

# The Solutions of $\sin x = x$ in the Complex Domain

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## Abstract

We give a complete elementary treatment of the equation  $\sin(x) = x$  over the complex numbers. We show that it has infinitely many solutions—exactly one, up to conjugation and sign, in each strip  $2k\pi < \Re z < (4k+1)\pi/2$ ,  $k \geq 1$ —among them only one real root,  $x = 0$ , and we supply an elementary proof of the uniqueness in each strip. Separating real and imaginary parts reduces the equation to two one-dimensional curve families whose intersections are precisely the solutions. We further derive explicit asymptotic expansions for the  $k$ -th solution: the real parts approach  $(4k+1)\pi/2$  and the imaginary parts grow only logarithmically,  $b_k \sim \ln(4\pi k)$ . The leading term of the latter goes back to Hardy (1902); the higher-order terms, whose coefficients are polynomials in  $\ln(2A_k)$  and which alternate in parity (odd powers for  $a_k$ , even powers for  $b_k$ ), appear to be new, and are verified against roots recomputed at 40-digit precision. As a by-product of this verification we correct misprints in the classical tables of Hillman and Salzer (1943) and of Fettis (1976).

## 1 Introduction

The equation

$$\sin(x) = x \tag{1}$$

is deceptively simple. Its only real solution is  $x = 0$ ; yet over the complex numbers it possesses infinitely many solutions, all of them non-real except the origin. Equations of this transcendent type have a long numerical history. The real root of  $\cos x = x$  was computed by iteration as early as 1890 by Miller [1]. For  $\sin z = z$  itself, the first ten complex roots were tabulated by Fadde [3] to five decimals, and—to six decimals, by the reduction to two one-variable equations described in Section 2—by Hillman and Salzer [4], who took their starting values from an approximation of Hardy [2]. Further tabulations followed: Mittelman and Hillman [5] (seven decimals, for  $z + \sin z$ ), Robbins and Smith [6] (six decimals, for  $\sin z = -z$ ), Coghlan and Smith [7] (for  $\sin z = kz$ ), and most extensively Ling [8], who computed the first hundred roots to eleven decimals; Fettis [9] tabulated  $\sin z = az$  over a range of  $a$  including  $a = 1$ . Roots of  $\sin z = z$  also occur as the multiple roots of the dispersion equation of an impedance-wall waveguide [13], and the first two of them are recorded, to 105 digits, in the On-Line Encyclopedia of Integer Sequences [16]. The classical series of Euler [10] and Rayleigh [11] for the roots of  $\tan x = x$  is the analogue of the expansions derived in Section 4.

What is *not* in this literature is a complete statement with proof. The tables give numbers; Hardy gives the leading asymptotic term; but we have found no explicit statement—still less an elementary proof—that each strip contains exactly one solution, nor any asymptotic expansion beyond the first term. The purpose of this paper is to supply both, using tools no heavier than one-variable calculus. Our contributions are:

- a *graphical-analytic method* (Section 2): separating real and imaginary parts splits (1) into two one-dimensional curve families in the  $(a, b)$ -plane whose intersections are precisely the solutions (the reduction to a one-variable equation appears already in [4]; we draw it as two curves and read the qualitative picture off them);

- a *complete classification theorem* (Theorem 1): the solutions are  $z = 0$  together with exactly one solution per strip  $2k\pi < \Re z < 2k\pi + \pi/2$  ( $k \geq 1$ ) and its conjugates and negatives, the uniqueness proved via the monotonicity of  $b/\sinh b$  and the intermediate value theorem;
- *asymptotic expansions* (Theorem 2) for the  $k$ -th solution  $(a_k, b_k)$ , derived by computer algebra and verified for all  $k \leq 500$ : the leading term is Hardy's, the higher orders are explicit rational-coefficient polynomials in  $\ln(2A_k)$  with a parity structure mirroring Euler's classical series for  $\tan x = x$ ;
- a *correction* (Table 3) of four misprints in the table of Hillman and Salzer [4], found in the course of the verification.

