

# A Spatial-Intersection Criterion for Global Smoothness of Three-Dimensional Euler Flow

Thomas Birnie

September 2026

## Abstract

We prove a global smoothness criterion for three-dimensional incompressible Euler flow on  $\mathbb{R}^3$ . For a smooth finite-energy solution, whole-interval  $L_t^2 H_x^K$  membership of the velocity at every finite spatial order generates every temporal velocity and canonical pressure derivative. The proof combines the full differentiated Euler recurrence with an energy interpolation inequality; each output requires only finitely many spatial orders. Adjacent temporal rows give strong Sobolev traces and continuity of every mixed derivative on the closed finite slab. One local continuation through a proposed finite maximal time then proves global smoothness under the stated membership hypothesis. Neither a sum over derivative orders nor an infinite sum of local lifespans is required. We also construct the full intersection on one common local interval from arbitrary smooth data. Extending that construction through an entire candidate finite maximal lifespan remains unresolved.

Further results examine the fluid's material and pressure evolution. From energy alone we construct a unique weak terminal velocity, a terminal transport isometry with an exact orthogonal energy defect, and a measure-preserving particle limit whose secants recover terminal momentum. Circulation reaches a terminal current; at strict times it combines with the deformed material metric to reconstruct velocity. The first temporal response has a sharp energy-level weak topology, but an explicit Fourier-packet construction makes the second response unbounded at a fixed output frequency. Separate pressure arguments give continuation under absolute and directional inverse-square bounds with coefficient below 2, and under a small signed weighted matrix primitive along vortical trajectories. We compare these deductions with existing pressure criteria and retain the spherical contact term in the coupled Hessian evolution. We describe normalized clocks, material anisotropy, cross-Leray exchange, and circulating surface stacks recruited by localized receivers. Fixed-mass recurrence at escaping frequencies forces infinite accumulated activity, and fixed-mass shrinking spatial cores are excluded under a Type-I gradient bound. Fragmentation and exact counterexamples identify the limits of that hypothesis and of proposed scalar budgets. These constructions and obstructions retain their stated hypotheses; they do not remove the spatial membership condition from the global theorem.

# Contents

|    |                                                                        |    |
|----|------------------------------------------------------------------------|----|
| 1  | From viscous flow to Euler flow . . . . .                              | 3  |
| 2  | The spatial intersection and global continuation . . . . .             | 6  |
| 3  | What reaches the terminal time from energy alone . . . . .             | 11 |
| 4  | A terminal transport isometry and its spatial intersection . . . . .   | 13 |
| 5  | Particle positions, circulation, and the changing metric . . . . .     | 15 |
| 6  | Spatial reconstruction of the material temporal coefficients . . . . . | 17 |
| 7  | The limits of reconstruction from energy . . . . .                     | 19 |
| 8  | Pressure and material separation . . . . .                             | 21 |
| 9  | Continuation from material pressure . . . . .                          | 22 |
| 10 | How the velocity generates pressure curvature . . . . .                | 26 |
| 11 | Energy normalization and changing material shape . . . . .             | 29 |
| 12 | Coupled transfer and material recruitment . . . . .                    | 33 |
| 13 | What the conserved structure does not determine . . . . .              | 41 |
| 14 | Global continuation and the remaining spatial condition . . . . .      | 45 |
| A  | Spatial tails and the full increment equation . . . . .                | 46 |
| B  | Equivalent forms of the endpoint construction . . . . .                | 48 |
| C  | Pressure tests with explicit realizability limits . . . . .            | 49 |
| D  | Periodic receiver and conservative-defect tests . . . . .              | 51 |

# 1 From viscous flow to Euler flow

The preceding Navier–Stokes paper [1] develops its global smoothness argument by constructing a compatible intersection of spatial Sobolev spaces. Each temporal derivative of velocity is a spatial field, and each of those fields has its own spatial derivatives. The proof follows one fluid through this mixed hierarchy, constructs the common intersection, and reads finite-coordinate bounds from it. This paper asks what that method gives when viscosity is removed.

The fluid can still stretch, turn, and transport its vorticity. Pressure still couples distant regions and keeps the motion incompressible. What disappears is diffusion of spatial variation. The Euler equations are

$$\partial_t u + (u \cdot \nabla)u = -\nabla p, \quad \nabla \cdot u = 0. \quad (1.1)$$

We work on  $\mathbb{R}^3$ , starting from a divergence-free velocity with finite Sobolev norm at every order. The question is whether one evolving fluid retains that complete spatial regularity through a finite candidate singular time. The Navier–Stokes argument supplies a way to formulate the spatial intersection. Its positive-viscosity recovery step requires separate examination before it can say anything about Euler.

Diffusion gives fine spatial structure a direct temporal response. A velocity variation on length  $\ell$  feels the Laplacian at order  $\ell^{-2}$ , so its diffusive time is of order  $\ell^2/\nu$ . As the length decreases, this response accelerates. It enters every row of the temporal tower: acceleration contains the Laplacian of velocity, jerk contains the Laplacian of acceleration, and the relation continues through all higher responses. Transport and pressure act throughout this hierarchy as well.

To retain all of these terms, write  $u^\nu$  for a classical solution with viscosity  $\nu \geq 0$ , and put  $V_n^\nu = \partial_t^n u^\nu$ . The Leray projector  $\mathbb{P}$  incorporates the pressure that makes each response divergence-free. Differentiation gives

$$\begin{aligned} V_{n+1}^\nu &= \nu \Delta V_n^\nu - \mathcal{B}_n^\nu, \\ \mathcal{B}_n^\nu &= \mathbb{P} \sum_{j=0}^n \binom{n}{j} (V_j^\nu \cdot \nabla) V_{n-j}^\nu. \end{aligned} \quad (1.2)$$

Here  $\mathcal{B}_n^\nu$  is the complete differentiated nonlinear response. For  $\nu > 0$ , the equation also reads

$$\nu \Lambda^2 V_n^\nu = -V_{n+1}^\nu - \mathcal{B}_n^\nu, \quad \Lambda = (-\Delta)^{1/2}.$$

Once the two terms on the right are realized at a spatial order, this identity recovers two additional spatial derivatives of row  $n$ . For an inhomogeneous Sobolev norm, the low frequencies of  $V_n^\nu$  also require control, for instance through its  $L^2$  norm. The nonlinear term has to be accounted for in that realization; the identity does not provide its control. What viscosity supplies is the nonzero coefficient through which temporal information can recover that higher spatial order.

The corresponding energy identity makes the sign of this connection visible. In viscous flow, put  $E_s = \frac{1}{2} \|\Lambda^s u\|_2^2$ . Then

$$E'_s = -2\nu E_{s+1} + \mathcal{P}_s, \quad \mathcal{P}_s = -\langle \Lambda^s u, \Lambda^s \mathbb{P}(u \cdot \nabla u) \rangle. \quad (1.3)$$

The spatial derivatives at height  $s+1$  contribute with a definite dissipative sign to the response at height  $s$ . At positive viscosity,  $2\nu E_{s+1} = \mathcal{P}_s - E'_s$  recovers that higher energy from the complete response and transfer. Setting  $\nu = 0$  deletes its coefficient. The identity becomes  $E'_s = \mathcal{P}_s$ , with no  $E_{s+1}$  left to recover.

For Euler, put  $V_n = V_n^0$  and  $\mathcal{B}_n = \mathcal{B}_n^0$ . The nonlinear relation is  $V_{n+1} = -\mathcal{B}_n$ . Its rows remain linked through the products of earlier responses and their pressure projection. What has disappeared

is the linear spatial recovery operation running through every row. Increasing temporal order now follows the differentiated transport and pressure interactions. It cannot recover the missing Laplacian by dividing the resulting equation by  $\nu$ . Pressure still has its elliptic determination from the velocity products. That reconstructs pressure for a given velocity; it does not reinstate the missing equation for  $\Delta V_n$ .

This loss also changes how space and time can be rescaled. Change velocity amplitude by  $b$  and length by  $\lambda^{-1}$ . Balancing temporal response with transport fixes the new clock factor at  $b\lambda$ , giving

$$\begin{aligned} u^{b,\lambda}(x, t) &= b u(\lambda x, b\lambda t), \\ p^{b,\lambda}(x, t) &= b^2 p(\lambda x, b\lambda t). \end{aligned} \tag{1.4}$$

Temporal response, transport, and pressure gradient each acquire the factor  $b^2\lambda$ . The Laplacian of the transformed velocity acquires  $b\lambda^2$ . A Navier–Stokes solution of viscosity  $\nu$  is consequently transformed into a solution of viscosity

$$\nu^{b,\lambda} = \frac{b}{\lambda} \nu.$$

Retaining a fixed positive viscosity forces  $b = \lambda$ , so the clock factor is  $\lambda^2$ . At  $\nu = 0$ , every positive pair  $(b, \lambda)$  preserves the equation. The coefficient that supplied the two-spatial-derivative link also supplied the restriction on scaling; both are lost together. Every finite positive rescaling preserves whether  $\nu$  is positive or zero. The inviscid equation lies outside the finite scaling orbit of a positive-viscosity solution.

