

Asymptotics of the Variance in the Collector's Problem with Group Drawings

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Abstract

In the collector's problem with *group drawings*, one run draws a uniformly random q -subset of an n -element set, all q coupons being kept, and T denotes the number of runs needed to complete the set. The distribution of T and all of its factorial moments are due to Stadge (1990); we do not claim them, and we say explicitly where our formulas reproduce his—in particular his relation (2.15) at $p = 2$ already yields the second moment, so item (i) below is a consolidation rather than a new formula. The asymptotics of $\mathbb{E}[T]$ as $n \rightarrow \infty$ with q fixed are due to Berend and Sher (2026) and, in a sharper form, to Doumas and Spektor (2026); we do not claim those either, and Section 8 downgrades our own expectation expansion to an independent rediscovery accordingly. What we add is fourfold, of which the first is a consolidation rather than a new formula. (i) An exact closed form for the second moment, hence for the variance, obtained by summing the inclusion–exclusion tail in closed form, together with a uniform one-line derivation of every higher moment. Having now read [1] in full we record that his general factorial-moment relation [1, eq. (2.15)], specialised to $A = S$ and $p = 2$, is equivalent to our second-moment formula; item (i) is therefore an explicit specialisation of Stadge's result and is not claimed as new (Remark 3). (ii) Two *subtraction-free* recursions—a backward recursion for $\mathbb{E}[T]$ in $O(nq)$ time and $O(q)$ memory, and a forward recursion for the whole distribution—which keep full double-precision accuracy up to $n = 1000$, whereas the inclusion–exclusion closed form loses all significant digits from $n \approx 40$ onwards: at $(n, q) = (60, 5)$ its relative error is 4.9×10^{-4} and at $(80, 5)$ it returns 1.99×10^6 for the true value 77.836. (iii) A quantitative explanation: the relative error of the closed form is predicted by $\varepsilon\kappa$, where ε is the unit roundoff and $\kappa = \sum_i |t_i| / |\sum_i t_i|$ is the condition number of the alternating sum; measured and predicted errors agree to within two orders of magnitude over twenty-six decades. (iv) *Asymptotics of the variance, with a complete proof.* Writing $A = H_n - H_{n-q}$, $\Sigma_2 = \sum_{j < q} (n-j)^{-2}$ and B, C for the second and third elementary symmetric functions of the numbers $(n-j)^{-1}$, $0 \leq j < q$, Theorem 3 proves

$$\mathrm{Var}[T] = \frac{H_n^{(2)}}{A^2} - \frac{H_n \Sigma_2}{A^3} + \frac{5B^2}{A^4} - \frac{4C}{A^3} - \frac{B}{A^2} + R, \quad |R| = O(e^{-\rho n}) \text{ with an explicit } \rho = \rho_q > 0,$$

hence the $n \log n$ form

$$\mathrm{Var}[T] = \frac{\pi^2}{6} \left(\frac{n}{q} \right)^2 - \frac{n}{q^2} (\log n + \gamma) - \frac{\frac{\pi^2}{6}(q-1) + 1}{q^2} n + \frac{q-1}{2q^2} \log n + c_0(q) + O\left(\frac{\log n}{n} \right).$$

For $q = 1$ everything degenerates *exactly* to the classical $\mathrm{Var}[T] = n^2 H_n^{(2)} - n H_n$. Doumas and Spektor obtain for the variance only the leading term $\frac{\pi^2}{6} (n/q)^2 (1 + o(1))$, so everything beyond it—the $n \log n$ term, the n -coefficient, the constant, the closed form and the exponential rate—is new for $q \geq 2$ as far as we have been able to determine (Remark 4); at

$q = 1$ the expansion collapses to the known result of Doumas and Papanicolaou (2012). We also record the exact support $[\lceil n/q \rceil, \infty)$ and point out that the widely quoted formula of Sharif–Hassibi and Xu–Tang belongs to a *different* model (draw d , keep one), which is why it gives different numbers.

1 Introduction

In the classical coupon collector problem one draws one coupon at a time, each of n types being equally likely, and asks for the number T of draws needed to see every type. The answer is $\mathbb{E}[T] = nH_n = n \log n + \gamma n + \frac{1}{2} + O(1/n)$, with $H_n = \sum_{k \leq n} k^{-1}$. Generalisations abound; see [5, 3, 8] and the references there.

This note concerns the *group drawing* version. Fix $1 \leq q \leq n$. A single run draws a uniformly random subset of size q (all $\binom{n}{q}$ subsets equally likely), all q coupons being kept; runs are independent. Let T be the number of runs needed until every coupon has appeared at least once. For $q = 1$ this is the classical problem, and for $q = n$ one has $T \equiv 1$.

Priority. This model—sampling with replacement of equiprobable groups of a fixed size—was treated in full generality by Stadje [1]. Specialising his subset $A \subseteq S$ to $A = S$, his results give both the distribution of the number of runs needed to complete the set and neat formulas for the associated expectations [1, p. 866]. His Theorem 2 is in fact not confined to first moments: it gives the probability mass function of the waiting time and adds that “the moments of $Z_n(A)$ can be computed from the relation” [1, eq. (2.15)], a formula for the p -th factorial moment valid for every $p \geq 1$, derived from the mass function by the geometric-series identity $\sum_{k \geq p} k(k-1) \cdots (k-p+1)x^{k-p} = p!(1-x)^{-p}$. In the notation of Section 2 below, the case $A = S$, group size $m = q$ reads

$$\mathbb{E}[T(T-1) \cdots (T-p+1)] = p! \sum_{i=1}^n (-1)^{i-1} \binom{n}{i} \frac{\lambda_i^{p-1}}{(1-\lambda_i)^p}. \quad (1)$$

At $p = 1$ this is the classical expectation formula $\sum_i (-1)^{i-1} \binom{n}{i} (1-\lambda_i)^{-1}$, and at $p = 2$ it gives $\mathbb{E}[T(T-1)] = 2 \sum_i (-1)^{i-1} \binom{n}{i} \lambda_i / (1-\lambda_i)^2$, which is algebraically identical to the second-moment formula of Theorem 2 (we checked the two forms against each other numerically for $(n, q) = (4, 1), (4, 2), (4, 3), (6, 1), \dots, (8, 3)$; they agree to machine precision). We were not aware of (1) when we first wrote down our formulas and had claimed the second moment as new; we withdraw that claim. The distribution is attributed to him in the later literature: Schilling and Henze write that Stadje “derived the probability distribution” of the waiting time for a complete series [2], and Flajolet, Gardy and Thimonier note that “this formula was obtained by Stadje by means of a subtle use of the inclusion–exclusion principle” [3]. *Consequently the tail formula of Theorem 1 and the moment formulas of Theorem 2 are not new*, and we do not present them as such; we re-derive them only because the tail form is the scaffolding for the subtraction-free recursions of Section 5 and for the residue computation of Section 8, and because we need the notation uniform.

Asymptotics: who did what. For fixed q and $n \rightarrow \infty$ one has $\mathbb{E}[T] = \frac{n}{q} \log n + O(n)$. Berend and Sher [10] prove the sharp two-sided bound

$$\frac{H_n}{H_n - H_{n-q}} < \mathbb{E}[T] < 1 + \frac{H_{n-1}}{H_n - H_{n-q}},$$

and in a second paper [11] obtain, for constant q , $\mathbb{E}[T] = \frac{n}{q} H_n - \frac{q-1}{2q} H_n + O(1)$. Subsequently Doumas and Spektor [14]—whose “Problem I” is exactly the present model—proved, by the

Nørlund–Rice method applied to the very same alternating binomial sum that we use below,

$$\mathbb{E}[T] = \frac{H_n}{A} + \frac{B}{A^2} + O(e^{-c_q n}), \quad A = \sum_{j=0}^{q-1} \frac{1}{n-j}, \quad B = \sum_{0 \leq i < j \leq q-1} \frac{1}{(n-i)(n-j)}, \quad (2)$$

together with a full asymptotic expansion and the Gumbel limit law. For the *variance* they record only the leading term $\text{Var}[T] = \frac{\pi^2}{6} \frac{n^2}{q^2} (1 + o(1))$; see also [14, Section 5]. In Section 8 we recover (2) independently, as a by-product of the second-moment computation, and we make no claim for it.

Two neighbouring lines of work deserve to be separated from this one. Betken and Thäle [9] study the same group-drawing process by Stein’s method, but their random variable is $Z_{n,s}(k_n)$, the number of coupons not seen during a *deterministic* number k_n of drawings; they say so themselves, contrasting their second question with the first (“how many packages does one need to buy to get the full set”), which they attribute to Stadje. Their quantitative CLT and Poisson bounds therefore do not concern the hitting time T . Athreya, Mukherjee and Mukherjee [8] consider the model in which a random r -subset is drawn and all of its s -subsets are marked as collected; the specialisation $s = 1$, $r = q$ is the present model, and for it their theorem gives only the leading term $\frac{n}{q} \log n (1 + o(1))$ together with a Gumbel limit [8, Remark on p. 3]. Finally, Giam [15] proves that the sequence $a_n = \frac{\pi^2}{6} + nw_n - n^2 c_n^2$ of alternating binomial sums is strictly decreasing in n , where a_n is defined by $\text{Var}[M_N(n)] \sim a_n N^2$ ($N \rightarrow \infty$) for the *multi-player* variant of Dumas [16]: n independent collectors each draw *one* coupon per trial and $M_N(n)$ is the fastest of their n completion times. That is a different process from ours, and only the leading term of the variance is involved. (Consistently, its one-player case $n = 1$ gives $a_1 = \pi^2/6$, the classical leading coefficient, as does our $q = 1$.)

A neighbouring model that must not be confused with this one. In the opportunistic-scheduling literature one draws d coupons per step but keeps only one—the “best” one. For that model Sharif and Hassibi [6], and following them Xu and Tang [7], obtain

$$\mathbb{E}(D_{1,n}^{(d)}) = \sum_{i=0}^{n-1} \frac{1}{1 - \binom{i}{d} / \binom{n}{d}}.$$

It is a different process and gives different numbers: for $(n, q) = (5, 2)$ it yields 7.0397 against 5.3254 here, and for $(40, 5)$ it yields 57.8713 against 32.8717. Confusing the two models is easy and leads to wrong tables.

What is new here.

- (C1) *Exact moments in closed form.* Theorem 2 gives closed forms for $\mathbb{E}[T^2]$ and $\text{Var}[T]$, and Remark 2 the general m -th moment. Because the tail of Theorem 1 is a finite linear combination of geometric sequences, no moment is out of reach: each is a finite sum of rational functions of the λ_i , obtained in one line from Lemma 1. *This is not a priority claim.* By (1) the same quantities are already available from [1]; what we add here is the observation of the equivalence, the compact rational form that Sections 6 and 8 are run on, and the uniform derivation of all moments from one lemma.
- (C2) *Subtraction-free algorithms.* Section 5 gives a backward recursion for $\mathbb{E}[T]$ ($O(nq)$ time, $O(q)$ memory) and a forward recursion for the entire distribution. Both use only additions of non-negative numbers. The use of Markov-chain/state-transition equations for collector problems is of course standard [4]; we claim no novelty for the idea, only for the stability analysis below, which we have not found documented.