Throughout,  $z = a + ib$  with  $a, b \in \mathbb{R}$ , and  $k$  denotes a non-negative integer.

## 2 The real roots and the graphical–analytic method

### 2.1 The unique real root

**Proposition 1.** *The equation  $\sin x = x$  has the unique real solution  $x = 0$ .*

*Proof.* For  $x > 0$ ,

$$x - \sin(x) = \int_0^x (1 - \cos t) dt > 0,$$

since the integrand is continuous, non-negative, and positive on  $(0, x)$ . Likewise  $x - \sin(x) < 0$  for  $x < 0$ . Hence  $x = 0$  is the unique real root.  $\square$

### 2.2 Separation of real and imaginary parts

Let  $z = a + ib$ . Since

$$\sin(z) = \sin(a) \cosh(b) + i \cos(a) \sinh(b),$$

equation (1) is equivalent to the system

$$\sinh(b) \cos(a) = b, \tag{2}$$

$$\cosh(b) \sin(a) = a. \tag{3}$$

Each equation defines a one-parameter family of curves in the  $(a, b)$ -plane:

- *Family I* (from (2)): where  $b > 0$ , we may solve for  $a$  as

$$a = 2k\pi + \arccos\left(\frac{b}{\sinh b}\right), \quad 0 < \frac{b}{\sinh b} < 1,$$

giving one branch per strip  $(2k\pi, 2k\pi + \pi/2)$ , rising monotonically from  $(2k\pi, 0)$  towards the vertical asymptote  $a = 2k\pi + \pi/2$ .

- *Family II* (from (3)): where  $\sin a > 0$ ,

$$b = \operatorname{arccosh}\left(\frac{a}{\sin a}\right),$$

giving one U-shaped branch per interval  $(2k\pi, (2k+1)\pi)$ , descending from  $+\infty$  at the left end, attaining its valley where  $\tan a = a$  (at which point  $\cosh b = a/\sin a = \sec a = \sqrt{1+a^2}$ ), and ascending back to  $+\infty$ .

