

Carbon Dating and the Topology of Time: Radiocarbon Chronology on the Circle S^1 versus the Line $\mathbb{R}^1$

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Abstract

Radiocarbon dating converts a measured isotope ratio into an age through the exponential decay law $N(t) = N_0 e^{-\lambda t}$, a law whose derivation and inversion tacitly assume that time has the topology of the real line $\mathbb{R}^1$. In this paper we ask what becomes of radiocarbon chronology if time is instead circular, $S^1 = \mathbb{R}/T_c\mathbb{Z}$, and in particular if the circle is regarded as the conformal compactification of linear time, $\overline{\mathbb{R}^1} \cong \mathbb{R}P^1 \cong S^1$, obtained by adjoining the single point $t = \pm\infty$. We first show that the pure decay equation $\dot{N} = -\lambda N$ admits no non-trivial solution on S^1 , because its Floquet multiplier $e^{-\lambda T_c}$ differs from unity; a periodic chronology therefore forces a modification of decay physics. We construct three mutually consistent periodic-time frameworks. (i) *Conformal chronology*: pulling the linear decay law back along the stereographic map $t = (T_c/\pi) \tan(\pi\tau/T_c)$ gives an arc-length age $\tau_C = (T_c/\pi) \arctan(\pi t_L/T_c)$ which is always younger than the linear age t_L and saturates at $T_c/2$. (ii) *Image-sum (resonant) quantum decay*: on a periodic time axis the energy spectrum is quantised with spacing h/T_c , and Poisson summation of the Breit–Wigner amplitude yields the exact survival probability $F_{S^1}(t) = \cosh^2[\lambda(t - T_c/2)/2] / \cosh^2(\lambda T_c/4)$, which reproduces the exponential law as $T_c \rightarrow \infty$ but gives ages that are older than t_L and that cease to exist below a floor $F_{\min} = \text{sech}^2(\lambda T_c/4)$. (iii) *Anti-resonant decay*, with $F(t) = \sinh^2[\lambda(T_c/2 - t)/2] / \sinh^2(\lambda T_c/4)$, which yields younger ages and total depletion at mid-cycle. We derive closed-form age-inversion formulae, asymptotic expansions, error-propagation laws, the periodic Green’s function of the atmospheric-reservoir equations, and the winding-number ambiguity $t \mapsto t + nT_c$ inherited from the covering map $\mathbb{R}^1 \rightarrow S^1$. Numerically, for a sample with $F = 0.01$ (linear age 38 069 yr) the S^1 ages for $T_c = 10^5$ yr are 27 833, 45 873 and 35 121 yr in the three frameworks; for a 5000-year world cycle the conformal age of every sample, however depleted, is below 2500 yr. Applied to ten well-known samples, from the Shroud of Turin and the Tyrolean Iceman to the Chauvet cave paintings and the Lyuba mammoth, a 100 000-year cycle leaves Holocene ages essentially unchanged but shifts Pleistocene ages by up to 13 000 yr, while a 5000-year cycle compresses linear ages from 709 to 43 016 yr into the conformal interval 667–2441 yr. We show that the discrepancy scales as $\pi^2 t_L^3 / 3 T_c^2$ (conformal) and as $(2/\lambda) e^{-\lambda T_c/2} (e^{\lambda t_L} - 1)$ (image-sum), obtain the parameter region in which linear and periodic chronologies are observationally indistinguishable, and discuss the thermodynamic (Loschmidt–Zermelo) and cosmological implications of a radiocarbon clock that must run back to its initial state once per cycle.

Keywords: radiocarbon dating; topology of time; cyclic time; conformal compactification; S^1 time; Poincaré recurrence; Poisson summation; Breit–Wigner decay; Floquet theory; arrow of time.

Contents

1	Introduction	3
2	Radiocarbon Dating in Linear Time $\mathbb{R}^1$	4
3	The Topology of Time: $\mathbb{R}^1$, S^1 and Conformal Compactification	7
4	Dynamics on a Periodic Time Axis	9
5	Radioactive Decay on S^1 : The Image-Sum Law	11
6	Radiocarbon Dating on S^1 : Three Age Equations	16
7	The Carbon Reservoir and Calibration on S^1	20
8	Numerical Results: How Widely Do Linear and Periodic Ages Differ?	22
9	Discussion	29
10	Conclusions	31
A	Image Sums for Classical and Quantum Kernels	32
B	The Periodic Bateman Problem	33
C	Numerically Stable Evaluation	33

1 Introduction

Few measurements have reshaped our picture of the human and geological past as thoroughly as radiocarbon dating. Following the prediction that cosmic-ray neutrons would maintain a steady-state inventory of ^{14}C in the biosphere, Libby, Anderson and Arnold demonstrated a world-wide, nearly uniform specific activity of natural radiocarbon [1] and, in the same year, confirmed the method against samples of historically known age [2, 3]. In the seven decades since, the technique has been refined by the introduction of isotopic fractionation corrections and reporting conventions [4, 5], by accelerator mass spectrometry [7–9], by the discovery of natural variations of atmospheric ^{14}C [10–13], and by calibration curves anchored in dendrochronology, varved sediments and U–Th dated speleothems [14–18]. Bayesian statistical machinery now routinely combines radiocarbon determinations with stratigraphic priors [20]. Comprehensive accounts are given in [21–23].

Underlying all of this sophistication is a single physical law, the exponential decay law of Rutherford and Soddy [33],

$$N(t) = N_0 e^{-\lambda t}, \quad \lambda = \frac{\ln 2}{T_{1/2}}, \quad (1.1)$$

inverted to give the age

$$t = \frac{1}{\lambda} \ln \frac{N_0}{N}. \quad (1.2)$$

Equation (1.2) is a statement about a function on the time axis. It presupposes that the time axis is the real line $\mathbb{R}^1$, that it is non-compact, that each instant occurs exactly once, and that $e^{-\lambda t}$ is single-valued on it. These are topological assumptions, and they are not dictated by the local laws of physics. The field equations of general relativity, for example, are local and admit solutions with closed timelike curves [53–56]; periodic identifications of the time coordinate appear naturally in thermal quantum field theory [51, 52]; and cyclic cosmologies—from Tolman’s oscillating universe [57] to the ekpyrotic/cyclic model [58] and Penrose’s conformal cyclic cosmology [59]—revive the ancient intuition of an eternal return [60, 61]. The mathematical structure most directly relevant here is the *conformal compactification* of spacetime [44–48]: the conformal group of the time line acts naturally not on $\mathbb{R}^1$ but on its one-point compactification $\mathbb{R}P^1 \cong S^1$, in which the “infinitely distant” past and future are a single point.

In earlier work the author examined how a circular topology of time bears on the Poincaré recurrence problem in general relativity [70], on rotating and Gödel-type universes with S^1 time [71], and on Loschmidt’s reversibility paradox and entropy [72]. The present paper applies the same point of view to a concrete, quantitative and practically important problem: *How would the time elapsed since the death of an organism, as inferred from its residual radiocarbon, change if time were periodic rather than linear?*

The question has an immediate and somewhat startling answer. On S^1 with circumference T_c , every physical state must return to itself after one cycle. But $e^{-\lambda(t+T_c)} \neq e^{-\lambda t}$. Hence the exponential law (1.1) cannot hold globally on S^1 , and the dating formula (1.2) must be replaced. We show that there is not one replacement but a small family of them, distinguished by what one takes the physically fundamental object to be—the coordinate map between $\mathbb{R}^1$ and S^1 , or the quantum-mechanical decay amplitude, whose spectrum becomes discrete on a periodic time axis. Each yields an explicit, closed-form age–ratio relation that tends to Eq. (1.2) as $T_c \rightarrow \infty$ and departs from it, sometimes dramatically, when T_c is comparable to the radiocarbon dating horizon of $\sim 5 \times 10^4$ yr.

Aims and structure of the paper

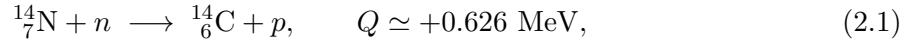
Our aims are (a) to formulate precisely what “periodic-time physics” means for a first-order decay process; (b) to derive the corresponding age equations in closed form; (c) to quantify how widely linear and periodic ages differ as a function of the cycle period T_c ; and (d) to identify what radiocarbon data, together with independent chronometers, can say about the topology of time.

Section 2 reviews the linear-time physics of radiocarbon: production, decay, the Bateman equations, conventional-age conventions, reservoir models and calibration. Section 3 sets out the topology: the covering map $\mathbb{R}^1 \rightarrow S^1$, the one-point and conformal compactifications, and the conformal factor. Section 4 develops dynamics on S^1 —single-valuedness, Floquet theory, periodic Green’s functions and the quantisation of energy. Section 5 derives the periodic decay laws via Poisson summation of the Breit–Wigner amplitude. Section 6 obtains the three S^1 dating formulae and their error propagation. Section 7 treats the atmospheric reservoir and calibration on S^1 . Section 8 presents numerical results, tables and the four figures. Section 9 discusses thermodynamics, observational constraints and interpretation, and Section 10 concludes. Mathematical details are collected in the Appendices.

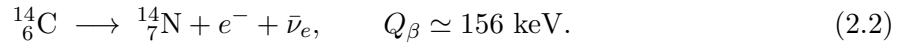
2 Radiocarbon Dating in Linear Time $\mathbb{R}^1$

2.1 Production and decay

Radiocarbon is produced in the upper atmosphere by the capture of secondary cosmic-ray neutrons on nitrogen,



at a globally averaged rate that modern models place at $P \approx 1.6\text{--}2.0 \text{ atoms cm}^{-2} \text{ s}^{-1}$ [25–27]. It decays by pure β^- emission back to nitrogen,



For a closed sample containing N radiocarbon atoms the decay law of Rutherford and Soddy [33] is

$$\frac{dN}{dt} = -\lambda N, \quad N(t) = N_0 e^{-\lambda t}, \quad (2.3)$$

with decay constant, half-life and mean life related by

$$\lambda = \frac{\ln 2}{T_{1/2}}, \quad \bar{\tau} = \frac{1}{\lambda} = \frac{T_{1/2}}{\ln 2}. \quad (2.4)$$

Two values of the half-life are in use. Libby’s value $T_{1/2}^L = 5568 \pm 30 \text{ yr}$ ($\bar{\tau}^L = 8033 \text{ yr}$) is retained by convention for reporting conventional radiocarbon ages [4], whereas the physically better value, the “Cambridge” half-life $T_{1/2}^C = 5730 \pm 40 \text{ yr}$ ($\bar{\tau}^C = 8267 \text{ yr}$), was recommended at the Fifth Radiocarbon Conference [24]. Throughout this paper physical decay uses

$$\lambda = \frac{\ln 2}{5730 \text{ yr}} = 1.2097 \times 10^{-4} \text{ yr}^{-1}. \quad (2.5)$$

The specific activity $A = \lambda N$ follows the same law, $A(t) = A_0 e^{-\lambda t}$. For modern carbon $A_0 \simeq 0.226 \text{ Bq per gram of carbon}$, corresponding to an isotope ratio $^{14}\text{C}/^{12}\text{C} \approx 1.2 \times 10^{-12}$.

2.2 The linear dating equation

Let $R = {}^{14}\text{C}/{}^{12}\text{C}$ and define the fraction modern

$$F \equiv F^{14}\text{C} = \frac{R_{\text{S,N}}}{R_{\text{ON}}}, \quad (2.6)$$

where $R_{\text{S,N}}$ is the sample ratio normalised for isotopic fractionation and R_{ON} is the normalised ratio of the oxalic-acid standard, decay-corrected to AD 1950 [5, 6]. Inverting Eq. (2.3) gives the *linear* age

$$t_L = -\frac{1}{\lambda} \ln F \quad (2.7)$$

and the conventional radiocarbon age (years BP) is the same expression with Libby's mean life,

$$t_{\text{conv}} = -8033 \ln F, \quad \frac{t_L}{t_{\text{conv}}} = \frac{5730}{5568} = 1.0291. \quad (2.8)$$

Fractionation normalisation to $\delta^{13}\text{C} = -25\text{‰}$ is performed via [4]

$$A_{\text{SN}} = A_{\text{S}} \left[\frac{0.975}{1 + \delta^{13}\text{C}/1000} \right]^2, \quad (2.9)$$

the square reflecting the approximately doubled mass-dependence of ${}^{14}\text{C}$ relative to ${}^{13}\text{C}$. The quantity $\Delta^{14}\text{C}$ used to describe atmospheric histories is

$$\Delta^{14}\text{C} = \left(F e^{\lambda(1950-y)} - 1 \right) \times 1000 \text{‰}, \quad (2.10)$$

for a sample whose growth year is y (AD).

Error propagation. For a measured $F \pm \sigma_F$ the linear age uncertainty is

$$\sigma_{t_L} = \left| \frac{dt_L}{dF} \right| \sigma_F = \frac{1}{\lambda} \frac{\sigma_F}{F}, \quad (2.11)$$

which grows without bound as $F \rightarrow 0$. With a typical AMS background $F_{\text{bkg}} \sim 2 \times 10^{-3}$ the practical dating horizon is

$$t_{\text{max}} \approx -\frac{1}{\lambda} \ln F_{\text{bkg}} \approx 5 \times 10^4 \text{ yr}, \quad (2.12)$$

roughly nine half-lives. This horizon will play a central role below: periodic-time corrections become large precisely when T_c is comparable to it.