Write  $\mu = b\lambda$  for the clock factor. Fixed-viscosity NS permits only  $\mu = \lambda^2$ . Euler permits independent positive choices of  $\mu$  and  $\lambda$ , with amplitude  $b = \mu/\lambda$ . The difference can be drawn in logarithmic coordinates: NS occupies a line in the plane of allowed clock and length changes, while Euler occupies the plane.

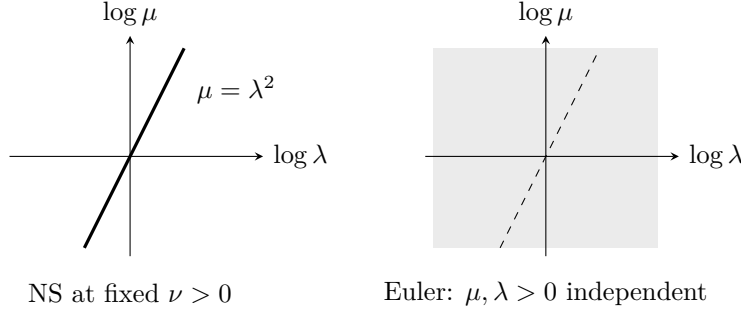


Figure 1: Allowed changes of clock and length. The shaded window is part of Euler’s full scaling plane. The dashed NS line remains an allowed Euler choice, with no preferred status imposed by the inviscid equation.

This is a change in the relations across the derivative hierarchy. Both equations still have temporal rows and spatial orders; viscosity does not create those indices. In NS, one temporal step and two spatial steps carry equal scaling weight, in accord with the Laplacian connection. Euler retains two independent weights. Section 2 follows these weights through the complete pressure-coupled hierarchy and keeps its nonlinear compatibility explicit after viscous recovery has vanished.

This freedom makes concentration inexpensive in energy. A rescaled velocity has  $L^2$  norm  $b\lambda^{-3/2}\|u\|_2$ , while its gradient has supremum norm  $b\lambda\|\nabla u\|_\infty$ . Taking  $b = \lambda^{3/2}$  preserves energy

and accelerates the local clock by  $\lambda^{5/2}$ . These rescaled fields are different initial data. A finite-time singularity would require one evolving datum to generate its own succession of concentrations. The dynamics connecting those stages is where a regularity argument has to act.

Section 2 proves the global continuation argument from whole-interval spatial membership of the velocity. The differentiated Euler equation generates each temporal row as a finite polynomial in spatial derivatives. Energy interpolation turns each polynomial bound into one higher spatial norm. The adjacent temporal row then gives a strong trace, and Sobolev embedding makes every mixed derivative continuous on the closed slab. Pressure and the full equation survive there. One continuation of the same solution excludes a finite maximal time. The spatial membership hypothesis is explicit: the section proves it on a common local interval from arbitrary smooth data, and proves global smoothness when it holds through any proposed finite maximal time.

The next constructions identify the endpoint information supplied by energy without that spatial hypothesis. Energy gives a unique weak terminal velocity, a terminal isometry for projected transport, and a measure-preserving limit of particle positions. These objects are linked: the isometry measures the exact energy defect, particle secants recover terminal momentum, and circulation reaches a distributional current. At strict times the initial circulation and the deformed material metric reconstruct the velocity. The unresolved attachment to the spatial intersection is thus expressed in concrete properties of the evolving transport and geometry. An explicit second-row counterexample in Section 7 excludes an attempted shortcut: energy cannot bound every temporal response in successively weaker fixed Sobolev spaces. Its second response can grow at a fixed output frequency.

The later sections pursue additional approaches to the Euler problem. They are connected by the fluid they describe, but they are not all instances of the intersection construction. Material pressure gives direct continuation criteria in Section 9. The absolute and directional inverse-square tests follow the Volterra inequality of Chae–Constantin [4, 5], with the coefficient below 2 obtained by an additional time integration. A separate signed matrix primitive retains cancellation before taking the norm. Its coupled energy identity allows noncommuting pressure histories. Neither condition is shown to hold for arbitrary smooth data. The pressure evolution is then calculated with its spherical contact term and complete material coupling retained.

Normalization and transfer examine how a hypothetical concentration would have to be generated. Section 11 keeps energy and peak vorticity fixed, recovers the physical clock, and identifies the anisotropic deformation left after isotropic rescaling. Section 12 retains cross-Leray work between donors and receivers. A fixed positive amount of energy recurring at finer scales forces infinite accumulated activity; a scalar fragmentation model explains why shrinking amounts require a different argument. A localized annular receiver also recruits a positive-measure stack of circulating material surfaces. Kelvin’s theorem and coarea convert it into an explicit requirement on the initial fluid’s deformation. Physical energy paths and frequency paths then reveal both the available transport laws and the correlation absent from their independent product. Combining finite physical flux with Chae–Wolf nonconcentration [16] excludes fixed-mass shrinking cores under a Type-I gradient bound.

Finally, exact tests distinguish these consequences from stronger claims. They include compact steady flows with large material shear, finite-energy data with opposite remote pressure responses, and the information lost by scalar pressure coordinates and conserved integrals. Auxiliary matrix counterexamples are kept separate from Euler realizations. The paper’s contribution includes the spatial membership theorem and its global continuation corollary, endpoint constructions, coupled identities, and checked obstructions. The global corollary keeps its whole-interval hypothesis; an unconditional arbitrary-data smoothness theorem for three-dimensional Euler is not established here.

Throughout the paper,  $H^m = H^m(\mathbb{R}^3)$  is the inhomogeneous  $L^2$  Sobolev space, with vector or matrix values understood, and

$$H^\infty = \bigcap_{m \in \mathbb{N}_0} H^m, \quad H_\sigma^\infty = \{v \in H^\infty : \operatorname{div} v = 0\}.$$

The intersection has the Fréchet topology of its finite Sobolev seminorms. We fix  $u_0 \in H_\sigma^\infty$  and a classical Euler solution on  $[0, T)$ ,  $0 < T < \infty$ , smooth on every strict compact subinterval. Write  $N = \|u_0\|_2$ , so its kinetic energy is  $N^2/2$ . The case  $N = 0$  is stationary. Pressure always has the canonical normalization

$$p = R_i R_j (u_i u_j), \quad R_i = \partial_i (-\Delta)^{-1/2}, \quad (1.5)$$

with repeated spatial indices summed. Local existence, uniqueness, persistence of regularity, and the vorticity continuation criterion are used in their classical Sobolev forms [9, 11, 2].

## 2 The spatial intersection and global continuation

The velocity, its spatial derivatives, its temporal responses, and pressure belong to one evolving fluid. The equation relates these fields at every strict classical time. A spatial intersection records their simultaneous regularity. To prove smoothness on a closed finite interval, the relevant membership must hold on that whole interval; no bound common to all derivative orders is needed.

Put  $V_n = \partial_t^n u$  and  $Q_n = \partial_t^n p$ . Differentiating the equation distributes the temporal derivatives across both velocity factors:

$$V_{n+1} = -\mathbb{P} \sum_{j=0}^n \binom{n}{j} (V_j \cdot \nabla) V_{n-j}, \quad (2.1)$$

$$Q_n = R_i R_k \sum_{j=0}^n \binom{n}{j} (V_j)_i (V_{n-j})_k. \quad (2.2)$$

Pressure is fixed by these velocity products at every row. It is an order-zero operation on the product, though the product itself requires spatial regularity. If  $\alpha$  is a spatial multi-index, the full mixed equation reads

$$\partial_t \partial^\alpha V_n = -\mathbb{P} \sum_{j=0}^n \sum_{\beta \leq \alpha} \binom{n}{j} \binom{\alpha}{\beta} (\partial^\beta V_j \cdot \nabla) \partial^{\alpha-\beta} V_{n-j}. \quad (2.3)$$

Retaining only a term with all derivatives on one factor would change the equation being followed.

