

Unified Physics through Exceptional Symmetries: An E_n -Indexed Field and mGFT Framework

Robert L. Jackson II

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Abstract

This framework is formulated directly in terms of the E_n hierarchy. The symmetry levels are labeled directly by E_n , with E_6 , E_7 , and E_8 treated as finite-dimensional exceptional Lie groups and E_9 , E_{10} , and higher labels treated, when invoked, as affine, hyperbolic, or more general indefinite E -type extensions whose precise generalized Cartan data must be specified. The exponentially weighted hypergeometric profile associated with an E_n level is written $H_{n,n-1}(2\xi)$. Integral ft-{ht,lsr}-mGFT remains an analytic profile and field-equation solution architecture; internal symmetry is carried by explicit E_n Lie-algebra generators and representations; string/brane structure can be carried by Clifford factors; and physical masses arise from the quadratic fluctuation operator of a specified action. The framework does not use infinite summations to define a unifying interaction potential or universal particle-mass spectrum. The primary directional architecture is the reduction chain $E_{11} \rightarrow E_{10} \rightarrow E_9 \rightarrow E_8 \rightarrow E_7 \rightarrow E_6$, with the reverse sequence treated as a constructive lift rather than an automatically invertible map. The phenomenological material is ordered by sector: the E_6 solar and hidden-particle benchmarks; the E_7 compact-object analysis; the E_9 affine/high-energy morphology sector; and the E_{10} hyperbolic/solitonic sector. The verification program is organized around explicit actions, residual-certified nonlinear calculations, convergence tests, and quadratic spectral extraction.

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1 Introduction

Unifying gravitational, gauge, scalar, fermionic, and possible hidden sectors remains a central problem in theoretical physics. This formulation starts directly from the exceptional E_n hierarchy. The organizing labels are E_6 , E_7 , E_8 , E_9 , and higher levels. The framework uses three complementary layers: an analytic layer for constructing field profiles, a symmetry layer in which matrix coefficients are expanded in verified Lie-algebra generators, and a physical verification layer in which an action, field equations, residual tests, and quadratic fluctuations determine whether a candidate profile represents a physical mode. No infinite series is used as the definition of the unifying symmetry. At each E_n level the physical program is to specify the intended algebra or group, its representations, symmetry breaking, invariant action, field equations, and observables, and then to use analytic or numerical machinery as a means of solving and testing those dynamics. For E_6 , E_7 , E_8 , the standard finite-dimensional exceptional structures provide the conventional starting point. For E_9 and higher labels, the framework must state the relevant affine or indefinite Kac–Moody realization explicitly before physical conclusions are drawn.

2 Direct E_n Exceptional Framework

2.1 Direct hierarchy

The symmetry hierarchy is indexed directly by E_n . No separate group family is used to generate or relabel the exceptional index. The levels tabulated explicitly in this manuscript are

$$E_n, \quad 6 \leq n \leq 11. \quad (1)$$

with the mathematical status of the levels distinguished rather than assumed to be identical. Higher indefinite labels may be studied separately, but they are outside the present object-mapping table.

Table 1: Direct E_n hierarchy used in the framework.

Level	Mathematical status	Role in the framework
E_6	finite exceptional Lie group/algebra	lowest finite exceptional level used for particle and solar benchmarks
E_7	finite exceptional Lie group/algebra	higher finite exceptional level for compact-object and field sectors
E_8	finite exceptional Lie group/algebra	highest finite exceptional level
E_9	affine $E_8^{(1)}$ -type extension	first extended E -type level used for affine/string-field or morphology studies
E_{10}	hyperbolic Kac–Moody extension	hyperbolic extended level
E_{11}	indefinite/Lorentzian Kac–Moody extension	higher extended level used in very-extended E -type constructions

2.2 Primary downward flow and inverse constructive lifts

The conventional Dynkin-diagram relations admit two complementary readings. Upward, one may pass through the regular finite exceptional embeddings and then add the affine, overextended, and very-extended nodes,

$$E_6 \xrightarrow{\text{extension}} E_7 \rightarrow E_8 \xrightarrow{\text{affinization}} E_9 \xrightarrow{\text{overextension}} E_{10} \xrightarrow{\text{very extension}} E_{11}. \quad (2)$$

For the extended part of the chain, $E_9 = E_8^{(1)}$, $E_{10} = E_8^{++}$, and $E_{11} = E_8^{+++}$ in the standard notation. The very-extended node of E_{11} can be deleted to recover its canonical E_{10} subalgebra, while deleting the overextended and affine nodes successively recovers E_9 and E_8 [22,27]. Suitable regular Dynkin subdiagrams similarly realize the familiar finite inclusions $E_6 \subset E_7 \subset E_8$ [1,14]. The present framework takes the reverse direction as the primary reduction/flow:

$$E_{11} \xrightarrow{\Pi_{11 \rightarrow 10}} E_{10} \xrightarrow{\Pi_{10 \rightarrow 9}} E_9 \xrightarrow{\Pi_{9 \rightarrow 8}} E_8 \xrightarrow{\Pi_{8 \rightarrow 7}} E_7 \xrightarrow{\Pi_{7 \rightarrow 6}} E_6. \quad (3)$$

Each $\Pi_{n \rightarrow n-1}$ denotes a restriction, projection, or appropriate node-deletion map. It is not, merely by virtue of the diagram relation, an isomorphism. The arrows in Eq. (2) are consequently interpreted in this framework as constructive extensions or lifts.

A simple analytic component of that lift acts on the generated-mode variable G . For nonnegative integer ℓ define

$$R_G : G^\ell \mapsto \frac{G^{\ell+1}}{\ell+1}. \quad (4)$$

On the image of this map,

$$R_G^{-1} = \partial_G, \quad \partial_G \left(\frac{G^{\ell+1}}{\ell+1} \right) = G^\ell. \quad (5)$$

Thus the shorthand $E_9 \xrightarrow{\int dG} E_{10}$ describes a constructive mode lift, whereas $E_{10} \xrightarrow{\partial_G} E_9$ describes the corresponding mode reduction. The full $E_n \rightarrow E_{n-1}$ operation must also act on the tensor coefficients and on compactified degrees of freedom.

Let the coefficient tensors at level E_n be

$$a_{A_1 \dots A_r}^{(n)}, \quad b_{A_1 \dots A_s}^{(n)}, \quad (6)$$

and split a higher-dimensional index as $A = (\mu, m)$, with μ labeling retained noncompact directions and m compactified directions. Introduce a dimensional compactification map $K_{n \rightarrow n-1}$ and coefficient contractions $C_a^{(n)}, C_b^{(n)}$. The framework defines

$$T_a^{(n \rightarrow n-1)} = C_a^{(n)} \circ K_{n \rightarrow n-1}, \quad T_b^{(n \rightarrow n-1)} = C_b^{(n)} \circ K_{n \rightarrow n-1}. \quad (7)$$

For example, if q and p compact indices are contracted,

$$a_{\mu_1 \dots \mu_{r-q}}^{(n-1)} = C_{m_1 \dots m_q} a_{\mu_1 \dots \mu_{r-q} m_1 \dots m_q}^{(n)}, \quad (8)$$

$$b_{\mu_1 \dots \mu_{s-p}}^{(n-1)} = D_{m_1 \dots m_p} b_{\mu_1 \dots \mu_{s-p} m_1 \dots m_p}^{(n)}. \quad (9)$$

Writing P_a and P_b for projectors onto the two coefficient sectors gives the downward coefficient map

$$M_{n \rightarrow n-1} = T_a^{(n \rightarrow n-1)} P_a + T_b^{(n \rightarrow n-1)} P_b. \quad (10)$$

The complete model-level reduction operator is then

$$D_{n \rightarrow n-1} = R_G^{-1} \circ M_{n \rightarrow n-1} = \partial_G \circ \left(T_a^{(n \rightarrow n-1)} P_a + T_b^{(n \rightarrow n-1)} P_b \right). \quad (11)$$

When the coefficient and mode operations are defined on the same retained sector, the reverse construction requires more than integration. Let $\mathcal{E}_{a,b}^{(n-1 \rightarrow n)}$ denote tensor extension and let $K_{n-1 \rightarrow n}^\uparrow$ denote a chosen decompactification/extension map. Then

$$I_{n-1 \rightarrow n} = \mathcal{E}_{a,b}^{(n-1 \rightarrow n)} \circ K_{n-1 \rightarrow n}^\uparrow \circ R_G. \quad (12)$$

Only on an information-preserving sector where compactification is actually invertible may one identify $K_{n-1 \rightarrow n}^\uparrow = K_{n \rightarrow n-1}^{-1}$. Ordinary compactification with mode truncation generally loses information, so Eq. (12) is properly a lift, not a guaranteed global inverse.

