

Three-Branch Structure of Operator Dynamics in a Two-Qubit Model

Maksim Lipyanchik

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Abstract

We investigate the dynamics of local operators in the simplest non-trivial model of two qubits governed by the generator $K(J, h) = JX_1X_2 + h(Z_1 + Z_2)$. Based on the maximized commutator magnitude $D_{12}(s) = \max_{a,b \in \{X,Y,Z\}} \|[\sigma_s(a_1), b_2]\|_{\text{op}}$, where $\sigma_s(A) = e^{isK} A e^{-isK}$, we construct a scalar characteristic of the operator dynamics. A precise three-branch formula is obtained: $D(x, r) = \max\{F_0(x), F_1(x, r), F_{\text{mix}}(x, r)\}$, where $x = Js$ and $r = h/J$. The branches F_0 , F_1 , and F_{mix} possess explicit analytical forms and correspond to the inter-qubit, field, and mixed contributions, respectively. We analytically describe the phenomenon of early blindness: at a fixed $r > 0$ and sufficiently short times, $D(x, r) = F_0(x)$ remains completely independent of the field parameter r . Furthermore, we identify blind lines at $x = (2n + 1)\pi/4$, where $D = 2$ for all $r > 0$. These results provide a complete analytical description of the considered observable in the minimal two-qubit model.

1 Introduction

Operator dynamics in the Heisenberg picture and its associated quantities (commutator norms, out-of-time-order correlators (OTOCs), and operator growth) are widely used to diagnose quantum chaos, scrambling, and thermalization [4–6]. In the general case, the analytical evaluation of such quantities is substantially more complex than for simple integrable systems, and frequently demands asymptotic or numerical methods.

In this work we consider the minimal non-trivial model in which the evolution of local operators admits a complete analytical description. Despite its simplicity, the model exhibits a rich structure characterized by a three-branch maximum, an early loss of sensitivity to the field parameter, and blind lines.

The primary result of this study is an explicit three-branch formula for the maximized commutator magnitude $D_{12}(s)$, along with a detailed characterization of its properties. This formula reduces the complex maximization over nine operator commutators to the maximum of three scalar functions with a transparent analytical structure.

This study is motivated, on the one hand, by the general interest in exactly solvable examples of operator dynamics, and on the other hand, by problems of sensitivity analysis for quantum generators under limited time observations. The latter requires a clear understanding of which aspects of the dynamics carry information about parameters and which remain blind.

A note on terminology: we use the term “operator dynamics” in the Heisenberg representation with a generator K . In a broader context, K can be interpreted as the modular Hamiltonian of a certain state [1–3]; however, in the present paper we do not tie ourselves to any specific state and employ the standard Heisenberg evolution.

2 Model and Observables

2.1 Generator and Heisenberg Evolution

We consider a system of two qubits with the Hamiltonian-generator:

$$K(J, h) = JX_1X_2 + h(Z_1 + Z_2),$$

where X_i, Z_i denote the Pauli matrices acting on the i -th qubit. The evolution of a local operator A in the Heisenberg picture is given by:

$$\sigma_s(A) = e^{isK} A e^{-isK}.$$

Our primary interest lies in tracking the growth of the non-commuting part of local operators under the action of this flow.

2.2 Maximized Commutator Characteristic

For distinct sites $1 \neq 2$, we introduce the quantity:

$$D_{12}(s) = \max_{a, b \in \{X, Y, Z\}} \|[\sigma_s(a_1), b_2]\|_{\text{op}}.$$

This observable measures the maximum non-commutativity of the evolved local operator at site 1 with respect to a local operator at site 2. It serves as a natural scalar characteristic of the operator dynamics in a two-particle system.

By construction, $D_{12}(s)$ is close in spirit to OTOC-like quantities used to characterize scrambling, but differs crucially in that it is maximized over the local basis and does not require any state-averaging procedure.

3 Dimensionless Reduction

3.1 Equivalence Under Parameter Sign Inversion

We show that the simultaneous or separate inversion of signs $J \rightarrow -J$ and $h \rightarrow -h$ leaves $D_{12}(s)$ invariant up to a redefinition of the local operator set.

