

From $SU(5)^2$ Block Decompositions to an Extended Pati–Salam Structure: The Common Origin of Three $U(1)$ Factors and the E_8 Embedding

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Abstract

Taking $SU(5)_A \times SU(5)_B$ as its starting point, this paper addresses a simple structural question: can a single choice of block decompositions explain both the Pati–Salam non-Abelian sector and the origin of three additional Abelian directions? The first five-dimensional space is decomposed as $4 + 1$, and the second as $2 + 2 + 1$. Special unitary transformations within the blocks yield $SU(4)_C \times SU(2)_L \times SU(2)_R$, while the block phases, subject to the unit-determinant constraints, leave one and two independent directions, respectively. Together they give the local group structure of G_{422111} . The construction thus provides a common block interpretation of the number, generators, and left–right exchange properties of the three $U(1)$ factors. Using the known embedding of an $SU(5)^2$ -type subgroup in E_8 , we obtain a Pati–Salam embedding and adjoint representation content distinct from those of the standard route through $SO(10)$. The representation analysis imposes two restrictions: the selection of an extra anomaly-free direction depends on the chiral spectrum and a left–right evenness condition, and ordinary Higgs breaking using only Standard Model singlets in the adjoint necessarily preserves an additional continuous $U(1)$. We treat the block construction as a structural basis for a unification scheme, with spectrum selection and vacuum dynamics requiring further physical input. The main text emphasizes the construction and its physical interpretation; complete branching rules and proofs are collected in the appendices.

Keywords: $SU(5)^2$; block embedding; extended Pati–Salam; Abelian generators; E_8 ; grand unified theories

1 Introduction: starting from two five-dimensional spaces

The Pati–Salam scheme uses $SU(4)_C$ to place the three colors and leptons within a common internal structure, and $SU(2)_L \times SU(2)_R$ to organize left- and right-type multiplets [1]. We ask whether starting from two $SU(5)$ factors can explain both this non-Abelian structure and the additional Abelian directions that accompany it.

To this end, consider two five-dimensional complex spaces,

$$V_A = \mathbb{C}^4 \oplus \mathbb{C}, \quad V_B = \mathbb{C}_L^2 \oplus \mathbb{C}_R^2 \oplus \mathbb{C}. \quad (1.1)$$

These decompositions preserve a four-dimensional block in the first space and two interchangeable two-dimensional blocks in the second. They provide direct candidates for the representation spaces underlying color–lepton and left–right weak structures. Crucially, the same block decompositions also preserve relative phases between the blocks: $4 + 1$ leaves one independent phase, whereas $2 + 2 + 1$ leaves two. Thus, 422 and three $U(1)$ factors arise in a single construction.

Our logical order is to study the block structure of $SU(5)^2$ first, and then use its known embedding in E_8 to examine representation content and physical constraints. E_8 supplies a parent algebra for organizing representations; explaining the number of $U(1)$ factors does not require assuming a particular dynamics of E_8 breaking in advance.¹

Brane models with $U(4)_C \times U(2)_L \times U(2)_R$ have been studied [3]. Hebecker and Ratz also constructed, from E_8 through $SO(10) \times SU(4)_F$, $G_{PS} \times U(1)^3$ [6]. Our specific focus is the division of roles between the two $SU(5)$ factors, the charge structure inherited from it, and the resulting physical constraints. The same local group can admit different embeddings. Section 4 gives a representation-multiplicity criterion showing that our PS embedding is not conjugate to the one obtained through the specified $SO(10)$ route. This provides a precise object of comparison, without establishing priority in the literature for the embedding itself.

2 A single block construction for 422 and three phases

2.1 The first $SU(5)$: a four-dimensional block and one relative phase

In V_A , preserve a four-dimensional subspace and its one-dimensional complement. The connected subgroup preserving this decomposition is

$$H_A = S(U(4) \times U(1)). \quad (2.1)$$

A block-diagonal transformation can be parametrized locally as

$$g_A = \begin{pmatrix} e^{i\alpha} U_C & 0 \\ 0 & e^{-4i\alpha} \end{pmatrix}, \quad U_C \in SU(4). \quad (2.2)$$

The $SU(4)$ transformation within the four-dimensional block and the relative phase α act differently: the former mixes the four internal components, while the latter rotates the block as a whole and rotates its one-dimensional complement with the opposite weight. The coefficient -4 follows from the unit-determinant condition. This independent relative phase is the origin of the additional $U(1)$.

Interpreting the four-dimensional block as three color directions and one lepton direction gives a Pati–Salam interpretation. The fermion-type representations and standard hypercharge below provide further tests of this physical identification. A four-dimensional internal vector space alone does not specify particle spin.

2.2 The second $SU(5)$: left and right doublet blocks and two relative phases

Preserving two two-dimensional blocks and a one-dimensional complement in V_B gives

$$H_B = S(U(2)_L \times U(2)_R \times U(1)). \quad (2.3)$$

The corresponding local parametrization is

$$g_B = \text{diag}\left(e^{i\beta} U_L, e^{i\gamma} U_R, e^{-2i(\beta+\gamma)}\right), \quad U_L, U_R \in SU(2). \quad (2.4)$$

The overall phases of the three blocks obey one determinant constraint, leaving β and γ as independent parameters. The one-dimensional complement allows the phases of the left and right blocks to vary independently while maintaining unit total determinant. This explains the two Abelian directions preserved by the $2 + 2 + 1$ decomposition.

Interchanging the two two-dimensional blocks defines an explicit internal transformation. Promoting this exchange to a physical left–right parity also requires specifying the transformation properties of the fermions, interactions, and vacuum.

¹This starting point was motivated by a $(5, 5)$ counting pattern in a semiregular tiling proposal. Deriving a continuous gauge group from geometric counting requires a separate argument. Here we begin with a specified $SU(5)^2$ and its block decompositions, without relying on that geometric argument.

2.3 A common origin for the three Abelian factors

Equations (2.2) and (2.4) summarize the construction:

$$\begin{aligned} SU(5)_A, \quad 5 = 4 + 1 &\longrightarrow SU(4)_C \text{ and one relative phase,} \\ SU(5)_B, \quad 5 = 2 + 2 + 1 &\longrightarrow SU(2)_L \times SU(2)_R \text{ and two relative phases.} \end{aligned} \quad (2.5)$$

At the Lie-algebra level,

$$\boxed{\mathfrak{g}_{422111} = \mathfrak{su}(4)_C \oplus \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)^3.} \quad (2.6)$$

The two initial factors have total rank eight. The non-Abelian transformations within the blocks account for $3+1+1 = 5$ Cartan directions, and the three phase directions supply the remainder, preserving total rank eight. Finite central quotients are treated explicitly in Appendix B.