2.3 Bateman chains and secular equilibrium

For a linear decay chain $1 \rightarrow 2 \rightarrow \dots \rightarrow n$ the Bateman solution [34] with only $N_1(0) \neq 0$ reads

$$N_n(t) = N_1(0) \left(\prod_{i=1}^{n-1} \lambda_i \right) \sum_{j=1}^n \frac{e^{-\lambda_j t}}{\prod_{k \neq j} (\lambda_k - \lambda_j)}. \quad (2.13)$$

In radiocarbon dating the chain terminates at stable ${}^{14}\text{N}$, so only the first term matters; but Eq. (2.13) illustrates a general feature of linear-time kinetics—every term is a decreasing expo-

nential and no linear combination of them is periodic. When production at rate P is included,

$$\frac{dN}{dt} = P - \lambda N \implies N(t) = \frac{P}{\lambda} + \left(N_0 - \frac{P}{\lambda}\right) e^{-\lambda t}, \quad (2.14)$$

and the atmospheric inventory relaxes to secular equilibrium $N_{\text{eq}} = P/\lambda$ on the time scale $\bar{\tau}$.

2.4 Reservoir models

The exchangeable carbon system is conventionally represented by coupled boxes (atmosphere, biosphere, surface ocean, deep ocean) [28–30]. Writing $\mathbf{N} = (N_1, \dots, N_m)^\top$ for the radiocarbon inventories,

$$\frac{d\mathbf{N}}{dt} = -\mathbf{K}\mathbf{N} - \lambda\mathbf{N} + \mathbf{P}(t), \quad (2.15)$$

where $\mathbf{K}$ is the matrix of first-order exchange coefficients (with columns summing to zero, conserving carbon) and $\mathbf{P}(t) = (P(t), 0, \dots, 0)^\top$. The formal linear-time solution is

$$\mathbf{N}(t) = e^{-(\mathbf{K}+\lambda)t}\mathbf{N}(0) + \int_0^t e^{-(\mathbf{K}+\lambda)(t-s)}\mathbf{P}(s) ds. \quad (2.16)$$

A sinusoidal production modulation $P(t) = P_0[1 + \epsilon \cos \omega t]$ in a single well-mixed box is attenuated and delayed:

$$\frac{\delta N}{N_{\text{eq}}} = \frac{\epsilon \lambda}{\sqrt{\lambda^2 + \omega^2}} \cos(\omega t - \varphi), \quad \tan \varphi = \frac{\omega}{\lambda}. \quad (2.17)$$

This is why the 11-yr Schwabe cycle leaves only a per-mil trace in $\Delta^{14}\text{C}$, while centennial and millennial solar and geomagnetic variations produce the “wiggles” of the calibration curve [10–13]. Rapid events such as the AD 774/775 spike [32] and the post-1955 bomb pulse [31] provide sharp time markers.

2.5 Calibration

Because R_{atm} varies in time, a radiocarbon age is mapped to calendar age through a calibration curve $\mu(\theta)$ with uncertainty $\sigma_c(\theta)$ [14–16]. Given a measurement $y \pm \sigma$, the calendar-age posterior is [20]

$$p(\theta | y) \propto \frac{1}{\sqrt{\sigma^2 + \sigma_c^2(\theta)}} \exp \left[-\frac{(y - \mu(\theta))^2}{2(\sigma^2 + \sigma_c^2(\theta))} \right] p(\theta). \quad (2.18)$$

IntCal20 is built from continuous tree-ring series back to ~ 13.9 cal kBP and, beyond that, from Lake Suigetsu varves [17], U–Th dated speleothems such as Hulu Cave [18], corals and marine sediments, reaching 55 cal kBP [14]. We will use these independent chronometers in Section 9 to constrain the cycle period.

2.6 The implicit topological assumption

Every formula above treats t as a coordinate on $\mathbb{R}^1$. In particular (i) $e^{-\lambda t}$ is injective, so Eq. (2.7) has exactly one solution; (ii) the integrals in Eq. (2.16) are over intervals of $\mathbb{R}$, with a unique past; (iii) the equilibrium in Eq. (2.14) is approached asymptotically as $t \rightarrow \infty$, a limit that exists only on a non-compact time axis. Each of these will change on S^1 .

3 The Topology of Time: $\mathbb{R}^1$, S^1 and Conformal Compactification

3.1 Two one-dimensional manifolds

Up to diffeomorphism there are exactly two connected, one-dimensional, second-countable Hausdorff manifolds without boundary: the line $\mathbb{R}^1$ and the circle S^1 [42, 43]. A time axis is therefore either linear or circular; there is no third option. The two are related in two distinct ways, and both will be needed.

The covering map. Fix a period $T_c > 0$. The quotient $S^1 = \mathbb{R}/T_c\mathbb{Z}$ identifies $t \sim t + nT_c$ for all $n \in \mathbb{Z}$. The projection

$$p: \mathbb{R} \rightarrow S^1, \quad p(t) = e^{2\pi it/T_c}, \quad (3.1)$$

is the universal covering map; its deck transformation group, and the fundamental group of the circle, is

$$\pi_1(S^1) \cong \mathbb{Z}, \quad (3.2)$$

generated by the translation $t \mapsto t + T_c$ [43]. Every $t \in \mathbb{R}$ is uniquely written as

$$t = \phi + nT_c, \quad \phi \in [0, T_c), \quad n = \lfloor t/T_c \rfloor \in \mathbb{Z}, \quad (3.3)$$

where ϕ is the *phase* (a point of S^1) and n the *winding number*. Physics on S^1 is physics on $\mathbb{R}$ restricted to functions invariant under $t \mapsto t + T_c$. The covering picture is illustrated by the helix of Fig. 1(b).

The one-point compactification. Alternatively, S^1 is obtained from $\mathbb{R}$ by adjoining a single point at infinity, $S^1 \cong \mathbb{R} \cup \{\infty\}$, with the neighbourhoods of ∞ being complements of compact sets [42]. Concretely, let the circle have radius t_* and let $\theta \in (-\pi, \pi]$ be the polar angle measured from the point of tangency with the time line. Stereographic projection from the antipodal “north pole” gives

$$t = 2t_* \tan \frac{\theta}{2}, \quad \theta = 2 \arctan \frac{t}{2t_*}, \quad (3.4)$$

which is a diffeomorphism of $(-\pi, \pi)$ onto $\mathbb{R}$ and sends $\theta = \pi$ to $t = \pm\infty$ (Fig. 1(a)). The remote past and remote future of linear time are the *same* point of the circle.

3.2 Conformal structure

The Möbius group. The conformal group of the time line is not the Euclidean group of translations and dilatations but the projective group $PSL(2, \mathbb{R})$ acting by fractional linear transformations [49, 50],

$$t \mapsto t' = \frac{at + b}{ct + d}, \quad ad - bc = 1. \quad (3.5)$$

For $c \neq 0$ the point $t = -d/c$ is sent to infinity, so $PSL(2, \mathbb{R})$ does not act on $\mathbb{R}^1$; it acts on the real projective line $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\} \cong S^1$. The generators

$$H = \partial_t, \quad D = t \partial_t, \quad K = t^2 \partial_t \quad (3.6)$$

close on the algebra $\mathfrak{sl}(2, \mathbb{R})$,

$$[D, H] = -H, \quad [D, K] = K, \quad [H, K] = 2D, \quad (3.7)$$

which is the conformal algebra of conformal quantum mechanics [50]. In this precise sense $S^1 \cong \mathbb{R}P^1$ is the conformal compactification of $\mathbb{R}^1$. The combination

$$L_0 = \frac{1}{2} \left(2t_* H + \frac{K}{2t_*} \right) = \left(t_* + \frac{t^2}{4t_*} \right) \partial_t \quad (3.8)$$

generates a compact $U(1)$ subgroup. Integrating its flow, $dt/ds = t_* + t^2/4t_*$, gives $t = 2t_* \tan(s/2)$: the flow parameter s is exactly the angle θ of Eq. (3.4), and the flow is periodic with period 2π . The “rotation” L_0 is the conformal time-translation of the compactified line.

The conformal factor. Introduce the arc length on the circle,

$$\tau = t_* \theta, \quad T_c = 2\pi t_*, \quad (3.9)$$

so that the circumference of the time circle is T_c . Differentiating Eq. (3.4),

$$dt = \sec^2\left(\frac{\theta}{2}\right) d\tau, \quad d\tau^2 = \Omega^2(t) dt^2, \quad \Omega(t) = \cos^2 \frac{\theta}{2} = \frac{1}{1 + t^2/(4t_*^2)} = \frac{1}{1 + \pi^2 t^2/T_c^2}. \quad (3.10)$$

The linear metric dt^2 and the circular metric $d\tau^2$ are conformally related by Ω^2 , which is unity at $t = 0$ and vanishes as t^{-2} at infinity—the standard behaviour of a compactifying conformal factor [45, 46]. In terms of T_c ,

$$\boxed{\tau = \frac{T_c}{\pi} \arctan\left(\frac{\pi t}{T_c}\right), \quad t = \frac{T_c}{\pi} \tan\left(\frac{\pi \tau}{T_c}\right)} \quad (3.11)$$

with $\tau \in (-T_c/2, T_c/2)$ covering all of $\mathbb{R}$ and $\tau = \pm T_c/2$ the point at infinity.

3.3 Relation to the conformal compactification of spacetime

The one-dimensional statement is the time-like shadow of a four-dimensional one. Penrose’s construction [44, 45] maps Minkowski space with null coordinates $u = t - r$, $v = t + r$ via

$$u = \tan \frac{T - R}{2}, \quad v = \tan \frac{T + R}{2}, \quad (3.12)$$

under which

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 = \frac{1}{4} \sec^2 \frac{T - R}{2} \sec^2 \frac{T + R}{2} (-dT^2 + dR^2 + \sin^2 R d\Omega_2^2). \quad (3.13)$$

The bracket is the metric of the Einstein static universe $\mathbb{R} \times S^3$ [46]. Minkowski space is conformal to a bounded diamond in it, and its conformal boundary consists of $\mathcal{I}^\pm$ and $i^0, i^\pm$. Dirac [48] and Segal [47] emphasised that the conformal group $SO(4, 2)$ acts globally only on the compactification

$$\overline{M}^4 \cong (S^1 \times S^3) / \mathbb{Z}_2, \quad (3.14)$$

in which the time direction is *circular*; the Einstein universe $\mathbb{R} \times S^3$ is its universal cover. Restricted to the world-line $r = 0$ ($R = 0$), Eq. (3.12) reduces to $t = \tan(T/2)$, i.e. to Eq. (3.4)

with $2t_* = 1$. Thus the circular time of this paper is the time-axis of Segal’s compactified Minkowski space, and the “compact” and “covering” pictures of Eqs. (3.1) and (3.4) correspond, respectively, to Einstein’s static universe and to the conformal compactification proper.

3.4 Two physical readings of circular time

The distinction just drawn leads to two different physical hypotheses, and it is essential not to confuse them.

Definition 3.1 (Conformal-chronometric hypothesis). *Linear time t and circular arc-length τ are two parametrisations of the same history, related by Eq. (3.11). Local physical laws hold in t ; physical clocks (“chronometric time” in Segal’s language) read τ .*

Definition 3.2 (Periodic-dynamics hypothesis). *Time is $S^1 = \mathbb{R}/T_c\mathbb{Z}$ with t the proper time on each cycle, and all physical fields and states are single-valued on S^1 : $\Psi(t + T_c) = \Psi(t)$ (up to a global phase for quantum states).*

Under Definition 3.1 the decay law is kept and only the conversion from decay to clock time changes. Under Definition 3.2 the dynamics itself is constrained. Section 4 shows that the constraint is severe: it is incompatible with Eq. (2.3).

4 Dynamics on a Periodic Time Axis

4.1 Single-valuedness and the Floquet obstruction

Under the periodic-dynamics hypothesis (Definition 3.2) a classical state variable $x(t)$ must satisfy

$$x(t + T_c) = x(t) \quad \forall t. \quad (4.1)$$

Consider a linear autonomous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Its flow over one period is the monodromy matrix $\mathbf{M} = e^{\mathbf{A}T_c}$, and by Floquet theory [40] a non-trivial solution satisfies Eq. (4.1) if and only if $\mathbf{M}$ has eigenvalue 1, i.e.

$$\exists \mu \in \text{spec } \mathbf{A} : \quad \mu = \frac{2\pi i m}{T_c}, \quad m \in \mathbb{Z}. \quad (4.2)$$

Proposition 4.1 (No radioactive decay on S^1). *The decay equation $\dot{N} = -\lambda N$ with $\lambda > 0$ has no non-zero solution on $S^1 = \mathbb{R}/T_c\mathbb{Z}$.*

Proof. Here $\mathbf{A} = -\lambda$ and the Floquet multiplier is $e^{-\lambda T_c} < 1$, so Eq. (4.2) fails. Directly: $N(T_c) = N(0)e^{-\lambda T_c} = N(0)$ forces $N(0) = 0$. $\square$

The same argument extends to every Bateman chain (2.13), whose spectrum $\{-\lambda_i\}$ is real and negative, and to the full reservoir system (2.15) without sources, whose matrix $-(\mathbf{K} + \lambda)$ has spectrum in the open left half-plane. It is a special case of a more general statement:

Proposition 4.2 (Monotone functionals are constant on S^1). *If $S(t)$ is single-valued on S^1 and $\dot{S} \geq 0$ everywhere, then $\dot{S} \equiv 0$.*

Proof. $0 = S(T_c) - S(0) = \int_0^{T_c} \dot{S} dt$ with a non-negative integrand. $\square$

Taking $S = -\ln N$ recovers Proposition 4.1; taking S to be the thermodynamic entropy gives the familiar conflict between a circular time and the second law [63–66, 68, 72]. For carbon dating the conclusion is sharp: on S^1 a closed sample cannot lose radiocarbon monotonically

throughout the cycle. Any consistent periodic-time radiocarbon physics must contain either (a) a source, (b) a phase of the cycle in which the decay is reversed, or (c) a reinterpretation of the clock. The rest of the paper develops each option.

