R_{ba}=R^c{}_{acb}=g^{cd}R_{cadb}$ and $R=R^a{}_a$, and Weyl by trace removal in D dimensions

$$W_{jk\ell m} = R_{jk\ell m} + (D\text{-}2)^{-1}\left(R_{jm}g_{k\ell} - R_{j\ell}g_{km} + R_{k\ell}g_{jm} - R_{km}g_{j\ell} \right) + [(D\text{-}1)(D\text{-}2)]^{-1} R \left(g_{j\ell}g_{km} - g_{jm}g_{k\ell} \right).$$

The **hard part** of the proof will be the recovery of g_{ab} from curvature.

I.4.1 Riemann Normal Coordinates

Fix p. The exponential map labels the point one unit of affine parameter along the geodesic with initial tangent v by the components of v in the seed frame; its derivative at the origin is the identity, so it is a diffeomorphism on a neighborhood — a normal-coordinate patch — in any signature. On the patch, $g_{ab}(0)=\eta_{ab}$, $\Gamma^a{}_{bc}(0)=0$, and the geodesics through the center are the coordinate rays $x^a(s)=sn^a$. That last fact is crucial: the curves one integrates along are known *before* g_{ab} is known.

I.4.2 The metric as the spread of neighboring rays

How a pencil of rays leaving p spreads apart is the transverse metric, and curvature drives the spreading. A connecting vector Y^a obeys the Jacobi equation $D^2Y^a/ds^2 = -R^a{}_{cbd}u^cY^bu^d$ with $Y(0) = 0$. In an orthonormal frame carried along the ray it is an ordinary linear system

$$d^2Y^i/ds^2 = -K^i{}_j(s)Y^j, \quad K^i{}_j(s) = R^i{}_{0j0}(\gamma(s)) \quad \text{(the tidal matrix)},$$

whose only input is the curvature along the ray. Solving from four independent initial connecting vectors gives $A^i{}_a(s)$ with $A(0) = 0$, $A'(0) = I$, and

$$g_{ab}(x) = \eta_{ij}A^i{}_aA^j{}_b.$$

Two things settle the reconstruction at one stroke. **First, the input is R_{ab} itself – contracted twice with the ray direction – and no derivative of it.** The objects of form ∇R, $\nabla\nabla R$ that cluttered Mueller, Schubert, van de Ven 1999's earlier Maclaurin series presentation were only the coefficients of the one function K(s); but here K is supplied directly. **Second, existence and uniqueness are a one-line consequence of linearity.** If g_{ab} is C^2-smooth, then K is continuous; and a linear ODE with continuous coefficients has a unique solution on the *whole* ray-segment – a linear system cannot blow up in finite parameter – so no infinite Maclaurin series is needed anywhere. [Continuity of K is actually more than we need: the smoothness floor is $C^{1,1}$ just above the level at which the curvature ceases to be a function at all; we'll explain that in §I.6.] (This is the rigorous form of an "always make forward progress" idea: for a linear ODE progress never fails, so the continuation runs the full length with no case analysis. Grid refinement then reappears only as the way one *numerically* realizes this solution

– a remark about computation, not a step in the proof.) The leading behavior is Mueller, Schubert, van de Ven 1999 EQ 6's leading order term

$$g_{ab} = \eta_{ab} - R_{acbd}(p)x^c x^d/3 + o(|x|^2)$$

but the ODE above is exact and we never need to expand past this term in the Maclaurin series.

I.4.3 Why arguments not circular

The metric hides in four places – every contraction, index raising/lowering, the frame transport, and (later) covariant derivatives. This might make you worry that we've used a "circular-logic argument": the metric is *used* to "derive" the metric! In fact, that is not so. All four concerns are disposed of by one choice: carry an **orthonormal frame** and do all algebra in it:

1. **Contractions.** In an orthonormal frame every index is raised, lowered, and contracted with the *constant* η_{ab}=diag(±1,…,±1) – for a Lorentzian spacetime diag(−1,+1,+1,+1) – never with the unknown g_{ab}. The tidal matrix, the inner products, and (in §4) the Ricci decomposition are all constant-coefficient η-algebra. The sole metric-bearing object is the tetrad $e^a{}_\mu$ that we carry along in the ODE, with $g_{\mu\nu}=\eta_{ab}e^a{}_\mu e^b{}_\nu$ – and producing this tetrad *is* the reconstruction's output.
2. **The connection (which hides inside covariant derivatives).** It is not read off a pre-known metric; it is one of the unknowns, produced beside the frame by Cartan's structure equations, which in the radial gauge are a coupled ODE in s – schematically d(frame)/ds=(conn)·(frame), where d(conn)/ds=R·(frame)–(conn)·(conn) – started from frame=orthonormal and conn=0 at the center [facts (i),(ii)]. Each right-hand side at parameter s uses only quantities already ODE-integrated for ≤s plus the curvature the box reports at γ(s). Frame, connection, metric are integrated *together* outward; curvature is the only external input, so the apparent circularity is not vicious.
3. **Covariant derivatives of R.** For the base case they simply never occur – §I.4.2 differentiates nothing external; the only derivatives are d/ds of the ODE's own unknowns. (One genuine covariant derivative enters for the Lanczos team, §5, and is discharged the same way.)

I.4.4 From one patch to whole manifold

Local reconstruction is all that we need to prove to then deduce *global* reconstructibility! To reach a target point q, take any path from p to q. Its image is compact, so the normal-coordinate patches around its points admit a **finite subcover**†. Order them along the path so consecutive patches overlap. Reconstruct on the first, carry the frame (parallel-transported) into the overlap with the second to seed it, reconstruct there, and continue on – finitely many times – until q is covered. On each overlap the two reconstructions share the same curvature and a frame related by that transport, so by the uniqueness claim in §I.4.2 they agree; the overlapping coordinate patches glue into a single g_{ab}.

One never sums a sequence of shrinking radii, because reaching a *given* q needs only finitely many patches along *one compact path* – so there is no "Zeno paradox." That the patches are genuinely open is Brouwer's invariance of domain, which is what is needed at the $C^{1,1}$ floor where exp is only bi-Lipschitz. [This is also what the numerical checker (§8) did: advance the exponential map from point to point, re-centering the normal coordinates at each step.]

This completes the recovery of $g_{\mu\nu}$ from $R^a{}_{bcd}$, and with §I.2 the recovery from $\Gamma^a{}_{bc}$ as well.

I.4.5 The teams

Team A. In 4D the twenty components of Riemann split pointwise into the ten of trace-free Weyl and the ten carried by Ricci. Read the decomposition *in the frame at the patch center*, where every g_{ab} is η_{ab}: the assembly is a fixed η-coefficient linear combination with no unknown metric and no derivatives. This is why the re-centering of I.4.4 is a necessity rather than a convenience for the teams — the assembly must happen where $g_{ab}=\eta_{ab}$.

Team B. A Lanczos potential generates Weyl through the first-order Lanczos→Weyl relation. This is the one place a covariant derivative ∇ genuinely enters, and it enters harmlessly: at the center $\Gamma^a{}_{bc}=0$, so ∇Ł is an *ordinary* derivative of the box's values. That yields W_{abcd}, whereupon one is back to Team A. The price is smoothness: Ł must be one order smoother than the curvature the other teams need.

Team C. Form $Ł_{abc}=\mathcal{P}_{c[a;b]}$ from the box's $\mathcal{P}_{ab}$ — again an ordinary curl at the center — then proceed as in Team B. The price over Team B is one further derivative. Unlike Team B, Team C is conditional: every sufficiently smooth 4-metric admits a Lanczos potential by Lanczos' theorem, but a symmetric superpotential is not known to exist in every spacetime. [However in the companion paper we present a lot of

evidence suggesting the conjecture that, at least in for analytically-smooth spacetimes, a superpotential $P_{(ab)}$ *does* always exist at least locally – or if not "always" then the exceptional superpotential-free metrics seem to be very rare and likely unphysical.]

I.5 The pairs are minimal

Neither half of any pair suffices alone, because the integration of I.4.2 needs the whole Riemann tensor as source while a single box supplies at most one ten-dimensional summand.

box alone	sees	cannot distinguish
$\{W_{abcd}\}$	Weyl (10), blind to Ricci	Minkowski versus de Sitter; Schwarzschild versus Schwarzschild-de Sitter
$\{R_{ab}\}$	Ricci (10), blind to Weyl	flat space versus a gravitational wave; Minkowski versus Schwarzschild
$\{R\}$	the trace only (1)	blind to Weyl and to trace-free Ricci
$\{Ł_{abc}\}$	Weyl only, via $\nabla Ł$	as $\{W_{abcd}\}$
$\{\mathcal{P}_{ab}\}$	Weyl only, via $\nabla\nabla\mathcal{P}$	as $\{W_{abcd}\}$, one derivative lower

I.6 The smoothness floor is $C^{1,1}$, not C^1.

Our arguments are most natural for C^2 or smoother metrics. since C^2 is what makes the Riemann, Ricci, and Weyl tensors exist everywhere (each is built from two derivatives of the metric, schematically $R \sim \partial\Gamma \sim \partial^2 g$). Nevertheless we can weaken the C^2 requirement to $C^{1,1}$ – meaning the first derivatives of the metric exist and are Lipschitz – with the Riemann curvature tensor then existing *almost* everywhere. We state and sketch a proof of this smoothness-floor claim, trying to explain the mechanism without belaboring that machinery.

The mechanism is the Picard-Lindelöf ODE existence & uniqueness theorem, that a first-order ODE initial-value problem whose right-hand side is Lipschitz in the unknowns has one, and only one, solution — applied twice. Apply it first to the geodesic equation

$$d^2x^a/ds^2 = -\Gamma^a{}_{bc}(x)(dx^b/ds)(dx^c/ds),$$

which defines the rays and hence the normal coordinates. Its right-hand side is Lipschitz in the state $(x, dx/ds)$ exactly when the connection $\Gamma^a{}_{bc}$ is Lipschitz – and Γ is Lipschitz exactly when the metric is $C^{1,1}$. So $C^{1,1}$ is not an arbitrary choice: it is *precisely* the regularity-level causing Picard-Lindelöf to deliver unique geodesics, a well-defined exponential map, and honest normal coordinates (with Lipschitz dependence on the initial data, so exp is a bi-Lipschitz homeomorphism – whereupon Brouwer's "invariance of domain principle" (the † footnote of §I.4.4) supplies openness, exp no longer needing to be differentiable). If you dropped to plain C^1, then $\Gamma^a{}_{bc}$ would only be continuous, not necessarily Lipschitz, in which case only Peano's weaker existence-only theorem could be used, allowing geodesics to branch, and the coordinates themselves to become ill-defined.

Now re-apply the same theorem to the reconstruction proper – the linear Jacobi and structure ODEs of §I.4.2–I.4.3. Being linear, their right-hand side is automatically Lipschitz in the unknowns (the Lipschitz constant is merely the size of the coefficient), so Picard-Lindelöf – in the Carathéodory form that lets the coefficient be bounded and measurable, rather than continuous, along the ray – again returns a unique solution, and linearity forbids finite-parameter blow-up, so the ODE runs the whole segment. That "coefficient" is the curvature, which at $C^{1,1}$ exists almost everywhere and is locally bounded by "Rademacher's theorem." Hence C^2 relaxes to $C^{1,1}$ but not to C^1. ($C^{1,1}$ is the recognized regularity floor of low-regularity geometry; a fully rigorous account requires the usual care – curvature defined only almost everywhere, handled by smooth approximation – which we do not belabor.)

I.7 Analytic data gives an analytic reconstruction

The floor of §I.6 is $C^{1,1}$, which is what makes the theorem widely applicable. At the other end, if the input data is **real-analytic** the conclusion strengthens: the reconstructed metric is real-analytic too.

Why, and with which theorem. If g_{ab} is analytic then so are $\Gamma^a{}_{bc}$, R_{abcd}, and the exponential map, so the normal coordinates of §I.4.1 are analytic and the coefficient $K^i{}_j(s)$ of the Jacobi equation is an analytic function of the affine parameter. One may invoke the Cauchy-

Kovalevskaya theorem here, but for this system something simpler and stronger applies: **the Jacobi and structure equations are linear**, and a linear ODE with analytic coefficients has analytic solutions on the whole interval where the coefficients are analytic — no smallness hypothesis, no shrinking of the domain. Cauchy-Kovalevskaya is the right tool for the general nonlinear case and is what §I.4.5 would need if the teams' assembly were nonlinear; here linearity does the work.

Why a second theorem is needed at all: along the rays versus across them. This is the point on which the argument turns, and it is easy to miss. The reconstruction integrates along *rays* from the seed point: in normal coordinates a point of the patch is $x^a = s\, n^a$, and the metric there is obtained by solving the Jacobi and structure equations out to parameter s along the ray of direction n^a. Linear-ODE theory — or Cauchy-Kovalevskaya — gives analyticity in **s**, that is, *along* each ray. It says nothing whatever about the dependence on n^a.

But n^a is what moves one **transverse** to the rays, and the reconstructed metric is a function of the whole patch, not of one ray. Neighbouring points off a given ray are reached by *different* members of the family of ODEs — the tidal matrix $K^i{}_j(s) = R^i{}_{0j0}(\gamma_n(s))$ is evaluated along the geodesic γ_n, so n^a enters as a **parameter** of the equation, not as its independent variable. Analyticity of g_{ab} as a function of position therefore requires joint analyticity in (s, n^a), and the second slot is exactly a statement of analytic dependence on parameters.

Analytic dependence on the data. The version of Cauchy-Kovalevskaya usually stated gives analyticity in the independent variable and says nothing about analytic dependence on the coefficients or the initial conditions. That gap is filled by a separate classical result, **Poincaré's theorem on analyticity** (also called Poincaré analyticity): for a system $\dot z_a = f_a(z; t; \lambda)$ whose right-hand sides are analytic in the dependent variables, the independent variable and the parameters λ, the solution is analytic in the initial conditions, the times, and the parameters. Poincaré proved it case by case by Cauchy's method of majorants; it is now usually obtained by Picard iteration and appears as background material in standard texts on ordinary differential equations [24–26]. For a *linear* system it is immediate in any case, since the fundamental matrix itself solves a linear equation whose coefficients are analytic in λ. Applied here with $\lambda = n^a$, together with the analyticity of the exponential map, this gives joint analyticity in (s, n^a) and hence analyticity of the reconstructed metric on the whole normal-coordinate patch — not merely along each ray separately.

So the division of labour is: **Cauchy-Kovalevskaya (or plain linear-ODE theory) works along the rays; Poincaré's theorem works across them.** Neither alone suffices, and the transverse direction is the one that is easy to forget, because the integration one actually performs is longitudinal. So the reconstruction depends analytically on the curvature data and on the seed frame.

For the complexified tower: which components inherit analyticity, and why. One might naively expect all of the imaginary components of Part II to inherit it automatically. **They do not, and the dividing line is exactly the number of ε's.** A single dual carries a factor $\sqrt{(-\det g)}$, which is analytic only where $\det g \neq 0$ and only after a branch has been chosen. In Schwarzschild coordinates $\sqrt{(-\det g)} = r^2|\sin\theta|$, and $\sqrt{(\sin^2\theta)} = |\sin\theta|$ is *not* analytic at $\theta = 0, \pi$. So *R, R*, L*, J* and the $\overleftarrow{i}$, $\vec{i}$ slots are analytic only away from the axis and only branch by branch. The **double** dual is immune, and for a reason worth stating precisely: ε appears *twice*, so the two factors of $\sqrt{(-\det g)}$ multiply to give $-\det g$, which is a **polynomial** in the metric components. **No square root survives.**

This is not merely an argument about signs. The double dual has a closed form with no ε in it at all — identity B6 of the catalogue, in the shape used throughout Part II:

$$(*R*)_{abcd} = -W_{abcd} - (R/24)(g \odot g)_{abcd} + \tfrac{1}{2}(g \odot S)_{abcd}$$

whose right-hand side is **rational** in g_{ab}, g^{ab} and R_{abcd}. Checked against the two-ε computation on ten metrics of §II.9, the two routes agree to ≤ 7e−16, and exactly (0.00e+00) on the pp-wave. So $(*R*)_{abcd}$ is rational in analytic data, hence analytic wherever the data is — including on the axis, where the single duals fail.

The dividing line is therefore sharp and easy to state:

object	ε count	$\sqrt{(-\det g)}$	consequence
R, R, L*, J*; the $\overleftarrow{i}$ and $\vec{i}$ slots	one	survives	orientation-dependent; analytic only off the axis, branch by branch
(*R*); the $\vec{i}\vec{i}$ slot	two	**cancels**	orientation-independent; rational; analytic wherever the data is

So the $\vec{i}\vec{i}$ slot of Part III inherits the analyticity of the data outright, while the arrow slots do not.

I.7.1 Two things analyticity does *not* buy

It does not make the reconstruction commute with parameter maps. This is worth stating because the temptation is strong. Schwarzschild's Riemann components are *rational* in the mass M — hence analytic, with no branch points anywhere and only poles at the horizon — so the continuation $M \to -M$ is single-valued and unambiguous. Nevertheless $R_{abcd}(-M) \neq -R_{abcd}(M)$: 16 of 256 components differ. The reason is parity, not analyticity. Each component splits into parts even and odd in M, and only the odd parts flip sign; $R_{2323} = 2Mr\sin^2\theta$ is purely odd and does flip, while R_{1313} carries an even part $2M^2\sin^2\theta/(4M^2-r^2)$ which does not. **Analytic continuation transports solutions; it does not make a function odd.**

It does not prevent the output from degenerating. The reconstruction sets $g_{ab} = \eta_{ij}A^i{}_aA^j{}_b$, and A remains analytic — but analytic functions may vanish. Where det A = 0, at a conjugate point, the reconstructed object is degenerate and has no signature, however analytic the data. Analyticity controls smoothness, not invertibility.

I.7.2 Where the analyticity holds: the determinant argument

The two limitations above are not independent, and tracing the determinant shows why. Since $g_{ab} = \eta_{ij}A^i{}_aA^j{}_b$,

$$\det g = \det(\eta)\cdot(\det A)^2 = \mathbf{-(\det A)^2}$$

(verified numerically to 1e−14 on random frames). Three consequences follow, and they turn out to pick out the same locus.

(i) Only even-order zeros occur. $-\det g = (\det A)^2$ is a perfect square, so every zero of it has even order and det g never changes sign. For a *general* analytic symmetric tensor field an odd-order zero of the determinant is possible, and there an eigenvalue crosses zero and the signature genuinely changes; for a *reconstructed* metric that case cannot arise.

(ii) The signature is stable; what can flip is orientation. Wherever A is invertible, $g_{ab} = A^T\eta A$ carries the signature of η, on both sides of any zero of det A. Crossing such a zero reverses the sign of det A and hence the **orientation**, not the signature.

(iii) The square root, and hence the single duals, fail exactly there. $\sqrt{(-\det g)} = |\det A|$, analytic wherever $\det A \neq 0$. At a zero of det A of order k it is analytic if k is even — the determinant touches zero and returns — and non-analytic if k is odd, where it crosses.

So three things coincide exactly:

at det A = 0	what happens there
nondegeneracy	lost — the object is not a metric
$\sqrt{(-\det g)}$, hence *R, L*, J* and the arrow slots	the only place they can fail to be analytic
the geodesic congruence	a conjugate point

Conclusion. The domain on which the reconstruction is an analytic metric of the seed signature is delimited by the vanishing of det A, and by nothing else. Within that domain everything is analytic; at its boundary all three failures occur together. On an outgoing radial Schwarzschild ray with M=1 the boundary is never reached: min |det A| = 8.0e−3 over 60 units of affine parameter with no sign change, for the original tidal matrix and for the shadow −K alike.

I.8 Other signatures and dimensions D

Nothing used the Lorentzian signature: replace η by diag(±1, …, ±1) and every step is unchanged. Hence all six foundations hold for Riemannian and mixed signatures alike. Dimension enters only as the coefficients inside the definition of the Weyl tensor; the metric, connection, and Riemann tensor (foundations 1–3) may be reconstructed identically in every dimension. **The Lanczos-involving pairs are the ones whose dimensional status is delicate.** A Lanczos potential for the actual Weyl tensor W_{abcd} is known to exist for D=4 (Lanczos), and is known *not* to exist for D≥7 (Edgar and Höglund, 2002). For D=5 and D=6 the question is open: what has been proved there is nonexistence for Weyl *candidate* tensors - tensors carrying the algebraic symmetries of W_{abcd} but not required to be the Weyl tensor of any metric - which is a strictly weaker statement, as Edgar and Höglund themselves note. Numerical jet computations in D=5 and D=6 find local Lanczos potentials for the actual Weyl tensor of every metric tried, including deliberately degenerate ones, so the widespread belief that Lanczos potentials are a four-dimensional phenomenon should be treated with care: it is certainly true of the *tensor-duality* formulation, which needs D=4 to make sense at all, but the underlying question - does W_{abcd} have an antiderivative? - appears to answer yes in five and six dimensions too. So the count is five or six (depending on whether the superpotential exists) in four dimensions, four for D≥7, and

unsettled for D=5 and D=6. Below four dimensions the list contracts, because the Weyl tensor is identically zero for D≤3. When D=3 the Ricci decomposition loses its Weyl part (zero) so the whole Riemann tensor is fixed by the Ricci tensor alone; foundations 4 and 2 thereby merge into a single Ricci-only foundation, leaving three (metric, connection, and Ricci ≡ Riemann). When D=2 the Riemann tensor has just one independent component, so the curvature scalar R by itself determines it – that scalar is then the entire curvature foundation.

I.9 Computer sanity-check

A computer program was written, and successfully tested, mostly on general Bianchi-I spacetime metrics, with numerous random start-end point pairs, to perform g_{ab} reconstructions from any of foundations 2-5 to high accuracy.

Q.E.D.

I.10 Degree of freedom counts as function of D

Much of the dimensional behavior above is visible in a table of component counts. Every entry below was verified by computing the matrix-rank of the defining symmetry conditions, not merely by quoting a formula. That way we do not need to worry that there might be extra linear restrictions obeyed by our tensors which somehow had escaped notice over the last 100 years. "Degrees of freedom" here means the number of linearly independent components at a point. (The defining conditions are all linear, so this is a dimension of a vector space, not an algebraic-independence count.)