- (C3) *A quantitative stability analysis.* The inclusion–exclusion closed forms are unusable in double precision well before n gets large; Section 6 measures this, explains it through the condition number κ , and shows that the recursions do not suffer from it (Table 2, Table 3, Figure 2, Figure 3).
- (C4) *Verification.* Section 7 reports six independent checks, including agreement with Stadje’s formula to ninety digits and the reduction at $q = 1$ to nH_n and $n^2H_n^{(2)} - nH_n$.
- (C5) *Asymptotics of the variance for $q \geq 2$, with a self-contained proof.* (At $q = 1$ the expansion is [12, eq. (1.10)]; see Remark 5.) Theorem 3 gives a closed form for $\text{Var}[T]$ that is accurate to $O(e^{-\rho n})$ with an *explicit* rate $\rho = \rho_q > 0$, and Corollary 2 its expansion in powers of n . The proof uses only the residue theorem applied to the rational Rice kernel, Rouché’s theorem, the Gauss–Lucas theorem and Stirling’s inequality. Two ingredients appear to be new even for the expectation: the residue identity is stated and proved for a rational integrand, which removes all growth and contour-deformation hypotheses, and the exponential rate is made explicit and proved positive ($\rho_q = \min_j \rho(2\pi j/q)$ with ρ of Lemma 8); [14] leaves the corresponding constant unspecified.

Throughout, $\binom{a}{b} = 0$ when $b < 0$ or $b > a$, and λ_i always denotes the ratio defined in (5). All high-precision computations were done with MPMATH [18] at 60–160 significant digits; all double-precision computations in IEEE binary64, for which the unit roundoff is $\varepsilon = 2^{-53} \approx 1.11 \times 10^{-16}$.

2 Model, notation and the support of T

Let $[n] = \{1, \dots, n\}$ be the set of coupon types and let $1 \leq q \leq n$. A *run* is a uniformly random q -subset of $[n]$, drawn independently of all previous runs; all q coupons are kept. Put

$$T = \min\{r \geq 1 : \text{the union of the first } r \text{ runs is } [n]\}.$$

Proposition 1 (support). *T takes values in $\{\lceil n/q \rceil, \lceil n/q \rceil + 1, \dots\}$ and $\mathbb{P}(T = \lceil n/q \rceil) > 0$. Equivalently $r_{\min} := \lfloor (n-1)/q \rfloor + 1 = \lceil n/q \rceil$. If $q = n$ then $T \equiv 1$ and $\text{Var}[T] = 0$.*

Proof. After r runs at most rq distinct coupons can have appeared, so $\mathbb{P}(T \leq r) = 0$ for $rq < n$, i.e. for $r < \lceil n/q \rceil$; and $\lfloor (n-1)/q \rfloor + 1 = \lceil n/q \rceil$ for integers. Conversely there is a positive probability that $r_{\min}$ prescribed disjoint blocks partition $[n]$ —each of the first $r_{\min} - 1$ runs contributes exactly q fresh coupons—so $\mathbb{P}(T = r_{\min}) > 0$. For $q = n$ a single run is the whole set. $\square$

For $0 \leq x \leq n$ let

$$p_{x,y} = \frac{\binom{x}{q-y} \binom{n-x}{y}}{\binom{n}{q}}, \quad 0 \leq y \leq \min(q, n-x), \quad (3)$$

the probability that, conditional on exactly x distinct coupons having been collected so far, the next run adds exactly y fresh ones: the run must choose $q-y$ of the x already held and y of the $n-x$ missing. By Vandermonde’s identity $\sum_y \binom{x}{q-y} \binom{n-x}{y} = \binom{n}{q}$, so

$$\sum_{y=0}^{\min(q, n-x)} p_{x,y} = 1 \quad (0 \leq x \leq n), \quad (4)$$

where for $x = n$ the sum is the empty convention $p_{n,0} := 0$ (see Section 5).

It is sometimes convenient to write the special value

$$\lambda_i := \frac{\binom{n-i}{q}}{\binom{n}{q}} = \prod_{j=0}^{i-1} \frac{n-q-j}{n-j}, \quad 1 \leq i \leq n-q, \quad (5)$$

the probability that a single run avoids a fixed set of i coupons. Note $\lambda_i = 0$ for $i > n-q$, and $\lambda_1 = (n-q)/n$. The product form on the right is the numerically safe way to evaluate λ_i : it never forms large binomial coefficients.

3 The exact distribution

Define the tail (“bad luck”) function

$$B(r) := \mathbb{P}(T > r), \quad r \geq 0.$$

Theorem 1 (tail and mass function). *For every integer $r \geq 1$,*

$$B(r) = \sum_{i=1}^{n-q} (-1)^{i-1} \binom{n}{i} \lambda_i^r, \quad \lambda_i = \frac{\binom{n-i}{q}}{\binom{n}{q}}, \quad (6)$$

with the convention that the sum is empty (so $B(r) = 0$) when $q = n$. Moreover $B(0) = 1$, and $B(r) = 1$ for every $0 \leq r < r_{\min}$. Consequently

$$\mathbb{P}(T = r) = B(r-1) - B(r) \quad (r \geq 1). \quad (7)$$

Proof. Fix $r \geq 1$ and, for $M \subseteq [n]$, let A_M be the event that no coupon of M has been seen in the first r runs. A run avoids M with probability $\binom{n-|M|}{q}/\binom{n}{q}$, and runs are independent, so $\mathbb{P}(A_M) = \lambda_{|M|}^r$; this vanishes when $|M| > n-q$. Since $\{T > r\}$ is the event that at least one coupon is still missing, inclusion–exclusion over the missing coupons gives

$$\mathbb{P}(T > r) = \sum_{\emptyset \neq M} (-1)^{|M|-1} \mathbb{P}(A_M) = \sum_{i=1}^{n-q} (-1)^{i-1} \binom{n}{i} \lambda_i^r.$$

Finally $\mathbb{P}(T > 0) = 1$, and for $r < r_{\min}$ the event $\{T > r\}$ is certain by Proposition 1; both statements are consistent with (6) because the terms with $i > n-q$, which are absent from (6) only through $\lambda_i^r = 0$ for $r \geq 1$, contribute $\sum_{i=1}^n (-1)^{i-1} \binom{n}{i} = 1$ at $r = 0$. $\square$

Remark 1 (attribution). Theorem 1 is the case $A = S$ of [1]: with his notation, $\mathbb{P}(X_k(S) = n)$ is the probability that k group drawings suffice, and

$$\mathbb{P}(T \leq k) = \sum_{j=0}^n (-1)^j \binom{n}{j} \left[\frac{\binom{n-j}{q}}{\binom{n}{q}} \right]^k, \quad (8)$$

which is $1 - B(k)$. We checked (8) against $1 - B(k)$ of (6) at $(n, q, k) = (40, 5, 20)$, $(40, 5, 28)$, $(39, 6, 28)$ and $(10, 3, 15)$: the two agree to at least 90 significant digits (the discrepancies are $\leq 3 \times 10^{-91}$, i.e. working-precision noise). We therefore regard Theorem 1 as known and use it as a reference point, not as a contribution.

Figure 1 shows $\mathbb{P}(T = r)$ for four pairs (n, q) . Note that (7) is a difference of two nearly equal numbers for small r ; this is already a hint of the numerical trouble quantified in Section 6.

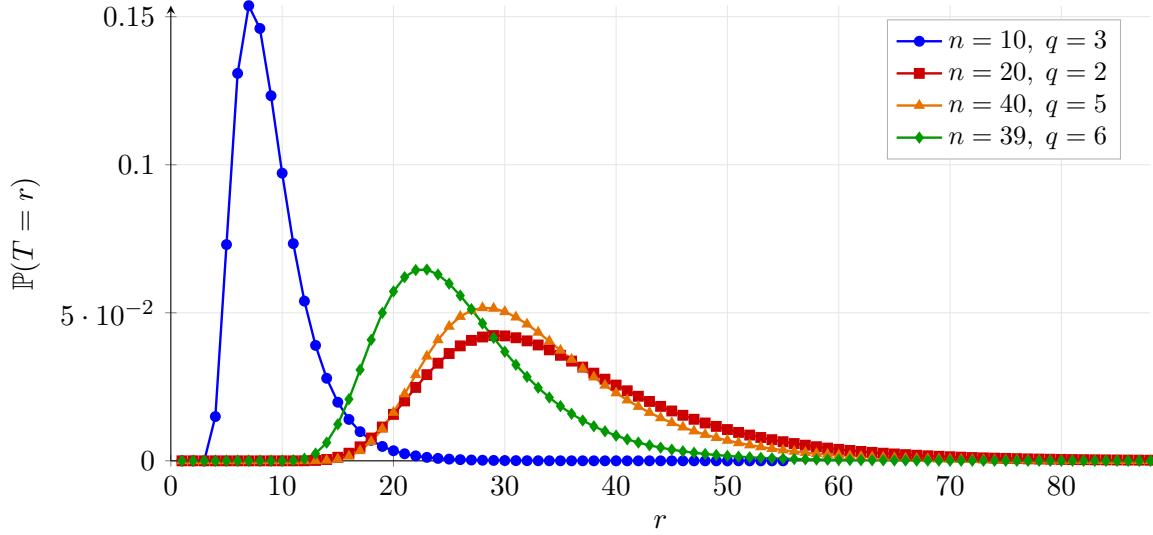


Figure 1: The probability mass function $\mathbb{P}(T = r)$ of Theorem 1 for four pairs (n, q) . Each curve starts at its own $r_{\min} = \lceil n/q \rceil$ and is right-skewed; the means are $\mathbb{E}[T] = 9.0462$ (10, 3), 35.3047 (20, 2), 32.8717 (40, 5) and 26.2355 (39, 6).

4 Exact moments

The tail of Theorem 1 is a finite linear combination of geometric sequences, so every moment can be summed in closed form. We use the following elementary identity.

Lemma 1. *For a random variable T with values in $\{1, 2, 3, \dots\}$ and any integer $m \geq 1$,*

$$\mathbb{E}[T^m] = \sum_{r \geq 0} ((r+1)^m - r^m) \mathbb{P}(T > r),$$

each series being absolutely convergent whenever the moment exists.