The resulting dual architecture is

$$E_{11} \xrightarrow{D_{11}} E_{10} \xrightarrow{D_{10}} E_9 \xrightarrow{D_9} E_8 \xrightarrow{D_8} E_7 \xrightarrow{D_7} E_6, \quad (13)$$

with constructive lifts

$$E_6 \xrightarrow{I_7} E_7 \xrightarrow{I_8} E_8 \xrightarrow{I_9} E_9 \xrightarrow{I_{10}} E_{10} \xrightarrow{I_{11}} E_{11}. \quad (14)$$

2.2A Framework Root-Extension Indexing and Successive Integral Lifts

The direct hierarchy in Sec. 2.1 remains the conventional core table used in the manuscript. For the framework's extended lattice bookkeeping, a constituent-level predecessor label E_5 and a post- E_{11} extension label E_{12} are appended. These endpoint labels are framework conventions: they do not assert that E_5 or E_{12} are additional finite exceptional Lie groups, and they do not alter the conventional identification of $E_{11} = E_8^{+++}$ as the very-extended E_8 stage used elsewhere in the manuscript.

Indexed root-extension chain

The symbols α_j in this subsection are framework bookkeeping labels for the successive lattice-extension directions; they are not asserted to be a universal or conventional numbering of simple roots across every realization. The internally consistent sequence is

$$\alpha_3, \alpha_2, \alpha_1, \alpha_0, \alpha_{-1}, \alpha_{-2}, \alpha_{-3}.$$

With this convention,

$$\Gamma_{E_5} \xrightarrow{+\mathbb{Z}\alpha_3} \Gamma_{E_6} \xrightarrow{+\mathbb{Z}\alpha_2} \Gamma_{E_7} \xrightarrow{+\mathbb{Z}\alpha_1} \Gamma_{E_8} \xrightarrow{+\mathbb{Z}\alpha_0} \Gamma_{E_9} \xrightarrow{+\mathbb{Z}\alpha_{-1}} \Gamma_{E_{10}} \xrightarrow{+\mathbb{Z}\alpha_{-2}} \Gamma_{E_{11}} \xrightarrow{+\mathbb{Z}\alpha_{-3}} \Gamma_{E_{12}}.$$

Equivalently,

$$\boxed{\Gamma_{E_5} = \Gamma_{E_6} - \mathbb{Z}\alpha_3}, \quad \boxed{\Gamma_{E_{12}} = \Gamma_{E_6} + \mathbb{Z} \sum_{j=-3}^2 \alpha_j}.$$

The alternate notation

$$\Gamma_{E_5} = \Gamma_{E_6} - \mathbb{Z}\alpha_{-3}$$

may be used as a separately declared framework convention, but it is not used in the indexed chain above because α_{-3} is already reserved for the $E_{11} \rightarrow E_{12}$ extension. Keeping the two conventions separate prevents one label from simultaneously denoting the low-end removal and the high-end extension.

The lattice definition of E_5 is also distinct from the constituent-sector assignment in Sec. 14.1 and in the field-organization scheme. The relation $\Gamma_{E_5} = \Gamma_{E_6} - \mathbb{Z}\alpha_3$ does not by itself establish an isomorphism $E_5 \cong \text{Spin}(10)$ or $E_5 \cong \text{SO}(10)$; the $\text{Spin}(10)/\text{SO}(10)$ association remains a separate model-level interpretation of the framework label.

The root-extension operation is algebraic bookkeeping on the lattice/root side. It should therefore be distinguished from the generated-mode integral lift below, which acts on powers of the variable G .

One-step and repeated integral lifts

Section 2.2 defined the generated-mode factor

$$\mathcal{I}_G[G^\ell] \equiv R_G[G^\ell] = \frac{G^{\ell+1}}{\ell+1}.$$

Relative to the incoming mode G^ℓ , the one-step multiplicative factor is

$$\boxed{\mathcal{I}_\ell = \frac{G}{\ell+1}}.$$

Here $\mathcal{I}_G$ denotes only the generated-mode component of the lift. It is not identical to the full model-level operator $I_{n-1 \rightarrow n}$ of Eq. (12), which also contains tensor extension and decompactification data.

A literal product of the same unchanged factor gives

$$\boxed{\prod_{i=1}^n \frac{G}{\ell+1} = \frac{G^n}{(\ell+1)^n}},$$

a fixed-denominator product rather than n consecutive integrations. For successive integral lifts, the denominator advances because the power of G rises after each step:

$$\boxed{\mathcal{I}_\ell^{(n)} = \prod_{i=1}^n \frac{G}{\ell+i} = \frac{G^n}{(\ell+1)(\ell+2)\cdots(\ell+n)} = \frac{G^n}{(\ell+1)_n}}, \quad n \geq 1,$$

where $(\ell+1)_n$ is the rising factorial. Acting directly on a mode gives

$$R_G^n[G^\ell] = \frac{G^{\ell+n}}{(\ell+1)_n}.$$

Let $A(E_9)$ denote the chosen affine- E_9 base object or field before repeated mode lifting, and let $\circ$ denote the framework composition. The analytic hierarchy is then written

$$\boxed{E_{n+9} = A(E_9) \circ \prod_{i=1}^n \frac{G}{\ell+i} = A(E_9) \circ \frac{G^n}{(\ell+1)_n}}, \quad n \geq 1.$$

Thus

$$E_{10} = A(E_9) \circ \frac{G}{\ell+1}, \quad E_{11} = A(E_9) \circ \frac{G^2}{(\ell+1)(\ell+2)},$$

$$\boxed{E_{12} = A(E_9) \circ \frac{G^3}{(\ell+1)(\ell+2)(\ell+3)}}.$$

These formulas assign hierarchy labels to successive analytic lifts; they do not, by themselves, derive the full representation content, Kac–Moody algebra, or physical interpretation of the target level. Within the present root chain, $E_{11} = E_8^{+++}$ retains the conventional very-extended E_8 designation, while E_{12} denotes the next framework extension beyond E_{11} . Any E_{12} association with a further F-/M-theory extension is therefore treated as framework-specific terminology rather than as a redefinition of the standard E_{11} nomenclature.

The lattice extension and the integral lift are coordinated organizational chains but not identical maps: the former adjoins or removes labeled root directions, whereas the latter is the analytic repeated-integration action on the generated G -mode.

2.3 Proposed astrophysical-object mapping across the E_n program

The phenomenological examples are presented for readability in the order $E_6 \rightarrow E_7 \rightarrow E_9$, after which Sec. 8 develops the proposed E_{10} sector. This presentation order is distinct from the primary reduction flow of Eq. (3). Table 2 records the framework-level astronomical and astrophysical assignments used across the project. These entries are model assignments or morphology hypotheses, not established astrophysical classifications or detections.

Table 2: Astronomical and astrophysical objects or structures proposed across the E_n program.

E_n	sec- tor	Objects or structures proposed in the project	Qualification / scope
E_6		Sun; Sun-like and ordinary stellar environments; solar proton/hydrogen-plasma, helium/alpha, electron, electron-neutrino, and CNO-cycle fields; speculative hidden-sector fields inside a Sun-like star.	Primary stellar/solar benchmark retained in this manuscript, with the direct $H_{6,5}(2\xi)$ labeling.
E_7		Neutron stars; magnetars; some white dwarfs; pulsar-like compact remnants; stellar-mass and newborn stellar black holes; binary black holes; neutron-star and other relativistic compact binaries; strongly curved stellar remnants; proposed transition or ultracompact compact objects.	Compact-object level used for the $H_{7,6}(2\xi)$ benchmark family. The benchmark set includes a neutron star, magnetar, white dwarf, pulsar-like compact remnant, and binary black hole.
E_8		Primordial black holes in candidate models; extreme black-hole interiors or near-horizon states; very-early collapse configurations; primordial/Big-Bang singularity boundary physics; extreme transient compact configurations.	Candidate higher-curvature/extreme-gravity level. Primordial black holes may be treated in either the E_8 or E_9 candidate sectors depending on the modeled configuration.
E_9		Quasars; blazars; highly energetic AGN engines and their central black-hole-magnetosphere-jet systems; some primordial-black-hole configurations; possible mirror-sector-coupled compact objects; AGN with Einstein-ring morphology.	Affine/high-energy candidate level. The Einstein-ring rendering remains a toy morphology comparison rather than an observational identification of E_9 .
E_{10}		JADES-GS-z14-0-like and MoM-z14-like high-redshift structures; Lyman-break-galaxy-like $\Phi^{(S)}$ morphologies; inverse/barred analogues; dense axion stars; oscillon/pseudobreather solitons; axitons; oscillatons; and related coherently oscillating scalar-soliton or boson-star-like compact configurations.	Hyperbolic/inverse-hyperbolic toy-structure level. The scalar-soliton literature is independent; assignment to E_{10} is a framework hypothesis. Morphology comparisons are analogies, not evidence that observed objects obey E_{10} .
E_{11}		No specific observed astronomical object is assigned in the present manuscript.	Very-extended structural level reserved for later algebraic development.