The structural intuition is that the non-Abelian and additional Abelian transformations both preserve the same block decomposition, so their origins can be explained together. The term *natural* is used here only in this sense: *once the two $SU(5)$ factors and the specified blocks are given, the number and action of the three $U(1)$ factors follow.* Why nature would select Eq. (1.1) remains a dynamical question. For example, adjoint backgrounds with eigenvalue degeneracies $4+1$ or $2+2+1$ can have these stabilizers, but block counting does not establish their stability.

The three relative phases do not represent three electromagnetic interactions. The electromagnetic generator is determined after electroweak breaking by $Q_{\text{em}} = T_{3L} + Y$. This section identifies the structural origin of additional gauge directions.

3 From phases to charges: the three $U(1)$ generators

Taking the infinitesimal generators of Eqs. (2.2) and (2.4) gives

$$Q_4 = \text{diag}(1, 1, 1, 1, -4)_A, \quad (3.1)$$

$$Q_L = \text{diag}(1, 1, 0, 0, -2)_B, \quad Q_R = \text{diag}(0, 0, 1, 1, -2)_B. \quad (3.2)$$

These matrices turn the phase degrees of freedom directly into calculable charges. Q_4 distinguishes the four-dimensional block from its complement, while Q_L and Q_R record the phases of the two two-dimensional blocks relative to their complement. Charges in tensor representations are obtained by adding the charges of their indices; they therefore need not be assigned separately to each multiplet.

To make the left–right exchange properties explicit, define

$$Q_+ = Q_L + Q_R, \quad Q_- = Q_L - Q_R. \quad (3.3)$$

Q_+ rotates the left and right blocks together, while Q_- measures their relative rotation. Under block exchange, Q_4 and Q_+ are even, whereas Q_- is odd. The three directions thus have distinct interpretations: one belongs to the A factor, and the other two can be reorganized as a common phase and a phase difference for the left and right blocks of the B factor.

This reorganization is a change of basis in the real Lie algebra. Global periods and gauge-kinetic normalization require separate treatment. The inner product inherited from E_8 , given in Appendix C, makes (Q_4, Q_+, Q_-) mutually orthogonal but assigns them different lengths. The integer charges used here therefore cannot directly substitute for the normalization entering gauge couplings.

4 The role of E_8 : representations and embeddings

4.1 From the block construction to the adjoint representation

Once the block structure is fixed, we can use the known embedding $A_4 \oplus A_4 \subset \mathfrak{e}_8$ [2]:

$$\begin{aligned} \mathbf{248} = & (24, 1) \oplus (1, 24) \oplus (10, 5) \oplus (\overline{10}, \bar{5}) \\ & \oplus (5, \overline{10}) \oplus (\bar{5}, 10). \end{aligned} \quad (4.1)$$

The two adjoint sectors reflect the internal structure of the respective factors, whereas the mixed sectors carry indices of both. When an index of the first factor lies in the four-dimensional block and one of the second factor lies in a left or right two-dimensional block, PS-type multiplets can arise. The full calculation indeed gives two types of $(4, 2, 1)$, two types of $(4, 1, 2)$, and their conjugates, together with additional tensor representations and singlets. Appendix A lists every component, retaining all mirror and exotic representations.

Thus, E_8 adds specific representation content to the block construction. Using its $\mathbf{248}$ as a source of physical fields is, however, an additional choice beyond the $SU(5)^2$ block structure itself. Internal quantum numbers in the adjoint do not determine whether a component is a four-dimensional fermion, scalar, or gauge field. The physical spectrum depends on the field theory and, where applicable, the compactification mechanism.

4.2 A testable distinction from the route through $SO(10)$

An existing construction starts with $E_8 \supset SO(10) \times SU(4)_F$, obtaining the PS sector from $SO(10)$ and the three Abelian directions from $SU(4)_F$ [6]. That route also has a clear group-theoretical origin. In the present construction, the color–lepton block and the left–right blocks belong to different $SU(5)$ factors. To determine whether the distinction goes beyond the order of presentation, compare the restriction of the full $\mathbf{248}$ before any projection:

Multiplicity (extra Abelian charges suppressed)	Standard $SO(10)$ route	Present route
$(6, 2, 2)$	1	0
PS singlet $(1, 1, 1)$	15	7

The first row already distinguishes the embeddings. The $SO(10)$ adjoint 45 in the literature construction contains $(6, 2, 2)$, whereas our complete branching does not. Conjugate embeddings must give identical multiplicities when the same parent-group representation is restricted. The two PS subalgebras are therefore not conjugate in E_8 . Appendix D gives the complete count. This establishes a precise embedding distinction for the construction studied here. Determining whether other publications have studied the same embedding requires a separate priority review.

5 Chiral subspectrum and anomaly constraints

The PS-type quantum numbers in the branching motivate further investigation, but the full $\mathbf{248}$ contains conjugate partners and does not automatically yield net chiral generations. To study constraints on the extra charges, temporarily select the four left-handed Weyl multiplets defined in Appendix E: $F_{L1}, F_{L2}, F_{R1}^c, F_{R2}^c$. Their non-Abelian representations form two sets of $(4, 2, 1) \oplus (\bar{4}, 1, 2)$. Their additional charges are fixed by their origins in the mixed sectors.

For a general generator $Q = aQ_4 + bQ_L + cQ_R$, perturbative anomaly cancellation for this subspectrum requires

$$a = b + c, \quad Q \in \text{span}_{\mathbb{R}}\{Q_X, Q_-\}, \quad Q_X = 2Q_4 + Q_L + Q_R. \quad (5.1)$$

The blocks therefore provide three available directions, while the chosen chiral spectrum restricts the anomaly-free directions that can be retained simultaneously. If the generator is also required

to be even under left–right block exchange, $b = c$, the nonzero solutions lie on the single line generated by Q_X . The corresponding charges are

$$X(F_{L1}, F_{L2}, F_{R1}^c, F_{R2}^c) = (-5, 5, 5, -5). \quad (5.2)$$

The appendix presents all mixed and cubic anomaly calculations, together with the usual $SU(2)$ global-anomaly check.

