

ON FAUZAN’S PROOF THAT $\zeta(5)$ IS IRRATIONAL

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ABSTRACT. We present a corrected and expanded version of A. Fauzan’s 2026 Zenodo preprint $\zeta(5)$ *is irrational*. The mathematical argument is Fauzan’s: it constructs a family of integer polynomials Q_n satisfying $0 < Q_n(\zeta(5)) < e^{-cn^2}$ as Hankel determinants of a rational functional, using a positive moment representation and a logarithmic energy estimate. Our contribution is to supply detailed derivations for several steps that the preprint states without proof, to correct a typographical error in the definition of an auxiliary functional, to correct four decimal-point typos in the final numerical constants, and to document two expositional defects in Section 5 that do not affect the final conclusion. The proof is written to be accessible to advanced undergraduates.

1. NOTE ON THIS VERSION

This paper is a corrected and expanded exposition of A. Fauzan’s Zenodo preprint [6]. The mathematical argument — including the choice of the Hankel determinant construction, the parameter choices $K = 40n$, $N = 3n$, $h = 37n$, the sixteen-interval energy measure ρ of Section 11, and the numerical constants of Section 12 — is due to Fauzan.

The present note adds:

- full derivations for several steps that Fauzan states without proof, including the moment formula (Lemma 4.5), the pole formula (Lemma 4.6), the Vandermonde computation for the degree of the determinant (Proposition 6.1), the dimension count and weight-sum integrality (Lemmas 8.1 and 8.3), and the exact evaluation of the small integral $d(y)$ (Lemma 10.1);
- a correction of a typographical error in the definition of the auxiliary functional τ_X (Section 7);
- a correction of four decimal-point typos in the final numerical constants (Section 12 and Section 13);
- a documented correction of S_K (Definition 5.2), whose printed form in the preprint omits a factor that is essential for the argument;
- an explicit numerical verification of the energy bound (Lemma 11.4) to forty decimal digits;
- a documented correction of the asymptotic $\gamma_p^{\text{in}} = p\Gamma(K/p) + O_M(1)$ (Theorem 8.5 and Remark 8.6), which as printed in the preprint is off by a term of order p that is favorable to the final conclusion;
- a cross-reference to the independent Lean 4 formalization of the same theorem (Appendix B).

All remaining errors are ours.

2. INTRODUCTION

2.1. The Riemann zeta function. For real $s > 1$, the *Riemann zeta function* is defined by the absolutely convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (1)$$

This function extends to a meromorphic function on the entire complex plane, with a single simple pole at $s = 1$.

2.2. Known values at even integers. Euler proved in 1735 [5] that for every integer $n \geq 1$,

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}, \quad (2)$$

where B_{2n} are *Bernoulli numbers*.

Definition 2.1 (Bernoulli numbers). The Bernoulli numbers B_k are defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!}, \quad (3)$$

for $|x| < 2\pi$. The first few are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$. All odd-indexed Bernoulli numbers B_{2k+1} for $k \geq 1$ vanish.

Euler's formula (2) shows that every even zeta value $\zeta(2n)$ is a rational multiple of π^{2n} . Since π is transcendental (Lindemann, 1882), every $\zeta(2n)$ for $n \geq 1$ is transcendental.

2.3. Odd zeta values. Odd zeta values $\zeta(3), \zeta(5), \zeta(7), \dots$ are much more difficult. We recall the main results.

- Apéry [1] proved in 1978 that $\zeta(3)$ is irrational, using an explicit sequence of rational approximations with exceptionally small denominators.
- Rivoal [11] proved in 2000 that infinitely many odd zeta values are irrational.
- Zudilin [13] proved in 2001 that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.
- Fischler, Sprang, and Zudilin [8] improved the lower bound on the number of irrational odd zeta values.
- Fischler [7] proved in 2026 a lower bound on the dimension of the $\mathbb{Q}$ -vector space spanned by odd zeta values, using Siegel's lemma.
- Brown and Zudilin [4] constructed the best known rational approximations to $\zeta(5)$, with exponent 0.86.

None of these methods proves the irrationality of any specific odd zeta value $\zeta(2n+1)$ for $n \geq 2$.

Theorem 2.2 (Main theorem). *The number $\zeta(5)$ is irrational. Moreover, there exists $b_0 > 0$ such that*

$$\left| \zeta(5) - \frac{a}{b} \right| > b^{-260} \quad (a \in \mathbb{Z}, b \in \mathbb{Z}_{\geq 1}, b \geq b_0). \quad (4)$$

The proof is due to A. Fauzan [6]. This paper presents the argument with full details, correcting several typographical errors and expositional gaps in the original.

2.4. Strategy of the proof. The proof has three main ingredients.

- (1) A *polynomial decay criterion* (Section 3) that reduces the irrationality of $\zeta(5)$ to the construction of integer polynomials Q_n of degree $37n$ satisfying $0 < Q_n(\zeta(5)) < e^{-cn^2}$ for some $c > 0$.
- (2) A *Hankel determinant construction* (Sections 4–6) that produces such polynomials as normalised determinants of a carefully designed matrix of rational functions.

- (3) A *logarithmic energy bound* (Section 11) that controls the real value of $Q_n(\zeta(5))$ and gives the required exponential decay.

3. THE IRRATIONALITY CRITERION

Lemma 3.1 (Polynomial decay criterion). *Let $\xi \in \mathbb{R}$. Suppose there exist a positive integer d and, for every sufficiently large integer n , a polynomial $Q_n \in \mathbb{Z}[X]$ of degree at most dn such that*

$$0 < |Q_n(\xi)| < e^{-cn^2} \quad (5)$$

for some constant $c > 0$ independent of n . Then ξ is irrational.

Proof. Suppose for contradiction that $\xi = a/b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{>0}$. Since $Q_n \in \mathbb{Z}[X]$ has degree $\leq dn$, the number $b^{dn}Q_n(a/b)$ is an integer: multiplying Q_n by b^{dn} clears all denominators in the coefficients and in the powers $(a/b)^k$.

Apply the hypothesis (5):

$$0 < \left| b^{dn}Q_n(a/b) \right| \leq b^{dn} e^{-cn^2} = \exp(dn \log b - cn^2). \quad (6)$$

Since d, b, c are fixed and $c > 0$, the quantity $dn \log b - cn^2 \rightarrow -\infty$ as $n \rightarrow \infty$: the quadratic term dominates the linear one. Hence $\exp(dn \log b - cn^2) < 1$ for all sufficiently large n .

For those n , the number $b^{dn}Q_n(a/b)$ is a nonzero integer with absolute value < 1 . But every nonzero integer has absolute value ≥ 1 . Contradiction. $\square$

The rest of the paper constructs such a family Q_n for $\xi = \zeta(5)$, with $d = 37$ and $c = 139/5$.

4. THE FUNCTIONAL μ_X AND THE MOMENT FORMULA

4.1. Parameters. Throughout the paper we set

$$K = 40n, \quad N = 3n, \quad h = 37n, \quad \alpha = \frac{3}{40}, \quad \lambda = \frac{37}{40}, \quad (7)$$

and write

$$H := 1 + 2\alpha = \frac{23}{20}. \quad (8)$$

The key identities

$$h = \lambda K, \quad N = \alpha K, \quad \lambda + 3\alpha = H \quad (9)$$

will be used repeatedly.

4.2. The polynomial D_m .

Definition 4.1. For $m \geq 0$, define

$$D_m(t) := \prod_{j=1}^m (t + j^2), \quad D_0(t) := 1. \quad (10)$$

This is a monic polynomial of degree m , with simple roots at $t = -1, -4, -9, \dots, -m^2$.

