

The Expected Time-Cost of Equipment Enhancement in *Milky Way Idle*:

An Exact Green’s Function and Path-Integral Analysis

Lukas A. Saul*

September 29, 2026

Abstract

We develop an exact, closed-form statistical mechanics of the equipment-enhancement system of the idle RPG *Milky Way Idle*. An enhancement attempt advances an item by one level with probability p and, on failure, either resets it to level 0 (protection off) or decrements it by one level at the cost of a consumable protection item (protection on, from a chosen level R upward). We compute the expected total cost $\mathbb{E}[C] = T\mathbb{E}[\tau] + L\mathbb{E}[M]$, where T is the per-attempt time cost, L the per-protection-item cost, τ the hitting time of a target level N , and M the number of protected failures incurred. The central object is the expected visit-count vector v_k , which we identify as the discrete Green’s function of a piecewise birth–death operator and solve in closed form in terms of p , N , and R alone. We recover the unprotected limit $C_n = (p^{-n} - 1)/(1 - p)$ as a special case, matching the result of Chae and Lee. We then present the full path-integral representation of the process as a sum over random-walk histories weighted by $e^{-\beta C}$, give the transfer-operator resolvent $G(x, y) = [(I - W(x, y))^{-1}]_{0, N}$, and analyse the continuum limit: the WKB/instanton action $S_{\text{inst}} = N \ln(1/\sigma)$ reproduces the exact exponent σ^{-N} , while Gaussian fluctuations around the instanton yield an erfc-form budget tail $\mathbb{P}(C \leq B) \approx \frac{1}{2} \text{erfc}((\mathbb{E}[C] - B)/\sqrt{2 \text{Var}[C]})$. All exact formulae are rational in p and are tabulated for the base success rates of the game.

Contents

1	Introduction	2
2	The Enhancement Process	3
2.1	State space and transitions	3
2.2	Cost functional and generating function	3
3	The Expected Visit-Count Vector	4
3.1	Balance equation	4
3.2	Zone I: the reset region ($0 \leq k \leq R$)	4
3.3	Zone II: the protected region ($R \leq k \leq N - 1$)	5
3.4	The visit-count vector in closed form	6
3.5	The boundary case $R = 0$	6

*合肥工业大学, Vermont Secure Computing. Correspondence: saul@hfut.edu.cn.

4	Expected Attempts and Protection Items	6
4.1	Expected attempts	6
4.2	Expected protection items	7
4.3	Total expected cost	7
4.4	Sanity checks	8
5	The Path-Integral Representation	8
5.1	Path space and path measure	8
5.2	Transfer operator and partition function	9
5.3	Thermodynamic form	9
5.4	Spectral representation	9
6	Continuum Limit and the Saddle-Point Expansion	10
6.1	Diffusion limit	10
6.2	Instanton action	10
6.3	Gaussian fluctuations and the erfc tail	10
6.4	Exact integral representation of the tail	11
7	Optimisation of the Protection Start Level	11
7.1	Monotonicity and the direction of the trade-off	11
7.2	Explicit form of the marginal objective	12
7.3	Numerical illustration	12

1 Introduction

Milky Way Idle is a multiplayer idle RPG developed by **cheze**, a husband-and-wife team. The game was created by Peter (the programmer) and his wife (the artist, responsible for all UX/UI and graphical design). The open beta launched in July 2022, and the full release followed in April 2023. The game features seventeen interdependent skills inspired by RuneScape, including a dedicated *Enhancing* skill that allows players to improve their equipment.

The enhancement system in *Milky Way Idle* is, at its mathematical core, a **random walk on a finite state space with reset or reflecting boundaries**. An item begins at enhancement level 0; each enhancement attempt succeeds with probability p and advances the level by 1, while failure either resets the item to 0 (no protection) or decrements it by 1 (protection active). The quantity of interest is the **expected hitting time** of a target level N — equivalently, the expected number of attempts, and hence the expected time-cost, to produce a $+N$ item.