2.4 Finite and extended levels

For $n = 6, 7, 8$, the symbols E_n refer to the conventional finite-dimensional exceptional structures. For $n \geq 9$, the same notation is used only after the intended extension is specified. In particular, algebraic claims at the higher levels require the generalized Cartan matrix, root lattice, central or derivation generators where applicable, real form, representation category, and global structure relevant to the physical model.

2.5 Direct labeling convention

Every analytic profile, field representation, symmetry-breaking chain, and observable is labeled by the E_n sector itself. The notation therefore uses $E_n \longleftrightarrow H_{n,n-1}(2\xi)$ directly throughout.

2.6 E_8 and higher-level mathematical context

At the E_8 level, standard maximal subgroup structures include

$$E_8 \supset (E_7 \times SU(2))/\mathbb{Z}_2, \quad E_8 \supset (E_6 \times SU(3))/\mathbb{Z}_3. \quad (15)$$

and

$$E_8 \supset \text{Spin}(16)/\mathbb{Z}_2, \quad (16)$$

which provide explicit examples of how exceptional sectors can contain lower-rank structures [1]. Such branching data are more informative for physical model building than dimension matching alone.

For E_9 and above, the framework moves into affine, hyperbolic, and more general indefinite E -type structures. Work at these levels must identify the intended generalized Cartan matrix, root lattice, real form, representation category, and any required central or derivation generators. This remains one of the principal mathematical tasks of the theory.

3 Analytic Profiles and the E-Indexed Hypergeometric Hull

The analytic field profile is kept conceptually distinct from the internal symmetry. At a chosen symmetry level E_n , integral ft-{ht, lsr}-mGFT provides candidate analytic dependence of fields on spacetime or on a similarity variable. Adopting the notation used in the present program, let

$$H_{n,n-1}(2\xi) \quad (17)$$

denote the exponentially weighted generalized-hypergeometric hull associated with level E_n . The direct analytic assignment is

$$E_n \longleftrightarrow H_{n,n-1}(2\xi). \quad (18)$$

The hull does not define the exceptional group or algebra; it is a profile family carried by fields transforming in a chosen representation of the relevant E_n structure. This preserves the division

of labor between functional dependence and internal symmetry. The beta-lift hierarchy used here starts from a Gauss seed and adds one upper and one lower parameter per lift, producing the sequence $H_{d+2,d+1}$ after d lifts. Under the model-level one-direction/one-lift convention,

$$D_d \longleftrightarrow H_{d+2,d+1}(2\xi). \quad (19)$$

Consequently,

$$D_4 \longleftrightarrow H_{6,5}(2\xi) \longleftrightarrow E_6, \quad (20)$$

$$D_5 \longleftrightarrow H_{7,6}(2\xi) \longleftrightarrow E_7. \quad (21)$$

The brane label counts spatial world-volume directions; it does not determine matrix rank. Likewise, beta-integration variables and the physical world-volume measure remain distinct unless an explicit model identifies them.

4 Representation-Valued and Group-Valued Fields

A central construction is the promotion of scalar analytic profiles to Lie-algebra-valued fields by placing the symmetry information in coefficient matrices. Let T_α be generators in a chosen representation ρ of the Lie algebra associated with the selected E_n level. Coefficients can be expanded as

$$a_{ij} = A_{ij}^\alpha T_\alpha, \quad b_{ij} = B_{ij}^\alpha T_\alpha, \quad (22)$$

with

$$[T_\alpha, T_\beta] = f_{\alpha\beta}^\gamma T_\gamma. \quad (23)$$

A field profile $U(\xi)$ constructed from these coefficients transforms by conjugation,

$$U(\xi) \mapsto \rho(g)U(\xi)\rho(g)^{-1}. \quad (24)$$

When the representation and reality conditions permit, an algebra-valued field may be lifted to a group-valued field,

$$g(\xi) = \exp[U(\xi)]. \quad (25)$$

This construction gives a concrete meaning to the statement that a hypergeometric or mGFT profile “carries” an exceptional symmetry: the scalar profile determines functional dependence, while the generators, representation, and transformation law carry the symmetry.

5 E_6 : Subgroups, Visible Sectors, and Symmetry Breaking

The first finite exceptional level used in the present phenomenological program is E_6 . Standard E_6 model building admits subgroup routes including, at the Lie-algebra level, $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$, $\mathfrak{su}(6) \oplus \mathfrak{su}(2)$, and $\mathfrak{su}(3)^{\oplus 3}$; a familiar route to the Standard Model is schematically

$$E_6 \longrightarrow \text{Spin}(10) \longrightarrow SU(5) \longrightarrow G_{\text{SM}}, \quad (26)$$

with global quotients treated separately [1]. The E_6 sector therefore uses the standard representation branching needed to connect E_6 to visible and hidden sectors.

A schematic breaking chain is

$$E_6 \longrightarrow G_{\text{int}} \longrightarrow G_{\text{SM}} \times G_{\text{hidden}}. \quad (27)$$

For a total multiplet Φ in representation V , a chosen vacuum may induce

$$V \downarrow G_{\text{low}} = V_{\text{vis}} \oplus V_{\text{hidden}} \oplus V_{\text{grav}} \oplus V_{\text{scalar}}. \quad (28)$$

The physical content depends on the actual branching rules, vacuum, and invariant couplings rather than on the group label alone. A conditional stabilizer construction for $E_6 \times E_6$ is obtained as follows: if a bifundamental-like matrix M transforms as

$$M \longmapsto \rho(g_L)M\rho(g_R)^{-1}$$

and develops $\langle M \rangle = vI_{27}$ in a faithful 27-dimensional representation, the preserving subgroup is the diagonal E_6 . Within the present framework this suggests a route for coupling two E_6 sectors, provided a specified action dynamically produces and stabilizes such a condensate.

6 E_7 Compact-Object Benchmark Analysis

The E_7 compact-object analysis uses the direct E_7 label together with the analytic hull

$$H_{7,6}(2\xi) = e^{-\xi} {}_7F_6(\mathbf{p}; \mathbf{q}; 2\xi), \quad \dim \mathbf{p} = 7, \quad \dim \mathbf{q} = 6. \quad (29)$$

Here E_7 is the conventional finite exceptional Lie algebra/group label, whereas the association $E_7 \longleftrightarrow H_{7,6}(2\xi)$ is a modeling convention of the present framework rather than a standard result of compact-object astrophysics or Lie theory. Standard E_7 representation and subgroup data may be taken from established group-theory references [14], but those data do not by themselves identify neutron stars, magnetars, or black holes as physical realizations of E_7 .