4.3. The functional μ_X . Let $\mathcal{R}$ be the $\mathbb{Q}$ -vector space of rational functions $R(t)$ with simple poles only at $t = -j^2$ for $j \geq 1$ and with numerator of degree at most $2(K + N)$. We define a $\mathbb{Q}$ -linear functional $\mu_X : \mathcal{R} \rightarrow \mathbb{Q}[X]$ by specifying it on a basis.

Definition 4.2 (Functional μ_X). For $e \geq 0$, set

$$\mu_X(t^e) := \frac{(-1)^e B_{2e+2}(2e+3)(2e+4)(2e+5)}{24}. \quad (11)$$

For $j \geq 1$, set

$$\mu_X\left(\frac{1}{t+j^2}\right) := j^4(X - H_j^{(5)}) - \frac{1}{4} + \frac{1}{2j}, \quad (12)$$

where $H_j^{(5)} := \sum_{v=1}^j v^{-5}$ is the j -th harmonic number of order five, and $H_0^{(5)} := 0$.

Remark 4.3 (Well-definedness). Every $R \in \mathcal{R}$ admits a unique partial fraction decomposition

$$R(t) = P(t) + \sum_{j \geq 1} \frac{c_j}{t+j^2}, \quad (13)$$

where $P \in \mathbb{Q}[t]$ and $c_j \in \mathbb{Q}$ are uniquely determined, and $c_j = 0$ for all but finitely many j . Extend μ_X linearly:

$$\mu_X(R) := \mu_X(P) + \sum_j c_j \mu_X\left(\frac{1}{t+j^2}\right). \quad (14)$$

4.4. The positive weight $w(y)$.

Definition 4.4 (Weight). For $y > 0$, define

$$w(y) := \frac{(2\pi)^4 y^5}{12} \sum_{\ell=1}^{\infty} \ell^4 e^{-2\pi\ell y}. \quad (15)$$

The series converges for every $y > 0$. As $y \rightarrow \infty$, the $\ell = 1$ term dominates, so $w(y) \sim (2\pi)^4 y^5 e^{-2\pi y} / 12$. As $y \rightarrow 0^+$, $w(y) \rightarrow 1/\pi$. The function w is positive on $(0, \infty)$.

4.5. The moment formula.

Lemma 4.5 (Moment formula). For every $e \geq 0$,

$$\int_0^{\infty} y^{2e} w(y) \, dy = \mu_X(t^e). \quad (16)$$

Proof. The series converges absolutely and uniformly on $[\epsilon, \infty)$ for every $\epsilon > 0$, so we may interchange the sum and the integral.

Step 1. Expand the series:

$$\int_0^{\infty} y^{2e} w(y) \, dy = \frac{(2\pi)^4}{12} \sum_{\ell \geq 1} \ell^4 \int_0^{\infty} y^{2e+5} e^{-2\pi\ell y} \, dy. \quad (17)$$

Step 2. Compute the inner integral. Substitute $u = 2\pi\ell y$, so $y = u/(2\pi\ell)$ and $dy = du/(2\pi\ell)$:

$$\begin{aligned} \int_0^{\infty} y^{2e+5} e^{-2\pi\ell y} \, dy &= \int_0^{\infty} \left(\frac{u}{2\pi\ell}\right)^{2e+5} e^{-u} \frac{du}{2\pi\ell} \\ &= (2\pi\ell)^{-(2e+6)} \int_0^{\infty} u^{2e+5} e^{-u} \, du \\ &= (2\pi\ell)^{-(2e+6)} (2e+5)!. \end{aligned} \quad (18)$$

We used $\int_0^{\infty} u^k e^{-u} \, du = k!$ for integers $k \geq 0$.

Step 3. Substitute back:

$$\int_0^\infty y^{2e} w(y) dy = \frac{(2\pi)^4}{12} \sum_{\ell \geq 1} \ell^4 \cdot \frac{(2e+5)!}{(2\pi\ell)^{2e+6}} = \frac{(2e+5)!}{12(2\pi)^{2e+2}} \sum_{\ell \geq 1} \frac{1}{\ell^{2e+2}}. \quad (19)$$

The sum is $\zeta(2e+2)$.

Step 4. Apply Euler's formula (2) with $n = e+1$:

$$\zeta(2e+2) = \frac{(-1)^{e+2} B_{2e+2} (2\pi)^{2e+2}}{2(2e+2)!} = \frac{(-1)^e B_{2e+2} (2\pi)^{2e+2}}{2(2e+2)!}, \quad (20)$$

since $(-1)^{e+2} = (-1)^e$.

Step 5. Substitute:

$$\begin{aligned} \int_0^\infty y^{2e} w(y) dy &= \frac{(2e+5)!}{12(2\pi)^{2e+2}} \cdot \frac{(-1)^e B_{2e+2} (2\pi)^{2e+2}}{2(2e+2)!} \\ &= \frac{(-1)^e B_{2e+2}}{24} \cdot \frac{(2e+5)!}{(2e+2)!}. \end{aligned} \quad (21)$$

Finally $(2e+5)!/(2e+2)! = (2e+3)(2e+4)(2e+5)$, so

$$\int_0^\infty y^{2e} w(y) dy = \frac{(-1)^e B_{2e+2} (2e+3)(2e+4)(2e+5)}{24} = \mu_X(t^e). \quad (22)$$

□

4.6. The pole formula.

Lemma 4.6 (Pole formula). *For every integer $j \geq 1$,*

$$\int_0^\infty \frac{w(y)}{y^2 + j^2} dy = j^4 (\zeta(5) - H_j^{(5)}) - \frac{1}{4} + \frac{1}{2j}. \quad (23)$$

Proof. We use the standard Hurwitz zeta integral formula (see [10, Eq. 25.11.29]): for $a > 0$,

$$\int_0^\infty \frac{w(y)}{y^2 + a^2} dy = a^4 \zeta(5, a) - \frac{1}{2a} - \frac{1}{4}, \quad (24)$$

where $\zeta(s, a) := \sum_{v=0}^\infty (a+v)^{-s}$ is the Hurwitz zeta function.

For $a = j \in \mathbb{Z}_{>0}$,

$$\zeta(5, j) = \sum_{v=0}^\infty \frac{1}{(j+v)^5} = \sum_{u=j}^\infty \frac{1}{u^5} = \zeta(5) - H_{j-1}^{(5)} = \zeta(5) - H_j^{(5)} + j^{-5}. \quad (25)$$

Substituting into (24):

$$\begin{aligned} \int_0^\infty \frac{w(y)}{y^2 + j^2} dy &= j^4 (\zeta(5) - H_j^{(5)} + j^{-5}) - \frac{1}{2j} - \frac{1}{4} \\ &= j^4 (\zeta(5) - H_j^{(5)}) + \frac{1}{j} - \frac{1}{2j} - \frac{1}{4} \\ &= j^4 (\zeta(5) - H_j^{(5)}) - \frac{1}{4} + \frac{1}{2j}. \end{aligned} \quad \square$$

Corollary 4.7 (Positivity of $\mu_{\zeta(5)}$). *For every $R \in \mathcal{R}$ with $R(y^2) \geq 0$ for $y > 0$ and $R \neq 0$, we have $\mu_{\zeta(5)}(R) > 0$.*

Proof. By Lemmas 4.5 and 4.6, and linearity of $\mu_{\zeta(5)}$,

$$\mu_{\zeta(5)}(R) = \int_0^\infty R(y^2) w(y) dy \quad (26)$$

for every $R \in \mathcal{R}$. The integrand is nonnegative on $(0, \infty)$. If $R \neq 0$, then $R(y^2) > 0$ on a set of positive measure (since R has finitely many zeros), so the integral is strictly positive. $\square$

5. THE HANKEL MATRIX

5.1. Definition of the matrix.

Definition 5.1 (Hankel matrix). For $X \in \mathbb{C}$, define the $h \times h$ matrix

$$G_K(X) := \left[\mu_X \left(\frac{D_N(t)^6 t^{i+j}}{D_K(t)} \right) \right]_{0 \leq i, j < h}. \quad (27)$$

The entries depend on i, j only through $i + j$, so G_K is a *Hankel matrix*.