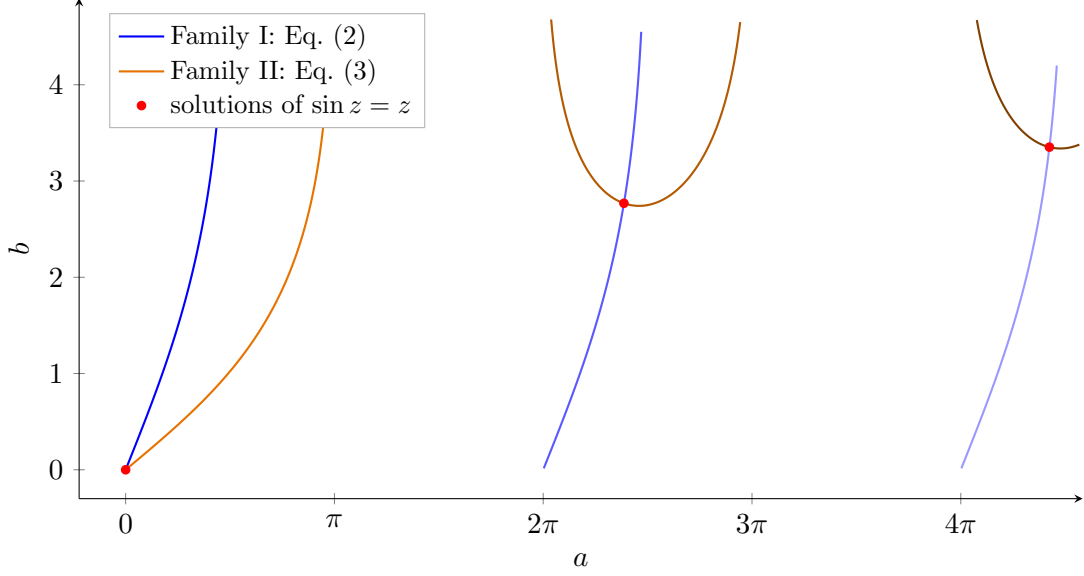


Figure 1: The two curve families defined by (2) (blue, one branch per strip  $(2k\pi, 2k\pi + \pi/2)$ ) and (3) (orange, one U-shaped branch per interval  $(2k\pi, (2k+1)\pi)$ ). Their intersections (red dots) are the non-real solutions of  $\sin z = z$  with  $b > 0$ ; the tangency of the  $k = 0$  branches at the origin corresponds to the real root  $z = 0$ . Colour distinguishes the two families; within each family the three plotted branches ( $k = 0, 1, 2$ ) are distinguished by decreasing shade, and all curves are drawn solid. The legend shows one representative branch for each family. The marked points in the strips  $k = 1, 2$  are  $a_1 + ib_1 \approx 7.4977 + 2.7687i$  and  $a_2 + ib_2 \approx 13.9000 + 3.3522i$ .

Solutions of (1) with  $b > 0$  are exactly the intersection points of the two families. Figure 1 displays the branches with  $k = 0, 1, 2$ : the first intersection is the first non-real solution  $a_1 + ib_1 \approx 7.4977 + 2.7687i$ , and the tangency of the two  $k = 0$  branches at the origin reflects the triple zero of  $\sin z - z$  there (Remark 2).

*Remark 1.* The valley of Family II in the  $k$ -th interval occurs where  $\tan a = a$ , because the valley minimizes  $a/\sin a$ , and  $\frac{d}{da}(a/\sin a) = 0$  is equivalent to  $\tan a = a$ ; there  $\cosh b = \sqrt{1 + a^2}$ . This observation localizes the search for  $a_k$  to a neighbourhood of the  $k$ -th positive root of  $\tan a = a$ , and provides the initial values used in the numerical computations of Section 5.

### 3 Classification of all solutions

The analytic engine behind Figure 1 is the following elementary lemma.

**Lemma 1.** *For every  $b > 0$ :*

- (i)  $0 < b/\sinh b < 1$ , and  $b/\sinh b$  is strictly decreasing;
- (ii)  $b \cosh b > \sinh b$ .

*Proof.* (i) We have

$$\frac{d}{db} \left( \frac{b}{\sinh b} \right) = \frac{\sinh b - b \cosh b}{\sinh^2 b} = \frac{\cosh b (\tanh b - b)}{\sinh^2 b} < 0,$$

since  $b - \tanh b$  is increasing on  $(0, \infty)$  (its derivative is  $\tanh^2 b > 0$ ) and vanishes at 0. Hence  $b/\sinh b$  strictly decreases from 1 (limit as  $b \rightarrow 0^+$ ) to 0.

(ii) The function  $b \cosh b - \sinh b$  has derivative  $b \sinh b > 0$  on  $(0, \infty)$  and vanishes at 0; hence it is positive for  $b > 0$ .  $\square$

**Theorem 1.** *The solutions of  $\sin(z) = z$  in  $\mathbb{C}$  are  $z = 0$ , together with, for each integer  $k \geq 1$ , the four numbers*

$$\pm a_k \pm i b_k,$$

where  $(a_k, b_k)$  is the unique pair of real numbers satisfying

$$2k\pi < a_k < 2k\pi + \frac{\pi}{2}, \quad b_k > 0, \quad \sinh(b_k) \cos(a_k) = b_k, \quad \cosh(b_k) \sin(a_k) = a_k. \quad (4)$$

*In particular, each strip  $2k\pi < \Re z < 2k\pi + \pi/2$  ( $k \geq 1$ ) contains exactly one solution with  $\Im z > 0$ , and the strip  $0 < \Re z < \pi/2$  contains none; consequently the equation has infinitely many solutions.*

*Proof.* Let  $z = a + ib$  with  $a, b \in \mathbb{R}$  satisfy  $\sin(z) = z$ , i.e. the system (2)–(3). If  $b = 0$  then  $\sin(a) = a$  and  $a = 0$  by Proposition 1. Since  $\sin(\bar{z}) = \overline{\sin z}$  and  $\sin(-z) = -\sin(z)$ , the conjugate and the negative of a solution are solutions; we may therefore assume  $b > 0$  and, as will follow,  $a > 0$ .

