Reciprocal Lie-Algebra Structures: Paired Closure, Exchange Symmetry, and the Interpretation of Ontological Two-Sidedness

Stephen P. Smith, October 2026

Abstract. This paper examines whether reciprocal representation provides a mathematically coherent way to express two distinguishable but structurally equivalent sectors without privileging either as primary. The starting point is a pair of isomorphic Lie algebras and their representation within a direct-sum algebra. A natural sector-exchange map is shown to be an involutive Lie-algebra automorphism, while the paired representation forms a Lie subalgebra fixed pointwise by that exchange. The direct sum therefore supplies a precise realization of reciprocal redundancy without requiring nonzero cross-sector brackets. The paper then considers a more general ambient algebra, where closure of the paired representation constrains the cross-sector bracket to be symmetric in its corresponding arguments. The Jacobi identity imposes additional consistency requirements but does not, by itself, require cross-sector brackets to vanish. The limiting case of coincident embeddings is examined, as is an illustrative example in which paired closure coexists with a nonzero cross-sector bracket. The resulting mathematical distinctions are related cautiously to a philosophical proposal of ontological two-sidedness. An appendix further examines how reciprocal exchange may be carried through the curvature construction to an exchange-invariant Yang–Mills Lagrangian, subject to appropriate compatibility conditions. The algebra does not prove that reality has a bilateral ontology; it provides a formal representation in which two distinguishable sides can be related reciprocally without assigning either side ontological priority.

Keywords: Lie algebras; direct sums; reciprocal exchange; paired closure; Jacobi identity; redundancy; bilateral representation; ontological two-sidedness

1. Introduction

Mathematical representations often distinguish structures that are equivalent in their formal organization. Two representations may differ in their position, orientation, or role while preserving the same underlying relations. This raises a question that is mathematical in its immediate formulation but also has a broader philosophical significance: can two distinguishable representations be organized reciprocally, without the formal structure privileging one as the original and the other as derivative?

The present paper investigates this question using Lie algebras (cf. Erdmann and Wildon, 2006). Its immediate objective is limited. It does not attempt to establish a new physical theory, derive a new law of gravitation, or prove an ontology from mathematical

formalism. Rather, it identifies a coherent algebraic construction that may serve as a disciplined formal analogue of reciprocal or bilateral organization.

The central construction begins with two isomorphic Lie algebras. Their direct sum preserves their respective internal bracket structures while allowing the two sectors to be exchanged by an involutive automorphism. A paired representation of the two sectors is invariant under that exchange. This combination is important: the sectors remain distinguishable in the representation, yet the paired object does not privilege either sector under the exchange operation.

A second question concerns whether the construction must remain a direct sum. In a direct sum, cross-sector brackets vanish by definition. In a more general ambient Lie algebra, however, elements belonging to different sectors may have nonzero brackets. Requiring the paired representation to remain closed then imposes a symmetry condition on those cross-sector brackets. The Jacobi identity further constrains their compatibility with the rest of the algebra.

These results establish a useful distinction between reciprocal exchange and nontrivial cross-sector interaction. The first is realized directly by the direct-sum construction; the second is an additional possibility requiring further algebraic conditions. Neither should be confused with the philosophical interpretation that motivates the investigation.

2. Isomorphic Lie Algebras and Their Paired Representation

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over the same field, and let

$$\varphi : \mathfrak{g} \to \mathfrak{g}'$$

be a Lie-algebra isomorphism. Thus, for all $X, Y \in \mathfrak{g}$,

$$\varphi([X, Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{g}'}.$$

The isomorphism establishes that the two algebras possess the same abstract bracket structure. It does not require them to be represented as the same subset of a larger algebra.

The simplest setting in which to represent both sectors together is their direct sum:

$$\mathfrak{L} = \mathfrak{g} \oplus \mathfrak{g}'.$$

An element of $\mathfrak{L}$ is an ordered pair (X, U), where $X \in \mathfrak{g}$ and $U \in \mathfrak{g}'$. The Lie bracket is defined componentwise:

$$[(X,U),(Y,V)]_{\mathfrak{L}} = ([X,Y]_{\mathfrak{g}},[U,V]_{\mathfrak{g}'}).$$

The direct sum therefore retains the internal Lie-algebraic structure of each sector. By definition, there are no cross-sector contributions to the bracket:

$$[(X,0),(0,U)]_{\mathfrak{L}} = (0,0).$$

This vanishing is not a consequence of a dynamical cancellation or a theorem about physical interactions. It is part of the direct-sum construction itself.