Proof. Since $T^m = \sum_{r=0}^{T-1} ((r+1)^m - r^m)$ (a telescoping sum), taking expectations and exchanging the (non-negative) sums gives the claim. $\square$

Theorem 2 (first two moments and the variance). *Let $1 \leq q < n$ and λ_i be as in (5). Then*

$$\mathbb{E}[T] = 1 + \sum_{i=1}^{n-q} (-1)^{i-1} \binom{n}{i} \frac{\lambda_i}{1 - \lambda_i}, \quad (9)$$

$$\mathbb{E}[T^2] = 1 + \sum_{i=1}^{n-q} (-1)^{i-1} \binom{n}{i} \frac{\lambda_i(3 - \lambda_i)}{(1 - \lambda_i)^2}, \quad (10)$$

$$\text{Var}[T] = \mathbb{E}[T^2] - \mathbb{E}[T]^2. \quad (11)$$

For $q = n$, $\mathbb{E}[T] = 1$ and $\text{Var}[T] = 0$.

Proof. For $m = 1$ the weight in Lemma 1 is $(r+1) - r = 1$, so

$$\mathbb{E}[T] = \sum_{r \geq 0} B(r) = 1 + \sum_{r \geq 1} \sum_{i=1}^{n-q} (-1)^{i-1} \binom{n}{i} \lambda_i^r = 1 + \sum_{i=1}^{n-q} (-1)^{i-1} \binom{n}{i} \frac{\lambda_i}{1 - \lambda_i},$$

the interchange being legitimate because the i -sum is finite and $0 \leq \lambda_i < 1$. For $m = 2$ the weight is $(r+1)^2 - r^2 = 2r+1$, and

$$\sum_{r \geq 1} (2r+1) \lambda^r = \frac{2\lambda}{(1-\lambda)^2} + \frac{\lambda}{1-\lambda} = \frac{\lambda(3-\lambda)}{(1-\lambda)^2},$$

whence (10). Finally $\text{Var}[T] = \mathbb{E}[T^2] - \mathbb{E}[T]^2$ by definition. $\square$

Remark 2 (higher moments). The same computation works for every m : the inner sum $\sum_{r \geq 1} ((r+1)^m - r^m) \lambda^r$ is a rational function of λ with denominator $(1-\lambda)^m$. For instance

$$\mathbb{E}[T^3] = 1 + \sum_{i=1}^{n-q} (-1)^{i-1} \binom{n}{i} \frac{\lambda_i(7 - 2\lambda_i + \lambda_i^2)}{(1 - \lambda_i)^3}.$$

At $(n, q) = (20, 4)$ the distribution of Theorem 1 gives $\sum_r r^3 \mathbb{P}(T = r) = 6599.2280949313578525$, identical to the closed form.

Remark 3 (what is and is not claimed). The expectation (9) is equivalent to the “neat formula for the expected value” announced in [1]; we do not claim it. In an earlier draft we also claimed the second moment (10), having only the abstract of [1] (which advertises formulas “especially for the expected values”) and the secondary sources [3, 2, 4] to go on. *We now withdraw that claim.* Having read [1] in full we find that his Theorem 2 supplies (1) for *every* factorial moment—his words are “the moments of $Z_n(A)$ can be computed from the relation”—and that at $p = 2$ it is algebraically identical to (10). The second moment and the variance in exact closed form are therefore Stadje’s, and (10) is an independent rediscovery of them. We keep the formula because everything downstream is built on it, and we flag the equivalence because we have not seen it noted anywhere.

The same applies, *mutatis mutandis*, to Section 8. We searched arXiv for the strings “coupon collector” (all fields, 100 newest), “coupon collector” + “variance”, “coupon collector” + “group drawing”, “coupon collector” + “batch”, “multi-draw” + “coupon”, “uniform group drawing”, and abs: “group drawings” + “variance”; we also listed the recent papers of Doumas, Spektor, Berend, Betken and Thäle, and we read in full the sources of [9] and [8]. As of 1 October 2026 this produced no variance result beyond the leading term for the present model. Since then we have also read [15] in full: it concerns the multi-player variant of Doumas [16], in which n independent collectors each draw one coupon per trial and the object of study is the *fastest* of their n completion times; there $\text{Var}[M_N(n)] \sim a_n N^2$ with $a_n = \frac{\pi^2}{6} + nw_n - n^2 c_n^2$, and the paper proves that (a_n) decreases strictly. That is a different process, and again only the leading term is at stake.

Four further sources have since been read in full, and none changes the picture. Doumas and Papanicolaou [12] develop techniques for the asymptotics of the first and second moments, hence of the variance, of the *single-draw* collector’s problem with arbitrary coupon probabilities. Their model is our case $q = 1$, and at equal probabilities they obtain [12, eq. (1.10)]

$$\text{Var}[T_N] = \frac{\pi^2}{6} N^2 - N \ln N - (\gamma + 1)N + O\left(\frac{\ln N}{N}\right),$$

which is *exactly* what Corollary 2 returns at $q = 1$ (there $A = 1/n$, $B = C = 0$, $\kappa_2 = 0$, $c_0(1) = 0$). Their method is integral representations plus Laplace-type estimates—no residues, no Rice kernel—so the agreement is a genuinely independent confirmation of our $q = 1$ degeneration, and it also means we claim nothing for that special case. For $q \geq 2$, which is where the group structure enters, there is no overlap. Anceaume, Busnel and Sericola [4] again treat single draws, with a null coupon, and obtain the distribution and the moments of the time needed to collect c of n coupons by Markov-chain arguments; their paper contains no numerical-stability discussion, which is the point of (C3). In [7] the variant of the “draw d , keep one” model in which all d coupons are kept—that is, the present model—appears as an auxiliary process inside a lower-bound argument, but it is not evaluated: the authors write that “it is not so straightforward to directly get an estimate for this new process” and only bound it. Finally, the monograph of Kolchin, Sevast’yanov and Chistyakov [13] treats in its Chapter 7 the scheme in which the particles are grouped in blocks of size m , no two particles of one block falling into

the same cell and all $\binom{N}{m}$ positions equiprobable—the allocation scheme underlying the present model. This monograph (Russian original 1976) is the blind spot we were most concerned about, because it predates every English-language source; we have not read it, but three independent pieces of evidence indicate that it does not contain our results. First, the zbMATH review [13, Zbl 0376.60003] describes the book as being centred on “the many limit theorems” for the occupancy counts μ_r ; second, the *Encyclopedia of Mathematics* entry on random allocation, written for this exact block scheme, reports only that the μ_r have asymptotically a normal or Poisson distribution as the number of blocks tends to infinity—an assertion about counts at a deterministic number of blocks, not about the moments of the hitting time; third, Stadje, who surveyed this literature carefully and cites the book, states in his introduction that for group drawings asymptotics were known only for $m = 1$ (Pólya 1930, “with a very sketchy proof”; Rényi 1962; Baum and Billingsley 1965), and Stadje’s own asymptotics concern his discrete-partition variant, not the case $A = S$ studied here. We therefore regard the risk as low but recorded.

One caveat remains: our arXiv search used abstracts and metadata, not full text.

Corollary 1 (the classical case $q = 1$). *For $q = 1$, (9)–(11) reduce to*

$$\mathbb{E}[T] = nH_n, \quad \text{Var}[T] = n^2 H_n^{(2)} - nH_n, \quad H_n^{(2)} := \sum_{k=1}^n \frac{1}{k^2}.$$

Proof. For $q = 1$, $\lambda_i = (n - i)/n$. The classical coupon collector writes $T = \sum_{k=1}^n G_k$ with independent geometric variables G_k of success parameter k/n (the waiting time to see a new coupon when $n - k$ are still missing), whence $\mathbb{E}[T] = \sum_{k \leq n} n/k = nH_n$ and $\text{Var}[T] = \sum_{k \leq n} (1 - k/n)(n/k)^2 = n^2 H_n^{(2)} - nH_n$. Both agree with (9)–(11) to working precision (checked at $n = 5, 20, 50$; the differences are 0 and $\leq 4 \times 10^{-85}$). $\square$

Table 1 lists exact moments for a range of (n, q) , computed at 60–160 significant digits and rounded. Note the entry $(n, q) = (10, 10)$: $T \equiv 1$, $\text{Var}[T] = 0$, as it must be.

5 Subtraction-free recursions

5.1 Backward recursion for the expectation

For $0 \leq x \leq n$ let E_x be the expected number of further runs needed when exactly x distinct coupons have already been collected, so that $\mathbb{E}[T] = E_0$. Conditioning on the next run and using (4),

$$E_x = \sum_y p_{x,y} (1 + E_{x+y}) = 1 + \sum_y p_{x,y} E_{x+y}, \quad 0 \leq x < n, \quad (12)$$

with the boundary condition $E_n = 0$ (equivalent to the convention $p_{n,0} := 0$ used in (4)). Moreover $E_0 = 1 + E_q$, because after one run exactly q coupons are held.

Since E_x depends only on E_{x+y} with $y \geq 1$, the system (12) is *upper triangular* in the unknowns $E_q, \dots, E_{n-1}$ and is solved by back substitution:

$$E_n := 0, \quad E_x = \frac{1 + \sum_{y=1}^{\min(q, n-x)} p_{x,y} E_{x+y}}{1 - p_{x,0}}, \quad x = n-1, n-2, \dots, q, \quad (13)$$

and then $\mathbb{E}[T] = 1 + E_q$.

Table 1: Exact expectation, variance and standard deviation of T , rounded to eight decimals.

n	q	$\mathbb{E}[T]$	$\text{Var}[T]$	$\sqrt{\text{Var}[T]}$
10	1	29.28968254	125.68709058	11.21102540
10	3	9.04621300	10.86218152	3.29578238
10	10	1.00000000	0.00000000	0.00000000
20	1	71.95479314	566.51050442	23.80148114
20	3	23.07659147	56.26594983	7.50106325
20	5	13.27093960	17.91978156	4.23317630
30	4	28.79524248	74.55176014	8.63433611
39	6	26.23546336	55.44287446	7.44599721
40	5	32.87170715	86.90250650	9.32215139
40	8	19.88094594	31.10337054	5.57703959
60	5	54.65414885	208.08033393	14.42498991
100	5	102.05140246	607.57927405	24.64912319
100	20	23.85217869	32.11462864	5.66697703
200	5	233.15814521	2524.96217326	50.24900171

Proposition 2 (no cancellation). *In (13) every numerator is a sum of non-negative terms and is ≥ 1 ; every coefficient $p_{x,y}$ is non-negative. The only subtraction is the formation of $1 - p_{x,0}$, and since $p_{x,0} = \binom{x}{q} / \binom{n}{q} \leq \binom{n-1}{q} / \binom{n}{q} = (n-q)/n$ we have $1 - p_{x,0} \geq q/n$, so that subtraction costs at most $p_{x,0}/(1 - p_{x,0}) \leq n/q$ in relative accuracy, i.e. at most $\log_{10}(n/q)$ decimal digits.*

Thus (13) is *subtraction-free* up to a single, explicitly bounded subtraction—in sharp contrast with (9), whose summands alternate in sign.