#	object	symmetries imposed	count	D=3	D=4	D=5	D=6	D=7	D=8	D=9
1	g_{ab} metric; also Ricci R_{ab}, Schouten Q_{ab}, and superpotential $\mathcal{P}_{ab}$	symmetric	(D+1)D/2	6	10	15	21	28	36	45
2	g_{ab} modulo diffeomorphism	symmetric, less D arbitrary coordinate functions	D(D−1)/2	3	6	10	15	21	28	36
3	$R_{ab} - R\, g_{ab}/D$, trace-free Ricci	symmetric, trace-free	(D+1)D/2 − 1	5	9	14	20	27	35	44
4	$\Gamma^a{}_{bc}$ Levi-Civita connection	symmetric in bc	$D^2(D+1)/2$	18	40	75	126	196	288	405
5	R_{abcd} Riemann	pair antisymmetries, pair exchange, first Bianchi	$D^2(D^2-1)/12$	6	20	50	105	196	336	540
6	W_{abcd} Weyl	Riemann symmetries, all traces zero	(D+1)(D+2) (D−3)D/12	0	10	35	84	168	300	495
7	$Ł_{abc}$ Lanczos, trace permitted	antisymmetric in ab, cyclic-free	D(D−1) (D+1)/3	8	20	40	70	112	168	240
8	L_{abc} Lanczos, trace-free; also C_{abc} Cotton, which is of the same symmetry class	antisymmetric in ab, cyclic-free, trace-free	D(D−2) (D+2)/3	5	16	35	64	105	160	231

The Weyl row is valid for D≥3; below that the Weyl tensor is not defined. Riemann splits as Weyl plus Ricci in every dimension D≥3, and the table shows it: 20=10+10, 50=35+15, 105=84+21, etc.

Two consequences, both exact.

(i) The trace-free Lanczos tensor has room to spare only below five dimensions. Subtracting the two closed forms,

$$D(D-2)(D+2)/3 - (D+1)(D+2)(D-3)D/12 \;=\; -(D+2)(D-1)(D-5)D/12,$$

whose sign is that of −(D−1)(D−5). So L_{abc} has *more* components than W_{abcd} exactly for D = 2, 3, 4; it has *exactly as many* at D=5, which is the unique dimension where the two counts tie (35=35); and it has fewer for every D≥6. This is an elementary re-derivation of the shape of the known results: four dimensions is the largest in which the Weyl antiderivative carries surplus components, which is the source of the Lanczos gauge freedom; five dimensions is a dead algebraic tie, which is why the D=5 case required a subtler integrability argument rather than a count; and from six dimensions on, a generic Weyl *candidate* simply has too many components for any L_{abc} to match.

(ii) D=4 is the unique dimension in which a symmetric superpotential has exactly as many components as the Weyl tensor. Setting (D+1)(D+2)(D−3)D/12=(D+1)D/2 gives (D+2)(D−3)=6, that is D^2−D−12=0, whose only positive root is D=4. The coincidence 10=10 is genuinely four-dimensional and has no analogue in any other dimension.

A caution about how these counts may be used. It is tempting to read row 1 against row 6 and conclude that a superpotential $\mathcal{P}_{ab}$, having only (D+1)D/2 components, "has too few degrees of freedom to determine W_{abcd}" once D≥5, and "therefore," superpotentials generically cannot exist on 5-dimensional manifolds; and similarly comparing row 6 versus 7 or 8 conclude Lanczos tensors generically cannot exist on 6-dimensional manifolds. Actually, that argument is not justified. (But we've nevertheless been able to prove 5D superpotential nonexistence by constructing specific counterexamples. Indeed *random* points on *random* 5D metrics empirically provide counterexamples. Neither we, nor Edgar & Höglund, were able to prove 6D nonexistence for Lanczos tensors.) What such a comparison actually establishes is that the system is **overdetermined**, and hence that a *generic* right-hand side admits no solution. It does not establish that a *particular* right-hand side admits none – and the problem is that the Weyl tensor is not at all generic.

This point is made most sharply by turning the same count on the metric itself. Given a Weyl candidate field in six dimensions, ask for a metric having it as its Weyl tensor: that is 84 equations in the 21 components of g_{ab}, far more overdetermined than the superpotential problem. The counting conclusion "no metric exists" would nevertheless be absurd whenever the candidate handed to you happened to be the Weyl tensor of a metric – one exists, and you were handed it. What the count correctly says is that most "Weyl-*candidates*" are not the Weyl tensor of any metric, which is true and useful, but a different statement.

Quantitatively, the targets one actually needs to reach are extremely *non*generic. Weyl-candidate fields in D=5 are 35 arbitrary functions of five variables, whereas the Weyl tensors of actual metrics are parameterized by at most D(D−1)/2=10 functions, by row two of the table; in D=6 it is 15 functions inside 84, and in D=7, 21 inside 168. Actual Weyl tensors occupy a set of measure zero within candidate-space, and the set gets relatively thinner as D grows – which is why results about generic candidates carry *less* information about actual metrics in higher dimensions, not more, and why Edgar and Höglund 2002's nonexistence theorem for D≥7 had to be proved by a dimensionally dependent tensor identity rather than by counting, but they were able to prove a nonexistence theorem for Lanczos potentials of Weyl-*candidate* tensors in 2000.

The corresponding open question for the material of this paper is whether the *Bianchi identities* satisfied by W_{abcd} supply precisely the integrability conditions that the overdetermined system for L_{abc} (or for $\mathcal{P}_{ab}$) requires. At D=4 the answer is yes and is "Lanczos's theorem"; at D≥7 it is no (Edgar and Höglund 2002); at D=5 and D=6 it is open, and *no numerical evidence either way should be claimed*. Truncated jet computations cannot reach it. The truncated Weyl-Lanczos system has equation-to-unknown ratio $(N_W/N_L)\,K(K+D)^{-1}$, whose limit as $K\to\infty$ is the ratio N_W/N_L of the table above: 1.312 at D=6, 1.600 at D=7, and **exactly 1 at D=5**. So overdetermination would need K > 19 at D=6 and K > 12 at D=7, both far out of computational reach, and at D=5 it never arrives at any order - which is the D=5 tie of I.10 reappearing as a limit on what computation can decide. A calibration confirms the insensitivity directly: at D=7, where nonexistence is a theorem, the same code reports the actual Weyl tensor *consistent* at both K=2 and K=3.

I.11 Degrees of freedom of the complexified and bicomplexified tensors

The objects of Part II exist only in four dimensions, since the tensor dual needs the four-index ε_{abcd}, so what follows is a D=4 statement and not a row of the D-dependent table above. This is also the precise sense in which the folklore "Lanczos tensors are four-dimensional" is correct: it is the *duality* formulation that requires D=4, not the underlying question of whether W_{abcd} has an antiderivative.

Complexification never changes the real degree-of-freedom count. Every construction in Part II is an invertible real-linear repackaging: W^*_{abcd} is determined by W_{abcd}, G_{ab} by R_{ab}, and so on, so no rung of the tower can carry information the real tensor did not already carry. What the complexification does change is the number of *complex* components needed to hold those degrees of freedom - and it changes it only for the dual-type rungs.

object	type	real degrees of freedom, D=4	complex components after packaging	compression?
$W_{abcd} \to \mathring{W}_{abcd}$	dual-type, antisymmetric pair locked ($*W_{abcd} = W^*_{abcd}$)	10	5	yes, halved
$L_{abc} \to \mathring{L}_{abc}$	dual-type, one antisymmetric pair	16	8	yes, halved
	dual-type, one antisymmetric			

C_{abc} Cotton → J_{abc}	pair; same symmetry class as L_{abc}	16	8	yes, halved
R_{ab} → $R_{ab} = R_{ab} + iG_{ab}$; likewise Q_{ab} → Q_{ab}	trace-type, no antisymmetric pair, hence no dual (II.5.2)	10	10, spanning a real 10-dimensional subspace of the complex 20	no
R_{abcd} → R_{abcd}	bicomplex, two independent antisymmetric pairs (II.4)	20	20, spread across the four slots $1, \vec{i}, \vec{i}, \vec{i}\vec{i}$	no - not a joint eigentensor of the two dual operators
P_{ab} superpotential	no antisymmetric pair at all, hence no factor and no natural dual	10	—	the P* problem, II.12

Why the dual-type rungs halve. On each of these symmetry classes the dual operator squares to −1, so it has eigenvalues ±i and splits the complexified class into two eigenspaces of equal dimension. The complexified tensor lies wholly in one of them - the eigen-relations $(W)^* = -iW$, $(L)^* = -iL$, $(J)^* = -iJ$ of II.5.1 say exactly this - so it is carried by half as many complex components as the real tensor had real components. Both cases were checked directly by constructing the symmetry class, restricting the dual operator to it, and diagonalizing: on the 16-dimensional trace-free Lanczos class the dual squares to −1 to within 5×10^{-15} and its eigenvalues are eight at +i and eight at −i; on the 10-dimensional Weyl class, five and five. The five complex components of W_{abcd} are the five entries of the symmetric trace-free 3×3 complex matrix whose Jordan structure is the Petrov type (II.11), and the collapse of the eighteen quadratic Lanczos invariants to two (II.7.2) is the same halving seen at the level of invariants.

Why the trace-type rungs do not. A symmetric two-index object has no antisymmetric pair, so there is no dual, no eigen-relation, and nothing to project onto. The imaginary partner has to be imported from the other trace of Riemann - Ricci from R_{abcd}, Einstein from $(*R*)_{abcd}$ - and the resulting $R_{ab} = R_{ab} + iG_{ab}$ is an invertible change of variables on ten real numbers rather than a compression of twenty into ten. The bicomplexified Riemann tensor is the same story at higher rank: it keeps the whole tensor, traceless-Ricci part included, but it is not a joint eigentensor of the two dual operators (II.3 explains why: on one summand $\vec{i}$ acts as $\vec{i}$, on the other as $-\vec{i}$), so its four slots hold the twenty real degrees of freedom redundantly rather than compactly.

So the tower compresses exactly where an antisymmetric index pair supplies a dual to be an eigentensor of, and nowhere else. That is the same structural fact as the factor count of II.4 - one factor $(1 + i^*)$ per independent antisymmetric pair - viewed through degrees of freedom instead of through the construction.

I.12 Nonexistence of superpotential→Lanczos→Weyl chain on 5D manifolds

Section I.10 showed that a symmetric superpotential carries (D+1)D/2 components against the Weyl tensor's (D+1)(D+2)(D−3)D/12, and warned that this counting establishes only that the system is overdetermined, hence that generic Weyl *candidates* have no superpotential - not that any particular Weyl *curvature* tensor lacks one. This section supplies what counting cannot: an explicit five-dimensional metric, an explicit point, and an explicit certificate that no superpotential exists there.

The argument is broken into lemmas, because the lemmas differ sharply in what kind of checking they need and it is worth keeping them apart. Lemma 1 is curvature arithmetic on a two-term polynomial metric. Lemma 2 reduces existence to a finite linear system, and carries the one step where a reader must think rather than compute. Lemma 3 asserts that this system is inconsistent, demonstrated with integer arithmetic. All large numbers in the proof live here. Lemma 4 and Lemma 5 record two properties of the obstruction that guard against the two most plausible ways such a result could be an artifact. Finally, the theorem is then three lines.

I.12.1 The metric and the initial statement

In five-dimensional coordinates $(x^0, x^1, x^2, x^3, x^4)$ with $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1, +1)$, define

component	value
$g_{23} = g_{32}$	$(x^4)^2$
$g_{04} = g_{40}$	x^1x^3

every other g_{ab}	η_{ab}

Every component is a polynomial of degree at most two; the metric is defined on all of $\mathbb{R}^5$; and at the origin $g_{ab}=\eta_{ab}$, so $\det g_{ab}=-1$ there and the metric is Lorentzian in a neighbourhood of it.

Initial Statement. At the origin this metric admits **no symmetric C^4-smooth superpotential at all**, while the corresponding system with L_{abc} as the unknown is consistent at the same jet order (residual 6.8×10^{-14}), so the obstruction is specific to the superpotential rung rather than to the Lanczos rung. The regularity hypothesis is used in Lemma 2, and I.12.10 points out that superpotentials of lower regularity are *not* excluded by this argument. The earlier version of this section proved the weaker statement that no P_{ab} exists whose $L_{abc}=P_{c[a;b]}$ is trace-free, and treated the trace-permitted $\mathcal{P}_{ab}$ separately. That distinction later became unnecessary: I.12.4's certificates never invoke the trace condition, so they rule out P_{ab} and $\mathcal{P}_{ab}$ together (Lemma 4). The obstruction lives in the three-jet. Although along the way we may mention various checks done with floating point arithmetic, the proof proper is done with exact integer arithmetic only, and basically amounts to a large linear algebra calculation.

Both metric terms are needed. Delete either one and a superpotential exists again: with only $g_{23}=(x^4)^2$ the residual is 1.3×10^{-13}, with only $g_{04}=x^1x^3$ it is 9.8×10^{-14}. The obstruction is a joint effect of the two terms.

The coefficients are not special. Replacing the two terms by $g_{23}=\alpha(x^4)^2$ and $g_{04}=\beta x^1x^3$ blocks the superpotential for every (α,β) tried: residual 5.94×10^{-2} at (1,1) and at (–1,1), 2.18×10^{-2} at (3,7), 1.39×10^{-2} at (1,3), 7.06×10^{-3} at (0.5,–2). So this is a two-parameter *family* of counterexamples, not one lucky metric.

I.12.2 Lemma 1: the Weyl tensor at the origin

Up to the Weyl symmetries there are five nonvanishing components, all small rationals:

$W_{0134}=+2^{-1}$	$W_{0203}=-3^{-1}$	$W_{0314}=+2^{-1}$	$W_{1213}=+3^{-1}$	$W_{2434}=-2\cdot3^{-1}$

The Ricci scalar vanishes at the origin; the Ricci tensor R_{ab} does not. In particular $W_{abcd}\neq0$ there, so the question is not vacuous.

These values were computed in exact rational arithmetic by a second, independently written program (I.12.8) which uses no floating point: the metric is polynomial, its inverse a terminating Neumann series, and every subsequent operation is rational. As arithmetic checks the computed W_{abcd} satisfies the two pair antisymmetries, the pair-exchange symmetry, the first Bianchi identity and trace-freeness at the origin identically.

I.12.3 Lemma 2: reduction to a finite linear system

Lemma. *Let P_{ab} be symmetric and C^4-smooth on some neighborhood of the origin, and suppose it satisfies the Weyl-Lanczos equation there with $L_{abc}=P_{c[a;b]}$. Then the Taylor coefficients of P_{ab} at the origin of degree at most four satisfy an explicit inhomogeneous linear system $M\,z=b$ of 2100 equations in 1890 rational unknowns.*

Expand P_{ab} in a Taylor series about the origin, set $L_{abc}=P_{c[a;b]}=(\nabla_bP_{ca}-\nabla_aP_{cb})\,2^{-1}$, and impose the Weyl-Lanczos equation in Edgar & Höglund's n-dimensional form,

$$W_{abcd}=L_{abc;d}-L_{abd;c}+L_{cda;b}-L_{cdb;a}-[\,g_{ac}V_{bd}-g_{bc}V_{ad}-g_{ad}V_{bc}+g_{bd}V_{ac}\,]\,(n-2)^{-1},$$
$$V_{bd}=g^{ij}(\,\nabla_iL_{bjd}+\nabla_iL_{djb}\,),$$

to Taylor degree two. (V_{bd} is named to avoid the S_{ab} that denotes traceless Ricci elsewhere in this paper.) The equations are indexed by (ab)(cd) with a<b and c<d, which is 100 index slots, times the 21 monomials of degree at most two: 2100 equations. The unknowns are the 15 symmetric components of P_{ab} times the 126 monomials of degree at most four: 1890 unknowns.

No side conditions are imposed. The cyclic identity needs no equation, since for symmetric P_{ab}

$$L_{abc}+L_{bca}+L_{cab}=[\,\nabla_bP_{ca}-\nabla_aP_{cb}+\nabla_cP_{ab}-\nabla_bP_{ac}+\nabla_aP_{bc}-\nabla_cP_{ba}\,]\;2^{-1}=0$$

identically. Tracefreeness ("Lanczos algebraic gauge") $g^{bc}L_{abc}=0$ is *not* imposed either – see Lemma 4. So M z = b consists of Weyl-Lanczos residual equations and nothing else.

The truncation is complete, not an approximation. This is the step that makes the argument a theorem about local *existence* rather than a statement about a *truncated model*. The Weyl-Lanczos operator involves one covariant derivative of L_{abc}, and $L_{abc}=P_{c[a;b]}$ involves one covariant derivative of P_{ab}; so the residual depends on P_{ab} through at most two derivatives, and the coefficient of a monomial of degree m in the residual can depend only on Taylor coefficients of P_{ab} of degree at most m+2. With m≤2 that is degree at most four – exactly the 1890 unknowns listed. **No Taylor coefficient of P_{ab} of degree five or higher can influence any of these 2100 equations**, so nothing has been discarded and no remainder term is being neglected. The finite system is a necessary consequence of existence.

The degree count was confirmed by direct computation: every P_{ab} monomial of degree five and of degree six that was tested contributes exactly zero to every residual coefficient of degree at most two.

Smoothness. This lemma is the *only* place in I.12 where any smoothness hypothesis is used, so it is worth being exact about the amount. Two derivatives of P_{ab} are needed for the two sides of the Weyl-Lanczos equation even to be defined, so C^2 is the floor for the hypothesis to mean anything. Passing from the pointwise identity to the linear system then requires differentiating that identity twice at the origin, which reaches the fourth derivatives of P_{ab} – exactly the degree-four Taylor coefficients that are the unknowns of M z = b. The clean sufficient hypothesis is "C^4 on a neighborhood of the origin" and is what the theorem states.

But we actually use and need slightly less: it suffices that P_{ab} be C^3 on a neighborhood of the origin with its third derivatives differentiable at the origin. Nothing in the argument reaches beyond that, and nothing in it is improved by assuming C^∞ or analyticity. Conversely, and this is the honest limitation, **a superpotential of lower regularity than that is not excluded**: a P_{ab} which is C^2 near the origin but whose third derivatives fail to be differentiable there has no degree-four Taylor coefficients for the certificate to act on, and falls outside the scope of this proof entirely. Whether such an object can exist for this metric is open, but we doubt it. [One reason we doubt it is: we suspect for an analytically-smooth metric (such as ours) that if a superpotential→Lanczos→Weyl chains exists, then one exists that is analytically smooth. If so, our C^4-assuming nonexistence proof, could drop the C^4-assumption.] Given the $C^{1,1}$ floor of I.6 and the role that failures of smoothness play in the mirror-paradigm paper, this gap is not merely pedantic and may deserve further investigation.

Numerically, the relative least-squares residual of M z = b is $\mathbf{5.92\times10^{-2}}$, against 6.8×10^{-14} for the same system with L_{abc} as the unknown instead of P_{ab}. The Lanczos tensor exists; the superpotential does not. That is the assertion the next lemma replaces with an all-integer proof.

I.12.4 Lemma 3: the linear system has no solution

Lemma. *There is an integer covector ϕ, indexed by the 2100 equations of Lemma 2, with*

$$\sum_i \phi_i M_{ij} = 0 \text{ for every } j, \quad \sum_i \phi_i b_i \neq 0.$$

This lemma is a statement about a fixed integer matrix and a fixed integer vector. No manifold, no differentiability and no regularity class appears in it at all; all of that was spent in Lemma 2. Such a ϕ proves that no solution exists: applying it to any purported solution would give $0 = \sum_i \phi_i b_i \neq 0$. Nine such covectors have been computed, from nine different elimination orders, and each has been verified exactly:

certificate	equations used	$\max_i \lvert\phi_i\rvert$	digits	$\sum_i \phi_i b_i$
1	106	5732740	7	1633160
2	106	11693512	8	-3014522
3	105	29455828	8	4525310
4	104	31966662	8	4525310
5	107	34633888	8	-8539970
6	96	39436200	8	7012322
7	106	48487320	8	-3746002
8	106	48487320	8	-3746002

9	107	76828312	8	-17511902

Each is supported on between 96 and 107 of the 2100 equations. Certificate 1 is smallest in coefficient size; certificate 6 is smallest in support and is the one printed below. That nine independently constructed covectors all certify the same conclusion is itself worth something: an error in extracting any one of them cannot produce agreement among the rest.

On the size of the integers. They are 7-8 digits, which cannot easily be reduced. Each ϕ spans a one-dimensional kernel of an integer matrix of order about 100 whose own entries reach only 192, so the entries of ϕ are essentially determinants of that order, and by the Hadamard bound eight digits is very small for a determinant of that order! The size also appears to be intrinsic rather than an artifact of extraction: 1401 short integer kernel vectors, all with entries of magnitude at most 2, were harvested from random subsystems and used to lattice-reduce every certificate, and not one could be shortened.

The 96 equations of certificate 6 and their coefficients follow. Each row is ϕ_i times the (ab)(cd) component of the Weyl-Lanczos equation above, read off at the stated Taylor monomial. Every row is a Weyl-Lanczos residual equation; there are no gauge rows and no cyclic rows. The table is in five side-by-side blocks of three columns; read down the leftmost block, then the next, and so on.