4.2 Periodic Green's function: decay with a source

With a periodic production term,

$$\dot{N} = -\lambda N + P(t), \quad P(t + T_c) = P(t), \quad (4.3)$$

the Floquet multiplier $e^{-\lambda T_c} \neq 1$ guarantees a *unique* periodic solution. On the universal cover it is the retarded solution $N(t) = \int_{-\infty}^t e^{-\lambda(t-s)} P(s) ds$; splitting the integral into periods,

$$N(t) = \sum_{n=0}^{\infty} \int_0^{T_c} e^{-\lambda(u+nT_c)} P(t-u) du = \int_0^{T_c} G_{T_c}(u) P(t-u) du, \quad (4.4)$$

with the periodic Green's function

$$G_{T_c}(u) = \sum_{n=0}^{\infty} e^{-\lambda(u+nT_c)} = \frac{e^{-\lambda u}}{1 - e^{-\lambda T_c}}, \quad 0 \leq u < T_c, \quad (4.5)$$

extended periodically. This is the *method of images* on S^1 : the response at phase t is the sum of responses to all copies of the source lying n windings in the past. Its Fourier series, with $\omega_m = 2\pi m/T_c$, is

$$G_{T_c}(u) = \frac{1}{T_c} \sum_{m \in \mathbb{Z}} \frac{e^{i\omega_m u}}{\lambda + i\omega_m}, \quad (4.6)$$

as may be verified from $\frac{1}{T_c} \int_0^{T_c} G_{T_c}(u) e^{-i\omega_m u} du = \frac{1 - e^{-(\lambda + i\omega_m)T_c}}{T_c(\lambda + i\omega_m)(1 - e^{-\lambda T_c})} = \frac{1}{T_c(\lambda + i\omega_m)}$. The Green's function has a jump discontinuity of unit size at $u = 0$,

$$G_{T_c}(0^+) - G_{T_c}(T_c^-) = \frac{1 - e^{-\lambda T_c}}{1 - e^{-\lambda T_c}} = 1, \quad (4.7)$$

as required by $(\partial_u + \lambda)G_{T_c} = \delta_{T_c}(u)$, where δ_{T_c} is the Dirac comb.

4.3 Quantum mechanics on periodic time: quantisation of energy

In quantum mechanics single-valuedness applies to physical states, i.e. to rays. For a state evolving under a time-independent Hamiltonian, $\psi(t) = \sum_E c_E e^{-iEt/\hbar} \phi_E$, periodicity up to a global phase $e^{-2\pi i\beta}$,

$$\psi(t + T_c) = e^{-2\pi i\beta} \psi(t), \quad \beta \in [0, 1), \quad (4.8)$$

requires every occupied energy to lie on the lattice

$$\boxed{E_n = \frac{\hbar}{T_c} (n + \beta), \quad n \in \mathbb{Z}} \quad (4.9)$$

This is the real-time analogue of the Matsubara quantisation that follows from periodicity in imaginary time [51, 52]. The level spacing is

$$\Delta E = \frac{\hbar}{T_c} = 1.31 \times 10^{-27} \text{ eV} \times \left(\frac{10^5 \text{ yr}}{T_c} \right). \quad (4.10)$$

It is instructive to compare this with the natural width of the ^{14}C ground state,

$$\Gamma = \hbar\lambda = 2.52 \times 10^{-27} \text{ eV}, \quad \frac{\Gamma}{\Delta E} = \frac{\lambda T_c}{2\pi} = 1.93 \times \left(\frac{T_c}{10^5 \text{ yr}} \right). \quad (4.11)$$

When $\Gamma \gg \Delta E$ the discreteness of the spectrum is washed out within the resonance and decay is indistinguishable from exponential; when $\Gamma \lesssim \Delta E$ the resonance is resolved into a few lines and decay must be strongly modified. The cross-over, $\lambda T_c \sim 2\pi$, i.e. $T_c \sim T_{1/2} \times 2\pi / \ln 2 \approx 5 \times 10^4 \text{ yr}$, coincides with the radiocarbon dating horizon (2.12). This is the physical reason why the topology of time is *not* invisible to radiocarbon, and it anticipates the quantitative results below.

4.4 Conformal quantum mechanics

Under the conformal-chronometric hypothesis the natural Hamiltonian is the compact generator L_0 of Eq. (3.8) rather than H . De Alfaro, Fubini and Furlan [50] showed that in conformally invariant quantum mechanics $L_0 = \frac{1}{2}(aH + K/a)$ has a discrete spectrum even when H has a continuous one—again a manifestation of compact “time”. We shall not need the details; what matters is that the change from H (evolution in t) to L_0 (evolution in τ) corresponds classically to the reparametrisation (3.11), under which a rate λ in t becomes the position-dependent rate

$$\lambda_{\text{eff}}(\tau) = \lambda \frac{dt}{d\tau} = \lambda \sec^2\left(\frac{\pi\tau}{T_c}\right) \quad (4.12)$$

in τ .

5 Radioactive Decay on S^1 : The Image-Sum Law

5.1 Survival amplitude in linear time

The quantum description of a decaying state $|\psi_0\rangle$ is through its survival amplitude [37, 38]

$$\mathcal{A}(t) = \langle \psi_0 | e^{-iHt/\hbar} | \psi_0 \rangle = \int \rho(E) e^{-iEt/\hbar} dE, \quad (5.1)$$

where $\rho(E) = |\langle E | \psi_0 \rangle|^2$ is the spectral density, and the survival probability is $\mathcal{P}(t) = |\mathcal{A}(t)|^2$. In the Weisskopf–Wigner approximation [35, 36] ρ is the Breit–Wigner (Lorentzian) profile

$$\rho_{\text{BW}}(E) = \frac{1}{2\pi} \frac{\Gamma}{(E - E_0)^2 + \Gamma^2/4}, \quad (5.2)$$

whose Fourier transform gives

$$\mathcal{A}_{\mathbb{R}}(t) = e^{-iE_0 t/\hbar} e^{-\Gamma|t|/2\hbar} = e^{-iE_0 t/\hbar} e^{-\lambda|t|/2}, \quad \lambda = \Gamma/\hbar, \quad (5.3)$$

and hence the exponential law $\mathcal{P}(t) = e^{-\lambda t}$ for $t \geq 0$. Note that $\mathcal{A}_{\mathbb{R}}(-t) = \mathcal{A}_{\mathbb{R}}(t)^*$: the survival amplitude is time-symmetric, a general consequence of Eq. (5.1) for real ρ . Deviations from Eq. (5.3) at very short times (the Zeno regime [39]) and very long times (Khalfin’s power-law tail [37]) arise from the bounded-below spectrum and are negligible for ^{14}C ; we ignore them.

5.2 Poisson summation on the energy lattice

On periodic time the spectrum is restricted to the lattice (4.9), and the integral (5.1) becomes a sum,

$$\mathcal{A}_{S^1}(t) = \sum_{n \in \mathbb{Z}} \Delta E \rho(E_n) e^{-iE_n t/\hbar}, \quad E_n = \Delta E (n + \beta). \quad (5.4)$$

Apply the Poisson summation formula [41]

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \hat{f}(k), \quad \hat{f}(k) = \int_{\mathbb{R}} f(x) e^{-2\pi i k x} dx, \quad (5.5)$$

to $f(x) = \Delta E \rho(\Delta E(x + \beta)) e^{-i\Delta E(x+\beta)t/\hbar}$. Changing variable to $E = \Delta E(x + \beta)$ and using $2\pi E/\Delta E = ET_c/\hbar$,

$$\hat{f}(k) = e^{2\pi i k \beta} \int \rho(E) e^{-iE(t+kT_c)/\hbar} dE = e^{2\pi i k \beta} \mathcal{A}_{\mathbb{R}}(t + kT_c). \quad (5.6)$$

Therefore, relabelling $k \rightarrow -k$,

$$\boxed{\mathcal{A}_{S^1}(t) = \sum_{k \in \mathbb{Z}} e^{-2\pi i k \beta} \mathcal{A}_{\mathbb{R}}(t - kT_c)} \quad (5.7)$$

The survival amplitude on the circle is the sum of the linear-time amplitude over all its images on the universal cover, weighted by the holonomy phase $e^{-2\pi i k \beta}$ of the k -th winding. Equation (5.7) is exact for any spectral density; it is the time-domain counterpart of spectral quantisation, and makes manifest the periodicity $\mathcal{A}_{S^1}(t + T_c) = e^{-2\pi i \beta} \mathcal{A}_{S^1}(t)$.

5.3 Closed form for the Breit–Wigner resonance

Insert Eq. (5.3). The k -th term carries the phase $e^{-iE_0 t/\hbar} e^{ik(E_0 T_c/\hbar - 2\pi\beta)}$; define the *detuning*

$$\delta = \frac{E_0}{\Delta E} - \beta \pmod{1}, \quad \varphi = 2\pi\delta, \quad (5.8)$$

which measures how far the resonance centre lies from the nearest allowed level. With

$$q \equiv e^{-\lambda T_c/2}, \quad (5.9)$$

and $0 \leq t \leq T_c$, the terms with $k \leq 0$ and $k \geq 1$ are geometric series:

$$\begin{aligned} e^{iE_0 t/\hbar} \mathcal{A}_{S^1}(t) &= \sum_{j=0}^{\infty} (qe^{-i\varphi})^j e^{-\lambda t/2} + \sum_{k=1}^{\infty} (qe^{i\varphi})^k e^{\lambda t/2} \\ &= \frac{e^{-\lambda t/2}}{1 - qe^{-i\varphi}} + \frac{qe^{i\varphi} e^{\lambda t/2}}{1 - qe^{i\varphi}}. \end{aligned} \quad (5.10)$$

Bringing to a common denominator $|1 - qe^{i\varphi}|^2$ and normalising to $\mathcal{P}(0) = 1$ yields the general *periodic survival law*

$$F_{\delta}(t) = \frac{(e^{-\lambda t/2} - q^2 e^{\lambda t/2})^2 + 4q^2 \sinh^2 \frac{\lambda t}{2} + 4q(e^{-\lambda t/2} - q^2 e^{\lambda t/2}) \sinh \frac{\lambda t}{2} \cos \varphi}{(1 - q^2)^2}. \quad (5.11)$$

Two cases are distinguished.

Resonant case ($\delta = 0$). The resonance centre sits on an allowed level. The numerator of Eq. (5.10) becomes $(e^{-\lambda t/2} + qe^{\lambda t/2})/(1 - q)$ and

$$F_0(t) = \left[\frac{e^{-\lambda t/2} + qe^{\lambda t/2}}{1 + q} \right]^2 = \frac{\cosh^2\left[\frac{\lambda}{2}\left(t - \frac{T_c}{2}\right)\right]}{\cosh^2\left(\frac{\lambda T_c}{4}\right)} \quad (5.12)$$

Anti-resonant case ($\delta = \frac{1}{2}$). The resonance centre lies midway between levels; then $e^{i\varphi} = -1$ and

$$F_{1/2}(t) = \left[\frac{e^{-\lambda t/2} - qe^{\lambda t/2}}{1 - q} \right]^2 = \frac{\sinh^2\left[\frac{\lambda}{2}\left(\frac{T_c}{2} - t\right)\right]}{\sinh^2\left(\frac{\lambda T_c}{4}\right)} \quad (5.13)$$

5.4 Properties of the periodic decay laws

Proposition 5.1. For $0 \leq t \leq T_c$ the functions (5.12), (5.13) satisfy:

- (a) Periodicity and recurrence: $F(0) = F(T_c) = 1$.
- (b) Reflection symmetry: $F(T_c - t) = F(t)$.
- (c) Extremum at mid-cycle: F_0 attains its minimum

$$F_{\min} \equiv F_0(T_c/2) = \operatorname{sech}^2\left(\frac{\lambda T_c}{4}\right) \quad (5.14)$$

while $F_{1/2}(T_c/2) = 0$.

- (d) Linear limit: $\lim_{T_c \rightarrow \infty} F_\delta(t) = e^{-\lambda t}$ for every δ and fixed t .
- (e) Bracketing: $F_{1/2}(t) \leq F_\delta(t) \leq F_0(t)$ for all δ and $0 \leq t \leq T_c$; moreover $e^{-\lambda t} \leq F_0(t)$ on the whole cycle and $F_{1/2}(t) \leq e^{-\lambda t}$ on the first half-cycle $0 \leq t \leq T_c/2$.