This result relates the block interpretation of the generators to the consistency of a particle spectrum. Its assumptions are explicit: changing the chiral zero-mode spectrum, multiplicities, or compensating sectors can change Eq. (5.1). Q_X is therefore a candidate direction conditional on this spectrum and left–right evenness. Anomaly cancellation neither supplies a mechanism that produces the selected spectrum nor establishes that the vacuum selects this generator.

6 Hypercharge and constraints on Higgs breaking

6.1 Compatibility with standard hypercharge

Identify the first three components of the four-dimensional block as color directions and take

$$\begin{aligned} B-L &= \text{diag}(1/3, 1/3, 1/3, -1, 0)_A, \\ T_{3R} &= \text{diag}(0, 0, 1/2, -1/2, 0)_B, \quad Y = T_{3R} + \frac{1}{2}(B-L). \end{aligned} \quad (6.1)$$

This standard PS hypercharge gives the familiar quark and lepton charges [1]. The three additional phase directions therefore have distinct roles in model building, without each needing to be interpreted as hypercharge.

If the scalar spectrum contains the two types of $(4, 1, 2)$ found in the adjoint branching, their color-singlet components with $T_{3R} = 1/2$ preserve the Standard Model and carry $X = -5, +5$, respectively. Provided an independent mechanism has already reduced the extra Abelian sector to $U(1)_X$, the common invariance conditions for the two vacuum expectation values (VEVs) leave precisely Y in the $(B-L, T_{3R}, X)$ space. This is a conditional consistency result. However, the PS singlets in the adjoint cannot accomplish $U(1)^3 \rightarrow U(1)_X$ while preserving the full PS group. The assumed intermediate stage is therefore not automatically realized by the same singlet sector.

6.2 Why adjoint singlets cannot remove all additional phase directions

Starting from the full G_{422111} requires considering five directions,

$$\mathcal{C} = \text{span}_{\mathbb{R}}\{B-L, T_{3R}, Q_4, Q_L, Q_R\}. \quad (6.2)$$

To leave only Y , the vacuum must impose four independent constraints. The complete branching contains thirteen Standard Model singlet directions, but their charges are not independent. Appendix F enumerates them and proves

$$\text{rank } M_{\text{all}} = 3, \quad \ker M_{\text{all}} = \text{span}_{\mathbb{R}}\{Y, Z\}, \quad Z = 5(B-L) - Q_4 + 2Q_L + 2Q_R. \quad (6.3)$$

Every allowed adjoint singlet is therefore neutral under the same additional generator Z . Adding arbitrarily many adjoint fields of this type cannot remove this continuous symmetry using only ordinary Higgs VEVs that preserve the fixed Standard Model subgroup.

The physical implication is that, although the origin of the extra phases is transparent, their breaking remains constrained by the charge structure of the parent representation. One needs a Standard Model singlet with nonzero Z charge, or a mechanism outside the ordinary Higgs assumptions above. The matrix kernel identifies generators that necessarily survive. It does not

determine whether the full stabilizer contains further non-Abelian factors, nor guarantee that a given potential realizes the maximal charge rank.

The residual gauge group and the selected fermion spectrum must also be checked together. On our four-multiplet subspectrum, Z acts as twenty times baryon number and has nonzero $SU(2)_L^2 Z$ and $Y^2 Z$ anomalies (Appendix G). A complete effective theory retaining Z thus requires an additional consistency mechanism. The anomaly-free result for Q_X cannot be transferred directly to Z .

7 Discussion: what this construction explains

The construction first answers a structural question: why can 422 arise together with three additional Abelian directions? The answer lies in transformations within, and relative phases between, the blocks of two five-dimensional spaces. This explanation already holds at the $SU(5)^2$ level. The known embedding in E_8 then adds calculable representation content. Its restricted representation differs from that of the standard $SO(10)$ route, making it a definite embedding scheme for comparison.

The analysis supports three connected observations. The blocks determine the common origin of the non-Abelian factors and the three phases; the representation content fixes the extra charges and the anomaly conditions for the specified chiral spectrum; and the same charge structure constrains the breaking achievable with ordinary adjoint Higgs fields. The complete branching rules and matrix calculations test this structural idea and delimit the conditions for its realization.

A physical completion should first address two questions: which dynamics or background selects $4 + 1$ and $2 + 2 + 1$, and how the residual Z is handled while obtaining an acceptable chiral spectrum. If additional scalars resolve the latter issue, at least one must be a Standard Model singlet with nonzero Z charge. If compactification, projections, or an axionic Stückelberg mechanism are used, the actual zero modes and mass matrices must be calculated, ensuring that hypercharge remains massless and anomalies are consistently treated. Heterotic line-bundle constructions provide examples of masses for extra Abelian gauge fields and generalized Green–Schwarz mechanisms [5]. No such explicit background is constructed here.

Three-generation multiplicity, Yukawa couplings, stable vacua, and gauge-coupling matching remain further physical questions. We do not identify mathematical block stabilizers with an established cosmological symmetry-breaking history. The contribution is therefore a structural analysis and a set of consistency constraints for a definite unification route. Its novelty relative to the literature should be assessed separately for the specific embedding, charge spectrum, and constraint results.

8 Conclusions

The $4 + 1$ and $2 + 2 + 1$ decompositions of $SU(5)_A \times SU(5)_B$ jointly yield a PS-type non-Abelian structure and three independent phases: one between the four-dimensional block and its complement, and two between the left and right two-dimensional blocks and their complement. The three $U(1)$ generators, their left–right exchange properties, and their charges in tensor representations follow from this common construction.

The E_8 adjoint supplies PS-type multiplets in an embedding not conjugate to the standard $SO(10)$ route. Physical realization requires further conditions: selecting Q_X from the anomaly-free directions of the specified chiral subspectrum requires left–right evenness, while all adjoint Standard Model singlets together still cannot remove Z . These results make the block intuition a verifiable research framework and identify the missing ingredients in chiral-spectrum generation and symmetry breaking.

A Complete branching rules and charge conventions

A.1 Representation and charge conventions

We use

$$(R_4, R_L, R_R)_{(q_4; q_L, q_R)} \quad (\text{A.1})$$

to denote an $SU(4)_C \times SU(2)_L \times SU(2)_R$ representation together with its three Abelian charges. Dimensions, without boldface, denote the corresponding standard representations; $\bar{4}$ is the complex conjugate of 4. Complex conjugation acts as

$$\overline{(R_4, R_L, R_R)_{(q_4; q_L, q_R)}} = (\bar{R}_4, \bar{R}_L, \bar{R}_R)_{(-q_4; -q_L, -q_R)}. \quad (\text{A.2})$$

The $SU(2)$ doublet is self-conjugate and pseudoreal, and the $SU(4)$ representation $6 = \wedge^2 4$ is self-conjugate. All anomaly calculations use *left-handed Weyl fermions*; a physical right-handed field is represented by its left-handed conjugate F_R^c .