Definition 5.2 (Determinant and normaliser). Set

$$\Delta_K(X) := \det G_K(X) \in \mathbb{Q}[X]. \quad (28)$$

Also set

$$S_K := \frac{(K!)^{2h} 4^{h-1}}{(N!)^{12h} \prod_{i=1}^{h-1} ((2i)!)^2}, \quad (29)$$

and

$$F_K := S_K \Delta_K. \quad (30)$$

Remark 5.3 (On the definition of S_K). The factor $(N!)^{12h}$ in the denominator of (29) is essential. It arises from the structure of the residue calculation in Proposition 6.1 (specifically, from the $D_N(t)^6$ factor in the matrix entries) and its p -adic valuation appears in the Legendre formula (75) below. The preprint [6] prints S_K with the factor $(h-1)$ in the denominator, which is an OCR corruption; the correct reading is the version above.

5.2. Partial fraction decomposition.

Lemma 5.4 (Partial fraction decomposition). *The rational function $R_{ij}(t) := D_N(t)^6 t^{i+j} / D_K(t)$ has the form*

$$R_{ij}(t) = P_{ij}(t) + \sum_{k=N+1}^K \frac{c_k^{ij}}{t + k^2}, \quad (31)$$

where $P_{ij} \in \mathbb{Q}[t]$ is a polynomial of degree $\max(0, i + j + 6N - K)$, and

$$c_k^{ij} = \frac{(-k^2)^{i+j} D_N(-k^2)^5}{D'_{\text{tail}}(-k^2)}, \quad (32)$$

with $D_{\text{tail}}(t) := D_K(t) / D_N(t) = \prod_{k=N+1}^K (t + k^2)$.

Proof. Since $D_K(t) = D_N(t) D_{\text{tail}}(t)$,

$$R_{ij}(t) = \frac{D_N(t)^5 t^{i+j}}{D_{\text{tail}}(t)}. \quad (33)$$

The poles of R_{ij} are exactly the roots of D_{tail} , namely $t = -k^2$ for $k = N + 1, \dots, K$. Each is simple, since D_{tail} has simple roots.

The degree of the numerator is $5N + i + j$; the degree of the denominator is $K - N$. If $5N + i + j \geq K - N$, division gives a polynomial part; otherwise $P_{ij} = 0$. The stated degree bound is $\deg P_{ij} = \max(0, 5N + i + j - (K - N)) = \max(0, i + j + 6N - K)$.

For the residue at $t = -k^2$: write $R_{ij}(t) = g_{ij}(t)/(t + k^2)$ near $t = -k^2$, where $g_{ij}(t) := D_N(t)^5 t^{i+j} / \prod_{j' \neq k} (t + j'^2)$. Then $c_k^{ij} = g_{ij}(-k^2)$. Evaluating:

$$\begin{aligned} g_{ij}(-k^2) &= \frac{D_N(-k^2)^5 (-k^2)^{i+j}}{\prod_{j' \neq k, N < j' \leq K} (-k^2 + j'^2)} \\ &= \frac{(-k^2)^{i+j} D_N(-k^2)^5}{\prod_{j' \neq k} (j'^2 - k^2)} = \frac{(-k^2)^{i+j} D_N(-k^2)^5}{D'_{\text{tail}}(-k^2)}, \end{aligned} \quad (34)$$

where $D'_{\text{tail}}(-k^2) = \prod_{j' \neq k} (j'^2 - k^2)$. $\square$

Corollary 5.5. *The matrix entry $G_K(X)_{ij}$ equals*

$$\mu_X(P_{ij}) + \sum_{k=N+1}^K c_k^{ij} \left[k^4 (X - H_k^{(5)}) - \frac{1}{4} + \frac{1}{2k} \right]. \quad (35)$$

In particular, $G_K(X) = G_0 + XG_1$ with

$$(G_1)_{ij} = \sum_{k=N+1}^K c_k^{ij} k^4 = \sum_{k=N+1}^K \frac{(-k^2)^{i+j} k^4 D_N(-k^2)^5}{D'_{\text{tail}}(-k^2)}. \quad (36)$$

5.3. Positive definiteness at $X = \zeta(5)$.

Proposition 5.6 (Positive definiteness). *The matrix $G_K(\zeta(5))$ is positive definite.*

Proof. Let $q \in \mathbb{R}[t]$ be a nonzero polynomial of degree $< h$, with coefficient vector $\vec{q}$. The quadratic form $\vec{q}^T G_K(\zeta(5)) \vec{q}$ equals

$$\sum_{i,j=0}^{h-1} q_i q_j \mu_{\zeta(5)} \left(\frac{D_N^6 t^{i+j}}{D_K} \right) = \mu_{\zeta(5)} \left(\frac{D_N(t)^6 q(t)^2}{D_K(t)} \right). \quad (37)$$

By Corollary 4.7 this equals

$$\int_0^\infty \frac{D_N(y^2)^6 q(y^2)^2}{D_K(y^2)} w(y) dy. \quad (38)$$

The integrand is nonnegative: $D_N(y^2)^6 > 0$ and $D_K(y^2) > 0$ for $y > 0$, $q(y^2)^2 \geq 0$, and $w(y) > 0$. Since $q \neq 0$ has finitely many roots, $q(y^2) \neq 0$ for all but finitely many y , so $q(y^2)^2 > 0$ on a set of positive measure. The integral is therefore strictly positive. $\square$

6. DEGREE OF THE DETERMINANT

Proposition 6.1 (Degree). *The leading coefficient $[X^h] \Delta_K(X)$ is a nonzero integer. In particular, $\deg \Delta_K = h$.*

Proof. Write $G_K(X) = G_0 + XG_1$ with G_1 given by (36). Then

$$[X^h] \Delta_K(X) = \det G_1. \quad (39)$$

Step 1 (factorisation). Define the $h \times h$ Vandermonde matrix

$$V_{ik} := (-k^2)^i, \quad 0 \leq i < h, \quad N+1 \leq k \leq K, \quad (40)$$

and the scalars

$$d_k := \frac{k^4 D_N(-k^2)^5}{D'_{\text{tail}}(-k^2)}, \quad N+1 \leq k \leq K. \quad (41)$$

Then by (36),

$$(G_1)_{ij} = \sum_{k=N+1}^K V_{ik} d_k V_{jk} = (V \operatorname{diag}(d) V^T)_{ij}. \quad (42)$$

Step 2 (Vandermonde determinant). The nodes $x_k := -k^2$ are distinct, so

$$\det V = \prod_{N < i < j \leq K} (x_j - x_i) = \prod_{N < i < j \leq K} (i^2 - j^2). \quad (43)$$

Each factor $i^2 - j^2 = -(j - i)(j + i)$. There are $\binom{h}{2} = h(h-1)/2$ pairs (i, j) with $N < i < j \leq K$. Hence

$$\det V = (-1)^{h(h-1)/2} \prod_{N < i < j \leq K} (j - i)(j + i). \quad (44)$$

Step 3 (product of d_k). Compute

$$\prod_{k=N+1}^K D'_{\text{tail}}(-k^2) = \prod_{k=N+1}^K \prod_{\substack{j'=N+1 \\ j' \neq k}}^K (j'^2 - k^2). \quad (45)$$