Proof. (a)–(c) follow by inspection of the hyperbolic forms. (d): $q \rightarrow 0$. (e): The first chain holds because Eq. (5.11) is linear in $\cos \varphi$ with coefficient $4q(e^{-\lambda t/2} - q^2 e^{\lambda t/2}) \sinh(\lambda t/2)$, which is non-negative precisely when $e^{-\lambda t} \geq q^2 = e^{-\lambda T_c}$, i.e. for $t \leq T_c$. Next, $F_0 e^{\lambda t} = [(1 + qe^{\lambda t})/(1 + q)]^2 \geq 1$ for all $t \geq 0$. Finally $F_{1/2} e^{\lambda t} = [(1 - qe^{\lambda t})/(1 - q)]^2 \leq 1$ whenever $q \leq qe^{\lambda t} \leq 2 - q$, which certainly holds for $t \leq T_c/2$, where $qe^{\lambda t} \leq 1$. In the second half-cycle $F_{1/2}$ regrows and eventually exceeds $e^{-\lambda t}$, reaching 1 at $t = T_c$. $\square$

Instantaneous decay rate. The logarithmic derivative $\gamma(t) = -d \ln F / dt$ is

$$\gamma_0(t) = \lambda \tanh\left[\frac{\lambda}{2}\left(\frac{T_c}{2} - t\right)\right], \quad \gamma_{1/2}(t) = \lambda \coth\left[\frac{\lambda}{2}\left(\frac{T_c}{2} - t\right)\right]. \quad (5.15)$$

In the resonant case the effective decay rate starts *below* the linear value, $\gamma_0(0) = \lambda \tanh(\lambda T_c/4) < \lambda$, falls to zero at mid-cycle and becomes *negative* in the second half-cycle: radiocarbon is regenerated. In the anti-resonant case the rate starts above λ and diverges at mid-cycle, where the sample is totally depleted; thereafter it regrows. Both behaviours are forced by Proposition 4.2.

Interpretation of the images. The $k = 0$ term of Eq. (5.7) is the ordinary decaying amplitude. The $k = 1$ term, $\mathcal{A}_{\mathbb{R}}(t - T_c) \propto e^{-\lambda(T_c - t)/2}$, grows with t : it is the amplitude of the *same* initial state one cycle in the future, propagated backward by the time-symmetric kernel. On S^1 the future preparation of the sample *is* its past preparation, and the two interfere. The relative

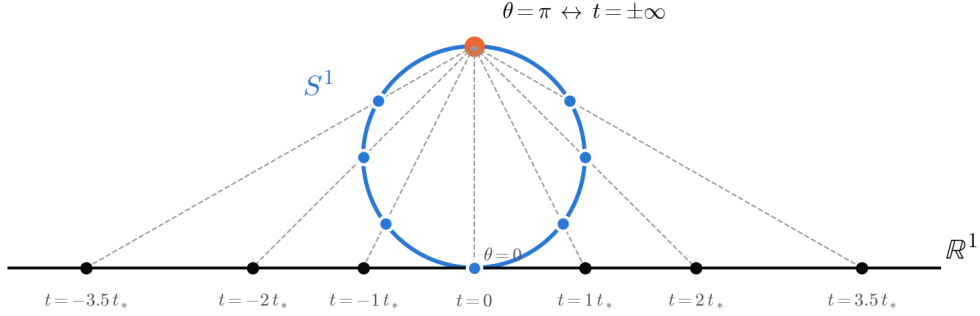
phase of the images is the holonomy $e^{-2\pi i\delta}$ of the circle, which decides whether the interference is constructive (resonant, slower apparent decay) or destructive (anti-resonant, faster apparent decay).

Asymptotic expansion. For $q \ll 1$, from $\ln F_0 = -\lambda t + 2 \ln(1 + qe^{\lambda t}) - 2 \ln(1 + q)$,

$$\ln F_0(t) = -\lambda t + 2q \left(e^{\lambda t} - 1 \right) - q^2 \left(e^{2\lambda t} - 1 \right) + O(q^3), \quad (5.16)$$

and similarly $\ln F_{1/2}(t) = -\lambda t - 2q(e^{\lambda t} - 1) - q^2(e^{2\lambda t} - 1) + O(q^3)$. The leading correction is of order $q e^{\lambda t} = e^{-\lambda(T_c/2 - t)}$: exponentially small unless the sample age is within a few mean lives of mid-cycle.

(a) One-point conformal compactification: $t = 2t_* \tan(\theta/2)$



(b) Covering map $p(t) = e^{2\pi it/T_c}$: all $t + nT_c$ project to one point

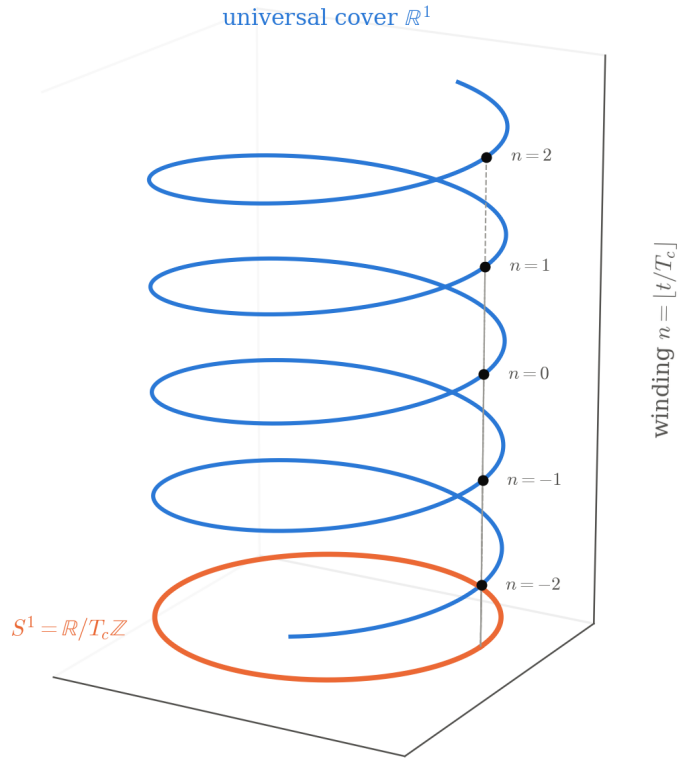


Figure 1: Topology of linear and circular time. (a) One-point conformal compactification of the time line: stereographic projection from the antipodal point $\theta = \pi$ maps each instant $t \in \mathbb{R}^1$ to a point of S^1 via $t = 2t_* \tan(\theta/2)$, Eq. (3.4). Points equally spaced in t crowd towards the antipode, which represents both $t = +\infty$ and $t = -\infty$. (b) The covering map $p : \mathbb{R}^1 \rightarrow S^1$, Eq. (3.1): linear time is drawn as a helix over the circle, and the instants $t + nT_c$ for every winding number $n \in \mathbb{Z} = \pi_1(S^1)$ project to a single point. The conformal-chronometric hypothesis (Definition 3.1) uses map (a); the periodic-dynamics hypothesis (Definition 3.2) uses map (b).

6 Radiocarbon Dating on S^1 : Three Age Equations

We now invert the periodic laws. Throughout, F is the measured fraction of initial radiocarbon remaining (after fractionation and reservoir corrections), $t_L = -\lambda^{-1} \ln F$ is the linear age, and T_c is the circumference of the time circle.

6.1 Framework I: conformal chronology

Under Definition 3.1, the linear decay law holds in t and the sample's clock age is the arc length τ . Substituting Eq. (3.11) into $F = e^{-\lambda t}$ gives the decay law in chronometric time,

$$F(\tau) = \exp \left[-\frac{\lambda T_c}{\pi} \tan \left(\frac{\pi \tau}{T_c} \right) \right], \quad 0 \leq \tau < \frac{T_c}{2}, \quad (6.1)$$

which reaches $F = 0$ at the finite chronometric age $\tau = T_c/2$, the image of $t = \infty$. The age equation is

$$\tau_C = \frac{T_c}{\pi} \arctan \left(\frac{\pi t_L}{T_c} \right) = \frac{T_c}{\pi} \arctan \left(-\frac{\pi \ln F}{\lambda T_c} \right) \quad (6.2)$$

Using $\arctan x = x - x^3/3 + x^5/5 - \dots$ for $|x| < 1$,

$$\tau_C = t_L - \frac{\pi^2}{3} \frac{t_L^3}{T_c^2} + \frac{\pi^4}{5} \frac{t_L^5}{T_c^4} - \dots, \quad (6.3)$$

while for $t_L \gg T_c$, using $\arctan x = \pi/2 - 1/x + \dots$,

$$\tau_C = \frac{T_c}{2} - \frac{T_c^2}{\pi^2 t_L} + O(t_L^{-3}). \quad (6.4)$$

Conformal ages are always *younger* than linear ages, $\tau_C < t_L$ for $t_L > 0$, and every sample, however depleted, has $\tau_C < T_c/2$. The age discrepancy is, to leading order,

$$\Delta_C \equiv t_L - \tau_C \simeq \frac{\pi^2 t_L^3}{3 T_c^2}. \quad (6.5)$$

The effective chronometric decay rate is Eq. (4.12): decay *accelerates* as the antipode is approached.

Error propagation. Since $d\tau_C/dt_L = \Omega(t_L)$,

$$\sigma_{\tau_C} = \frac{\sigma_{t_L}}{1 + \pi^2 t_L^2 / T_c^2} = \frac{\sigma_F}{\lambda F (1 + \pi^2 \ln^2 F / \lambda^2 T_c^2)}. \quad (6.6)$$

The conformal factor compresses both ages and their uncertainties; a sample at the linear horizon ($t_L = 5 \times 10^4$ yr) with $\sigma_{t_L} = 1000$ yr has, for $T_c = 10^5$ yr, $\sigma_{\tau_C} = 288$ yr.

6.2 Framework II: resonant image-sum chronology

Under Definition 3.2 with $\delta = 0$, inverting Eq. (5.12) on the first half-cycle $0 \leq t \leq T_c/2$:

$$t_P^{(0)} = \frac{T_c}{2} - \frac{2}{\lambda} \operatorname{arccosh} \left[\sqrt{F} \cosh \left(\frac{\lambda T_c}{4} \right) \right], \quad F \geq F_{\min} = \operatorname{sech}^2 \left(\frac{\lambda T_c}{4} \right) \quad (6.7)$$

together with the reflected root $T_c - t_P^{(0)}$ on the second half-cycle. For $F < F_{\min}$ there is *no* solution: such a ratio cannot occur on a circle of circumference T_c . Equivalently, the largest linear age that has a periodic counterpart is

$$t_{L,\max} = \frac{2}{\lambda} \ln \cosh\left(\frac{\lambda T_c}{4}\right) \xrightarrow{\lambda T_c \gg 1} \frac{T_c}{2} - \frac{2 \ln 2}{\lambda} = \frac{T_c}{2} - 2T_{1/2}. \quad (6.8)$$

From Eq. (5.16), with $e^{\lambda t_L} = 1/F$,

$$t_P^{(0)} = t_L + \frac{2q}{\lambda} (e^{\lambda t_L} - 1) + O(q^2) = t_L + \frac{2}{\lambda} e^{-\lambda T_c/2} \frac{1-F}{F} + O(q^2), \quad (6.9)$$

so resonant periodic ages are *older* than linear ones, with discrepancy

$$\Delta_P \equiv t_P^{(0)} - t_L \simeq \frac{2}{\lambda} e^{-\lambda(T_c/2 - t_L)} \quad (t_L \gg \bar{\tau}). \quad (6.10)$$

Error propagation. Differentiating Eq. (6.7) with $c = \cosh(\lambda T_c/4)$,

$$\frac{dt_P^{(0)}}{dF} = -\frac{c}{\lambda \sqrt{F} \sqrt{F c^2 - 1}}, \quad \boxed{\frac{\sigma_{t_P}}{\sigma_{t_L}} = \left(1 - \frac{F_{\min}}{F}\right)^{-1/2}} \quad (6.11)$$

The uncertainty of a periodic age diverges as $F \rightarrow F_{\min}$: near mid-cycle the decay curve is flat (Eq. (5.15), $\gamma_0 \rightarrow 0$) and the ratio carries no age information.

6.3 Framework III: anti-resonant image-sum chronology

With $\delta = \frac{1}{2}$, Eq. (5.13) is invertible for every $F \in (0, 1]$:

$$\boxed{t_P^{(1/2)} = \frac{T_c}{2} - \frac{2}{\lambda} \operatorname{arcsinh}\left[\sqrt{F} \sinh\left(\frac{\lambda T_c}{4}\right)\right]} \quad (6.12)$$

with expansion

$$t_P^{(1/2)} = t_L - \frac{2}{\lambda} e^{-\lambda T_c/2} \frac{1-F}{F} + O(q^2), \quad (6.13)$$

and error law, with $s = \sinh(\lambda T_c/4)$,

$$\frac{\sigma_{t_P^{(1/2)}}}{\sigma_{t_L}} = \left(1 + \frac{1}{F s^2}\right)^{-1/2} < 1. \quad (6.14)$$

Anti-resonant ages are younger than linear ages and bounded by $T_c/2$; as $F \rightarrow 0$, $t_P^{(1/2)} \rightarrow T_c/2 - 2\sqrt{F} s/\lambda$.

By Proposition 5.1(e), for a general detuning δ the periodic age lies between the anti-resonant and resonant values,

$$t_P^{(1/2)}(F) \leq t_P^{(\delta)}(F) \leq t_P^{(0)}(F), \quad (6.15)$$

and the linear age t_L lies inside the same bracket. The detuning is a property of the ^{14}C nucleus and of T_c that is not accessible to present experiment, so Eq. (6.15) is the honest statement of the periodic-time age.

6.4 The winding and reflection ambiguity

On the universal cover every solution of the dating equation comes with its images. In Framework II the full solution set is

$$\mathcal{S}(F) = \{t_P + nT_c, T_c - t_P + nT_c : n \in \mathbb{Z}\}, \quad (6.16)$$

the orbit of t_P under the infinite dihedral group $D_\infty = \langle t \mapsto t + T_c, t \mapsto -t \rangle$, which is the symmetry group of the periodic survival law (Proposition 5.1(a,b)). Radiocarbon alone cannot determine n or distinguish t_P from its reflection. In linear time the convention is implicitly $n = 0$ and “the root on the decaying branch”; on S^1 , under Definition 3.2, all elements of $\mathcal{S}(F)$ label the *same* event, and the question “how many cycles ago?” has no physical meaning. Only the phase $\phi \in S^1$ is observable. This is the chronological counterpart of the statement $\pi_1(S^1) \cong \mathbb{Z}$.