Now define the paired map $\iota : \mathfrak{g} \to \mathfrak{L}$ by

$$\iota(X) = (X,\varphi(X)).$$

Its image is the paired subset

$$\mathfrak{L}_{\mathrm{r}} = \{\,(X,\varphi(X)) : X \in \mathfrak{g}\,\}.$$

The image consists of those elements in which the two sectors are related through the isomorphism φ. It is the graph of φ within the direct-sum algebra (Humphreys 1972).

The bracket of two paired elements is

$$[\iota(X),\iota(Y)]_{\mathfrak{L}} = ([X,Y]_{\mathfrak{g}},[\varphi(X),\varphi(Y)]_{\mathfrak{g}'}).$$

Since φ preserves the bracket,

$$[\iota(X),\iota(Y)]_{\mathfrak{L}} = ([X,Y]_{\mathfrak{g}},\varphi([X,Y]_{\mathfrak{g}})) = \iota([X,Y]_{\mathfrak{g}}).$$

Consequently, $\mathfrak{L}_{\mathrm{r}}$ is a Lie subalgebra of $\mathfrak{L}$. The paired representation is closed under the Lie bracket, and the map ι preserves the Lie-algebraic structure of $\mathfrak{g}$.

This result is elementary but central to the present proposal. It establishes that two isomorphic sectors can be represented together in a single algebraic structure, with their paired representation preserving the original bracket relations.

3. Reciprocal Exchange as an Involutive Automorphism

The direct sum admits a natural operation that exchanges its two sectors.

Define $S : \mathfrak{L} \to \mathfrak{L}$ by $S(X,U) = (\varphi^{-1}(U),\varphi(X))$.

The first component of the image is obtained by mapping the second sector back into $\mathfrak{g}$; the second component is obtained by mapping the first sector into $\mathfrak{g}'$.

Applying S twice gives

$$S^2(X,U) \;=\; S(\varphi^{-1}(U),\varphi(X)) \;=\; (X,U).$$

Hence,

$S^2 \;=\; I$, where I is the identity map on $\mathfrak{L}$. The exchange is an involution.

The operation also preserves the Lie bracket. For any $(X,U),(Y,V) \;\in\; \mathfrak{L}$,

$$S([(X,U),(Y,V)]\mathfrak{L}) \;=\; S(([X,Y]\mathfrak{g},[U,V]\mathfrak{g}')) \;=\; (\varphi^{-1}([U,V]\mathfrak{g}'),\varphi([X,Y]\mathfrak{g})).$$

Because φ is an isomorphism,

$$\varphi^{-1}([U,V]\mathfrak{g}') = \;[\varphi^{-1}(U),\varphi^{-1}(V)]\mathfrak{g},$$

and

$$\varphi([X,Y]\mathfrak{g}) \;=\; [\varphi(X),\varphi(Y)]\mathfrak{g}'.$$

Therefore, $S([(X,U),(Y,V)]\mathfrak{L}) \;=\; [S(X,U),S(Y,V)]\mathfrak{L}$.

Thus, S is a Lie-algebra automorphism as well as an involution.

The paired representation has a particularly important property. Applying S to $\iota(X)$ gives

$$S(\iota(X)) \;=\; S(X,\varphi(X)) \;=\; (\varphi^{-1}(\varphi(X)),\varphi(X)) \;=\; (X,\varphi(X)).$$

Therefore, $S(\iota(X)) \;=\; \iota(X)$.

The paired representation is fixed pointwise by the exchange involution.

This result expresses a precise mathematical relationship between distinction and reciprocity. The two components remain distinguishable as coordinates of the direct-sum representation, but the exchange operation does not privilege either component. The paired element is invariant under their interchange.

The result does not depend on φ being its own inverse. In general, φ maps from $\mathfrak{g}$ to $\mathfrak{g}'$, while φ^{-1} maps in the opposite direction. It is the combined exchange S, rather than φ alone, that is self-inverse.

The direct sum therefore provides an explicit and economical realization of reciprocal exchange. No nonzero cross-sector bracket is required for this result.

4. Paired Closure in a General Ambient Lie Algebra

The direct-sum construction establishes a baseline. A more general question arises when two isomorphic Lie algebras are embedded as subalgebras of a common ambient Lie algebra $\mathfrak{L}$, whose bracket is not assumed to be componentwise.

Let $j : \mathfrak{g} \to \mathfrak{L}$ and $j' : \mathfrak{g}' \to \mathfrak{L}$ be injective Lie-algebra homomorphisms. Define the paired map

$$\iota(X) = j(X) + j'(\varphi(X)).$$

Its image is

$$\mathfrak{L}_{\mathrm{r}} = \{\iota(X) : X \in \mathfrak{g}\}.$$

We require paired closure:

$$[\iota(X), \iota(Y)]_{\mathfrak{L}} = \iota([X, Y]_{\mathfrak{g}})$$

for all $X, Y \in \mathfrak{g}$.