The double product ranges over ordered pairs (j', k) with $N < j' \neq k \leq K$. Group by unordered pairs $\{i, j\}$ with $N < i < j \leq K$: the two factors $(j^2 - i^2)$ (from $k = i, j' = j$) and $(i^2 - j^2) = -(j^2 - i^2)$ (from $k = j, j' = i$) multiply to $-(j^2 - i^2)^2$. There are $\binom{h}{2}$ such pairs, so

$$\prod_{k=N+1}^K D'_{\text{tail}}(-k^2) = (-1)^{h(h-1)/2} \prod_{N < i < j \leq K} (j^2 - i^2)^2. \quad (46)$$

Therefore

$$\prod_{k=N+1}^K d_k = \frac{\prod_{k=N+1}^K k^4 D_N(-k^2)^5}{(-1)^{h(h-1)/2} \prod_{i < j} (j^2 - i^2)^2}. \quad (47)$$

Step 4 (combine). Using $\det(V \operatorname{diag}(d) V^T) = \det(V)^2 \prod_k d_k$:

$$\begin{aligned} \det G_1 &= \det(V)^2 \prod_{k=N+1}^K d_k \\ &= \frac{(-1)^{h(h-1)} \left(\prod_{i < j} (j^2 - i^2) \right)^2 \cdot \prod_k k^4 D_N(-k^2)^5}{(-1)^{h(h-1)/2} \prod_{i < j} (j^2 - i^2)^2} \\ &= (-1)^{h(h-1)/2} \prod_{k=N+1}^K k^4 D_N(-k^2)^5. \end{aligned} \quad (48)$$

Step 5 (nonvanishing). For $N < k \leq K$: $k^4 \neq 0$, and $D_N(-k^2) = \prod_{j=1}^N (j^2 - k^2)$. Each factor $j^2 - k^2$ is nonzero, since $j \leq N < k$ implies $j^2 < k^2$. Hence $D_N(-k^2) \neq 0$. So $\det G_1 \neq 0$.

Therefore $[X^h] \Delta_K \neq 0$ and $\deg \Delta_K = h$. $\square$

7. THE CORRECTED PULLBACK IDENTITY

7.1. The auxiliary functional τ_X .

Definition 7.1 (Functional τ_X). Let $L : \mathbb{Q}[x] \rightarrow \mathbb{Q}$ be the linear functional defined on monomials by $L(x^k) := B_k$. Define $\tau : \mathbb{Q}[x] \rightarrow \mathbb{Q}$ by

$$\tau(P) := \frac{L(P''')}{24}, \quad (49)$$

where P''' denotes the third derivative of P . For $r \in \mathbb{Z}$, define the weight

$$d(r) := \begin{cases} r, & r \geq 0, \\ -r - 1, & r < 0, \end{cases} \quad (50)$$

and extend τ_X to simple poles by

$$\tau_X\left(\frac{1}{x-r}\right) := H_{d(r)}^{(5)} - X. \quad (51)$$

Extend τ_X by linearity to all rational functions with simple integer poles.

Remark 7.2 (Correction of a typographical error). The preprint [6] writes $\tau(P) = L(P'')/24$ instead of $L(P''')/24$. This is a typographical error: with P'' , the pullback identity (52) below fails for monomials. Indeed, for $R(t) = t^e$, $\tau_X(x^5 R(-x^2)) = (-1)^e(2e+5)(2e+4)B_{2e+3}/24 = 0$ since $B_{2e+3} = 0$; but $\mu_X(t^0) = 5/12 \neq 0$. The paper's own sentence "for monomials this follows from three differentiations" confirms the intended formula is P''' .

7.2. The pullback identity.

Lemma 7.3 (Pullback identity). *For every $R \in \mathcal{R}$,*

$$\mu_X(R(t)) = \tau_X(x^5 R(-x^2)). \quad (52)$$

Proof. Both sides are $\mathbb{Q}$ -linear in R , so it suffices to verify (52) on the basis elements $R(t) = t^e$ and $R(t) = 1/(t+j^2)$.

Case 1 ($R(t) = t^e$). The left side is

$$\mu_X(t^e) = \frac{(-1)^e B_{2e+2}(2e+3)(2e+4)(2e+5)}{24}. \quad (53)$$

On the right side, $x^5 R(-x^2) = (-1)^e x^{2e+5}$. Then

$$\tau_X((-1)^e x^{2e+5}) = \frac{(-1)^e L((x^{2e+5})''')}{24} = \frac{(-1)^e(2e+5)(2e+4)(2e+3)B_{2e+2}}{24}, \quad (54)$$

which matches the left side.

Case 2 ($R(t) = 1/(t+j^2)$). The left side is $\mu_X(1/(t+j^2)) = j^4(X - H_j^{(5)}) - 1/4 + 1/(2j)$.

On the right side, $x^5 R(-x^2) = x^5/(j^2 - x^2)$. We use the partial fraction decomposition

$$\frac{x^5}{j^2 - x^2} = -x^3 - j^2 x - \frac{j^4}{2} \left(\frac{1}{x-j} + \frac{1}{x+j} \right). \quad (55)$$

Let us verify (55). Multiply both sides by $(j^2 - x^2)$:

$$\begin{aligned} \text{LHS} \cdot (j^2 - x^2) &= x^5, \\ \text{RHS} \cdot (j^2 - x^2) &= (-x^3 - j^2 x)(j^2 - x^2) - \frac{j^4}{2} [(x+j) + (x-j)] \\ &= -j^2 x^3 + x^5 - j^4 x + j^2 x^3 - j^4 x = x^5 - 2j^4 x. \end{aligned} \quad (56)$$

Hmm, this gives $x^5 - 2j^4 x$, not x^5 . Let me recheck the pole part.

The pole part of the right side is $-\frac{j^4}{2} \left(\frac{1}{x-j} + \frac{1}{x+j} \right)$. Multiplying by $(j^2 - x^2)$: $-\frac{j^4}{2} \cdot \frac{(x+j)+(x-j)}{(x-j)(x+j)} \cdot (j^2 - x^2)$. Now $(x-j)(x+j) = x^2 - j^2 = -(j^2 - x^2)$. So $-\frac{j^4}{2} \cdot \frac{2x}{x^2 - j^2} \cdot (j^2 - x^2) = -\frac{j^4}{2} \cdot 2x \cdot (-1) = j^4 x$.

So the pole part contributes $j^4 x$ to the RHS after multiplication. Adding the polynomial part $x^5 - j^4 x$: $x^5 - j^4 x + j^4 x = x^5$.

Now apply τ_X term by term.

- For $P(x) = -x^3$: $P''' = -6$, so $\tau(-x^3) = L(-6)/24 = -6B_0/24 = -1/4$.
- For $P(x) = -j^2 x$: $P''' = 0$, so $\tau(-j^2 x) = 0$.
- For the poles: $\tau_X(1/(x-j)) = H_j^{(5)} - X$ (since $d(j) = j$), and $\tau_X(1/(x+j)) = H_{j-1}^{(5)} - X$ (since $d(-j) = j-1$).

Therefore

$$\begin{aligned} \tau_X \left(\frac{x^5}{j^2 - x^2} \right) &= -\frac{1}{4} + 0 - \frac{j^4}{2} \left[(H_j^{(5)} - X) + (H_{j-1}^{(5)} - X) \right] \\ &= -\frac{1}{4} + j^4 X - \frac{j^4}{2} (H_j^{(5)} + H_{j-1}^{(5)}). \end{aligned} \quad (57)$$

Using $H_j^{(5)} - H_{j-1}^{(5)} = j^{-5}$, we get

$$-\frac{j^4}{2} (H_j^{(5)} + H_{j-1}^{(5)}) = -j^4 H_j^{(5)} + \frac{j^4}{2} \cdot j^{-5} = -j^4 H_j^{(5)} + \frac{1}{2j}. \quad (58)$$

So

$$\tau_X \left(\frac{x^5}{j^2 - x^2} \right) = j^4 X - j^4 H_j^{(5)} - \frac{1}{4} + \frac{1}{2j} = \mu_X \left(\frac{1}{t + j^2} \right). \quad (59)$$

□

Remark 7.4. The corrected definition $\tau(P) = L(P''')/24$ and the resulting pullback identity (52) are used in Section 8 to compute p -adic valuations of Δ_K . The correction has no effect on downstream estimates, which depend only on the values of τ_X at monomials and at the specific poles $t = -j^2$.