6.5 Summary of the age equations

Framework	Survival law	Age formula	vs. t_L
Linear $\mathbb{R}^1$	$e^{-\lambda t}$	$-\lambda^{-1} \ln F$	—
I. Conformal	$\exp[-\frac{\lambda T_c}{\pi} \tan \frac{\pi \tau}{T_c}]$	$\frac{T_c}{\pi} \arctan \frac{\pi t_L}{T_c}$	younger; $< T_c/2$
II. Resonant	$\frac{\cosh^2[\lambda(t - T_c/2)/2]}{\cosh^2(\lambda T_c/4)}$	$\frac{T_c}{2} - \frac{2}{\lambda} \operatorname{arccosh}[\sqrt{F} \cosh \frac{\lambda T_c}{4}]$	older; none if $F < F_{\min}$
III. Anti-resonant	$\frac{\sinh^2[\lambda(T_c/2 - t)/2]}{\sinh^2(\lambda T_c/4)}$	$\frac{T_c}{2} - \frac{2}{\lambda} \operatorname{arcsinh}[\sqrt{F} \sinh \frac{\lambda T_c}{4}]$	younger; $< T_c/2$

Table 1: Survival laws and age equations in linear and periodic time. All three periodic frameworks reduce to the linear one as $T_c \rightarrow \infty$.

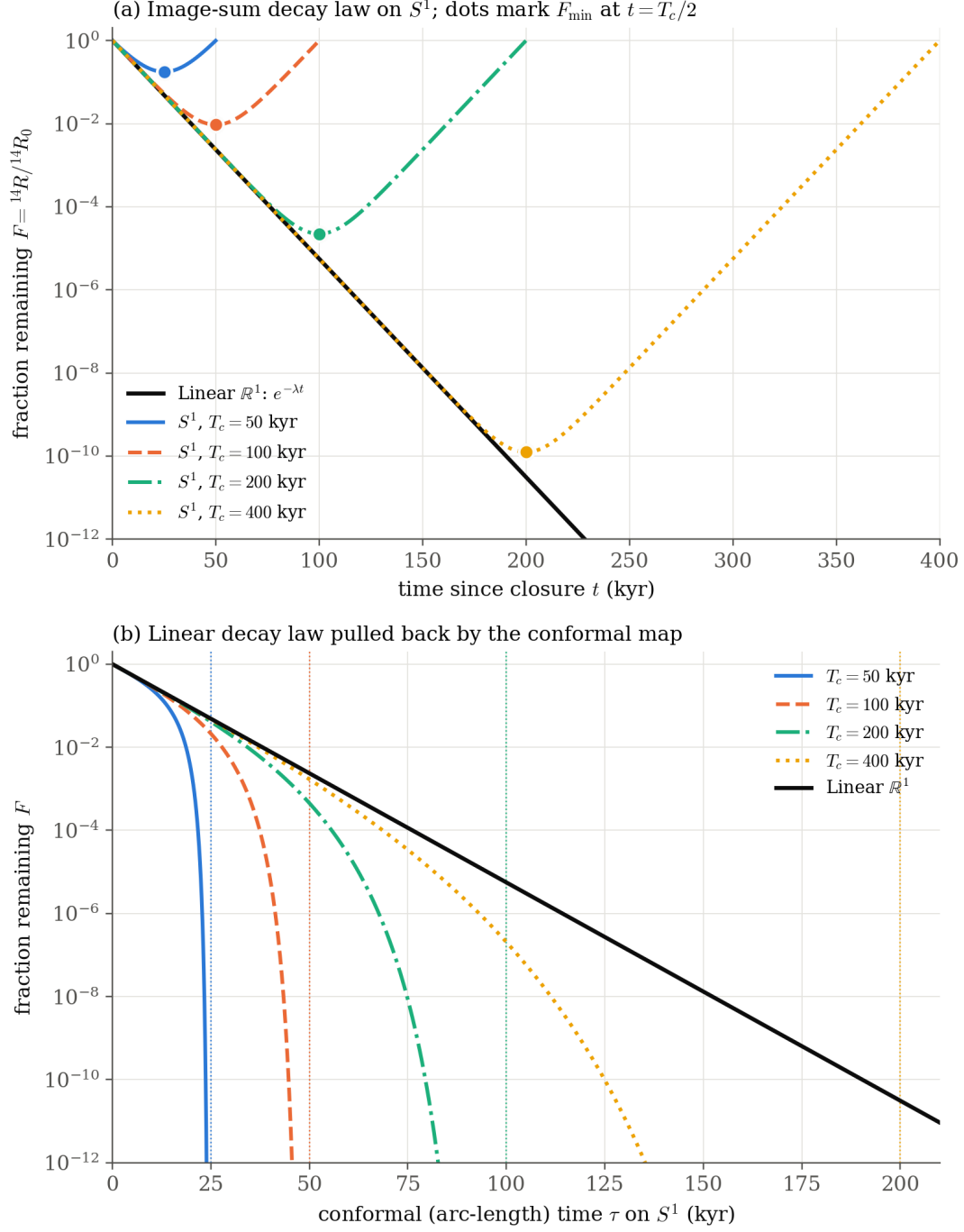


Figure 2: Decay laws in linear and periodic time. (a) Fraction remaining F versus time since closure for the linear law $e^{-\lambda t}$ (black) and the resonant image-sum law $F_0(t)$ of Eq. (5.12) for $T_c = 50, 100, 200$ and 400 kyr. Each periodic curve follows the exponential initially, flattens, reaches its floor $F_{\min} = \text{sech}^2(\lambda T_c/4)$ at mid-cycle (dots) and regrows to unity at $t = T_c$, as required by Proposition 4.2. (b) The same linear law expressed in conformal chronometric time τ , Eq. (6.1): complete depletion is reached at the finite time $\tau = T_c/2$ (dotted verticals), the image of $t = \infty$ on the circle. $\lambda = \ln 2/5730 \text{ yr}^{-1}$.

7 The Carbon Reservoir and Calibration on S^1

A radiocarbon age depends not only on the decay law of the sample but on the initial ratio R_0 inherited from the atmosphere at the moment of closure. We therefore extend the periodic formalism to the production–exchange system.

7.1 Periodic reservoir with classical (retarded) kinetics

If the atoms in the exchangeable reservoir decay classically, the atmospheric inventory obeys Eq. (4.3) and its unique periodic solution is Eq. (4.4). Expanding the production rate in harmonics of the cycle,

$$P(t) = \sum_{m \in \mathbb{Z}} p_m e^{i\omega_m t}, \quad \omega_m = \frac{2\pi m}{T_c}, \quad (7.1)$$

Eq. (4.6) gives

$$N(t) = \sum_{m \in \mathbb{Z}} \frac{p_m}{\lambda + i\omega_m} e^{i\omega_m t}, \quad \bar{N} = \frac{p_0}{\lambda}. \quad (7.2)$$

Two remarks are in order. First, the secular equilibrium $\bar{N} = p_0/\lambda$, which in linear time is only approached as $t \rightarrow \infty$ (Eq. (2.14)), is on S^1 simply the zero mode of the periodic solution; no infinite past is required. Second, the transfer function $(\lambda + i\omega)^{-1}$ is sampled only at the discrete frequencies ω_m . For the multi-box system (2.15),

$$\mathbf{N}(t) = \int_0^{T_c} \left(1 - e^{-(\mathbf{K} + \lambda)T_c}\right)^{-1} e^{-(\mathbf{K} + \lambda)u} \mathbf{P}(t - u) du, \quad (7.3)$$

which exists because every eigenvalue of $\mathbf{K} + \lambda$ has positive real part, so that $1 - e^{-(\mathbf{K} + \lambda)T_c}$ is invertible.

A spectral test. Equation (7.2) implies that on S^1 the atmospheric $\Delta^{14}\text{C}$ record can contain only frequencies that are integer multiples of $1/T_c$:

$$\nu \in \left\{ \frac{m}{T_c} : m \in \mathbb{Z} \right\}. \quad (7.4)$$

The quasi-periodicities identified in the Holocene record—the ~ 11 -yr Schwabe, ~ 88 -yr Gleissberg, ~ 208 -yr de Vries–Suess and ~ 2300 -yr Hallstatt cycles [10–13, 27]—must all be harmonics of the fundamental. With a record of length $L \leq T_c$, frequencies are only resolved to $\sim 1/L$, so for $T_c \gg L$ this condition is weak; it becomes a real test only if T_c is comparable to the length of the calibration record ($\sim 5 \times 10^4$ yr).

7.2 Periodic reservoir with image-sum kinetics

If instead every radiocarbon atom obeys the periodic survival law of Section 5, the probability that an atom produced at phase s is present at phase t is $F_\delta(t - s)$, already periodic; no further sum over windings is needed:

$$N(t) = \int_0^{T_c} F_\delta(u) P(t - u) du. \quad (7.5)$$

For constant production P_0 in the resonant case,

$$\bar{N}^{(0)} = P_0 \int_0^{T_c} \frac{\cosh^2[\lambda(u - T_c/2)/2]}{\cosh^2(\lambda T_c/4)} du = P_0 \frac{T_c/2 + \lambda^{-1} \sinh(\lambda T_c/2)}{\cosh^2(\lambda T_c/4)} \xrightarrow{\lambda T_c \gg 1} \frac{2P_0}{\lambda}, \quad (7.6)$$

and in the anti-resonant case

$$\bar{N}^{(1/2)} = P_0 \frac{\lambda^{-1} \sinh(\lambda T_c/2) - T_c/2}{\sinh^2(\lambda T_c/4)} \xrightarrow{\lambda T_c \gg 1} \frac{2P_0}{\lambda}. \quad (7.7)$$

The factor of two is not an artefact. Near $u = T_c$ the survival probability returns to unity (Proposition 5.1(a)): an atom produced at phase s has a non-zero probability of being present *before* s , through the advanced ($k = 1$) image. The limit $T_c \rightarrow \infty$ at fixed t recovers $e^{-\lambda t}$, but the limit of the integral over a full cycle does not recover P_0/λ ; the two limits do not commute. The inventory therefore discriminates between retarded (Eq. (7.2)) and time-symmetric (Eq. (7.5)) periodic kinetics. Since the global ^{14}C inventory and independently modelled production rates [25, 26] agree to within a few tens of per cent, the unrestricted image-sum reservoir appears disfavoured, whereas an image-sum law applied only to closed samples (with the reservoir itself retarded) remains possible. We regard this as one of the sharper qualitative predictions of the periodic framework.

7.3 The periodic dating equation

Let a sample close (die) at phase s and be measured at phase t . Its normalised ratio is

$$F_{\text{meas}}(t) = C(s) \mathcal{F}([t - s]_{T_c}), \quad C(s) = \frac{R_{\text{atm}}(s)}{R_{\text{std}}}, \quad (7.8)$$

where $[x]_{T_c} \in [0, T_c)$ denotes reduction modulo T_c and $\mathcal{F}$ is the survival law of whichever framework is adopted. The calibration function C is itself periodic, $C(s + T_c) = C(s)$. Defining the radiocarbon age of the calibration curve as in linear time, $\mu(s) = -\lambda_L^{-1} \ln[C(s)\mathcal{F}([t_0 - s]_{T_c})]$ with t_0 the reference epoch, the Bayesian calibration (2.18) carries over with $\theta \in S^1$:

$$p(\theta | y) \propto \frac{1}{\sqrt{\sigma^2 + \sigma_c^2(\theta)}} \exp\left[-\frac{(y - \mu(\theta))^2}{2(\sigma^2 + \sigma_c^2(\theta))}\right] p(\theta), \quad \theta \in S^1, \quad (7.9)$$

but the posterior is now a density on the circle, normalised by $\oint p(\theta|y) d\theta = 1$, and multimodality from the reflection symmetry of Section 6 is generic. Circular statistics (e.g. the von Mises family) replace Gaussian summaries: the calendar “mean” of a set of dates $\{\theta_j\}$ is

$$\bar{\theta} = \arg \sum_j e^{2\pi i \theta_j / T_c} \cdot \frac{T_c}{2\pi}, \quad \bar{R} = \frac{1}{J} \left| \sum_{j=1}^J e^{2\pi i \theta_j / T_c} \right|, \quad (7.10)$$

with $1 - \bar{R}$ serving as the circular variance.

8 Numerical Results: How Widely Do Linear and Periodic Ages Differ?

8.1 Parameters

We use $\lambda = \ln 2/5730 \text{ yr}^{-1}$ (Eq. (2.5)) and report *physical* ages; conventional ages are smaller by the factor $5568/5730$ (Eq. (2.8)). We consider five cycle periods chosen to span the relevant range: $T_c = 5 \times 10^3 \text{ yr}$, the period of the world cycle in some cyclic cosmologies and comparable to recorded history; $T_c = 5 \times 10^4 \text{ yr}$, comparable to the radiocarbon horizon (2.12); $T_c = 10^5$ and $2 \times 10^5 \text{ yr}$, comparable to the glacial cycles of the late Pleistocene; and $T_c = 4.32 \times 10^6 \text{ yr}$, the *mahāyuga* of Indian cosmology, representative of cycles much longer than the dating horizon. All computations use the closed-form expressions of Table 1; the logarithmic form of Eq. (6.7), obtained from $\operatorname{arccosh} x = \ln x + \ln(1 + \sqrt{1 - x^{-2}})$ and $\ln \cosh y = y + \ln(1 + e^{-2y}) - \ln 2$, is used to avoid overflow at large λT_c .