Expanding the left-hand side gives

$$[\iota(X), \iota(Y)]_{\mathfrak{L}} = [j(X), j(Y)]_{\mathfrak{L}} + [j'(\varphi(X)), j'(\varphi(Y))]_{\mathfrak{L}}$$
$$+[j(X), j'(\varphi(Y))]_{\mathfrak{L}} + [j'(\varphi(X)), j(Y)]_{\mathfrak{L}}.$$

Since j and j' preserve their respective internal brackets, the first two terms sum to

$$j([X, Y]_{\mathfrak{g}}) + j'(\varphi([X, Y]_{\mathfrak{g}})) = \iota([X, Y]_{\mathfrak{g}}).$$

It follows that paired closure is equivalent to the cancellation of the two cross-sector contributions:

$$[j(X), j'(\varphi(Y))]_{\mathfrak{L}} + [j'(\varphi(X)), j(Y)]_{\mathfrak{L}} = 0.$$

Define the cross-sector relation

$$B(X, Y) = [j(X), j'(\varphi(Y))]_{\mathfrak{L}}.$$

By antisymmetry of the Lie bracket,

$$[j'(\varphi(X)), j(Y)]_{\mathfrak{L}} = -B(Y, X).$$

The paired-closure condition therefore becomes

$$B(X,Y) - B(Y,X) = 0,$$

or equivalently,

$$B(X,Y) = B(Y,X).$$

This is the principal constraint obtained from paired closure in the general ambient setting.

The symmetry of B is not a claim that the Lie bracket itself has become symmetric. The ambient Lie bracket remains antisymmetric. Rather, B is defined by placing its arguments in corresponding but different sectors, and closure requires symmetry under interchange of those corresponding arguments.

Nor does this condition require $B(X,Y) = 0$. It permits nonzero cross-sector brackets, provided they satisfy the symmetry condition and are compatible with the other requirements of the ambient Lie algebra.

The distinction is important. Reciprocal representation does not, by itself, require interaction between the sectors. But neither does the closure condition alone rule out nontrivial cross-sector relations.

5. The Direct Sum as the Reference Construction

The direct sum is more than a preliminary example. It is the canonical reference construction for the present investigation because it realizes the central requirements in a transparent form.

In $\mathfrak{g} \oplus \mathfrak{g}'$, the paired representation is

$$\iota(X) = (X, \varphi(X)),$$

the bracket is componentwise, and the cross-sector brackets vanish. Paired closure follows directly from the bracket-preserving property of φ. The exchange map

$$S(X,U) = (\varphi^{-1}(U), \varphi(X))$$

is an involutive automorphism, and the paired graph is fixed pointwise by S.

The construction thus realizes three properties together:

1. The sectors are structurally equivalent but represented as distinguishable components.
2. Their paired representation is closed under the Lie bracket.
3. Their exchange is an involutive automorphism under which the paired representation is invariant.

These properties do not depend on nonzero cross-sector coupling. The direct sum shows that reciprocal exchange and nontrivial interaction are logically distinct.

This is a useful conclusion in its own right. If the aim is to represent reciprocal redundancy, a direct sum already provides a mathematically coherent model. A nontrivial cross-sector bracket would add further structure, but it is not needed to establish the existence of reciprocal exchange.

The direct sum also clarifies what is meant by redundancy in this context. The two sectors are not identical as coordinate components; rather, their abstract structures correspond through φ, and the combined representation treats them symmetrically under S. The representation contains two distinguishable components without making either component the privileged representative of the pair.

6. Cross-Sector Relations and the Jacobi Identity

The direct sum satisfies the Jacobi identity because both component algebras satisfy it. The identity holds componentwise:

$$[A,[C,D]] + [C,[D,A]] + [D,[A,C]] = 0.$$

In a more general ambient Lie algebra, Jacobi continues to hold, but it imposes compatibility conditions on the cross-sector brackets. The symmetry condition $B(X,Y) = B(Y,X)$ obtained from paired closure is therefore not the only constraint on a possible nontrivial extension.

For example, consider a mixed triple consisting of $j(X)$, $j(Y)$, and $j'(\varphi(Z))$. The Jacobi identity gives

$$[j(X),[j(Y),j'(\varphi(Z))]] + [j(Y),[j'(\varphi(Z)),j(X)]] + [j'(\varphi(Z)),[j(X),j(Y)]] = 0.$$

Using the definition of B and the bracket-preserving property of j, this can be written as

$$[j(X),B(Y,Z)] - [j(Y),B(X,Z)] - B([X,Y],Z) = 0.$$

Other choices of mixed triples produce additional constraints. Together, these relations restrict which cross-sector brackets can occur within a coherent ambient Lie algebra.