Suppose  $b > 0$ . From (2) and Lemma 1(i),  $\cos(a) = b/\sinh b \in (0, 1)$ . From (3),  $\sin(a) = a/\cosh b$  has the sign of  $a$ . If  $a = 0$ , then (2) gives  $\sinh b = b$ , impossible for  $b > 0$ ; hence  $a \neq 0$ , and by the odd symmetry we take  $a > 0$ , so  $\sin(a) > 0$  and  $\cos(a) > 0$ , which forces  $a \in (2k\pi, 2k\pi + \pi/2)$  for some integer  $k \geq 0$ . Note that no solution lies on the boundary of these strips:  $a = 2k\pi$  would give  $b = 0$  by (2), and  $a = 2k\pi + \pi/2$  would give, via (2) with  $\cos a = 0$ , again  $b = 0$ , whence  $1 = \sin a = a$ , a contradiction.

Fix  $k \geq 0$  and let  $I = (2k\pi, 2k\pi + \pi/2)$ . For  $a \in I$ , equation (2) has exactly one solution  $b_1(a) > 0$  by Lemma 1(i). Implicit differentiation of  $\cos a = b_1/\sinh(b_1)$  gives

$$b_1'(a) = \frac{\sin(a) \sinh^2(b_1)}{b_1 \cosh b_1 - \sinh b_1} > 0,$$

the denominator being positive by Lemma 1(ii). Define

$$G(a) = \cosh(b_1(a)) \sin(a) - a.$$

Then

$$G'(a) = \sinh(b_1) b_1'(a) \sin(a) + \cosh(b_1) \cos(a) - 1.$$

By Lemma 1(ii) applied to  $b_1 > 0$ ,

$$\cosh(b_1) \cos(a) = \frac{b_1 \cosh(b_1)}{\sinh(b_1)} = b_1 \coth(b_1) > 1$$

on  $I$ ; since also  $\sinh(b_1) > 0$ ,  $b_1'(a) > 0$  and  $\sin(a) > 0$ , we get  $G'(a) > 0$ , so  $G$  is strictly increasing on  $I$ .

If  $k \geq 1$ , then as  $a \rightarrow 2k\pi^+$  we have  $b_1(a) \rightarrow 0$  and  $G(a) \rightarrow -2k\pi < 0$ , while as  $a \rightarrow (2k\pi + \pi/2)^-$  we have  $\cos a \rightarrow 0^+$ , hence  $b_1(a) \rightarrow +\infty$  by Lemma 1(i), and  $G(a) \rightarrow +\infty$ . By the intermediate value theorem  $G$  has exactly one zero  $a_k$  in  $I$ , and  $z = a_k + i b_1(a_k)$  is the unique solution of  $\sin(z) = z$  in the strip with  $b > 0$ .

If  $k = 0$ , then  $G(0) = 0$  and  $G$  is strictly increasing, so  $G > 0$  on  $(0, \pi/2)$  and the strip  $(0, \pi/2)$  contains no solution with  $b > 0$ .  $\square$

*Remark 2.*  $z = 0$  is a triple zero of  $\sin(z) - z$ , since

$$\sin(z) - z = -\frac{z^3}{6} + \frac{z^5}{120} - \cdots;$$

this is why the two  $k = 0$  branches of Figure 1 are tangent rather than transversal at the origin.