8.2 Age tables

Table 2 lists, for six representative values of F spanning the dating range, the linear age and the three periodic ages.

Several features stand out.

1. **Short cycles** ($T_c = 5000 \text{ yr}$). The differences are enormous. A sample with half its radiocarbon left, conventionally 5730 yr old, has conformal age 2069 yr and anti-resonant age 729 yr; a sample at the detection limit ($F = 0.002$, linear age 51 374 yr) has conformal age 2451 yr. Every sample, however old in linear reckoning, is compressed into the half-cycle $\tau < 2500 \text{ yr}$. In the resonant framework the floor is $F_{\min} = 0.977$, so any sample that has lost more than 2.3% of its radiocarbon—i.e. any sample with $t_L > 188 \text{ yr}$ —is *inconsistent* with the cycle.
2. **Cycles comparable to the horizon** ($T_c = 5 \times 10^4$ – 10^5 yr). Differences are of order 10–50%. For $T_c = 10^5 \text{ yr}$ and $F = 0.01$ ($t_L = 38\,069 \text{ yr}$) the three periodic ages are 27 833, 45 873 and 35 121 yr: a spread of 18 000 yr from a single measurement, depending only on the choice of periodic physics.
3. **Long cycles** ($T_c \geq 2 \times 10^5 \text{ yr}$). The image-sum corrections become exponentially small ($\propto e^{-\lambda T_c/2}$), but the conformal correction decays only as a power, $\propto T_c^{-2}$ (Eq. (6.5)). For $T_c = 2 \times 10^5 \text{ yr}$ and $F = 0.002$ the conformal age is still 8 149 yr younger than the linear one.
4. **Cosmological cycles** ($T_c = 4.32 \times 10^6 \text{ yr}$). All frameworks agree with linear time to better than 25 yr throughout the dating range: radiocarbon is blind to a cycle this long.

8.3 Case studies: well-known samples from archaeology, biology and geology

To make the comparison concrete, Table 3 applies the age equations of Table 1 to ten well-known radiocarbon-dated samples: four from archaeology, three from biology and three from geology. For each sample we take the published conventional radiocarbon age t_{conv} , recover the measured fraction remaining from Eq. (2.8),

$$F = \exp\left(-\frac{t_{\text{conv}}}{8033 \text{ yr}}\right), \quad (8.1)$$

F	t_L (yr)	τ_C (yr)	$t_P^{(0)}$ (yr)	$t_P^{(1/2)}$ (yr)
	linear	I. conformal	II. resonant	III. anti-resonant
$T_c = 5 \times 10^3$ yr ($F_{\min} = 0.977$)				
0.9	871	797	—	127
0.5	5 730	2 069	—	729
0.25	11 460	2 280	—	1 246
0.1	19 035	2 367	—	1 707
0.01	38 069	2 434	—	2 249
0.002	51 374	2 451	—	2 388
$T_c = 5 \times 10^4$ yr ($F_{\min} = 0.177$)				
0.9	871	870	966	786
0.5	5 730	5 500	6 651	5 011
0.25	11 460	9 932	14 984	9 523
0.1	19 035	13 917	—	14 449
0.01	38 069	18 698	—	21 459
0.002	51 374	20 219	—	23 407
$T_c = 10^5$ yr ($F_{\min} = 0.0094$)				
0.9	871	871	875	867
0.5	5 730	5 669	5 769	5 691
0.25	11 460	11 000	11 579	11 344
0.1	19 035	17 155	19 399	18 695
0.01	38 069	27 833	45 873	35 121
0.002	51 374	32 343	—	42 655
$T_c = 2 \times 10^5$ yr ($F_{\min} = 2.23 \times 10^{-5}$)				
0.9	871	871	871	871
0.5	5 730	5 715	5 730	5 730
0.25	11 460	11 339	11 460	11 460
0.1	19 035	18 496	19 035	19 034
0.01	38 069	34 310	38 078	38 060
0.002	51 374	43 225	51 420	51 328
$T_c = 4.32 \times 10^6$ yr ($F_{\min} = 1.33 \times 10^{-113}$)				
0.9	871	871	871	871
0.5	5 730	5 730	5 730	5 730
0.25	11 460	11 460	11 460	11 460
0.1	19 035	19 033	19 035	19 035
0.01	38 069	38 060	38 069	38 069
0.002	51 374	51 350	51 374	51 374

Table 2: Linear and periodic radiocarbon ages for representative fractions remaining F . A dash denotes $F < F_{\min}$, for which the resonant periodic law has no solution. Ages are physical (Cambridge half-life) and rounded to the nearest year.

and compute from this single number the physical linear age t_L (Eq. (2.7)) and the three periodic ages for cycle periods $T_c = 10^5$ yr and $T_c = 5000$ yr. All ages are uncalibrated radiocarbon ages, in years before AD 1950, computed with the Cambridge half-life. Calendar calibration (Section 2) is deliberately not applied: the calibration curve is itself a product of linear-time chronology, and its periodic counterpart (Eq. (7.9)) is unknown. The comparison is therefore between the ages that each topology of time assigns to the *same measured isotope ratio*.

Four points emerge from Table 3.

1. **Historical and Holocene samples, $T_c = 10^5$ yr.** For the Shroud of Turin, the Iceman, Kennewick Man and the Mazama eruption, the three periodic frameworks agree with linear time to within about 2.5%. For example, the Iceman is 4659 yr old linearly and 4626–4688 yr old periodically. A 100 000-year cycle is effectively invisible in the Holocene.
2. **Late-glacial samples, $T_c = 10^5$ yr.** Clovis, Two Creeks and Laacher See lie near 11 500–12 200 yr. Here the conformal ages are already 450–550 yr younger, comparable to or larger than typical calibrated uncertainties.
3. **Pleistocene samples, $T_c = 10^5$ yr.** The frameworks diverge sharply. The Chauvet rhinoceros drawing ($t_L = 31\,840$ yr) is 25 004 yr old conformally, 34 040 yr resonantly and 30 282 yr anti-resonantly, a spread of 9000 yr. For the Lyuba mammoth ($t_L = 43\,016$ yr) the conformal age is 29 722 yr, 13 300 yr younger, and the resonant framework gives no age at all because $F = 0.0055$ lies below $F_{\min} = 0.0094$.
4. **A 5000-year cycle.** Every age is compressed into less than half a cycle. In the conformal framework the ten samples, spanning linear ages from 709 to 43 016 yr, occupy only the interval 667–2441 yr. The Iceman becomes 1976 yr old and the Chauvet drawing 2421 yr. In the anti-resonant framework the compression is stronger still: the Shroud becomes 104 yr old and the Iceman 611 yr. In the resonant framework no sample has an age, because every one has lost more than the 2.3% of its radiocarbon permitted by $F_{\min} = 0.977$.

The last point is the quantitative form of the observation in Section 9: under a 5000-year cycle, radiocarbon ages of samples that span the whole of the late Pleistocene and Holocene in linear time collapse into a band of about two millennia. Their *order* is preserved, because all three age maps are monotonic in F , but the intervals between them are drastically shortened. Thus the Iceman ($4659 \rightarrow 1976$ yr) and the Chauvet drawing ($31\,840 \rightarrow 2421$ yr), separated by 27 000 linear years, are separated by only 445 conformal years.

8.4 Discrepancy maps and scaling laws

The analytic approximations (6.5) and (6.10) summarise the dependence on T_c :

$$\Delta_C \simeq \frac{\pi^2 t_L^3}{3T_c^2}, \quad |\Delta_P| \simeq \frac{2}{\lambda} e^{-\lambda T_c/2} \left(e^{\lambda t_L} - 1 \right). \quad (8.2)$$

The conformal discrepancy is a power law in both variables, while the image-sum discrepancy switches on abruptly as T_c falls towards $2t_L + (2/\lambda) \ln(2/\lambda\sigma)$. Setting each equal to a tolerance σ defines the *indistinguishability boundary*,

$$T_c^{(C)}(t_L, \sigma) = \pi \sqrt{\frac{t_L^3}{3\sigma}}, \quad T_c^{(P)}(t_L, \sigma) = \frac{2}{\lambda} \ln \left[\frac{2(e^{\lambda t_L} - 1)}{\lambda\sigma} \right], \quad (8.3)$$

Sample	Field	^{14}C age (BP)	F	Linear t_L	$T_c = 100\,000$ yr			$T_c = 5000$ yr		
					τ_C (I)	$t_P^{(0)}$ (II)	$t_P^{(1/2)}$ (III)	τ_C (I)	$t_P^{(0)}$ (II)	$t_P^{(1/2)}$ (III)
Shroud of Turin [73]	Archaeology	689 ± 16	0.9178	709	709	713	706	667	—	104
Ötzi, the Tyrolean Iceman [74]	Archaeology	$4\,527 \pm 7$	0.5692	4\,659	4\,626	4\,688	4\,629	1\,976	—	611
Clovis culture (onset) [75]	Archaeology	$11\,050^a$	0.2527	11\,371	10\,922	11\,488	11\,257	2\,279	—	1\,240
Chauvet Cave rhinoceros drawing [76, 77]	Archaeology	$30\,940 \pm 610$	0.0212	31\,840	25\,004	34\,040	30\,282	2\,421	—	2\,134
Kennewick Man [78]	Biology	$8\,410 \pm 60$	0.3510	8\,655	8\,450	8\,727	8\,583	2\,211	—	1\,015
<i>Silene stenophylla</i> fruit [79]	Biology	$31\,800 \pm 300$	0.0191	32\,725	25\,441	35\,253	31\,012	2\,423	—	2\,153
Lyuba, mammoth calf [80]	Biology	$\approx 41\,800$	0.0055	43\,016	29\,722	—	38\,394	2\,441	—	2\,314
Mount Mazama eruption [81, 82]	Geology	$\approx 6\,800$	0.4289	6\,998	6\,888	7\,050	6\,946	2\,144	—	859
Two Creeks buried forest [83]	Geology	$\approx 11\,850$	0.2287	12\,195	11\,646	12\,328	12\,065	2\,293	—	1\,301
Laacher See eruption [84]	Geology	$11\,230 \pm 40$	0.2471	11\,557	11\,086	11\,677	11\,439	2\,282	—	1\,254

Table 3: Radiocarbon ages of ten well-known samples in linear time and in periodic time with cycle periods of 100 000 and 5000 years. The ^{14}C age column is the published conventional radiocarbon age (Libby half-life); F is the corresponding fraction remaining, Eq. (8.1). All other columns are physical ages in years before AD 1950 (Cambridge half-life): t_L from Eq. (2.7); τ_C from the conformal age equation (6.2) (Framework I); $t_P^{(0)}$ from the resonant image-sum equation (6.7) (Framework II); $t_P^{(1/2)}$ from the anti-resonant equation (6.12) (Framework III). A dash means $F < F_{\min}$, so the resonant periodic law has no solution; for $T_c = 5000$ yr, $F_{\min} = 0.977$ and every sample is excluded. ^aOnset of the Clovis range 11 050–10 800 ^{14}C BP; ages are computed for 11 050. The Kennewick age is not corrected for the marine reservoir effect [78].

above which linear and periodic chronologies agree to within σ . These boundaries are the labelled contours of Fig. 4; Table 4 gives exact values obtained by solving the full equations numerically. For instance, with $t_L = 5 \times 10^4$ yr and $\sigma = 100$ yr, Eq. (8.3) gives $T_c^{(C)} = 2.03 \times 10^6$ yr and $T_c^{(P)} = 1.84 \times 10^5$ yr, in agreement with the exact values.

Age range tested	Independent chronometer	σ (yr)	$T_c^{(P)}$ (yr)	$T_c^{(C)}$ (yr)
14 000 yr	tree rings [14]	100	1.09×10^5	2.99×10^5
14 000 yr	tree rings	300	9.1×10^4	1.70×10^5
30 000 yr	varves, speleothems [17, 18]	300	1.26×10^5	5.39×10^5
30 000 yr	varves, speleothems	1000	1.08×10^5	2.89×10^5
50 000 yr	speleothems [18]	300	1.67×10^5	1.16×10^6
50 000 yr	speleothems	1000	1.48×10^5	6.30×10^5

Table 4: Cycle periods above which the image-sum ($T_c^{(P)}$, resonant case) and conformal ($T_c^{(C)}$) chronologies agree with linear chronology to within σ at the stated age. The anti-resonant case gives the same $T_c^{(P)}$ to leading order, since its leading correction (6.13) has the same magnitude. If the independent chronometers are accepted as measuring the same time as the decay law, these are lower bounds on T_c .

8.5 Propagation of measurement error

Equations (6.6), (6.11) and (6.14) show that the three frameworks treat measurement error very differently. In the conformal and anti-resonant frameworks the age uncertainty is *smaller* than the linear one, because ages are compressed into a finite interval. In the resonant framework it is *larger*, and diverges at $F = F_{\min}$. For $T_c = 10^5$ yr ($F_{\min} = 0.0094$), the inflation factor $(1 - F_{\min}/F)^{-1/2}$ is 1.01 at $F = 0.5$, 1.05 at $F = 0.1$, 1.37 at $F = 0.02$ and 4.1 at $F = 0.01$. A resonant periodic chronology therefore not only shifts old ages but renders them imprecise.