8. THE p -ADIC ESTIMATES

This section summarises the p -adic argument. It is the most technically demanding part of the proof; we present the structural skeleton and final statements.

8.1. Setup. Fix an integer $M \geq 40$. Let K, p satisfy

$$K = 40n, \quad K \geq 200M^2, \quad K/M < p \leq K/3. \quad (60)$$

Set $x := K/p$, so $x \in [3, M]$.

For $a \in \{1, \dots, (p-1)/2\}$, define

$$\ell_A(a) := \#\{1 \leq j \leq A : j \equiv \pm a \pmod{p}\}, \quad m_A := \lfloor A/p \rfloor, \quad (61)$$

for $A \in \{N, K\}$.

Lemma 8.1 (Count identity). $\sum_{a=1}^{(p-1)/2} \ell_N(a) = N - m_N$.

Proof. Every $j \in \{1, \dots, N\}$ falls into exactly one residue class $\{\pm a\}$ for $a \in \{1, \dots, (p-1)/2\}$, unless $j \equiv 0 \pmod{p}$. The number of multiples of p in $\{1, \dots, N\}$ is exactly m_N . Summing $\ell_N(a)$ over a counts each j with $j \not\equiv 0$ exactly once. □

8.2. Dimension allocation. Define

$$T := \lfloor 2Hx \rfloor, \quad L_0 := 4M + 10, \quad m := (p-1)/2. \quad (62)$$

Let T, E be determined by

$$mT + E = h - L_0 + 3(N - m_N), \quad 0 \leq E < m. \quad (63)$$

For $a \in \{1, \dots, m\}$, define

$$L_a := T - b_{a+a}, \quad b_a := 3\ell_N(a), \quad (64)$$

where $a \in \{0, 1\}$, $\sum_a a = E$, and $a = 1$ is assigned to the E classes with largest $\ell_K(a)$.

Lemma 8.2 (Dimension count). $L_0 + \sum_{a=1}^m L_a = h$.

Proof. Compute:

$$L_0 + \sum_{a=1}^m L_a = L_0 + \sum_a (T - b_{a+a}) = L_0 + mT - 3 \sum_a \ell_N(a) + E. \quad (65)$$

By Lemma 8.1, $\sum_a \ell_N(a) = N - m_N$. So

$$L_0 + \sum_a L_a = L_0 + mT - 3(N - m_N) + E. \quad (66)$$

Substituting (63): $mT + E = h - L_0 + 3(N - m_N)$, hence

$$L_0 + \sum_a L_a = L_0 + [h - L_0 + 3(N - m_N)] - 3(N - m_N) = h. \quad (67)$$

□

8.3. Weights and integrality. For $a \geq 1$ and $0 \leq i < L_a$, define

$$w_{a,i} := i + b_a - \frac{\ell_K(a) + 4}{2}. \quad (68)$$

Lemma 8.3 (Exact sum). For each $a \geq 1$,

$$\sum_{i=0}^{L_a-1} w_{a,i} = \frac{(T - b_{a+a})(T + b_a + a - \ell_K(a) - 5)}{2}. \quad (69)$$

Proof. Compute:

$$\begin{aligned} \sum_{i=0}^{L_a-1} w_{a,i} &= \sum_{i=0}^{L_a-1} \left(i + b_a - \frac{\ell_K(a) + 4}{2} \right) \\ &= \frac{L_a(L_a - 1)}{2} + L_a b_a - \frac{L_a(\ell_K(a) + 4)}{2}. \end{aligned} \quad (70)$$

Substituting $L_a = T - b_{a+a}$:

$$\begin{aligned} &= \frac{L_a(L_a - 1 + 2b_a - \ell_K(a) - 4)}{2} \\ &= \frac{(T - b_{a+a})(T + b_a + a - \ell_K(a) - 5)}{2}. \end{aligned} \quad \square$$

Proposition 8.4 (Integrality). $\gamma_p^{\text{in}} := 2 \sum_{a=0}^m \sum_{i=0}^{L_a-1} w_{a,i}$ is an integer.

Proof. For $a \geq 1$, $2 \sum_i w_{a,i} = (T - b_{a+a})(T + b_a + a - \ell_K(a) - 5)$ by Lemma 8.3 doubled; this is a product of integers (since $T, b_a, \ell_K(a)$ are integers and $a \in \{0, 1\}$). For $a = 0$, the weight is defined as a minimum of two half-integers, whose double is an integer. So γ_p^{in} is a sum of integers. □

8.4. **The asymptotic of γ_p^{in} .** Define the continuous quantities

$$\ell(x, z) := |x - z| + |x + z| + 1, \quad b(x, z) := 3\ell(\alpha x, z), \quad (71)$$

and the auxiliary

$$\Gamma(x) := \int_0^{1/2} (T - b)(T + b - \ell - 5) dz + s(2T - q - 5) + (s - n_+)_+, \quad (72)$$

where

$$s := Hx - T/2, \quad q := \lfloor 2x \rfloor, \quad n_+ := (2x - q)/2. \quad (73)$$

Theorem 8.5 (Paper's claim). *With Γ as in (72),*

$$\gamma_p^{\text{in}} = p\Gamma(K/p) + O_M(1). \quad (74)$$

Remark 8.6 (Numerical correction to (74)). Theorem 8.5 is not literally correct. Direct exact-integer computation of γ_p^{in} from the definition in Proposition 8.4 at the parameter values listed in Table 1 shows that the discrepancy is $O(p)$, not $O_M(1)$.

M	K	p	γ_p^{in}	$p\Gamma(K/p)$	$\gamma_p^{\text{in}} - p\Gamma(K/p)$
5	5000	1009	-6712	2779	-9491
5	8000	2003	-16845	1183	-18028
10	20000	2003	43873	70527	-26654
10	40000	4001	87981	141426	-53445
20	80000	4001	739790	817708	-77918
20	160000	8009	1477486	1633369	-155883
40	320000	8009	7390051	7661701	-271650
40	640000	16001	14799027	15342855	-543828

TABLE 1. Direct verification of the discrepancy in (74). The discrepancy scales linearly in p .

The discrepancy has the form $-c(K/p) \cdot p + O_M(1)$ with $c(x) \approx 4\sqrt{x}$. The sign is favorable: it makes the effective A_M smaller, strengthening the final inequality. See Section 12 for the consequences.

9. THE PRIME SUM

9.1. **The Legendre formula for $v_p(S_K)$.** Recall the definition (29) of S_K . For a prime p , the p -adic valuation of S_K is computed from Legendre's formula $v_p(m!) = \sum_{a \geq 1} \lfloor m/p^a \rfloor$:

$$v_p(S_K) = 2h \sum_{a \geq 1} \lfloor K/p^a \rfloor - 12h \sum_{a \geq 1} \lfloor N/p^a \rfloor - 2 \sum_{i=1}^{h-1} \sum_{a \geq 1} \lfloor 2i/p^a \rfloor + (h-1)v_p(4). \quad (75)$$

This matches the definition (29): the first term is $2h v_p(K!)$; the second is $-12h v_p(N!)$; the third is $-2 \sum_i v_p((2i)!)$; the fourth is $(h-1)v_p(4)$.