**Corollary 1.** *Equation (1) has infinitely many solutions in  $\mathbb{C}$ ; exactly one solution (with positive imaginary part) lies in each strip  $2k\pi < \Re z < 2k\pi + \pi/2$ ,  $k \geq 1$ , and no other solutions exist.*

## 4 Asymptotic expansions of the solutions

Theorem 1 settles existence, uniqueness, and location. We now describe the solutions quantitatively. Let

$$A_k = \frac{(4k+1)\pi}{2}, \quad L_k = \ln(2A_k).$$

**Theorem 2.** *As  $k \rightarrow \infty$ , the solution  $(a_k, b_k)$  of Theorem 1 satisfies*

$$a_k = A_k - \frac{L_k}{A_k} + \frac{4L_k^3 - 18L_k^2 + 6L_k + 3}{12A_k^3} - \frac{96L_k^5 - 1000L_k^4 + 2320L_k^3 - 840L_k^2 - 420L_k + 15}{480A_k^5} + O\left(\frac{L_k^7}{A_k^7}\right), \quad (5)$$

$$b_k = L_k + \frac{2L_k^2 - 4L_k - 1}{4A_k^2} - \frac{24L_k^4 - 176L_k^3 + 216L_k^2 + 24L_k - 15}{96A_k^4} + O\left(\frac{L_k^6}{A_k^6}\right). \quad (6)$$

In particular,  $b_k \sim \ln(4\pi k)$ : the imaginary parts grow only logarithmically, while  $a_k - 2k\pi \rightarrow \pi/2$ .

*Proof.* Put  $a = A - \varepsilon$  with  $A = A_k$ . Since  $A = 2k\pi + \pi/2$ , we have  $\cos(a) = \sin(\varepsilon)$  and  $\sin(a) = \cos(\varepsilon)$ , and the system (2)–(3) becomes

$$\sinh(b) \sin(\varepsilon) = b, \quad \cosh(b) \cos(\varepsilon) = A - \varepsilon. \quad (7)$$

The second equation gives  $\cosh(b) = (A - \varepsilon) \sec(\varepsilon) =: x$ , hence  $b = \operatorname{arccosh}(x)$  and  $\sinh(b) = \sqrt{x^2 - 1}$ ; substituting into the first equation yields the single equation

$$\sin(\varepsilon) \sqrt{x^2 - 1} = \operatorname{arccosh}(x), \quad x = (A - \varepsilon) \sec(\varepsilon). \quad (8)$$

For large  $k$ ,  $x$  is large and  $\varepsilon$  is small. Using the expansions

$$\operatorname{arccosh}(x) = \ln(2x) - \frac{1}{4x^2} - \frac{3}{32x^4} - \dots, \quad \sqrt{x^2 - 1} = x \left(1 - \frac{1}{2x^2} - \dots\right),$$

and  $\ln(2x) = L + \ln((1 - \varepsilon/A) \sec \varepsilon)$ , and expanding (8) in powers of  $1/A$  with  $L = \ln(2A)$  carried as a logarithmic gauge—the technique employed by Euler for the roots of  $\tan x = x$  [10]—one obtains  $\varepsilon_k = L/A + O(L^3/A^3)$  and  $b_k = \operatorname{arccosh}(x) = L + O(L^2/A^2)$ . Continuing the expansion order by order gives the displayed coefficients; the expansion of  $\varepsilon_k$  contains only odd powers of  $1/A$  and that of  $b_k$  only even powers.  $\square$

*Remark 3* (higher-order terms). The next terms are

$$\begin{aligned} a_k : & + \frac{960L^7 - 16464L^6 + 80864L^5 - 132720L^4 + 46200L^3 + 23730L^2 - 3990L - 805}{6720A^7}, \\ b_k : & + \frac{80L^6 - 1096L^5 + 3900L^4 - 3720L^3 - 210L^2 + 495L + 35}{480A^6} \\ & - \frac{40320L^8 - 836352L^7 + 5318656L^6 - 12800256L^5}{322560A^8} \\ & + \frac{10251360L^4 + 440160L^3 - 1809360L^2 - 78960L + 32025}{322560A^8}. \end{aligned}$$

With all terms through  $A^{-7}$  (for  $a_k$ ) and  $A^{-8}$  (for  $b_k$ ), the absolute errors, measured against roots recomputed by Newton's method at 40-digit precision, are below  $1.2 \times 10^{-6}$  for every  $k \geq 1$  (worst case  $k = 1$ ), below  $4 \times 10^{-13}$  for  $k \geq 10$ , and decrease like  $A_k^{-9}$ ; see Table 2. One step of Newton's method started from the series value refines any of these to machine precision.