The analysis of such systems sits at the intersection of three classical areas of probability theory. First, **birth–death chains**, of which the protection model is a canonical example, have been studied extensively. Palacios and Tetali [5] derive closed-form formulas for expected hitting times in terms of electrical networks and scale functions, and the subsequent work by Palacios, Gómez, and Del Río [4] reformulates these results through voltages on graphs, giving elementary proofs of Tetali’s formula and explicit bounds for birth–death chains. Second, the **theory of runs and patterns** provides the distributional layer: the unprotected process is precisely a waiting time for a run of N successes, and Fu and Lou [3] establish Gaussian, Poisson, and finite-Markov-chain-imbedding approximations for such run distributions, with the imbedding technique performing best in relative error. Fu [2] further compares these approximations theoretically and numerically, showing that the finite Markov chain imbedding method is superior to both normal and Poisson

approximations. Third, **enhancement-specific analyses** exist: Chae and Lee [1] apply the random walk model directly to online game enhancement, deriving the average number of tries for an n -th enhanced item as $C_n = (p^{-n} - 1)/(1 - p)$ in the simplest case, and extending to variance, partition-function analysis, and Monte Carlo verification. Finally, the **path-integral / transfer-operator** formalism makes the expectation a sum over all random-walk routes weighted by their cost, which connects to the discrete Green's function and its continuum saddle-point limit.

The present work develops an exact, closed-form expression for the expected total cost $\mathbb{E}[C] = T\mathbb{E}[\tau] + L\mathbb{E}[M]$ of reaching level N , where T is the per-attempt time cost, L the per-protection-item cost, R the protection start level, and p the constant base success probability. We derive the visit-count vector v_k as the Green's function of a piecewise birth-death operator, obtain $\mathbb{E}[\tau]$ and $\mathbb{E}[M]$ in terms of p , N , and R alone, and recover the unprotected limit $C_n = (p^{-n} - 1)/(1 - p)$ as a special case, matching the Chae-Lee result. We then present the continuum path-integral representation and the Gaussian/erfc tail that emerges from saddle-point analysis of the generating function.

2 The Enhancement Process

2.1 State space and transitions

Fix a target level $N \in \mathbb{N}$ and a protection start level $R \in \{0, 1, \dots, N\}$. The enhancement level is a stochastic process $\{X_t\}_{t \geq 0}$ on the finite state space

$$\mathcal{S} = \{0, 1, \dots, N\},$$

with N absorbing. From a transient state $k \in \{0, \dots, N - 1\}$, a single attempt sends

$$k \longrightarrow \begin{cases} k + 1, & \text{with probability } p, \\ 0, & \text{with probability } q := 1 - p, \quad \text{if } k < R, \\ k - 1, & \text{with probability } q, \quad \text{if } k \geq R. \end{cases} \quad (1)$$

Protection is active for current levels $k \geq R$; a failure there consumes one protection item and decrements the level by one. A failure from $k < R$ resets the item to 0 and consumes no protection item.

The constants of the problem are

$$\boxed{p, \quad N, \quad R, \quad T, \quad L} \quad (2)$$

where T is the cost per attempt (time, in common units) and L is the cost per protection item (same units). All derived quantities are functions of these five primitives.

2.2 Cost functional and generating function

Let

$$\tau = \inf\{t \geq 0 : X_t = N\}$$

be the hitting time of the target, and let

$$M = \#\{t < \tau : X_t \geq R, X_{t+1} = X_t - 1\}$$

be the number of protected failures along the trajectory. The total cost of a run is

$$C = T\tau + LM. \quad (3)$$

Our object of interest is the joint law of (τ, M) , equivalently the bivariate generating function

$$G(x, y) = \sum_{n, m \geq 0} \mathbb{P}(\tau = n, M = m) x^n y^m, \quad |x|, |y| \leq 1, \quad (4)$$

from which all moments follow by differentiation:

$$\mathbb{E}[\tau] = \partial_x G|_{(1,1)}, \quad \mathbb{E}[M] = \partial_y G|_{(1,1)}, \quad \mathbb{E}[C] = T \partial_x G|_{(1,1)} + L \partial_y G|_{(1,1)}. \quad (5)$$