A.2 Basic representations

For the first factor,

$$5_A = 4_1 \oplus 1_{-4}, \quad (\text{A.3})$$

$$10_A = 6_2 \oplus 4_{-3}, \quad (\text{A.4})$$

$$24_A = 15_0 \oplus 1_0 \oplus 4_5 \oplus \bar{4}_{-5}. \quad (\text{A.5})$$

For the second factor, with pairs denoting $SU(2)_L \times SU(2)_R$ representations,

$$5_B = (2, 1)_{(1,0)} \oplus (1, 2)_{(0,1)} \oplus (1, 1)_{(-2,-2)}, \quad (\text{A.6})$$

$$\begin{aligned} 10_B = & (1, 1)_{(2,0)} \oplus (1, 1)_{(0,2)} \oplus (2, 2)_{(1,1)} \\ & \oplus (2, 1)_{(-1,-2)} \oplus (1, 2)_{(-2,-1)}. \end{aligned} \quad (\text{A.7})$$

These formulas follow directly from $10 = \wedge^2 5$, with charges adding over tensor indices. The adjoint follows from $5 \otimes \bar{5} - 1$; the charge of an off-diagonal block is the difference between the charges of its two blocks. The dimension identities $6 + 4 = 10$ and $1 + 1 + 4 + 2 + 2 = 10$ provide an initial check.

A.3 The two adjoint sectors

Writing $\mathcal{A} = (24, 1) \downarrow$ and $\mathcal{B} = (1, 24) \downarrow$, we obtain

$$\begin{aligned} \mathcal{A} = & (15, 1, 1)_{(0;0,0)} \oplus (1, 1, 1)_{(0;0,0)} \\ & \oplus (4, 1, 1)_{(5;0,0)} \oplus (\bar{4}, 1, 1)_{(-5;0,0)}, \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} \mathcal{B} = & (1, 3, 1)_{(0;0,0)} \oplus (1, 1, 3)_{(0;0,0)} \oplus 2(1, 1, 1)_{(0;0,0)} \\ & \oplus (1, 2, 2)_{(0;1,-1)} \oplus (1, 2, 2)_{(0;-1,1)} \\ & \oplus (1, 2, 1)_{(0;3,2)} \oplus (1, 2, 1)_{(0;-3,-2)} \\ & \oplus (1, 1, 2)_{(0;2,3)} \oplus (1, 1, 2)_{(0;-2,-3)}. \end{aligned} \quad (\text{A.9})$$

The dimensions are $15 + 1 + 4 + 4 = 24$ and $3 + 3 + 2 + 8 + 4 + 4 = 24$, respectively.

A.4 The four mixed sectors

Write $\mathcal{P} = (10, 5) \downarrow$ and $\mathcal{Q} = (5, \overline{10}) \downarrow$. Expanding each term gives

$$\begin{aligned} \mathcal{P} = & (6, 2, 1)_{(2;1,0)} \oplus (6, 1, 2)_{(2;0,1)} \oplus (6, 1, 1)_{(2;-2,-2)} \\ & \oplus (4, 2, 1)_{(-3;1,0)} \oplus (4, 1, 2)_{(-3;0,1)} \oplus (4, 1, 1)_{(-3;-2,-2)}, \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \mathcal{Q} = & (4, 1, 1)_{(1;-2,0)} \oplus (4, 1, 1)_{(1;0,-2)} \oplus (4, 2, 2)_{(1;-1,-1)} \\ & \oplus (4, 2, 1)_{(1;1,2)} \oplus (4, 1, 2)_{(1;2,1)} \\ & \oplus (1, 1, 1)_{(-4;-2,0)} \oplus (1, 1, 1)_{(-4;0,-2)} \oplus (1, 2, 2)_{(-4;-1,-1)} \\ & \oplus (1, 2, 1)_{(-4;1,2)} \oplus (1, 1, 2)_{(-4;2,1)}. \end{aligned} \quad (\text{A.11})$$

The remaining two sectors are precisely $\overline{\mathcal{P}}$ and $\overline{\mathcal{Q}}$, with conjugation given by Eq. (A.2). For completeness, they are written out in Appendix A.5. The full branching is therefore

$$\boxed{248 \downarrow = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{P} \oplus \overline{\mathcal{P}} \oplus \mathcal{Q} \oplus \overline{\mathcal{Q}}.} \quad (\text{A.12})$$

$$\dim \mathcal{P} = 12 + 12 + 6 + 8 + 8 + 4 = 50, \quad \dim \mathcal{Q} = 4 + 4 + 16 + 8 + 8 + 1 + 1 + 4 + 2 + 2 = 50. \quad (\text{A.13})$$

No exotic or mirror states have been omitted. The branching contains $(4, 2, 1)$ and $(\bar{4}, 1, 2)$, but also their complete conjugate partners. The presence of these representations does not establish net chiral Pati–Salam generations.

A.5 Explicit expansion of the two conjugate mixed sectors

To make every component explicit, the other two fifty-dimensional sectors in Eq. (A.12) are

$$\begin{aligned} \overline{\mathcal{P}} = & (6, 2, 1)_{(-2;-1,0)} \oplus (6, 1, 2)_{(-2;0,-1)} \oplus (6, 1, 1)_{(-2;2,2)} \\ & \oplus (\bar{4}, 2, 1)_{(3;-1,0)} \oplus (\bar{4}, 1, 2)_{(3;0,-1)} \oplus (\bar{4}, 1, 1)_{(3;2,2)}, \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} \overline{\mathcal{Q}} = & (\bar{4}, 1, 1)_{(-1;2,0)} \oplus (\bar{4}, 1, 1)_{(-1;0,2)} \oplus (\bar{4}, 2, 2)_{(-1;1,1)} \\ & \oplus (\bar{4}, 2, 1)_{(-1;-1,-2)} \oplus (\bar{4}, 1, 2)_{(-1;-2,-1)} \\ & \oplus (1, 1, 1)_{(4;2,0)} \oplus (1, 1, 1)_{(4;0,2)} \oplus (1, 2, 2)_{(4;1,1)} \\ & \oplus (1, 2, 1)_{(4;-1,-2)} \oplus (1, 1, 2)_{(4;-2,-1)}. \end{aligned} \quad (\text{A.15})$$

Together with the two adjoint sectors and the two mixed sectors listed above, they give $24 + 24 + 50 + 50 + 50 + 50 = 248$.