These relations also define a jet from a single spatial velocity. Starting with  $\mathcal{V}_0(v) = v$ , use (2.1) recursively to define  $\mathcal{V}_n(v)$ , and then (2.2) to define  $\mathcal{Q}_n(v)$ . For each fixed  $n, m$ , a sufficiently high finite Sobolev norm of  $v$  controls the requested output in  $H^m$ . One may, for example, use  $H^{m+n+3}$  for  $\mathcal{V}_n$  and  $H^{m+n+3}$  for  $\mathcal{Q}_n$ . This follows by induction from the Sobolev algebra estimate above dimension divided by two and boundedness of the Riesz transforms. The maps are continuous homogeneous polynomials of degrees  $n+1$  and  $n+2$ , respectively.

Consequently the generated jet space

$$\mathcal{J}_E = \{(\mathcal{V}_n(v), \mathcal{Q}_n(v))_{n \geq 0} : v \in H_\sigma^\infty\} \quad (2.4)$$

is homeomorphic to  $H_\sigma^\infty$  in the product of the spatial projective topologies. The inverse is projection onto velocity row zero. This is a statement about a spatial datum and its equation-generated derivatives. It does not assert convergence of a Taylor series in time.

The next theorem obtains all temporal rows from the velocity's spatial column. It separates the production of derivatives by the Euler equation from the passage to a closed time interval.

**Theorem 2.1** (Spatial membership and smooth continuation). *Let  $u$  be the classical Euler solution from  $u_0 \in H_\sigma^\infty$  on  $[0, T)$ , where  $T < \infty$ . Suppose*

$$u \in \bigcap_{K \in \mathbb{N}_0} L^2(0, T; H^K). \quad (2.5)$$

*Then every velocity row  $V_n$  and canonical pressure row  $Q_n$  belongs to  $C^r([0, T]; H^m)$  for every finite  $n, r, m$ . Every mixed space-time derivative is continuous on  $\mathbb{R}^3 \times [0, T]$ . There is one velocity  $u_* = u(T) \in H_\sigma^\infty$ , with*

$$V_n(T) = \mathcal{V}_n(u_*), \quad Q_n(T) = \mathcal{Q}_n(u_*),$$

*and the same Euler solution extends smoothly through  $T$ .*

*Proof.* Fix finite  $n, m$ , and choose the integer  $L = m + n + 3$ . For  $0 \leq j \leq n$ , put

$$A(t) = \|u(t)\|_{H^L}, \quad D_j(t) = \|V_j(t)\|_{H^{L-j}}.$$

All products below lie above the Sobolev algebra threshold. Boundedness of  $\mathbb{P}$ , one spatial derivative on the second factor, and (2.1) give, for  $j < n$ ,

$$D_{j+1} \leq C_L \sum_{a=0}^j \binom{j}{a} D_a D_{j-a}.$$

Here  $C_L \geq 1$  may be enlarged to cover the finitely many product and multiplier estimates being used. Induction from  $D_0 = A$  gives

$$D_j \leq j! C_L^j A^{j+1}, \quad \|Q_n\|_{H^m} \leq (n+1)! C_L^{n+1} A^{n+2}. \quad (2.6)$$

Indeed, each summand in the induction has coefficient  $\binom{j}{a} a! (j-a)! = j!$ , and there are  $j+1$  summands. The same calculation in (2.2) proves the pressure bound. Only finitely many orders have been used.

Energy conservation converts the powers in these polynomial estimates into higher spatial orders. For  $N = \|u_0\|_2 > 0$ , use the probability measure  $N^{-2} |\hat{u}(\xi, t)|^2 d\xi$ , with a unitary Fourier transform. Jensen's inequality gives, for every integer  $k \geq 1$ ,

$$\|u(t)\|_{H^L}^k \leq N^{k-1} \|u(t)\|_{H^{kL}}. \quad (2.7)$$

For  $N = 0$ , the solution is zero and the conclusion is immediate. Combining (2.6) and (2.7) with (2.5) yields

$$\|V_n\|_{L^2(0, T; H^m)} \leq n! C_L^n N^n \|u\|_{L^2(0, T; H^{(n+1)L})}, \quad (2.8)$$

$$\|Q_n\|_{L^2(0, T; H^m)} \leq (n+1)! C_L^{n+1} N^{n+1} \|u\|_{L^2(0, T; H^{(n+2)L})}. \quad (2.9)$$

In particular, the equation has produced the full mixed membership

$$V_n, Q_n \in L^2(0, T; H^m) \quad \text{for every finite } n, m. \quad (2.10)$$

Each assertion uses a finite input from the same spatial column. No series over the orders, and no bound uniform in the orders, is taken.

The distributional identities  $\partial_t V_n = V_{n+1}$  and  $\partial_t Q_n = Q_{n+1}$  now place each row in  $H^1(0, T; H^m)$ . Its continuous representative satisfies, for example,

$$\|V_n(t) - V_n(s)\|_{H^m} \leq |t - s|^{1/2} \|V_{n+1}\|_{L^2(0, T; H^m)}. \quad (2.11)$$

The fundamental theorem of calculus, applied successively to adjacent rows, gives  $C^r([0, T]; H^m)$  for every finite  $r$ , with one-sided derivatives at the endpoints. Choosing  $m > k + 3/2$  embeds each row continuously into  $C_b^k(\mathbb{R}^3)$ . Thus every mixed derivative is jointly continuous on the closed slab. This proves smoothness directly on the finite domain.

Uniqueness of limits in distributions identifies the endpoint values across all spatial orders. Passing each finite recurrence (2.1)–(2.2) in sufficiently high Sobolev spaces identifies the full endpoint jet with the jet generated by  $u_*$ . In particular pressure retains its canonical normalization.

Local Euler theory at  $u_* \in H_\sigma^4$  gives a solution  $v$  on  $(T - \varepsilon, T + \varepsilon)$ , with  $v(T) = u_*$ . Persistence of regularity gives all spatial orders on the same smaller interval. To verify attachment, let  $w = u - v$  and  $D(t) = \|w(t)\|_2^2$  on its left-hand overlap with  $[0, T)$ . Incompressibility cancels transport and pressure in the difference energy identity, leaving

$$D'(t) = -2 \int_{\mathbb{R}^3} w \cdot (w \cdot \nabla) v \, dx, \quad |D'(t)| \leq 2 \|\nabla v(t)\|_\infty D(t).$$

For  $t < s < T$ , backward Gronwall therefore gives

$$D(t) \leq D(s) \exp\left(2 \int_t^s \|\nabla v(r)\|_\infty \, dr\right).$$

Both fields converge strongly to  $u_*$  in  $L^2$ , so letting  $s \uparrow T$  gives  $D(t) = 0$ . The extension is the same history. The row recurrences then give smooth velocity and pressure across  $T$ .  $\square$

**Corollary 2.2** (Global smoothness under whole-interval membership). *Let  $u_0 \in H_\sigma^\infty$ , and let  $[0, T_{\max})$  be the maximal classical Euler lifespan. Assume that, if  $T_{\max} < \infty$ ,*

$$\int_0^{T_{\max}} \|u(t)\|_{H^K}^2 \, dt < \infty \quad \text{for every finite } K. \quad (2.12)$$

*Then  $T_{\max} = \infty$ . The velocity and canonical pressure have every mixed derivative continuously on every finite closed time slab.*

*Proof.* If  $T_{\max}$  were finite, Theorem 2.1 would extend the same solution through  $T_{\max}$ , contradicting maximality. There is one continuation at the proposed finite endpoint. No sequence of local lifespans has to be summed. On every finite interval of the resulting global solution, persistence of spatial regularity and the row recurrences give the stated smoothness.  $\square$

A weakly continuous realization of the spatial intersection is another way to supply the theorem's membership. Let

$$\mathcal{J}_T^w = \bigcap_{m \in \mathbb{N}_0} C_w([0, T]; H^m), \quad (2.13)$$

where all paths and endpoint values agree as distributions.

**Corollary 2.3** (Weak spatial intersection). *If the velocity on  $[0, T)$  extends as one member of  $\mathcal{J}_T^w$ , then all conclusions of Theorem 2.1 hold. It is enough to prescribe weak spatial membership of velocity row zero.*

*Proof.* Weak continuity on the compact interval and uniform boundedness give, for each fixed  $m$ , a finite bound  $B_m = \sup_{0 \leq t \leq T} \|u(t)\|_{H^m}$ . Hence  $\|u\|_{L^2(0,T;H^m)} \leq \sqrt{T}B_m$ , which supplies (2.5). The theorem generates all remaining rows and proves their strong continuity.  $\square$

One can also start from weak membership of every temporal row in  $\mathcal{J}_T^w$ . Compactness and uniform boundedness then give  $B_{n,m} = \sup_{[0,T]} \|V_n\|_{H^m} < \infty$ , and the adjacent row implies the stronger estimate

$$\|V_n(t) - V_n(s)\|_{H^m} \leq B_{n+1,m}|t - s|. \quad (2.14)$$

For mixed integrability alone, a useful weaker coordinate pair is  $V_n \in L^2 H^{m+1}$ ,  $V_{n+1} \in L^2 H^{m-1}$ ; the Hilbert-triple trace theorem also gives  $V_n \in CH^m$  in that case [10].