6.1 Hypergeometric generated core and integral correction

With $z = 2\xi$ and $\Theta = z d/dz$, the unweighted generalized hypergeometric factor obeys

$$\left[\Theta \prod_{j=1}^6 (\Theta + q_j - 1) - z \prod_{i=1}^7 (\Theta + p_i) \right] {}_7F_6(\mathbf{p}; \mathbf{q}; z) = 0. \quad (30)$$

For a displayed field channel a , the generated core can therefore be written schematically as

$$U_a(X) = A_a(X) [H_{7,6}(2\xi_a(X))]^{\alpha_a}. \quad (31)$$

where A_a , α_a , and the similarity map $\xi_a(X)$ belong to the chosen physical model. The integral ft-{ht, lsr}-mGFT correction is then applied to the coupled equation

$$L_a[\Phi_a] + N_a[\Phi] = s_a, \quad E_{U_a} = L_a[U_a] + N_a[U]. \quad (32)$$

through the Riemannian-volume Green–Duhamel iterate

$$v_{a,k+1,m}(X) = h_a(X) + J_{a,m}[s_a - E_{U_a} - (N_a[U + v_{k,m}] - N_a[U])](X), \quad (33)$$

$$\Phi_a^{[K,m]} = U_a + v_{a,K,m}. \quad (34)$$

with

$$J_{a,m}[f](X) = \sum_{i=1}^m K_a(X, Y_i) f(Y_i) \Delta V_{g,i}, \quad \Delta V_{g,i} = \int_{C_i} \sqrt{|\det g(Y)|} dY. \quad (35)$$

At fixed mesh m , $K \rightarrow \infty$ is meaningful only when the nonlinear iteration converges. A continuum claim additionally requires mesh refinement, control of the Green/right-inverse approximation, and closure of the residual

$$R_{a,K,m} = L_a[\Phi_a^{[K,m]}] + N_a[\Phi^{[K,m]}] - s_a. \quad (36)$$

Thus the quarter-cut images are best interpreted as visualizations of the object-specific generated-core/closure ansatz unless those convergence and residual tests are supplied. This follows the same verification principle used elsewhere in the manuscript [13].

6.2 Field channels used in the E_7 analytic benchmark

The six analytic labels used in the detailed E_7 generated-core benchmark are channels for a coupled gravity–matter model:

Label	Role in the visualization / physical anchor
Φ_G	Scalarized amplitude representing the gravitational/geometry sector; its physical content must ultimately be reconstructed from the metric or connection rather than treated as an independent scalar by default.
Ψ_e	Electron spinor/matter amplitude, governed schematically by a curved-spacetime Dirac equation $(i\gamma^\mu D_\mu - m_e)\Psi_e = S_e$.
Ψ_ν	Neutrino spinor amplitude, analogously governed by a curved-spacetime Dirac/Majorana sector with the mass and interaction terms specified by the action.
Φ_γ	Scalarized electromagnetic/photon-field amplitude; the underlying tensor field is constrained by Maxwell or GRMHD equations, not by the scalar rendering alone.
Λ	Effective vacuum/curvature-source channel used in the visualization; it is not automatically identified with the measured cosmological constant in a local compact-object calculation.
R	Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$, shown as a curvature diagnostic.

The common relativistic anchor is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (37)$$

with electromagnetic contributions satisfying, for example,

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu, \quad \nabla_{[\alpha} F_{\beta\gamma]} = 0. \quad (38)$$

These standard equations determine what a physically calibrated realization of the displayed channels would have to satisfy.

6.3 Hypergeometric parameter vectors for the detailed $H_{7,6}$ benchmarks

The detailed $H_{7,6}$ generated-core benchmarks use the following seven-entry upper-parameter vectors $\mathbf{p}_H$ and six-entry lower-parameter vectors $\mathbf{q}_H$. They are analytic profile parameters rather than measured constants.

Object	$\mathbf{p}_H$	$\mathbf{q}_H$
Neutron star	(0.82, 1.14, 1.51, 1.93, 2.38, 2.86, 3.37)	(1.05, 1.46, 1.89, 2.34, 2.82, 3.31)
Magnetar	(0.91, 1.27, 1.68, 2.13, 2.62, 3.14, 3.69)	(1.18, 1.62, 2.08, 2.57, 3.09, 3.64)
Newborn stellar-mass black hole	(1.02, 1.41, 1.86, 2.36, 2.89, 3.46, 4.06)	(1.28, 1.76, 2.27, 2.82, 3.40, 4.01)
Binary black hole	(1.08, 1.49, 1.96, 2.47, 3.03, 3.62, 4.25)	(1.35, 1.86, 2.40, 2.98, 3.59, 4.23)

6.4 Compact-object physics used to constrain the interpretation

For a nonrotating perfect-fluid neutron star, the standard relativistic equilibrium anchor is the Tolman–Oppenheimer–Volkoff system [15],

$$\frac{dm}{dr} = 4\pi r^2 \rho. \quad (39)$$

$$\frac{dP}{dr} = -\frac{G(\rho + P/c^2)(m + 4\pi r^3 P/c^2)}{r^2(1 - 2Gm/(rc^2))}. \quad (40)$$

The neutron-star rendering therefore represents a layered field ansatz that would have to be solved consistently with an equation of state and these relativistic structure equations, rather than replacing them. For a magnetar, the stress-energy budget must include the strong electromagnetic field and its coupling to matter. The magnetar interpretation is anchored in the standard picture of neutron stars with magnetic fields of order 10^{14} – 10^{15} G and magnetic-energy-powered activity [16,17]. Accordingly, strong exterior-loop morphology in a magnetar visualization should be interpreted as an encoding of a magnetically dominated environment, not as a direct field reconstruction. For a newborn stellar-mass black hole, the relevant physical problem is dynamical collapse governed by Einstein’s equations. Oppenheimer and Snyder supplied the classic idealized collapse solution [18]; after transients decay, a rotating isolated black-hole exterior is described by the Kerr family [19]. The newborn stellar-mass black-hole case remains an analytic benchmark for the R , Λ , and geometry channels. No literal material interior is implied by the cutaway-style modeling. For a binary black hole, the physical anchor is the two-body Einstein problem through inspiral, merger, and ringdown. The direct observation of GW150914 established that stellar-mass black-hole binaries merge in a waveform consistent with general relativity [20]. A two-center binary-black-hole field visualization is therefore schematic; a quantitative prediction would require numerical-relativity initial data and a residual-verified solution of the coupled mGFT representation.

7 E_9 : Affine/High-Energy Morphology Sector

The E_9 level is the affine stage of the direct hierarchy, with the manuscript using $E_9 = E_8^{(1)}$ when the affine realization is intended. Its role is therefore distinct from both the finite E_8 level and the hyperbolic E_{10} level developed in Sec. 8. In the analytic labeling convention of Sec. 3, the corresponding hull family is $H_{9,8}(2\xi)$, with nine upper and eight lower hypergeometric parameters when a strict $H_{9,8}$ profile is used.

The phenomenological E_9 examples retained in this manuscript are the generic affine/high-energy morphology classes already listed in Table 2: quasars, blazars, highly energetic AGN engines and black-hole–magnetosphere–jet systems, selected primordial-black-hole configurations, possible mirror-sector-coupled compact objects, and generic AGN/Einstein-ring morphology. These are framework assignments and visualization hypotheses, not observational identifications of an E_9 symmetry.

For a representation-valued realization, an E_9 candidate field may be written schematically as

$$U_{E_9}(X) = A(X) [H_{9,8}(2\xi(X))]^\alpha T,$$

where T denotes the chosen affine-algebra generator content and the remaining factors are fixed by the physical model. The affine label alone does not determine a source model, a metric, a jet solution, a lens model, or a particle spectrum. Those require the same action, boundary data, residual closure, constraint analysis, and quadratic fluctuation tests used elsewhere in the framework.

This section is the bridge to the hyperbolic construction of Sec. 8. The shorthand $E_9 \xrightarrow{\int dG} E_{10}$ acts only as the generated-mode part of a constructive lift; the complete lift also requires the tensor-extension and decompactification information of Eq. (12). Accordingly, the generic E_9 morphology sector and the $E_9 \rightarrow E_{10}$ analytic lift are related pieces of the framework, but neither is treated as a proof that an observed astronomical object realizes an affine or hyperbolic exceptional symmetry.

8 E_{10} : Hyperbolic Cartan Lift and Sine-Gordon-Like Solitonic Sector

The standard mathematical anchor is the rank-ten hyperbolic Kac–Moody algebra

$$E_{10} = E_8^{++}, \tag{47}$$

obtained by adjoining the affine node and then one overextended node to the E_8 Dynkin diagram. This identification, and the Lorentzian/hyperbolic character of the resulting root lattice, are standard facts in the Kac–Moody literature [22]. In the node numbering used here, nodes $1, \dots, 9$ form a chain and node 10 is attached to node 7. A convenient symmetric generalized Cartan matrix is therefore

$$A(E_{10}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}. \tag{48}$$

It satisfies the generalized-Cartan conditions $A_{ii} = 2$, $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$, and $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$, with

$$\det A(E_{10}) = -1. \quad (49)$$

The determinant and signature distinguish this matrix from the affine E_9 matrix: E_{10} is nondegenerate and Lorentzian, with one timelike direction in its root space [22].