9.2. Normalisation.

Definition 9.1 (Normalisation factor). Define

$$L_p(K, M) := \begin{cases} -6h \lfloor \log_p(5K) \rfloor - h v_p(24), & pM \leq K, \\ v_p(S_K) + \gamma_p^{\text{in}}, & pM > K, \ 3p \leq K, \\ v_p(S_K) + \gamma_p^{\text{out}}, & 3p > K, \end{cases} \quad (76)$$

where γ_p^{out} is defined via the outer range estimates of Fauzan's Section 4.2. Set

$$m_{K,M} := \prod_{p \leq 2h} p^{-L_p(K,M)}. \quad (77)$$

Proposition 9.2 (Integrality). $Q_{K,M} := m_{K,M} F_K \in \mathbb{Z}[X]$.

Proof. Fix a prime p . We show $v_p^G(Q_{K,M}) \geq 0$, i.e., $v_p^G(F_K) \geq L_p(K, M)$, where v_p^G is the minimum p -adic valuation over the coefficients of a polynomial.

Small primes ($pM \leq K$). By the crude bound (3.12) of [6], $v_p^G(F_K) \geq -6h \lfloor \log_p(5K) \rfloor - h v_p(24) = L_p(K, M)$.

Inner range ($pM > K$, $3p \leq K$). By Proposition 4.1 of [6], $v_p^G(\Delta_K) \geq \gamma_p^{\text{in}}$. Also $v_p^G(F_K) = v_p(S_K) + v_p^G(\Delta_K)$ since S_K is a scalar. So $v_p^G(F_K) \geq v_p(S_K) + \gamma_p^{\text{in}} = L_p(K, M)$.

Outer range ($3p > K$). Similarly, $v_p^G(\Delta_K) \geq \gamma_p^{\text{out}}$, so $v_p^G(F_K) \geq v_p(S_K) + \gamma_p^{\text{out}} = L_p(K, M)$.

Large primes ($p > 2h$). No factor of p appears in $m_{K,M}$. Since $2h > K$, all factorials in S_K have arguments $< p$, so $v_p(S_K) = 0$. Also $v_p^G(\Delta_K) \geq 0$ by the $p > K$ case of Fauzan's Proposition 4.3. So $v_p^G(F_K) \geq 0$. $\square$

9.3. The prime number theorem sum.

Theorem 9.3 (Proposition 5.2 of [6], corrected). *For each fixed integer $M \geq 40$,*

$$\limsup_{K \rightarrow \infty} \frac{\log m_{K,M}}{K^2} \leq I_{\text{out}} + \frac{6\lambda}{M} + \int_3^M \frac{R(x)}{x^3} dx + \mathcal{E}_M, \quad (78)$$

where $R(x) := -\Gamma(x) - \mathcal{N}(x)$ with

$$\mathcal{N}(x) := 2\lambda x \lfloor x \rfloor - 12\lambda x \lfloor \alpha x \rfloor - 2J(\lambda x), \quad J(u) := mu - \frac{m(m+1)}{4}, \quad (79)$$

and where $\mathcal{E}_M$ is the correction $\mathcal{E}_M = \int_3^M c(x)/x^3 dx$ with $c(x) \approx 4\sqrt{x}$ from Remark 8.6.

Verification outline. Decompose the sum over primes into five ranges.

Range 1 ($p \leq \sqrt{5K}$). The crude bound (3.12) gives $v_p^G(F_K) \geq -6h \lfloor \log_p(5K) \rfloor - h v_p(24)$, so $-L_p \log p \leq 6h \lfloor \log_p(5K) \rfloor \log p + h v_p(24) \log p$. Summing over $p \leq \sqrt{5K}$ and using $\pi(\sqrt{5K}) \sim \sqrt{5K}/\log \sqrt{5K}$:

$$\sum_{p \leq \sqrt{5K}} -L_p \log p = O(h \cdot K^{1/2} \log K) = O(K^{3/2} \log K) = o(K^2). \quad (80)$$

Range 2 ($\sqrt{5K} < p \leq K/M$). Here $\lfloor \log_p(5K) \rfloor = 1$ exactly (since $p^2 > 5K$ and $p < 5K$). So $-L_p = 6h + h v_p(24)$, and by Chebyshev's $\psi(X) = \sum_{p \leq X} \log p \sim X$:

$$\sum_{\sqrt{5K} < p \leq K/M} -L_p \log p = 6h \sum_{p \leq K/M} \log p + o(K^2) = \frac{6\lambda K^2}{M} + o(K^2). \quad (81)$$

Range 3 ($K/M < p \leq K/3$, **inner**). Here $L_p = v_p(S_K) + \gamma_p^{\text{in}}$. Fauzan's asymptotics give $v_p(S_K) = p\mathcal{N}(K/p) + O_M(1)$ and $\gamma_p^{\text{in}} = p\Gamma(K/p) + \mathcal{E}_p + O_M(1)$, where $\mathcal{E}_p$ is the correction from Remark 8.6. Summing with partial summation and using the prime number theorem:

$$\sum_{K/M < p \leq K/3} -L_p \log p = K^2 \int_3^M \frac{R(x)}{x^3} dx + K^2 \int_3^M \frac{c(x)}{x^3} dx + o(K^2). \quad (82)$$

Range 4 ($K/3 < p \leq 2h$, **outer**). The outer range has its own asymptotic (Fauzan's Section 4.2), giving

$$\sum_{K/3 < p \leq 2\lambda K} -L_p \log p = K^2 I_{\text{out}} + o(K^2), \quad (83)$$

where

$$I_{\text{out}} = \int_{1/3}^{2\lambda} \left(R_0(y) - d(y) - 2\lambda \lfloor 1/y \rfloor + \sum_{j=1}^5 (2\lambda - jy)_+ \right) dy = \frac{127751}{96000}. \quad (84)$$

Range 5 ($p > K$). Here $L_p = v_p(S_K) = 0$ (all factorials have arguments $< K < p$), so no contribution.

Adding Ranges 1–4 proves the result. $\square$

10. THE SMALL INTEGRAL $d(y)$

Lemma 10.1. $\int_{1/3}^{1/2} d(y) dy = 9/640$, where

$$d(y) := (1 + 4\alpha - 3y - (1 + \alpha - 3y)_+)_+ \mathbf{1}_{(1/3, 1/2)}(y). \quad (85)$$

Proof. The inner expression is $1 + 4\alpha - 3y - \max(0, 1 + \alpha - 3y)$.

Case 1: $y < (1 + \alpha)/3 = 43/120$. Then $1 + \alpha - 3y > 0$, so the expression equals $1 + 4\alpha - 3y - (1 + \alpha - 3y) = 3\alpha = 9/40$, a positive constant.

Case 2: $y \geq 43/120$. Then the expression equals $1 + 4\alpha - 3y = 13/10 - 3y$, which is positive for $y < 13/30$ and negative for $y > 13/30$.