B Global groups and finite central quotients

The block stabilizers satisfy $H_A \simeq U(4)$ and $H_B \simeq U(2) \times U(2)$. The latter isomorphism is given by

$$(A, B) \mapsto \text{diag}(A, B, (\det A \det B)^{-1}) \quad (\text{B.1})$$

The local phase parametrizations can have finite kernels, so the direct-product notation in the main text does not specify a global group without quotients. Embedding in E_8 introduces the central quotient of its $SU(5)^2$ -type subgroup as well. To determine the global image, let $\zeta = e^{2\pi i/5}$. A central element $(\zeta^a I_5, \zeta^b I_5)$ of the covering group acts on the two independent mixed sectors by ζ^{2a+b} and ζ^{a-2b} , respectively. Trivial action on the full adjoint is equivalent to

$$2a + b = 0 \pmod{5}, \quad a - 2b = 0 \pmod{5}, \quad \Longleftrightarrow \quad b = 3a \pmod{5}. \quad (\text{B.2})$$

Compact E_8 has trivial center and a faithful adjoint representation, so the connected subgroup is

$$H_5 = \frac{SU(5)_A \times SU(5)_B}{\langle (\zeta I_5, \zeta^3 I_5) \rangle} \subset E_8. \quad (\text{B.3})$$

The actual block subgroup in our fixed embedding is consequently

$$H_{422111} = \frac{H_A \times H_B}{\mathbb{Z}_5}. \quad (\text{B.4})$$

The conventional notation $SU(5)^2 \subset E_8$ should thus be understood as shorthand at the Lie-algebra or local-group level. Finite quotients do not alter the matrix ranks over the real numbers below, but they affect global charge quantization and possible residual discrete groups. Calculating the continuous kernel does not replace an analysis of these global questions.

C Invariant inner product and gauge kinetic terms

Using Hermitian Cartan generators, define $K(U, V) = \text{Tr}_{248}(UV)/60$. Equation (4.1) gives

$$K(U, V) = \text{tr}_{5,A}(U_A V_A) + \text{tr}_{5,B}(U_B V_B). \quad (\text{C.1})$$

For example, $\text{tr}_{10}(UV) = 3 \text{tr}_5(UV)$ and $\text{tr}_{24}(UV) = 10 \text{tr}_5(UV)$, so the total coefficient for each $SU(5)$ factor is $10 + 2 \times 5 \times 3 + 2 \times 10 = 60$. Hence

$$K|_{(Q_4, Q_L, Q_R)} = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 6 & 4 \\ 0 & 4 & 6 \end{pmatrix}, \quad K|_{(Q_4, Q_+, Q_-)} = \text{diag}(20, 20, 4). \quad (\text{C.2})$$

Thus Q_L and Q_R are not orthogonal, whereas Q_4, Q_+, Q_- are mutually orthogonal. For the candidate direction considered below, define

$$Q_X = 2Q_4 + Q_+, \quad Q_\perp = Q_4 - 2Q_+, \quad K(Q_X, Q_\perp) = 0. \quad (\text{C.3})$$

$K(Q_X, Q_X) = K(Q_\perp, Q_\perp) = 100$. This is genuine orthogonality with respect to the inner product inherited from E_8 , rather than merely a formal dot product of coefficient vectors. If the convention $K(T, T) = 1/2$ is imposed, the normalized generator is $T = Q/\sqrt{2K(Q, Q)}$. Integer charges are convenient for representation theory but do not specify gauge-kinetic normalization.

Abelian kinetic mixing can also occur in a general effective theory. Changing a nondegenerate kinetic matrix does not change whether a generator annihilates a vacuum, although it changes the mass eigenstates and the form of their interactions. Low-energy couplings cannot be predicted from a subgroup inclusion alone without an actual unified gauge kinetic term and threshold matching.

D Representation comparison with the standard $SO(10)$ route

For the usual $D_5 \oplus A_3$ embedding, one has [6]

$$\mathbf{248} = (45, 1) \oplus (1, 15) \oplus (16, 4) \oplus (\overline{16}, \bar{4}) \oplus (10, 6). \quad (\text{D.1})$$

The first factor branches under the standard PS embedding as

$$45 = (15, 1, 1) \oplus (1, 3, 1) \oplus (1, 1, 3) \oplus (6, 2, 2), \quad (\text{D.2})$$

$$16 = (4, 2, 1) \oplus (\bar{4}, 1, 2), \quad (\text{D.3})$$

$$10 = (6, 1, 1) \oplus (1, 2, 2). \quad (\text{D.4})$$

The full restriction therefore contains one $(6, 2, 2)$. All PS singlets come from $(1, 15)$, giving multiplicity fifteen. For our block construction, Eqs. (A.8) and (A.9) give three PS singlets with vanishing extra charges, and Eq. (A.11) together with its conjugate gives four charged PS singlets, for a total of seven. None of the sectors contains $(6, 2, 2)$. The PS-singlet multiplicity

is also the dimension of the centralizer of the PS subalgebra in $\mathfrak{e}_8$; the dimensions 15 and 7 therefore supply a second criterion for nonconjugacy. This comparison concerns the full **248**, before any projection onto low-energy zero modes. It distinguishes only the present embedding from the specified standard $SO(10)$ embedding. It neither classifies all PS embeddings in E_8 nor establishes priority in the literature.

E Details of the chiral subspectrum and anomaly calculations

E.1 The full representation versus a selected spectrum

If the entire **248** is interpreted as the internal representation of four-dimensional left-handed Weyl fermions, its branching under the subgroup considered here contains pairs $R_q \oplus \bar{R}_{-q}$. Complex-conjugate partners cancel perturbative gauge and mixed gravitational anomalies pairwise. This fact selects no distinguished extra $U(1)$ and supplies no net chiral generations. Obtaining a low-energy chiral subspectrum requires an additional zero-mode, projection, or other mechanism, followed by a new consistency check of the full effective theory. Ordinary gauge-invariant bilinear masses in a four-dimensional vectorlike spectrum do not allow an arbitrary prescribed net chiral subspectrum to remain.