Table 1: The first nine non-real solutions  $a_k + ib_k$  of  $\sin z = z$ , rounded to four decimals. The first row reproduces the 105-digit OEIS values A138282/A138283.

| $k$ | $a_k$ (real part) | $b_k$ (imaginary part) |
|-----|-------------------|------------------------|
| 1   | 7.4977            | 2.7687                 |
| 2   | 13.9000           | 3.3522                 |
| 3   | 20.2385           | 3.7168                 |
| 4   | 26.5545           | 3.9831                 |
| 5   | 32.8597           | 4.1933                 |
| 6   | 39.1588           | 4.3668                 |
| 7   | 45.4541           | 4.5146                 |
| 8   | 51.7468           | 4.6434                 |
| 9   | 58.0377           | 4.7575                 |

*Remark 4* (priority of the leading term). The leading term  $b_k \approx L_k = \ln(2A_k) = \ln((4k+1)\pi)$  is not new: Hardy [2] already used

$$\operatorname{Im} z_k \approx \log((4n+1)\pi)$$

as a starting approximation, and Hillman and Salzer [4] computed their table with it. What Theorem 2 adds is the systematic expansion beyond the leading term: the coefficients of  $A_k^{-3}, A_k^{-5}, A_k^{-7}$  (and  $A_k^{-2}, A_k^{-4}, A_k^{-6}, A_k^{-8}$  for  $b_k$ ) as explicit polynomials in  $L_k$ , together with the parity statement and the observed error decay  $A_k^{-9}$ .

*Remark 5* (comparison with the classical series for  $\tan x = x$ ). Writing the large roots of  $\tan x = x$  as  $\alpha_n = A_n - \varepsilon_n$  with  $A_n = (n + \frac{1}{2})\pi$ , one has

$$\varepsilon_n = \frac{1}{A_n} + \frac{2}{3A_n^3} + O(A_n^{-5}),$$

a series in *pure* odd powers of  $1/A_n$  with numerical coefficients—this is the expansion going back to Euler [10] and Rayleigh [11]. The series (5)–(6) share the parity structure (odd powers for the abscissa-like quantity, even powers for the ordinate-like one) but their coefficients involve the logarithmic gauge  $L_k = \ln(2A_k)$ . This is forced by the geometry of (1): the non-real roots have an ordinate  $b_k$  that itself grows like  $\ln A_k$ , whereas the roots of  $\tan x = x$  all lie on the real axis. Theorem 2 may thus be viewed as a two-dimensional, logarithmically gauged descendant of the classical one-dimensional series.

## 5 Numerical results

The roots were computed by Newton’s method applied to the system (2)–(3), with initial values read off Figure 1 (namely, near the valley  $\tan a = a$  of Family II); all computations were carried out in 40-digit precision arithmetic and cross-checked against the 105-digit values of the OEIS [16]. We emphasize that the numerical values themselves are not new: a hundred roots to eleven decimals are tabulated by Ling [8], and the first five roots to ten decimals for  $a = 1$  are given by Fettis [9]. Table 1 lists the first nine non-real solutions (with  $\Re z, \Im z > 0$ ) for reference; the remaining solutions follow by the symmetries of Theorem 1.

Table 2 tests the full asymptotic series (all terms of Remark 3) against the exact roots. The accuracy is remarkable: already for  $k = 1$  the truncated series gives six correct digits, for  $k \geq 10$  it gives twelve or more, and the error decays like  $A_k^{-9}$  (Figure 2).