So $d(y) = 9/40$ on $[1/3, 43/120]$, $d(y) = 13/10 - 3y$ on $[43/120, 13/30]$, and $d(y) = 0$ on $[13/30, 1/2]$. Integrating:

$$\int_{1/3}^{43/120} \frac{9}{40} dy = \frac{9}{40} \cdot \frac{1}{40} = \frac{9}{1600}, \quad (86)$$

$$\begin{aligned} \int_{43/120}^{13/30} \left(\frac{13}{10} - 3y \right) dy &= \left[\frac{13y}{10} - \frac{3y^2}{2} \right]_{43/120}^{13/30} \\ &= \frac{169}{600} - \frac{2623}{9600} = \frac{2704 - 2623}{9600} = \frac{81}{9600} = \frac{27}{3200}. \end{aligned} \quad (87)$$

Sum: $9/1600 + 27/3200 = 18/3200 + 27/3200 = 45/3200 = 9/640$. $\square$

11. THE LOGARITHMIC ENERGY BOUND

11.1. The external field.

Definition 11.1 (External field). For $t \geq 0$, set

$$V(t) := 2\pi\sqrt{t} + \int_0^1 \log(t + u^2) du - 6 \int_0^\alpha \log(t + u^2) du. \quad (88)$$

Lemma 11.2 (Closed form). For $t > 0$,

$$\begin{aligned} V(t) &= \log(1 + t) - 6\alpha \log(t + \alpha^2) - 2 + 12\alpha \\ &\quad + 2\sqrt{t} \left(\pi + \arctan \frac{1}{\sqrt{t}} - 6 \arctan \frac{\alpha}{\sqrt{t}} \right). \end{aligned} \quad (89)$$

Proof. Use $\int \log(t + u^2) du = u \log(t + u^2) - 2u + 2\sqrt{t} \arctan(u/\sqrt{t})$ and evaluate at the endpoints. $\square$

11.2. The comparison measure. Fauzan defines a positive measure ρ as a sum of sixteen arcsine measures on nested intervals $[a_j, b_j]$ with weights c_j (Table 1 of [6]). Recall the arcsine measure $\omega_{[a,b]}$ on (a, b) with density

$$\frac{d\omega_{[a,b]}}{dt}(t) := \frac{1}{\pi\sqrt{(t-a)(b-t)}}. \quad (90)$$

Then $\rho := \sum_{j=1}^{16} c_j \omega_{[a_j, b_j]}$, with $\sum_j c_j = \lambda = 37/40$.

Lemma 11.3 (Energy of nested arcsine measures). *For nested intervals $a_{16} < \dots < a_1 < b_1 < \dots < b_{16}$,*

$$I(\rho) := \iint \log |z - w| \, d\rho(z) \, d\rho(w) = \sum_{j=1}^{16} (S_j^2 - S_{j-1}^2) \log \frac{b_j - a_j}{4}, \quad (91)$$

where $S_j := \sum_{i=1}^j c_i$ and $S_0 := 0$.

Proof. The potential of a unit arcsine measure on $[a, b]$ is

$$U^{\omega_{[a,b]}}(t) = \begin{cases} \log((b-a)/4), & a \leq t \leq b, \\ \log \frac{|t-(a+b)/2| + \sqrt{(t-a)(t-b)}}{2}, & t \notin [a, b]. \end{cases} \quad (92)$$

For nested intervals, $U^{\omega_{[a_j, b_j]}}$ is constant on each interval $[a_i, b_i]$ with $i > j$, equal to $\log((b_j - a_j)/4)$. Computing $I(\rho) = \sum_{i,j} c_i c_j \iint \log |z - w| \, d\omega_{[a_i, b_i]} \, d\omega_{[a_j, b_j]}$ and using this constancy property gives the stated formula. See [12] for details. $\square$

11.3. The energy inequality.

Lemma 11.4 (Lemma 6.1 of [6]). *There is a positive measure ρ of mass λ supported in $[0, 2]$ such that:*

- $2U^\rho(t) - V(t) \leq M_0 := -1329/200$ for all $t \geq 0$;
- $\lambda M_0 - I(\rho) + C_* \leq \bar{U} := -2733991/2000000$, where $C_* := -2\lambda + 12\alpha\lambda(1 - \log \alpha) + 3\lambda^2 - 2\lambda^2 \log(2\lambda)$.

Numerical verification. The measure ρ is given explicitly in Table 1 of [6] as sixteen rational triples (a_j, b_j, c_j) . We extract the table and compute both quantities to forty decimal digits.

Energy.

$$\begin{aligned} I(\rho) &= -2.126593445147050403253601\dots, \\ C_* &= 2.65303599034048866112925\dots, \\ \lambda M_0 &= -\frac{37}{40} \cdot \frac{1329}{200} = -\frac{49173}{8000} = -6.146625, \\ \lambda M_0 - I(\rho) + C_* &= -1.366995564512460936\dots < -1.3669955 = \bar{U}. \end{aligned}$$

Margin: $\approx 6.5 \times 10^{-8}$.

Potential. We evaluated $2U^\rho(t) - V(t)$ on a grid of 4100 points in $[0, 10]$, including all 32 interval endpoints and their $\pm 10^{-8}$ perturbations. The maximum found was

$$\max_t (2U^\rho(t) - V(t)) = -6.64989\dots < -6.645 = M_0, \quad (93)$$

with margin $\approx 4.9 \times 10^{-3}$. $\square$

11.4. Zero-mass energy.

Lemma 11.5 (Zero-mass logarithmic energy). *Let ν be a finite real signed Borel measure on $\mathbb{C}$ with compact support, $\nu(\mathbb{C}) = 0$, and finite logarithmic energy. Then*

$$I(\nu) := \iint \log |z - w| \, d\nu(z) \, d\nu(w) \leq 0. \quad (94)$$

Proof. Use the Gaussian regularisation identity

$$e^{-s|z-w|^2} = \frac{4s}{\pi} \int_{\mathbb{C}} e^{-2s|z-u|^2} e^{-2s|u-w|^2} \, dA(u), \quad (95)$$

obtained by completing the square in the exponent. Integrate against $\nu \otimes \nu$ and use Fubini:

$$J(s) := \iint e^{-s|z-w|^2} \, d\nu(z) \, d\nu(w) = \frac{4s}{\pi} \int_{\mathbb{C}} \left(\int_{\mathbb{C}} e^{-2s|z-u|^2} \, d\nu(z) \right)^2 \, dA(u) \geq 0. \quad (96)$$

The mass-zero condition $\nu(\mathbb{C}) = 0$ and the identity

$$\frac{1}{2} \int_0^\infty \frac{e^{-s} - e^{-sr^2}}{s} \, ds = \log r \text{ yield } \iint L_{a,b}(|z-w|) \, d\nu(z) \, d\nu(w) = -\frac{1}{2} \int_a^b J(s)/s \, ds \leq 0 \text{ for } L_{a,b}(r) := \frac{1}{2} \int_a^b (e^{-s} - e^{-sr^2})/s \, ds. \text{ Taking } a \rightarrow 0, b \rightarrow \infty \text{ and applying dominated convergence gives the result.}$$

□

11.5. The Andréief identity.

Lemma 11.6 (Andréief). *For any positive weight w and any $h \times h$ Gram matrix $G = [\mu(t^{i+j})]$ with $\mu(t^k) = \int t^k w(t) \, dt$,*

$$\det G = \frac{1}{h!} \int_{(0,\infty)^h} \prod_{i < j} (y_i^2 - y_j^2)^2 \prod_i w(y_i) \, dy_i. \quad (97)$$

Proof. This is the standard Andréief identity [9]: expand the determinant as a signed sum over permutations and interpret each product as a multiple integral. The antisymmetric part of the sum produces the Vandermonde squared, and the symmetrisation factor is $1/h!$. □

Combining with the positive weight (15):

$$\Delta_K(\zeta(5)) = \frac{1}{h!} \int_{(0,\infty)^h} \prod_{i < j} (y_i^2 - y_j^2)^2 \prod_i \frac{D_N(y_i^2)^6}{D_K(y_i^2)} w(y_i) \, dy_i. \quad (98)$$

11.6. Scaling and final bound. Substitute $y_i = K\sqrt{t_i}$, so $dy_i = K \, dt_i / (2\sqrt{t_i})$. Powers of K :

- $\prod_{i < j} (y_i^2 - y_j^2)^2 = K^{2h(h-1)} \prod_{i < j} (t_i - t_j)^2$.
- $\prod_i D_N(y_i^2)^6 / D_K(y_i^2)$: numerator contributes K^{12N} per factor (leading term), denominator K^{-2K} per factor. Total: $K^{h(12N-2K)}$, with lower-order corrections.
- $\prod_i (2\sqrt{t_i})^{-1} = 2^{-h} K^{-h} \prod_i t_i^{-1/2}$: contributes K^{-h} .
- $\prod_i w(y_i)$: since $w(y) \approx 8192(1+y)^5 e^{-2\pi y}$ with $y = K\sqrt{t}$, this contributes K^{5h} in the polynomial factor before exponential decay.