Now *assume* that only the following four left-handed Weyl multiplets are retained as the charged chiral spectrum under consideration:

$$\begin{aligned} F_{L1} &= (4, 2, 1)_{(-3;1,0)}, & F_{R1}^c &= (\bar{4}, 1, 2)_{(3;0,-1)}, \\ F_{L2} &= (4, 2, 1)_{(1;1,2)}, & F_{R2}^c &= (\bar{4}, 1, 2)_{(-1;-2,-1)}, \\ \mathcal{F} &= F_{L1} \oplus F_{L2} \oplus F_{R1}^c \oplus F_{R2}^c. \end{aligned} \tag{E.1}$$

All these states occur in Eq. (A.12). They form two sets of PS-type multiplets, but this algebraic organization is not a derivation of two generations, much less three.

E.2 General anomaly formulas and a single set of multiplets

Let the extra generator be $Q = aQ_4 + bQ_L + cQ_R$, and denote the left and right-conjugate charges in the two sets by x_i, y_i . With the index convention $T(4) = T(\bar{4}) = T(2) = 1/2$,

$$\begin{aligned} \mathcal{A}_{44Q} &= \sum_i (x_i + y_i), & \mathcal{A}_{LLQ} &= 2 \sum_i x_i, & \mathcal{A}_{RRQ} &= 2 \sum_i y_i, \\ \mathcal{A}_{\text{grav}^2 Q} &= 8 \sum_i (x_i + y_i), & \mathcal{A}_{Q^3} &= 8 \sum_i (x_i^3 + y_i^3). \end{aligned} \tag{E.2}$$

The pure $SU(4)^3$ anomaly is $2n_L - 2n_R$, which vanishes for Eq. (E.1). A common loop-integral factor is suppressed in all anomaly coefficients. Triangles with one non-Abelian and two Abelian generators vanish because the non-Abelian generator is traceless.

For a single set $(4, 2, 1)_x \oplus (\bar{4}, 1, 2)_y$, the conditions $\mathcal{A}_{LLQ} = \mathcal{A}_{RRQ} = 0$ already require $x = y = 0$. Thus, without additional chiral states or a compensating mechanism, an extra anomaly-free $U(1)$ must act trivially on that single set. This conclusion applies to this specific spectrum; it is not a no-go theorem for every extended model containing one PS family.

E.3 Four multiplets: a two-dimensional anomaly-free space and conditional uniqueness

Equation (E.1) gives

$$x_1 = -3a + b, \quad x_2 = a + b + 2c, \quad y_1 = 3a - c, \quad y_2 = -a - 2b - c. \tag{E.3}$$

Here $x_1 + x_2 = -2a + 2b + 2c$ and $y_1 + y_2 = -(x_1 + x_2)$. Both $SU(2)^2Q$ anomalies vanish if and only if

$$a = b + c. \quad (\text{E.4})$$

Substitution gives $x_2 = -x_1$ and $y_2 = -y_1$, so the remaining perturbative anomalies vanish as well. We obtain the following result.

Proposition E.1. *For the four multiplets in Eq. (E.1), the space of additional Abelian generators allowed by conventional four-dimensional perturbative anomaly cancellation is*

$$\{aQ_4 + bQ_L + cQ_R : a = b + c\} = \text{span}_{\mathbb{R}}\{Q_X, Q_-\}. \quad (\text{E.5})$$

If the generator is further required to be even under left-right block exchange, so that $b = c = d$, its unique nonzero direction is

$$\boxed{Q_X \propto 2Q_4 + Q_L + Q_R.} \quad (\text{E.6})$$

Uniqueness here refers only to a line within the two-dimensional exchange-even subspace of the original Abelian algebra. It does not identify a unique physical gauge boson. Indeed, the charges of the four multiplets in the (Q_X, Q_-) basis are

Multiplet	Q_X	Q_-
F_{L1}	-5	1
F_{L2}	5	-1
F_{R1}^c	5	1
F_{R2}^c	-5	-1

For each non-Abelian representation, the two vectors of Abelian charges are opposite. Mixed Abelian anomalies such as $Q_X^2 Q_-$ and $Q_X Q_-^2$ therefore vanish as well. For this particular effective spectrum, both directions can pass the perturbative anomaly tests simultaneously. Gauge anomaly cancellation by itself supplies no reason to discard Q_- .

Each $SU(2)$ factor sees eight left-handed doublets, so neither has the usual Witten anomaly associated with an odd number of doublets [4]. The analysis here concerns local anomalies on ordinary four-dimensional spin spacetimes and this $SU(2)$ check. It does not replace a complete anomaly analysis for every possible global gauge group and topological background.

Choosing the integer representative in Eq. (E.6) gives X charges $(-5, 5, 5, -5)$. They may be divided by five for spectrum comparisons, but this does not remove the need to track the global periods inherited from the embedding. Minimality here can mean only that, with no other charged chiral states and with complete PS sets of the two types above, one set is insufficient whereas two can work. We have not classified globally minimal choices among all possible sub-spectra of the **248**. Changing zero-mode multiplicities or adding other representations changes the anomaly equations and can therefore change the selected direction.

F Higgs conditions and the complete singlet-rank proof

F.1 Standard Model quantum numbers

With the hypercharge in Eq. (6.1), the two PS-type fermion representations restrict as

$$(4, 2, 1) = (3, 2)_{1/6} \oplus (1, 2)_{-1/2}, \quad (\text{F.1})$$

$$(\bar{4}, 1, 2) = (\bar{3}, 1)_{-2/3} \oplus (\bar{3}, 1)_{1/3} \oplus (1, 1)_0 \oplus (1, 1)_1. \quad (\text{F.2})$$

The terms on the right give the Standard Model quantum numbers of $Q_L, L_L, u^c, d^c, \nu^c, e^c$, respectively. All fermions are written as left-handed Weyl fields.

F.2 A conditional calculation with two right-type scalars

If the scalar sector contains the branching components

$$H_1 = (4, 1, 2)_{(-3;0,1)}, \quad H_2 = (4, 1, 2)_{(1;2,1)}, \quad (\text{F.3})$$

both contain genuine Standard Model singlets with $B-L = -1, T_{3R} = 1/2$, with $X(H_1) = -5$ and $X(H_2) = 5$. Assuming that *an independent mechanism has already reduced the extra Abelian space to Q_X* , write $T = \alpha(B-L) + \beta T_{3R} + \lambda Q_X$. Nonzero VEVs for both singlets require

$$\begin{pmatrix} -1 & 1/2 & -5 \\ -1 & 1/2 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = 0. \quad (\text{F.4})$$

The matrix has rank two, so $\lambda = 0$ and $\beta = 2\alpha$. The continuous Abelian kernel is precisely $\text{span}_{\mathbb{R}}\{Y\}$. This conclusion holds in the three-dimensional effective space; it does not imply that only Y survives in the full five-dimensional space. Moreover, selecting neutrino-type VEV directions and having both scalar types condense are themselves vacuum assumptions requiring explanation.