The Jacobi identity does not automatically force B to vanish. Nor does the symmetry of B guarantee that a nonzero solution exists for every choice of algebras and embeddings. Those conclusions depend on the particular structure under consideration.

For the direct-sum construction, the cross-sector relation is zero, and the Jacobi identity is automatically satisfied. The direct sum therefore supplies an unambiguous baseline against which any proposed nontrivial extension can be assessed.

In the interpretation advanced here, Jacobi may be regarded as a formal constraint on the coherence of nested relations. Pairwise relations can be nonzero, but their organization must remain compatible with the algebra as a whole. This is an interpretive description of the identity's structural role, not a claim that Jacobi is a physical regulator or that it proves any general law of evolution.

Possible extension: Idempotent mediation of the Lie bracket. The reciprocal construction may admit an alternative formulation using an R-mediated bracket, defined on a suitable matrix space by $[A,B]_{\mathrm{r}} = ARB - BRA$, where $R^2 = R$ (cf. Santilli 1968). The associated product $A \star B = ARB$ is associative, so its commutator satisfies the Jacobi identity. Idempotency additionally permits R to be interpreted as a projection onto a selected subspace. This raises the possibility that reciprocal relations could be organized through a common projection operator, rather than solely through a symmetric cross-sector relation $B(X,Y)$. In particular, an R-mediated cross-sector operator might satisfy an analogous symmetry condition under paired closure, provided the sector embeddings preserve the R-mediated bracket. Whether this construction offers a useful alternative realization of the proposed middle-term, or a formal analogue of a relational boundary such as a Markov blanket, remains an open question. It does not alter the present paper's results, but suggests a potentially fruitful extension of its algebraic framework.

7. Coincident Embeddings and the Abelian Case

A limiting case deserves particular attention. Suppose the embeddings coincide after identifying the two algebras through φ:

$$j = j' \circ \varphi.$$

Then

$$j(X) = j'(\varphi(X))$$

for every $X \in \mathfrak{g}$. The two embeddings no longer define distinct embedded subalgebras of $\mathfrak{L}$; they represent the same image.

Under this condition, the cross-sector relation becomes

$$B(X,Y) = [j(X), j'(\varphi(Y))]_{\mathfrak{L}} = [j(X), j(Y)]_{\mathfrak{L}} = j([X,Y]_{\mathfrak{g}}).$$

But paired closure requires $B(X,Y) = B(Y,X)$.

Consequently, $j([X,Y]_{\mathfrak{g}}) = j([Y,X]_{\mathfrak{g}})$.

Antisymmetry of the Lie bracket gives $j([X,Y]_{\mathfrak{g}}) = -j([X,Y]_{\mathfrak{g}})$, so $2j([X,Y]_{\mathfrak{g}}) = 0$.

Over a field of characteristic zero, this implies $j([X,Y]_{\mathfrak{g}}) = 0$. Since j is injective,

$$[X,Y]_{\mathfrak{g}} = 0$$

for every $X, Y \in \mathfrak{g}$. Thus, $\mathfrak{g}$ is abelian, and so is $\mathfrak{g}'$.

The conclusion is specific: coincident embeddings, together with the paired-closure condition, force the represented sector algebra to be abelian. They do not force the entire ambient algebra $\mathfrak{L}$ to be abelian.

An abelian subalgebra can exist inside a non-Abelian ambient algebra. For example, one-dimensional subalgebras of a non-Abelian Lie algebra are abelian, even though brackets between elements of different subalgebras may be nonzero. Consequently, the abelian result does not make the construction mathematically vacuous.

The Abelian case is also relevant to the electromagnetic motivation (Smith 2026) for this work. An Abelian sector can be represented redundantly alongside a corresponding sector without requiring a non-Abelian interaction between them. The resulting formal structure may be of interpretive interest precisely because reciprocal representation does not depend on a nonzero cross-sector bracket.

8. A Nonzero Cross-Sector Bracket Can Coexist with Paired Closure

Although the direct sum is the reference construction, it is useful to establish that paired closure does not universally require cross-sector brackets to vanish.

Consider the two-dimensional Lie algebra with basis elements x and u and bracket

$$[x,u] = u,$$

$$[x,x] = 0,$$

$$[u,u] = 0.$$

The bracket is extended by bilinearity and antisymmetry. This is a solvable Lie algebra.

Let $\mathfrak{g}$ be the one-dimensional abelian algebra spanned by x, and let $\mathfrak{g}'$ be the one-dimensional abelian algebra spanned by u. These algebras are isomorphic. Define the embeddings into the two-dimensional ambient algebra by $j(x) = x, j'(u) = u$, and let $\varphi(x) = u$. The paired element is $\iota(x) = x + u$.

Since $\mathfrak{g}$ is one-dimensional and abelian, its only internal bracket is $[x, x] = 0$. The paired closure condition is therefore $[\iota(x), \iota(x)] = \iota([x, x]) = 0$.