Inversion of the sign of J . Conjugation by Z_1 at site 1 yields $Z_1X_1Z_1 = -X_1$, $Z_1Y_1Z_1 = -Y_1$, and $Z_1Z_1Z_1 = Z_1$, while operators at site 2 remain untouched. Thus:

$$Z_1K(J, h)Z_1 = -JX_1X_2 + h(Z_1 + Z_2) = K(-J, h).$$

Inversion of the sign of h . Conjugation by X_1X_2 gives $(X_1X_2)X_1(X_1X_2) = X_1$, $(X_1X_2)Y_1(X_1X_2) = -Y_1$, and $(X_1X_2)Z_1(X_1X_2) = -Z_1$, and identically for Z_2 . Thus:

$$(X_1X_2)K(J, h)(X_1X_2) = JX_1X_2 - h(Z_1 + Z_2) = K(J, -h).$$

Both conjugations map the set $\{X_1, Y_1, Z_1\}$ onto itself up to signs and leave the operators on site 2 unaltered. Consequently, the set of operator norms $\|[\sigma_s(a_1), b_2]\|_{\text{op}}$ is preserved, and $D_{12}(s)$ is invariant under sign changes of J and h . It is therefore sufficient to restrict the analysis to the fundamental sector:

$$J > 0, \quad r = h/J > 0.$$

3.2 Dimensionless Variables

We introduce the dimensionless parameters:

$$x = Js, \quad \Omega = \sqrt{1 + 4r^2}.$$

Consequently, $D_{12}(s)$ depends solely on the pair (x, r) :

$$D_{12}(s) = D(x, r).$$

4 The Three-Branch Formula

4.1 Derivation of the Formula

The direct evaluation of all nine commutators $[\sigma_s(a_1), b_2]$ and the computation of their operator norms lead to the following exact result:

$$D(x, r) = \max\{F_0(x), F_1(x, r), F_{\text{mix}}(x, r)\},$$

where

$$\begin{aligned} F_0(x) &= 2|\sin 2x|, \\ F_1(x, r) &= \frac{4|\sin(\Omega x)|}{\Omega^2} \sqrt{\Omega^2 - \sin^2(\Omega x)}, \\ F_{\text{mix}}(x, r) &= \frac{2}{\Omega} \sqrt{(|\cos x \sin(\Omega x)| + |\Omega \cos(\Omega x) \sin x|)^2 + 4r^2 \sin^2 x \sin^2(\Omega x)}. \end{aligned}$$

Expanding the squares yields an equivalent representation for the mixed branch:

$$F_{\text{mix}}^2(x, r) = 4 \sin^2 x + \frac{2|\sin 2x|}{\Omega} |\sin(2\Omega x)| + \frac{4 \cos 2x}{\Omega^2} \sin^2(\Omega x).$$

The general behavior of $D(x, r)$ for various values of r , along with the three competing branches F_0, F_1, F_{mix} at $r = 1.0$, is illustrated in Fig. 1.

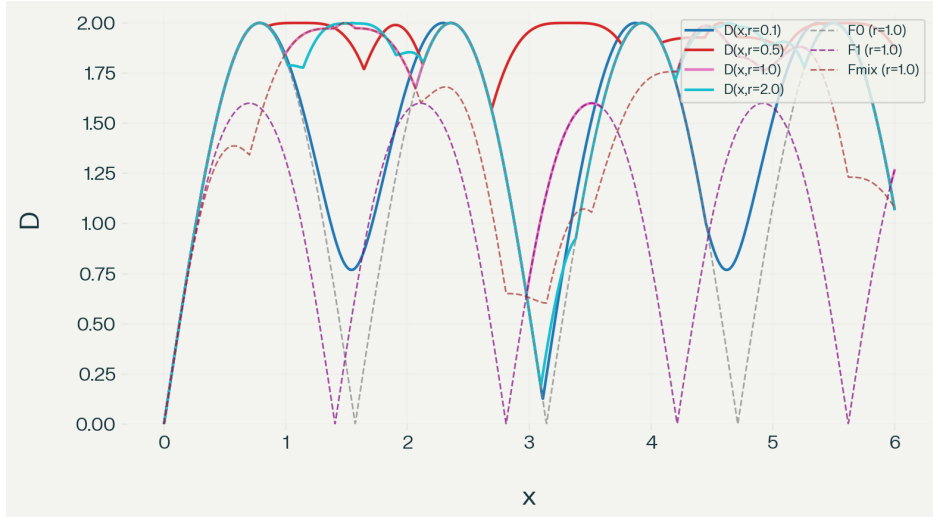


Figure 1: The maximized commutator magnitude $D(x, r)$ for several values of r (solid lines) and the three competing branches F_0, F_1, F_{mix} at $r = 1.0$ (dashed lines). It is evident that D tracks the upper envelope of the three branches.