**Proposition 2.4** (A common local interval from smooth data). *Every  $u_0 \in H_\sigma^\infty$  generates a solution satisfying (2.5) and (2.10) on one common positive interval  $[0, \tau]$ .*

*Proof.* Choose  $\tau$  strictly within the local  $H^4$  lifespan supplied by classical Euler theory [9, 11]. For every integer  $K \geq 4$ , its persistence estimate reads

$$\|u(t)\|_{H^K} \leq \|u_0\|_{H^K} \exp\left(C_K \int_0^t \|\nabla u(r)\|_\infty dr\right).$$

The time integral is finite on  $[0, \tau]$  because the  $H^4$  solution is continuous there. Regularized solutions satisfy the estimate on this same interval; passage to the local solution gives every finite  $K$ . Lower spatial orders follow by embedding. Thus the spatial column is finite on one common interval, and (2.9) supplies every temporal and pressure row there.  $\square$

The distinction in interval is essential. Proposition 2.4 constructs the full intersection on a common local interval from the datum. Corollary 2.2 requires membership through a putative finite maximal lifespan. Finiteness on every strictly shorter interval does not itself prove (2.12). The global conclusion is proved under that explicit hypothesis; its derivation for arbitrary smooth data is not supplied by the local result.

The Euler scaling acts on the complete rows by

$$V_n^{b,\lambda}(x, t) = b^{n+1} \lambda^n V_n(\lambda x, b\lambda t), \quad (2.15)$$

$$Q_n^{b,\lambda}(x, t) = b^{n+2} \lambda^n Q_n(\lambda x, b\lambda t), \quad (2.16)$$

$$\begin{aligned} \|V_n^{b,\lambda}(t)\|_{\dot{H}^m} &= b^{n+1} \lambda^{n+m-3/2} \\ &\quad \times \|V_n(b\lambda t)\|_{\dot{H}^m}. \end{aligned} \quad (2.17)$$

The two factors distinguish motion up the temporal tower from motion across a spatial row. With  $\mu = b\lambda$ , the velocity multiplier is

$$\mu^{n+1} \lambda^{m-5/2}.$$

One additional temporal derivative multiplies this scaling factor by  $\mu$ ; one additional spatial derivative multiplies it by  $\lambda$ . These are ratios of scaling multipliers, not ratios of the physical derivative norms. In fixed-viscosity NS,  $\mu = \lambda^2$  combines the two weights into

$$\lambda^{2n+m-1/2}.$$

The positions  $(n+1, m)$  and  $(n, m+2)$  then carry equal weight. They are exactly the positions linked by the viscous Laplacian in (1.2). Euler allows  $\mu$  and  $\lambda^2$  to differ. A temporal step no longer has a prescribed equivalence to two spatial steps.

This distinction also changes the meaning of a critical spatial height. For the NS velocity row,  $n = 0$ , the homogeneous  $\dot{H}^{1/2}$  norm is invariant under fixed-viscosity scaling. Euler has no spatial Sobolev order whose norm is invariant under every clock and length rescaling: the factor  $\mu^{n+1}$  remains at every temporal row. At a fixed clock, the velocity multiplier is independent of length when  $m = 5/2$ , but changing the clock multiplies it by  $\mu^{n+1}$ . That value is a property of the chosen fixed-clock slice. It supplies neither a universal Euler critical height nor a whole-interval regularity conclusion. Choosing the NS line inside Euler's scaling plane would restore the NS weights by choice, rather than by a restriction of Euler's dynamics.

The extra scaling freedom leaves the nonlinear compatibility intact. Every term  $(V_j \cdot \nabla)V_{n-j}$  carries the velocity-response multiplier  $\mu^{n+2}\lambda^{-1}$ , regardless of how the temporal derivatives are split. Pressure has multiplier  $\mu^{n+2}\lambda^{-2}$ , so its gradient participates at the required degree as well. At spatial order  $m$ , the pressure norm has multiplier  $\mu^{n+2}\lambda^{m-7/2}$ . The full pressure-complete recurrence preserves both degrees; freedom to rescale the hierarchy does not make its entries independently assignable.

Each row still carries every spatial Sobolev order in the common intersection. The change from NS to Euler concerns the equation connecting those orders and rows. The finite binomial calculation and energy interpolation above generate the temporal rows from the spatial column. They leave the production of that column over an entire candidate maximal interval as the precise hypothesis of the global continuation result.

A finite positive rescaling preserves membership in each spatial Sobolev space. If its parameters depend on a sequence of times, recovering a physical coordinate also recovers its inverse scale factor. Bounded normalized coordinates establish physical boundedness only when those recovery factors are controlled. The scaling plane identifies the freedom that an Euler construction has to retain; it does not construct the whole-interval spatial intersection by itself.

### 3 What reaches the terminal time from energy alone

Whole-interval spatial membership gives more than a distributional endpoint. It is useful to construct the endpoint that the equation already supplies, because every stronger realization will have to agree with it.

Energy controls how rapidly velocity can change when tested against a fixed smooth spatial field. For  $\varphi \in H^3$ ,

$$\begin{aligned} |\langle \partial_t u, \varphi \rangle| &= |\langle u \otimes u, \nabla \mathbb{P} \varphi \rangle| \\ &\leq N^2 \|\nabla \mathbb{P} \varphi\|_\infty \leq CN^2 \|\varphi\|_{H^3}. \end{aligned} \quad (3.1)$$

The projector is bounded on  $H^3$ , and  $H^3(\mathbb{R}^3)$  embeds in  $W^{1,\infty}$ . Integrating in time constructs one terminal distribution, independently of a subsequence.

**Proposition 3.1** (Unconditional terminal velocity). *There is a unique  $u_* \in L^2_\sigma$  such that*

$$\|u(t) - u_*\|_{H^{-3}} \leq CN^2(T - t), \quad u(t) \rightharpoonup u_* \quad \text{in } L^2. \quad (3.2)$$

Moreover  $\|u_*\|_2 \leq N$ , and the convergence is strong in  $H^{-r}$  for every  $r > 0$ .

*Proof.* Estimate (3.1) makes  $u(t)$  Cauchy in  $H^{-3}$ . Every sequence approaching  $T$  has a weakly convergent  $L^2$  subsequence by the energy bound. Its limit agrees with the  $H^{-3}$  limit. This identifies all subsequences and proves the full weak limit. Weak lower semicontinuity gives the norm bound. For  $0 < r < 3$ , interpolate the bounded  $L^2$  difference with its vanishing  $H^{-3}$  norm. Larger negative orders follow by continuous embedding.  $\square$

The weak endpoint can lose kinetic energy in this limiting description. This statement permits a defect; it does not assert that an Euler solution with such a defect exists. Define its size by

$$d_*^2 = N^2 - \|u_*\|_2^2. \quad (3.3)$$

Weak convergence shows that  $\|u(t) - u_*\|_2^2 \rightarrow d_*^2$ . Thus  $d_* = 0$  is exactly strong  $L^2$  convergence from times below  $T$ . Smoothness of the terminal distribution alone would not remove this difference between the distribution and the approaching motion.

Once that difference vanishes, a smooth endpoint attaches to the original flow with a comparison that uses the gradient of the smooth restart.

**Lemma 3.2** (Backward attachment). *If  $u_* \in H^\infty_\sigma$  and  $d_* = 0$ , the Euler solution extends smoothly through  $T$ .*

*Proof.* Let  $v$  be the local smooth Euler solution through  $v(T) = u_*$ , defined on a two-sided neighborhood of  $T$ . On the overlap, put  $w = u - v$ . The projected difference equation is

$$\partial_t w + \mathbb{P}(u \cdot \nabla w + w \cdot \nabla v) = 0.$$

The first transport term has zero  $L^2$  pairing with  $w$ . With  $D(t) = \|w(t)\|_2^2$ ,

$$D'(t) = -2 \int w \cdot (w \cdot \nabla v) \, dx, \quad |D'(t)| \leq 2 \|\nabla v(t)\|_\infty D(t).$$

For  $t < s < T$ , backward integration gives

$$D(t) \leq D(s) \exp \left( 2 \int_t^s \|\nabla v(r)\|_\infty \, dr \right). \quad (3.4)$$

The coefficient is bounded through  $T$ , while  $D(s) \rightarrow 0$  by the strong  $L^2$  limit as  $t \uparrow T$ . Hence  $u = v$  on the overlap.  $\square$

Spatial information can now be assembled without forcing all heights to use one chosen sequence of times.