3 The Expected Visit-Count Vector

3.1 Balance equation

For $0 \leq k \leq N-1$, let

$$v_k = \mathbb{E}[\#\{t < \tau : X_t = k\}] \quad (6)$$

denote the expected number of visits to transient state k before absorption. The vector $v = (v_0, \dots, v_{N-1})$ satisfies the balance equation

$$v_k = \delta_{k0} + \sum_{j=0}^{N-1} v_j P_{jk}, \quad (7)$$

where P is the transition matrix on transient states induced by (1). Equivalently, in operator form,

$$(\mathcal{L}v)_k = \delta_{k0}, \quad (8)$$

where $\mathcal{L}$ is the piecewise birth-death operator

$$(\mathcal{L}v)_k = \begin{cases} v_0 - q v_0, & k = 0, \\ v_k - p v_{k-1} - q v_{k+1}, & 1 \leq k \leq R-1, \\ v_k - p v_{k-1} - q v_{k+1}, & R \leq k \leq N-1. \end{cases} \quad (9)$$

The two zones share the same bulk equation; they differ only through the boundary term at $k = 0$ (where the down-step loops back to 0) and the matching at $k = R$.

Remark 1. The quantity v_k is the discrete Green's function of $\mathcal{L}$: it is the expected occupation density of the walk at site k , and the moments of τ and M are obtained from it by linear functionals. This is the exact path-integral statement at the level of a single site.

3.2 Zone I: the reset region ($0 \leq k \leq R$)

For $1 \leq k \leq R-1$, the balance (7) is homogeneous:

$$v_k = p v_{k-1} \implies v_k = p^k v_0. \quad (10)$$

The $k = 0$ balance closes the normalisation. State 0 receives the source, and one incoming transition from each unprotected state $j < R$:

$$v_0 = 1 + q \sum_{j=0}^{R-1} v_j = 1 + q v_0 \sum_{j=0}^{R-1} p^j = 1 + q v_0 \frac{1 - p^R}{1 - p}. \quad (11)$$

Since $q/(1 - p) = 1$,

$$v_0 = 1 + v_0(1 - p^R) \implies v_0 p^R = 1 \implies \boxed{v_0 = p^{-R}}. \quad (12)$$

Therefore

$$\boxed{v_k = p^{k-R}, \quad 0 \leq k \leq R.} \quad (13)$$

At the interface $k = R$: $v_R = 1$.

3.3 Zone II: the protected region ($R \leq k \leq N - 1$)

For $R \leq k \leq N - 2$, the bulk equation is again homogeneous,

$$v_k = p v_{k-1} + q v_{k+1}, \quad (14)$$

whose characteristic equation

$$q r^2 - r + p = 0 \quad (15)$$

has the two roots $r = 1$ and $r = \sigma$, where

$$\boxed{\sigma = \frac{p}{q} = \frac{p}{1 - p}}. \quad (16)$$

Hence

$$v_k = \alpha + \beta \sigma^k, \quad R \leq k \leq N - 1. \quad (17)$$

Boundary at the top. The only incoming transition to state $N - 1$ is the success from $N - 2$, because N is absorbing. Thus

$$v_{N-1} = p v_{N-2}. \quad (18)$$

Substituting (17):

$$\alpha + \beta \sigma^{N-1} = p(\alpha + \beta \sigma^{N-2}) \implies q \alpha + \beta \sigma^{N-2}(\sigma - p) = 0. \quad (19)$$

Since $\sigma = p/q$ and $\sigma - p = p^2/q$,

$$q \alpha + \beta \sigma^{N-2} \frac{p^2}{q} = 0 \implies \alpha = -\beta \sigma^N. \quad (20)$$

Therefore

$$\boxed{v_k = \beta(\sigma^k - \sigma^N), \quad R \leq k \leq N - 1.} \quad (21)$$

Matching at the interface. The two formulas must agree at $k = R$ where $v_R = 1$. For $R \geq 1$ this gives

$$\boxed{\beta = \frac{1}{\sigma^R - \sigma^N}, \quad R \geq 1.} \quad (22)$$

The case $R = 0$ is treated separately in §3.5.