The exhaustive derivation featuring the explicit forms of all nine commutators is provided in Appendix A.

4.2 Interpretation of the Branches

$F_0(x) = 2|\sin 2x|$ is the branch independent of r , originating purely from the inter-qubit interaction JX_1X_2 .

$F_1(x, r)$ is the branch containing the dependence on the local field, governed by the frequency $\Omega = \sqrt{1 + 4r^2}$. It is through this channel that the parameter r enters one of the competing components of the maximum. At $r = 0$, we have $\Omega = 1$ and $F_1(x, 0) = 4|\sin x|\sqrt{1 - \sin^2 x} = 2|\sin 2x| = F_0(x)$; the branching occurs exclusively in the presence of a non-zero field.

$F_{\text{mix}}(x, r)$ is the mixed branch, which carries the richest phase-amplitude structure. It captures the interference between the inter-qubit and local-field dynamical channels.

The individual branches and their maximum at $r = 1.0$ are displayed in Fig. 2.

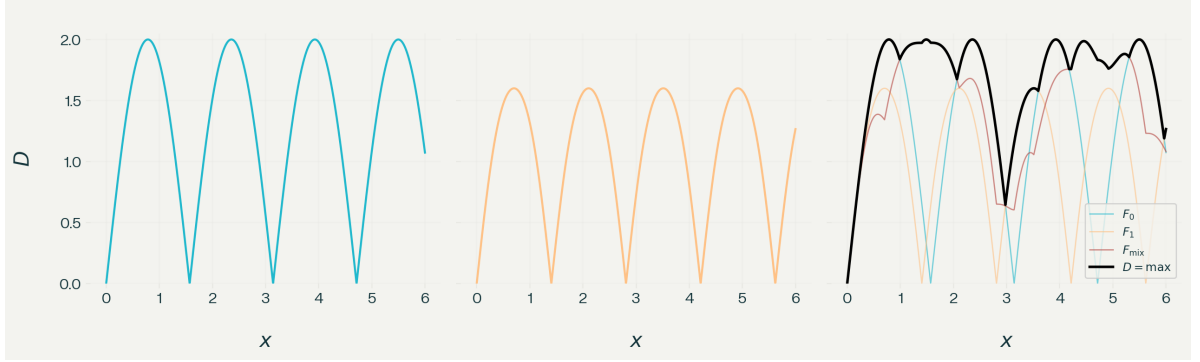


Figure 2: The branches F_0, F_1, F_{mix} shown separately (at $r = 1.0$) and their maximum (the bold line on the right). The right panel demonstrates the alternation of the branches that realize D as a function of x .

We interpret these three branches as manifestations of the inter-qubit, field, and mixed contributions to the maximized commutator magnitude. This interpretation serves as our working framework for this model; we do not claim that an identical decomposition into three distinct branches has been previously established in the literature on operator growth and scrambling.

5 Early Blindness and Blind Lines

A comprehensive picture of which branch realizes the maximum $D(x, r)$ across the parameter domain is provided by the active branch map in Fig. 3.