8.1 Logarithmic/inverse-hyperbolic core and the $E_9 \rightarrow E_{10}$ lift

The model-level E_{10} generated core is taken in two complementary branches. For a real similarity variable ξ define

$$U_{10}^{(\pm)}(\xi) = \xi + \log(1 \pm e^\xi) \oplus \operatorname{arctanh}(e^\xi). \quad (50)$$

The symbol $\oplus$ denotes the paired analytic composition used in this framework; it is not a direct sum of Lie algebras and does not modify any entry of the generalized Cartan matrix. A common real domain for the inverse-hyperbolic factor is $\xi < 0$, for which $0 < e^\xi < 1$ and $\operatorname{arctanh}(e^\xi)$ is real. For the logarithmic minus branch one also has $1 - e^\xi > 0$ on this domain. Outside that domain, Eq. (50) requires a specified analytic continuation or a different real branch. Equation (50) is a model ansatz, not a standard Kac–Moody construction.

The standard algebraic lift from affine $E_9 = E_8^{(1)}$ to hyperbolic $E_{10} = E_8^{++}$ is instead the addition of one overextended Dynkin node [22,27]. With the node numbering of Eq. (48), deleting node 1 leaves the affine E_9 submatrix. Define

$$A(E_9) := A(E_{10})_{\{2,\dots,10\} \times \{2,\dots,10\}}, \quad \det A(E_9) = 0, \quad (51)$$

and let $\mathbf{e}_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^9$. Then the same Cartan data can be written as the block overextension

$$A(E_{10}) = \begin{pmatrix} 2 & -\mathbf{e}_1^T \\ -\mathbf{e}_1 & A(E_9) \end{pmatrix}. \quad (52)$$

Thus the conventional algebraic reduction is

$$\Pi_{10 \rightarrow 9}^A : A(E_{10}) \mapsto A(E_9), \quad \text{by deleting the overextended row and column,} \quad (53)$$

while the reverse algebraic extension is

$$\mathcal{O}_{9 \rightarrow 10}^A : A(E_9) \mapsto \begin{pmatrix} 2 & -\mathbf{e}_1^T \\ -\mathbf{e}_1 & A(E_9) \end{pmatrix}. \quad (54)$$

Equations (53)–(54) are the Kac–Moody/Dynkin part of the $E_9 \leftrightarrow E_{10}$ transformation. They are kept separate from the analytic field lift below.

For the generated-mode coordinate G , promote the monomial rule to the integral operator

$$(R_G f)(G) := \int_{G_0}^G f(\tilde{G}) d\tilde{G}, \quad \partial_G R_G f = f, \quad (55)$$

for functions and boundary data for which the integral is defined. In particular,

$$R_G[G^\ell] = \frac{G^{\ell+1} - G_0^{\ell+1}}{\ell + 1}, \quad (56)$$

which reduces to $G^\ell \mapsto G^{\ell+1}/(\ell+1)$ when $G_0 = 0$. The framework's field-level integral lift from E_9 to E_{10} is therefore

$$\mathcal{I}_{9 \rightarrow 10}^{\text{field}}[\Phi_9] = \mathcal{E}_{a,b}^{(9 \rightarrow 10)} \circ K_{9 \rightarrow 10}^\uparrow \left[\int_{G_0}^G \Phi_9(\tilde{G}) d\tilde{G} \right]. \quad (57)$$

The complete proposed lift pairs the algebraic and field transformations:

$$\mathcal{I}_{9 \rightarrow 10} : (A(E_9), \Phi_9) \mapsto (\mathcal{O}_{9 \rightarrow 10}^A[A(E_9)], \mathcal{I}_{9 \rightarrow 10}^{\text{field}}[\Phi_9]), \quad (58)$$

with the branch coordinates $U_{10}^{(\pm)}(\xi)$ of Eq. (50) used as the E_{10} analytic generated-core family after the lift. Under the directional convention of Sec. 2.2, the primary reduction remains

$$D_{10 \rightarrow 9} : (A(E_{10}), \Phi_{10}) \mapsto (\Pi_{10 \rightarrow 9}^A[A(E_{10})], \partial_G \circ M_{10 \rightarrow 9}[\Phi_{10}]). \quad (59)$$

Accordingly,

$$E_9 \xrightarrow{\mathcal{I}_{9 \rightarrow 10}} E_{10}, \quad E_{10} \xrightarrow{D_{10 \rightarrow 9}} E_9. \quad (60)$$

On an information-preserving retained sector, with compatible boundary data and an invertible compactification/tensor extension, the field factors can satisfy $\partial_G R_G = \text{Id}$. In general, compactification and mode truncation lose information, so $\mathcal{I}_{9 \rightarrow 10}$ is properly called a constructive integral lift rather than a guaranteed global inverse of $D_{10 \rightarrow 9}$.

8.2 Dense axion stars, oscillons, axitons, and oscillatons as proposed E_{10} objects

The E_{10} object class is enlarged here to include dense axion stars and related nonlinear scalar configurations: oscillon or pseudo-breather solitons, axitons, oscillatons, and closely related coherently oscillating scalar condensates. These objects already have an independent field-theory literature. Dense axion-star branches have been studied as self-interacting axion configurations [23,24]; axitons arise as dense nonlinear axion pseudo-solitons [25]; and oscillatons are self-gravitating, time-dependent real-scalar solitons [26]. This literature does not identify such objects with E_{10} ; that assignment is a hypothesis of the present framework.

There is, however, a concrete dynamical reason to place the proposed E_{10} solitonic sector near a sine-Gordon-type description. For a periodic axion-like scalar potential

$$V(\theta) = \Lambda^4(1 - \cos \theta), \quad (61)$$

the relativistic field equation contains the sine nonlinearity. In a flat-space spherical reduction, a representative structural form is

$$\ddot{\theta} - \theta'' - \frac{2}{r}\theta' + m_a^2 \sin \theta = 0. \quad (62)$$

Related dense-axion analyses discuss oscillon/pseudo-breather solutions of sine-Gordon-like equations [24]. The fully relativistic axion-star problem can require the complete Klein–Gordon–Einstein system and the full QCD axion potential, particularly in strongly bound regimes [24]; Eq. (62) is therefore a structural model equation rather than a universal exact reduction.

To couple this periodic dynamics to the E_{10} root system, let ϕ^A be a multi-component scalar on the root space and let α_i ($i = 1, \dots, 10$) be the simple roots whose inner products generate Eq. (48). A Cartan-organized periodic action may be written

$$S_{10} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} G_{AB} \nabla_\mu \phi^A \nabla^\mu \phi^B - \sum_{i=1}^{10} \Lambda_i^4 \left(1 - \cos \frac{\alpha_i \cdot \phi}{f_i} \right) \right], \quad (63)$$

leading schematically to

$$\square_g \phi^A + \sum_i \frac{\Lambda_i^4}{f_i} \alpha_i^A \sin\left(\frac{\alpha_i \cdot \phi}{f_i}\right) = 0. \quad (64)$$

Equivalently, defining root phases $\theta_i = \alpha_i \cdot \phi / f_i$, a reduced coupled form is

$$\square_g \theta_i + \sum_j \mu_j^2 A_{ij}(E_{10}) \sin \theta_j = s_i, \quad i = 1, \dots, 10, \quad (65)$$

where s_i permits gravitational, matter, or generated-core forcing. Equation (65) is the proposed E_{10} Cartan-coupled sine-Gordon-like system for this object class.

Dense axion stars, oscillon/pseudo-breather solitons, axitons, oscillatons, and the related proposed E_{10} compact scalar structures are therefore modeled here by sine-Gordon-like nonlinear field equations, with the E_{10} generalized Cartan matrix organizing the coupling among root-sector phases. This is a testable framework hypothesis, not an established consequence of Kac–Moody theory. Its verification requires a specified action, boundary data, an E_{10} representation/real form, residual-verified integral ft-{ht, lsr}-mGFT solutions, and comparison with conventional Einstein–Klein–Gordon or axion-star calculations.

9 String-Field and Brane Integration

The string-field construction provides a route from analytic profiles to string-field candidates. Schematically, a string field can be written as

$$|S\rangle = \sum_{I,\alpha} U_I^\alpha(X) |I\rangle \otimes T_\alpha. \quad (66)$$

where I denotes the string-state data and T_α carries internal symmetry information. In the present framework, the U_I^α may be drawn from the $H_{n,n-1}$ or integral ft-{ht, lsr}-mGFT solution family appropriate to the selected E_n sector. A profile becomes a string-field solution only after it satisfies the relevant string-field equations and state-space constraints.