Implementation notes. (a) Only $E_{x+1}, \dots, E_{x+q}$ are needed when computing E_x , so the recursion runs in $O(q)$ memory, not $O(nq)$. (b) The coefficients should not be formed from large binomials; use the stable product $p_{x,0} = \prod_{j=0}^{q-1} (x-j)/(n-j)$ and, starting from the first non-zero $p_{x,y}$, the recurrence

$$\frac{p_{x,y+1}}{p_{x,y}} = \frac{(q-y)(n-x-y)}{(x-q+y+1)(y+1)},$$

which follows directly from (3). Both are free of overflow and of cancellation. (c) Time is $O(nq)$: each of the $n-q$ rows costs $O(q)$.

Table 2: Double precision, $q = 5$ unless stated: reference value, the inclusion–exclusion closed form (9), the backward recursion (13), their relative errors, and the condition number (15).

n	q	reference	(9)	(13)	rel. err. (9)	rel. err. (13)	κ
20	3	23.07659147	23.07659147	23.07659147	1.87e-13	2.22e-16	7.47e+03
20	5	13.27093960	13.27093960	13.27093960	2.27e-14	2.21e-16	2.97e+03
40	5	32.87170715	32.87170520	32.87170715	5.94e-08	9.50e-17	1.14e+09
40	8	19.88094594	19.88094589	19.88094594	2.82e-09	4.06e-17	2.31e+08
60	5	54.65414885	54.62756210	54.65414885	4.86e-04	7.28e-16	7.05e+14
80	5	77.83597400	1990709.68891430	77.83597400	2.56e+04	3.19e-16	5.14e+20
100	5	102.05140246	500522505910.00555420	102.05140246	4.90e+09	6.59e-17	4.08e+26
100	20	23.85217869	-14554875.13784097	23.85217869	6.10e+05	9.88e-16	5.29e+22

5.2 Forward recursion for the whole distribution

The same idea gives the distribution with no subtraction at all. Let

$$v_x^{(r)} := \mathbb{P}(T > r \text{ and exactly } x \text{ coupons are held after } r \text{ runs}), \quad 0 \leq x < n.$$

Then $v_0^{(0)} = 1$, $v_x^{(0)} = 0$ for $x > 0$, and

$$v_{x+y}^{(r+1)} = v_x^{(r)} p_{x,y} \quad (x+y < n), \quad \mathbb{P}(T = r+1) = \sum_{x=\max(0, n-q)}^{n-1} v_x^{(r)} p_{x, n-x}, \quad (14)$$

together with $\mathbb{P}(T > r) = \sum_{x < n} v_x^{(r)}$. Every operation adds non-negative reals, so *no* cancellation can occur; the mass that reaches $x = n$ is not subtracted from anything, it is simply not propagated. The cost is $O(nq r_{\max})$ time and $O(n)$ memory if the pmf is wanted up to $r_{\max}$.

6 Numerical stability: the closed form versus the recursions

The right-hand side of (9) is an alternating sum $\sum_i (-1)^{i-1} t_i$ of positive terms $t_i = \binom{n}{i} \lambda_i / (1 - \lambda_i)$. Its condition number is

$$\kappa(n, q) := \frac{\sum_{i=1}^{n-q} t_i}{\left| \sum_{i=1}^{n-q} (-1)^{i-1} t_i \right|} = \frac{\sum_i t_i}{\mathbb{E}[T] - 1}. \quad (15)$$

Standard roundoff analysis predicts a relative error of order $\varepsilon \kappa$: the summands are computed to relative accuracy ε , their absolute errors are of size εt_i , and they add up to $\varepsilon \sum_i t_i$ against a result of size $\mathbb{E}[T] - 1$. Since $\max_i t_i$ is of order $\binom{n}{\lfloor n/2 \rfloor} \approx 2^n \sqrt{2/(\pi n)}$, one expects $\log_{10} \kappa \approx 0.3 n$: *each increment of n by about 3.3 costs one more decimal digit*. That is the practical rule behind everything in this section.

Table 2 and Figure 2 test the prediction against measurement. For each n the reference value was computed by (13) at 80 significant digits (a cancellation-free route, by Proposition 2), so the measured errors are those of the double-precision computations alone.

Three observations.

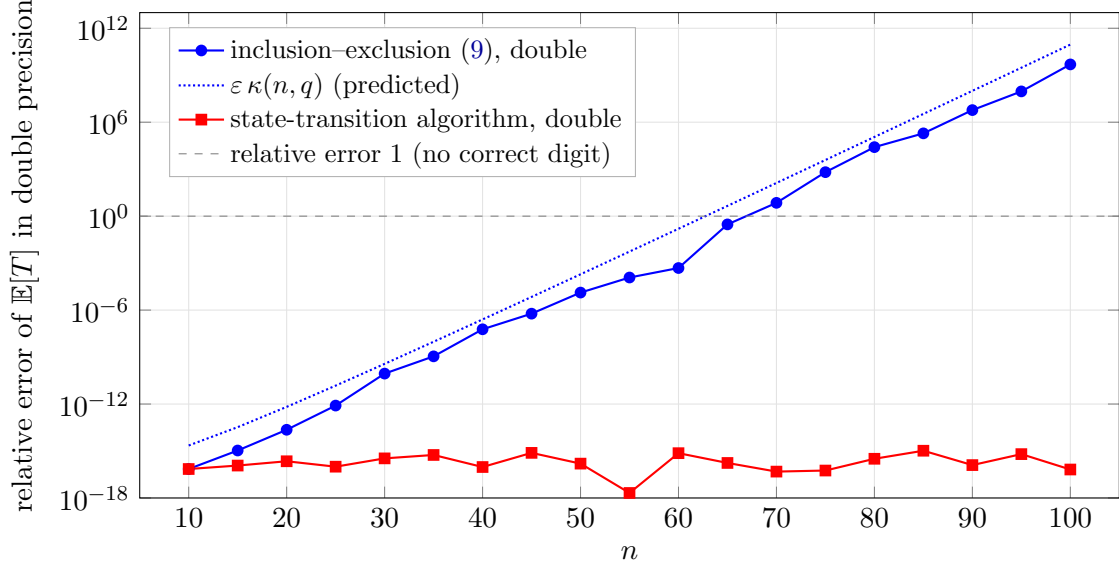


Figure 2: Relative error of $\mathbb{E}[T]$ in IEEE double precision as a function of n at $q = 5$. The inclusion–exclusion closed form (9) (dots) loses one significant decimal digit for every ≈ 3.3 increment of n : 7×10^{-17} at $n = 10$ but 4.9×10^9 at $n = 100$, where it returns 5.005×10^{11} instead of 102.05. The dotted curve is the prediction $\varepsilon \kappa(n, q)$ of (15), which tracks the measurement to within two orders of magnitude across twenty-six decades. The backward recursion (13) (squares) stays at the roundoff level.

1. *The closed form fails early and silently.* At $(n, q) = (20, 5)$ the relative error is already 2.3×10^{-14} ; at $(40, 5)$ it is 5.9×10^{-8} , so that the ninth significant digit—the first one a user is likely to doubt—is already wrong. At $(80, 5)$ the closed form returns 1990 709.69 against the true 77.8360: wrong by four orders of magnitude, and wrong in a way that no sanity check on the output (positivity, order of magnitude) would catch, since the value is positive and finite.
2. *The failure is intrinsic, not an artefact of evaluating λ_i .* We recomputed (9) with λ_i obtained from exact integer binomials $\binom{n-i}{q} / \binom{n}{q}$ instead of the stable product (5); the errors are the same size (compare the two columns of Table 2 with the run at $n = 50$: 1.3×10^{-5} against 2.0×10^{-6}). The damage is done by the alternating signs, not by the arguments.
3. *The recursions do not degrade.* Table 3 pushes the backward recursion (13) to $n = 1000$ at $q = 5$: the relative error never leaves the interval $[2 \times 10^{-17}, 8 \times 10^{-16}]$. This is exactly what Proposition 2 predicts—the single subtraction costs at most $n/q = 200$ in relative accuracy, i.e. about 2×10^{-14} in the worst case, and empirically the error does not even approach that bound.

The same phenomenon hits the distribution. Table 4 and Figure 3 compare, at $(n, q) = (60, 5)$, the pmf obtained from (6)–(7) in double precision with the pmf obtained from the forward recursion (14) in double precision. The inclusion–exclusion route is worst precisely in the *lower* tail—at $r = 15$ it returns -7.16×10^{-11} for a probability of 1.007×10^{-13} , i.e. not merely inaccurate but of the wrong sign—because small r is where the powers λ_i^r are largest and the cancellation is most violent. For $r \gtrsim 40$ it recovers, as the geometric factors have then damped the large summands. The forward recursion holds 10^{-16} throughout.

Table 3: Backward recursion (13) in double precision at $q = 5$ against the 80-digit reference, for large n .

n	reference $\mathbb{E}[T]$	recursion	relative error
20	13.2709395994	13.2709395994	2.21e-16
50	43.5547680996	43.5547680996	1.56e-16
100	102.0514024553	102.0514024553	6.59e-17
200	233.1581452095	233.1581452095	7.03e-16
300	374.8383317748	374.8383317748	2.35e-17
500	676.9597567067	676.9597567067	5.63e-16
1000	1494.4969833721	1494.4969833721	4.60e-16

Table 4: The pmf at $(n, q) = (60, 5)$: exact value, inclusion–exclusion (7) in double precision, forward recursion (14) in double precision.

r	exact $\mathbb{P}(T = r)$	(7)	(14)	rel. err. (7)	rel. err. (14)
15	1.0071e-13	-7.1595e-11	1.0071e-13	7.1e+02	8.8e-17
20	2.0146e-07	2.0146e-07	2.0146e-07	6.4e-06	3.2e-16
25	1.0548e-04	1.0548e-04	1.0548e-04	4.7e-10	6.4e-17
30	2.3386e-03	2.3386e-03	2.3386e-03	2.6e-12	2.0e-16
40	2.4032e-02	2.4032e-02	2.4032e-02	6.9e-16	2.6e-16
50	3.2562e-02	3.2562e-02	3.2562e-02	3.9e-15	4.5e-16
60	2.1577e-02	2.1577e-02	2.1577e-02	9.0e-16	9.0e-16
70	1.0869e-02	1.0869e-02	1.0869e-02	2.9e-15	1.0e-15

7 Verification and numerical results

Six independent checks were run; all pass.