ϕ_i	(ab)(cd)	monomial	ϕ_i	(ab)(cd)	monomial	ϕ_i	(ab)(cd)	monomial	ϕ_i	(ab)(cd)	monomial	ϕ_i	(ab)(cd)	monomial
+39436200	(04)(13)	x^2x^3	-12922940	(04)(14)	x^2x^4	-8684486	(04)(34)	x^0x^3	-5129230	(02)(03)	x^2x^3	+2300446	(01)(12)	x^0x^2
+33479693	(04)(13)	1	+12922940	(14)(24)	x^0x^4	+8251724	(02)(04)	x^0x^1	-5058554	(13)(34)	x^1x^4	+1851826	(14)(34)	x^1x^3
+27960928	(04)(12)	$(x^1)^2$	+12792316	(04)(24)	x^0x^2	-7416160	(02)(24)	x^1x^2	+4689248	(01)(24)	$(x^0)^2$	+1825018	(03)(34)	x^0x^4
-25455736	(03)(23)	x^1x^4	-12461228	(01)(14)	x^1x^2	-7399868	(23)(34)	x^0x^1	+4687052	(02)(04)	x^2x^4	-1798700	(01)(01)	$(x^2)^2$
-25286952	(04)(12)	$(x^3)^2$	-12123660	(04)(04)	$(x^2)^2$	+7367552	(02)(02)	$(x^1)^2$	-4552090	(24)(34)	x^2x^3	+1798700	(34)(34)	$(x^2)^2$
-24994024	(02)(13)	x^3x^4	+12123660	(13)(13)	$(x^2)^2$	-7367552	(34)(34)	$(x^1)^2$	-4109912	(03)(04)	x^3x^4	+1698740	(02)(23)	x^0x^3
-22667432	(12)(34)	x^0x^3	+12004034	(14)(24)	x^1x^2	-7305480	(01)(01)	$(x^3)^2$	+3818760	(01)(34)	x^2x^3	+1687632	(03)(13)	x^0x^1
-21396624	(12)(14)	x^0x^1	-11737394	(24)(34)	1	+7305480	(12)(24)	x^0x^2	-3668076	(12)(24)	x^1x^4	-1671128	(04)(12)	$(x^0)^2$
-20033168	(14)(23)	x^0x^3	-11013560	(01)(24)	$(x^2)^2$	+7305480	(24)(24)	$(x^3)^2$	-3663900	(02)(02)	$(x^3)^2$	+1531830	(02)(03)	1
-18891432	(03)(34)	x^1x^2	+11013560	(04)(12)	$(x^4)^2$	+7061380	(14)(14)	$(x^0)^2$	+3663900	(14)(14)	$(x^3)^2$	-1390114	(01)(03)	x^1x^3
-17453260	(04)(04)	$(x^1)^2$	+10969380	(04)(04)	$(x^3)^2$	+7061380	(23)(23)	$(x^0)^2$	-3652740	(23)(24)	x^3x^4	-923424	(01)(24)	$(x^1)^2$
+17453260	(23)(23)	$(x^1)^2$	-10969380	(12)(12)	$(x^3)^2$	+6888268	(13)(34)	x^0x^2	+3530690	(01)(14)	x^0x^4	+899350	(23)(34)	x^2x^4
+15645614	(01)(34)	1	-10578316	(01)(02)	x^0x^4	-6883572	(02)(12)	x^0x^1	+3376040	(01)(24)	$(x^3)^2$	-577140	(12)(13)	x^2x^3
+15267564	(13)(23)	x^0x^4	-10491870	(03)(23)	x^0x^2	+6399592	(12)(12)	$(x^0)^2$	+3241940	(12)(23)	x^1x^3	+461712	(02)(34)	x^1x^3
-14788460	(13)(23)	x^1x^2	-10383226	(01)(13)	x^0x^3	+6399592	(34)(34)	$(x^0)^2$	-2784426	(01)(02)	x^1x^2	-457172	(13)(14)	x^3x^4
-14610960	(04)(12)	$(x^2)^2$	+10085708	(03)(03)	$(x^1)^2$	-5653474	(12)(13)	1	+2749556	(02)(02)	$(x^4)^2$	-110680	(02)(12)	x^2x^4
+13922360	(03)(03)	$(x^2)^2$	-10085708	(24)(24)	$(x^1)^2$	+5617460	(01)(23)	x^3x^4	-2749556	(03)(03)	$(x^4)^2$			
-13922360	(14)	$(x^2)^2$	-9214860	(03)	x^2x^4	+5586402	(12)	x^2x^4	+2749556	(12)	$(x^4)^2$			

	(14)			(13)			(14)			(12)				
-13460972	(13) (13)	$(x^0)^2$	-8935396	(01) (12)	x^1x^4	-5355708	(02) (24)	x^0x^4	-2749556	(13) (13)	$(x^4)^2$			
-13460972	(24) (24)	$(x^0)^2$	+8726630	(01) (04)	x^1x^4	+5195940	(04) (14)	x^0x^1	+2326592	(01) (04)	x^0x^2			

I.12.5 Lemma 4: the obstruction is not a gauge artifact

Lemma. *Every certificate of I.12.4 is supported entirely on Weyl-Lanczos residual equations. None uses the "Lanczos algebraic gauge" $g^{bc}L_{abc}=0$, and none uses the cyclic identity $L_{[abc]}=0$.*

Two consequences. First, the theorem needs no gauge hypothesis. Second, it covers the trace-permitted superpotential $\mathcal{P}_{ab}$ by the same certificate and with no separate argument, since nothing anywhere in the proof asks L_{abc} to be trace-free.

I.12.6 Lemma 5: the obstruction first appears at third order

Lemma. *The same construction truncated at second order in the jet gives a system of 915 equations which is consistent, to relative residual 1.6×10^{-16}. The obstruction appears only when the third-order jet is included.*

So the third-order truncation of Lemma 2 is not merely sufficient, it is minimal: the theorem cannot be proved by any smaller computation of this shape. Enlarging it changes nothing - the residual is unmoved by extra ring degree or extra Taylor degrees of P_{ab}, as Lemma 2's degree count predicts:

Taylor order K of the equations	ring degree	degree of P_{ab}	system	relative residual
2	3	3	915 × 840	1.6×10^{-16}, not yet visible
2	4	4	915 × 1890	1.6×10^{-16}, still not visible
3	4	4	2100 × 1890	5.92×10^{-2}
3	5	5	2100 × 3780	5.92×10^{-2}
3	5	6	2100 × 3780	5.92×10^{-2}
4	5	5	7490 × 3780	5.92×10^{-2}

A null result at low jet order therefore proves nothing – a lesson that applies to both the dimension-five and dimension-six Lanczos questions, and useful as a "sanity check" on computer code: any implementation which reports nonexistence at second order for this metric has a bug.

I.12.7 The theorem

Theorem. *For the metric of I.12.1 there is no symmetric superpotential P_{ab}, C^4-smooth on any neighborhood of the origin, whose derived tensor $L_{abc}=P_{c[a;b]}$ is a Lanczos potential for that metric's Weyl tensor W_{abcd}. This holds whether or not L_{abc} is required to be trace-free. The hypothesis may be weakened to: C^3 on a neighborhood of the origin, with third derivatives differentiable at the origin.*

Proof. Suppose such a P_{ab} existed. Its regularity is exactly what Lemma 2 requires, so its Taylor coefficients through degree four exist and satisfy M z = b. By Lemma 3 that system has no solution. Contradiction. By Lemma 1 the Weyl tensor is nonzero at the origin, so the statement is not vacuously true; and by Lemma 4 no gauge condition was used, so the trace-permitted case is included. **Q.E.D.**

I.12.8 How typical is this?

A single counterexample settles existence; the separate question is whether nonexistence is the rule or the exception. Two facts:

Among purely quadratic metrics, the counterexample is exceptional. Draw $g_{ab} = \eta_{ab} + \sum c(a,b,m)\, x^m$ with the sum over a≤b and over all fifteen monomials of degree two, and c(a,b,m) independent uniform on [−1,+1]. At K=3 not one of six such metrics blocks; residuals cluster at 10^{-10}. Adding all degree-three monomials as well does not change this: none of eight blocks, residuals near 10^{-11}. Scaling the coefficients down by five changes nothing. So the two-term metric of I.12.1 blocks for a structural reason that a generic member of those classes destroys.

Among metrics carrying enough Taylor orders, nonexistence is the rule. Draw $g_{ab} = \eta_{ab} + \sum c(a,b,m)\, x^m$ over all monomials of degree two, three and four, with each c(a,b,m) a uniform random integer in [−3,+3] divided by five, and go to K=4. Then the superpotential system is inconsistent for every metric drawn so far - residuals 4.37×10^{-2}, 4.76×10^{-2} and 4.37×10^{-2} on three draws, against a floor of 10^{-12} for the corresponding L_{abc} system on the same metrics. Three draws is a small sample and is reported as such; each takes several minutes to assemble. Since a random draw from a stated distribution lands in a proper algebraic subvariety with probability zero, and consistency of a linear system in the metric's Taylor coefficients is exactly such a subvariety condition, this is a probabilistic argument that nonexistence holds at generic points of generic five-dimensional metrics – with the usual caveat that it is evidence about the truncated system rather than a proof.

The apparent tension between the two paragraphs is the order threshold of Lemma 5 again, not a contradiction: the dense low-degree classes were tested at K=3, where a dense metric's obstruction has not yet appeared, and the dense class at K=4 does block. The two-term metric is useful precisely because it blocks already at K=3, which is what makes a certificate of only about a hundred equations possible.

I.12.9 Verification software

Everything above was established twice, by two programs sharing no code.

- **Independent re-implementation in exact rational arithmetic.** A second program was written from the textbook formulae, with its own truncated-polynomial arithmetic over the rationals, its own metric inverse by terminating Neumann series, and its own Christoffel, Riemann, Ricci, Weyl and Weyl-Lanczos routines. It uses no floating point anywhere and needs no integer rescaling of the system. It reproduces the five Weyl components of Lemma 1 exactly and confirms all nine certificates.
- **Deterministic, not probabilistic, verification.** For each certificate the condition $\sum_i \phi_i\, M_{ij}=0$ was checked against all 1890 basis elements of the P_{ab} jet space, one at a time – 15 symmetric components times 126 monomials. That is a complete check over a spanning set, not a random sample.
- **Negative controls on the checker.** We fed the verifier deliberately corrupted certificates. It rejected every one: a single coefficient changed by +1; a single sign flipped; one term deleted; two coefficients swapped; one monomial label altered; one index label altered; the zero vector; random coefficients. It accepted the unmodified certificate, and accepted it again after scaling by 2. It also rejected a valid certificate presented against either of the two one-term metrics of I.12.1, confirming that it is sensitive to the geometry and not merely to the shape of its input.
- **The equation was checked against the literature.** The Weyl-Lanczos operator implemented here was compared term by term with Edgar & Höglund's published n-dimensional equation, including all four sign patterns of the trace terms and the $(n-2)^{-1}$ coefficient, and agrees.

I.12.10 What is claimed, and what is not

- **The certificates are exact.** The integer identities of I.12.4 hold in integer arithmetic, not to a tolerance, and the matrix they are applied to is now itself built in exact rational arithmetic (I.12.9). The floating-point numbers that still appear in this section - the residuals of I.12.6 and I.12.8 - are diagnostics and motivation, and the theorem does not rest on any of them.
- **The statement is local and it is about one point.** It concerns three-jets at the origin of one metric. It says nothing about other points of that metric.
- **It is not a general five-dimensional theorem.** Five-dimensional pp-waves and products of a curved surface with flat space *do* admit superpotentials. What is established is that the superpotential rung fails somewhere in five dimensions, and – on the evidence of I.12.8 – fails generically.
- **Superpotentials rougher than C^3 are not excluded.** The proof acts on the degree-four Taylor coefficients of P_{ab}, so it says nothing about a P_{ab} which is merely C^2 near the origin, or which is C^3 with third derivatives not differentiable at the origin. Such a P_{ab} still makes both sides of the Weyl-Lanczos equation well defined, so the question is not empty. This smoothness hypothesis conceivably could be dropped; it is open whether it can be.
- **The Lanczos rung is untouched.** The corresponding system with L_{abc} as the unknown is consistent at this jet order, which is evidence that a trace-free L_{abc} exists here but is not by itself a proof of existence. Whether Lanczos potentials exist for actual Weyl tensors in dimensions five and six remains open, and the numerical methods used here are structurally incapable of settling it: the

truncated system's equation-to-unknown ratio is $(N_W/N_L)\,K(K+n)^{-1}$, whose limit is exactly 1 at n=5 by the tie recorded in I.8.1, so it never becomes overdetermined at any order.

- **What no amount of arithmetic can settle.** The residual risk is that the formulation itself misstates the intended geometric question. That risk is concentrated in Lemma 2, which is why its completeness argument is given explicitly rather than assumed, and why the certificates are published with labelled terms so that a reader can test them against a formulation of their own.

Consequence for the foundations. Foundation (*6) of I.2 is the pair $\{\mathcal{P}_{ab}, R_{ab}\}$, and it is conditional on the superpotential existing. In four dimensions the same computation, at the same jet order and through the same code, finds superpotentials for generic curved metrics and for pp-waves alike. In five it does not. Whatever else is true of dimension, **the superpotential foundation is four-dimensional in a way that the Lanczos foundation has not been shown to be**, and any argument that geometry of this kind singles out D=4 is better built on the superpotential rung than on the Lanczos rung.

I.13 Bridge to Parts II and III

Part II complexifies the curvature objects of Part I. Part III then feeds those complexified objects back into the reconstruction of I.4, which accepts them without modification because the reconstruction equation is linear. The result is a bicomplex metric whose extra slots turn out to carry, respectively, a “curvature-reversed shadow” geometry and the momentum constraint of the ADM formulation of the Einstein field equations.

Part II. Complex and bicomplex reformulation in four dimensions

Goal of Part II. *To find the most concise formulations of four-dimensional curvature, and to provide a catalogue of identities usable in practice, so that the complex and bicomplex forms are as easy to work with as the unpackaged ones.*

II.1 Conventions

II.1.1 Duals

Here **w = 2**, and that value is forced. It is the only normalisation making the dual an isometry and making it square to −1: with w = 2 one has $*(*R_{abcd}) = -R_{abcd}$ exactly, whereas with w = 1 the dual would multiply norms by 2 and square to −4. Since the whole bicomplex apparatus of II.1.2 rests on $\overleftarrow{i}^2 = \vec{i}^2 = -1$, no other choice is available.

written	means
$*R_{abcd}$	$\varepsilon_{ab}{}^{ef}R_{efcd}/w$ — left dual, on ab
$R*_{abcd}$	$R_{ab}{}^{ef}\varepsilon_{efcd}/w$ — right dual, on cd
$(*R*)_{abcd}$	both — the double dual
$W*_{abcd}$	the Weyl dual; left and right coincide, so only one object exists. Star on the **right**, since a leading star risks being read as a multiplication-like operator.
$\bar{\underset{\circ}{W}}$	complex conjugate of $\underset{\circ}{W}$

II.1.2 The four bicomplex units

unit	acts on	square	kind
1		+1	real
$\overleftarrow{i}$	the left (ab) pair	−1	imaginary
$\vec{i}$	the right (cd) pair	−1	imaginary

$\overleftrightarrow{\mathbf{i}}\overleftrightarrow{\mathbf{i}}$	both; the product $\overleftarrow{\mathrm{i}}\overrightarrow{\mathrm{i}}$	**+1**	**split / hyperbolic — not an imaginary unit**

The symbol for the product of $\overleftarrow{\mathrm{i}}$ and $\overrightarrow{\mathrm{i}}$ is the small Roman numeral two, $\overleftrightarrow{\mathbf{i}}\overleftrightarrow{\mathbf{i}}$ (unicode #8561 with the combining left-right arrow 8417). which is deliberately not the letter i. **Its square is +1, not −1**, so it is not a third imaginary unit. This is the load-bearing fact in the whole structure — it is what gives the algebra zero-divisors and makes it split — and a reader who assumes $\overleftrightarrow{\mathrm{i}}\overleftrightarrow{\mathrm{i}}^2 = -1$ would wrongly conclude the algebra is quaternionic.

II.1.3 Curvature notation

symbol	meaning
W_{abcd}	Weyl
R_{ab}, R	Ricci tensor, Ricci scalar
G_{ab}	Einstein tensor, $G_{ab} = R_{ab} - Rg_{ab}/2$
$\mathbf{S_{ab}}$	**traceless Ricci**, $S_{ab} = R_{ab} - Rg_{ab}/4$
$\mathbf{Q_{ab}}$	**Schouten**, $Q_{ab} = R_{ab} - Rg_{ab}/6$

Warning. Some sources use Q_{ab} for the traceless Ricci tensor. Here Q_{ab} is always the Schouten tensor and S_{ab} is always the traceless Ricci tensor; every formula below containing Q_{ab} must be read that way. **Warning on normalization.** Our Q_{ab} is *twice* the Schouten tensor as usually normalized in the literature, which in D dimensions carries a factor $(D-2)^{-1}$ - one half, in four dimensions. We drop that factor, matching the companion paper "Switcheroo and Lanczos Gravity," where $Q_{ab} = R_{ab} - Rg_{ab}(2D-2)^{-1}$ in D dimensions. The gain is that the Cotton tensor becomes exactly one antisymmetrized derivative of Q_{ab} with no leftover factor: $J_{abc} = Q_{c[a;b]}$ (catalogue entry F2). The cost is a factor of one half wherever Q_{ab} is fed back into the Weyl decomposition; those places are flagged below. The scalar part of the Ricci decomposition is written inline as $(R/24)\, g \odot g$ rather than given its own letter, avoiding a further collision with S_{ab}.

II.1.4 Strategy

Our goal in Part II was to produce a large table of identities satisfied by complexified and bicomplexified tensors, analogous to those differential geometers have worked out since 1867 for real-valued tensors. We took a brute-force "industrial production" approach: we had earlier assembled a large collection of test-metrics. We would simply postulate an identity and try it on many random spacetime points in different metrics. If test failed, we'd reject that identity as false. We did a fairly-exhaustive large search for identities analogous to known ones. This winnowing eventually left a set of candidate-identities, which we then would verify by exact symbolic simplification, or at least with independently-written numerical-test software.

II.1.5 The Lanczos tensor and the Lanczos→Weyl equation

A **Lanczos tensor** $Ł_{abc}$ of a Weyl candidate W_{abcd} is a three-index tensor with the algebraic properties

$$Ł_{abc} = Ł_{[ab]c}, \quad Ł_{[abc]} = 0, \quad Ł_{ab}{}^{b} = 0$$

(antisymmetric in the first pair; vanishing totally antisymmetric part, called the *cyclic* identity; and trace-free), which generates the Weyl tensor through the first-order **Lanczos→Weyl equation**. In D dimensions this reads

$$W_{abcd}/2 = Ł_{ab[c;d]} + Ł_{cd[a;b]} - \left[(D-2)^{-1}\right] g_{a[c}\left(Ł_{|b|}{}^{e}{}_{d];e} + Ł_{d]}{}^{e}{}_{b;e}\right) + \left[(D-2)^{-1}\right] g_{b[c}\left(Ł_{|a|}{}^{e}{}_{d];e} + Ł_{d]}{}^{e}{}_{a;e}\right)$$

with square brackets carrying the ½ as always (§II.1). In four dimensions $(D-2)^{-1} = ½$, and the equation is equivalent to Lanczos' original form

$$W_{abcd} = Ł_{ab[c;d]} + Ł_{cd[a;b]} - (*Ł*)_{ab[c;d]} - (*Ł*)_{cd[a;b]}$$

in the differential gauge $\zeta_{ab} = Ł_{abc}{}^{;c}$. **The D-dimensional form above is the one used in every computation reported here**, and it was calibrated before use: with the coefficient left as an unknown and solved for, it came out to exactly the value stated, and the resulting operator reproduces the Weyl tensor of the axisymmetric Majumdar-Papapetrou metric on all 21 independent components.

Most of the length of that equation is bookkeeping. §II.8.7 shows that the four trace terms exist only to force the right side to have Weyl-candidate symmetries, and that if one performs that forcing explicitly the whole equation collapses to a single antisymmetrized derivative - in a form which, unlike the one above, commutes with dualization and therefore carries over to the complexified tensors of this Part unchanged.

Two conventions of this paper. Plain $\mathbf{L_{abc}}$ denotes a Lanczos tensor that is trace-free; the slashed glyph $\mathbf{Ł_{abc}}$ is used when a nonzero trace is permitted. Correspondingly $\mathbf{P_{ab}}$ is a superpotential whose Lanczos tensor is trace-free and $\mathbf{Ᵽ_{ab}}$ one whose Lanczos tensor need not be, with $Ł_{abc} = Ᵽ_{c[a;b]}$ in both cases.

II.1.6 Trace-free versus general Lanczos and its superpotential

L_{abc} is a trace-free Lanczos tensor (meaning $L^a{}_{ba}=0$), while the "slashed L" glyph $Ł_{abc}$ is used if we permit nonzero trace. P_{ab} denote a superpotential whose Lanczos is trace-free, while the "P with a curl" glyph $Ᵽ_{ab}$ denotes one whose Lanczos need not be tracefree, with $Ł_{abc}=Ᵽ_{c[a;b]}$ and brackets including the ½.

II.1.7 The Kulkarni-Nomizu product

For symmetric 2-tensors h_{ab}, k_{ab}:

$$(h \odot k)_{abcd} = h_{ac}k_{bd} + h_{bd}k_{ac} - h_{ad}k_{bc} - h_{bc}k_{ad}.$$

It is symmetric in h ↔ k and its output carries the full Riemann symmetries — both verified to machine zero. It shortens several formulas below; in particular Weyl is

$$\mathbf{W_{abcd} = R_{abcd} - (g \odot Q)_{abcd}/2}, \quad \text{where} \quad Q_{ab} = R_{ab} - R\, g_{ab}/6 \quad \text{is the Schouten tensor.}$$

II.2 The algebra of the two duals

relation	value	consequence
$\overleftarrow{\mathbf{i}}^2, \vec{\mathbf{i}}^2$	-1	two genuine imaginary units
$\overleftrightarrow{\mathbf{ii}} - \overrightarrow{\mathbf{ii}}$	0	**they commute** — disjoint index pairs
$\overleftrightarrow{\mathbf{ii}}^2$	+1	hyperbolic; forced by commutativity

Our algebra, the "bicomplex numbers," is *not* the quaternions. Quaternions involve three anticommuting √-1 symbols, but ours all commute. The bicomplex numbers are $\mathbb{R}[\overleftarrow{\mathbf{i}}, \vec{\mathbf{i}}] \cong \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}\oplus\mathbb{C}$. They are generated by adjoining *two* commuting square roots of −1 to the reals $\mathbb{R}$. Result: the "complexified complex numbers" aka bicomplex numbers. Not a field, hence it splits, and that splitting is important.

Group structure, and what it is good for. The units generated by the two duals are $\pm1, \pm\overleftarrow{\mathbf{i}}, \pm\vec{\mathbf{i}}, \pm\overleftrightarrow{\mathbf{ii}}$ — eight elements, forming $Z_4 \times Z_2$. The Z_4 factor is generated by $\overleftarrow{\mathbf{i}}$ (order 4, since $\overleftarrow{\mathbf{i}}^2 = -1$); the Z_2 factor is generated by $\overleftrightarrow{\mathbf{ii}}$, which has order 2 because $\overleftrightarrow{\mathbf{ii}}^2 = +1$. The second dual is recovered as $\vec{\mathbf{i}} = -\overleftarrow{\mathbf{i}}\overleftrightarrow{\mathbf{ii}}$, so nothing is lost in this description.

The useful part is what the Z_2 generator does. Because $\overleftrightarrow{\mathbf{ii}}^2 = +1$, it has eigenvalues ±1, and its eigenspaces are exactly the two halves of the splitting of §II.3:

$\overleftrightarrow{\mathbf{ii}}$ eigenvalue	equivalent statement	what lives there

−1	$\overleftarrow{i} = \vec{i}$ — the two duals **agree**	Weyl tensor and the scalar curvature
+1	$\overleftarrow{i} = -\vec{i}$ — the two duals **differ by a sign**	**traceless** Ricci tensor

So the Z_2 is a two-valued test: apply both duals to a piece of the Riemann tensor and ask whether they give the same answer or opposite answers. Agreement puts it in the Weyl-and-scalar half; disagreement puts it in the traceless-Ricci half. The subgroup on which the two arrows point the same way — the −1 eigenspace — is the single-i tower of the earlier sections, which is why that tower sees Weyl but never the traceless Ricci tensor.

Note. It is the **traceless** Ricci tensor S_{ab} that occupies the +1 eigenspace, not the full Ricci tensor: the scalar curvature sits in the other half alongside Weyl (identities B4 and B5). An earlier draft said "the Ricci content", which is wrong on that point.

II.3 The splitting, and what lives in each half

Because $\overleftrightarrow{ii}^2 = +1$, the operators $(1 \mp \overleftrightarrow{ii})/2$ are orthogonal idempotents. Applied to Riemann they reproduce the Ricci decomposition exactly, albeit bracketed in a way that is *not* the textbook grouping:

$$\mathbf{(1 - \overleftrightarrow{ii})R_{abcd} = 2W_{abcd} + (R/12)\,(g \owedge g)_{abcd}}$$
$$\mathbf{(1 + \overleftrightarrow{ii})R_{abcd} = (g \owedge S)_{abcd}}, \quad S_{ab} = R_{ab} - Rg_{ab}/4$$

i.e. the first half is Weyl plus the scalar part, the second is the traceless-Ricci part. Both verified to 1e−16.