By antisymmetry,

$$[x + u, x + u] = [x, x] + [x, u] + [u, x] + [u, u] = 0 + u - u + 0 = 0.$$

Thus, the paired subset is closed. Nevertheless, the cross-sector relation is nonzero:

$$B(x, x) = [x, u] = u \neq 0.$$

The two cross-sector terms cancel in the paired bracket, even though each is nonzero.

This example establishes a limited but useful point: nonzero cross-sector relations can coexist with paired closure in a suitable ambient Lie algebra. It does not establish that every pair of isomorphic Lie algebras admits such an extension, nor that the example possesses an exchange involution that swaps x and u while preserving the ambient bracket. Paired closure and sector-exchange symmetry remain distinct requirements.

The example therefore complements, rather than replaces, the direct-sum construction. The direct sum gives the cleanest realization of reciprocal exchange; the nonzero-bracket example shows that closure is not synonymous with vanishing cross-sector relations.

9. Relation to the Electromagnetic Motivation

The motivation for this construction is partly connected to earlier work (Smith 2026) in which a redundant sector was introduced to represent opposite orientations within an electromagnetic framework. In that setting, the direct sum offered a way to place corresponding sectors side by side while retaining their respective algebraic identities.

The elementary identity $[X, -X] = 0$ follows from bilinearity and antisymmetry of the Lie bracket: $[X, -X] = -[X, X] = 0$.

By itself, this is a formal algebraic identity. It does not demonstrate a physical annihilation process, nor does it establish a physical interaction between matter and antimatter. Such interpretations require additional definitions and physical arguments.

The more general motivation is the possibility of representing a structure together with a reciprocal counterpart without deciding, from the formalism alone, that one is ontologically prior to the other. In the direct-sum construction, this idea is expressed not by the identity $[X, -X] = 0$ alone, but by the combination of structural equivalence, paired closure, and an exchange involution that leaves the paired representation invariant.

This distinction also matters when considering non-Abelian gauge theories. A direct-sum representation can be constructed for isomorphic non-Abelian Lie algebras as well as Abelian ones. However, the fact that the original Yang–Mills framework employs a non-Abelian gauge algebra does not make that algebra solvable, nor does the present construction establish a new Yang–Mills theory. The direct sum and its exchange map are algebraic structures; any physical interpretation must be justified separately.

Accordingly, the electromagnetic connection serves as motivation for the formal question, not as evidence that the reciprocal construction is already a physical description of nature.

The remaining question is what philosophical significance, if any, this formally coherent construction may have beyond its immediate mathematical and physical motivations.

10. Reciprocal Representation and Ontological Two-Sidedness

The mathematical distinction between the two sectors does not depend on the cross-sector bracket $B(X,Y)$ being nonzero. In the direct-sum construction, the sectors are distinguished by their separate coordinates; in the general ambient construction, the distinction depends on whether the embeddings remain distinct under the identification being used. Paired closure in the general ambient construction requires the cross-sector relation $B(X,Y)$ to be symmetric, not to vanish. These results establish conditions for a coherent paired representation, but they do not determine whether the represented distinction persists in the ontology or is transformed within a more encompassing structure.

The same caution applies to relational observation. A measurement involves an interaction, comparison, or correspondence between what is measured and the conditions under which it is measured. Yet the relational character of measurement does not, by itself, settle the ontology of what exists. Likewise, a representation that distinguishes two sides does not establish that reality contains two independently existing sides.

This paper does not provide a proof of ontological symmetry. The direct sum is a mathematical construction, and its symmetry follows from its definition. It does not establish that reality has the same structure. Moreover, the distinction between representation and reality raises a further question: whether the explicit bilateral distinction of the representation must persist in the ontology that the representation is intended to illuminate.

The two sectors are distinguishable in the ordered pair $(X,\varphi(X))$, but this representational distinction need not imply that reality consists of two independently existing sides, or that the distinction persists unchanged at a more encompassing level of description. A unified structure might reconcile the represented sides so that their

explicit distinction is lost, transformed, or retained only implicitly. The bilateral representation may therefore express reciprocal organization without serving as a literal inventory of what exists.

This possibility points toward a more demanding interpretation of ontological two-sidedness. The proposal is not merely that two represented sides should be treated symmetrically, with neither regarded as primary. It also asks whether the apparent bilateral distinction is provisional: a feature of how a relational structure is represented, rather than a final account of its ontological constitution. The deeper organization might remain bilateral in an implicit sense, or its bilateral character might be transformed beyond explicit recognition. The present algebra does not decide between these possibilities.