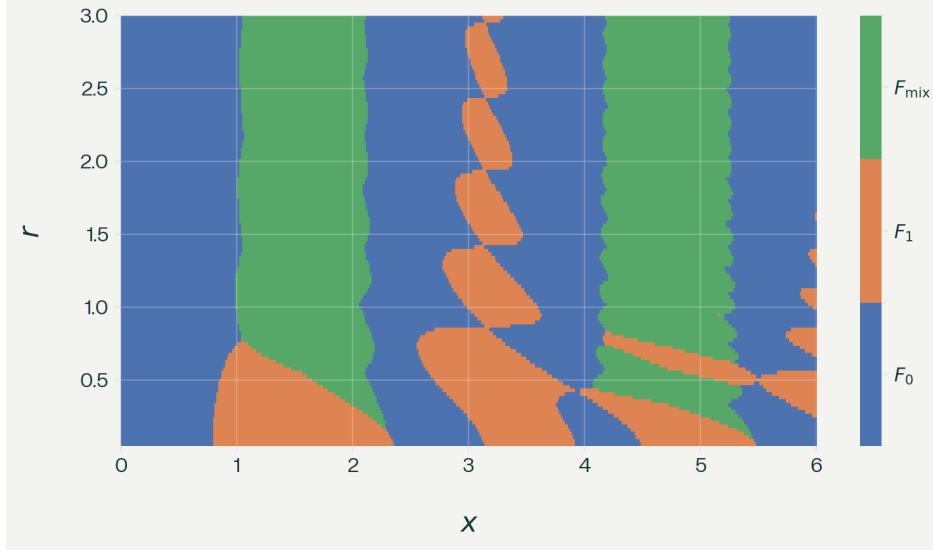


Figure 3: The active branch map: at each point (x, r) in the plane, the map indicates which of the functions F_0, F_1, F_{mix} realizes the maximum $D(x, r)$. The dominance domains of each branch and their boundaries are clearly visible.

5.1 Early Blindness

For a fixed field parameter $r > 0$ and in the limit $x \rightarrow 0^+$, the Taylor expansions of the squared branches take the form:

$$\begin{aligned} F_0^2 &= 16x^2 - \frac{64}{3}x^4 + O(x^6), \\ F_1^2 &= 16x^2 - \frac{64}{3}(1 + r^2)x^4 + O(x^6), \\ F_{\text{mix}}^2 &= 16x^2 - \left(\frac{64}{3} + \frac{80}{3}r^2\right)x^4 + O(x^6). \end{aligned}$$

Their respective differences:

$$F_0^2 - F_1^2 = \frac{64}{3}r^2x^4 + O(x^6), \quad F_0^2 - F_{\text{mix}}^2 = \frac{80}{3}r^2x^4 + O(x^6)$$

are strictly positive for any $r > 0$ as $x \rightarrow 0^+$. Consequently, for each fixed $r > 0$ there exists a finite threshold $\varepsilon(r) > 0$ such that:

$$D(x, r) = F_0(x) = 2 \sin 2x, \quad 0 < x < \varepsilon(r).$$

Within this region $\partial_r D = 0$: the early dynamics is completely insensitive to the field parameter. This phenomenon is termed early blindness. We note that $\varepsilon(r)$ depends explicitly on

r ; no uniform bound over all $r > 0$ is implied. A detailed visualization of the early blindness domain and the difference asymptotics is shown in Fig. 4.

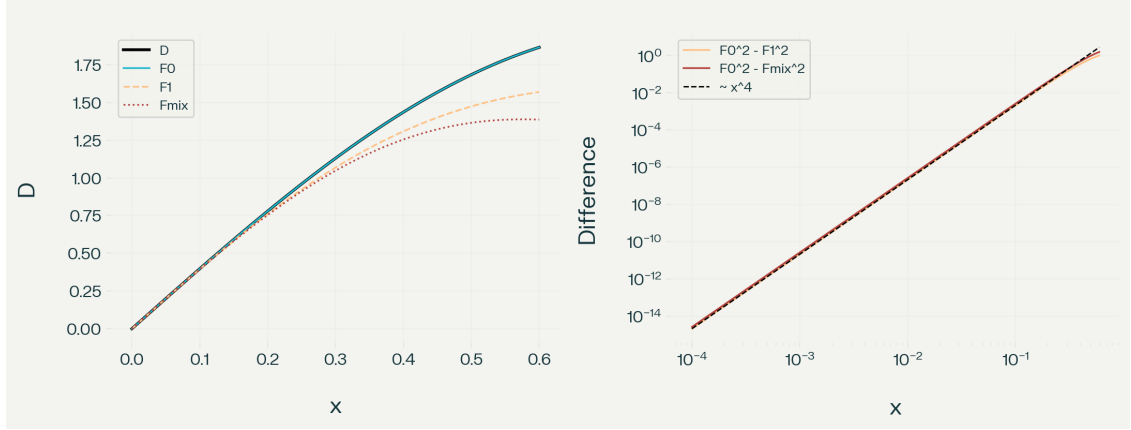


Figure 4: Early blindness at $r = 1.0$. Left: in the region $0 < x < \varepsilon(r)$ all three branches virtually coincide, and $D = F_0$. Right: the differences $F_0^2 - F_1^2$ and $F_0^2 - F_{\text{mix}}^2$ grow as x^4 , ensuring the existence of $\varepsilon(r) > 0$.