Table 2: Accuracy of the full asymptotic series (terms through  $A^{-7}$  for  $a_k$  and  $A^{-8}$  for  $b_k$ ) against exact roots computed at 40-digit precision.

| $k$ | $a_k$ (exact)   | $b_k$ (exact) | error in $a_k$        | error in $b_k$        |
|-----|-----------------|---------------|-----------------------|-----------------------|
| 1   | 7.4976762778    | 2.7686782830  | $1.1 \times 10^{-6}$  | $4.2 \times 10^{-7}$  |
| 2   | 13.8999597140   | 3.3522098849  | $8.6 \times 10^{-9}$  | $2.8 \times 10^{-9}$  |
| 3   | 20.2385177078   | 3.7167676798  | $1.7 \times 10^{-9}$  | $5.6 \times 10^{-12}$ |
| 5   | 32.8597410051   | 4.1932514704  | $7.3 \times 10^{-11}$ | $4.4 \times 10^{-12}$ |
| 10  | 64.3272337133   | 4.8599166479  | $4.2 \times 10^{-13}$ | $3.0 \times 10^{-14}$ |
| 20  | 127.1909739047  | 5.5397691118  | $4.8 \times 10^{-16}$ | $7.9 \times 10^{-17}$ |
| 50  | 315.7096400349  | 6.4481761403  | $2.5 \times 10^{-18}$ | $3.2 \times 10^{-21}$ |
| 100 | 629.8779939924  | 7.1387369112  | $2.0 \times 10^{-20}$ | $9.2 \times 10^{-23}$ |
| 500 | 3143.1606673312 | 8.7461351812  | $8.3 \times 10^{-26}$ | $1.6 \times 10^{-28}$ |

Table 3: Errata in previously published tables, against roots recomputed at 40-digit precision. For Hillman and Salzer [4] the remaining six entries are correct to the last printed digit; the errors are transpositions of one or two digits. Fettis [9] gives ten-decimal values for several values of  $a$ ; for  $a = 1$  the ordinate of the fifth root is correct, but the abscissa was printed as 32.85971 005 instead of 32.85974 1005.

| source         | $n$ | exact $a_n$   | exact $b_n$  | printed $a_n$      | printed $b_n$     |
|----------------|-----|---------------|--------------|--------------------|-------------------|
| Hillman–Salzer | 6   | 39.158817     | 4.366795     | 39.158817          | 4.369 <b>795</b>  |
| Hillman–Salzer | 7   | 45.454071     | 4.514640     | 45.454 <b>971</b>  | 4.51 <b>6410</b>  |
| Hillman–Salzer | 9   | 58.037662     | 4.757515     | 58. <b>637</b> 062 | 4.757515          |
| Hillman–Salzer | 10  | 64.327234     | 4.859917     | 64.327234          | 4. <b>838</b> 917 |
| Fettis         | 5   | 32.8597410051 | 4.1932514704 | 32.85971005        | 4.1932514700      |

## 6 Concluding remarks

### 6.1 No closed form; series as the natural description

The roots of (1) do not belong to the range of any classical special function. Contrast the sister equation  $e^x = x$ , all of whose complex solutions are  $-W_k(-1)$  in terms of the branches of the Lambert  $W$  function [15]—but the function  $W$  was invented precisely to solve that equation, and  $\sin z = z$  admits no analogous inverse. (A rigorous proof of “no closed form” is not available for any transcendental equation of this type; the statement is one about the current state of mathematics, and  $\tan x = x$  has likewise been treated by series since Euler [10].) The asymptotic series of Theorem 2 is therefore the natural description of the general solution: for each  $k$ ,  $a_k$  and  $b_k$  admit explicit rational-coefficient expansions in  $A_k$  and  $L_k$ , the former containing only odd powers and the latter only even ones. We emphasize that these are *asymptotic* expansions, valid as  $k \rightarrow \infty$  and generally divergent if summed to infinity—as is also the case for Euler’s series for  $\tan x = x$ —but truncation after the last displayed term is extraordinarily accurate (six digits already at  $k = 1$ ), and a single Newton step from the series value reaches machine precision.