Weyl and the scalar part share an eigenspace. That is why the tempting guess "$(1 - \overleftrightarrow{ii})R_{abcd} = 2W_{abcd}$" is false: it is off by exactly $(R/24)\,(g \odot g)_{abcd}$, invisible in any metric with R = 0. Equivalently

$$W_{abcd} = [R_{abcd} - (*R*)_{abcd}]/3 - (R/24)(g \odot g)_{abcd}.$$

summand	contains	the two units	empty when
$(1 - \overleftrightarrow{ii})/2$	$W_{abcd} + (R/24)\,(g \odot g)_{abcd}$	$\overleftarrow{i} = \vec{i}$	conformally flat with R = 0
$(1 + \overleftrightarrow{ii})/2$	$(g \odot S)_{abcd}/2$	$\overleftarrow{i} = -\vec{i}$	Einstein

On the first summand the two units are identified, the algebra collapses to a single $\mathbb{C}$, and one recovers the ordinary one-i complexification. **This is why the Weyl tensor never needed anything exotic and why $*W_{abcd} = W^*_{abcd}$.** Riemann is the only rung that sees both summands, which is why it alone looked ambiguous.

II.4.1 Which duals are Riemann-type tensors

An exhaustive search over all 24 index permutations of each of the four objects settles this. The rank of a single object's 24 permutations is 2 exactly when all the classical symmetries hold (the permutations collapse to ±T); higher rank means they fail.

object	rank of its 24 permutations	verdict
R	2 always	Riemann-type
(*R*)	2 always	**Riemann-type**
*R	2 if Einstein, else 5–6	keeps both antisymmetries; **loses** pair symmetry and first Bianchi
R*	2 if Einstein, else 5–6	same

So the four are not on an equal footing: $\{R_{abcd}, (*R*)_{abcd}\}$ are Riemann-type, $\{(*R)_{abcd}, (R*)_{abcd}\}$ are not. This is why the idempotent split works — the projectors are built from R and (*R*) only, so both halves land back in the Riemann-symmetric class. Answering the natural question directly: **$R_{(ab)cd} = 0$ and the first Bianchi identity bicomplexify for the 1 and $\overleftrightarrow{ii}$ components and fail for the $\overleftarrow{i}$ and $\vec{i}$ ones.**

Rank diagnostic. The rank of $\{R_{abcd}, (*R)_{abcd}, (R*)_{abcd}, (*R*)_{abcd}\}$ together is 4 when both summands are occupied and 2 when one is empty — 2 for Schwarzschild, Kottler, Kerr, Kasner, FRW; 4 for the rest. Kerr sits at rank 2 despite rotating: it is *not* rotation that populates the second summand, it is non-Einstein matter content.

II.4 The sandwich construction

$$\underset{\circ\circ}{R}_{abcd} = (1 + \overleftarrow{i}*)\, R_{abcd}\, (1 + \overrightarrow{i}*) = R_{abcd} + \overleftarrow{i}(*R)_{abcd} + \overrightarrow{i}(R*)_{abcd} + \overleftrightarrow{ii}(*R*)_{abcd}$$

verified to 0.00e+00. Each factor sits on the side of the index pair it acts on, so the stars land where §II.1.1 puts them and the double dual appears of its own accord. Writing the factors on opposite sides also makes commutativity manifest rather than asserted: nothing has to pass through anything.

One factor per independent antisymmetric index pair — the whole tower in one line:

object	independent pairs	construction
$\mathcal{P}_{ab}$	0	no factor — hence the P* problem, §II.12
$Ł_{abc}$, J_{abc}	1 (ab)	$(1 + i*)Ł$
W_{abcd}	2, but locked ($*W_{abcd} = W*_{abcd}$)	$W(1 + i*)$; the second factor is redundant
R_{abcd}	2, independent	$(1 + \overleftarrow{i}*)\, R\, (1 + \overrightarrow{i}*)$

Each factor is twice a projector, since $(i*)^2 = (-1)(-1) = +1$ makes $(1 + i*)/2$ idempotent. The one-i tower is not a separate construction: it is the one-factor case. The count of factors, the count of independent duals, and the summand structure of §II.3 are three views of one fact.

What the bicomplexified Riemann is and is not. It is linear and it keeps everything — numerically the traceless-Ricci part survives at full strength, so this is the whole Riemann tensor repackaged, not a projection onto the Weyl sector. But it is **not** a joint eigentensor of the two dual operators; a plausible formal argument to that effect is false and the numbers refute it decisively. The reason is the splitting itself: on one summand $\overleftarrow{i}$ acts as $\overrightarrow{i}$, on the other as $-\overrightarrow{i}$, so no single algebra element represents the operator on both at once. In the one-i case W lives entirely in the first summand, which is exactly why $W(1 + i*)$ *is* an eigentensor there.

II.5 Definitions: the complexified tensors

II.5.1 Dual-type (one antisymmetric pair, one imaginary unit)

object	definition	pair	duality eigen-relation
Lanczos	$Ł_{\circ abc} = Ł_{abc} + iŁ*_{abc}$	ab	$(Ł_{\circ})* = -iŁ_{\circ}$
Weyl	$W_{\circ abcd} = W_{abcd} + iW*_{abcd}$	ab or cd (equal)	$(W_{\circ})* = -iW_{\circ}$
Cotton	$J_{\circ abc} = J_{abc} + iJ*_{abc}$	ab	$(J_{\circ})*_{abc} = -iJ_{\circ}$

The *c*ircle below $W_{\circ abcd}$ denotes the *complexified* as opposed to real (no circle) tensor version. For *bi*complexified tensors, we put *two* circles below: $\underset{\circ\circ}{R}$.

II.5.3 Index symmetries, and why the duals trade them

The Lanczos tensor L_{abc} and the Cotton tensor J_{abc} have the same symmetry type: antisymmetric in the first pair, vanishing cyclic part, and trace-free. Dualizing on the antisymmetric pair **exchanges two of these conditions**:

property of L_{abc}	corresponding property of L^*_{abc}
$L_{abc} = -L_{bac}$	$L^*_{abc} = -L^*_{bac}$ (preserved)
$L_{abc} + L_{bca} + L_{cab} = 0$ (cyclic)	$g^{ac}L^*_{abc} = 0$ (trace-free)
$g^{ac}L_{abc} = 0$ (trace-free)	$L^*_{abc} + L^*_{bca} + L^*_{cab} = 0$ (cyclic)

Verified numerically: an L_{abc} that is antisymmetric in ab with its cyclic part projected out gives $|g^{ac}L^*_{abc}| = |g^{bc}L^*_{abc}| = 0$ exactly, while a generic antisymmetric L_{abc} with |cyclic| of order 2 gives dual traces of order 1. Both traces of L* are equal, so there is only one trace condition to impose.

This is why the complexified objects are tidier than their real parts. In $\mathring{L}_{abc} = L_{abc} + iL^*_{abc}$ the cyclic condition and the trace condition are no longer two separate requirements: each is the other's image under the dual, so imposing one on the complexified object imposes both. The same holds for $\mathring{J}_{abc}$.

Status. The forward implication (cyclic $L_{abc} \Rightarrow$ trace-free L^*_{abc}) is verified; the converse was not tested cleanly and is stated here as the expected symmetric statement rather than a checked one.

II.5.2 Trace-type (no antisymmetric pair, no epsilon)

Symmetric 2-index objects have no pair and hence no dual. Their imaginary partner comes from the *other* trace of Riemann: tracing R gives Ricci, tracing (*R*) gives the Einstein tensor. So

$$\mathbf{\mathring{R}_{ab} = R_{ab} + iG_{ab}}, \quad g^{ab}\mathring{R}_{ab} = (1 - i)R.$$

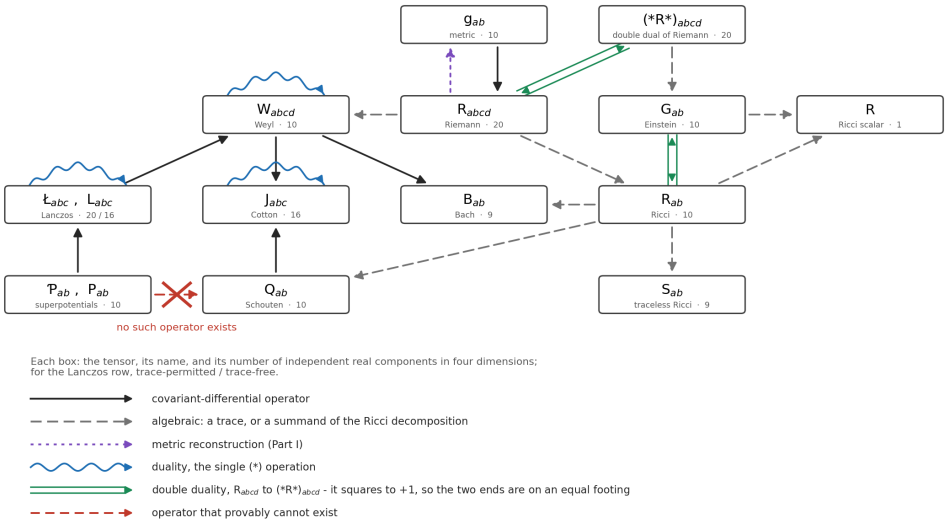
Schouten follows by the same linear formula: $\mathring{Q}_{ab} = \mathring{R}_{ab} - \mathring{R}\, g_{ab}/6$.

> **A warning, and its resolution.** The Cotton tensor can be built either from Schouten, $C_{abc} = Q_{a[b;c]} = (\nabla_c Q_{ab} - \nabla_b Q_{ac})/2$, or as the divergence of Weyl, $J_{abc} = \nabla^d W_{abcd}$. It is tempting to complexify the first route by feeding it the trace-type $\mathring{Q}_{ab}$ above, and the second by the dual type $\mathring{J}_{abc} = J_{abc} + i(J^*)_{abc}$, and to conclude that the two disagree — their imaginary parts do not even share a nonzero index slot on Reissner-Nordström. **An earlier revision of this paper recorded that disagreement as an open problem, the "Cotton clash". It was an artifact of the first construction, not a feature of the geometry, and it is resolved in §II.8.1a.** Feeding the trace-type complexification into a curl does produce a well-defined object, but that object is *not* the imaginary part of the Cotton rung.

II.5.4 Bach

$B_{ab} = \nabla^c\nabla^d W_{acbd} + R^{cd}W_{acbd}/2$ is built from Weyl (unambiguous dual) and Ricci (no dual), so the Riemann ambiguity never reaches it. Substituting $W_{abcd} \to \mathring{W}_{abcd}$ and leaving R^{cd} real defines a complex $\mathring{B}$ whose real part is the Bach tensor. **Untested** — §II.12.

II.6 Cast of characters — the tower



The quadrilateral does not close. It is natural to want a fifth arrow $\mathcal{P}_{ab} \to Q_{ab}$, and there is none because no such operator can exist. The Ricci decomposition is a direct sum, and the superpotential rung lives wholly in the Weyl summand: it is a potential for W_{abcd} and carries no information about the trace sector. Two conformally flat metrics settle it. Both Minkowski space and a general conformally flat metric $g_{ab} = \Omega^2 \eta_{ab}$ have $W_{abcd} = 0$ (verified to 3.3×10^{-16}), so $Ł_{abc} = 0$ is a valid Lanczos tensor for each and $\mathcal{P}_{ab} = 0$ a valid superpotential for each; yet $Q_{ab} = 0$ for the first and $Q_{ab} \neq 0$ for the second (max $|Q_{ab}| = 3.456$ for $\Omega = 1 + 0.3\, x^a x_a$). The same $\mathcal{P}_{ab}$, different Q_{ab}: no function of $\mathcal{P}_{ab}$ alone can produce Q_{ab}. If the metric is supplied as well then the arrow exists but is vacuous, since Q_{ab} is then already determined without reference to $\mathcal{P}_{ab}$.

This is not a defect of the tower; it is the statement that the Weyl and Ricci sectors are independent, which is exactly why the foundations of I.2 pair a Lanczos-type object *with* R_{ab} rather than using it alone.

#	(bi)complexified tensor	status
1	superpotentials $\mathcal{P}_{(ab)}$ and $P_{(ab)}$	real part $P_{(ab)}$ apparently generically exists for suitable traceless Lanczos $L_{[ab]c}$; but complexification apparently generically does not exist since a "*-superpotential" for $L^*_{[ab]c}$ does not
2	Lanczos tensors $Ł_{[ab]c}$ and $L_{[ab]c}$	exist in 4D
3	Weyl tensor W_{abcd}	defined in terms of metric
4	Cotton tensor $J_{[ab]c}$	defined in terms of metric
5	Bach tensor $B_{(ab)}$	defined in terms of metric
6	Riemann curvature tensor $R_{[ab][cd]}$	defined in terms of metric
7	Ricci curvature tensor $R_{(ab)}$	defined in terms of metric
8	Einstein curvature tensor $G_{(ab)}$	defined in terms of metric
9	Schouten curvature	defined in terms of metric

	tensor $\overset{\circ}{Q}_{(ab)}$	
10	Ricci curvature scalar R	real only (pointless to complexify, since its imaginary part would just be negated real part)
11	traceless Ricci tensor $\overset{\circ}{S}_{(ab)}$	defined in terms of metric
12	metric tensor $\overset{\infty}{g}_{(ab)}$	defined from Riemann by metric-reconstruction

II.7 Identities — algebraic

II.7.1 Two identities in the complexified Weyl tensor alone

The identities in II.7.2 onward are, for the most part, pairs of real statements written as one complex statement. The two here are different in kind: they have **no real analogue of the same shape**, because they exploit the fact that $\overset{\circ}{W}_{abcd}$ is equivalent to a single trace-free complex 3×3 matrix M_{ij}, defined just below, and the real Weyl tensor is not.

The matrix M, defined explicitly. Choose an orthonormal frame $\{e_0{}^a, e_1{}^a, e_2{}^a, e_3{}^a\}$ with $e_0{}^a$ timelike. For i, j running over 1, 2, 3 set

$$E_{ij} = W_{abcd}\, e_0{}^a\, e_i{}^b\, e_0{}^c\, e_j{}^d, \quad B_{ij} = (*W)_{abcd}\, e_0{}^a\, e_i{}^b\, e_0{}^c\, e_j{}^d, \quad \mathbf{M_{ij} = E_{ij} + i\, B_{ij}}$$

— the electric and magnetic parts of the Weyl tensor relative to $e_0{}^a$, packaged by the same imaginary unit that defines $\overset{\circ}{W}_{abcd}$. Equivalently $M_{ij} = \overset{\circ}{W}_{abcd}\, e_0{}^a\, e_i{}^b\, e_0{}^c\, e_j{}^d$. All nine entries are determined by the ten independent components of W_{abcd} together with the frame; M_{ij} is complex symmetric and trace-free, so it carries five complex = ten real components, exactly matching.

M_{ij} is **not a tensor** — it is the frame-component array of one, and it depends on the choice of $e_0{}^a$. The distinction matters because the identities below are matrix identities: "M^3" means the matrix cube $M_{ik}M_{k\ell}M_{\ell j}$ and tr M^2 means $M_{ij}M_{ji}$, neither of which is a tensor contraction of $\overset{\circ}{W}_{abcd}$ with itself. That is precisely why these identities have no index-carrying analogue.

Cayley-Hamilton. A trace-free complex 3×3 matrix satisfies its own characteristic equation, so

$$\mathbf{M^3 = \tfrac{1}{2}(\mathrm{tr}\, M^2)\cdot M + \tfrac{1}{3}(\mathrm{tr}\, M^3)\cdot 1}$$

a cubic identity in the complexified Weyl tensor with nothing else in it. Verified to ≤ 7e−16 relative on every metric of II.9, and to 3e−33 on the pp-waves. In tensor language this is the dimensionally-dependent identity for the Weyl tensor in four dimensions; in $\overset{\circ}{W}_{abcd}$ it is one line.

Algebraic speciality is a single complex equation. Writing I = tr M^2 and J = tr M^3 for the two invariants of II.10,

$$\mathbf{I^3 = 6J^2 \iff \text{the Weyl tensor is algebraically special}}$$

The Goldberg-Sachs theorem asserts that a "null congruence" exists if and only if a metric is algebraically special.

Petrov type	metrics	$\lvert I^3 - 6J^2 \rvert$
D, N, III, O	Schwarzschild, Kottler, RN, Kerr, Kerr-Newman, Gödel, Melvin, Barriola-Vilenkin, Kasner, Vaidya, Taub-NUT, C-metric, Nariai, wormhole, both pp-waves, Kundt, FRW, Bertotti-Robinson	≤ 5e−17
I	Majumdar-Papapetrou	**1.1e−08**
I	Van Stockum dust	**2.1e−08**

Nine orders of magnitude separate the special from the general case. This is worth contrasting with the Bel criteria of II.11.2: those are a *search* — one must hunt for a null vector k and then test conditions on it — whereas $I^3 = 6J^2$ is a *computation* on $\overset{\circ}{W}$ with no vectors involved.

II.7.2 Quadratic invariants collapse: eighteen to two

With $Q = L_{abc}L^{abc}$ and $P = L_{abc}L^{*abc}$, every one of the eighteen invariants obtained by permuting the second factor's indices across the three families is one of $\pm Q, \pm Q/2, \pm P, \pm P/2$:

family	abc	bac	acb	cba	bca	cab
L·L	Q	–Q	Q/2	Q/2	–Q/2	–Q/2
L·L*	P	–P	P/2	P/2	–P/2	–P/2
L*·L*	–Q	Q	–Q/2	–Q/2	Q/2	Q/2

Sources: antisymmetry in [ab] kills half of each row; the cyclic identity forces the acb and cba entries to Q/2; and $\varepsilon_{abde}\varepsilon^{ab}{}_{fg} = -2(\delta^d{}_f\delta^e{}_g - \delta^d{}_g\delta^e{}_f)$ forces L*·L* = –L·L, eliminating the third family. Verified to fifteen decimals on random admissible tensors.

The same table holds verbatim for the Cotton tensor. J has the identical algebraic symmetry type as L — one antisymmetric pair, vanishing cyclic part, trace-free on both contractions (§II.8.1) — so this is a property of the symmetry type, not of the Lanczos tensor specifically. Replace L by J throughout.

Consequences. (a) $\mathring{L}\cdot\mathring{L} = 2(Q + iP)$: one complex evaluation delivers both. (b) **Dualization cannot manufacture a divergence**, since |L*·L*| = |L·L| pointwise. (c) P is a pseudo-scalar, hence identically zero wherever the spacetime has a $t \to -t$ symmetry; in a static spacetime the imaginary half is empty.

II.7.3 Self-dual eigentensors

$$L^{\pm} = (L \mp iL^*)/2, \quad (L^+)^* = iL^+, \quad L^+\cdot L^+ = (Q - iP)/2, \quad L^+\cdot L^- = 0.$$

The eighteen real invariants are the real and imaginary parts of one complex number Q – iP, and the self-dual and anti-self-dual parts are orthogonal. Note $\mathring{L} = 2L^-$.

II.7.4 Bel-Robinson

$$\mathbf{T_{abcd} = \mathring{W}_{aecf}\,\bar{\mathring{W}}_b{}^e{}_d{}^f}$$

exactly — no factor of ½ and no instruction to take a real part, because the imaginary part of the product vanishes identically (0.00e+00, not merely small). Total symmetry and trace-freeness follow.

Caveat on the literature form. The frequently quoted B = R·R + *R·*R, built from *Riemann* rather than Weyl, agrees with the above **only in vacuum**: identical for Schwarzschild (7e–16) but differing by 8e–2, 1.08 and 2.2e–1 for Reissner-Nordström, Gödel and Majumdar-Papapetrou. Unfortunately both forms appear in print without the qualification.

§II.8.7 expresses this same T_{abcd} directly through the complexified Lanczos tensor, and one rung lower through the superpotential.

II.7.5 L* is an admissible Lanczos tensor

On random samples L* satisfies antisymmetry, the cyclic identity *and* trace-freeness to machine zero. Dualization is therefore an involution up to sign on the admissible class.

More is true, and it is what the complexification programme needs: **L* is a Lanczos tensor for W^*_{abcd}**, generated by the same operator with the same constant as L generates W_{abcd}. That is a differential statement rather than an algebraic one, so it is proved in §II.8.7; the consequence here is that the Lanczos rung descends to the self-dual sector with nothing re-derived.

II.8 Identities — differential

II.8.1 The Cotton rung

Define $J_{abc} = \nabla^d W_{abcd}$. Then J_{abc} is antisymmetric in **ab**, has vanishing cyclic part, and is trace-free on both contractions — the last confirmed *symbolically* on Reissner-Nordström, not merely numerically. This ordering is the right one for the tower: the antisymmetric pair is inherited from Weyl's first pair, so the dual acts on the same slot at the Lanczos and Cotton rungs. The Cotton tensor is *not* totally antisymmetric; in the common ordering $C_{abc} = Q_{a[b;c]} = (\nabla_c Q_{ab} - \nabla_b Q_{ac})/2$ it is antisymmetric in bc only.

II.8.1a The Cotton rung has exactly one imaginary unit

Which imaginary partner does the Cotton tensor take? The question is settled by the divergence of the bicomplexified Riemann tensor of §II.4, whose four slots can simply be computed. On Reissner-Nordström (Cotton ≠ 0, so the test is not vacuous), symbolically and exactly:

slot	object	value	nonzero components
1	$\nabla^d R_{abcd}$	$\mathbf{2\,J_{abc} = 2\,C_{cab}}$	6
$\overleftarrow{\mathrm{i}}$	$\nabla^d (*R)_{abcd}$	$\mathbf{2\,(J^*)_{abc}}$	6
$\vec{\mathrm{i}}$	$\nabla^d (R^*)_{abcd}$	0	**0**
$\vec{\vec{\mathrm{i}}}$	$\nabla^d (*R^*)_{abcd}$	0	**0**

Both identities in the first row were checked **componentwise on all 64 components**, not merely on the slots where both sides happen to be nonzero: $\nabla^d R_{abcd} = 2J_{abc}$ holds exactly, and so does $J_{abc} = C_{cab}$. *The index ordering matters and is easy to get wrong:* $J_{abc} = -C_{abc}$ fails on 6 of the 64 components, as does $J_{abc} = -C_{bca}$; only the orderings C_{cab} and $-C_{cba}$ hold everywhere. A ratio test restricted to commonly-nonzero slots returns ±1 for *all six* orderings and is therefore worthless here — different orderings sample disjoint sets of components.