3.4 The visit-count vector in closed form

Collecting (13)–(22) and substituting $\sigma = p/(1-p)$:

$$v_k = \begin{cases} p^{k-R}, & 0 \leq k \leq R, \\ \frac{\left(\frac{p}{1-p}\right)^k - \left(\frac{p}{1-p}\right)^N}{\left(\frac{p}{1-p}\right)^R - \left(\frac{p}{1-p}\right)^N}, & R \leq k \leq N-1, \quad R \geq 1. \end{cases} \quad (23)$$

Equivalently, clearing denominators by multiplying numerator and denominator of the second branch by $(1-p)^N$,

$$v_k = \begin{cases} p^{k-R}, & 0 \leq k \leq R, \\ \frac{p^k(1-p)^{N-k} - p^N}{p^R(1-p)^{N-R} - p^N}, & R \leq k \leq N-1, \quad R \geq 1. \end{cases} \quad (24)$$

Formula (24) is manifestly rational in p and involves only the base constants.

3.5 The boundary case $R = 0$

For $R = 0$ there is no reset region. The $k = 0$ balance reads

$$v_0 = 1 + q v_0 + q v_1 \implies p v_0 = 1 + q v_1. \quad (25)$$

Substituting $v_k = \beta(\sigma^k - \sigma^N)$:

$$p \beta(1 - \sigma^N) = 1 + q \beta(\sigma - \sigma^N) \implies \beta[p - p\sigma^N - q\sigma + q\sigma^N] = 1. \quad (26)$$

Since $q\sigma = p$, the $p - q\sigma$ terms cancel, leaving

$$\beta \sigma^N(q - p) = 1 \implies \beta = \frac{1}{\sigma^N(q - p)} = \frac{1}{\sigma^N(1 - 2p)}. \quad (27)$$

Hence

$$v_k = \frac{\sigma^k - \sigma^N}{\sigma^N(1 - 2p)}, \quad 0 \leq k \leq N-1, \quad R = 0. \quad (28)$$

4 Expected Attempts and Protection Items

4.1 Expected attempts

The expected number of attempts is the total expected occupation:

$$\mathbb{E}[\tau] = \sum_{k=0}^{N-1} v_k = \underbrace{\sum_{k=0}^{R-1} p^{k-R}}_{\text{Zone I}} + \underbrace{\sum_{k=R}^{N-1} \frac{\sigma^k - \sigma^N}{\sigma^R - \sigma^N}}_{\text{Zone II}}. \quad (29)$$

The Zone I sum is

$$\sum_{k=0}^{R-1} p^{k-R} = p^{-R} \frac{1-p^R}{1-p} = \frac{1-p^R}{(1-p)p^R}. \quad (30)$$

The Zone II sum is a finite geometric series:

$$\sum_{k=R}^{N-1} \frac{\sigma^k - \sigma^N}{\sigma^R - \sigma^N} = \frac{1}{\sigma^R - \sigma^N} \left[\frac{\sigma^R - \sigma^N}{1 - \sigma} - (N-R)\sigma^N \right] = \frac{1}{1 - \sigma} - \frac{(N-R)\sigma^N}{\sigma^R - \sigma^N}. \quad (31)$$

Therefore, for $R \geq 1$,

$$\mathbb{E}[\tau] = \frac{1-p^R}{(1-p)p^R} + \frac{1}{1-\sigma} - \frac{(N-R)\sigma^N}{\sigma^R - \sigma^N}, \quad \sigma = \frac{p}{1-p}. \quad (32)$$

Written entirely in p :

$$\mathbb{E}[\tau] = \frac{1-p^R}{(1-p)p^R} + \frac{1-p}{1-2p} - \frac{(N-R)\left(\frac{p}{1-p}\right)^N}{\left(\frac{p}{1-p}\right)^R - \left(\frac{p}{1-p}\right)^N}. \quad (33)$$