**Theorem 3.3** (Reconstruction from spatial snapshots). *Assume that, for each integer  $m \geq 0$ , there is a sequence  $t_j^{(m)} \uparrow T$  with*

$$\sup_j \|u(t_j^{(m)})\|_{H^m} < \infty. \quad (3.5)$$

*Then  $u_* \in H_\sigma^\infty$ ,  $d_* = 0$ , and the same solution extends smoothly through  $T$ . Every velocity and canonical pressure row reaches its full spatial intersection there.*

*Proof.* At each fixed height, weak compactness in  $H^m$  produces a subsequential limit. Proposition 3.1 identifies it with  $u_*$ . Thus all heights belong to one distribution and  $u_* \in H_\sigma^\infty$ .

Choose any positive integer  $m$  and its bounded sequence. For  $f \in H^m$ , Fourier-space Hölder gives

$$\|f\|_2 \leq \|f\|_{H^{-3}}^{m/(m+3)} \|f\|_{H^m}^{3/(m+3)}. \quad (3.6)$$

Apply this to  $u(t_j^{(m)}) - u_*$ . The low norm tends to zero and the high norm stays bounded, so convergence is strong in  $L^2$ . Conservation of the velocity norm gives  $\|u_*\|_2 = N$ . Lemma 3.2 attaches the smooth restart. The row recurrences then give all terminal velocity and pressure derivatives.  $\square$

Global interpolation in (3.6) is important on  $\mathbb{R}^3$ : no extra spatial-tightness hypothesis enters this proof. The theorem also makes no optimality claim. A bounded approaching sequence at one classical height  $H^s$ ,  $s > 5/2$ , already gives continuation by the lifespan estimate in local theory. The all-height formulation identifies directly how distinct spatial observations recover the intersection, with one low terminal distribution fixing their compatibility.

A possible failure of strong convergence can also be recorded quadratically. Along any sequence approaching  $T$ , weak-\* compactness of the finite matrix measures  $u(t) \otimes u(t) dx$  yields a subsequential local limit of the form

$$u_* \otimes u_* dx + R, \quad R \geq 0. \quad (3.7)$$

Positivity follows by testing a quadratic form against a nonnegative compact function and using weak lower semicontinuity in  $L^2$ . The trace mass of  $R$  is at most  $d_*^2$ ; any remaining mass can escape compact sets. Pressure is still slaved to the velocity product, so its limiting distribution includes  $R_i R_j R_{ij}$ . The pressure formula alone does not set  $R$  to zero. If  $d_* = 0$ , strong  $L^2$  convergence removes both the local defect and escape, and the quadratic products converge in  $L^1$ .

The acceleration has a corresponding subsequential limit. Write  $\mu = u_* \otimes u_* dx + R$ . The extracted sequence satisfies

$$\begin{aligned} p(t_j) &\rightharpoonup R_i R_j \mu_{ij} \quad \text{in } H^{-s}, \quad s > 3/2, \\ \partial_t u(t_j) &\rightharpoonup -\mathbb{P} \operatorname{div} \mu \quad \text{in } H^{-s}, \quad s > 5/2. \end{aligned} \quad (3.8)$$

One extracted measure works for every displayed Sobolev index. Indeed, the dual maps  $R_i R_j : H^s \rightarrow C_0$  and  $\nabla \mathbb{P} : H^s \rightarrow C_0$  are bounded at the indicated indices. Weak-\* measure convergence can be tested against their images. The pressure and acceleration limits may depend on the subsequence; (3.8) does not assert differentiability at the terminal time.

The energy split is exact:

$$d_*^2 = \operatorname{tr} R(\mathbb{R}^3) + E_\infty, \quad E_\infty = N^2 - \operatorname{tr} \mu(\mathbb{R}^3) \geq 0.$$

The weak velocity trace determines neither contribution. Thus energy constructs a pressure-complete first response only with its possible quadratic defect retained.

### 3.1 Additional pressure contact removes the local stress defect

Pressure can distinguish terminal stresses that weak velocity alone cannot. This requires an additional hypothesis, separate from existence of the low trace. Define  $p[u_*] = R_i R_j(u_{*,i} u_{*,j})$  as a distribution. If

$$-\Delta(p(t) - p[u_*]) \longrightarrow 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3), \quad (3.9)$$

then  $u(t) \rightarrow u_*$  strongly in  $L^2_{\text{loc}}$ .

Indeed, every subsequential stress defect  $R$  above is a finite positive-semidefinite matrix measure, and the pressure equation gives  $\partial_i \partial_j R_{ij} = 0$ . Choose  $\Phi_L(x) = \frac{1}{2}|x|^2 \chi(x/L)$ , with a smooth compact cutoff equal to one near zero. Its Hessians are uniformly bounded and converge pointwise to  $I$ . Testing gives

$$0 = \int R : \nabla^2 \Phi_L \longrightarrow \text{tr } R(\mathbb{R}^3).$$

Dominated convergence uses finiteness of the positive trace measure. Positivity now forces  $R = 0$ . Local weak convergence plus convergence of the local squared norms proves strong local convergence, and the argument applies to every terminal subsequence.

Only spatial escape can remain in the energy defect. Terminal spatial tightness removes it and gives  $\|u_*\|_2 = N$ ; if in addition  $u_* \in H^\infty_\sigma$ , the attachment lemma gives smooth continuation. On the torus a constant positive matrix defect has zero double divergence and finite mass, so the cutoff-Hessian argument is unavailable. The periodic construction in Appendix D exhibits this distinction with exact Euler histories.

## 4 A terminal transport isometry and its spatial intersection

The energy defect has an exact representation that retains the pressure projection throughout transport. Fix the actual Euler velocity as a coefficient and transport a divergence-free test field  $f$  by

$$\partial_t f + L_t f = 0, \quad L_t f = \mathbb{P}(u(t) \cdot \nabla f). \quad (4.1)$$

This equation is linear in  $f$ . It is distinct from the linearized Euler equation, which would also contain  $\mathbb{P}(f \cdot \nabla u)$ . Its pressure keeps  $f$  divergence-free, and  $\langle L_t f, f \rangle = 0$  conserves its  $L^2$  norm.

On every strict interval, standard regularization gives a unitary propagator  $\mathcal{U}(t, s)$  on  $\mathcal{H} = L^2_\sigma$ . For completeness, take self-adjoint smooth Fourier cutoffs  $J_\varepsilon$  and solve the bounded-operator equations with

$$L_t^\varepsilon = J_\varepsilon \mathbb{P}(u(t) \cdot \nabla J_\varepsilon).$$

They are skew-adjoint bounded operators and conserve  $L^2$ . Differentiating spatially, cancelling the principal transport, and retaining the commutators gives, for smooth data,

$$\left| \frac{d}{dt} \|f(t)\|_{H^m}^2 \right| \leq C_m \|u(t)\|_{W^{m,\infty}} \|f(t)\|_{H^m}^2 \quad (m \geq 1).$$

All coefficient bounds are finite on a strict prefix. Difference estimates give convergence of the regularizations in  $C_t L^2$ ; higher uniform bounds and interpolation give convergence in every requested Sobolev space. Solving backward gives the inverse propagator. Density extends the resulting isometries to unitary maps on  $\mathcal{H}$ . This construction uses the PDE and its energy estimates, rather than inferring an operator-domain statement from skew symmetry alone.

Write  $\mathcal{U}_t = \mathcal{U}(t, 0)$  and  $A_t = \mathcal{U}_t^*$ . Since Euler itself satisfies (4.1),  $u(t) = \mathcal{U}_t u_0$ . The adjoint acts especially well on a fixed smooth test field:

$$\frac{d}{dt} A_t \varphi = A_t L_t \varphi, \quad \|A_t \varphi - A_s \varphi\|_2 \leq N|t - s| \|\nabla \varphi\|_\infty. \quad (4.2)$$

The derivative falls on the fixed field, so energy alone controls this limit even if derivatives of the evolving velocity become large.

**Theorem 4.1** (Terminal isometry and exact defect). *There is a strong operator limit  $W = \lim_{t \uparrow T} \mathcal{U}_t^*$  on  $\mathcal{H}$ , with  $W^*W = I$ . It satisfies*

$$\begin{aligned} u_* &= W^* u_0, \\ u_0 &= W u_* + g, \quad g = (I - WW^*) u_0, \\ g &\perp \text{ran } W, \quad \|g\|_2^2 = d_*^2. \end{aligned} \quad (4.3)$$

Furthermore  $\mathcal{U}_t W u_* \rightarrow u_*$  strongly in  $L^2$ , while  $\mathcal{U}_t g \rightarrow 0$  with constant norm  $\|g\|_2$ .