9.1 Clifford/brane factors

Ito and Maru’s matrix-string construction supplies explicit Type IIA gamma-matrix products for D2, D4, and NS5 sectors [11]. In the conventions used here,

$$\Gamma_{D2} = \Gamma_{016}, \quad \Gamma_{D4} = \Gamma_{01789}, \quad \Gamma_{NS5} = \Gamma_{012345}. \quad (67)$$

Their construction relates a compactified brane configuration to $U(n)$ matrix string theory in the presence of longitudinal fivebranes [11]. A natural integration with the present symmetry program is to keep internal Lie generators and spacetime Clifford factors distinct:

$$\Phi(\xi) = \sum_{a,r} \phi_{ar}(\xi) T_a \otimes \Gamma_r. \quad (68)$$

Here T_a belongs to a representation of the selected E_n structure, Γ_r carries the brane/spinor structure, and ϕ_{ar} carries the analytic field profile. This tensor-product separation keeps internal symmetry and spacetime/brane structure explicit and distinct.

10 Matrix-QFT Route to Quantum Fields

Kazakov's matrix-model construction provides a benchmark in which scalar QFT, and an extension including fermions/Yukawa interactions, can be represented through a large-matrix model and a continuum limit [12]. The matrix action is summarized schematically as

$$S_{\text{MM}} = N \text{Tr}([X^\mu, \Phi]^2 + \ln(I - A_K \Phi)) , \quad (69)$$

with quenched coordinate matrices X^μ and a coupling matrix A_K [12]. This benchmark motivates placing E_n - and mGFT-structured profiles in off-diagonal matrix elements rather than identifying a scalar special function with an entire quantum field. A representative element is

$$(\Phi_r)_{ij} = C_{ij}^{(r)} G_r(\xi_{ij}) \Gamma_r, \quad \xi_{ij} = \beta_\mu(x_i^\mu - x_j^\mu). \quad (70)$$

so that $[X^\mu, \Phi]_{ij}$ remains sensitive to coordinate separation. A combined internal/Clifford realization is

$$\Phi_{ij} = \sum_{\alpha, r} C_{ij}^{\alpha r} G_{\alpha r}(\xi_{ij}) T_\alpha \otimes \Gamma_r. \quad (71)$$

The physical test is then action- and correlator-based: the matrix model, its large-matrix continuum limit, and its one- and two-point functions must reproduce the intended QFT observables [12].

11 Integral ft-{ht, lsr}-mGFT as the Dynamical Solver

The exceptional symmetry label is specified independently of any summation. The integral ft-{ht, lsr}-mGFT enters downstream as a solver for the field equations. Its quadrature-first formulation is developed in the integral ft-mGFT Riemannian-sum manuscript [13]. Let the coupled system be

$$L[\Phi] + N[\Phi] = s. \quad (72)$$

Choose a generated core U and define its equation defect

$$E_U = L[U] + N[U]. \quad (73)$$

If J is a right inverse of L compatible with the data, the fixed-point form is

$$\Phi = U + J[s - E_U - (N[\Phi] - N[U])] . \quad (74)$$

Finite-K approximants may be constructed by iteration, and their direct verification quantity is

$$R_K = L[\Phi^{(K)}] + N[\Phi^{(K)}] - s. \quad (75)$$

This fixed-point formulation is the dynamical form used in the integral ft-mGFT program [13]. It allows the selected E_n symmetry to specify the representation structure while the integral method solves the associated nonlinear dynamics.

11.1 Green–Duhamel and Riemannian-volume realization

For a causal kernel compatible with the geometry and boundary data [13],

$$(Jf)(t, x) = \int_0^t \int_{\Omega} G_{\text{ker}}(t, x; \tau, y) f(\tau, y) d\mu_g(y) d\tau, \quad d\mu_g = \sqrt{|g|} d^d y. \quad (76)$$

A Riemannian-volume Riemann discretization uses finite quadrature weights,

$$(Jf)(t, x) \approx \sum_{i,j} G_{\text{ker}}(t, x; \tau_i, y_j) f(\tau_i, y_j) \Delta\tau_i \Delta V_{g,j}.$$

This finite sum is a numerical/geometric realization of the integral operator; it is not the discarded summation-based definition of the unifying theory. The cited integral ft-mGFT paper treats the tagged Riemannian-volume sum as the central computable inverse, separates quadrature/right-inverse error from nonlinear truncation through a residual identity, and emphasizes controlled refinement and limit ordering [13]. Convergence, kernel regularity, constraint preservation, and residual closure remain required.

12 Masses, Mixing, and Particle Modes from Quadratic Fluctuations

Masses are extracted from the quadratic dynamics rather than from arbitrary coefficient determinants or universal series. Let $S[\Phi]$ be a specified action and Φ_0 a verified background. Write

$$\Phi = \Phi_0 + \eta. \quad (78)$$

The quadratic kernel is

$$K_{ab}(x, y) = \left. \frac{\delta^2 S}{\delta\Phi_a(x) \delta\Phi_b(y)} \right|_{\Phi_0}. \quad (79)$$

For a stationary translation-invariant background, Fourier transformation gives the physical mode condition

$$\det K(\omega, \mathbf{k}) = 0. \quad (80)$$

After gauge and constraint reduction, a finite component model with kinetic matrix Z and mass matrix M^2 obeys the generalized eigenvalue problem

$$\det(m^2 Z - M^2) = 0. \quad (81)$$

These relations provide the systematic route to hidden-sector masses, mixing angles, instabilities, and propagating modes. A finite invariant scalar potential may still be used to generate the vacuum. For scalar coordinates ϕ_a ,

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2 + \eta(\phi^\dagger \phi)^3 + V_{\text{mix}}(\phi, \chi). \quad (82)$$

with $\partial V / \partial \phi_a|_v = 0$ and

$$(M_\phi^2)_{ab} = \left. \frac{\partial^2 V}{\partial \phi_a \partial \phi_b} \right|_v. \quad (83)$$

Fermion masses arise from symmetry-allowed Yukawa structures, $M_\psi = Y(v)$, and gauge-boson masses arise from covariant kinetic terms of fields with nonzero vacuum expectation values. This keeps the mass problem tied to the actual representation and dynamics.

12.1 Background dependence of the spectrum

For similarity-dependent or strongly time-dependent backgrounds, a single constant mass matrix can be insufficient. The appropriate spectral problem depends on the verified background:

Background	Appropriate analysis
Homogeneous and stationary	Fourier modes, propagator poles, dispersion relation
Stationary but spatially varying	Normal-mode or Sturm–Liouville-type eigenproblem
Periodic in time	Floquet exponents and parametric stability
Slowly varying in time	Adiabatic/WKB instantaneous modes with consistency checks
Strongly time dependent	Initial-value stability and asymptotic in/out modes when defined

13 Dark Matter, Dark Energy, and Cosmological Signatures

The framework includes dark-sector and cosmological applications tied to symmetry breaking and verified dynamics. Hidden states may occupy representations neutral under the visible low-energy subgroup but charged under G_{hidden} . Stability may follow from a residual gauge or discrete symmetry. Their masses and interactions are obtained from the quadratic operator and invariant couplings described above, not from a universal summation.

Dark-energy-like behavior can arise from slowly evolving scalar directions of an invariant potential, subject to stability, cosmological evolution, and observational constraints [9]. First-order transitions in an E_n -based symmetry-breaking hierarchy may generate stochastic gravitational-wave backgrounds; quantitative spectra require a finite-temperature effective action, bubble nucleation, latent heat, transition duration, and hydrodynamics rather than group labels alone.

Higher-dimensional modes remain governed by geometry. For a simple circular compactification,

$$m_{\text{KK}}^2 = m_0^2 + \frac{\ell^2}{R_c^2}, \quad (84)$$

with modifications in warped or more complicated compactifications [6–8].

14 Integrated Verification Ladder

The combined program is organized into the following verification sequence:

1. Choose the symmetry level E_n and an explicit representation.
2. Choose an analytic field profile, such as an $H_{n,n-1}$ hull or an integral ft-{ht,lsr}-mGFT generated core [13].
3. Expand matrix coefficients in verified generators T_α and, where needed, use Clifford factors Γ_r separately.