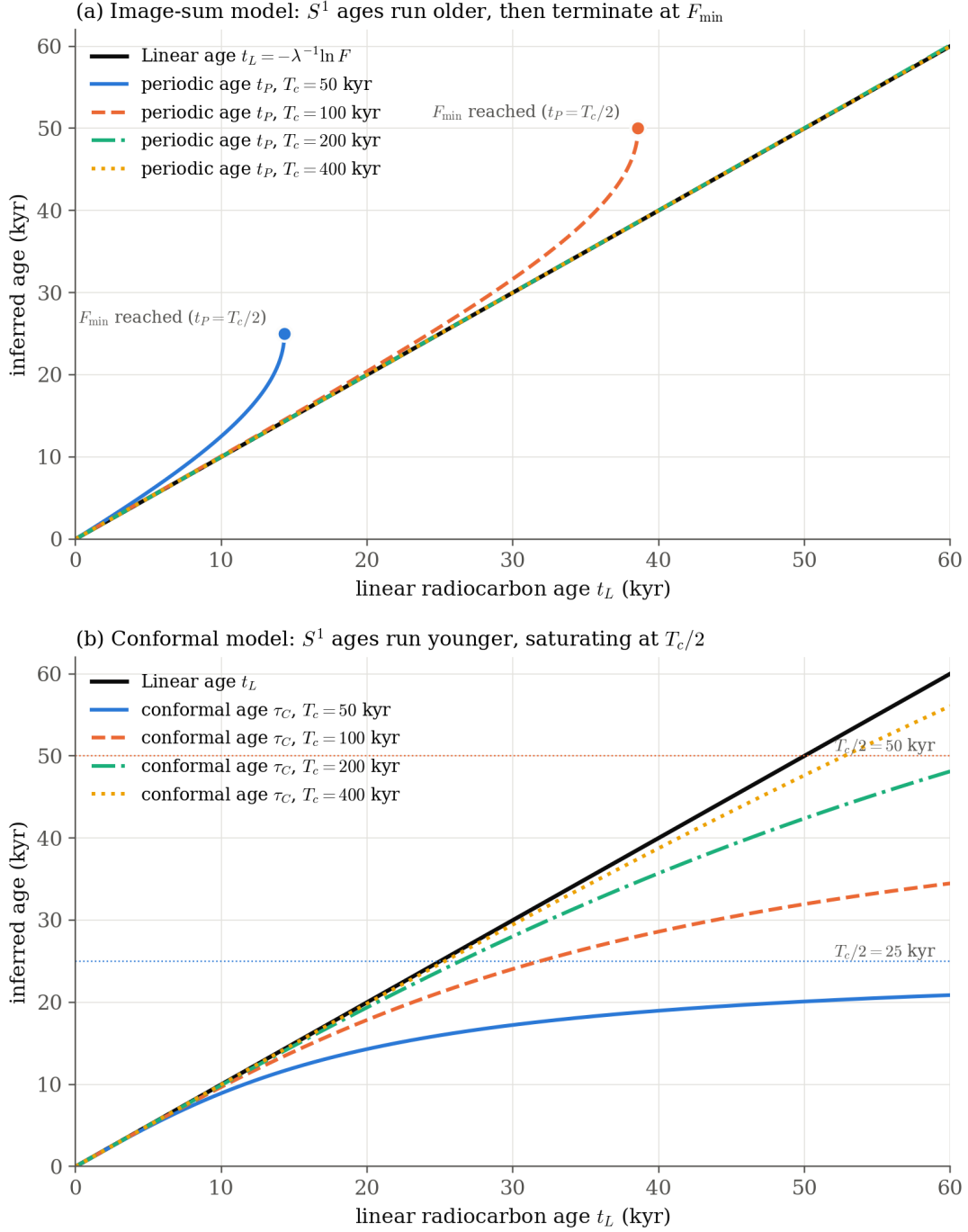


Figure 3: Periodic ages versus linear ages. (a) Resonant image-sum age $t_P^{(0)}$, Eq. (6.7), as a function of the linear age $t_L = -\lambda^{-1} \ln F$ of the same sample, for $T_c = 50, 100, 200, 400$ kyr. Periodic ages are older and each curve terminates (dots) where F reaches $F_{\min}$ and $t_P = T_c/2$; beyond this point the measured ratio has no periodic interpretation. For $T_c \geq 200$ kyr the curves are indistinguishable from the diagonal. (b) Conformal age τ_C , Eq. (6.2): periodic ages are younger and saturate at $T_c/2$ (dotted horizontals), with the discrepancy growing as t_L^3/T_c^2 .

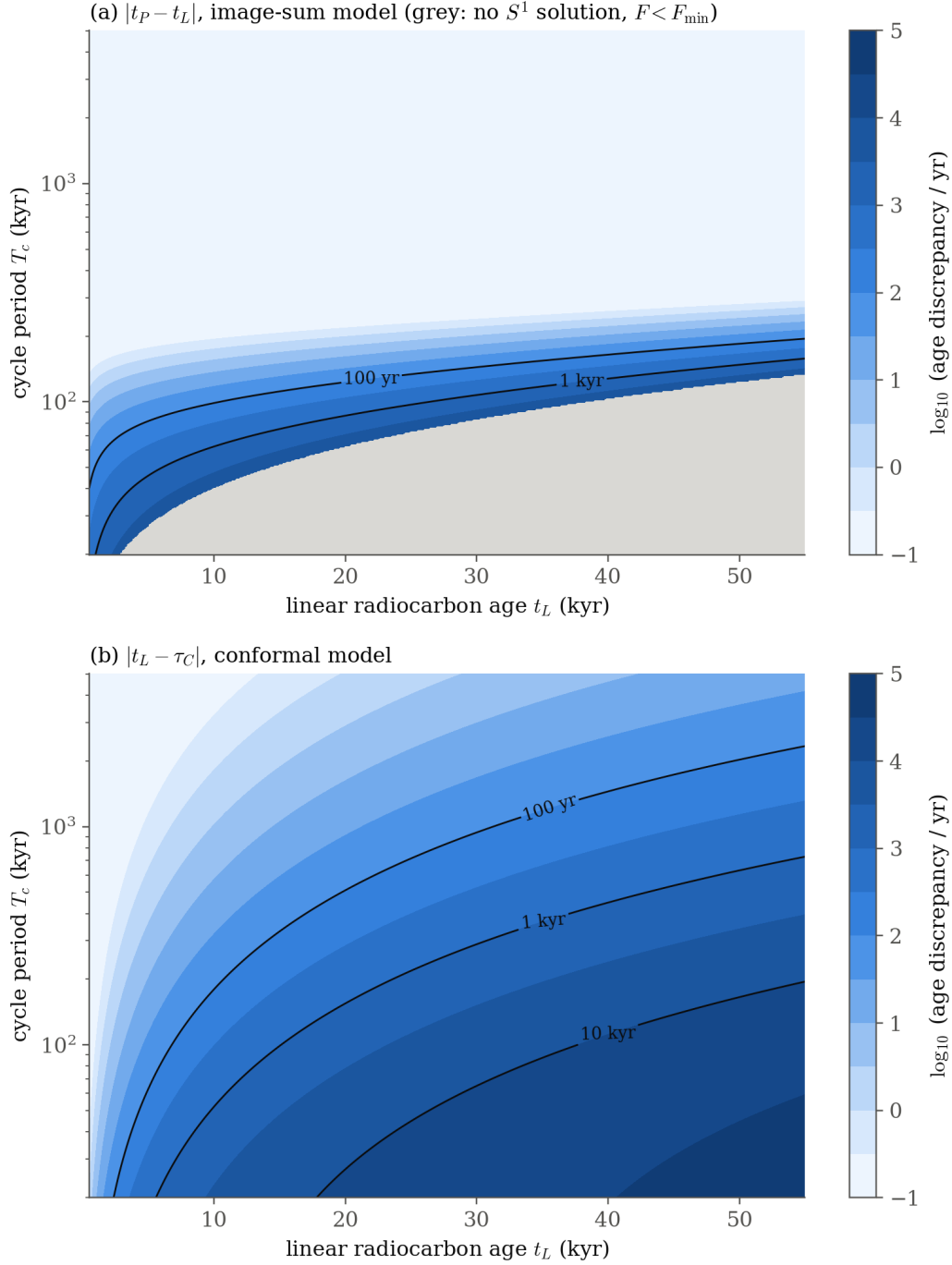


Figure 4: Where does the topology of time matter? Absolute age discrepancy between periodic and linear chronologies as a function of linear age t_L (horizontal) and cycle period T_c (vertical, logarithmic). (a) Resonant image-sum framework, $|t_P^{(0)} - t_L|$; the grey region is excluded because $F < F_{\min}$. The discrepancy switches on abruptly as T_c decreases, reflecting the $e^{-\lambda T_c/2}$ scaling of Eq. (8.2). (b) Conformal framework, $|t_L - \tau_C|$; the discrepancy falls off only as T_c^{-2} , so conformal compactification remains detectable for cycles of order 10^6 yr. Contours mark 100 yr, 1 kyr and 10 kyr.

9 Discussion

9.1 Which time does a clock measure?

The three periodic frameworks differ not only in their formulae but in what they claim to be observable, and this must be made explicit before any comparison with data.

In the *conformal* framework (Definition 3.1) the laws of physics hold in linear time t , and τ is a second, compactified parametrisation of the same history. If *every* physical process—radioactive decay, tree growth, sediment deposition, the Earth’s orbit—runs in t , then all local chronometers agree with one another and with t_L , and the conformal age τ_C is an unobservable relabelling. The framework acquires empirical content only if some class of clocks reads τ while radioactive decay proceeds in t (or vice versa). This is precisely the structure of Segal’s chronometric cosmology [47], in which cosmological redshift arises from the difference between the chronometric (compact) and local (linear) time generators. The bounds $T_c^{(C)}$ of Table 4 should therefore be read conditionally: *if* dendrochronological years are chronometric while decay is linear, then $T_c \gtrsim 10^6$ yr.

In the *image-sum* frameworks (Definition 3.2) the modification is dynamical. Tree rings and varves are counted by processes whose own time scales (one year) are negligible compared with T_c , so local counting is unaffected, while the radiocarbon survival law is changed at order $e^{-\lambda(T_c/2-t)}$. The comparison of radiocarbon with counted chronologies is then a genuine test, and the bounds $T_c^{(P)}$ of Table 4 ($T_c \gtrsim 1.5 \times 10^5$ yr) apply directly, modulo the unknown detuning δ which only moves the result within the bracket (6.15).

9.2 A multi-isotope concordance test

A universal reparametrisation of time cannot be detected by comparing clocks that all transform in the same way. What *can* be detected is a relative effect between clocks that transform differently. In the image-sum frameworks the correction depends on the decay constant through λT_c . For a set of isotopes $\{i\}$ with decay constants λ_i measured on the same sample of true periodic age t , the apparent linear ages are

$$t_L^{(i)} = -\frac{1}{\lambda_i} \ln F_\delta^{(i)}(t) = t - \frac{2}{\lambda_i} \ln \frac{1 + q_i e^{\lambda_i t}}{1 + q_i} \quad (\delta = 0), \quad q_i = e^{-\lambda_i T_c/2}, \quad (9.1)$$

which differ from isotope to isotope. For long-lived chronometers with $\lambda_i T_c \ll 1$, such as ^{230}Th ($T_{1/2} = 75.6$ kyr) or ^{10}Be (1.39 Myr), $q_i \rightarrow 1$ and the periodic law is far from exponential over the whole cycle, whereas for ^{14}C the correction is exponentially small when $T_c \gg 2t$. The observed concordance of ^{14}C and U–Th ages in speleothems and corals [14, 18] is therefore a sensitive probe of periodic dynamics, sharper than any single-isotope measurement. Carrying out this test quantitatively requires a treatment of the U-series disequilibrium chain on S^1 via the periodic Bateman problem (Appendix B), which we leave for future work.

9.3 Short cycles and the continuity of the tree-ring record

A cycle of $T_c = 5000$ yr, as posited in some cyclic cosmologies, produces the most dramatic differences in Table 2: all conformal ages below 2500 yr, and all resonant periodic ages undefined beyond $t_L \approx 188$ yr. It is important to state clearly what such a cycle would require of other evidence. Under Definition 3.2, a continuous annual record longer than T_c is impossible without repetition: the ring-width sequence of year y must coincide with that of year $y + T_c$. The central European oak–pine chronology extends continuously over 12460 years [19], and the IntCal20 tree-ring data reach ~ 13.9 cal kBP [14]; an exact 5000-yr repetition of ring-width

and $\Delta^{14}\text{C}$ patterns within these records would be striking and, to our knowledge, has not been reported. Under Definition 3.1 the conflict is avoided only if annual layers are counted in chronometric time, in which case at most $T_c/2$ layers could be observed in any archive. A short world cycle therefore implies not merely a reinterpretation of radiocarbon ages but a correlated reinterpretation of every counted chronology. The equations of this paper make that reinterpretation quantitative; whether it is tenable is an empirical question on which the present evidence weighs heavily.