*Proof.* Estimate (4.2) constructs the limit on a dense smooth subspace. If  $h \in \mathcal{H}$  and  $\varphi$  approximates it in  $L^2$ , then

$$\|A_t h - A_s h\|_2 \leq 2\|h - \varphi\|_2 + N|t - s| \|\nabla \varphi\|_\infty.$$

Thus the limit exists for every  $h$ , and norm preservation gives  $W^*W = I$ . Pairing  $u(t) = \mathcal{U}_t u_0$  against fixed tests identifies its weak limit as  $W^* u_0$ , hence as  $u_*$ . The operator  $WW^*$  is the orthogonal projection onto the closed range of the isometry. Orthogonality gives (4.3). Finally, for every  $h \in \mathcal{H}$ ,

$$\|\mathcal{U}_t W h - h\|_2 = \|W h - A_t h\|_2 \rightarrow 0,$$

and  $\langle \mathcal{U}_t g, h \rangle \rightarrow \langle g, W h \rangle = 0$ . □

The terminal operator need not be declared onto. The result only constructs an isometry. Its range makes the attachment question a spatial intersection with an exact physical meaning:

$$\bigcap_{m \geq 0} W(H_\sigma^m) = W(H_\sigma^\infty). \quad (4.4)$$

Indeed, if a vector is represented as  $W h_m$  at each height, injectivity forces every  $h_m$  to equal its unique preimage under  $W$ . This does not assert that  $W$  preserves physical Sobolev regularity.

**Corollary 4.2** (Equivalent terminal membership). *The same Euler history extends in  $C_t H_\sigma^\infty$  through  $T$  if and only if*

$$u_0 \in W(H_\sigma^\infty). \quad (4.5)$$

Equivalently,  $u_* \in H_\sigma^\infty$  and  $g = 0$ .

*Proof.* Membership gives  $g = 0$  and a smooth preimage  $u_*$ , so Lemma 3.2 applies. Conversely, an  $H^\infty$  extension makes the coefficient in (4.1) smooth through  $T$ . The propagator then exists there, and  $u_0 = \mathcal{U}_T^* u(T) = W u(T)$  with  $u(T) \in H_\sigma^\infty$ . □

The terminal isometry is constructed from arbitrary smooth Euler data, but its construction does not establish (4.5). That membership is equivalent to continuation, rather than a proved smaller condition. It separates two exact requirements: the terminal physical velocity must have all spatial orders, and no part of the conserved energy may disappear from its strong limit.

## 5 Particle positions, circulation, and the changing metric

Projected transport retains pressure in physical space. Following particles instead makes the deformation of the fluid explicit. Let

$$\partial_t X(a, t) = u(X(a, t), t), \quad X(a, 0) = a, \quad F = D_a X, \quad v = \partial_t X. \quad (5.1)$$

At each strict time,  $X$  is a smooth volume-preserving diffeomorphism:  $\det F = 1$ . This follows from  $\partial_t F = (\nabla u \circ X)F$  and incompressibility. Volume preservation turns conservation of kinetic energy into a bound on the total displacement of the particles:

$$\|X(\cdot, t) - X(\cdot, s)\|_2 \leq \int_s^t \|u(X(\cdot, r), r)\|_2 dr = N|t - s|. \quad (5.2)$$

The maps themselves are not in  $L^2$ , but their displacements from  $\text{Id}$  are. Completeness gives a measurable map  $X_*$  such that

$$X(\cdot, t) - \text{Id} \longrightarrow X_* - \text{Id} \quad \text{in } L^2, \quad \|X(\cdot, t) - X_*\|_2 \leq N(T - t). \quad (5.3)$$

**Proposition 5.1** (Volume and terminal momentum). *The terminal map preserves Lebesgue measure:  $(X_*)_{\#} da = dx$ . Composition  $C_* f = f \circ X_*$  is an isometry from  $L_x^2$  to  $L_a^2$ . Without requiring an inverse map at  $T$ , the terminal velocity satisfies*

$$u_* = \lim_{h \downarrow 0} \mathbb{P} C_*^* \frac{X_* - X(\cdot, T - h)}{h}. \quad (5.4)$$

The convergence has error at most  $CN^2h$  in  $H^{-3}$ .

*Proof.* For  $\varphi \in C_c(\mathbb{R}^3)$  supported in  $B_R$ , and  $L > R$ ,

$$\int_{|a| > L} |\varphi(X(a, t))| da \leq \|\varphi\|_{\infty} \frac{N^2 T^2}{(L - R)^2}.$$

The event  $X(a, t) \in B_R$ ,  $|a| > L$ , requires displacement larger than  $L - R$ ; Chebyshev and (5.2) give the estimate. It also holds for  $X_*$ . On the finite label ball, convergence in measure and boundedness of the integrand pass the integral to  $X_*$ . Letting  $L \rightarrow \infty$  in the strict-time volume identity proves measure preservation globally. It gives the composition isometry.

Write  $C_t f = f \circ X(\cdot, t)$ . For compact smooth  $f$ ,

$$\|(C_t - C_*)f\|_2 \leq N(T - t)\|\nabla f\|_{\infty}.$$

Density and the isometry property imply strong convergence on every fixed  $L^2$  field. Define

$$D_h = \frac{X_* - X(\cdot, T - h)}{h} = \frac{1}{h} \int_{T-h}^T C_s u(s) ds, \quad M_h = C_*^* D_h.$$

These are Bochner integrals with  $\|D_h\|_2, \|M_h\|_2 \leq N$ . For  $f \in H^3$ ,

$$\begin{aligned} \langle M_h - u_*, f \rangle &= \frac{1}{h} \int_{T-h}^T \langle C_s u(s), (C_* - C_s)f \rangle ds \\ &\quad + \frac{1}{h} \int_{T-h}^T \langle u(s) - u_*, f \rangle ds. \end{aligned}$$

The first term is bounded by  $N^2 h \|\nabla f\|_{\infty} / 2$ . Proposition 3.1 bounds the second by  $CN^2 h \|f\|_{H^3}$ . This proves the asserted rate and, after the bounded projection, (5.4).  $\square$

Thus terminal positions and actual final displacement quotients recover momentum in a weak spatial topology. Positions alone have not determined momentum, and volume preservation alone has not controlled derivatives of  $X_*$ .

Circulation exposes what those derivatives would contribute. The pullback of the Euler velocity one-form obeys Weber's identity

$$F^T v = u_0 + \nabla_a q. \quad (5.5)$$

Indeed,  $\partial_t(F^T v) = \nabla_a(\frac{1}{2}|v|^2 - p \circ X)$ , so  $q(a, t) = \int_0^t (\frac{1}{2}|v|^2 - p \circ X)(a, s) ds$  is one choice of potential. Closed-loop integration gives Kelvin circulation; taking the exterior derivative gives Cauchy's vorticity formula

$$\omega(X(a, t), t) = F(a, t)\omega_0(a). \quad (5.6)$$

The conserved circulation constrains a velocity one-form carried by a deforming loop. It does not prescribe the physical length or orientation of that loop.

The circulation also has an unconditional terminal meaning as a current. For a compact smooth vector test  $\psi$ , define

$$\mathcal{C}_t(\psi) = \int F(a, t)\omega_0(a) \cdot \psi(X(a, t)) da.$$

Cauchy's formula, volume preservation, and integration by parts give  $\mathcal{C}_t(\psi) = \langle u(t), \text{curl } \psi \rangle$ . Hence

$$\mathcal{C}_t \longrightarrow \mathcal{C}_* := \text{curl } u_* \quad \text{in } H^{-4}, \quad |\mathcal{C}_*(\psi)| \leq N \|\text{curl } \psi\|_2, \quad u_* = \text{curl}(-\Delta)^{-1}\mathcal{C}_*. \quad (5.7)$$

The convergence rate in  $H^{-4}$  is at most  $CN^2(T - t)$ . The homogeneous bound places  $\mathcal{C}_*$  in  $\dot{H}^{-1}$ , and the last identity follows from  $\text{div } u_* = 0$  by Fourier inversion. There is no nonzero harmonic ambiguity in  $L^2$ .