It is important to emphasize that early blindness pertains specifically to the observable $D_{12}(s)$ in this model. Other observables (e.g., individual unmaximized commutators or correlators of a different type) may exhibit sensitivity to the field parameter even at early times. In more complex systems, the initial stages of operator growth can also be predominantly governed by the local interaction structure.

5.2 Blind Lines

Along the lines defined by:

$$x = \frac{(2n+1)\pi}{4}, \quad n \in \mathbb{Z},$$

we have $F_0(x) = 2$. Since for any unitary operators the inequality $\|[A, B]\|_{\text{op}} \leq 2\|A\|_{\text{op}}\|B\|_{\text{op}} = 2$ holds, we obtain:

$$D\left(\frac{(2n+1)\pi}{4}, r\right) = 2 \quad \forall r > 0.$$

Thus, these specific time points are entirely blind to the parameter r . The corresponding lines are marked in Fig. 5.

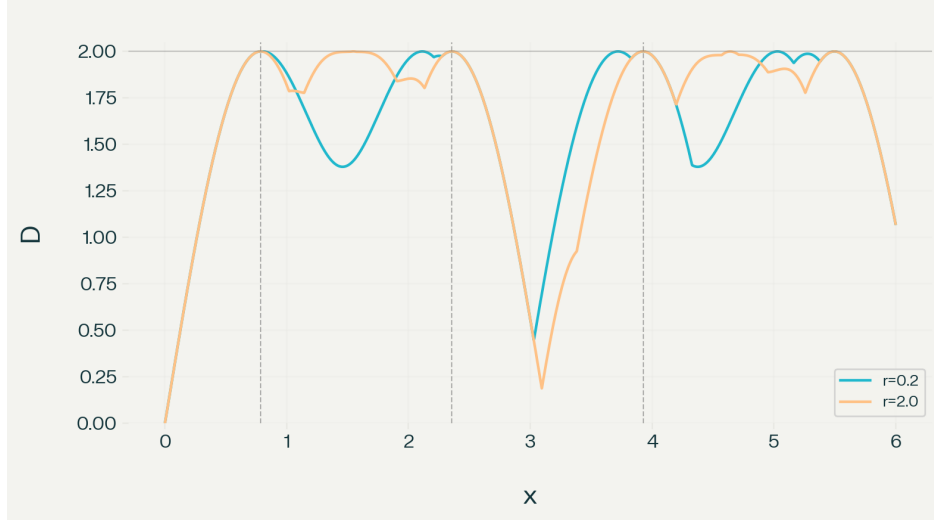


Figure 5: Blind lines at $x = (2n + 1)\pi/4$: at these coordinates, $D(x, r) = 2$ for all values of $r > 0$. Curves for $r = 0.2$ and $r = 2.0$ are displayed.

The term “blind” here signifies independence from the value of r within the framework of the considered model for all $r > 0$; cross-model universality is not claimed. On these lines, the branch F_0 saturates the absolute upper bound of 2, forcing $D(x, r) = 2$ for all r . This does not imply that the underlying dynamics halts, but rather that the maximized observable saturates its upper bound and loses its capacity to resolve the field parameter.

6 Spectral Reduction for Two Subsystems

From the quantity $D_{12}(s)$, we construct a weighted graph Laplacian:

$$L(s) = \begin{pmatrix} w & -w \\ -w & w \end{pmatrix}, \quad w = e^{-D_{12}(s)/\tau}.$$

Its spectrum is explicitly given by:

$$\Sigma_K(s) = \{0, 2e^{-D_{12}(s)/\tau}\}.$$

The non-zero eigenvalue as a function of time is plotted in Fig. 6.