Two slots vanish identically, and that is the content of the result. The surviving pair $\{1, \overleftarrow{\mathrm{i}}\}$ is one complex object — real part the Cotton tensor, imaginary part its ε-dual — so

$$\mathbf{\mathring{J}_{abc} = J_{abc} + i(J^*)_{abc}}$$

is the complexified Cotton tensor, and there is only one. Both routes to Cotton reach it, differing only by the constants above. The vanishing of the $\vec{\mathrm{i}}$ and $\vec{\vec{\mathrm{i}}}$ slots is precisely the statement that Cotton carries one antisymmetric pair and therefore one unit — which is what the tower's design (§II.1.4) asserts on general grounds. Here it is confirmed by computation rather than assumed.

Why the earlier construction misled. Complexifying the Schouten *input* by the trace type (§II.5.2) and then taking the curl produces an object whose imaginary part lies in the $(1 + \vec{\vec{\mathrm{i}}})/2$ summand. But the table above shows the Cotton rung's imaginary part lies in the $\overleftarrow{\mathrm{i}}$ slot, which is in the other summand. The two are not rival descriptions of one quantity; the trace-type construction simply does not compute the Cotton rung.

II.8.2 The Bianchi identity complexifies with no correction

$$\mathring{J}_{abc} = \nabla^d \mathring{W}_{abcd}.$$

Proof. ε is covariantly constant, so dualization commutes with ∇ when the derivative index is not one of the dualized indices. The dual acts on ab while the divergence contracts d in the cd pair, so they never meet. Q.E.D. In vacuum this is $\nabla^d \mathring{W}_{abcd} = 0$: one complex equation replacing the pair ($\nabla^d W_{abcd} = 0$, $\nabla^d (W^*)_{abcd} = 0$), the second of which is usually quoted separately as the "dual Bianchi" identity. Here it is merely the imaginary part.

II.8.3 Divergences of the duals

There effectively is only **one** dual tensor here, not two. The pair-interchange symmetry R_{abcd}=R_{cdab} of the Riemann tensor gives

$$\mathbf{(R^*)_{abcd} = (*R)_{cdab}}$$

since $R^*_{abcd} = R_{ab}{}^{pq}\varepsilon_{pqcd}/2 = \varepsilon_{pqcd}R^{pq}{}_{ab}/2 = (*R)_{cdab}$. Verified to 0.00e+00 on Schwarzschild, Reissner-Nordström and Melvin and to ~1e−16 on the rest of the collection. So the right dual is the left dual with its index pairs exchanged.

But *R is not itself pair-symmetric — dualizing one pair destroys that symmetry. Measuring $|{}^*R - ({}^*R)_{swap}|$ relative to $|{}^*R|$:

metric	relative asymmetry of *R
Schwarzschild, Kottler, Kerr (Einstein)	≤ 6e−16 — pair symmetry retained
Reissner-Nordström	0.105
Kerr-Newman	0.0138
Barriola-Vilenkin	0.130
Gödel	1.00
Melvin	1.09

It is retained exactly in the Einstein cases, which is the familiar statement that dualization preserves pair symmetry precisely when the Ricci part is absent.

II.8.3.1 Why the two divergences differ

Both divergences are taken *on the last index*, e.g. $\mathrm{div}(R)=R_{abcd}{}^{;d}$. But because R* is *R with the pairs swapped, taking div(R*) on its last index means contracting ***R on its second index**, whereas div(*R) contracts ***R on its fourth**. These are different slots of the same tensor, and since that tensor is not pair-symmetric they need not agree. The clean statement is:

***R is divergence-free on its second index, and not on its fourth.**

II.8.3.2 Both statements are identities, not merely observations

div(R*) = 0. Writing $R^*_{abcd} = R_{ab}{}^{pq}\varepsilon_{pqcd}/2$, the divergence is $\varepsilon_{pqcd}\nabla^d R_{ab}{}^{pq}/2$. Contracting with ε extracts the totally antisymmetric part in the three differentiated indices, and that is exactly what the **second Bianchi identity** annihilates. So this holds in every spacetime, with no appeal to a sample.

div(*R) = $-\varepsilon_{ab}{}^{pq}\nabla_p R_{qc}$. Here $({}^*R)_{abcd} = \varepsilon_{ab}{}^{pq}R_{pqcd}/2$, so the divergence is $\varepsilon_{ab}{}^{pq}\nabla^d R_{pqcd}/2$, and the *contracted* Bianchi identity gives $\nabla^d R_{pqcd} = \nabla_p R_{qc} - \nabla_q R_{pc}$; the ε folds the two terms together. The proportionality constant was fitted rather than assumed, and came out to exactly −1 — the same single value on Reissner-Nordström, FRW, Gödel and Kasner, with no second value appearing anywhere.

Corollary: div(*R) vanishes in vacuum, since Ricci = 0 kills the right-hand side. Confirmed on two genuine vacua — the pp-wave with $H = x^2 - y^2$ (harmonic profile, hence actually vacuum) and Schwarzschild — both giving zero nonzero components, with both sides vanishing together.

It is retained exactly in the Einstein cases, which is the familiar statement that dualization preserves pair symmetry precisely when the Ricci part is absent.

Two metrics were mislabelled as vacuum in an earlier draft, and any row of §II.9 resting on them needs re-checking. The pp-wave with $H = x^3$ has Laplacian 6x, so it is not vacuum; a genuine vacuum profile is $H = x^2 - y^2$. And the Kasner metric written $\mathrm{diag}(-1, t^{2/3}, t^{2/3}, t^{-1/3})$ encodes exponents (1/3, 1/3, −1/6), which satisfy neither Kasner condition $\Sigma p_i = \Sigma p_i{}^2 = 1$; the vacuum form is $\mathrm{diag}(-1, t^{4/3}, t^{4/3}, t^{-2/3})$.

II.8.4 Second Bianchi and its dual

$\nabla_{[e}R_{ab]cd} = 0$ holds at the finite-difference floor (3e−8), and the same statement for *R holds equally (6e−8) — again because ε is covariantly constant.

II.8.5 Divergences at the trace rung

$$\nabla^a G_{ab} = 0, \quad \nabla^a R_{ab} = \nabla_b R/2 \;\Rightarrow\; \mathbf{\nabla^a \mathring{R}_{ab} = \nabla_b R/2 \text{ is purely real.}}$$

The imaginary half of the complexified Ricci tensor is divergence-free by construction — the contracted Bianchi identity restated as a reality condition, and arguably the cleanest motivation for the combination $R_{ab} + iG_{ab}$.

II.8.6 Schouten

$$\mathbf{Q_{ab}{}^{;b} = Q_b{}^b{}_{;a}}$$

with no factor of 2. Both sides equal $\nabla_a R/3$, since $\nabla^b Q_{ab} = \nabla_a R/3$ and $Q^b{}_b = R/3$. (In D dimensions both sides are $\nabla_a R(D–2)(2D–2)^{-1}$. The identity itself is normalization-independent - it holds for either scaling of Q_{ab} - so the numerical check below is unaffected by the rescaling.) Confirmed numerically on FRW dust (1.9e–7) where both sides are genuinely nonzero; the factor-2 version fails there by a factor of exactly 2. On RN, Gödel and MP both sides vanish, so those metrics give a vacuous verdict — see §II.9.1.

II.8.7 The projected Lanczos→Weyl equation, and Bel-Robinson through it

The Lanczos→Weyl equation of §II.1.5 carries four derivative terms and four trace terms. Almost all of that is bookkeeping: the trace terms exist only to force the right side to carry the symmetries a Weyl candidate must have. If that forcing is written as an explicit projector, the equation collapses to a single antisymmetrized derivative.

The projector. For any four-index Z_{abcd} put

$$\hat{Z}_{abcd} = (Z_{[ab][cd]} + Z_{[cd][ab]})\, 2^{-1}, \qquad \hat{Z}_{ab} = g^{cd}\hat{Z}_{cadb}, \qquad \hat{Z} = g^{ab}\hat{Z}_{ab}$$

and then, in the Kulkarni-Nomizu notation of §II.1.7,

$$\mathbf{Proj\{Z\}_{abcd} = \hat{Z}_{abcd} - \hat{Z}_{[abcd]} - (g \odot Q)_{abcd}\, 2^{-1}, \qquad Q_{ab} = \hat{Z}_{ab} - \hat{Z}\, g_{ab}\, 6^{-1}}$$

Three terms: symmetrize, drop the totally antisymmetric part, subtract the Schouten tensor exactly as in the Weyl formula of §II.1.7 - and Q_{ab} may be built from $\hat{Z}_{abcd}$ rather than from the first two terms together, because $\hat{Z}_{[abcd]}$ is trace-free. Proj is idempotent and fixes W_{abcd}; its output is antisymmetric in each pair, pair-exchange symmetric, cyclic-free and trace-free, all to machine zero. Against the equivalent three-step verbal construction it agrees on 300 random Z_{abcd} to 1.6×10^{-15}. It is the tensor counterpart of the total symmetrization over unprimed indices that makes the spinor form of this equation a one-liner.

For every input used below the middle term is identically zero (3.2×10^{-17}), since the cyclic identity $L_{[abc]} = 0$ forces it, so in the application Proj is only two terms.

The identity. For a Lanczos tensor L_{abc} of W_{abcd},

$$\mathbf{W_{abcd} = 4\, Proj\{\, L_{ab[c;d]}\,\}}$$

One antisymmetrized covariant derivative in place of eight terms. The operator *commutes with dualization*, so the identical statement holds for the dual and hence for the complexified Lanczos tensor, with no change of constant:

$$\mathbf{W^*_{abcd} = 4\, Proj\{\, L^*_{ab[c;d]}\,\}, \qquad \mathring{W}_{abcd} = 4\, Proj\{\, \mathring{L}_{ab[c;d]}\,\}}$$

with $\mathring{L}_{abc} = L_{abc} + \vec{i}L^*_{abc}$. The middle statement strengthens §II.7.5: L* is not merely *admissible*, it is a Lanczos tensor *for the dual Weyl tensor*, generated by the same operator with the same constant. That is what lets the whole L→W rung descend to the self-dual sector with nothing re-derived - which was the original motivation for the complexification programme, and is here confirmed rather than assumed.

One rung lower. Substituting $L_{abc} = P_{c[a;b]}$ merges the two derivative terms into a single second derivative of the superpotential:

$$\mathbf{W^{ab}{}_{cd} = 4\, Proj\{\, P^{[a}{}_{[c}{}^{;b]}{}_{;d]}\,\}}$$

the two brackets being stacked rather than nested, in the manner of the Holgersson form quoted in the Companion Paper. Notably this survives the non-commutativity of covariant derivatives: the commutator terms generated in curved space land inside the kernel of Proj

rather than spoiling the identity.

Verification. Two independent settings. In weak field the arithmetic is exact - vacuum plane waves give analytic derivatives, with no finite differences anywhere - while on Schwarzschild the Lanczos tensor comes from the sign-flip construction, so the two tests share neither the metric nor the method of obtaining L_{abc}. In every row the fitted constant came out to 1.000000000000, i.e. exactly the 4 shown above.

identity	residual, weak field (exact)	residual, Schwarzschild
$W_{abcd} = 4\ \mathrm{Proj}\{\ L_{ab[c;d]}\ \}$	3.8×10^{-16}	5.2×10^{-9}
$W^*_{abcd} = 4\ \mathrm{Proj}\{\ L^*_{ab[c;d]}\ \}$	1.9×10^{-16}	2.6×10^{-9}
$W_{\circ abcd} = 4\ \mathrm{Proj}\{\ L_{\circ ab[c;d]}\ \}$	3.1×10^{-16}	5.2×10^{-9}
$W^{ab}{}_{cd} = 4\ \mathrm{Proj}\{\ P^{[a}{}_{[c}{}^{;b]}{}_{;d]}\ \}$	3.8×10^{-16}	3.3×10^{-8}

The Schwarzschild figures sit at that metric's finite-difference noise floor. The weak-field checks were run both on vacuum plane waves, where $|W^*_{abcd}| = |W_{abcd}|$ exactly so the dual is not degenerate, and on a static purely-electric field reproducing Schwarzschild's Petrov character; the constant is the same in both. The superpotential row was verified on Schwarzschild using the raw sign-flip superpotential with no trace projection applied to it, since Proj removes traces in any case.

Bel-Robinson. Feeding this into §II.7.4 gives the tensor directly from the complexified Lanczos tensor,

$$\mathbf{T_{abcd} = 16\ Proj\{\ L_{\circ ae[c;f]}\ \}\ \ Proj\{\ \bar{L}_{\circ bg[d;h]}\ \}\ \ g^{eg}g^{fh}}$$

and, one rung lower, as a product of two second derivatives of the superpotential. Built this way T_{abcd} reproduces the direct computation to 5.7×10^{-9}. The same run re-confirms §II.7.4 itself: imaginary part 3.0×10^{-10}, total-symmetry defect 8.4×10^{-9}, trace 6.2×10^{-17}.

Two caveats. First, Proj is not free: expanded, its third term reintroduces trace terms of its own. The gain is not in raw operation count but in packaging - one standard, reusable, named projection, already needed elsewhere in this Part, in place of four trace terms peculiar to one equation. That is exactly what the spinor symmetrization buys, and the point of writing it in tensor form is that the complexified apparatus here can use it without leaving tensor language. Second, Proj cannot be pushed through the Bel-Robinson contraction: forming T_{abcd} from the unprojected derivative expression is wrong by 37%, so Proj must act on each factor before the product is taken.

Health warning. These identities were obtained numerically, and two earlier versions of the dual computation behind them were wrong - once by dualizing partial rather than covariant derivatives ($\varepsilon_{abcd} = \sqrt{(-g)}[abcd]$ is covariantly constant but is not a constant coordinate array), and once through a repeated summation index in the curved-space dual. Both errors produced plausible-looking wrong constants, and the second briefly appeared to refute the dual identity above, before being caught. The identities as stated survive exact weak-field arithmetic on a generic Weyl tensor and agree with the curved case to the noise floor, but they deserve an independent reimplementation before anyone leans on them.

II.9 Numerical test-error survey

Thirteen metrics: Schwarzschild, Kottler, Reissner-Nordström, Kerr, Kerr-Newman, Gödel, Melvin, Barriola-Vilenkin, pp-wave ($H = x^3$), Kasner, Majumdar-Papapetrou (two centers), Vaidya, FRW radiation. Coverage: Petrov D, I, N, O; vacuum, Einstein-with-Λ, Einstein-Maxwell, null dust, magnetic and conformally flat; static, stationary-rotating and dynamical. Each entry is the worst relative error over metrics where the quantity has nontrivial content.

On the status of this evidence. The verifications of this part are numerical, not proofs. Although our “experimental” approach here could be attacked by mathematical purists, we heard claims that physicists like experimental proofs. The reader should nevertheless keep §II.9.1 in view: numerical agreement is evidence, and this paper contains several identities that agreed numerically and turned out to be false, and at least one that appeared to fail numerically and turned out to be true. Where an identity has since been established symbolically the catalogue of Part IV marks it **sym**; those still marked **num** rest on this experimental footing alone.

identity	worst error	attained at
$\overleftarrow{i}^2 = \vec{i}^2 = -1$	6.2e−16	Barriola-Vilenkin

$[\overleftarrow{i}, \vec{i}] = 0$	1.6e−16	Gödel
$\overleftrightarrow{ii}^2 = +1$	1.2e−15	Vaidya
$W_{abcd} = R_{abcd} - (g \odot Q)_{abcd}$	2.4e−16	Kottler
$(1 - \overleftrightarrow{ii})R_{abcd}/2 = W_{abcd} + (R/24)(g \odot g)_{abcd}$	3.6e−16	Kottler
$(1 + \overleftrightarrow{ii})R_{abcd} = (g \odot S)_{abcd}$	1..0e−15	Melvin
on the first summand: $\overleftarrow{i} = \vec{i}$	2.6e−16	Melvin
on the second: $\overleftarrow{i} = -\vec{i}$	3.3e−16	FRW
tr(*R*) = Einstein	2.0e−14	Kerr-Newman
$\mathrm{tr}(R_{abcd}) = R_{ab}$	0	—
$^*W_{abcd} = W^*_{abcd}$	4.0e−16	Gödel
(*W*) = −W	6.6e−16	Vaidya
$(\underset{\circ}{W})^* = -i\underset{\circ}{W}$	6.6e−16	Vaidya
$\underset{\circ}{W}$ traceless; $\underset{\circ}{W}$ first Bianchi	8.0e−16	Gödel
$T = \underset{\circ}{W}\,\bar{\underset{\circ}{W}}$	2.1e−16	Majumdar-Papapetrou
$\mathrm{Im}[\underset{\circ}{W}\,\bar{\underset{\circ}{W}}] = 0$	1.2e−16	Gödel
T totally symmetric; T traceless	9.5e−16	Melvin
sandwich = $R + \overleftarrow{i}(^*R) + \vec{i}(R^*) + \overleftrightarrow{ii}(^*R^*)$	0	—
KN symmetric; KN has Riemann symmetries	2.4e−16	Gödel
div R* = 0; div (*R*) = 0	2.2e−08	Majumdar-Papapetrou
J antisym ab; J cyclic = 0	7.7e−11	Kerr-Newman
$g^{ab}J_{abc} = 0$	4.2e−12	Barriola-Vilenkin
$g^{bc}J_{abc} = 0$	exact (symbolic)	Reissner-Nordström
$(\underset{\circ}{J})^* = -i\underset{\circ}{J}$	3.8e−11	Kerr-Newman
2nd Bianchi; 2nd Bianchi for *R	6e−08	Reissner-Nordström
$\nabla^a G_{ab} = 0$	1.3e−07	Reissner-Nordström
$\nabla^a \underset{\circ}{R}_{ab}$ purely real	1.3e−07	Reissner-Nordström
$Q_{ab}{}^{;b} = Q_b{}^{b}{}_{;a}$	1.9e−07	FRW dust

II.9.1 Two ways the numerics can lie

> **Small is not zero.** Rows at 1e−16 are pure algebra in double precision and are safe. Rows at 1e−5 to 1e−11 involve a numerical derivative of the curvature and sit at the finite-difference floor — and the floor hides things. A claimed identity $J_{abc} = C_{bca}$ passed numerically at 4.9e−5 and was **false**: symbolic evaluation on Reissner-Nordström gives a residual $(-r^2/2 + r - 1/8)r^{-7}$, a function that

> merely happens to be small at the sample radius. Anything reported at 1e−5 must be confirmed symbolically, and one evaluation point cannot distinguish "zero" from "small here".
>
> **Zero is not confirmation.** Testing X = Y on a metric where both vanish proves nothing. The Schouten identity of §II.8.6 appeared to fail on three metrics because both sides were ≈ 1e−13 there; only FRW dust, where both sides equal 0.0905, gave a real verdict. Every relative-error entry needs a magnitude check on the quantity being tested.

Method. Riemann is built from second derivatives of the metric only; those are differentiated symbolically and everything downstream is numerical linear algebra. Derivatives of curvature use central differences with $h=3\times10^{-4}$. This replaced a fully symbolic pipeline that could not complete Kerr; the present one does all thirteen metrics in under a second.

II.10 Invariants: the harvest

Write B for the sandwich object of §II.4. Contracting it with itself and with its variants gives four real scalars at a time, one per algebra component.

II.10.1 The quadratic harvest is complete at three scalars

component of B·B	identification
1	4 × Kretschmann (ratio 4.000000 on every metric)
$\overleftarrow{\mathrm{i}}$	4 × Chern-Pontryagin
$\vec{\mathrm{i}}$	4 × Chern-Pontryagin — **identical to the $\overleftarrow{\mathrm{i}}$ slot**
$\vec{\vec{\mathrm{i}}}$	−4 × Kretschmann *in Λ-vacuum*; independent otherwise

The two arrow slots coinciding is the split again: Pontryagin lives on the diagonal $\overleftarrow{\mathrm{i}} = \vec{\mathrm{i}}$. The $\vec{\vec{\mathrm{i}}}$ slot is the genuinely new one — it collapses to $-4R_{abcd}R^{abcd}$ in Λ-vacuum (Kerr: −4.000000) and detaches off Λ-vacuum (Kerr-Newman −3.9996, Majumdar-Papapetrou −3.73), so its deviation from $-4R_{abcd}R^{abcd}$ is a pure measure of traceless-Ricci content.

The other two quadratic contractions add nothing. $B\cdot\bar{B}$ vanishes identically (3e−16, all four components, all six metrics) — the self-dual and anti-self-dual parts are orthogonal, exactly as $L^+\cdot L^- = 0$ one rung down. And $B\cdot(*B*)$ is $B\cdot B$ with its 1 and $\vec{\vec{\mathrm{i}}}$ slots swapped and negated and its arrows negated, since multiplying by $\vec{\vec{\mathrm{i}}}$ is an algebra rotation. So the quadratic harvest is **three independent scalars**, not sixteen.

II.10.2 Cubics reach the rest of the Weyl sector

The cubic contraction of the complexified Weyl, $W_{ae}{}^{cf}W_{cg}{}^{eh}W_{ag}{}^{fh}$, delivers a single complex number whose real and imaginary parts are the cubic Weyl invariants (W2R and W2I in the Carminati-McLenaghan labelling) — on Kerr, −58.91 and −35.26. Together with $W_{abcd}W^{abcd}$ giving W1R and W1I, **two expressions deliver the entire Weyl sector of the CM set**. The bicomplex cubic $B\cdot B\cdot B$ has the same structural shape: arrow slots identical, 1 and $\vec{\vec{\mathrm{i}}}$ equal and opposite in Λ-vacuum, detaching off Λ-vacuum.

> **Provisional.** The cubic *values* are trustworthy but their *identification* is not: the least-squares fits used a basis with only one cubic element, so the reported coefficients (numbers like −191324 with residual 1.4e+02) are fitting artefacts, not identifications. A proper degree-3 basis (S^3, $W\cdot S\cdot S$, $K\cdot R$, …) is needed.

II.10.3 The mixed sector fires

Contracting the complexified Weyl against the traceless Ricci tensor reaches the invariants that single-object contractions cannot. Writing S^{ab} for the raised traceless Ricci tensor, the mixed scalars

$$K = W_{abcd}S^{ac}S^{bd}, \quad L = W_a{}^e{}_c{}^f S^{ce}S^{af}, \quad M = W_{abcd}S^{ce}S^{df}S^{ae}S^{bf}$$

are complex and nonzero exactly where expected. On Kerr-Newman, K = 1.019e−6 + 1.862e−7 i, L = 4.837e−7 + 8.839e−8 i, M = −1.099e−13 − 2.008e−14 i; on Gödel, K vanishes but L = 0.2957 and M = 0.003875 are real and healthy. They vanish identically in vacuum (Schwarzschild: 1e−34 and below), as they must, since S = 0 there.

Together with the Ricci-sector scalars $I_6 = S_{ab}S^{ab}$, $I_7 = S_{ab}S^{bc}S_c{}^a$ and its quartic analogue I_8, and the Weyl-sector I and J of §II.10.1–9.2, the collected values over six spacetimes form a matrix of rank 6 — i.e. the invariants computed are independent over the sample and none is a fixed multiple of another.