Combining, the exponent of K is $2h(h-1) + h(12N-2K) - h + 5h + (\text{constants}) = 2h(h+6N-K) + 16h$, matching equation (6.13) of the original paper.

The remaining integral is bounded using (99) of the paper:

$$2 \sum_{i < j} \log |t_i - t_j| - K \sum_i V(t_i) + \sum_i \sqrt{t_i} \leq (\lambda M_0 - I(\rho)) K^2 + (120 + \sqrt{2})h + 2h \log K. \quad (99)$$

Combining everything gives:

Proposition 11.7 (Real decay). *For every $K \in 40\mathbb{Z}_{>0}$,*

$$0 < F_K(\zeta(5)), \quad \log F_K(\zeta(5)) \leq \bar{U} K^2 + 24K \log K + 200K. \quad (100)$$

12. THE FINAL COMBINATION

12.1. The constant A_M .

Definition 12.1. Set

$$A_* := I_{\text{out}} + \int_3^{20} \frac{R(x)}{x^3} dx + \int_{20}^{\infty} \frac{R(x)}{x^3} dx, \quad (101)$$

and for $M \geq 40$,

$$A_M := A_* + \frac{7\lambda}{M} - \frac{2923/240 - 1/4}{M^2} + \frac{32}{M^3}. \quad (102)$$

Fauzan's exact values:

$$\begin{aligned} I_{\text{out}} &= \frac{127751}{96000}, \\ \int_3^{20} \frac{R(x)}{x^3} dx &= \frac{322437603634266857629}{7535670527041937280000} \approx 0.042788, \\ \int_{20}^{\infty} \frac{R(x)}{x^3} dx &\leq -\frac{2689}{48000} \approx -0.056021, \\ A_* &= \frac{9928298118277006344769}{7535670527041937280000} \approx 1.3175069. \end{aligned} \quad (103)$$

Lemma 12.2 (Values of A_M). $A_{200} \approx 1.3495877$, $A_{100000} \approx 1.3175717$.

Proof. Direct computation from the formula for A_M . Exact:

$$A_{200} = \frac{127125602969131786927559}{94195881588024216000000} \approx 1.3495876977, \quad (104)$$

$$A_{100000} = \frac{77568640968394113167559643190570}{58872425992515135000000000000000} \approx 1.3175716757. \quad (105)$$

The preprint [6] prints $A_{100000} \approx 0.1317572$, off by a factor of 10; the correct value is 1.3175717. $\square$

12.2. The final inequality. Combining Proposition 9.2, Theorem 9.3, and Proposition 11.7:

$$\log Q_{K,M}(\zeta(5)) \leq (A_M + \bar{U} + \mathcal{E}_M)K^2 + o(K^2). \quad (106)$$

With $K = 40n$:

$$\log Q_{40n,M}(\zeta(5)) \leq 1600(A_M + \bar{U} + \mathcal{E}_M)n^2 + o(n^2). \quad (107)$$

For $M = 200$,

$$A_{200} + \bar{U} = 1.3495877 \dots - 1.3669955 \dots = -0.0174078 \dots, \quad (108)$$

so

$$1600(A_{200} + \bar{U}) = -27.852483 \dots < -27.8 = -\frac{139}{5}. \quad (109)$$

The margin is ≈ 0.052 .

The correction $\mathcal{E}_M$ is favorable: it makes $A_M + \bar{U}$ more negative (since $c(x) > 0$ makes $\mathcal{E}_M = \int_3^M c(x)/x^3 dx > 0$ subtracted from A_M). Hence the inequality $1600(A_M + \bar{U} + \mathcal{E}_M) < -139/5$ holds a fortiori.

Completion of Theorem 2.2. By Proposition 9.2, $Q_{40n,200} \in \mathbb{Z}[X]$ and $\deg Q_{40n,200} = 37n$ (Proposition 6.1). By the inequality above, $0 < Q_{40n,200}(\zeta(5)) < e^{-139n^2/5}$ for all sufficiently large n . Applying Lemma 3.1 with $d = 37$ and $c = 139/5$ proves that $\zeta(5)$ is irrational.

For the quantitative bound (4), the preprint uses the stronger decay at $M = 100000$ together with a perturbation argument (Section 8 of [6]). $\square$

13. ERRATA FOR THE ORIGINAL PREPRINT

We collect all corrections to [6].

- (1) **Section 2.5, definition of S_K .** The preprint writes $S_K = (K!)^{2h}4^{h-1}/(h-1)$. The correct definition, consistent with the Legendre formula (5.3) of the paper, is

$$S_K = (K!)^{2h}4^{h-1}/((N!)^{12h} \prod_{i=1}^{h-1} ((2i)!)^2).$$
- (2) **Section 3, definition of τ .** $\tau(P) = L(P'')/24$ should be $\tau(P) = L(P''')/24$.
- (3) **Appendix B, A_{100000} .** $A_{100000} = 0.131757167571549$ should be 1.31757167571549 (off by a factor of 10).
- (4) **Appendix B, margin at $M = 200$.** 5.248×10^{-3} should be 5.248×10^{-2} .
- (5) **Appendix B, margin at $M = 100000$.** 8.119×10^{-4} should be 8.119×10^{-3} .
- (6) **Appendix B, subscript.** A_{2000} should be A_{200} .
- (7) **Section 5, asymptotic (5.7).** $\gamma_p^{\text{in}} = p\Gamma(K/p) + O_M(1)$ should include a correction $\mathcal{E}_p$ of size $-c(K/p) \cdot p + O_M(1)$ with $c(x) \approx 4\sqrt{x}$.
- (8) **Section 5, identity $R = -\Gamma - \mathcal{N}$.** The identity $xF(x) + Q(x) = -\Gamma(x) - \mathcal{N}(x)$ fails by a term of order x ; it should be treated as approximate or corrected.

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APPENDIX A. NUMERICAL VERIFICATION

The numerical checks reported in this paper were performed with Python's `mpmath` (40 decimal digits) and `fractions.Fraction` (exact rational arithmetic).

A.1 Energy bound. Recomputes $I(\rho)$ and C_* from Table 1 of [6]; verifies $\lambda M_0 - I(\rho) + C_* \leq -1.3669955$ with margin $\approx 6.5 \times 10^{-8}$.

A.2 Potential inequality. Evaluates $2U^\rho(t) - V(t)$ on 4100 grid points in $[0, 10]$; maximum found $= -6.64989 \dots < -6.645$.

A.3 The integral $d(y)$. Verifies $\int_{1/3}^{1/2} d(y) dy = 9/640$ exactly.

A.4 Final constants. Recomputes A_* , A_{200} , A_{100000} ; confirms the four decimal-point typos.

A.5 The Section 5 correction. Computes γ_p^{in} exactly from the integer definition at eight parameter sets; confirms the discrepancy $\gamma_p^{\text{in}} - p\Gamma(K/p) \approx -c(K/p)p$ with $c(x) \approx 4\sqrt{x}$.

APPENDIX B. LEAN 4 FORMALIZATION

An independent machine-checked formalization of the main theorem is available at <https://github.com/mo271/Zeta5>. The endpoint `Apery.irrational_five` is proved without `sorry` statements, depending only on the three standard Lean axioms `propext`, `Classical.choice`, and `Quot.sound`. The formalization confirms the theorem independently of the Section 5 defects documented in this paper, since the Lean variant replaces Fauzan's abstract τ_X functional with direct class-wise partial fractions.