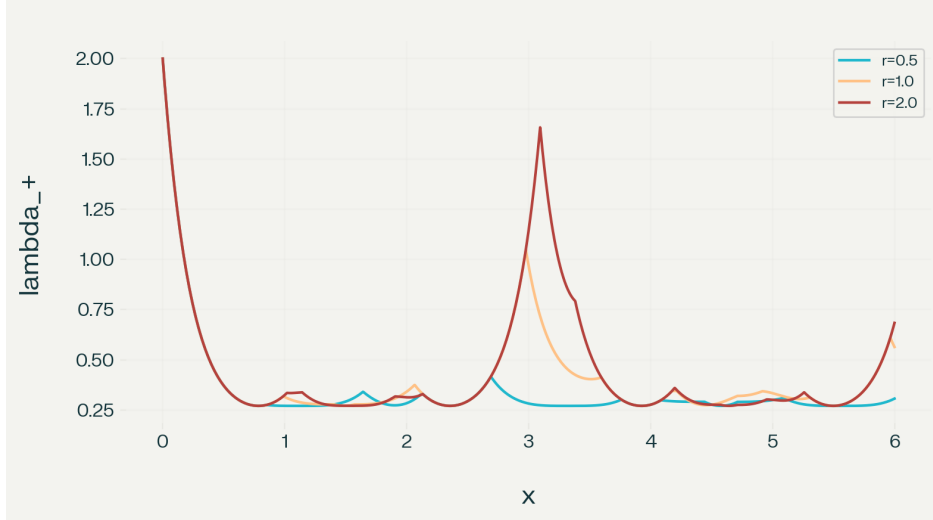


Figure 6: The non-zero eigenvalue of the weighted Laplacian $\lambda_+(s) = 2e^{-D_{12}(s)/\tau}$ at $\tau = 1$ for several values of r . For $N = 2$, the spectrum is completely equivalent to $D_{12}(s)$.

For a system of two subsystems, the spectrum $\Sigma_K(s)$ is fully equivalent to the scalar function $D_{12}(s)$, which follows trivially from the explicit form of $L(s)$. The spectral representation for $N = 2$ yields no additional information beyond D_{12} ; its role is strictly illustrative, demonstrating how a scalar commutator characteristic embeds into a spectral construction, which becomes non-trivial for $N > 2$.

7 Relation to Literature and Discussion

The derived three-branch structure provides a complete analytical account of the considered observable in the minimal non-trivial model.

Early blindness demonstrates that during the initial stage of the dynamics, the field parameter cannot be reconstructed from $D_{12}(s)$. This aligns qualitatively with the conventional understanding that early operator growth in many systems is predominantly governed by the local interaction structure, although cross-model universality is not implied.

The blind lines reveal that even at late times, isolated points exist that are completely insensitive to r . In more complex setups, analogous effects could manifest as temporal windows with diminished sensitivity to specific parameters.

The mixed branch F_{mix} carries the most intricate phase-amplitude structure and, depending on x and r , can determine the maximum of $D_{12}(s)$.

We view this model as a convenient, exactly tractable test case in which the description of operator dynamics can be carried out rigorously to the very end. A direct comparison with other exactly solvable models of operator growth is left for a separate work; we make no claims of cross-model universality for the derived structure here.

From the standpoint of further applications, the model allows one to cleanly separate structural blindness (early and along special lines) from local sensitivity (through the derivatives D_x, D_r).

8 Conclusion

In this study we have obtained an exact three-branch formula for the maximized commutator magnitude in a two-qubit model with the generator $K(J, h) = JX_1X_2 + h(Z_1 + Z_2)$, comprising the inter-qubit, field, and mixed branches. We have analytically characterized the phenomena of early blindness and blind lines. Furthermore, we showed that for two subsystems the spectrum of the weighted Laplacian is entirely equivalent to the scalar function $D_{12}(s)$.

The model serves as an exactly tractable test case for broader investigations into operator dynamics, scrambling, and parameter sensitivity analysis.

Appendix A. Derivation of the Three-Branch Formula

In this appendix we explicitly evaluate all nine commutators $[\sigma_s(a_1), b_2]$, $a, b \in \{X, Y, Z\}$, and demonstrate that their operator norms collapse onto three functional forms: $F_0(x)$, $F_1(x, r)$, and $F_{\text{mix}}(x, r)$.

A.1. Parity Decomposition

In the ordered basis $(|00\rangle, |11\rangle, |01\rangle, |10\rangle)$, the generator $K = X_1 X_2 + r(Z_1 + Z_2)$ (setting $J = 1$) is block-diagonal:

$$K_e = \begin{pmatrix} 2r & 1 \\ 1 & -2r \end{pmatrix}, \quad K_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $K_e^2 = \Omega^2 I$ and $K_o^2 = I$, with $\Omega = \sqrt{1 + 4r^2}$. Therefore, the matrix exponentials are:

$$e^{ixK_e} = \cos(\Omega x)I + i \frac{\sin(\Omega x)}{\Omega} K_e, \quad e^{ixK_o} = \cos x I + i \sin x K_o.$$

In what follows we denote $C = \cos(\Omega x)$, $S = \sin(\Omega x)$, $c = \cos x$, and $s = \sin x$.