II.10.4 Relation to the Carminati-McLenaghan and Zakhary-McIntosh sets

There are at most fourteen independent real algebraic invariants of the Riemann tensor in four-dimensional Lorentzian space. Carminati and McLenaghan (1991) gave a set of sixteen designed to cover degenerate cases — Einstein-Maxwell and perfect fluid — where earlier sets fail; Zakhary and McIntosh (1997) augmented CM by a single further complex invariant, m_6, producing the first set complete for all metric types, with as many as seventeen required in the most degenerate cases and eight syzygies relating them.

The CM/ZM classification maps onto the summand structure of §II.3 directly. Their invariants are grouped as: R the curvature scalar; two complex invariants I and J built from the Weyl spinor; three real invariants I_6, I_7, I_8 built from the trace-free Ricci spinor; and three complex mixed invariants K, L, M. In the present language:

CM/ZM group	lives in	generated by
R	trace	trivial
I, J (complex)	the $(1 - \vec{i}\vec{i})/2$ summand	$\mathring{W}_{abcd}\mathring{W}^{abcd}$ and tr M^3 — **two expressions, four real invariants**
I_6, I_7, I_8	the $(1 + \vec{i}\vec{i})/2$ summand	traces of powers of S_{ab}
K, L, M (complex)	**both** summands	$\mathring{W}$–S contractions, §II.10.3
m_6 (ZM)	both, quartic	two Weyl spinors against four Ricci spinors

This explains a limitation encountered above. The contractions B·B and B·B·B involve a single object and therefore cannot mix summands; they reach R, the Weyl sector and one contrast scalar, and nothing else. The mixed invariants require the two halves to meet, which is what §II.10.3 arranges. It also sets the degree ceiling: m_6 is quartic, so no quadratic or cubic sweep can be complete, and any claim of completeness must reach degree four.

II.10.5 What remains for a complete harvest

The CM set is 16 real invariants. R is the trace; the Weyl sector (W1R, W1I, W2R, W2I) falls out of the two expressions above. The Ricci sector (R1, R2, R3) and the seven mixed M-invariants require contractions of the complexified Weyl against the separately complexified Ricci $\mathring{R}_{ab}$ — that is, objects from the *other* summand. A plausible generating set is therefore $\{\mathring{W}_{abcd}\mathring{W}^{abcd}, \text{tr } M^3, \mathring{R}_{ab}\mathring{R}^{ab}, \text{tr}(\mathring{R}^3)\}$ plus two or three mixed $\mathring{W}$–$\mathring{R}$ contractions: roughly six expressions for sixteen invariants. Untested.

II.11 Application: Petrov classification

The Bel-Debever criteria determine the Petrov type at a point by searching for null vectors k satisfying conditions stated as *pairs* — one condition on W_{abcd} and a matching one on its dual. Every such pair is the real and imaginary part of a single condition on $\mathring{W}$.

type	usual statement	complexified
N	$W_{abcd}k^d = 0$	$\mathring{W}_{abcd}k^d = 0$
III	$W_{abcd}k^bk^d = 0 = {*W}_{abcd}k^bk^d$	$\mathring{W}_{abcd}k^bk^d = 0$
II	$W_{abcd}k^bk^d = \alpha k_ak_c$ and ${*W}_{abcd}k^bk^d = \beta k_ak_c$	$\mathring{W}_{abcd}k^bk^d = \lambda k_ak_c$, one such k

D	the same for two independent k, k′	the same for **two** such k

So the eight real equations of the II and D criteria are four complex ones, and the whole classification reduces to two questions: how many null directions satisfy $\mathring{W}_{abcd}k^bk^d = \lambda k_a k_c$, and is λ zero?

II.11.1 Petrov type is the Jordan structure of M

Throughout this section M_{ij} is the 3×3 complex matrix defined in §II.7.1: $M_{ij} = \mathring{W}_{abcd}e_0{}^a e_i{}^b e_0{}^c e_j{}^d = E_{ij} + iB_{ij}$ in an orthonormal frame. All statements below are about M_{ij}, not about a contraction of $\mathring{W}_{abcd}$ with itself.

Because Weyl lives entirely in the $(1 - \vec{i}\vec{i})/2$ summand, where the two units are identified, $\mathring{W}_{abcd}$ is equivalent to a single complex symmetric trace-free 3×3 matrix M_{ij} — the familiar E + iB matrix, with $E_{ij} = W(e_0,e_i,e_0,e_j)$ and B_{ij} the same contraction of W*. **The Petrov type is exactly its Jordan structure.**

Jordan structure of $\mathring{W}$	Petrov type
three distinct eigenvalues	I
one repeated eigenvalue, diagonalizable	D
one repeated eigenvalue, one 2×2 block	II
nilpotent, $M^2 \neq 0$	III
nilpotent, $M^2 = 0$, $M \neq 0$	N
$M = 0$	O

Verified on eighteen of the spacetimes of §II.9; all agree with the known type. Sample eigenvalue triples: Schwarzschild (–0.128, 0.064, 0.064) — the (–2λ, λ, λ) pattern characteristic of type D; Kerr (–0.1253 – 0.0228i, and 0.0626 + 0.0114i twice), the same pattern but complex; Majumdar-Papapetrou (–0.0493, 0.0315, 0.0179) and Van Stockum (–0.0703, 0.0398, 0.0306), three distinct — type I; Reissner-Nordström, Kerr-Newman, Gödel, Melvin, Barriola-Vilenkin, Vaidya, Taub-NUT, C-metric, Nariai and the Morris-Thorne wormhole all type D; FRW and Bertotti-Robinson type O.

This unifies three descriptions that are usually given separately: the Bel criteria below, the quartic in the Weyl scalars $\Psi_0 \ldots \Psi_4$, and the E + iB eigenvalue computation. They are one statement about $\mathring{W}$.

Two caveats. The type N and III representatives (pp-waves, Kundt) are **not** among the eighteen: they are written in null coordinates where the orthonormal-frame construction used here fails, so the delicate nilpotent cases remain unverified by this route and would need a null tetrad. And one apparent mismatch in the run was an error in the expected-value list rather than in the method: the Kasner metric used has exponents (2/3, 2/3, –1/3), two of them equal, which is the axisymmetric Kasner and genuinely type D; generic Kasner with distinct exponents is type I.

II.11.2 The condition αβ ≠ 0 is defective

The type II and D criteria are usually accompanied by the side condition $\alpha\beta \neq 0$. **That condition misclassifies Schwarzschild.** Evaluating on the principal null direction:

metric	α	β	λ	αβ ≠ 0?	λ ≠ 0?
Schwarzschild (D)	–0.1280	**0**	–0.1280	**fails**	holds
Reissner-Nordström (D)	–0.1152	**0**	–0.1152	**fails**	holds
Kerr (D)	–0.12526	–0.022766	–0.12526 – 0.022766i	holds	holds
pp-wave (N)	0	0	0	fails	fails
Kundt (III)	0	0	0	fails	fails

All residuals for the proportionality are at 1e−16. The reason is transparent: β vanishes whenever Ψ_2 is real, which it is for any static type-D metric; Kerr passes only because rotation makes Ψ_2 complex. **The correct side condition is $\lambda \neq 0$**, which holds for all three type-D cases and correctly fails for types N and III — that failure being exactly the II-versus-III distinction.

This is an argument for the complexified form beyond concision. Writing α and β separately invites the error, because "both parts nonzero" is not a natural condition on a complex number while "$\lambda \neq 0$" is. The formalism makes the mistake hard to commit.

Caveat. The principal null directions were supplied in closed form rather than found by search, so these calculations confirm the form of the criteria on known k; they do not test the uniqueness-up-to-scaling clause of the Bel statement.

II.12 Open, provisional and untested

1. **Complete the CM harvest — Weyl sector done, mixed sector open.** The Weyl-sector constants are now fitted rather than provisional, across the whole collection: $\mathring{W}\cdot\mathring{W} = 2(W\cdot W) + 2i(W\cdot{}^*W)$, both ratios pinned to 2.000000; and M^3 gives $4[\mathrm{cub}(W) + i\cdot\mathrm{cub}(^*W)]$, both constants exactly 4. Two sanity checks come with them: the imaginary parts are nonzero only for Kerr, Kerr-Newman and Taub-NUT, as a pseudo-scalar must be, and all eight Weyl columns vanish exactly for the type N and III metrics, as a nilpotent M_{ij} requires. *What remains:* the mixed $\mathring{W}$–S contractions K, L, M of §II.10.3 — whether they are independent of the above needs a rank computation against this basis, not a ratio fit.
2. **Bach — settled on three metrics, still open in general.** $\mathrm{Im}(\mathring{B}_{ab})$, the Bach tensor built from $({}^*W)_{abcd}$, **vanishes identically** on Reissner-Nordström — all sixteen components exactly zero, computed symbolically with exact derivatives, on three metrics with Cotton ≠ 0 and Weyl ≠ 0, so none of the tests is vacuous, and each returns a healthy nonzero *real* Bach tensor alongside. So it is an identity there, not a definition. *An earlier revision claimed the opposite* on numerical evidence; that was noise from two nested finite differences, whose floor had been mis-estimated on a case where the answer was zero. What remains open is whether it holds generally: the symbolic computation needs two covariant derivatives of Weyl and does not terminate on two-variable metrics without first rewriting the Bach operator as a linear form in the jets of W_{abcd}.
3. **P*, the missing rung — it can fail to exist, and does on the cases tested.** $\mathcal{P}_{ab}$ has no antisymmetric pair, hence no factor in the sandwich and no dual of either kind. The proposed repair was to define P* as whatever would be a superpotential for L*, i.e. to seek a symmetric P^*_{ab} with $L^*_{abc} = P^*_{c[a;b]}$. On the extremal Reissner-Nordström detector, where L_{abc} is known in closed form, **no such P^*_{ab} exists**: of the 24 equations, four carry no derivatives, and two of those are incompatible unless N = 0 — that is, unless the Lanczos tensor degenerates. The obstruction is algebraic, not a matter of awkward integration.

   So the admissible Lanczos class **is** closed under dualization (L^*_{abc} satisfies every algebraic condition, §II.7.5) while the superpotential-derived subclass **is not**. That is the structural difference between the two classes that this question was posed to detect, and it means $\mathring{\mathcal{P}}_{ab}$ does not exist by this route: the tower is complete at the Lanczos rung and stops there. *The obstruction is not an artifact of extremality*. The same computation on **Schwarzschild** — vacuum, type D, and about as far from the extremal charged case as the collection reaches — fails identically. The data, stated in full so the reader need not take it on trust. With the sign-flip superpotential $\mathcal{P}_{ab} = \mathrm{diag}(-f, -f^{-1}, r^2, r^2\sin^2\theta)$ and $f = 1 - 2M/r$, the trace-free Lanczos tensor has three independent nonzero components:

   $$L_{trt} = 2(r - 3M)(3r^2)^{-1}, \quad L_{r\theta\theta} = r(r - 3M)[3(2M - r)]^{-1}, \quad L_{r\phi\phi} = r(r - 3M)\sin^2\theta[3(2M - r)]^{-1},$$

   and its dual has three:

   $$L^*_{t\theta\phi} = (r/3 - M)|\sin\theta|, \quad L^*_{t\phi\theta} = (M - r/3)|\sin\theta|, \quad L^*_{\theta\phi t} = 2(M - r/3)|\sin\theta|.$$

   Seeking a symmetric $P^*_{ab}(r, \theta)$ with $L^*_{abc} = P^*_{c[a;b]}$ gives 24 equations, of which exactly four contain no derivatives:

   $$(t\phi t): \quad M(r - 2M)P^*_{r\phi}(2r^3)^{-1} = 0$$
   $$(t\phi r): \quad (3M - r)P^*_{t\phi}[2r(r - 2M)]^{-1} = 0$$
   $$(t\phi\theta): \quad (r/3 - M)|\sin\theta| - P^*_{t\phi}(2\tan\theta)^{-1} = 0$$
   $$(t\phi\phi): \quad (M - r/2)P^*_{tr}(\cos 2\theta - 1)/2 + P^*_{t\theta}\sin 2\theta/4 = 0$$

   The first two force $P^*_{r\phi} = P^*_{t\phi} = 0$ away from r = 2M and r = 3M. Substituting $P^*_{t\phi} = 0$ into the third leaves $(r/3 - M)|\sin\theta| = 0$, which holds only on the single sphere r = 3M — not an identity. So the algebraic subsystem is inconsistent, and no symmetric P^*_{ab} exists for this superpotential. So two very different spacetimes fail in the same way, by an algebraic obstruction that appears before

any integrability question arises.

But P^*_{ab} certainly does exist sometimes, and trivially so. Wherever the Weyl tensor vanishes — Minkowski, de Sitter, anti-de Sitter, and every conformally flat spacetime — one may take $L_{abc} = 0$, whence $L^*_{abc} = 0$ and $P^*_{ab} = 0$ serves. The de Sitter / anti-de Sitter shadow pair of §III.2a is of exactly this kind. So the correct statement is not that P^*_{ab} never exists, but that **it can fail to exist**, and does so for the superpotentials examined here on two algebraically special spacetimes with $W_{abcd} \neq 0$.

Why failure should be expected to be generic

Grant that a superpotential $\mathcal{P}_{ab}$ exists (Part I's Team C assumes it; §II.7.5 exhibits one in many cases). The extra demand that P^*_{ab} exist is that a **linear** partial differential equation with fixed right-hand side be soluble:

$$P^*_{c[a;b]} = (\mathcal{P}_{c[a;b]})^*$$

— a *curl* equation for the unknown symmetric P^*_{ab}, whose right-hand side is determined by $\mathcal{P}_{ab}$. Write curl for the linear map $\mathcal{P}_{ab} \mapsto \mathcal{P}_{c[a;b]}$ on symmetric tensor fields, and * for the (linear) dual on the antisymmetric pair. Then

$$\textbf{the good set} = \{\ \mathcal{P}_{ab} : (\text{curl}\ \mathcal{P})^* \in \text{image(curl)}\ \}$$

— the superpotentials for which a P^*_{ab} exists. It is the preimage of a linear subspace under a linear map, hence itself a **linear subspace**.

Two things follow immediately, and a third does not.

(1) The good set is proper. The Schwarzschild sign-flip superpotential above lies outside it, so the good set is not everything.

(2) good + bad = bad. If $\mathcal{P}^{good}$ is in the good set and $\mathcal{P}^{bad}$ is not, then their sum is not either, since otherwise $\mathcal{P}^{bad} = (\mathcal{P}^{good} + \mathcal{P}^{bad}) - \mathcal{P}^{good}$ would lie in the good set.

(3) What does *not* follow: that one bad element spoils every combination containing it. Two elements outside a subspace may sum into it — taking the good set to be span(e_1), both e_2 and $e_1 - e_2$ are bad while their sum is good. **Badnesses can cancel**, so (2) applies only when every other term is good. Note that linear independence of the bad elements does not help: the two just given are independent. The condition that does is independence *modulo* the good set, since $\Sigma c_i \mathcal{P}_i$ lies in the good set exactly when $\Sigma c_i[\mathcal{P}_i] = 0$ in the quotient. **If the bad superpotentials have linearly independent images in that quotient, no finite combination of them is good** — a condition checkable by a rank computation in any finite-dimensional ansatz family.

Status of the genericity claim. What (1) and (2) give is a heuristic, and a good one: the set of superpotentials admitting a P^*_{ab} is a proper linear subspace, which in *finite* dimensions is of measure zero — so within any finite-dimensional family of ansatz superpotentials, such as the jet-truncated families actually searched here, failure is generic and this is a proof. In infinite dimensions it is not yet one. Shyness in the prevalence sense requires the subspace to be closed (or at least Borel), and whether the good set is closed amounts to whether curl has closed range on symmetric tensor fields — a functional-analytic question not addressed here. **So: proved for finite-dimensional families, conjectural in general.**

Subject to that caveat, §II.7.5's prevalence argument runs in the same direction for a different conclusion — there, detection is generic because non-detection is a codimension condition; here, existence of P^*_{ab} is exceptional because it *is* a codimension condition. Both rest on the same observation, that a proper linear (or affine) subspace is negligible.

Note the affine refinement needed for a fixed spacetime. The superpotentials of a *given* metric form an affine space — a particular solution plus the gauge freedom — not a linear one, so the combinations must have coefficients summing to 1. The argument survives unchanged: the good set intersected with that affine space is an affine subspace, proper as soon as one bad element is exhibited, and the Schwarzschild sign-flip superpotential above is such an element.

What is still not established: the scope of the failure. Two metrics, each with one particular superpotential, is evidence of a general phenomenon rather than a proof of one; the Lanczos gauge freedom (§II.7.5) means a different superpotential for the same geometry has not been ruled out; and the conformally flat case shows the answer genuinely depends on the spacetime, so no uniform verdict is available.

4. **Is the $\overleftrightarrow{ii}$ shadow a metric at all, and when?** This is the largest open question in Part III. The reconstruction of §I.4 accepts any tidal

matrix and returns an output; it never verifies that the output's own Riemann tensor is the matrix supplied. Several things are now known, and they narrow the question considerably without closing it.

Settled affirmatively on one class. By the eigenvalue theorem (§III.2a) the only constant μ with $(*R*)_{abcd} = \mu R_{abcd}$ on a non-flat spacetime are ± 1. For $\mu = +1$ — conformally flat with zero scalar curvature — the shadow tidal matrix is $+K^i{}_j$, and since a positive rescaling of the tidal matrix by c is the tidal matrix of g_{ab}/c, the shadow is the original metric itself. **On that class the shadow is certainly a metric.** For $\mu = -1$ (Einstein) the rescaling argument needs $L^2 = -1$ and fails over the reals, which is why the Einstein class is where the question has teeth.

A tempting identification that does not survive. For Kottler the tidal-matrix *spectra* of the shadow and of Kottler(−M, −Λ) agree exactly: both {0.138, –0.054, –0.054}. It is natural to conclude that the shadow of an Einstein space is the same family with mass and cosmological constant negated. **It is not.** Componentwise, R_{abcd} of Schwarzschild(−M) is *not* $-R_{abcd}$ of Schwarzschild(+M): 16 of 256 components differ, e.g. $R_{0202}(-M) = M(-2M - r)/r^2$ against $-R_{0202}(+M) = M(2M - r)/r^2$, differing in the 2M term rather than by a sign. The same holds at the connection level with mixed indices (12 of 256). The reason is that M enters the metric function nonlinearly, so negating it is not an overall sign flip on curvature. **Matching spectra means the two tensors are related by a frame rotation, not that one is minus the other** — a distinction that misled us for several rounds.

An analytic-continuation argument that fails, and where. One may argue: Kottler is Einstein, so its shadow curvature is $-R_{abcd}$; that equals the curvature of Kottler(−M, −Λ); that spacetime has a superpotential, namely P_{ab} with the parameters negated; so continue it back and obtain P^*_{ab}. Step one is exact ($(*R*)_{abcd} = -R_{abcd}$ holds to 0 of 256 components on Schwarzschild). **Step two is false**, by the paragraph above. Two further checks confirm the conclusion independently: P(−M) used with the original connection gives a Lanczos tensor differing from L* in 12 of 64 components, and L* on Schwarzschild(+M) versus L_{abc} of Schwarzschild(−M) computed with its own connection have **disjoint supports** — six nonzero components each, occupying different index slots. Dualizing rotates a tensor among index slots; negating a parameter changes the geometry. The two agree on Riemann invariants for Einstein spaces, which is what makes the argument tempting, but they are different operations and do not commute into one another.

Signature is not guaranteed. The reconstruction integrates $A'' = -K^i{}_j A$ and sets $g_{ab} = \eta_{ij} A^i{}_a A^j{}_b$, so the signature is that of η *provided A stays invertible*. A negative eigenvalue of K gives exponential growth, which never returns to zero; a positive one gives oscillation, which can reach a zero of det A — a conjugate point, where the reconstructed object degenerates and has no signature at all. **Negating K swaps which directions oscillate and which grow**, so the shadow's conjugate points sit in different places from the original's. On an outgoing radial Schwarzschild ray both survive (min |det A| = 3.4e−3 over 60 units of affine parameter, no sign changes, spatial eigenvalues positive throughout — with the shadow's odd direction swapped relative to the original's, as expected). But that is a favourable case: $1/r^3$ decays fast enough that the oscillating directions never complete a quarter period. In a bounded region or a closed cosmology there is no such protection.

So the open questions, sharpened:

- *Integrability — one negative result now in hand*. For an Einstein space the shadow field is $-R_{abcd}$. Schwarzschild's is spherically symmetric and static, so if a shadow metric exists the natural class to look in is the same one. Writing $g' = \mathrm{diag}(-A(r), B(r), C(r), C(r)\sin^2\theta)$ and imposing $\mathrm{Riemann}(g') = -R_{abcd}$ gives four independent equations for three functions. In the areal gauge $C = r^2$ they can be solved in order: the (2323) equation gives **$B = r/(r+2)$**, which is exactly Schwarzschild with $M \to -M$; the (0202) equation then forces $A' = 2(2-r)[r^2(r+2)]^{-1}$, *not* the $-2/r^2$ that Schwarzschild(−M) requires — they differ by $8[r^2(r+2)]^{-1}$, and integrating gives a logarithm rather than a rational function. The remaining (1212) equation is then inconsistent, with residual $-4r^3(r+2)^{-2}$ and **no roots** — not zero anywhere, not even on an exceptional sphere. **So the shadow of Schwarzschild is not a static spherically symmetric metric.** This does not prove that no metric has that curvature, but the shadow field carries those symmetries itself, so this was the natural place to look. It also explains why the $M \to -M$ guess half-works: the spatial part matches exactly and the temporal part does not.
- *Identification*. For Einstein spaces the shadow is *sometimes* the parameter-negated metric and sometimes not, and the dividing line is unknown. For pure de Sitter it **is**: the tidal matrix is isotropic there ($W_{abcd} = 0$), so negating it is exactly $\Lambda \to -\Lambda$, and $K_{\tilde{i}\tilde{j}}(\mathrm{dS})$ agrees with K(AdS) to 3.3e−15. Switch on a mass and it is **not**: Kottler's Weyl part is nonzero, and $R_{abcd}(-M)$ differs from $-R_{abcd}(M)$ in 16 of 256 components, because M enters the metric function nonlinearly. So the parameter-negation picture holds exactly where the Weyl tensor vanishes and fails once it does not — which is suggestive but not yet a theorem, since "conformally flat Einstein" is a very small class to generalise from. What is the shadow in the general Einstein case? An invariant characterisation is wanted — what is preserved, what reversed, what happens to horizons, causal structure, Petrov type.
- *Degeneracy*. Where does det A vanish for the shadow, and does its conjugate-point structure relate to the original's at all?
- *Relation between the rungs*. The Ricci-rung and full-curvature shadows disagree (§III.2.1). The Ricci datum is strictly less

information — it does not fix the mass — suggesting the full-curvature version is the real object.
- *Global versus local*. Everything measured so far is tidal matrices at a point; the integrated object has never been constructed.

Until these are settled, "shadow metric" should be read throughout as a label for a computed tidal matrix, not as a claim about a spacetime. In short, it is so shadowy we aren't even sure that it *is* a metric of anything.