- V1** *Two derivations of $\mathbb{E}[T]$.* (9) and (13) agree to 1.2×10^{-63} or better at $(n, q) = (10, 3), (20, 5), (40, 5), (39, 6)$ —that is, to the full working precision, although the two routes share nothing but the model.
- V2** *The classical case.* At $q = 1$, $\mathbb{E}[T]$ reproduces nH_n to 4×10^{-87} ($n = 20$; see also $n = 5, 50$), and $\text{Var}[T]$ reproduces $n^2 H_n^{(2)} - nH_n$ exactly (Corollary 1).
- V3** *Internal consistency of the distribution.* At $(n, q) = (20, 4)$, $\sum_r \mathbb{P}(T = r)$ equals 1 to all twenty digits computed, and $\sum_r r\mathbb{P}(T = r)$, $\sum_r r^2\mathbb{P}(T = r)$, $\sum_r r^3\mathbb{P}(T = r)$ reproduce (9), (10) and Remark 2 to all digits computed.
- V4** *Agreement with Stadje.* See Remark 1: ninety digits.

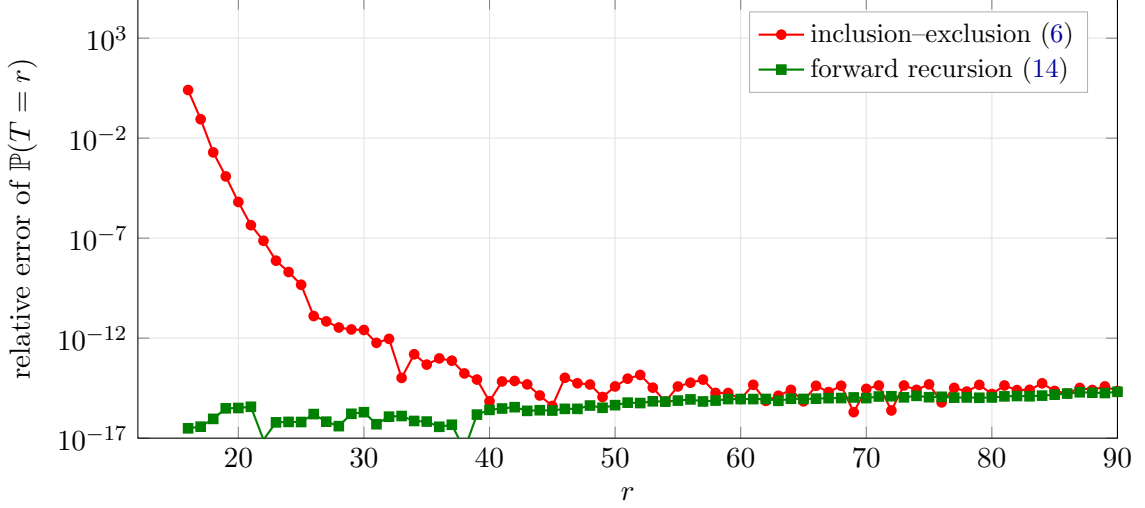


Figure 3: Relative error of $\mathbb{P}(T = r)$ in double precision at $(n, q) = (60, 5)$. The inclusion–exclusion route (7) is unusable in the lower tail ($r \lesssim 30$), where it can even produce negative “probabilities”; the forward recursion (14) is flat at the roundoff level.

Table 5: Monte-Carlo validation: exact moments against simulation (fixed seeds, 100 000–200 000 independent trials each).

n	q	$\mathbb{E}[T]$ exact	$\text{Var}[T]$ exact	MC mean	MC variance	s.e. of mean
10	3	9.046213	10.862182	9.03722	10.80871	0.00735
20	5	13.270940	17.919782	13.27778	18.07809	0.00951
40	5	32.871707	86.902506	32.84491	86.91587	0.02948
39	6	26.235463	55.442874	26.24941	55.16172	0.02349

V5 *Monte-Carlo.* Table 5. Four configurations were simulated with a fixed seed; the sample means fall within 1.3 standard errors of the exact expectations, and the sample variances match the closed-form variances to within the sampling error of a variance estimator from a right-skewed distribution.

V6 *Boundary.* $q = n$ gives $\mathbb{E}[T] = 1$, $\text{Var}[T] = 0$ through the empty sums; $r_{\min} = \lceil n/q \rceil$ is the first point of the support.

8 Asymptotics of the variance

Throughout this section $q \geq 1$ is *fixed* and $n \rightarrow \infty$. Put

$$A := \sum_{j=0}^{q-1} \frac{1}{n-j} = H_n - H_{n-q}, \quad \Sigma_r := \sum_{j=0}^{q-1} \frac{1}{(n-j)^r} \quad (r \geq 2),$$

and let B and C be the second and third elementary symmetric functions of the q numbers $(n-j)^{-1}$, $0 \leq j \leq q-1$. Newton’s identities give

$$B = \sum_{0 \leq i < j \leq q-1} \frac{1}{(n-i)(n-j)} = \frac{A^2 - \Sigma_2}{2}, \quad C = \frac{A^3 - 3A\Sigma_2 + 2\Sigma_3}{6}. \quad (16)$$

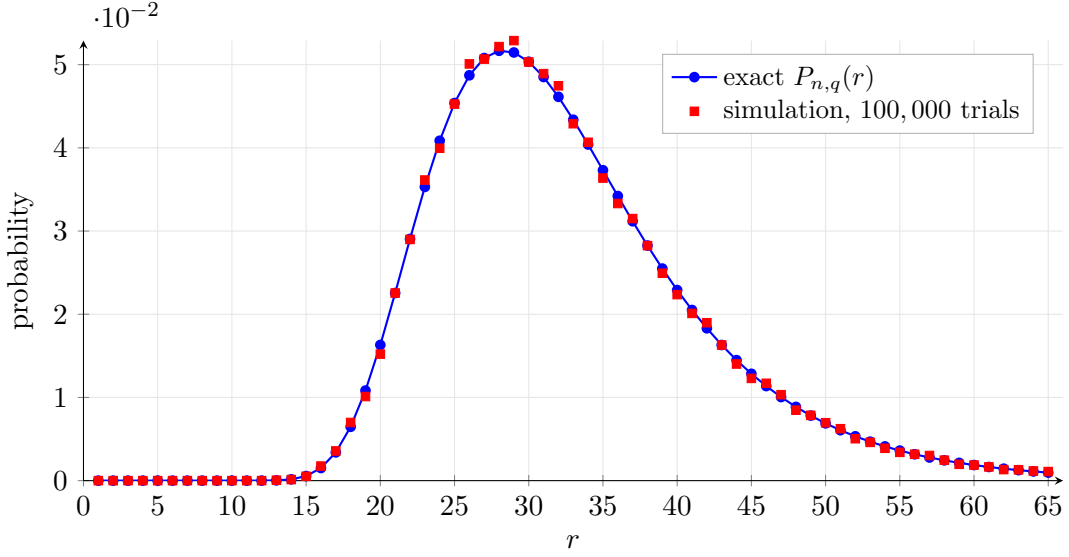


Figure 4: $(n, q) = (40, 5)$: the exact pmf of Theorem 1 against 100 000 simulated values of T . The simulated mean is 32.8449 against the exact 32.8717 (standard error 0.0295).

Define also the polynomial of degree q

$$Q(s) := 1 - \prod_{j=0}^{q-1} \left(1 - \frac{s}{n-j}\right), \quad \phi(s) := \frac{1}{Q(s)}. \quad (17)$$

By (5), $Q(k) = 1 - \lambda_k$ and hence $\phi(k) = (1 - \lambda_k)^{-1}$ for every integer $k \in [1, n]$; note $Q(k) > 0$ there. Expanding the product, $Q(s) = As - Bs^2 + Cs^3 - \dots$.

Theorem 3 (variance to exponential accuracy). *Let $q \geq 1$ be fixed. There is a constant $\rho_q > 0$, given explicitly by Definition 2 and proved positive in Lemma 8, such that for every $\rho < \rho_q$ there is $C_0 = C_0(q, \rho)$ with*

$$\mathbb{E}[T] = \frac{H_n}{A} + \frac{B}{A^2} + R_1, \quad |R_1| \leq C_0 e^{-\rho n}, \quad (18)$$

$$\text{Var}[T] = \frac{H_n^{(2)}}{A^2} - \frac{H_n \Sigma_2}{A^3} + \frac{5B^2}{A^4} - \frac{4C}{A^3} - \frac{B}{A^2} + R, \quad |R| \leq C_0 e^{-\rho n}. \quad (19)$$

For $q = 1$ one has $A = 1/n$, $\Sigma_2 = 1/n^2$, $B = C = 0$, the error terms vanish identically, and both formulas reduce exactly to $\mathbb{E}[T] = nH_n$ and $\text{Var}[T] = n^2 H_n^{(2)} - nH_n$.

Remark 4 (attribution). The expectation part (18) is the theorem of Doumas and Spektor [14], obtained here as a by-product; weaker $O(1)$ forms are due to Berend and Sher [10, 11]. We claim nothing for it. For the variance, [14] obtain—for their Problem II, in which the collector keeps the least-collected coupon—

$$\text{Var}[T_{N,1}] = \frac{\pi^2}{6} \frac{N^2}{d^2} - \frac{1}{d^2} N \log N + O(N)$$

by an Euler–Maclaurin argument, with the $O(N)$ term equal to $-K_d N + o(N)$ and $K_1 = 1 + \gamma$, $K_2 \approx 0.8538$ fitted numerically [14, Thm 5 and Rem. K]. For Problem I, which is the present model, they record only the leading term $\frac{\pi^2}{6} (N/d)^2$, remarking that the retention rule “enters only at order $N \log N$ ” without computing it there. Corollary 2 below supplies that coefficient for Problem I: it equals $-1/q^2$, the same value as in their Problem II, which is a genuinely

independent confirmation, since the two are obtained by different methods (residues here, Euler–Maclaurin there) on different processes. At the next order the two models do differ: our n -coefficient for Problem I is $-(\frac{\pi^2}{6}(q-1)+1)/q^2$, which at $q=2$ gives -0.6612 , whereas their fitted $K_2 = 0.8538$ belongs to Problem II. Everything beyond that—the closed form (19), the explicit rate, and the expansion of Corollary 2—we have not found in the literature, subject to the caveats recorded in Remark 3. In particular we have read [9], [8] and [15] in full; none of them treats the variance of the present model beyond its leading term.