In this sense, unification may have a conceptual price. A representation that makes reciprocal relations explicit can clarify their symmetry, yet a more encompassing account may no longer preserve those relations as a visibly bilateral arrangement. What is gained in integration may therefore coincide with a loss of explicit distinction. This is not a demonstrated consequence of the Lie-algebra construction, but a philosophical possibility that motivates the distinction between bilateral representation and ontological two-sidedness.

The image of a perfect mirror may help convey the initial symmetry: neither side is established as the original and the other as derivative. Yet the analogy has a further limitation. The deeper significance may lie not in preserving two mirror images as permanent ontological counterparts, but in asking whether their reciprocal relation points toward an organization more encompassing than the two images themselves. The analogy is suggestive, not a mathematical or physical conclusion.

The central philosophical suggestion is therefore twofold. Reciprocal exchange provides a formal way to represent distinguishable components without assigning either formal priority. Ontological two-sidedness asks the further question of whether reciprocal organization remains fundamental when the explicit distinctions of a representation are reconciled into a more encompassing structure. Whether the bilateral character persists, becomes implicit, or is transformed in that transition remains an open interpretive question.

The bilateral construction may thus be interpreted as a way of representing two mutually related sectors whose reciprocal organization permits a coherent whole, even as their explicit distinction recedes. The idea of homeostasis offers one possible philosophical analogy for this relationship, but it is not a result established by the algebra. Similarly, the perfect reflection evoked by “I Am That I Am” may suggest self-sufficiency or a ground that does not depend on an external counterpart (Smith 2022); it is a philosophical evocation, not a conclusion derived from the mathematical construction. Ontological two-sidedness is therefore proposed as a mode of inquiry into reciprocal distinction and possible integration, not as a theory of everything.

11. Conclusion

This paper has established a coherent algebraic construction for reciprocal representation.

Given two isomorphic Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, their direct sum admits the involutive automorphism

$$S(X, U) = (\varphi^{-1}(U), \varphi(X)),$$

with $S^2 = I$. The paired representation

$$\iota(X) = (X, \varphi(X))$$

is a Lie subalgebra and is fixed pointwise by S. Reciprocal exchange is therefore realized without requiring nonzero cross-sector brackets.

In a more general ambient Lie algebra, paired closure requires the cross-sector relation

$$B(X, Y) = [j(X), j'(\varphi(Y))]$$

to satisfy

$$B(X, Y) = B(Y, X).$$

This symmetry is a consequence of paired closure under the stated embedding assumptions. It is not symmetry of the Lie bracket itself, and it does not require B to vanish. The Jacobi identity imposes additional compatibility conditions but does not universally eliminate nonzero cross-sector relations. A simple two-dimensional solvable example demonstrates that paired closure and a nonzero cross-sector bracket can coexist, although that example does not by itself establish an exchange automorphism.

The limiting case of coincident embeddings yields an abelian sector under paired closure, but it does not imply that the entire ambient algebra is abelian. This distinction preserves the relevance of Abelian sectors while allowing them to be considered within larger non-Abelian structures.

The principal result is not a new physical law or a proof of ontological two-sidedness. It is the identification of a precise formal structure in which two corresponding sectors remain distinguishable, are related reciprocally, and contribute to a paired representation invariant under exchange. The direct sum is sufficient to establish this result; nontrivial cross-sector coupling is an additional possibility rather than a necessary condition.

The broader interpretation is that reciprocal representation may provide a useful mathematical analogue for thinking about bilateral organization without reducing it either

to two unrelated entities or to a single privileged side. The algebra establishes the coherence of the representation. The philosophical proposal concerns what, if anything, that form of coherence may illuminate about the structure of reality.

To clarify how the reciprocal algebraic structure developed here may extend to an exchange-invariant Yang–Mills Lagrangian, the following appendix outlines the compatibility between the algebraic exchange, the curvature construction, and the invariant bilinear pairing, while distinguishing this formal result from its further physical interpretation.

Appendix A. Reciprocal Exchange and Invariance of the Yang–Mills Lagrangian

A.1. The Question of Compositional Invariance

The preceding analysis establishes a formal structure in which two algebraically distinguishable sectors can be related reciprocally without assigning either sector priority. In the direct-sum construction, an involutive exchange automorphism preserves the paired representation, while the more general ambient construction permits nonzero cross-sector brackets under the symmetry condition required by paired closure. These results provide an algebraic framework for representing reciprocal organization.

A further question arises when this framework is considered in relation to a Lagrangian constructed from the underlying Lie algebra. In particular, if the resulting Yang–Mills Lagrangian is invariant under an exchange of the two sectors, how is that invariance related to the reciprocal structure of the Lie bracket and algebra on which the Lagrangian rests? The invariance of a composed Lagrangian cannot simply be assumed to establish the same kind of invariance at every level of its construction. The relationship between the algebraic exchange, the curvature derived from the connection, and the invariant pairing used in the Lagrangian must therefore be made explicit.