This limit can be obtained directly from the initial covector before estimating deformation. For compact solenoidal  $f$ , let  $h_{t,f} = F^{-1}(f \circ X)$ . Piola's identity gives  $\text{div}_a h_{t,f} = 0$ , so (5.5) gives

$$Q_t(f) := \int u_0 \cdot h_{t,f} da = \int u(t) \cdot f dx.$$

Differentiating the transported test gives  $\partial_t h_{t,f} = F^{-1}([u, f] \circ X)$ , where  $[u, f] = (u \cdot \nabla)f - (f \cdot \nabla)u$ . The bracket is compact and divergence-free. Pairing and integrating by parts yields

$$Q'_t(f) = \int u_i u_j \partial_j f_i dx, \quad |Q_t(f) - Q_s(f)| \leq N^2 |t - s| \|\nabla f\|_\infty.$$

Taking  $f = \text{curl } \psi$  recovers (5.7). The paired material expression converges even though its deformation factors have no established individual terminal bounds. Its limit is a distribution on test fields; it does not supply a finite vorticity measure or circulation on every terminal material loop. The strict-time covector formulation is classical [6]; the terminal passage here uses its pairing with the low trace of the same history.

At a strict time, however, circulation together with the current geometry does determine the entire velocity. Set  $C_X = F^{-1}F^{-T}$ . Piola's identity turns physical incompressibility into

$$\text{div}_a(C_X(u_0 + \nabla_a q_X)) = 0. \quad (5.8)$$

Since  $F$  and  $F^{-1}$  are bounded on strict prefixes, this is a uniformly elliptic equation for  $q_X$  in the homogeneous energy space  $\dot{H}^1$ , modulo constants. Lax–Milgram gives its unique gradient, and

$$\mathcal{V}[X; u_0] = F^{-T}(u_0 + \nabla_a q_X) = v. \quad (5.9)$$

The function  $q_X$  minimizes

$$\mathcal{E}_X(r) = \frac{1}{2} \int (u_0 + \nabla_a r) \cdot C_X(u_0 + \nabla_a r) \, da. \quad (5.10)$$

Testing (5.8) against  $r - q_X$  gives the exact positive difference

$$\mathcal{E}_X(r) - \mathcal{E}_X(q_X) = \frac{1}{2} \int \nabla_a(r - q_X) \cdot C_X \nabla_a(r - q_X) \, da.$$

This reconstruction uses an anisotropic metric produced by the flow. It does not introduce a freely chosen pressure field.

Let  $S = (\nabla u + \nabla u^T)/2$ . Differentiating that metric gives

$$\partial_t C_X = -2F^{-1}(S \circ X)F^{-T}.$$

The minimizing condition cancels derivatives of  $q_X$  when the minimum energy is differentiated. The remaining term is

$$\frac{d}{dt} \mathcal{E}_X(q_X) = - \int v \cdot (S \circ X)v \, da = - \int u \cdot Su \, dx = 0. \quad (5.11)$$

The last integral is the divergence of  $u|u|^2/2$ . The variational minimum recovers kinetic energy conservation, while the ellipticity constants still depend on deformation. Passing (5.9) to the endpoint requires information about  $F$  beyond the displacement limit (5.3).

## 6 Spatial reconstruction of the material temporal coefficients

The material description offers a different way to organize temporal depth. Instead of differentiating velocity at a fixed physical position, write the particle displacement as  $\xi(a, t) = X(a, t) - a$ . Its curl and divergence are constrained by circulation and volume. Those two constraints determine the displacement through a spatial Hodge reconstruction.

Taking the curl of (5.5), and expanding  $F = I + D_a \xi$ , gives

$$\text{curl}_a \dot{\xi} = \omega_0 + \sum_{k=1}^3 \nabla_a \xi_k \times \nabla_a \dot{\xi}_k. \quad (6.1)$$

The volume equation  $\det(I + D_a \xi) = 1$  gives

$$\text{div}_a \xi = -\frac{1}{2} [(\text{tr } D_a \xi)^2 - \text{tr}((D_a \xi)^2)] - \det D_a \xi. \quad (6.2)$$

These identities retain pressure implicitly through the actual flow. In particular, differentiating (5.5) also gives  $F^T X_{tt} = \nabla_a(q_t - |v|^2/2)$ , the pressure acceleration in material coordinates.

Define the signed sources

$$\mathcal{C}(t) = \int_0^t \sum_k \nabla_a \xi_k \times \nabla_a \dot{\xi}_k \, ds, \quad (6.3)$$

$$\mathcal{D}(t) = -\frac{1}{2} [(\text{tr } D_a \xi)^2 - \text{tr}((D_a \xi)^2)] - \det D_a \xi. \quad (6.4)$$

Then  $\text{curl } \xi = t\omega_0 + \mathcal{C}$  and  $\text{div } \xi = \mathcal{D}$ . With  $\mathcal{N} = (-\Delta_a)^{-1}$ , the Hodge formula is

$$\xi = \text{curl } \mathcal{N}(t\omega_0 + \mathcal{C}) - \nabla \mathcal{N} \mathcal{D}. \quad (6.5)$$

The  $L^2$  displacement excludes an undetermined harmonic vector field.

This construction reaches every finite material Taylor coefficient. Let  $\xi_s = \partial_t^s \xi(a, 0)/s!$ , and put  $A_s = D_a \xi_s$ . We use coefficient notation without assuming a positive radius for an infinite series. Write  $c_s = \text{curl } \xi_s$  and  $d_s = \text{div } \xi_s$ . Matching coefficients in (6.1) gives

$$c_s = \delta_{s1} \omega_0 + \frac{1}{2s} \sum_{m+n=s} (n-m) \sum_k \nabla(\xi_m)_k \times \nabla(\xi_n)_k, \quad (6.6)$$

where all summation indices are positive. The antisymmetric weight  $n-m$  results from pairing the ordered terms before estimating them. For the determinant, define its symmetric polarization by

$$\mathfrak{d}(A, B, C) = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} A_{ip} B_{jq} C_{kr}, \quad \mathfrak{d}(A, A, A) = \det A.$$

Then

$$\begin{aligned} d_s &= \frac{1}{2} \sum_{m+n=s} [\text{tr}(A_m A_n) - \text{tr } A_m \text{tr } A_n] \\ &\quad - \sum_{\ell+m+n=s} \mathfrak{d}(A_\ell, A_m, A_n), \\ \xi_s &= \text{curl } \mathcal{N} c_s - \nabla \mathcal{N} d_s. \end{aligned} \quad (6.7)$$

**Proposition 6.1** (Finite material coefficient reconstruction). *For every  $u_0 \in H_\sigma^\infty$ , recurrences (6.6)–(6.7) determine  $\xi_s \in H^\infty$  at every finite order. On the local classical Euler solution, these are exactly its material Taylor coefficients.*

*Proof.* At order one,  $c_1 = \omega_0$ ,  $d_1 = 0$ , and  $\xi_1 = u_0$ . All terms at order  $s > 1$  involve strictly lower coefficients. The Sobolev algebra property gives arbitrary positive spatial regularity for their products. The apparent low-frequency difficulty in the inverse derivative is removed by the divergence structure of the sources.

In the curl source,  $\nabla f \times \nabla g = \text{curl}(f \nabla g)$ . The quadratic determinant polarization has the identity

$$\text{tr}(Df Dg) - \text{div } f \text{div } g = \text{div}((f \cdot \nabla)g - f \text{div } g). \quad (6.8)$$

For the cubic terms, the cofactor identity  $\text{div}(\text{cof}(D\xi)^T \xi) = 3 \det D\xi$  and polarization give an analogous divergence of products with one undifferentiated factor. These products belong to  $L^2$ , using one factor in  $L^2$  and the remaining smooth factors in  $L^\infty$ . After Hodge inversion, the resulting compositions of derivatives and  $\mathcal{N}$  have order zero and are bounded on  $L^2$ . This establishes the missing low-frequency control as well as every higher Sobolev order. The classical flow satisfies the identical triangular recurrences; induction identifies the coefficients.  $\square$

The first pressure response appears at the next coefficient:

$$\xi_1 = u_0, \quad \xi_2 = -\frac{1}{2} \nabla p_0, \quad c_3 = -\frac{1}{6} \sum_k \nabla(u_0)_k \times \nabla \partial_k p_0. \quad (6.9)$$

Here  $d_3 = \text{tr}(A_1 A_2) - \det A_1$ . The recurrence does not replace pressure by a free material force. It reconstructs its spatially coupled responses from the initial velocity. Cauchy-invariant Taylor

recursions and local Lagrangian time analyticity are established by Zheligovsky–Frisch [12]; the differential-form formulation and its geometric extensions are developed by Besse–Frisch [3]. Proposition 6.1 uses that classical mechanism in the all-order Sobolev setting, with its low-frequency closure written explicitly. Its role here is to compare finite coefficient reconstruction with the actual terminal topology. A local analytic interval does not supply a uniform radius through an arbitrary later terminal approach.