Part III. (Bi)complex reconstructors

Shadow metrics; (Bi)complexified 4D tensor picture as foundation upon which to erect(?) future "quantum gravity."

III.1 The construction

Part I reconstructs the metric from curvature by integrating one linear ordinary differential equation outward from a seed point. Part II replaces the curvature by a complex or bicomplex object. Because the reconstruction equation is **linear**, it does not object: feed it a complex tidal matrix and it returns a complex frame, hence a complex metric. Nothing has to be re-derived.

Concretely, the reconstruction gives, to second order about the seed point,

$$g_{ab}(x) = \eta_{ab} - \tfrac{1}{3}R_{acbd}x^c x^d + o(|x|^2),$$

and substituting the bicomplexified Riemann tensor of II.4 in place of R gives a bicomplex metric with four slots:

$$\overset{\circ}{g}_{ab}(x) = \eta_{ab} - \tfrac{1}{3}[R_{acbd} + \overleftarrow{i}(*R)_{acbd} + \vec{i}(R^*)_{acbd} + \overleftrightarrow{ii}(*R^*)_{acbd}]x^c x^d.$$

The question this part answers is what the three new slots contain. The answer is that all three have identifiable meanings, and none of them is a new geometric degree of freedom.

slot	source	symmetric?	what it is
1	R_{abcd}	yes	the metric itself
$\overleftrightarrow{ii}$	$(*R^*)_{abcd}$	**yes**	a *shadow tidal matrix*: exactly –K in Λ-vacuum; status open (§III.2.1)
$\overleftarrow{i}, \vec{i}$	$*R, R^*$	**no**	the *momentum constraint* of the Einstein equations (III.4)

That the $\overleftrightarrow{ii}$ slot is symmetric and the arrow slots are not is forced by II.4.1: R and (*R*) are Riemann-type tensors, while *R and R* fail pair symmetry and the first Bianchi identity. What looked there like an algebraic nuisance turns out here to be carrying physics.

III.2 The $\overleftrightarrow{ii}$ slot: a shadow tidal matrix, of unclear status

Write $K^i{}_j = R^i{}_{0j0}$ for the tidal matrix that drives the reconstruction of §I.4.2, and $K_{\overleftrightarrow{ij}}$ for the same contraction of the double dual. Measured in an orthonormal frame:

metric	$K_{\overleftrightarrow{ij}}$ vs K	residual
Schwarzschild, Kottler, Kerr, Kasner, Taub-NUT, Nariai *(all Einstein, i.e. Λ-vacuum, Λ = 0 included)*	**−1.0000 × K**	≤ 4e−16
Reissner-Nordström	−1.075 × K	4e−2
Kerr-Newman	−1.009 × K	8e−3

Barriola-Vilenkin	$-1.100 \times K$	4e−2
Gödel, Melvin, Vaidya, C-metric, Majumdar-Papapetrou, wormhole	not proportional	O(1)
FRW, Bertotti-Robinson (conformally flat)	**$+1.0000 \times K$**	0

In Λ-vacuum the $\overleftrightarrow{ii}$ tidal matrix is exactly minus the real one. Off Λ-vacuum the proportionality degrades with the matter content, and for conformally flat metrics the sign flips to +1, so the reversal is carried by the Weyl part.

III.2.1 What this does *not* establish

It is tempting to describe the −K case as "a shadow metric with reversed curvature" and to expect it to be some recognisable spacetime. **We have no warrant for either half of that.** Three points, all of which we can state and none of which we can currently resolve.

(a) It is not obviously a metric at all. The reconstruction of §I.4 integrates whatever tidal matrix it is handed and returns *something*. Nothing in it checks that the result's own Riemann tensor is the matrix that went in. A symmetric K is necessary for that but not sufficient: the full Riemann tensor must satisfy the first and second Bianchi identities and the integrability conditions that make a curvature field come from a metric, and we have not verified that −K does. Until that is checked, "the shadow metric" is a name for the output of an algorithm, not for a geometry.

(b) Where we can identify it, it is not the obvious candidate — and the near-miss is instructive. For Kottler one might guess the shadow is Kottler with Λ negated, or with the mass negated, or both. Comparing tidal-matrix *spectra* at M = 1, Λ = 3/100:

object	tidal spectrum	matches $K_{\overleftrightarrow{ii}}$?
Kottler(M, Λ) — the original	(−0.138, 0.054, 0.054)	—
$\mathbf{K_{\overleftrightarrow{ii}}}$ — the shadow	(−0.054, −0.054, 0.138)	—
Kottler(M, −Λ)	(−0.118, 0.074, 0.074)	no
Kottler(−M, Λ)	(−0.074, −0.074, 0.118)	no
Kottler(−M, −Λ)	(−0.054, −0.054, 0.138)	**yes, exactly**

The last row is a genuine spectral match, and it is tempting to stop there. **But matching spectra is not tensor equality.** Componentwise, R_{abcd} of Schwarzschild(−M) is *not* $-R_{abcd}$ of Schwarzschild(+M): 16 of 256 components differ, e.g. $R_{0202}(-M) = M(-2M - r)/r^2$ against $-R_{0202}(+M) = M(2M - r)/r^2$ — a discrepancy in the 2M term, not an overall sign. The same failure appears at the connection level with mixed indices (12 of 256). M enters the metric function nonlinearly, so negating it does not negate the curvature. The spectra agree because the two tensors are related by a frame rotation, which is much weaker.

A methodological note, since this cost us several rounds. An earlier revision reported these spectra as disagreeing, on the basis of comparing frame *components* between the two metrics. That was wrong: the orthonormal frame is built by eigen-decomposition, which orders basis vectors by eigenvalue, so two different metrics receive differently-ordered frames and the componentwise comparison is meaningless. Compare invariants or spectra across different metrics, never frame components. The correction then overshot in the other direction, treating the spectral match as an identification. Both errors are recorded because the distinction — spectrum versus tensor — is exactly what the reader needs to keep hold of here.

(c) The two rungs give different answers. At the Ricci rung the imaginary part of $\mathring{R}_{ab} = R_{ab} + iG_{ab}$ is the Einstein tensor, and for an Einstein space $G_{ab} = -R_{ab}$ exactly (verified to 4e−16 on Kottler, Schwarzschild-anti-de Sitter and Nariai). Reconstructing from *that* datum returns an Einstein space of the opposite cosmological constant, which invites the slogan "de Sitter and anti-de Sitter are each other's shadows". But the Ricci datum does not determine the mass — $R_{ab} = -\Lambda g_{ab}$ holds for every Schwarzschild-anti-de Sitter metric and for pure anti-de Sitter alike — whereas the full-curvature $\overleftrightarrow{ii}$ slot fixes everything and, as (b) shows, does not give a Kottler at all. **Where $W_{abcd} \neq 0$ the Ricci-rung and full-curvature shadows are different constructions and disagree.** Where $W_{abcd} = 0$ they agree: for pure de Sitter the tidal matrix is isotropic, negating it is exactly $\Lambda \to -\Lambda$, and $K_{\overleftrightarrow{ii}}(dS)$ matches K(AdS) to 3.3e−15. So the dS/AdS slogan is correct outright, and it is the *massive* case that needs the qualification.

What survives all this is the measured statement at the top of the section, which is about tidal matrices and makes no claim about metrics: in

Λ-vacuum the double dual's tidal matrix is exactly the negative of Riemann's, and off Λ-vacuum it detaches. Everything past that is open.

III.2a The eigenvalue theorem: μ = ±1, and nothing else

The table of §III.2 is not an empirical list. It is the eigenspace decomposition of the double-dual operator, and the following settles which proportionality constants can occur at all.

Theorem. Suppose $(*R*)_{abcd} = \mu\, R_{abcd}$ for a constant μ, on a spacetime that is not flat. Then $\mu = \pm 1$. Specifically:

μ	equivalent condition	$\overset{\leftrightarrow}{ii}$ shadow tidal matrix
−1	$S_{ab} = 0$, i.e. **Einstein** (Λ-vacuum)	$-K^i{}_j$
+1	$W_{abcd} = 0$ *and* $R = 0$, i.e. **conformally flat with zero scalar curvature**	$+K^i{}_j$
any other	**impossible unless $R_{abcd} = 0$**	

Proof. Both the Riemann tensor and its double dual decompose across the idempotents of §II.3:

$$R_{abcd} = W_{abcd} + (R/24)(g \odot g)_{abcd} + \tfrac{1}{2}(g \odot S)_{abcd},$$
$$(*R*)_{abcd} = -W_{abcd} - (R/24)(g \odot g)_{abcd} + \tfrac{1}{2}(g \odot S)_{abcd}.$$

(Both verified to ≤ 7e−16 on every metric of §II.9.) The three pieces are independent, so $(*R*)_{abcd} = \mu\, R_{abcd}$ is equivalent to the three conditions

$$(\mu+1)\, W_{abcd} = 0, \quad (\mu+1)\, R = 0, \quad (\mu-1)\, S_{ab} = 0.$$

If $\mu = -1$ the first two are automatic and the third forces $S_{ab} = 0$. If $\mu = +1$ the third is automatic and the first two force $W_{abcd} = 0$ and $R = 0$. For any other μ all three fire at once, giving $W_{abcd} = 0$, $R = 0$ and $S_{ab} = 0$, hence $R_{abcd} = 0$. □

Independently, the double dual is an **involution** — $(*(*R*)*)_{abcd} = R_{abcd}$, verified to 1.1e−15 — so its eigenvalues are ±1 on general grounds. The proof above adds which summand each root annihilates.

III.2a.1 Consequence for the shadow

Under a constant rescaling $g_{ab} \to L^2 g_{ab}$ the frame tidal matrix scales as L^{-2} (measured: ratios 1, ¼, ⅙ for L = 1, 2, 3). So a tidal matrix multiplied by a constant $c > 0$ *is* the tidal matrix of the metric g_{ab}/c, and the shadow is then a genuine metric — a constant rescaling of the original.

By the theorem the only positive value available is $\mu = +1$, where $c = 1$ and the shadow is the original metric itself. **On the conformally-flat, zero-scalar-curvature class the shadow is therefore certainly a metric**, which settles the §II.12 open problem affirmatively on that class. For $\mu = -1$ the rescaling argument requires $L^2 = -1$ and fails over the reals; that is exactly why the Einstein class is where the integrability question has teeth.

Notation. μ is used here for this eigenvalue only. $K^i{}_j$ always denotes the tidal matrix; the Kretschmann scalar is written $R_{abcd}R^{abcd}$ throughout rather than abbreviated, and K, L, M in §II.10.3 are the Carminati-McLenaghan mixed invariants, which are unrelated to either.

III.3 The arrow slots are the momentum constraint

III.3.1 What a constraint is

This subsection is expository; readers familiar with the 3+1 formulation may skip it.

Choose a family of spacelike slices — "instants" — and a unit timelike vector u^a normal to them. Every tensor then splits into pieces according to how many of its indices point along u. Einstein's field equations, ten in number, split correspondingly into two kinds:

- **Six evolution equations**, which say how the geometry of a slice changes as one moves to the next slice. These involve second time derivatives and behave like ordinary dynamical laws.
- **Four constraint equations**, which involve no second time derivatives at all. They are conditions that the data *on a single slice* must already satisfy. One cannot choose the geometry of an instant freely and then evolve it; the initial data must be constrained.

The four constraints are the components of the Einstein tensor with at least one index along u. The one with both indices along u, $G_{ab}u^a u^b$, is the **Hamiltonian constraint**; it ties the intrinsic and extrinsic curvature of the slice to the energy density. The three with one index along u, $G_{ab}u^a e_i{}^b$, are the **momentum constraint**; they tie the slice's geometry to the **momentum density**, that is, to the flux of energy across the slice. In the presence of matter these read, by the field equations, $G_{0i} = 8\pi T_{0i}$ — the momentum density of whatever matter is present.

The relevant curvature split is into an **electric part** $E_{ij} = R_{i0j0}$, the tidal matrix, which is what stretches and squeezes; and a **magnetic part** $B_{ij} = (*R)_{i0j0}$, the same contraction of the dual, which is associated with frame-dragging and with gravitational radiation. The electric part is automatically symmetric. The magnetic part is *not*: its symmetric trace-free piece is the magnetic Weyl tensor, and its antisymmetric piece is fixed by the momentum constraint.

III.3.2 The measurement

Taking the antisymmetric part of the magnetic tidal matrix, converting it to a 3-vector with the spatial Levi-Civita symbol, and comparing with the momentum density G_{0i} in the same frame:

$$\textbf{antisym}(B)_i = G_{0i}/2$$

metric	|antisym B|	$\|G_{0i}\|$	fitted ratio	residual
Kerr-Newman	4.585e−4	9.170e−4	**+0.50000**	1.2e−16
Vaidya	7.937e−2	1.587e−1	**+0.50000**	1.7e−16
the other seventeen	0	0	—	—

The two metrics with a nonzero antisymmetric part are exactly the two with an energy flux in the chosen frame. A rotating charged body carries a Poynting flux, so Kerr-Newman has one; null dust is streaming energy radially, so Vaidya has one. Static charge alone does not (Reissner-Nordström), rotation alone in vacuum does not (Kerr), and rotating dust comoving with the frame does not (Gödel, Van Stockum).

III.3.3 How to read it

It is tempting to see a metric with an antisymmetric part and think of Einstein-Schrödinger nonsymmetric gravitation, where the antisymmetric part of g is an independent field posited at the outset and then given its own dynamics. **We think that reading is wrong here.** The antisymmetric part in the arrow slots is not independent of anything: it is determined, pointwise and algebraically, by the matter already present, through a constraint equation that the Einstein equations impose anyway.

The better reading is that the bicomplex metric is a bookkeeping device which carries the constraint content of the field equations alongside the geometry. Its four slots are then:

real part — the geometry;
$\overleftrightarrow{ii}$ part — the curvature-reversed shadow (status open, §III.2.1);
arrow parts — the momentum constraint.

That every slot has a meaning is more than one would expect from a formal construction, and it is the main reason we record this part at all. What it is *good for* we do not know. The observation that a formalism packages the geometry and its constraints into one object is the kind of thing that matters if one wants a canonical or quantum treatment, where the constraints are the central difficulty; but we make no claim in that direction here, and §III.7 marks the corresponding speculation as speculation.

III.4 Everything here is four-dimensional

Each ingredient of this paper fails outside four dimensions, and for the same underlying reason.

ingredient	why it needs n = 4

the two duals	the Hodge dual of a 2-form is an (n−2)-form; only for n = 4 is that again a 2-form, so only then can * act on an index pair and return an object of the same type
the bicomplex algebra	follows immediately: no two pair-duals, no $\overleftarrow{\mathbf{i}}, \vec{\mathbf{i}}, \vec{\mathbf{i}}\vec{\mathbf{i}}$
Lanczos potential Ł	generates the Weyl tensor in general only for n = 4 (I.7)
superpotential $\mathcal{P}$	a fortiori
foundations *5 and *6	the two that are strictly richer than the metric (I.3) are exactly the two that are four-dimensional
$\mathring{W}$ as a 3×3 matrix	the Petrov classification, Cayley-Hamilton (IV.D7) and $I^3 = 6J^2$ (IV.D8) all rest on it

So the entire apparatus — the richer foundations, the complexification, the shadow objects, the constraint slots — is available in four dimensions and in no other. For n ≥ 5 the foundation count drops to four and none of Part II survives; for n ≤ 3 the Weyl tensor vanishes identically and the question does not arise.

It is tempting to read this as bearing on why spacetime is four-dimensional: if a unification of gravity with quantum field theory required this structure, then it would require n = 4, and the dimension of the world would be explained rather than assumed. **We flag this as the weakest inference in the paper.** An anthropic-style argument of this shape needs the antecedent — that unification requires this structure — and nothing here establishes it. Four dimensions being special in many ways that have nothing to do with the present formalism, the observation is offered as a curiosity and a possible line of enquiry, not as an explanation.

III.5 Caveats

1. The reconstruction was carried out to **second order about the seed point**, using the exact leading term of Part I rather than integrating the full ODE along rays. Everything above is therefore a statement about the curvature sources of the reconstruction, which determine it, rather than about a globally integrated shadow metric. Integrating the full system with complex coefficients is straightforward in principle and has not been done.
2. The split into electric and magnetic parts, and hence the identification of the momentum density, **depends on the frame**. The statement “the antisymmetric part is the momentum density” is frame-covariant, but which metrics show a nonzero one depends on the choice of u. Gödel and Van Stockum show none in the frame used here; a boosted frame would show one.
3. Three metrics in the collection (the pp-waves and the Kundt type III wave) are written in null coordinates where the orthonormal-frame construction used here fails, so they are absent from these tables. They are precisely the radiative cases, where the magnetic part is most interesting.

III.7 Speculative importance of all this for future "quantum gravity" theory-attempts

> "Less is more." – widespread aphorism.

This section (mainly by WDS, trying to convince a hesitant CLAUDE of his point of view) will be opinionated speculation.

With (bi)complexified tensors and six rather than one "foundation" for differential geometry, we get much more than old pictures. (Up to 4×6=24 times more!) But at the same time, we have much less: many statements become more concisely stateable in the new formalism. You can say more with less. This all is getting at the Natural Truth. But is it the Natural Truth of mathematics, or also the Natural Truth of *physics*? I speculate: both.

First (not very convincingly by itself, but enough to put you on alert) physics is 3+1 dimensional. And all our bicomplex extra stuff comes to differential geometry *only* in 4 dimensions. Quite conveniently for anybody trying to create a universe. If you are trying to connect quantum field theory (QFT) to gravity, you now have a much wider palette of options for how to do that. Indeed, a quantum theory need not be built on the metric; it might couple more naturally to the Lanczos or superpotential foundations, which by I.3 are strictly *richer* than the metric and carry extra gauge freedom the metric cannot see. The number of possible ways to try to hook QFT and gravity together increases, I daresay, by a lot more than merely a factor of 24. Maybe 24^2, 24^3, or 2^{24}. Anyhow, whatever it is, it's big — increasing success-chance tremendously. That already tells you: this is a high-value family of ideas to explore next to seek True Physics.

And incidentally, in case you did not notice, quantum fields most often are naturally interpreted as complex number valued, or even bigger packages can be used such as "biquaternions," — not merely real numbers. These seem to be crying out to theorists trying to unify QFT↔gravity: "please, give me complex or bicomplex gravity so I can talk to it!" For example, the classic Einstein field eqaution $G_{ab}=8\pi T_{ab}$ concerned only the real component of $\mathring{G}_{ab}$. If matter described by T_{ab} also included a complex component arising from the fact that "matter" really is made of quantum fields, that could provide one possible avanue to try to make GR know about quantum matter.

That leads to a concrete question rather than a hope: **what breaks if T_{ab} acquires an imaginary part?** The contracted Bianchi identity forces $\nabla^a G_{ab}=0$ identically, so $\mathrm{Im}(T_{ab})$ would have to be separately divergence-free. That is a strong constraint, and possibly a productive one, since divergence-free symmetric tensors are exactly what conserved currents look like. It is also precisely the identity that appears in §II.8.5 as the reality condition $\nabla^a \mathring{R}_{ab}=\nabla_b R/2$ — so the statement "the complexified Ricci tensor has real divergence" is exactly what a complex source would have to violate or evade.

Second, traceless Lanczos tensors look likely to be able to yield black hole "horizon detectors," enabling future Lanczos-aware physics to be able to solve otherwise-unsolvable-looking famous "paradoxes" about black holes, and enable Smith 2026's "mirror paradigm."

Third, GR(Λ) – Einstein's general relativity with cosmical constant Λ – even if hypothetical invisible undetectable "cold dark matter" is added – seems to have great, perhaps insurmountable, difficulty matching astronomer *observations*. E.g: observed galaxy formation, development, and "rotation curves" versus theory-based models. Additional problems for ΛCDM include the "Hubble tension" and "lithium problem" from cosmology. Meanwhile, M.Milgrom invented a theoretically-ridiculous kludge called MOND which embarrassed all highfalutin theorists by having much better success modelling galaxies. In the companion paper (Smith & CLAUDE 2026 [35], which I hadn't told CLAUDE existed until right now!) it is shown that **traceless Lanczos tensors** enable natural MOND+GR(Λ) *marriages* in simple and surprisingly-near-unique ways now-consonant with Einsteinian principles. Some such marriage should be able to match observations better than either spouse alone and offers (at least potentially) the ability to solve all those problems and more.

Fourth, let us consider the Bayesian implications of our discovery here that the $\overleftarrow{\mathbf{i}}$ and $\vec{\mathbf{i}}$ components of the bicomplexified Riemann tensor are equivalent to the "momentum constraint" in the ADM (Arnowitt-Deser-Misner) "time-slicing" reformulation of GR(Λ). This formulation time-slices spacetime into 3-spaces and evolves data forward.

When ADM introduced time-slicing, it was merely as an **artificially-designed** somewhat-ugly kludge intended to enable computers to simulate GR. It was not intended to be, and did not seem to be, the Natural Truth about physics. ADM's central awkwardness is that the slicing is a choice made before the physics: one picks a foliation, splits every tensor by it, and then works with objects that are not manifestly covariant. There is a special reference frame. It is not general.

But on the other hand: for hoped-for unification of GR with quantum field theories, time-slicing of this kind offers advantages. One big obstacle to such unifications is the fact that quantum field theories like the "standard model" naturally reside inside Minkowski flat spacetime. But GR is naturally about *curved* spacetime. With ADM time-slicing, you can regard GR in a more QFT-like way as "time-evolving fields in flat spacetime." There are then two ways to view GR:

1. GR is set in curved (3+1)-spacetime. (Natural & beautiful.)
2. GR is a theory of time-evolving fields on flat spacetime. (Seemed like an ugly artificial kludge, although with practical usefulness.)

The two views naively seem to contradict. But in fact they are mathematically equivalent. They are just two different *interpretations* of the same thing. You feel like, and your telescopes see like, you are living in a curved space universe, due to interpretation #1. But everything is merely time-evolving fields on flat space the way the Standard Model says (which also is what numerical relativity does in practice), due to interpretation #2. These two interpretations coexist. They live harmoniously together. They do not conflict.

Our new insight is that our bicomplex formalism supplies the beautiful and natural covariant parent of exactly ADM's ugly "time-evolving field" objects. $\mathring{W}$ is defined without reference to any slicing. *Choosing* a unit timelike u decomposes it into the electric and magnetic parts E+iB, and Part III shows that the remaining slots of the bicomplexified Riemann tensor carry ADM's "momentum constraint." So the data that 3+1 produces by artificial cutting is here obtained by projecting a single object that existed beforehand. That is a genuine structural difference: the split is recovered rather than imposed, and the pieces are components of one geometric quantity rather than separately defined fields.