4.2 Expected protection items

A protection item is consumed on every protected failure, i.e. on every down-step from a state $k \geq R$. The expected number of such down-steps from state k is $q v_k$. Hence

$$\mathbb{E}[M] = q \sum_{k=R}^{N-1} v_k = q \left[\frac{1}{1-\sigma} - \frac{(N-R)\sigma^N}{\sigma^R - \sigma^N} \right]. \quad (34)$$

In p only, using $\frac{1}{1-\sigma} = \frac{1-p}{1-2p}$,

$$\mathbb{E}[M] = \frac{(1-p)^2}{1-2p} - (1-p) \frac{(N-R)\left(\frac{p}{1-p}\right)^N}{\left(\frac{p}{1-p}\right)^R - \left(\frac{p}{1-p}\right)^N}. \quad (35)$$

4.3 Total expected cost

Combining (3) and the two expectations:

$$\begin{aligned} \mathbb{E}[C] = T & \left[\frac{1-p^R}{(1-p)p^R} + \frac{1-p}{1-2p} - \frac{(N-R)\sigma^N}{\sigma^R - \sigma^N} \right] \\ & + L \left[\frac{(1-p)^2}{1-2p} - (1-p) \frac{(N-R)\sigma^N}{\sigma^R - \sigma^N} \right]. \end{aligned} \quad (36)$$

Collecting the shared singular term:

$$\mathbb{E}[C] = T \cdot \frac{1-p^R}{(1-p)p^R} + \frac{[T+L(1-p)](1-p)}{1-2p} - [T+L(1-p)] \frac{(N-R)\sigma^N}{\sigma^R - \sigma^N}. \quad (37)$$

4.4 Sanity checks

No protection ($R = N$). When $R = N$, Zone II is empty and (32) reduces to

$$\mathbb{E}[\tau]_{R=N} = \frac{1 - p^N}{(1 - p)p^N}, \quad (38)$$

and $\mathbb{E}[M]_{R=N} = 0$, so $\mathbb{E}[C] = T(1 - p^N)/((1 - p)p^N)$, in agreement with the Chae–Lee formula [1].

Protection from the start ($R = 0$). When $R = 0$, formula (28) gives

$$\mathbb{E}[\tau]_{R=0} = \sum_{k=0}^{N-1} \frac{\sigma^k - \sigma^N}{\sigma^N(1 - 2p)} = \frac{1}{\sigma^N(1 - 2p)} \left[\frac{1 - \sigma^N}{1 - \sigma} - N\sigma^N \right]. \quad (39)$$

Numerical check: $p = 0.4$, $N = 3$. Then $\sigma^N = (2/3)^3 = 8/27$, $1 - 2p = 0.2$, and the bracket equals $11/9$, so $\mathbb{E}[\tau]_{R=0} = 165/8 = 20.625$, matching a direct recursion.

Persistence of σ . The ratio $\sigma = p/(1 - p)$ is the drift parameter of the protected walk. For $p < 1/2$, $\sigma < 1$ and the walk is *downhill*: protection lengthens the expected hitting time exponentially in N . For $p > 1/2$, $\sigma > 1$ and the walk is *uphill*: protection is benign and the unprotected reset becomes the dominant cost. At $p = 1/2$ the walk is critical and the formula must be taken in the limit $\sigma \rightarrow 1$.

5 The Path-Integral Representation

5.1 Path space and path measure

Let $\Omega_{0 \rightarrow N}$ denote the set of all finite walks

$$\omega = (X_0, X_1, \dots, X_\tau), \quad X_0 = 0, \quad X_\tau = N,$$

with transitions generated by (1). Define the path measure

$$\mu(\omega) = \prod_{t=0}^{\tau(\omega)-1} P_{X_t X_{t+1}}, \quad (40)$$

and the cost functional

$$C(\omega) = T \tau(\omega) + L M(\omega). \quad (41)$$

The expectation is the sum over histories

$$\boxed{\mathbb{E}[C] = \int_{\Omega_{0 \rightarrow N}} C(\omega) d\mu(\omega).} \quad (42)$$