8.1 Moments as alternating binomial sums

Lemma 2 (inclusion–exclusion). *For every integer $r \geq 0$, $\mathbb{P}(T > r) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \lambda_k^r$.*

Proof. Let B_i be the event that coupon i is not seen during the first r runs, so $\{T > r\} = \bigcup_i B_i$. For $|S| = k$ the event $\bigcap_{i \in S} B_i$ occurs iff all r independent runs avoided S , which has probability λ_k per run. Inclusion–exclusion over the n events gives the claim. $\square$

Proposition 3. *For $m = 1, 2$ put $S_m := \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \phi(k)^m$, with ϕ as in (17). Then $\mathbb{E}[T] = S_1$ and $\mathbb{E}[T^2] = 2S_2 - S_1$.*

Proof. By Lemma 2 and $\phi(k) = (1 - \lambda_k)^{-1}$ we have $S_1 = \sum_{r \geq 0} \mathbb{P}(T > r) = \mathbb{E}[T]$ by Lemma 1 with weights $(r+1) - r = 1$. For the second moment the same two lemmas give

$$\mathbb{E}[T^2] = \sum_{r \geq 0} (2r+1) \mathbb{P}(T > r) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \sum_{r \geq 0} (2r+1) \lambda_k^r,$$

and since $\sum_{r \geq 0} (2r+1) \lambda^r = \frac{2}{(1-\lambda)^2} - \frac{1}{1-\lambda}$ for $|\lambda| < 1$, we get $\mathbb{E}[T^2] = 2S_2 - S_1$. (One can equally start from Theorem 2, using $\frac{\lambda}{1-\lambda} = \frac{1}{1-\lambda} - 1$ and $\frac{\lambda(3-\lambda)}{(1-\lambda)^2} = \frac{1+\lambda}{(1-\lambda)^2} - 1$ together with $\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} = 1$.) $\square$

8.2 The Rice kernel and an exact residue identity

Definition 1. $K_n(s) := \frac{n!}{s(s-1) \cdots (s-n)}$.

Lemma 3. (i) K_n has simple poles at $0, 1, \dots, n$ and $\text{Res}_{s=m} K_n = (-1)^{n-m} \binom{n}{m}$. (ii) For $|s| < 1$, $K_n(s) = \frac{(-1)^n}{s} \exp(\sum_{r \geq 1} s^r H_n^{(r)}/r) = \frac{(-1)^n}{s} [1 + H_n s + \frac{H_n^2 + H_n^{(2)}}{2} s^2 + O(s^3)]$. (iii) $K'_n/K_n = -\sum_{m=0}^n (s-m)^{-1}$.

Proof. (i) $\text{Res}_{s=m} K_n = n! / \prod_{\ell \neq m} (m - \ell) = (-1)^{n-m} \binom{n}{m}$. (ii) $\frac{1}{s(s-1) \cdots (s-n)} = \frac{(-1)^n}{s n!} \prod_{m=1}^n (1 - \frac{s}{m})^{-1}$ and $-\sum_m \log(1 - \frac{s}{m}) = \sum_{r \geq 1} s^r H_n^{(r)}/r$, convergent for $|s| < 1$ since $H_n^{(r)} \leq \zeta(2)$ for $r \geq 2$. (iii) is the logarithmic derivative. $\square$

Lemma 4 (exact residue identity). *Let ϕ be a rational function such that (H1) ϕ has no pole at any integer $k \in \{1, \dots, n\}$, and (H2) $\phi(s) = O(|s|^M)$ as $|s| \rightarrow \infty$ for some $M < n$. Put $G(s) := (-1)^{n+1} K_n(s) \phi(s)$ and let $\mathcal{P}$ be the set of poles of ϕ . Then*

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \phi(k) = -\text{Res}_{s=0} G(s) - \sum_{\substack{\sigma \in \mathcal{P} \\ \sigma \neq 0}} \text{Res}_{s=\sigma} G(s). \quad (20)$$

Proof. G is rational with poles contained in $\{0, 1, \dots, n\} \cup \mathcal{P}$. At $m \in \{1, \dots, n\}$ the function ϕ is holomorphic by (H1), so

$$\text{Res}_{s=m} G = (-1)^{n+1} \phi(m) (-1)^{n-m} \binom{n}{m} = (-1)^{m+1} \binom{n}{m} \phi(m),$$

since $(-1)^{2n+1-m} = (-1)^{1-m} = (-1)^{m+1}$. As $K_n(s) = n! s^{-(n+1)}(1 + O(1/s))$, hypothesis (H2) gives $G(s) = O(|s|^{M-n-1}) = o(|s|^{-1})$ at infinity, so $\text{Res}_{s=\infty} G = 0$. The residues of a rational function on the Riemann sphere sum to zero, which is (20). $\square$

Lemma 4 is the Rice (Nørlund–Rice) integral in residue form [17]. For a *rational* ϕ it is completely elementary: no contour deformation and no growth hypothesis beyond (H2). Our $\phi = 1/Q$ and $\phi = 1/Q^2$ are rational with $M = -q$ and $M = -2q$, and (H1) holds because $Q(k) = 1 - \lambda_k > 0$ for $1 \leq k \leq n$.

8.3 The poles of ϕ and an exponential bound

Lemma 5 (the poles are simple). *Let $n > 2q - 2$. Then Q has exactly q roots, all simple: $s = 0$ and $q - 1$ further roots $\omega_1, \dots, \omega_{q-1}$, none of them in the real interval $[n - q + 1, n]$.*

Proof. Put $P(s) := \prod_{j=0}^{q-1} (1 - \frac{s}{d_j})$ with $d_j := n - j$, a degree- q polynomial with distinct real roots $d_0 > \dots > d_{q-1}$; then $Q = 1 - P$ and $Q' = -P'$. A multiple root ω of Q satisfies $P(\omega) = 1$ and $P'(\omega) = 0$; by the Gauss–Lucas theorem the latter forces ω into the convex hull $[d_{q-1}, d_0] = [n - q + 1, n]$ of the roots of P . For real $\omega \in [n - q + 1, n]$ and every j we have $|1 - \omega/d_j| \leq 1$, because $0 \leq \omega \leq n \leq 2d_j$ (the last step uses $n \geq 2q - 2$) and $|1 - t| \leq 1$ for $t \in [0, 2]$; moreover equality would force $\omega = 0$ or $\omega = 2d_j$, and both are excluded since $\omega \geq n - q + 1 > 0$ and $2d_j \geq 2(n - q + 1) > n \geq \omega$ when $n > 2q - 2$. Hence $|P(\omega)| < 1$, contradicting $P(\omega) = 1$. Finally $Q(0) = 1 - P(0) = 0$ and $Q'(0) = \sum_j d_j^{-1} = A > 0$, so 0 is a simple root. $\square$

Lemma 6 (location of the non-zero poles). *After a suitable labelling, $\omega_j/n \rightarrow a_j := 1 - e^{2\pi i j/q}$ as $n \rightarrow \infty$, $j = 1, \dots, q - 1$. Consequently, for all large n , either (a) $|\Im \omega_j| \geq \frac{1}{2} |\sin(2\pi j/q)| n$ and $|\omega_j| \leq 3n$, or (b) $j = q/2$ with q even, ω_j is real and $\omega_j \geq \frac{3}{2}n$.*

Proof. Let $\mathcal{P}_n(z) := Q(nz) = 1 - \prod_{j < q} (1 - \frac{z}{1-j/n})$ and $\mathcal{P}(z) := 1 - (1 - z)^q$; these are polynomials of degree q whose coefficients converge, so $\mathcal{P}_n \rightarrow \mathcal{P}$ locally uniformly. The zeros $a_j = 1 - e^{2\pi i j/q}$ of $\mathcal{P}$ are simple; choose disjoint closed disks $D_j \ni a_j$, set $\eta_j := \min_{\partial D_j} |\mathcal{P}| > 0$, and note that $|\mathcal{P}_n - \mathcal{P}| < \eta_j$ on ∂D_j for large n . By Rouché's theorem $\mathcal{P}_n$ has exactly one zero in each D_j ; since $\deg \mathcal{P}_n = q$ and the disks are disjoint, these exhaust all its zeros. Writing that zero as ω_j/n and letting the disks shrink gives $\omega_j/n \rightarrow a_j$, whence (a) and (b), using $\Im a_j = -\sin(2\pi j/q)$, $|a_j| \leq 2$, $a_{q/2} = 2$. $\square$

Definition 2. For $a \in \mathbb{C} \setminus [0, 1]$ set $J(a) := \int_0^1 \log |a - x| dx$ and $\rho(a) := 1 + J(a)$. For $\theta \in (0, 2\pi)$ write $\tilde{\theta} = \min(\theta, 2\pi - \theta)$ and $\rho(\theta) := \rho(1 - e^{i\theta})$, and put $\rho_q := \min_{1 \leq j \leq q-1} \rho(2\pi j/q)$ for $q \geq 2$.

Lemma 7 (exponential bound at the non-zero poles). *Let $q \geq 2$ be fixed. For every $\rho < \rho_q$ there are n_0 and C with $|K_n(\omega_j)| \leq Ce^{-\rho n}$ and $|K'_n(\omega_j)| \leq Ce^{-\rho n}$ for all $n \geq n_0$ and all j .*

Proof. Write $\omega = \omega_j$, $a = \omega/n$, so $a \rightarrow a_j$ and $|a| \leq 3$ for large n (Lemma 6).

Case (a): $\Im a_j \neq 0$. Let $\delta := \frac{1}{2} |\Im a_j| > 0$, so $|\Im a| \geq \delta$ for large n , and set $f(x) := \log |a - x|$ on $[0, 1]$. Then $|f'(x)| = |x - \Re a|/|a - x|^2 \leq 4/\delta^2$ and $|f(x)| \leq \log 4 + |\log \delta| =: M_\delta$. Since $|\int_{m/n}^{(m+1)/n} f - \frac{1}{n} f(\frac{m}{n})| \leq n^{-2} \sup |f'|$, summation over $m = 0, \dots, n - 1$ and multiplication by n give

$$\left| \sum_{m=0}^n f\left(\frac{m}{n}\right) - nJ(a) \right| \leq \sup_{[0,1]} |f'| + |f(1)| \leq 4/\delta^2 + M_\delta =: C_1.$$

Hence $\prod_{m=0}^n |\omega - m| = n^{n+1} \exp(\sum_{m=0}^n f(\frac{m}{n})) \geq n^{n+1} e^{nJ(a) - C_1}$, and with Stirling's inequality $n! \leq e n^{n+1/2} e^{-n}$,

$$|K_n(\omega)| = \frac{n!}{\prod_{m=0}^n |\omega - m|} \leq e^{1+C_1} n^{-1/2} e^{-n\rho(a)}.$$

Because J is continuous away from $[0, 1]$ and $a_j \notin [0, 1]$, we get $\rho(a) \rightarrow \rho(a_j) \geq \rho_q$, hence $\rho(a) \geq \rho$ for large n .

Case (b): $a_j = 2$. Then ω is real with $\omega \geq \frac{3}{2}n$, so $|\omega - m| \geq \omega - n \geq n/2$ for all $m \leq n$ and $|K_n(\omega)| \leq e n^{n+1/2} e^{-n} (n/2)^{-(n+1)} = 2e n^{-1/2} (2/e)^n$, which decays geometrically; enlarging C absorbs the remaining rates $\rho < \log(e/2)$, and larger $\rho < \rho_q$ are covered because $\rho(a_j) = \rho(\pi) = \log 4$ dominates here.