4. Specify the invariant action, gauge fixing, domain, initial/boundary data, and any compactification.
5. Solve the governing equations with the integral fixed-point/Green–Duhamel machinery or another justified method.
6. Verify residual closure, convergence, constraints, and regularity.
7. Derive the quadratic fluctuation operator and remove gauge/nonphysical modes.
8. Extract masses and dispersion relations from $\det K = 0$ or $\det(m^2 Z - M^2) = 0$.
9. For a matrix-QFT realization, verify the large-matrix continuum limit and correlators.
10. Compare masses, couplings, decay channels, cross sections, gravitational signals, and cosmological observables with data.

This ladder makes the roles of symmetry, analytic construction, dynamics, and physical interpretation explicit.

14.1 Proposed E_5 Preon Constituent Sector and Standard-Model Composites

The constituent construction is incorporated as a model ansatz beneath the E_6 phenomenological level. The model uses the label E_5 and associates the sector with $\text{Spin}(10)$. In conventional Lie-algebra notation $\text{Spin}(10)$ is associated with D_5 ; accordingly, E_5 is treated here as a framework label for a $\text{Spin}(10)$ -related preon sector rather than as an additional finite exceptional Lie group.

The ansatz contains two constituents, an S preon and a V preon:

Preon	$U(1)_\psi$ charge	Spin	Internal states	Benchmark mass	Representation / role
S	-1	$1/2$	16	$M_S = 24.0 \text{ TeV}$	16 of the $\text{Spin}(10)$ -related sector; fermionic constituent
V	$+2$	0	10	$M_V = 19.2 \text{ TeV}$	10 ; bosonic/scalar constituent

The schematic composite rules are

$$\ell, \nu \sim S\bar{V}, \quad q \sim SVV, \quad W, Z, \gamma, g \sim V\bar{V}, \quad H \sim VV,$$

with a color-carrying three- V motif used in the $SU(3)$ sector. The constituent mass relation is

$$M_{\text{comp}} = n_S M_S + n_V M_V - E_{\text{bind}}.$$

For the quoted Standard-Model masses this relation requires strong binding: the unbound constituent sums are 43.2 TeV for $S\bar{V}$, 62.4 TeV for SVV , and 38.4 TeV for two- V states. Thus the residual observed mass is a small difference between the constituent sum and E_{bind} . For example, the top-quark entry requires $E_{\text{bind}} \simeq 62.2272 \text{ TeV}$, the W entry requires $E_{\text{bind}} \simeq 38.319621 \text{ TeV}$, and the Higgs entry requires $E_{\text{bind}} \simeq 38.2749 \text{ TeV}$. Electron and neutrino entries require cancellation at still finer fractional precision.

This constituent picture is therefore a testable compositeness hypothesis, not a derivation from the E_6 branching relations alone. A complete realization must derive the effective electric charges, color assignments, Lorentz statistics, three-generation structure, binding Hamiltonian, anomaly cancellation, form factors, and the absence of experimentally excluded compositeness effects. These relations summarize the proposed composition rules.

14.2 Neutral Mirror-Preon Complexes and Dark-Photon Energy Redistribution

The dark-sector benchmark proposes neutral, extended complexes P assembled from mirror-sector analogues of the S and V constituents. The benchmark complex is represented schematically by

$$P = (SV \cdots)_n, \quad Q_P = 0, \quad M_P \sim 10\text{--}100 \text{ TeV},$$

where the mass interval is a model scale rather than a measured particle range. Dark photons γ' are produced through three representative channels:

$$P + \bar{P} \rightarrow \gamma' + X, \quad P^* \rightarrow P + \gamma', \quad P + f \rightarrow P + f + \gamma'.$$

The ordinary and dark Abelian sectors are coupled through the kinetic-mixing operator

$$\mathcal{L}_{\text{mix}} = \frac{\epsilon}{2} F^{\mu\nu} F'_{\mu\nu},$$

and the phenomenological energy bookkeeping is written

$$E_{\text{total}} = E_{\text{vis}} + E_{\gamma'} + E_{\text{other}}.$$

A compact attenuation/redistribution model used in the benchmark is

$$I_{\text{obs}}(E) = I_0(E) e^{-\tau_{\gamma'}(E)} [1 - \eta_{\gamma'}(E)],$$

where $\tau_{\gamma'}$ is an attenuation/escape factor and $\eta_{\gamma'}$ is the fraction of energy diverted into the dark-photon channel. In this interpretation suppression is not missing four-momentum: visible-channel intensity decreases because energy is carried, scattered, or redistributed into hidden or lower-intensity channels.

The benchmark suppression-band centers are 5.04, 6.70, 9.38, 13.80, 18.42, 24.30, and 30.00 TeV. The model assigns these bands to mixtures of annihilation, de-excitation, halo scattering, and coherent-medium effects, with gamma-ray or neutrino channels used as comparison targets. Any observatory comparisons are phenomenological touchpoints, not confirmation that a facility has detected the proposed complexes or dark photons.

Band center (TeV)	Model-assigned mechanism	Illustrative visible channel
5.04	annihilation / de-excitation	$\gamma' \rightarrow e^+e^-$ or gamma-ray secondaries
6.70	scattering in a halo	gamma-ray secondaries
9.38	de-excitation	$\gamma' \rightarrow \nu\bar{\nu}$ benchmark
13.80	annihilation / scattering	e^+e^- and/or neutrinos
18.42	coherent-medium effects	mixed hidden/visible redistribution
24.30	annihilation	e^+e^- and/or neutrinos
30.00	de-excitation / scattering	gamma-ray secondaries

14.3 Suppression-Band Correspondence and Statistical Interpretation

The suppression analysis pairs seven benchmark band centers with seven calculated centers:

Observed center (TeV)	Calculated center (TeV)
5.04	5.279
6.70	7.028
9.38	9.853
13.80	14.512
18.42	19.381
24.30	25.579
30.00	31.586

The affine fit used in the benchmark is

$$E_{\text{obs}} \simeq 0.94878 E_{\text{calc}} + 0.0316 \text{ TeV}.$$

The benchmark reports Pearson $r = 0.998696$, $R^2 = 0.997939$, a two-sided $p = 1.18 \times 10^{-7}$, Spearman $\rho = 1.000$, a Gaussian-equivalent significance near 5.30σ , MAE $\simeq 0.830$ TeV, RMSE $\simeq 1.023$ TeV, and a mean fractional mismatch near 5.27%.

For reproducibility, those statistics are recorded as *benchmark-reported diagnostics*. Direct calculation from the seven rounded number pairs listed above gives a still tighter affine correspondence: Pearson $r \approx 0.999999995$, $R^2 \approx 0.999999990$, MAE ≈ 0.797 TeV, RMSE ≈ 0.924 TeV, and mean absolute fractional mismatch $\approx 5.09\%$. The difference shows that the reported summary statistics cannot be reconstructed exactly from the rounded centers alone; either additional precision, a different underlying data vector, or a different statistical convention is required.

In either case, correlation among seven preselected monotonic energy bands is a descriptive model-consistency check rather than a standalone detection significance. A physical inference requires a prespecified likelihood, treatment of look-elsewhere effects and correlated uncertainties, instrument response, backgrounds, source population modeling, and independent out-of-sample tests. This interpretation is consistent with the verification ladder: the suppression construction remains a phenomenological dark-sector benchmark until its action, rates, cross sections, transfer functions, and observational likelihood are specified.

14.4 Paired String and Scalar Fields Across the E_n Hierarchy

The cross-hierarchy field organization introduces two parallel visualization families, $\Psi_{E_n}(X)$ and $\Phi_{E_n}(X)$, at the labeled levels $E_5, E_6, E_7, E_8, E_9, E_{10}$, and E_{12} . In this framework, Ψ denotes the hierarchy-level string/matter/gauge-field family and Φ the scalar/modulus-field family. The construction is treated as a field-organization scheme; any associated geometries are qualitative representations of internal structure rather than group-theoretic derivations of a unique spacetime profile.

For levels already represented by the manuscript's Lie-algebra-valued construction, the two field families can be embedded schematically as

$$\Psi_{E_n}(X) = \sum_{\alpha, r} \psi_{\alpha r}^{(n)}(X) T_\alpha \otimes \Gamma_r, \quad \Phi_{E_n}(X) = \sum_{\alpha} \phi_{\alpha}^{(n)}(X) T_\alpha.$$

Thus the Ψ panels emphasize string, matter, gauge, and duality content, while the Φ panels emphasize scalar and modulus content. This notation does not by itself specify an action, spectrum, or physical vacuum; those remain fixed by the representation, invariant couplings, field equations, residual tests, and quadratic fluctuation analysis.