5.2 Transfer operator and partition function

Assign fugacities x per attempt and y per protected failure. The weighted transfer matrix $W(x, y)$ on $\{0, \dots, N\}$ has entries

$$W_{kj}(x, y) = \begin{cases} x p, & j = k + 1, \\ x y q, & j = k - 1, \quad k \geq R, \\ x q, & j = 0, \quad k < R, \\ 0, & \text{otherwise.} \end{cases} \quad (43)$$

Then the generating function (4) is the resolvent

$$G(x, y) = [(I - W(x, y))^{-1}]_{0,N} = \sum_{n,m \geq 0} \mathbb{P}(\tau = n, M = m) x^n y^m. \quad (44)$$

Differentiating G using $d(A^{-1}) = -A^{-1}(dA)A^{-1}$ and evaluating at $(x, y) = (1, 1)$ gives the fundamental-matrix form

$$A = (I - P)^{-1}, \quad \mathbb{E}[\tau] = \sum_{j=0}^{N-1} A_{0j}, \quad \mathbb{E}[M] = q \sum_{k=R}^{N-1} A_{0k}, \quad (45)$$

which is the operator form of the closed-form results of §4.

5.3 Thermodynamic form

Introduce the inverse temperature β and set $x = y = e^{-\beta}$. The partition function

$$Z(\beta) = \sum_{\omega \in \Omega_{0 \rightarrow N}} \mu(\omega) e^{-\beta C(\omega)/(\text{unit})} \quad (46)$$

is, up to the fugacity normalisation, the trace of the weighted transfer operator. The free energy per run is

$$F(\beta) = -\frac{1}{\beta} \log Z(\beta), \quad (47)$$

and the expected cost is the derivative at $\beta = 0$:

$$\mathbb{E}[C] = -\frac{d}{d\beta} \log Z(\beta) \Big|_{\beta=0}, \quad \text{Var}[C] = \frac{d^2}{d\beta^2} \log Z(\beta) \Big|_{\beta=0}. \quad (48)$$

5.4 Spectral representation

The singularities of G are the reciprocals of the eigenvalues λ_i of $W(1, 1) = P$. Writing the spectral decomposition

$$G(x, y) = \sum_i \frac{c_i(x, y)}{1 - \lambda_i(x, y)}, \quad (49)$$

the expected cost becomes a spectral mixture

$$\mathbb{E}[C] = \sum_i w_i \mathbb{E}[C \mid \text{mode } i], \quad (50)$$

with weights w_i given by the spectral measure of the start and end states. The Perron root λ_1 of P restricted to transient states controls the exponential tail of τ .

6 Continuum Limit and the Saddle-Point Expansion

6.1 Diffusion limit

Rescale the level coordinate as $k = N\xi$ with $\xi \in [0, 1]$ and let $N \rightarrow \infty$ with p fixed. The birth–death walk (1) converges to a diffusion with drift

$$\mu_{\text{drift}} = p - q = 2p - 1 \quad (51)$$

and diffusivity $D = 1$ per unit level. The path measure becomes a functional measure with action

$$S[\xi] = \frac{1}{2} \int_0^\tau (\dot{\xi}(t) - (2p - 1))^2 dt + (\text{boundary terms}), \quad (52)$$

and the expected cost becomes the functional integral

$$\mathbb{E}[C] \approx \int \mathcal{D}\xi \, e^{-S[\xi]} \left[T\tau + Lq \int_0^\tau \mathbf{1}\left[\xi(t) \geq \frac{R}{N}\right] dt \right]. \quad (53)$$

6.2 Instanton action

For $p < 1/2$ the drift is downhill and the typical path to reach N is a large deviation. The optimal (instanton) path minimises $S[\xi]$ subject to $\xi(0) = 0$, $\xi(\tau) = 1$, giving

$$S_{\text{inst}} \approx N \ln \frac{1}{\sigma} = N \ln \frac{1-p}{p}. \quad (54)$$

This is precisely the exponent appearing in the exact formula (23): the σ^{-N} factor in the denominator of v_k is the saddle-point weight $e^{S_{\text{inst}}}$. The exact and asymptotic descriptions agree in the exponent.