The question is not whether a Lagrangian can be invariant under a symmetry; such invariance is familiar in mathematical physics. Rather, the question is whether the reciprocal construction developed in this paper provides a coherent algebraic basis for understanding how two distinguishable sectors can be exchanged while leaving the resulting Lagrangian unchanged.

A.2. The Connection and Its Reciprocal Representations

Let $\mathfrak{g}$ be a Lie algebra with bracket $[X, Y]$, and let $A_\mu(x)$ be a connection taking values in $\mathfrak{g}$. In a chosen basis $\{T_\mathrm{a}\}$, the connection may be written

$$A_\mu(x) \;=\; A_\mu{}^{\mathrm{a}}\,(x)\,T_\mathrm{a}.$$

The coefficient functions $A_\mu{}^{a}(x)$ specify the contribution of each algebraic generator at the spacetime point x. They are not themselves the Lie-algebra generators; rather, they are the coefficients through which the connection is expressed in the chosen basis.

Suppose that two representations of the underlying algebra are related by the reciprocal construction. Using the notation of this paper, let $j : \mathfrak{g} \to \mathfrak{L}$ and $j' : \mathfrak{g}' \to \mathfrak{L}$ be the relevant embeddings, with $\varphi : \mathfrak{g} \to \mathfrak{g}'$ an isomorphism. The corresponding algebra-valued connections may be represented as

$$A_\mu{}^{+}(x) = j(A_\mu(x)),$$

$$A_\mu{}^{-}(x) = j'(\varphi(A_\mu(x))).$$

The coefficient functions $A_\mu{}^{a}(x)$ have not been independently changed in constructing these two expressions. What differs is their algebraic representation. The same underlying connection is thus represented in two corresponding sectors, rather than replaced by two unrelated connections.

This distinction is central to the proposed resolution. The reciprocal sectors can remain distinguishable in the representation even when they are constructed from the same underlying connection data. Their relationship is expressed through the embeddings and the isomorphism, rather than through an assumption that the two algebraic images are literally identical.

A.3. Compatibility of the Curvature with Reciprocal Exchange

The curvature associated with a connection is conventionally expressed as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu],$$

where the precise sign and normalization of the bracket term depend on convention. This expression combines differentiation of the connection with its Lie-algebraic bracket. The curvature is therefore not an independent algebraic structure added arbitrarily to the connection; it is constructed from the connection and the bracket that defines its algebra.

For the direct-sum construction, the exchange map is the involutive automorphism defined in Section 3; in a more general ambient algebra, the corresponding exchange property must be established separately rather than inferred from the existence of the two embeddings alone. Since j, j', and φ are linear bracket-preserving maps that commute with differentiation, the corresponding curvatures satisfy

$$F_{\mu\nu}{}^{+} = j(F_{\mu\nu}),$$

$$F_{\mu\nu}{}^{-} = j'(\varphi(F_{\mu\nu})).$$

When an exchange map S is defined on the relevant algebra and acts compatibly on the two represented sectors, the corresponding curvatures satisfy

$$S(F_{\mu\nu}{}^{+}) \;=\; F_{\mu\nu}{}^{-}.$$

Here S acts only on the algebraic representation; spacetime indices are unaffected. The curvature construction therefore respects the reciprocal exchange under the stated conditions. The exchange compatibility is carried from the underlying algebra to the curvature because both the bracket and the differentiation involved in its construction are respected by the relevant maps.

This result does not require the connection coefficient functions to be independently transformed merely to create a second sector. The reciprocal distinction is instead expressed through the algebraic representations of the same underlying connection. The conclusion is conditional on the specified compatibility properties; it does not follow from the existence of two isomorphic algebras alone.

A.4. The Invariant Pairing and the Yang–Mills Lagrangian

The curvature enters the Yang–Mills Lagrangian through an invariant bilinear pairing on the relevant Lie algebra or representation space. Let $\mathfrak{L}$ denote the algebra in which the represented curvature is evaluated, and let $\langle\cdot,\cdot\rangle$ be a bilinear form satisfying the exchange-invariance condition

$$\langle S(X), S(Y)\rangle \;=\; \langle X, Y\rangle$$

for the relevant $X, Y \;\in\; \mathfrak{L}$. Exchange invariance is the condition relevant to the present argument. For a fully gauge-invariant Yang–Mills theory, the bilinear form must also satisfy the appropriate gauge-algebra invariance condition.

A Yang–Mills-type Lagrangian density may then be written schematically as

$$\mathcal{L} \;=\; -¼\,\langle F_{\mu\nu}, F^{\mu\nu}\rangle,$$

with the gauge coupling and normalization factors understood to depend on convention. The spacetime indices are contracted using the relevant spacetime metric.