### 6.2 Related equations

The real root of  $\cos x = x$ , the Dottie number  $0.739085\dots$ , is the fixed point of the iteration  $x_{n+1} = \cos x_n$  and has an extensive literature of its own [14]; the imaginary roots of  $\cos x = x$  were studied by Miller in 1902 [12]. The method of Section 2 applies verbatim to both equations.

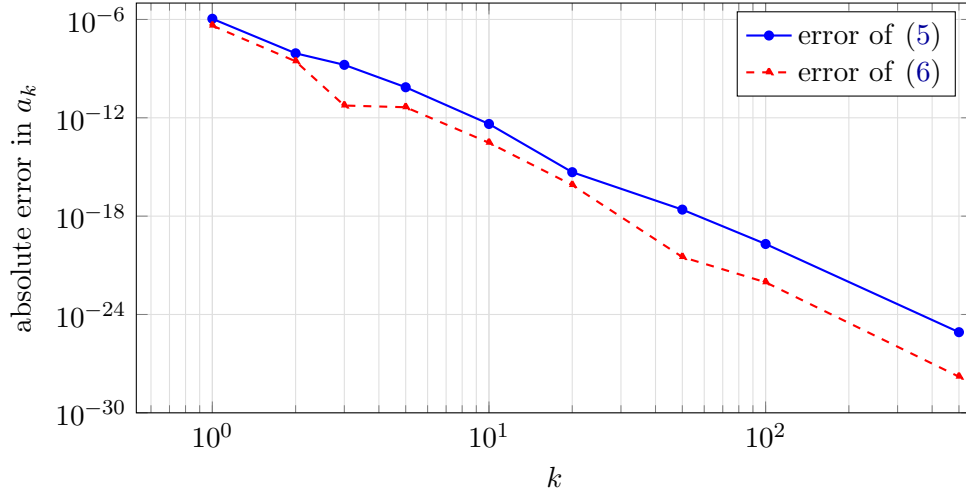


Figure 2: Absolute error of the truncated asymptotic series (5)–(6) versus  $k$  (log–log scale). The error decays like  $A_k^{-9}$ , in accordance with Remark 3.

### 6.3 Dynamical remark

Viewed as a fixed point equation of the entire function  $\sin z$ , the point 0 is a *parabolic* fixed point ( $\cos 0 = 1$ ), while every non-real fixed point  $z_k = a_k + ib_k$  is *repelling*: by (4),

$$|\cos z_k|^2 = \cos^2(a_k) \cosh^2(b_k) + \sin^2(a_k) \sinh^2(b_k) = b_k^2 \coth^2(b_k) + a_k^2 \tanh^2(b_k) > b_k^2,$$

and  $\cosh b_k = a_k / \sin a_k \geq a_k > 2\pi$  implies  $b_k > \operatorname{arccosh}(2\pi) \approx 2.53$ ; hence  $|\cos z_k| > 2.5$  for all  $k \geq 1$  (already  $|\cos z_1| \approx 7.95$ ), and asymptotically  $|\cos z_k| \sim |z_k|$ . This agrees with the OEIS note that the first non-real solution is a repelling fixed point of  $\sin z$  [16], and connects (1) to the dynamics of entire functions.

### Outlook

The graphical–analytic method of Section 2 extends without conceptual change to  $\cos z = z$ ,  $\tan z = z$ , and to the eigenvalue equations of quantum wells and impedance-wall waveguides [13, 11], where transcendental characteristic equations of precisely this shape govern the spectrum. A natural next step is a rigorous error bound for the asymptotic series of Theorem 2, which the computer-algebra derivation of Remark 3 makes plausible but does not yet supply.

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