Finally $K'_n/K_n = -\sum_{m=0}^n (\omega - m)^{-1}$ has modulus at most $(n+1)/(\delta n) \leq 2/\delta$ in case (a) and at most $2(n+1)/n \leq 4$ in case (b), so $|K'_n(\omega)| \leq C_2 |K_n(\omega)|$. $\square$

Lemma 8 (closed form and positivity of the rate). *For $\theta \in (0, 2\pi)$,*

$$\rho(\theta) = \frac{1}{2} \left[(1 - \cos \tilde{\theta}) \log(2(1 - \cos \tilde{\theta})) + \sin \tilde{\theta} (\pi - \tilde{\theta}) \right] > 0, \quad \tilde{\theta} = \min(\theta, 2\pi - \theta),$$

and consequently $\rho_q > 0$ for every $q \geq 2$. Numerically $\rho_2 = \log 4 = 1.3863$, $\rho_3 = 1.2774$, $\rho_5 = 1.0081$, $\rho_{10} = 0.6467$.

Proof. $J(\bar{a}) = J(a)$, so assume $\theta \in (0, \pi]$. For $a = \alpha + i\beta$, $\beta \neq 0$, integrating $\frac{1}{2} \log((x - \alpha)^2 + \beta^2)$ over $[0, 1]$ gives, with $u = 1 - \alpha$, $v = -\alpha$,

$$J(a) = \frac{1}{2} \left[u \log(u^2 + \beta^2) - v \log(v^2 + \beta^2) - 2 + 2\beta \left(\arctan \frac{u}{\beta} - \arctan \frac{v}{\beta} \right) \right]. \quad (21)$$

For $a = 1 - e^{i\theta}$ we have $\alpha = 1 - \cos \theta$, $\beta = -\sin \theta$, $u = \cos \theta$, $v = \cos \theta - 1$, $u^2 + \beta^2 = 1$, $v^2 + \beta^2 = 2(1 - \cos \theta)$. For $\theta \in (0, \pi)$, $\arctan \frac{\cos \theta}{-\sin \theta} = \theta - \frac{\pi}{2}$ and $\arctan \frac{\cos \theta - 1}{-\sin \theta} = \frac{\theta}{2}$, so the difference of arctangents is $\frac{\theta}{2} - \frac{\pi}{2}$ and $2\beta(\frac{\theta}{2} - \frac{\pi}{2}) = \sin \theta(\pi - \theta)$. Substituting into (21) and adding 1 (the -1 cancels) yields the displayed formula, which extends to $\theta = \pi$ by continuity.

For positivity put $t = \theta/2 \in (0, \pi/2]$ and write

$$\psi(\theta) := 2\rho(\theta) = 2 \sin t G(t), \quad G(t) := 2 \sin t \log(2 \sin t) + (\pi - 2t) \cos t,$$

using $4 \sin^2 t = 2(1 - \cos \theta)$ and $4 \sin^2 t \log(2 \sin t) = 2(1 - \cos \theta) \cdot \frac{1}{2} \log(2(1 - \cos \theta))$. Now $G(\frac{\pi}{2}) = 2 \log 2 > 0$ and

$$G'(t) = 2 \cos t \log(2 \sin t) + 2 \cos t - 2 \cos t - (\pi - 2t) \sin t = 2 \cos t \log(2 \sin t) - (\pi - 2t) \sin t.$$

Substituting $w = \frac{\pi}{2} - t$ (so $\cos t = \sin w$, $\sin t = \cos w$, $\pi - 2t = 2w$), the inequality $G'(t) < 0$ becomes $2 \sin w \log(2 \cos w) < 2w \cos w$, i.e. $h(w) := w - \tan w \log(2 \cos w) > 0$ on $(0, \frac{\pi}{2})$. Since $h(0) = 0$ and

$$h'(w) = 1 - \sec^2 w \log(2 \cos w) + \tan^2 w = \sec^2 w [1 - \log(2 \cos w)] > 0$$

there (because $2 \cos w \leq 2 < e$ implies $\log(2 \cos w) < 1$), we get $h > 0$, hence $G' < 0$, hence $G(t) \geq G(\frac{\pi}{2}) = 2 \log 2 > 0$, hence $\psi(\theta) > 0$ on $(0, \pi)$; and $\psi(\pi) = 4 \log 2 > 0$. $\square$

8.4 The residues at $s = 0$

Lemma 9 (local expansions at the origin). *As $s \rightarrow 0$,*

$$\begin{aligned} \frac{1}{Q(s)} &= \frac{1}{As} \left[1 + \frac{B}{A}s + \left(\frac{B^2}{A^2} - \frac{C}{A} \right) s^2 + O(s^3) \right], \\ \frac{1}{Q(s)^2} &= \frac{1}{A^2 s^2} \left[1 + 2 \frac{B}{A}s + \left(3 \frac{B^2}{A^2} - 2 \frac{C}{A} \right) s^2 + O(s^3) \right], \\ (-1)^{n+1} K_n(s) &= -\frac{1}{s} \left[1 + H_n s + \frac{H_n^2 + H_n^{(2)}}{2} s^2 + O(s^3) \right]. \end{aligned}$$

Proof. The first two come from $Q(s) = As(1 - \frac{B}{A}s + \frac{C}{A}s^2 - \dots)$ and the geometric series $(1 - u)^{-1} = 1 + u + u^2 + O(u^3)$ (squared for the second); the third is Lemma 3(ii) multiplied by $(-1)^{n+1}(-1)^n = -1$. $\square$

Proposition 4 (the two residues). *Let $G_1 := (-1)^{n+1}K_n\phi$ and $G_2 := (-1)^{n+1}K_n\phi^2$. Then*

$$-\text{Res}_{s=0} G_1 = \frac{H_n}{A} + \frac{B}{A^2}, \quad -\text{Res}_{s=0} G_2 = \frac{H_n^2 + H_n^{(2)}}{2A^2} + \frac{2H_nB}{A^3} + \frac{3B^2}{A^4} - \frac{2C}{A^3}.$$

Proof. By Lemma 9, $G_1(s) = -\frac{1}{As^2}[1 + H_ns + \dots][1 + \frac{B}{A}s + \dots]$, whose s^{-1} coefficient is $-\frac{1}{A}(\frac{B}{A} + H_n)$; and $G_2(s) = -\frac{1}{A^2s^3}[1 + H_ns + \frac{H_n^2 + H_n^{(2)}}{2}s^2 + \dots][1 + 2\frac{B}{A}s + (3\frac{B^2}{A^2} - 2\frac{C}{A})s^2 + \dots]$, whose s^{-1} coefficient is $-\frac{1}{A^2}[\frac{H_n^2 + H_n^{(2)}}{2} + 2\frac{H_nB}{A} + 3\frac{B^2}{A^2} - 2\frac{C}{A}]$. Negate both. $\square$

Lemma 10 (residues at the non-zero poles). *Let $q \geq 2$ be fixed and $\rho < \rho_q$. Then $\sum_j |\text{Res}_{s=\omega_j} G_1| = O(ne^{-\rho n})$ and $\sum_j |\text{Res}_{s=\omega_j} G_2| = O(n^2e^{-\rho n})$.*

Proof. By Lemma 5, $Q(s) = \kappa \prod_{\sigma}(s - \sigma)$ over its q roots, with $\kappa = (-1)^{q+1}(\prod_{j < q}(n - j))^{-1}$; thus $|\kappa|^{-1} \leq n^q$. Distinctness of the a_j and Lemma 6 give $|\omega - \sigma| \geq cn$ for all roots $\sigma \neq \omega$, with $c = c(q) > 0$.

For G_1 , ϕ has a simple pole at ω and $Q'(\omega) = \kappa \prod_{\sigma \neq \omega}(\omega - \sigma)$, so $|Q'(\omega)| \geq |\kappa|(cn)^{q-1} \geq c^{q-1}/n$ and, by Lemma 7, $|\text{Res}_{s=\omega} G_1| = |K_n(\omega)|/|Q'(\omega)| \leq Cne^{-\rho n}$.

For G_2 , write $\phi(s)^2 = \kappa^{-2}(s - \omega)^{-2}R(s)$ with $R(s) = \prod_{\sigma \neq \omega}(s - \sigma)^{-2}$ analytic near ω . Then

$$\text{Res}_{s=\omega} G_2 = \frac{(-1)^{n+1}}{\kappa^2} (K'_n(\omega)R(\omega) + K_n(\omega)R'(\omega)),$$

with $|R(\omega)| \leq (cn)^{-2(q-1)}$ and $|R'(\omega)| = 2|R(\omega)||\sum_{\sigma \neq \omega}(\omega - \sigma)^{-1}| \leq 2(q-1)c^{-1}n^{-1}|R(\omega)|$. Since $|\kappa|^{-2} \leq n^{2q}$, $n^{2q}(cn)^{-2(q-1)} = O(n^2)$ and $n^{2q}(cn)^{-2(q-1)}n^{-1} = O(n)$, Lemma 7 gives the stated bound. Summing over the $q-1$ poles, a number independent of n , finishes the proof. $\square$

Proof of Theorem 3. Let $q \geq 2$ and $\rho < \rho_q$ (the case $q = 1$ is treated at the end). Apply Lemma 4 with $\phi = 1/Q$: hypotheses (H1), (H2) hold, and $\mathcal{P} = \{0, \omega_1, \dots, \omega_{q-1}\}$, so by Propositions 3, 4 and Lemma 10,

$$\mathbb{E}[T] = S_1 = \frac{H_n}{A} + \frac{B}{A^2} + R_1, \quad R_1 = O(ne^{-\rho n}),$$

which is (18). Applying Lemma 4 again with $\phi = 1/Q^2$ (again (H1), (H2)) gives, with $M_2 := -\text{Res}_{s=0} G_2$,

$$S_2 = M_2 + R_2, \quad R_2 = O(n^2e^{-\rho n}), \quad \mathbb{E}[T^2] = 2S_2 - S_1 = 2M_2 - M_1 + (2R_2 - R_1),$$

where $M_1 := H_n/A + B/A^2$. Therefore, writing $\delta := 2R_2 - R_1$,

$$\text{Var}[T] = 2M_2 - M_1 + \delta - (M_1 + R_1)^2 = (2M_2 - M_1 - M_1^2) + \delta - 2M_1R_1 - R_1^2.$$

Since $M_1 = O(n \log n)$, the total error is $O(n^2 \log n e^{-\rho n})$, which is $O(e^{-\rho' n})$ for every $\rho' < \rho$; as $\rho < \rho_q$ was arbitrary, this yields (19) after renaming. Finally,

$$\begin{aligned} 2M_2 - M_1 - M_1^2 &= \left[\frac{H_n^2 + H_n^{(2)}}{A^2} + \frac{4H_nB}{A^3} + \frac{6B^2}{A^4} - \frac{4C}{A^3} \right] - \left[\frac{H_n}{A} + \frac{B}{A^2} \right] - \left[\frac{H_n^2}{A^2} + \frac{2H_nB}{A^3} + \frac{B^2}{A^4} \right] \\ &= \frac{H_n^{(2)}}{A^2} + \frac{2H_nB}{A^3} + \frac{5B^2}{A^4} - \frac{4C}{A^3} - \frac{H_n}{A} - \frac{B}{A^2}, \end{aligned}$$

and by (16), $2B = A^2 - \Sigma_2$, so $\frac{2H_nB}{A^3} - \frac{H_n}{A} = \frac{H_n(A^2 - \Sigma_2)}{A^3} - \frac{H_nA^2}{A^3} = -\frac{H_n\Sigma_2}{A^3}$.