A.2. Local Operator Evolution

Direct evaluation of the local operator evolution yields:

$$\begin{aligned} \sigma_x(X_1) &= \left(cC + \frac{sS}{\Omega}\right) X_1 - \frac{2rcS}{\Omega} Y_1 + \frac{2rsS}{\Omega} Z_1 X_2 + \left(sC - \frac{cS}{\Omega}\right) Z_1 Y_2, \\ \sigma_x(Y_1) &= \frac{2rcS}{\Omega} X_1 + \left(cC - \frac{sS}{\Omega}\right) Y_1 - \left(sC + \frac{cS}{\Omega}\right) Z_1 X_2 + \frac{2rsS}{\Omega} Z_1 Y_2, \\ \sigma_x(Z_1) &= aX_1 X_2 + bX_1 Y_2 + c_1 Y_1 X_2 - aY_1 Y_2 + (\text{local terms}), \end{aligned}$$

where the coefficients are defined as:

$$a = \frac{2rS^2}{\Omega^2}, \quad b = \frac{-\sin 2x + \Omega^{-1} \sin(2\Omega x)}{2}, \quad c_1 = \frac{\sin 2x + \Omega^{-1} \sin(2\Omega x)}{2}.$$

The local terms in $\sigma_x(Z_1)$ do not contribute to commutators with any b_2 ($b \in \{X, Y, Z\}$) and are therefore omitted.

A.3. The Nine Commutators and Their Norms

Direct calculation of the operator norms (the maximum singular value of the corresponding 4×4 matrix) yields:

- $\|[\sigma_x(X_1), X_2]\|_{\text{op}} = 2|sC - \frac{cS}{\Omega}|$
- $\|[\sigma_x(X_1), Y_2]\|_{\text{op}} = \frac{4r|sS|}{\Omega}$
- $\|[\sigma_x(X_1), Z_2]\|_{\text{op}} = 2\sqrt{(sC - \frac{cS}{\Omega})^2 + \frac{4r^2 s^2 S^2}{\Omega^2}}$

- $\|[\sigma_x(Y_1), X_2]\|_{\text{op}} = \frac{4r|sS|}{\Omega}$
- $\|[\sigma_x(Y_1), Y_2]\|_{\text{op}} = 2|sC + \frac{cS}{\Omega}|$
- $\|[\sigma_x(Y_1), Z_2]\|_{\text{op}} = 2\sqrt{(sC + \frac{cS}{\Omega})^2 + \frac{4r^2s^2S^2}{\Omega^2}}$
- $\|[\sigma_x(Z_1), X_2]\|_{\text{op}} = 2\sqrt{a^2 + b^2}$
- $\|[\sigma_x(Z_1), Y_2]\|_{\text{op}} = 2\sqrt{a^2 + c_1^2}$
- $\|[\sigma_x(Z_1), Z_2]\|_{\text{op}} = \max\{F_0, F_1\}$

A.4. Reduction to Three Branches

The Mixed Branch. Let $A = sC$, $B = cS/\Omega$, and $Q = 2rsS/\Omega$. Then:

$$\|[\sigma_x(X_1), Z_2]\| = 2\sqrt{(A - B)^2 + Q^2} =: f_{XZ}, \quad \|[\sigma_x(Y_1), Z_2]\| = 2\sqrt{(A + B)^2 + Q^2} =: f_{YZ}.$$

Employing the algebraic identity $\max\{(A - B)^2, (A + B)^2\} = (|A| + |B|)^2$, we obtain:

$$F_{\text{mix}} = \max\{f_{XZ}, f_{YZ}\} = \frac{2}{\Omega} \sqrt{(|\cos x \sin(\Omega x)| + |\Omega \cos(\Omega x) \sin x|)^2 + 4r^2 \sin^2 x \sin^2(\Omega x)}.$$

Expanding the terms under the square root yields the equivalent expression for F_{mix}^2 provided in Section 4.1.