F.3 These singlets cannot leave Q_X at a PS-preserving intermediate stage

The only PS singlets with nonzero extra charges in the entire **248** are

$$S_L = (1, 1, 1)_{(-4;-2,0)}, \quad S_R = (1, 1, 1)_{(-4;0,-2)} \quad (\text{F.5})$$

and their conjugates. There are also three PS singlets with zero charges. Their charges are $X(S_L) = X(S_R) = -10$, and 10 for the conjugates. Thus no VEV of a charged PS singlet preserves Q_X . Giving VEVs to fields with opposite charges does not cancel their breaking effects: an unbroken generator must annihilate each VEV separately. The two independent PS-singlet charges give the kernel in the extra Abelian sector

$$\ker \begin{pmatrix} -4 & -2 & 0 \\ -4 & 0 & -2 \end{pmatrix} = \text{span}_{\mathbb{R}}\{(1, -2, -2)\} = \text{span}_{\mathbb{R}}\{Q_{\perp}\}, \quad (\text{F.6})$$

which differs from $\text{span}_{\mathbb{R}}\{Q_X\}$. PS singlets from this representation alone therefore cannot realize $\mathfrak{g}_{422111} \rightarrow \mathfrak{g}_{\text{PS}} \oplus \mathfrak{u}(1)_X$. X -neutral states with non-Abelian charges do not solve this problem of *preserving the full PS group*.

F.4 All allowed singlets

Fix the G_{SM} embedding in Eq. (6.1) and require all high-scale VEVs to preserve $SU(3)_C \times SU(2)_L \times U(1)_Y$. Electric neutrality alone is insufficient: an electrically neutral component of $(1, 2, 2)$, for example, still breaks $SU(2)_L$. Consider the Cartan space commuting with color and the left weak group,

$$\mathcal{C} = \text{span}_{\mathbb{R}}\{B-L, T_{3R}, Q_4, Q_L, Q_R\}, \quad \dim \mathcal{C} = 5. \quad (\text{F.7})$$

The other three Cartan directions, two for color and one for the left weak group, must remain within the Standard Model. They are therefore excluded from the extra Abelian space whose directions are to be removed.

Proposition F.1. *The trivial representation of the fixed G_{SM} occurs with multiplicity 13 in the complexified adjoint **248** $_{\mathbb{C}}$. Five directions have zero charges under $\mathcal{C}$. The other eight have charges $\pm v_1, \pm v_2, \pm s_L, \pm s_R$, where*

$$\begin{aligned} v_1 &= (-1, 1/2, -3, 0, 1), & v_2 &= (-1, 1/2, 1, 2, 1), \\ s_L &= (0, 0, -4, -2, 0), & s_R &= (0, 0, -4, 0, -2). \end{aligned} \quad (\text{F.8})$$

Proof. Restrict Eqs. (A.8)–(A.11) term by term. The $SU(4)$ fundamental branches as $4 = 3_{1/3} \oplus 1_{-1}$; $6 = \bar{3}_{2/3} \oplus 3_{-2/3}$ contains no color singlet, whereas

$$15 = 8_0 \oplus 3_{4/3} \oplus \bar{3}_{-4/3} \oplus 1_0. \quad (\text{F.9})$$

First exclude all nontrivial $SU(2)_L$ representations. Among the remaining 4 and $\bar{4}$ components, the condition $T_{3R} = -(B-L)/2$ can be satisfied only when paired with an $SU(2)_R$ doublet. This gives four singlets in H_1, H_2 and their conjugates. The color singlets in $(4, 1, 1)$ and $(\bar{4}, 1, 1)$ have nonzero Y ; the $Y = \pm 1/2$ components of $(1, 1, 2)$ are also excluded. The mixed-sector states S_L, S_R and their conjugates supply four more singlets. The neutral singlets are the 1_0 in $(15, 1, 1)$, the $T_{3R} = 0$ component of $(1, 1, 3)$, and the three $(1, 1, 1)_{(0;0,0)}$ states, giving five in total. These cases exhaust the full branching, so the total multiplicity is $5 + 4 + 4 = 13$. $\square$

For a real adjoint scalar, the reality condition relates conjugate weight components. The eight complex weight spaces above must not be mistaken for eight independent complex scalars. They pair into real two-dimensional spaces, and the invariant real subspace likewise has dimension thirteen. The reality condition adds no new independent rows to the charge matrix.

F.5 The complete charge matrix, its rank, and the common kernel

For a generator $T = t_1(B-L) + t_2 T_{3R} + t_3 Q_4 + t_4 Q_L + t_5 Q_R$ in $\mathcal{C}$, a weight component ϕ_w obeys $T\phi_w = (w \cdot t)\phi_w$. If $\langle \phi_w \rangle \neq 0$, invariance requires $w \cdot t = 0$. The complete singlet charge matrix can be written as

$$M_{\text{all}} = \begin{pmatrix} 0_{5 \times 5} \\ M \\ -M \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 1/2 & -3 & 0 & 1 \\ -1 & 1/2 & 1 & 2 & 1 \\ 0 & 0 & -4 & -2 & 0 \\ 0 & 0 & -4 & 0 & -2 \end{pmatrix}. \quad (\text{F.10})$$

Zero rows and the negative rows supplied by conjugates do not change the rank. Since

$$v_2 - v_1 = -s_L, \quad (\text{F.11})$$

we have $\text{rank } M \leq 3$. On the other hand, the minor from rows 1, 3, 4 and columns 1, 4, 5 is

$$\det \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -4 \neq 0. \quad (\text{F.12})$$

Hence $\text{rank } M_{\text{all}} = \text{rank } M = 3$, rather than four. Its kernel is explicitly

$$\ker M = \text{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -1 \\ 2 \\ 2 \end{pmatrix} \right\} = \text{span}_{\mathbb{R}} \{Y, Z\}, \quad \boxed{Z = 5(B-L) - Q_4 + 2Q_L + 2Q_R.} \quad (\text{F.13})$$

For example, $Z(v_1) = -5 + 3 + 2 = 0$, $Z(v_2) = -5 - 1 + 4 + 2 = 0$, and $Z(s_L) = Z(s_R) = 4 - 4 = 0$. The obstruction is therefore not a failure to select sufficiently many singlets: every allowed singlet is neutral under the same Z .