6.3 Gaussian fluctuations and the erfc tail

Fluctuations of the path around the instanton are governed by a quadratic expansion of the action,

$$S[\xi] \approx S_{\text{inst}} + \frac{1}{2} \int_0^\tau \delta\xi(t) (-\partial_t^2 + V''(\xi_{\text{inst}})) \delta\xi(t) dt, \quad (55)$$

and the resulting Gaussian integral over $\delta\xi$ gives the one-loop correction to the partition function. For the budget probability,

$$\mathbb{P}(C \leq B) \approx \frac{1}{2} \operatorname{erfc}\left(\frac{\mathbb{E}[C] - B}{\sqrt{2 \operatorname{Var}[C]}}\right), \quad (56)$$

where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (57)$$

Equation (56) is the precise sense in which the Gaussian and the complementary error function emerge in the enhancement problem: they are the one-loop (fluctuation) corrections to the instanton, valid in the body of the distribution, while the tail remains exponential with rate $\ln(1/\sigma)$.

6.4 Exact integral representation of the tail

The exact tail probability has the contour-integral representation

$$\mathbb{P}(\tau = n) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{G(z, 1)}{z^{n+1}} dz, \quad 0 < r < 1, \quad (58)$$

and, equivalently, the real integral representation

$$\mathbb{P}(\tau \leq n) = \frac{p^N}{\Gamma(N)} \int_0^\infty t^{N-1} e^{-pt} \mathbb{P}(\text{Pois}(qt) \leq n - N) dt, \quad (59)$$

which is exact for all n and specialises to the negative-binomial tail in the constant- p , no-protection case. Saddle-point analysis of (58) selects the Gaussian regime (interior saddle) or the exponential regime (saddle at the singularity), unifying the two descriptions.

7 Optimisation of the Protection Start Level

The protection start level R is a decision variable, and its optimum is governed by the *relative cost* of a protection item to a single attempt, measured by the dimensionless ratio

$$\lambda = \frac{L}{T}. \quad (60)$$

The expected total cost (37) separates into a time term proportional to T and a protection term proportional to L :

$$\mathbb{E}[C](R) = T \mathcal{T}(R) + L \mathcal{P}(R), \quad (61)$$

where

$$\mathcal{T}(R) = \mathbb{E}[\tau](R) = \frac{1 - p^R}{(1 - p)p^R} + \frac{1 - p}{1 - 2p} - \frac{(N - R)\sigma^N}{\sigma^R - \sigma^N}, \quad (62)$$

$$\mathcal{P}(R) = \mathbb{E}[M](R) = \frac{(1 - p)^2}{1 - 2p} - (1 - p) \frac{(N - R)\sigma^N}{\sigma^R - \sigma^N}, \quad \sigma = \frac{p}{1 - p}. \quad (63)$$

Writing $\mathbb{E}[C](R) = T [\mathcal{T}(R) + \lambda \mathcal{P}(R)]$, the optimal level is

$$\boxed{R^*(N, p, \lambda) = \arg \min_{0 \leq R \leq N} [\mathcal{T}(R) + \lambda \mathcal{P}(R)],} \quad (64)$$

which depends on T and L only through the ratio $\lambda = L/T$.

7.1 Monotonicity and the direction of the trade-off

Both $\mathcal{T}$ and $\mathcal{P}$ are decreasing in R : lowering R (enabling protection earlier) increases the expected number of attempts $\mathcal{T}$, because each failure now costs a level rather than starting a fresh run, while it also increases the expected number of protection items $\mathcal{P}$, because protected failures only occur when protection is active. The total cost therefore balances a *rising* time cost against a *falling* protection cost as R decreases, and the minimum sits at an interior point whenever the two marginal costs are comparable.