For $q = 1$ we have $Q(s) = s/n$, so $\mathcal{P} = \{0\}$: both sums over non-zero poles are empty and $R_1 = R = 0$. With $A = 1/n$, $\Sigma_2 = 1/n^2$, $B = C = 0$ the formulas give $\mathbb{E}[T] = nH_n$ and $\text{Var}[T] = n^2H_n^{(2)} - H_n(\frac{1}{n^2} \cdot n^3) = n^2H_n^{(2)} - nH_n$, in agreement with Corollary 1. $\square$

8.5 Expansion in powers of n

Corollary 2. For fixed $q \geq 1$,

$$\text{Var}[T] = \frac{\pi^2}{6} \left(\frac{n}{q}\right)^2 - \frac{n}{q^2}(\log n + \gamma) - \frac{\frac{\pi^2}{6}(q-1) + 1}{q^2} n + \frac{q-1}{2q^2} \log n + c_0(q) + O\left(\frac{\log n}{n}\right), \quad (22)$$

where, with $\kappa_2 := \frac{3(q-1)^2}{4} - \frac{(q-1)(2q-1)}{3}$,

$$c_0(q) = \frac{1}{q^2} \left[\frac{\pi^2}{6} \kappa_2 + (q-1) \left(1 + \frac{\gamma}{2}\right) + \frac{q^2 - 1}{12} \right].$$

Dropping the logarithmic term into the remainder,

$$\text{Var}[T] = \frac{\pi^2}{6} \left(\frac{n}{q}\right)^2 - \frac{n}{q^2}(\log n + \gamma) - \frac{\frac{\pi^2}{6}(q-1) + 1}{q^2} n + \frac{q^2 - 1}{12q^2} + O(\log n). \quad (23)$$

Proof. With $\varepsilon := \frac{q-1}{2n} + \frac{(q-1)(2q-1)}{6n^2} + O(n^{-3})$ one has $A = \frac{1}{n} \sum_{j < q} (1 - \frac{j}{n})^{-1} = \frac{q}{n} (1 + \varepsilon)$, whence

$$\frac{1}{A^2} = \left(\frac{n}{q}\right)^2 \left[1 - \frac{q-1}{n} + \frac{\kappa_2}{n^2} + O(n^{-3}) \right], \quad \kappa_2 = \frac{3(q-1)^2}{4} - \frac{(q-1)(2q-1)}{3},$$

and $\Sigma_2 = \frac{q}{n^2} [1 + \frac{q-1}{n} + O(n^{-2})]$, $\Sigma_3 = \frac{q}{n^3} [1 + O(n^{-1})]$, so by (16)

$$B = \frac{q(q-1)}{2n^2} \left[1 + \frac{q-1}{n} + O(n^{-2}) \right], \quad \frac{B}{A^2} = \frac{q-1}{2q} + O(n^{-1}), \quad C = \frac{q(q-1)(q-2)}{6n^3} [1 + O(n^{-1})].$$

Using $H_n = \log n + \gamma + \frac{1}{2n} + O(n^{-2})$ and $H_n^{(2)} = \frac{\pi^2}{6} - \frac{1}{n} + O(n^{-2})$,

$$\begin{aligned} \frac{H_n^{(2)}}{A^2} &= \frac{\pi^2}{6} \left(\frac{n}{q}\right)^2 - \frac{\frac{\pi^2}{6}(q-1) + 1}{q^2} n + \frac{1}{q^2} \left[\frac{\pi^2}{6} \kappa_2 + (q-1) + \frac{1}{2} \right] + O(n^{-1}), \\ -\frac{H_n \Sigma_2}{A^3} &= -\frac{n}{q^2} (\log n + \gamma) + \frac{q-1}{2q^2} (\log n + \gamma) - \frac{1}{2q^2} + O\left(\frac{\log n}{n}\right), \end{aligned}$$

the second line because $\Sigma_2/A^3 = \frac{n}{q^2} [1 - \frac{q-1}{2n} + O(n^{-2})]$, and finally

$$\frac{5B^2}{A^4} - \frac{4C}{A^3} - \frac{B}{A^2} = \frac{5(q-1)^2}{4q^2} - \frac{2(q-1)(q-2)}{3q^2} - \frac{q-1}{2q} + O(n^{-1}) = \frac{q^2 - 1}{12q^2} + O(n^{-1}),$$

since $15(q-1)^2 - 8(q-1)(q-2) - 6q(q-1) = (q-1)(q+1)$ over the denominator $12q^2$. Adding the three displays gives the stated constant $c_0(q)$ and the single surviving logarithmic term $\frac{q-1}{2q^2} \log n$. $\square$

Remark 5 (the case $q = 1$ is not ours). At $q = 1$ one has $\kappa_2 = 0$ and $c_0(1) = 0$, so (22) collapses to

$$\text{Var}[T] = \frac{\pi^2}{6} n^2 - n \log n - (\gamma + 1)n + O\left(\frac{\log n}{n}\right),$$

which is precisely eq. (1.10) of Doumas and Papanicolaou [12], obtained there for the single-draw model by integral representations and Laplace-type estimates—no residues, no Rice kernel, no group structure. We therefore claim (22) only for $q \geq 2$; that two independent methods agree at the common boundary $q = 1$ is one of the checks of Section 7.

Table 6: Numerical check of (22): the residual $\text{Var}[T]_{\text{exact}} - (\text{four main terms} + \frac{q-1}{2q^2} \log n + c_0(q))$, with $\text{Var}[T]$ computed by the subtraction-free recursion of Section 5. According to Corollary 2 it should be $O(\log n/n)$; according to Corollary 3 it should equal $-\frac{1}{12n}$ exactly when $q = 1$ (e.g. $-\frac{1}{2400} = -4.1667 \times 10^{-4}$ at $n = 200$).

	q	$c_0(q)$	$n = 200$	$n = 400$	$n = 800$	$n = 1600$
	1	0.000000	-4.167e-4	-2.083e-4	-1.041e-4	-5.22e-5
	2	0.281844	-1.059e-4	-5.250e-5	-2.612e-5	-1.299e-5
	3	0.299508	-1.415e-4	-7.008e-5	-3.488e-5	-1.738e-5
	5	0.286177	-2.220e-4	-1.096e-4	-5.449e-5	-2.716e-5
	10	0.260160	-3.231e-4	-1.567e-4	-7.717e-5	-3.830e-5

Corollary 3 (the classical case). *For $q = 1$ the remainders vanish and (22) reads*

$$\text{Var}[T] = n^2 H_n^{(2)} - n H_n = \frac{\pi^2}{6} n^2 - n \log n - \gamma n - n - \frac{1}{12n} + O(n^{-3}),$$

consistently with $\kappa_2(1) = 0$ and $c_0(1) = 0$; this is the classical $n^2 H_n^{(2)} - n H_n$ of Corollary 1.

In addition, computing $\text{Var}[T]$ to 80 significant digits confirms the *shape* of the remainder in Theorem 3: for instance at $(q, n) = (2, 80)$ the exact variance is 2480.64030551 and $|\text{Var}[T]_{\text{exact}} - \text{Var}[T]_{\text{closed}}| = 3.7 \times 10^{-45}$, while the measured rate $-\frac{1}{n} \log |R|$ increases from 1.158 at $n = 30$ to 1.279 at $n = 80$ against the theoretical $\rho_2 = \log 4 = 1.386$; and the rescaled quantity $|R|e^{\rho_q n}/n^2$ stays bounded (between 0.5 and 1.8 in all runs) while $|R|$ itself ranges over 45 decades, in agreement with the n^2 prefactor predicted by the double poles in Lemma 10.

9 Discussion

Summary. The distribution of T , its mean and *all* of its factorial moments in closed form are Stadge’s [1]: his relation (2.15) at $p = 2$ is equivalent to our second-moment formula (10) (Remark 3), so that item is a consolidation, not an increment. The asymptotics of the mean are Berend and Sher’s and, in sharper form, Doumas and Spektor’s; those of the variance beyond the leading term are, we believe, new for $q \geq 2$ (Section 8, Remark 4), the case $q = 1$ being [12, eq. (1.10)]. The practically most useful outcome is negative: *the elegant closed forms are unusable in ordinary floating point for the very parameter ranges one cares about*, and the plain state-transition recursions—which every textbook treatment would write down first—are the ones that actually work. The condition number (15) turns this from an anecdote into a prediction: $\log_{10} \kappa \approx 0.3n$, so the closed form needs about $0.3n$ guard digits, and a practitioner can read off from n and q whether double precision will do.

Related work. Stadge also solves an optimal-stopping variant: if a group costs α and missing coupons may be bought at $\beta > \alpha/m$ each, he derives the optimal purchasing strategy [1]. Schilling and Henze [2] prove Poisson limit theorems for the number of types sampled at most $c - 1$ times and deduce limit laws for the waiting time; Anceaume, Busnel and Sericola [4] treat a generalised collector problem by Markov chains. Betken and Thäle [9] obtain rates for the normal and Poisson approximation of the number of coupons still missing after a deterministic number of group drawings. The asymptotics of $\mathbb{E}[T]$ and of $\text{Var}[T]$ for fixed q are treated in Section 8 above, sharpening the variance results of Doumas and Spektor [14]; we note once more that they do not cite Stadge, whereas [10, 11, 9] do.

Reproducibility. All numerical material in this note was produced by four scripts that are distributed with the source. `make_coupon_data.py` recomputes the high-precision reference values, the double-precision values of both routes, the condition numbers, the Monte-Carlo runs (fixed seeds) and the `pgfplots` data files of Sections 2–7; `prove_var.py` and `prove_var_mp.py` recompute the residuals of Theorem 3 in double and in 80-digit precision respectively, together with the measured rates $-\frac{1}{n} \log |R|$ and the rescaled quantity $|R|e^{\rho q n}/n^2$ of Table 6; `check_constants.py` recomputes the logarithmic coefficient, the constant $c_0(q)$ of Corollary 2 and its residual, including the exact $-1/(12n)$ residual at $q = 1$; `prior_art_sweep.py` reproduces the arXiv sweep reported in Remark 3.

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