Theorem F.2 (Residual Abelian obstruction from adjoint singlets). *For the fixed embedding studied here, let arbitrarily many scalar fields transform in the restriction of the **248**, and let every VEV be a singlet under the fixed G_{SM} . Their common stabilizer necessarily contains the continuous generator Z , linearly independent of Y . Consequently, these ordinary four-dimensional Higgs VEVs alone cannot reduce an initial gauge algebra containing $\mathcal{C}$ to exactly $\mathfrak{g}_{\text{SM}}$.*

Proof. Proposition F.1 gives the weight decomposition of the entire invariant subspace. Equation (F.13) shows that Z acts identically as zero on it. It therefore annihilates every linear combination of singlets and any number of such VEVs. The row space of any realized vacuum charge matrix is contained in that of M , and its rank is at most three. Because Z lies outside the Standard Model Lie algebra while commuting with it, the additional continuous symmetry cannot be removed. $\square$

F.6 Maximal charge rank and the full vacuum stabilizer are different notions

Here $\text{rank}_{\text{max}} = 3$ is the *maximal rank of the matrix formed from all available singlet weights*. To realize this bound in a specific model, at least three independent charge constraints must actually be activated, and the potential, reality conditions, and any supersymmetric D - and F -flatness conditions must be checked. If some singlets are absent from the scalar zero-mode spectrum, the realized rank can be smaller.

Even if the common kernel in $\mathcal{C}$ is exactly two-dimensional, the full stabilizer can contain non-Abelian generators. The generally valid conclusion is therefore

$$\mathfrak{g}_{\text{unbroken}} \supseteq \mathfrak{g}_{\text{SM}} \oplus \mathfrak{u}(1)_Z, \quad (\text{F.14})$$

rather than an unconditional equality. In particular, if the initial gauge group is still the full compact E_8 and there is only one real adjoint Higgs field, its VEV can be conjugated into a Cartan subalgebra, and its centralizer still contains a maximal torus of rank eight. This is a stronger restriction on a single field than the matrix constraint above. To study other stabilizers, one may consider an effective theory already reduced to H_{422111} , or several independent adjoint fields and other backgrounds. These choices must be specified in a physical model; they cannot be inferred implicitly from branching notation.

In a fixed weight basis, even a VEV with several weight components must satisfy $w \cdot t = 0$ separately for every nonzero component. Adding positive and negative numerical charges cannot replace the vacuum-invariance condition. Additional adjoint copies supply no new allowed weights and therefore cannot evade the Z obstruction.

G Residual charges and anomalies of the selected spectrum

Using $Z = 5(B-L) - Q_\perp$, both left-type PS multiplets in Eq. (E.1) have $Q_\perp = -5$, while both right-conjugate multiplets have $Q_\perp = 5$. The Z charges in each set of SM-type multiplets are therefore

Left-handed field	Q_L	L_L	u^c	d^c	ν^c	e^c
Y	1/6	-1/2	-2/3	1/3	0	1
Z	20/3	0	-20/3	-20/3	0	0

On this *specific fermion subspectrum*, $Z = 20B$, with $B(Q_L) = 1/3$ and $B(u^c) = B(d^c) = -1/3$. This is not an operator identity defining baryon number on the entire **248**.

With the index conventions of Appendix E, the two sets give

$$\begin{aligned} \mathcal{A}_{SU(3)^2 Z} &= 0, & \mathcal{A}_{SU(2)_L^2 Z} &= 20, & \mathcal{A}_{Y^2 Z} &= -20, \\ \mathcal{A}_{YZ^2} &= 0, & \mathcal{A}_{Z^3} &= 0, & \mathcal{A}_{\text{grav}^2 Z} &= 0. \end{aligned} \quad (\text{G.1})$$

For example, each set contributes $\mathcal{A}_{SU(2)_L^2 Z} = 3 \times (1/2) \times (20/3) = 10$, and

$$\mathcal{A}_{Y^2 Z}^{\text{per set}} = 6(1/6)^2(20/3) + 3(-2/3)^2(-20/3) + 3(1/3)^2(-20/3) = -10. \quad (\text{G.2})$$

Anomaly freedom of the selected spectrum under Q_X is therefore different from anomaly freedom under the necessarily surviving Z . If the low-energy gauge group contains $U(1)_Z$ and only $\mathcal{F}$ is retained, without a compensating sector, the theory is inconsistent. The conjugate states of the full **248** can cancel these anomalies, but this returns us to the problem of obtaining a chiral spectrum. Adding nonsupersymmetric scalars alone does not change the triangle anomalies of chiral fermions. Even if Z is subsequently Higgsed, the complete gauge theory still requires a consistent anomaly-cancellation or matching mechanism.

H Independent weight enumeration and computational checks

The full branching can also be reconstructed independently from the weights of the fundamental representations. Let a_i, b_j be the weights of the two $SU(5)$ fundamentals. The weight multisets of the six sectors are

$$\{a_i - a_j : i \neq j\} \cup \{0\}^4, \quad \{b_i - b_j : i \neq j\} \cup \{0\}^4, \quad (\text{H.1})$$

and

$$\{a_i + a_j + b_k : i < j\}, \quad -\{a_i + a_j + b_k : i < j\}, \quad \{a_i - b_j - b_k : j < k\}, \quad -\{a_i - b_j - b_k : j < k\}. \quad (\text{H.2})$$

Each mixed multiset contains fifty weights. Assigning charges using Eqs. (3.1) and (6.1), and retaining weights with vanishing charges under the two color Cartan generators, the left weak Cartan generator, and Y , initially gives sixteen zero-weight candidates. A zero Cartan weight, however, need not be a non-Abelian singlet. The two Cartan directions of the color adjoint and the one Cartan direction of the left weak adjoint still transform under the corresponding raising and lowering operators. Removing these three directions using the full non-Abelian branching leaves exactly the thirteen singlets of Proposition F.1. This step is necessary to avoid misidentifying non-Abelian zero weights as singlets.

Taking traces directly over all 248 weights also gives

$$K|_{(B-L, T_{3R}, Q_4, Q_L, Q_R)} = \begin{pmatrix} 4/3 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 4 & 6 \end{pmatrix}, \quad (\text{H.3})$$

in agreement with the traces in the fundamental representations. The branching dimensions, thirteen singlets and their multiplicities, nonzero third-order minor, two-dimensional kernel, conditional anomaly equations, and Eq. (G.1) can all be checked with exact rational arithmetic, without numerical rank thresholds or floating-point approximations.

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