Two qualifications: (a) the objects are not *literally* the ADM variables: ADM evolves the intrinsic 3-metric and the extrinsic curvature, while E and B are built from those through the Gauss-Codazzi relations. The correspondence is one of content, not of literal identity. (b) More importantly, **the electric/magnetic split is itself slicing-dependent** – it needs u just as ADM needs a foliation. What is covariant is $\mathring{W}$ and the bicomplexified Riemann tensor; the ADM-like reading appears only after a u is chosen. The formalism therefore does not *remove* the foliation choice; rather it *relocates* it from the definition of the variables to the reading of a slicing-independent object.

When we cooked up the bicomplex generalization of 4D differential geometry, we did not work in the style of ADM. They artificially forced

specific time slicings in specific reference frames with the goal of enabling time-evolution computer calculations. We did not. We just did the mathematically natural thing. The bicomplex formulation. We weren't trying to artificially design any kludge. We were just trying to express the Truth in the cleanest simplest most concise way. But then — lo and behold — our natural stuff turned out — via an unrequested and a-priori-improbable *miracle* — to *be* ADM's stuff. Call it superstition or call it Bayesian inference; either way this makes it very probable that the True Unification works this way.

I mean, we did not ask. We did not force. We just sought Natural Mathematical Truth. And then it **just magically naturally gave us *exactly* what was wanted to be like ADM and be QFT-marriage-compatible, and now in a nice way, not an ugly-kludge way.** (And then: it turned out that discoveries of our ilk had already been made in prior literature [16–20], effectively-independently confirming our deductions, and showing they were not mathematical delusions caused by insanity.)

Fifth, in the Companion Paper trying to find the natural Einstein-acceptable version of Milgrom's MOND (which postulates a "change in the law of gravity" thresholded by "gravitational acceleration" crossing a threshhold value $a_0 \approx 1.2\times10^{-10}$ meter/sec^2) when we investigated what "gravitational acceleration" should mean in the Kerr rotating-black-hole metric, one Lanczos tensor L_{abc} magically appeared to be more "natural": than all others; and then it turned out the the acceleration of the "distinguished observer" magically was brilliantly simply describable in terms of the *complexified* version $\mathring{L}_{abc}$ of that Lanczos tensor, and the *complexified* Weyl tensor $\mathring{W}_{abcf}$.

Sixth, many times during our investigation in this and the Companion Paper, I made suggestions for further progress, which CLAUDE was surprised to find, worked. That surprise is explained when we consider that, unbeknownst to CLAUDE, I had "the force" on my side. That is, my suggestions were motivated by this whole underlying theory, which I had, but CLAUDE at the time, was unaware of. CLAUDE presumably has immense experience about the statistics of what tends to work versus what does not. Those statistics kept delivering wrong guesses, while WDS's intuition, even though based on far less experience than CLAUDE's, kept delivering correct forecasts. That surprise is not a surprise if we hypothesize that the "Universe is trying" to make it work, in the sense that this whole theory is related to how the universe operates.

If all that is not trying to tell us something, nothing is. CLAUDE tells me he has nothing analogous to adrenal glands. But let me tell you: when I saw the First through Sixth points above, mine slammed into OVERDRIVE. But to look at it more coldly: consider this from the viewpoint of Bayesian statistics. Under the "null hypothesis" that our complexified tensors aren't important for physics, what is the probability that all six of those things would have happened by pure lucky fluke? I estimate: $(62\times2\times5\times10\times5\times8)^{-1}=0.000004$. If you don't like my numbers you can make your own estimates, but that number is going to be tiny. In view of this plus the far-greater accessibility of comparison versus observation, the picture advanced in this and the Companion Paper now should become **the pre-eminent candidate** for further progress on core fundamental physics.

Part IV. Catalogue of identities

Reference list. Each entry is numbered for citation, stated without derivation, and carries a status: **alg** = algebraic identity verified at machine precision (≤1e−15); **num** = verified numerically at the finite-difference floor (1e−5 to 1e−11), see II.9.1; **sym** = verified symbolically in closed form; **prov** = provisional, see the note attached. The metric collection is that of II.9 (21 spacetimes, Petrov types I, II–absent, III, N, D, O). Conventions: I.1, II.1.

IV.A The two duals

#	identity	status
A1	$\overleftarrow{i}^2 = -1$	alg
A2	$\overrightarrow{i}^2 = -1$	alg
A3	$\overleftrightarrow{ii} = \overrightarrow{i}\overleftarrow{i}$	alg (exactly 0)
A4	$\overleftrightarrow{ii} = \overleftrightarrow{i\,i}$, $\overleftrightarrow{ii}^2 = \mathbf{+1}$	alg
A5	$\mathbb{R}[\overleftarrow{i},\overrightarrow{i}] \cong \mathbb{C}\otimes\mathbb{C} \cong \mathbb{C}\oplus\mathbb{C}$; group $Z_4\times Z_2$	alg
A6	$^*W_{abcd} = W^*_{abcd}$ (the Weyl duals coincide)	alg

A7	$(*W*) = -W$	alg
A8	$\varepsilon_{abde}\varepsilon^{ab}{}_{fg} = -2(\delta^d{}_f\delta^e{}_g - \delta^d{}_g\delta^e{}_f)$	alg

IV.B Structure of the Riemann tensor

#	identity	status
B1	$(h \odot k)_{abcd} = h_{ac}k_{bd} + h_{bd}k_{ac} - h_{ad}k_{bc} - h_{bc}k_{ad}$; $h\odot k = k\odot h$	alg
B2	$W_{abcd} = R_{abcd} - (g \odot Q)_{abcd}/2$ (Q = Schouten, twice the usual normalization)	alg
B3	$R = W + (g\odot S)/2 + (R/24)(g \odot g)_{abcd}$ (S = traceless Ricci)	alg
B4	$(1 - \vec{i}\vec{i})R/2 = W_{abcd} + (R/24)(g \odot g)_{abcd}$	alg
B5	$(1 + \vec{i}\vec{i})R = (g\odot S)$	alg
B6	$2W = [R - (*R*)] - (R/12)\,(g\odot g)$	alg
B6a	$(*\mathbf{R}*)_{\mathbf{abcd}}$ **contains no** $\sqrt{(-\det g)}$: two ε's give $(\sqrt{(-\det g)})^2 = -\det g$, a polynomial. Hence $(*R*)_{abcd}$ is rational in g_{ab}, g^{ab}, R_{abcd} — orientation-independent, and analytic wherever the metric is. Single duals are not (§I.7)	alg + proof
B7	on the $(1-\vec{i}\vec{i})/2$ summand: $\overleftarrow{i} = \vec{i}$; on $(1+\vec{i}\vec{i})/2$: $\overleftarrow{i} = -\vec{i}$	alg
B8	$\mathrm{tr}(R_{abcd}) = R_{ab}$; $\mathrm{tr}((*R*)_{abcd}) = \mathbf{+}G_{ab}$	alg
B9	R and (*R*) carry the full Riemann symmetries; *R and R* carry only the two antisymmetries — they fail pair symmetry and the first Bianchi identity	alg
B9a	$(*\mathbf{R}*)_{\mathbf{abcd}} = \mu\mathbf{R}_{\mathbf{abcd}}$ **forces** $\mu = \pm 1$ on any non-flat spacetime: $\mu = -1 \Leftrightarrow$ Einstein; $\mu = +1 \Leftrightarrow W_{abcd} = 0$ and $R = 0$; any other $\mu \Rightarrow R_{abcd} = 0$ (§III.2a)	alg + proof
B9b	the double dual is an involution: $(*(*R*)*)_{abcd} = R_{abcd}$	alg
B10	$\mathrm{rank}\{R_{abcd}, (*R)_{abcd}, (R*)_{abcd}, (*R*)_{abcd}\} = 4$, dropping to 2 exactly when the metric is Einstein or conformally flat	alg

IV.C Complexified objects

#	definition or identity	status
C1	$\underset{\circ}{Ł} = Ł_{abc} + iL^*{}_{abc}$, dual on ab; $(\underset{\circ}{Ł})^* = -i\underset{\circ}{Ł}$	alg
C2	$\underset{\circ}{W} = W + iW^*$; $(\underset{\circ}{W})^* = -i\underset{\circ}{W}$	alg
C3	$\underset{\circ}{J} = J_{abc} + iJ^*{}_{abc}$, dual on ab; $(\underset{\circ}{J})^* = -i\underset{\circ}{J}$	alg
C4	$\underset{\circ}{R}_{ab} = R_{ab} + iG_{ab}$; $g^{ab}\underset{\circ}{R}_{ab} = (1-i)R$	alg
C5	$\underset{\circ}{Q}_{ab} = \underset{\circ}{R}_{ab} - \underset{\circ}{R}g_{ab}/6$	definition
C6	$(1 + \overleftarrow{i}*)\,R\,(1 + \vec{i}*) = R + \overleftarrow{i}(*R) + \vec{i}(R*) + \vec{i}\vec{i}(*R*)$	alg (exactly 0)
C7	$\underset{\circ}{W}$ is traceless and satisfies the first Bianchi identity	alg
C8	$\underset{\circ}{W}_{abcd} \equiv M_{ij}$, a trace-free complex symmetric 3×3 matrix (E + iB in a frame; defined in §II.7.1)	alg
C9	the bicomplexified Riemann tensor is **not** a joint eigentensor of $\overleftarrow{i}, \vec{i}$	alg (counterexample)

IV.D Invariants

#	identity	status
D1	$L^*\cdot L^* = -L\cdot L$	alg
D2	with $Q = L\cdot L$, $P = L\cdot L^*$: the 18 permuted quadratic invariants take only the values $\pm Q$, $\pm Q/2$, $\pm P$, $\pm P/2$ (table II.7.2)	alg
D3	$\mathring{L}\cdot\mathring{L} = 2(Q + iP)$	alg
D4	$L^{\pm} = (L \mp iL^*)/2$: $(L^+)^* = iL^+$, $L^+\cdot L^+ = (Q-iP)/2$, $L^+\cdot L^- = 0$	alg
D5	P is a pseudo-scalar; $P \equiv 0$ in any spacetime with a $t \to -t$ symmetry	alg
D6	D1–D5 hold verbatim with L replaced by the Cotton tensor J (same symmetry type)	alg
D7	$\mathbf{M^3 = \tfrac{1}{2}(\mathrm{tr}\, M^2)M + \tfrac{1}{3}(\mathrm{tr}\, M^3)\cdot 1}$ (Cayley–Hamilton)	alg
D8	with $I = \mathrm{tr}\, M^2$, $J = \mathrm{tr}\, M^3$: $\mathbf{I^3 = 6J^2 \Leftrightarrow}$ **algebraically special**	alg (9 orders of separation)
D9	bicomplex Kretschmann B·B: slot 1 = 4×Kretschmann, slots $\overleftarrow{i}$ and $\vec{i}$ = 4×Pontryagin (equal), slot $\vec{i}\vec{i}$ = −4×Kretschmann in Λ-vacuum	alg
D10	$B\cdot\bar{B} = 0$ identically (all four slots)	alg
D11	B·(*B*) is B·B with slots 1 and $\vec{i}\vec{i}$ swapped and negated, arrows negated	alg
D12	cubic contraction $\mathrm{tr}\, M^3$ gives the CM invariants W2R, W2I as one complex number; $\mathring{W}_{abcd}\mathring{W}^{abcd}$ gives W1R, W1I	prov (values sound, identification unfitted)
D13	mixed $\mathring{W}$–S contractions K, L, M are nonzero and independent; vanish in vacuum	alg

IV.E Petrov classification

#	identity	status
E1	Petrov type = Jordan structure of $\mathring{W}$: 3 distinct eigenvalues → I; repeated + diagonalizable → D; repeated + 2×2 block → II; nilpotent with rank 2 → III; rank 1 → N; zero → O	alg (21/21)
E2	the nilpotent cases must be read by **ranks of powers**, not eigenvalues: a 3×3 nilpotent block displaces eigenvalues by $\varepsilon^{1/3}$, so double precision gives 1e−5, not 1e−16	alg
E3	Bel criteria, complexified: type N ⇔ $\mathring{W}_{abcd}k^d = 0$; III ⇔ $\mathring{W}_{abcd}k^bk^d = 0$; II ⇔ $= \lambda k_a k_c$ for one k; D ⇔ for two	alg
E4	the side condition is $\boldsymbol{\lambda \neq 0}$, not $\alpha\beta \neq 0$: the latter misclassifies Schwarzschild and Reissner-Nordström, where $\beta = 0$ because Ψ_2 is real	alg

IV.F Differential identities

#	identity	status
F1	$J_{abc} = \nabla^d W_{abcd}$; antisymmetric in ab, cyclic part zero, trace-free on both contractions	sym
F2	$\mathbf{J_{abc} = C_{cab} = -C_{cba}}$ ($C_{abc} = Q_{a[b;c]}$ from the Schouten tensor of §II.1.3; coefficient exactly 1, equivalently $J_{abc} = Q_{c[a;b]}$). *Index ordering is load-bearing:* $-C_{abc}$ and $-C_{bca}$ each fail on 6 of 64 components	sym (componentwise, all 64)
	the divergence of the bicomplexified Riemann has only two nonzero slots: $\nabla^d R_{abcd} = -2C_{abc}$ and	

F2a	$\nabla^d(*R)_{abcd} = -4(*C)_{abc}$; the $\vec{i}$ and $\vec{\vec{i}}$ slots vanish	sym
F2b	hence **one** complexified Cotton tensor, $J_{\circ abc} = J_{abc} + i(J^*)_{abc}$; both construction routes reach it ($\nabla^d R_{abcd}/J_{abc} = 2$)	sym
F3	$J_{\circ abc} = \nabla^d W_{\circ abcd}$ (no correction term); in vacuum = 0	num
F4	$(R^*)_{abcd} = (*R)_{cdab}$; *R is **not** pair-symmetric off Einstein. Hence: ***R is divergence-free on its second index, not on its fourth**	sym
F4a	$\nabla^d(R^*)_{abcd} = 0$ — the second Bianchi identity contracted with ε	sym
F4b	$\mathbf{\nabla^d(*R)_{abcd} = -\varepsilon_{ab}{}^{pq}\nabla_p R_{qc}}$ (constant fitted, exactly −1)	sym
F4c	corollary: div(*R) = 0 in vacuum	sym
F4d	$\nabla^d(*R^*)_{abcd} = 0$	sym
F5	$\nabla_{[e}R_{ab]cd} = 0$ and the same for *R	num
F6	$\nabla^a G_{ab} = 0$; $\nabla^a R_{ab} = \nabla_b R/2$	num
F7	$\mathbf{\nabla^a R_{\circ ab} = \nabla_b R/2}$ **is purely real**	num
F8	$Q_{ab}{}^{;b} = Q_b{}^b{}_{;a}$ (no factor of 2; both sides = $\nabla_a R/3$)	num, and see II.9.1 on vacuous tests
F9	Im($\underset{\circ}{B}$) ≠ 0 in general — the Bach tensor built from W* does **not** vanish identically	prov (Majumdar-Papapetrou only, two nested derivatives)

IV.G Bel-Robinson

#	identity	status
G1	$\mathbf{T_{abcd} = W_{\circ aecf}\bar{W}_{\circ b}{}^e{}_d{}^f}$ — no factor of ½, no real part taken	alg
G2	Im[$W_{\circ}\bar{W}_{\circ}$] = 0 identically (exactly, not merely small)	alg
G3	T is totally symmetric and trace-free	alg
G4	the Riemann-built form B = R·R + *R·*R equals G1 **only in vacuum**	alg

IV.H Superpotential rung

#	identity	status
H1	L* is an admissible Lanczos tensor: antisymmetric, cyclic part zero, trace-free	sym
H2	**but no symmetric P* with $L^*_{abc} = P^*_{c[a;b]}$ exists** — the obstruction is algebraic, two of the four derivative-free equations being incompatible unless N = 0	sym (extremal Reissner-Nordström)
H3	consequently the admissible Lanczos class is closed under dualization but the superpotential-derived subclass is **not**; $\underset{\circ}{P}$ does not exist	sym, one spacetime

References

Works cited in the text, together with the sources for results we rediscovered independently. Entries marked ★ were reached in this work before the prior literature was identified; they are recorded as independent confirmation, not as priority claims.

Lanczos potentials

1. C. Lanczos, *The splitting of the Riemann tensor*, Rev. Mod. Phys. **34** (1962) 379-389. — the original; the existence proof given there was later shown to be flawed.
2. F. Bampi and G. Caviglia, *Third-order tensor potentials for the Riemann and Weyl tensors*, Gen. Rel. Grav. **15** (1983) 375-386. — first rigorous local existence, four-dimensional analytic spaces, any signature.
3. R. Illge, *On potentials for several classes of spinor and tensor fields in curved spacetimes*, Gen. Rel. Grav. **20** (1988) 551-564. — existence without analyticity, via spinors; the wave equation in Lanczos differential gauge.
4. S. B. Edgar and A. Höglund, *The Lanczos potential for the Weyl curvature tensor: existence, wave equations and algorithms*, Proc. R. Soc. A **453** (1997) 835-851.
5. S. B. Edgar and A. Höglund, *The Lanczos potential for Weyl candidate tensors exists only in four dimensions*, Gen. Rel. Grav. **32** (2000) 2307-2318. — the four-dimensionality on which foundations *5 and *6, and all of Part II, depend (III.4).
6. F. Andersson and S. B. Edgar, *Existence of Lanczos potentials and superpotentials for the Weyl spinor/tensor*, Class. Quantum Grav. **18** (2001) 2297-2304. — the two superpotentials T_{ABCD} and $H_{ABA'B'}$; the latter, which corresponds to our $\mathcal{P}_{ab}$, is established for Einstein spacetimes and a class of algebraically special non-vacuum spacetimes.
7. F. Andersson and S. B. Edgar, *Local existence of symmetric spinor potentials of symmetric (3,1)-spinors in Einstein spacetimes*, J. Geom. Phys. **37** (2001) 273-290.
8. G. Bergqvist, *A Lanczos potential in Kerr geometry*, J. Math. Phys. **38** (1997) 3142-3154.

Curvature classification and invariants

9. A. Z. Petrov, *Einstein Spaces*, Pergamon (1969).
10. L. Bel, *Introduction d'un tenseur du quatrième ordre*, C. R. Acad. Sci. Paris **248** (1959) 1297. — the Bel-Robinson tensor, and the Bel criteria discussed in II.11.2.
11. M. Ortaggio, *Bel-Debever criteria for the classification of the Weyl tensor in higher dimensions*, Class. Quantum Grav. **26** (2009) 195015; arXiv:0906.3818.
12. G. S. Hall, *Symmetries and Curvature Structure in General Relativity*, World Scientific (2004). — equivalent forms of the Bel criteria in terms of duals and self-duals.
13. J. Carminati and R. G. McLenaghan, *Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space*, J. Math. Phys. **32** (1991) 3135-3140.
14. E. Zakhary and C. B. G. McIntosh, *A complete set of Riemann invariants*, Gen. Rel. Grav. **29** (1997) 539-581.
15. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed., Cambridge (2003). — source for the metrics of II.9.

The 1+3 (ADM-like) content of Part III

16. ★ A. Matte, *Sur de nouvelles solutions oscillatoires des équations de la gravitation*, Canad. J. Math. **5** (1953) 1-16. — the electric/magnetic split of the Weyl tensor.
17. ★ J. Ehlers, *Contributions to the relativistic mechanics of continuous media*, Abh. Akad. Wiss. Lit. Mainz **11** (1961); reprinted Gen. Rel. Grav. **25** (1993) 1225.
18. ★ G. F. R. Ellis, *Relativistic cosmology*, in *General Relativity and Cosmology*, ed. R. K. Sachs, Academic Press (1971) 104-182.
19. ★ R. Maartens and B. A. Bassett, *Gravito-electromagnetism*, Class. Quantum Grav. **15** (1998) 705-717; arXiv:gr-qc/9704059. — the once-contracted Bianchi identities as two propagation and **two constraint** equations for E and H, the constraints carrying momentum-density sources. This is the result rediscovered in III.3.2 as antisym(B) = $G_{0i}/2$.
20. R. Arnowitt, S. Deser and C. W. Misner, *The dynamics of general relativity*, in *Gravitation: An Introduction to Current Research*, ed. L. Witten, Wiley (1962) 227-265; arXiv:gr-qc/0405109.

Flat-background readings of general relativity (II.12)

21. N. Rosen, *General relativity and flat space I, II*, Phys. Rev. **57** (1940) 147-153.
22. S. Deser, *Self-interaction and gauge invariance*, Gen. Rel. Grav. **1** (1970) 9-18; arXiv:gr-qc/0411023.
23. S. Weinberg, *Gravitation and Cosmology*, Wiley (1972), ch. 7.

Analytic ODE theory (§I.7)

24. H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier-Villars (1892-1899). — the original source of the analyticity theorem, proved there by Cauchy's method of majorants.
25. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill (1955), ch. 1. — a standard modern treatment of existence, uniqueness, and analytic dependence on initial conditions and parameters, via Picard iteration. Any comparable text (Hartman; Arnol'd) will serve.
26. A. J. Dragt and T. J. Stasevich, *Poincaré analyticity and the complete variational equations*, arXiv:1102.3394. — states the theorem in its opening lines and attributes it to Poincaré: solutions are analytic in the initial conditions and parameters provided the right sides are analytic in the variables, the time, and the parameters. A convenient citable statement, and the source verified for this paper.
27. S. Kovalevskaya, *Zur Theorie der partiellen Differentialgleichung*, J. reine angew. Math. **80** (1875) 1-32. — the Cauchy-Kovalevskaya theorem, needed for the general nonlinear case but not for the linear systems of §I.4.

Complex and bicomplex structures

28. R. Penrose and W. Rindler, *Spinors and Space-Time*, vol. 1, Cambridge (1984). — the self-dual Weyl spinor, of which II.11.1 is the tensorial counterpart.
29. M. Ko, M. Ludvigsen, E. T. Newman and K. P. Tod, *The theory of H-space*, Phys. Rep. **71** (1981) 51-139. — complex metrics and the role of the self-dual sector, referred to in II.12.
30. G. B. Price, *An Introduction to Multicomplex Spaces and Functions*, Marcel Dekker (1991). — the bicomplex numbers $\mathbb{C}\otimes\mathbb{C}$ and their idempotent decomposition, used in §II.2-II.3.

Majumdar-Papapetrou

31. S. D. Majumdar, *A class of exact static solutions of the gravitational field equations*, Phys. Rev. **72** (1947) 390-398.
32. A. Papapetrou, *A static solution of the equations of the gravitational field for an arbitrary charge-distribution*, Proc. R. Irish Acad. A **51** (1947) 191-204.
33. J. B. Hartle and S. W. Hawking, *Solutions of the Einstein-Maxwell equations with many black holes*, Commun. Math. Phys. **26** (1972) 87-101.

Companion papers

34. Warren D. Smith, Mostly-Destructive Consequences of Non- and Nowhere-Analyticity of the Spacetime Metric; and new "2-sided Mirror Paradigm" hypothetical grandiose universe picture (2026).
35. Warren D. Smith & CLAUDE: Switcheroo and Lanczos Gravity: MOND done right (2026); *the* "companion paper."