The Inter-Qubit and Field Branches. As shown in Section A.3, $\|[\sigma_x(Z_1), Z_2]\|_{\text{op}} = \max\{F_0, F_1\}$. The commutator matrix $[\sigma_x(Z_1), Z_2] = 2iM$, when squared, gives $M^2 = \alpha I + \beta Z_1 Z_2$, where $\alpha = b^2 + c_1^2 + 2a^2$ and $\beta = 2(bc_1 + a^2)$. Its operator norm equals the square root of the largest eigenvalue $\alpha + |\beta|$, which strictly reduces to $\max\{F_0, F_1\}$ via the trigonometric identities $4r^2 = \Omega^2 - 1$ and $\Omega^2 C^2 + 4r^2 S^2 = \Omega^2 - S^2$.

Absorption of the (Z, X) and (Z, Y) Pairs. The quantities $2\sqrt{a^2 + b^2}$ and $2\sqrt{a^2 + c_1^2}$ are bounded from above by F_{mix} due to the identity $(A \pm B)^2 + Q^2 - (a^2 + b^2/c_1^2) = (s^2 - \Omega^{-2}S^2)^2 \geq 0$. Consequently, out of the nine candidate norms, the global maximum is always realized either by the mixed branch F_{mix} or by the (Z, Z) pair, which yields $\max\{F_0, F_1\}$.

A.5. Numerical Validation

The analytical formula was validated by direct numerical construction of the 4×4 matrices K , $U = e^{ixK}$, and $\sigma_x(a) = UaU^\dagger$, without utilizing the derived branch equations. For each point (x, r) , all nine commutators were evaluated, their operator norms computed via singular value decomposition (SVD), and the maximum extracted.

The verification used a regular grid with $x \in [0.005, 6.0]$ (200 points) and $r \in [0.05, 3.0]$ (60 points), totaling 12,000 nodes, together with an additional random sample of 250 points uniformly distributed in the same rectangle. All computations were performed in double precision (`numpy.float64`) using `scipy.linalg.expm`.

The results of the numerical experiment versus the analytical formula are presented in Fig. 7.

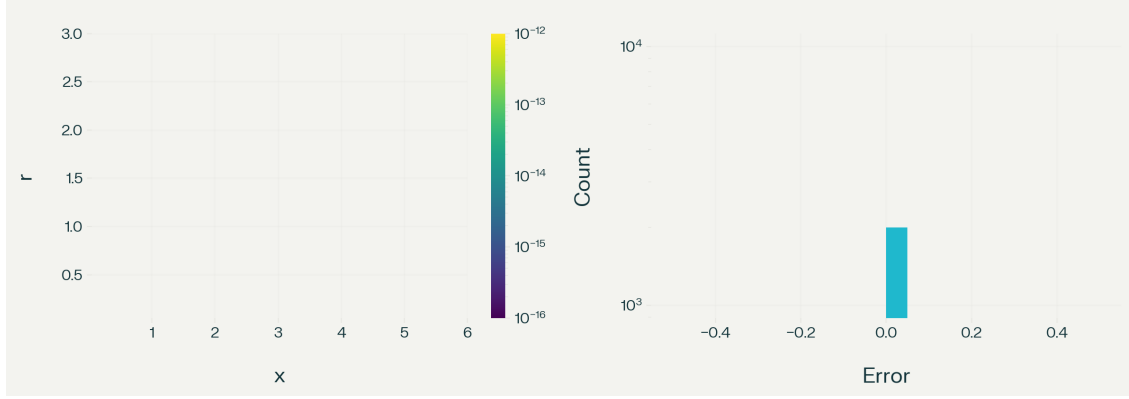


Figure 7: Numerical validation of the three-branch formula. Left: map of the discrepancy $|D_{\text{num}} - D_{\text{claim}}|$ over the (x, r) grid; the error is bounded everywhere by machine precision ($\leq 10^{-12}$), rendering the map visually uniform. Right: distribution of the discrepancies, showing a single sharp peak centered at zero with no long tails. The calculation is independent of the analytical branch formulas.

At all checked points, the brute-force maximum of the nine norms matched the value of $\max\{F_0, F_1, F_{\text{mix}}\}$ within a tolerance of 10^{-12} , consistent with the accumulated numerical error of `expm` and SVD. As an example, at $x = 0.5, r = 0.7$, the exact maximum is $D = 1.6829 = F_0$, realized by the (Z, Z) pair.

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