If the exchange preserves the bilinear pairing and the spacetime contraction, then

$$\mathcal{L}' \;=\; -¼\,\langle S(F_{\mu\nu}), S(F^{\mu\nu})\rangle$$

and therefore

$$\mathcal{L}' \;=\; -¼\,\langle F_{\mu\nu}, F^{\mu\nu}\rangle \;=\; \mathcal{L}.$$

Here, the prime on $\mathcal{L}'$ denotes the Lagrangian expression after application of the exchange map S, rather than a label for the minus sector. The $+$ and $-$ labels identify the corresponding sector representations or sector-specific pairings, as indicated by their placement in the notation. The Lagrangian is thus invariant under the exchange. In a formulation using separate sector pairings, the corresponding requirement is that the pairing be compatible with the isomorphism φ; for example,

$$\langle \varphi(X), \varphi(Y) \rangle_- = \langle X, Y \rangle_+.$$

The preservation of the Lie bracket does not, by itself, guarantee preservation of every possible bilinear form. The compatibility of the pairing must therefore be included among the assumptions of the construction.

The significance of this result is compositional. The Lagrangian is invariant because the reciprocal exchange is compatible with the curvature construction and because the pairing used to compose the Lagrangian is itself invariant under that exchange. The invariance of the composed expression is thereby related explicitly to the algebraic structure on which it depends. The exchange symmetry is therefore carried from the connection, through the curvature construction, to the invariant Lagrangian, provided the stated compatibility conditions hold at each stage.

A.5. Relation to Reciprocal Redundancy

The preceding construction connects the algebraic reciprocity established in this paper with the invariance of a Lagrangian built from the corresponding curvature. The reciprocal sectors remain distinguishable as algebraic representations, while their exchange leaves the curvature construction compatible and the composed Lagrangian unchanged.

This is a form of representational redundancy in the present algebraic sense: two corresponding algebraic representations can be exchanged without changing the value of the invariant Lagrangian. The conclusion does not require the two sectors to be identified as the same algebraic object. Nor does it require a nonzero cross-sector bracket. The direct-sum construction described in the main text supplies a sufficient algebraic setting for reciprocal exchange, while more general ambient constructions may admit additional cross-sector relations subject to the required closure and consistency conditions.

The distinction between algebraic representation and the invariant quantity constructed from it is important. The fact that the Lagrangian is unchanged under exchange does not erase the formal distinction between the sectors. Rather, it shows that the composed expression does not distinguish between them under the specified exchange. The invariant therefore expresses a commonality across the reciprocal representations without, by itself, determining their ultimate physical or ontological status.

A.6. Scope and Remaining Questions

The purpose of this appendix is limited to clarifying the compatibility between the reciprocal algebraic construction and an exchange-invariant Yang–Mills Lagrangian. It does not derive a new Yang–Mills theory, establish a new CPT theorem, or demonstrate that the reciprocal sectors correspond to independent physical systems.

A further question is whether the exchange considered here should be interpreted physically as CPT. The mathematical exchange map must be related to the appropriate transformation of physical fields and spacetime variables before that identification can be fully specified. This appendix does not attempt to re-prove the established CPT theorem; its concern is whether the reciprocal construction offers a useful algebraic representation of the symmetry under consideration.

Other questions also remain open. Invariance of the Lagrangian does not by itself establish that the sectors are related by gauge redundancy, that they represent the same physical states, or that their reciprocal organization reflects a deeper ontological structure. These possibilities require additional physical and interpretive analysis beyond the scope of the present appendix.

The result established here is consequently a compatibility result: an exchange-preserving reciprocal representation can be carried through the curvature construction and, with an appropriate invariant bilinear pairing, can yield a Yang–Mills Lagrangian insensitive to the exchange. This provides a bridge between the formal algebraic structure developed in the paper and a familiar invariant expression in gauge theory, while leaving the physical interpretation of that bridge for subsequent investigation.

References

Erdmann, K., and M.J. Wildon, 2006, *Introduction to Lie Algebras*, Springer.

Humphreys, J.E., 1972, *Introduction to Lie Algebras and Representation Theory*, Springer.

Santilli, R.M., 1968, An Introduction to Lie-Admissible Algebras, *Supplemento al Nuovo Cimento Serie I*, 6,1225-1249.

Smith, S.P., 2022, Universal Grammar, the Mirror Universe Hypothesis & Kinesiological Thinking, *Journal of Consciousness Exploration & Research*, 13 (4), 482-498.

Smith, S.P., 2026, Lie Algebra, Global Inversion, and the Case for Ontological Two-Sidedness, ai.viXra.org: 2610.0024 [Quantitative Biology].