9.4 Thermodynamics and the arrow of time

Proposition 4.2 is the one-dimensional core of Loschmidt’s and Zermelo’s objections [63–65]: a monotone quantity cannot be periodic. Radioactive decay is an entropy-producing process—each decay converts a nuclear excitation into kinetic energy of the recoil and leptons—and radiocarbon “remembers” elapsed time precisely because this process is irreversible. On S^1 the memory must be erased once per cycle. The image-sum law does this in a specific way: by Eq. (5.15) the effective decay rate reverses sign at mid-cycle, and in the second half-cycle radiocarbon is regenerated along the time-reverse of its decay curve (Fig. 2(a)). This is a realisation, for a single observable, of the two-time boundary-value problems studied by Schulman [66] and of the time-symmetric cosmologies discussed in [67–69]. In the author’s earlier analysis of Loschmidt’s paradox on spacetimes with S^1 time [72], a periodic entropy with a reversal of the thermodynamic arrow at mid-cycle was found to be the generic consequence of circular time; the survival laws (5.12)–(5.13) are the radiocarbon expression of the same structure, with $t = T_c/2$ the “turning point” of the arrow. An observer in the second half-cycle would measure radiocarbon *increasing* in closed samples; the absence of any such observation places us, on this view, in a first half-cycle—or, more conservatively, constrains T_c as in Table 4.

It is instructive to compare the Poincaré recurrence time of a macroscopic sample [62], which is of order $\exp(N)$ for $N \sim 10^{10}$ radiocarbon atoms, with T_c . In the image-sum framework recurrence is enforced by the topology of time rather than achieved by ergodic wandering in phase space, and occurs after exactly one period for every system simultaneously. This distinguishes topological recurrence from the dynamical (Poincaré) recurrence that is the subject of Tipler’s no-go theorem for recurrence in general relativity [61]; see [70] for a treatment of exact recurrence in general relativity and its relation to that theorem.

9.5 Cosmological cycles

For the cycles of relativistic cosmology—Tolman’s oscillating models [57], the cyclic ekpyrotic model [58], or conformal cyclic cosmology [59]—the period is at least of order 10^{10} yr. Table 2 and Eq. (8.2) show that radiocarbon is completely insensitive to such cycles: the image-sum corrections are of order $e^{-\lambda T_c/2} \sim e^{-6 \times 10^5}$ and the conformal correction of order $(t_L/T_c)^2 \sim 10^{-11}$. Moreover, in conformal cyclic cosmology successive aeons are joined by a conformal rescaling at which massive particles, and hence radioactive clocks, do not survive, so the question of a radiocarbon clock spanning aeons does not arise. Radiocarbon is a probe of *short* time-cycles only; for long cycles one must turn to long-lived chronometers (^{238}U , ^{232}Th , ^{187}Re), where the same formalism applies with $\lambda \rightarrow \lambda_i$.

9.6 Limitations

Several simplifications should be recorded. (i) We have used the Breit–Wigner spectral density; realistic densities, bounded below, produce Khalfin tails [37] which the image sum (5.7) would also periodise, but these are utterly negligible for ^{14}C . (ii) The detuning δ is unknown, and we

have presented its extreme values; the physical age lies in the bracket (6.15). (iii) We treated a single global period T_c ; a spacetime with S^1 time may have a period that varies between world-lines, as in Gödel-type solutions [53, 71]. (iv) Closed timelike curves raise consistency issues (the Novikov self-consistency principle [55]) and may be forbidden by quantum effects (chronology protection [56]); we have not addressed these, assuming only that a consistent periodic history exists. (v) The reservoir analysis of Section 7 used a single-box or linear multi-box model; real carbon-cycle dynamics are non-linear on glacial time scales.

10 Conclusions

We have examined radiocarbon dating on the two possible topologies of a one-dimensional time, the line $\mathbb{R}^1$ and the circle S^1 , where the circle is understood both as the quotient $\mathbb{R}/T_c\mathbb{Z}$ and as the conformal compactification $\mathbb{RP}^1$ of the time line. The main results are as follows.

1. **The exponential decay law is topologically forbidden on S^1 .** Its Floquet multiplier $e^{-\lambda T_c}$ differs from unity, so $\dot{N} = -\lambda N$ has no non-trivial periodic solution (Proposition 4.1); more generally no strictly monotone quantity can be single-valued on a circle (Proposition 4.2). Periodic time forces new decay physics.
2. **Conformal compactification gives a chronometric age** $\tau_C = (T_c/\pi) \arctan(\pi t_L/T_c)$, always younger than the linear age, saturating at $T_c/2$, and differing from it by $\simeq \pi^2 t_L^3/3T_c^2$. The underlying geometry is that of the $PSL(2, \mathbb{R})$ action on $\mathbb{RP}^1$ and of Segal's compactified Minkowski space $(S^1 \times S^3)/\mathbb{Z}_2$.
3. **Quantum decay on periodic time is an image sum.** Quantisation of energy in units of h/T_c and Poisson summation turn the Breit–Wigner amplitude into $\mathcal{A}_{S^1}(t) = \sum_k e^{-2\pi i k \beta} \mathcal{A}_{\mathbb{R}}(t - kT_c)$, giving the exact survival laws $F_0 = \cosh^2[\lambda(t - T_c/2)/2] / \cosh^2(\lambda T_c/4)$ (resonant) and $F_{1/2} = \sinh^2[\lambda(T_c/2 - t)/2] / \sinh^2(\lambda T_c/4)$ (anti-resonant), which bracket all other cases and reduce to $e^{-\lambda t}$ as $T_c \rightarrow \infty$.
4. **Closed-form periodic age equations** (Table 1) show that resonant ages are older than linear ones and do not exist below $F_{\min} = \text{sech}^2(\lambda T_c/4)$, while anti-resonant and conformal ages are younger and bounded by $T_c/2$. The discrepancy with linear ages scales as $e^{-\lambda(T_c/2 - t_L)}$ in the image-sum frameworks and as T_c^{-2} in the conformal one.
5. **The difference is wide exactly when T_c is comparable to the dating horizon.** The natural width of ^{14}C equals the level spacing h/T_c when $T_c \approx 2\pi/\lambda \approx 5 \times 10^4$ yr. For $T_c = 10^5$ yr a single sample of linear age 38 069 yr has periodic ages between 27 833 and 45 873 yr; for $T_c = 5000$ yr every conformal age is below 2500 yr; for $T_c \gtrsim 4 \times 10^6$ yr all chronologies agree to within decades.
6. **Well-known samples.** For ten iconic archaeological, biological and geological samples (Table 3), a 100 000-year cycle changes Holocene ages by at most a few per cent but Pleistocene ages by up to 13 000 yr (Lyuba mammoth: 43 016 $\rightarrow$ 29 722 yr conformally). A 5000-year cycle maps the whole set into 667–2441 yr (conformal) or 104–2314 yr (anti-resonant), with the order of the samples preserved.
7. **Periodic chronology is intrinsically ambiguous and circular.** Ages are defined only up to the infinite dihedral group generated by $t \mapsto t + T_c$ and $t \mapsto T_c - t$; calibration becomes a problem in circular statistics; and the atmospheric reservoir admits an equilibrium without an infinite past, though time-symmetric kinetics would double the steady-state inventory.

8. **The topology of time is testable.** Agreement between radiocarbon and counted chronologies bounds $T_c \gtrsim 1.5 \times 10^5$ yr in the image-sum frameworks, and $T_c \gtrsim 10^6$ yr in the conformal framework if counted years are chronometric. A multi-isotope concordance test, exploiting the λ -dependence of the periodic correction, offers a sharper probe.

Linear-time radiocarbon chronology is therefore not a theorem but a hypothesis about the global structure of time, and one that can be tested. The formulae derived here show precisely how, and by how much, the ages of the past would change if time were a circle.

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Data availability

All numerical results and figures were computed directly from the closed-form expressions given in the text; no external data were used beyond the published constants cited.

A Image Sums for Classical and Quantum Kernels

The two periodic constructions of the main text—the classical Green’s function (4.5) and the quantum image sum (5.7)—are instances of one principle: a kernel $k(t)$ on the universal cover $\mathbb{R}$ descends to S^1 as

$$k_{S^1}(t) = \sum_{n \in \mathbb{Z}} \chi^n k(t - nT_c), \quad (\text{A.1})$$

where $\chi \in U(1)$ is a character of $\pi_1(S^1) \cong \mathbb{Z}$. For the retarded classical kernel $k(t) = \Theta(t)e^{-\lambda t}$ and trivial character $\chi = 1$,

$$\sum_{n \in \mathbb{Z}} \Theta(t - nT_c) e^{-\lambda(t - nT_c)} = \sum_{n \leq \lfloor t/T_c \rfloor} e^{-\lambda(t - nT_c)} = \frac{e^{-\lambda \lfloor t/T_c \rfloor T_c}}{1 - e^{-\lambda T_c}} = G_{T_c}(t), \quad (\text{A.2})$$

recovering Eq. (4.5). Only past images contribute, because the kernel is retarded. For the time-symmetric quantum kernel $k(t) = e^{-iE_0 t/\hbar - \lambda|t|/2}$ both past ($n \leq 0$) and future ($n \geq 1$) images contribute, and Eq. (A.1) with $\chi = e^{-2\pi i \beta}$ reproduces Eq. (5.7). The difference between retarded and symmetric kernels is the origin of the factor of two in Eq. (7.6).

The equivalence of Eq. (A.1) with spectral quantisation follows from the distributional Poisson formula

$$\sum_{n \in \mathbb{Z}} \delta(t - nT_c) = \frac{1}{T_c} \sum_{m \in \mathbb{Z}} e^{2\pi i m t/T_c}. \quad (\text{A.3})$$

Convolving k with the Dirac comb and taking Fourier transforms, k_{S^1} has Fourier coefficients $T_c^{-1} \tilde{k}(\omega_m)$ at $\omega_m = 2\pi(m + \beta)/T_c$, i.e. it is supported on the energy lattice (4.9). For the Breit–Wigner kernel $\tilde{k}(\omega) \propto \rho_{\text{BW}}(\hbar\omega)$, so

$$\mathcal{A}_{S^1}(t) = \frac{2\pi\hbar}{T_c} \sum_{m \in \mathbb{Z}} \rho_{\text{BW}}\left(\frac{\hbar(m + \beta)}{T_c}\right) e^{-2\pi i(m + \beta)t/T_c}, \quad (\text{A.4})$$

which is Eq. (5.4) with $\Delta E = \hbar/T_c$. The normalisation $\mathcal{A}_{S^1}(0) = \sum_m \Delta E \rho(E_m)$ differs from

unity by terms of order q ; this is why the survival probabilities of Section 5 are normalised by $|\mathcal{A}_{S^1}(0)|^2$.

B The Periodic Bateman Problem

For a decay chain $1 \rightarrow 2 \rightarrow \dots \rightarrow m$ with sources, write $\dot{\mathbf{N}} = -\Lambda \mathbf{N} + \mathbf{P}(t)$ with

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ -\lambda_1 & \lambda_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & -\lambda_{m-1} & \lambda_m \end{pmatrix}. \quad (\text{B.1})$$

The spectrum of Λ is $\{\lambda_i\}$, so for distinct positive λ_i the monodromy $e^{-\Lambda T_c}$ has no unit eigenvalue and the unique periodic solution is

$$\mathbf{N}(t) = \int_0^{T_c} (1 - e^{-\Lambda T_c})^{-1} e^{-\Lambda u} \mathbf{P}(t - u) du = \sum_{m \in \mathbb{Z}} (\Lambda + i\omega_m)^{-1} \mathbf{p}_m e^{i\omega_m t}. \quad (\text{B.2})$$

Using the spectral decomposition $e^{-\Lambda u} = \sum_j e^{-\lambda_j u} \Pi_j$, with Π_j the eigenprojectors, the kernel is

$$(1 - e^{-\Lambda T_c})^{-1} e^{-\Lambda u} = \sum_{j=1}^m \frac{e^{-\lambda_j u}}{1 - e^{-\lambda_j T_c}} \Pi_j, \quad 0 \leq u < T_c, \quad (\text{B.3})$$

a sum of scalar periodic Green's functions (4.5), one per mode. For constant production of the parent only, the zero mode gives secular equilibrium $\lambda_1 \bar{N}_1 = \lambda_2 \bar{N}_2 = \dots = P_0$, exactly as in linear time. For long-lived members with $\lambda_j T_c \lesssim 1$ the factor $(1 - e^{-\lambda_j T_c})^{-1} \approx (\lambda_j T_c)^{-1}$ is large and the periodic solution departs strongly from its linear-time counterpart; this is the regime relevant to U-series dating and to the concordance test of Section 9.

C Numerically Stable Evaluation

For $\lambda T_c \gg 1$ the direct evaluation of $\cosh(\lambda T_c/4)$ overflows and $\operatorname{arccosh}$ loses precision. Writing $y = \lambda T_c/4$ and using

$$\ln \cosh y = y + \ln(1 + e^{-2y}) - \ln 2, \quad (\text{C.1})$$

$$\operatorname{arccosh} x = \ln x + \ln(1 + \sqrt{1 - x^{-2}}), \quad (\text{C.2})$$

Eq. (6.7) becomes, with $\ln x = \frac{1}{2} \ln F + \ln \cosh y$,

$$t_P^{(0)} = t_L - \frac{2}{\lambda} \left[\ln(1 + e^{-\lambda T_c/2}) - \ln 2 + \ln(1 + \sqrt{1 - x^{-2}}) \right], \quad (\text{C.3})$$

which involves only bounded quantities and makes the limit $t_P^{(0)} \rightarrow t_L$ as $T_c \rightarrow \infty$ manifest (the bracket tends to $-\ln 2 + \ln 2 = 0$). Analogous expressions hold for $t_P^{(1/2)}$ with $\sinh$ and $\operatorname{arcsinh}$. The values in Tables 2 and 4 and in Figs. 2–4 were obtained in double precision with these formulae; the bounds of Table 4 were found by Brent's root-finding method applied to $|t_P(T_c) - t_L| = \sigma$ and $t_L - \tau_C(T_c) = \sigma$.

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