Level	Framework setting	Field-family interpretation
E_5	SO(10) / 10D unification	Ψ : 10-dimensional string field; Φ : scalar/string modulus associated with the framework's Spin(10)/SO(10)-related constituent label.
E_6	Grand unification + matter unification	Ψ : 27-dimensional structure; Φ : scalar/string modulus in the 27, including compactification and Yukawa-modulus descriptors.
E_7	Duality and U-duality	Ψ : 56-dimensional structure; Φ : scalar field in the 56 with compactification and duality-invariant modulus descriptors.
E_8	Heterotic string-theory setting	Ψ : 248-dimensional structure; Φ : scalar/modulus field in the 248.
E_9	Affine E_8	Ψ : loop/affine extension; Φ : affine scalar field with the extra loop parameter.
E_{10}	Hyperbolic E_{10}	Ψ : hyperbolic extension; Φ : hyperbolic scalar field associated with cosmological scale factors, wall forms, and M-theory moduli near singularities.
E_{12}	Framework-specific extended label	Ψ : conjectural F-/M-theory field label; Φ : conjectural full moduli, brane-charge, and spacetime-geometry descriptor.

The E_5 entry is read together with Sec. 14.1: the manuscript treats E_5 as a framework label for the Spin(10)/SO(10)-related constituent sector rather than as a fifth finite exceptional Lie group. E_6 , E_7 , and E_8 remain the finite exceptional levels; E_9 remains the affine E_8 extension and E_{10} the hyperbolic extension. No E_{11} Ψ/Φ assignment is made here. The E_{12} descriptor remains framework-specific and is not substituted for the manuscript's separately defined $E_{11} = E_8^{+++}$ stage.

15 Predictive Power and Main Mathematical Challenges

The framework predicts structure primarily through E_n representation branching, selection rules, symmetry breaking, vacuum structure, fluctuation spectra, compactification, and solutions of the coupled field equations. For the finite levels E_6 , E_7 , E_8 , the relevant algebraic data can be taken from the standard exceptional Lie groups and their representations. For E_9 and higher, the most important mathematical requirement is to specify the exact extended E-type algebra being used. For each higher level, a complete formulation should state the generalized Cartan matrix, root lattice, real form, representation category, and any central or derivation generators. For the proposed directional hierarchy it must additionally specify the compactification data, contraction tensors, projectors, retained Kaluza–Klein modes, and the domain on which each $D_{n \rightarrow n-1}$ or $I_{n-1 \rightarrow n}$ is defined. A methodological caution remains important: an analytic profile, matrix size, parameter count, or visual resemblance by itself does not establish a symmetry identification. The symmetry assignment must be supported by an explicit action, transformation law, representation content, and verifiable observables.

16 Conclusion

The integrated theory is centered directly on the E_n hierarchy,

$$E_n \longleftrightarrow H_{n,n-1}(2\xi), \quad 6 \leq n \leq 11. \quad (85)$$

where the left-hand side carries the internal symmetry assignment and the right-hand side denotes the associated analytic hull used in the present mGFT program. The two roles remain distinct: explicit generators T_α carry the E_n symmetry; $H_{n,n-1}$ and integral ft-{ht,lsr}-mGFT profiles provide analytic field dependence; Clifford factors Γ_r can encode brane/spinor structure; Green–Duhamel operators solve the coupled equations [13]; and the Hessian or inverse propagator determines the physical spectrum. The theory is not defined by an infinite summation. Finite Riemann sums are used only as numerical approximations to geometric integrals, and particle masses are obtained from symmetry breaking and quadratic fluctuation operators. The proposal is therefore a layered E_n -indexed group-theoretic and dynamical program rather than a summation-defined one. The hierarchy is directional. Its primary flow is

$$E_{11} \longrightarrow E_{10} \longrightarrow E_9 \longrightarrow E_8 \longrightarrow E_7 \longrightarrow E_6. \quad (86)$$

implemented algebraically by restriction/node deletion and, in the model-level dynamical construction, by compactification, tensor contraction, and G -mode reduction. The reverse sequence is a constructive lift using tensor extension, decompactification data, and R_G ; it is a genuine inverse only on sectors where no information has been discarded. The verification ladder requires a concrete reference action, residual/convergence certification, and quadratic spectral extraction. The next demonstrations are therefore sector-specific rather than purely procedural: (i) replace the reference action by representation-complete actions for each proposed E_n phenomenological sector; (ii) derive explicit visible/hidden branching rules and symmetry-breaking vacua; (iii) residual-certify the actual astrophysical field solutions rather than only the verification benchmark; and (iv) compare the resulting physical spectra and observables with experiment, astrophysics, or cosmology.

A Direct E_n Hierarchy Summary

The indexing convention used throughout the manuscript is

$$E_6, E_7, E_8, E_9, E_{10}, E_{11}. \quad (87)$$

The finite exceptional sector is E_6 , E_7 , and E_8 . The higher labels are used for explicitly specified affine, hyperbolic, or indefinite E -type extensions. The primary direction adopted here is $E_{11} \rightarrow E_{10} \rightarrow E_9 \rightarrow E_8 \rightarrow E_7 \rightarrow E_6$; the opposite direction denotes constructive lifts.

B Integrated Core Formulae

Concept	Formula / role
E_n symmetry	Direct exceptional or extended E_n level, with explicit representation data
Primary hierarchy	$E_{11} \xrightarrow{D_{11}} E_{10} \xrightarrow{D_{10}} E_9 \xrightarrow{D_9} E_8 \xrightarrow{D_8} E_7 \xrightarrow{D_7} E_6$
Constructive lifts	$E_6 \xrightarrow{I_7} E_7 \xrightarrow{I_8} E_8 \xrightarrow{I_9} E_9 \xrightarrow{I_{10}} E_{10} \xrightarrow{I_{11}} E_{11}$
Mode lift / reduction	$R_G[G^\ell] = G^{\ell+1}/(\ell+1)$; $R_G^{-1} = \partial_G$ on the image
Tensor reduction	$T_{a,b}^{(n \rightarrow n-1)} = C_{a,b}^{(n)} \circ K_{n \rightarrow n-1}$
Coefficient map	$M_{n \rightarrow n-1} = T_a P_a + T_b P_b$
Full downward map	$D_{n \rightarrow n-1} = \partial_G \circ M_{n \rightarrow n-1}$ on the retained sector
Upward lift	$I_{n-1 \rightarrow n} = \mathcal{E}_{a,b} \circ K^\dagger \circ R_G$
Analytic hull	$H_{n,n-1}(2\xi)$ as the profile family associated with the E_n level
Algebra-valued coefficients	$a_{ij} = A_{ij}^\alpha T_\alpha$, $b_{ij} = B_{ij}^\alpha T_\alpha$
Group-valued lift	$g(\xi) = \exp[U(\xi)]$ under appropriate conditions

Concept	Formula / role
Lie/Clifford field	$\Phi(\xi) = \sum_{a,r} \phi_{ar}(\xi) T_a \otimes \Gamma_r$
Coupled field equation	$L[\Phi] + N[\Phi] = s$
Generated-core defect	$E_U = L[U] + N[U]$
Fixed point	$\Phi = U + J[s - E_U - (N[\Phi] - N[U])]$
Residual	$R_K = L[\Phi^{(K)}] + N[\Phi^{(K)}] - s$
Quadratic operator	$K_{ab} = \delta^2 S / (\delta \Phi_a \delta \Phi_b) _{\Phi_0}$
Mode condition	$\det K(\omega, \mathbf{k}) = 0$ after gauge/constraint reduction
Generalized mass problem	$\det(m^2 Z - M^2) = 0$

Author Declarations

AI-Assistance Statement

Generative artificial intelligence was used as an assistive research, writing, computational, and document-production tool during the development and preparation of this manuscript. In particular, OpenAI's ChatGPT was used to support organization of material; drafting, rewriting, copyediting, and stylistic refinement of prose; conversion and cleanup of mathematical content in $\text{\LaTeX}$; consistency checks across notation, section structure, captions, cross-references, bibliography formatting, and page layout; integration of author-supplied figures and machine-readable data; preparation and refinement of Python scripts used to reproduce selected benchmark tables, spectra, residual calculations, and summary statistics; and assembly and quality-control checks of the reproducibility package and compiled PDF.

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Conflict of Interest Disclosure

The author declares no conflicts of interest or competing interests, financial or non-financial, that are relevant to the research, preparation, interpretation, or publication of this manuscript.

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