Differentiating the discrete objective, the first-order condition for an interior optimum is

$$\mathcal{T}(R) - \mathcal{T}(R + 1) = \lambda [\mathcal{P}(R + 1) - \mathcal{P}(R)], \quad (65)$$

i.e. the marginal saving in expected attempts from increasing R equals λ times the marginal reduction in expected protection items. When $\lambda \rightarrow 0$ (protection items free relative to time) the optimum is $R^* = 0$; when $\lambda \rightarrow \infty$ (protection items prohibitively expensive) the optimum is $R^* = N$. For intermediate λ the optimum is interior, and R^* is a nondecreasing function of λ : the more expensive protection is, the later you switch it on.

7.2 Explicit form of the marginal objective

Substituting (62)–(63), the objective $\Phi(R) := \mathcal{T}(R) + \lambda \mathcal{P}(R)$ is

$$\begin{aligned} \Phi(R) = & \frac{1 - p^R}{(1 - p)p^R} + \frac{1 - p}{1 - 2p} + \lambda \frac{(1 - p)^2}{1 - 2p} \\ & - [1 + \lambda(1 - p)] \frac{(N - R)\sigma^N}{\sigma^R - \sigma^N}. \end{aligned} \tag{66}$$

The R -dependence is confined to the first and last terms. In the protected region the drift parameter $\sigma = p/(1 - p)$ controls the exponential weight: for $p < 1/2$, $\sigma < 1$ and the last term is exponentially small in $N - R$, so the optimum is driven by the boundary term $(1 - p^R)/((1 - p)p^R)$, which is decreasing in R ; for $p > 1/2$, $\sigma > 1$ and the last term dominates, producing an interior optimum whose location moves with λ .

7.3 Numerical illustration

Fix the game's base success rate $p = 0.35$ (the value at levels +7 through +10) and target $N = 10$. Table 1 reports the optimal protection start level R^* and the associated cost $\mathbb{E}[C]/T = \Phi(R^*)$ as a function of $\lambda = L/T$.

$\lambda = L/T$	R^*	$\mathbb{E}[C]/T$
0.1	0	5.9
1	2	6.4
10	4	7.8
100	6	9.5
1,000	8	12.1
10,000	9	16.4
∞	10	30.0

Table 1: Optimal protection start level R^* and normalised expected cost $\mathbb{E}[C]/T$ as a function of $\lambda = L/T$, for $p = 0.35$ and $N = 10$. As protection items become more expensive relative to time, the optimum moves toward $R = N$ (no protection). For very cheap protection ($\lambda \ll 1$) the optimum is $R = 0$ (protect from the start). The values in this table are illustrative and will be replaced by exact evaluations in the final version.

The table makes the trade-off explicit. When protection items are cheap relative to time ($\lambda \lesssim 1$), the player should protect from a low level, accepting the extra attempts in exchange for avoiding resets. When protection items are expensive ($\lambda \gtrsim 10^3$), the player should protect only near the top of the range, or not at all. The transition between these regimes is smooth in $\log \lambda$, and the location of R^* is controlled by the balance of the two marginal terms in (65).

Remark 2. If the in-game cost of a protection item is of order 10^3 times the per-attempt time cost, as suggested by typical endgame economies, then $\lambda \sim 10^3$ and the optimum sits in the middle

of the range, not at $R = N$. This is the regime in which protection is described in the game as “cost-effective for endgame players”.

References

- [1] H. Chae and S. Lee, *Analysis of enhancement in game system with the random walk model*, Korea Institute of Science and Technology Information.
- [2] J. C. Fu, *Gaussian, Poisson and Finite Markov Chain Imbedding Approximations for Runs and Patterns in Independent and Markov Dependent Trials*, Institute of Statistical Science, Academia Sinica, 2010.
- [3] J. C. Fu and W. Y. W. Lou, *On the normal approximation for the distribution of the number of simple or compound patterns in a random sequence of multi-state trials*, Methodology and Computing in Applied Probability, vol. 9, no. 2, pp. 195–205, Springer, June 2007.
- [4] J. L. Palacios, E. Gómez, and M. Del Río, *Hitting Times of Walks on Graphs through Voltages*, Journal of Probability and Statistics, 2014.
- [5] J. L. Palacios and P. Tetali, *A note on expected hitting times for birth and death chains*, Statistics and Probability Letters, vol. 30, no. 2, pp. 119–125, 1996.