The endpoint strength of (6.5) can be measured exactly. From (5.3),  $\xi(t) \rightarrow \xi_*$  in  $L^2$ . Consequently

$$\mathcal{C}(t) \rightarrow \mathcal{C}_* := \operatorname{curl} \xi_* - T\omega_0, \quad \mathcal{D}(t) \rightarrow \mathcal{D}_* := \operatorname{div} \xi_* \quad (6.10)$$

strongly in  $\dot{H}^{-1}$ . Fourier orthogonality gives

$$\begin{aligned} \|\xi(t) - \xi_*\|_2^2 &= \|(t - T)\omega_0 + \mathcal{C}(t) - \mathcal{C}_*\|_{\dot{H}^{-1}}^2 \\ &\quad + \|\mathcal{D}(t) - \mathcal{D}_*\|_{\dot{H}^{-1}}^2. \end{aligned} \quad (6.11)$$

The combined nonlinear sources do converge. At this level, Hodge reconstruction returns precisely the displacement topology already constructed by energy. It adds no terminal spatial derivative.

Material temporal depth must also be converted back to Eulerian temporal depth. Holding physical position fixed while labels change introduces

$$\begin{aligned} \mathcal{T}_X &= \partial_t - (F^{-1}v) \cdot \nabla_a, \\ V_n \circ X &= \mathcal{T}_X^n v. \end{aligned} \quad (6.12)$$

Both the inverse deformation and its differentiated coefficients enter this conversion. Smoothness of each finite  $\xi_s$  at the initial time does not remove that terminal requirement. Under (1.4),  $\xi_s^{b,\lambda}(a) = b^s \lambda^{s-1} \xi_s(\lambda a)$ , which also shows why the material coefficients and Eulerian velocity rows carry different characters.

## 7 The limits of reconstruction from energy

The endpoint constructions reach a velocity, a transport operator, and material currents without first bounding the spatial derivatives throughout the interval. They leave a tempting possibility: perhaps energy can control successive temporal responses in weak spaces, after which the equation could recover their spatial content. The first response permits such control. The second does not admit any estimate uniform over a fixed-energy class, even when the output is tested in an arbitrarily weak fixed Sobolev space.

### 7.1 The sharp first-response topology

**Proposition 7.1** (Energy-level time modulus). *For every  $s > 5/2$ , the classical Euler history satisfies*

$$\|\partial_t u(t)\|_{H^{-s}} \leq C_s N^2, \quad \|\partial_t u(t)\|_{B_{2,\infty}^{-5/2}} \leq C N^2. \quad (7.1)$$

*The path is Lipschitz in each of these spaces. Neither the strong  $H^{-5/2}$  endpoint nor  $B_{2,q}^{-5/2}$ ,  $q < \infty$ , has a bound depending only on energy over all smooth Euler initial data.*

*Proof.* Since  $u_i u_j \in L^1$ , its Fourier transform is bounded by  $N^2$ . The Leray-divergence symbol gives  $|\widehat{\partial_t u}(\zeta)| \leq CN^2|\zeta|$ . The square of the Sobolev bound is controlled by

$$CN^4 \int_0^\infty r^4 (1+r^2)^{-s} dr,$$

which is finite precisely when  $s > 5/2$ . On a dyadic block, the localized Leray-divergence kernel has  $L^2$  norm at most  $C2^{5j/2}$ . Young's inequality gives  $\|\Delta_j \partial_t u\|_2 \leq C2^{5j/2}N^2$ , including a bounded low block. Taking the supremum of the weighted block norms proves the Besov estimate. Time integration gives the moduli.

For sharpness, take a nonzero real  $\varphi \in C_c^\infty(\mathbb{R})$ , set  $\psi(x) = \varphi(2x_1)\varphi(x_2)\varphi(x_3)$ , and let  $v = (\partial_2 \psi, -\partial_1 \psi, 0)$ . Its kinetic tensor is

$$M = \int v \otimes v dx = \text{diag}(m/2, 2m, 0), \quad m = \left( \int \varphi^2 \right)^2 \int (\varphi')^2 > 0.$$

Multiply  $v$  by a constant to fix any prescribed positive energy. For the local history  $U$  starting at  $v$ , write  $a = \partial_t U(0)$ . In this proof we use  $\widehat{f}(\zeta) = \int e^{-ix \cdot \zeta} f(x) dx$ , absorbing Plancherel constants into the estimates. Then

$$r^{-1} \widehat{a}(rn) \longrightarrow -i\Pi(n)Mn \quad (r \downarrow 0), \quad \Pi(n) = I - n \otimes n,$$

uniformly on  $S^2$ . The limiting angular square integral is

$$\int_{S^2} |\Pi(n)Mn|^2 d\sigma(n) = \frac{4\pi}{15} (3 \text{tr}(M^2) - (\text{tr } M)^2) > 0.$$

Scale the exact Euler history by  $U_K(x, t) = K^{3/2}U(Kx, K^{5/2}t)$ . Its initial acceleration satisfies  $\widehat{a}_K(\zeta) = K\widehat{a}(\zeta/K)$ . For a sufficiently small fixed  $\varepsilon > 0$ , the angular square average of  $\widehat{a}_K(rn)$  is bounded below by  $cr^2$  on  $1 \leq r \leq \varepsilon K$ . Consequently

$$\|a_K\|_{H^{-5/2}}^2 \geq c \log K - C, \quad \|a_K\|_{H^{-s}} \geq c_s K^{5/2-s} \quad (s < 5/2). \quad (7.2)$$

Rapid decay of the unscaled stress transform supplies the matching upper bound  $\|a_K\|_{H^{-5/2}} \leq C\sqrt{1 + \log K}$ . Each dyadic block with  $1 \ll 2^j \ll K$  contributes a positive constant to its weighted  $B_{2,q}^{-5/2}$  norm. Summing over these blocks gives growth at least  $c_q(\log K)^{1/q}$  when  $q < \infty$ .  $\square$

The weak time modulus also permits a fixed  $L^2$  change in an arbitrarily short interval across different data. In the preceding example  $a \neq 0$ , so choose a small  $c > 0$  for which  $d = \|U(c) - U(0)\|_2 > 0$ . Exact scaling gives

$$\|U_K(cK^{-5/2}) - U_K(0)\|_2 = d. \quad (7.3)$$

This excludes an  $L^2$  modulus controlled by energy alone. A modulus depending on the higher norms of a fixed initial field is unaffected. The local lifespan is rescaled along with the solution, so every chord in (7.3) belongs to a classical history.

## 7.2 A fixed-output obstruction at the second temporal row

**Theorem 7.2** (No energy-only bound for row two). *Fix  $E > 0$  and  $\sigma \in \mathbb{R}$ . There are real solenoidal Schwartz data  $u_{0,K}$ , with  $\|u_{0,K}\|_2^2 = E$ , whose local classical Euler histories satisfy*

$$\|\partial_t^2 u_K(0)\|_{H^\sigma} \geq c_{E,\sigma} K \quad (K \geq K_0). \quad (7.4)$$

*The growth occurs in a fixed bounded Fourier region separated from zero.*

*Proof.* Write  $B(v, w) = -\mathbb{P}(v \cdot \nabla w)$ . Euler generates

$$V_2 = B(B(u, u), u) + B(u, B(u, u)).$$

Consider the three frequency–polarization pairs

$$\begin{array}{l|l} p = (K, 0, 0) & a = e_2 \\ r = (0, K, 0) & b = e_1 + e_3 \\ s = (-K, -K, 1) & c = e_1 - e_2 \end{array} \quad p + r + s = q = e_3.$$

Each polarization is perpendicular to its frequency. For nonzero  $\zeta$ , write  $\Pi(\zeta) = I - \zeta \otimes \zeta / |\zeta|^2$ . For a pair  $(k, d), (\ell, e)$ , the symmetrized quadratic coefficient is  $-iC_{k\ell}$ , where

$$C_{k\ell} = \Pi(k + \ell)((\ell \cdot d)e + (k \cdot e)d).$$

Put  $D = K^2 + 1$ . The three allocations of the remaining singleton to the inner pair contribute at  $q$

$$(-K, K, 0), \quad (K - 2K^2/D, -2K/D, 0), \quad (0, K(K^2 + 2K - 1)/D, 0).$$

Their sum is

$$\left( -\frac{2K^2}{K^2 + 1}, \frac{2K(K^2 + K - 1)}{K^2 + 1}, 0 \right). \quad (7.5)$$

This computes the coefficient with every ordering in both cubic terms retained. The second component divided by  $K$  tends to 2.

To obtain finite-energy fields, choose a fixed nonzero nonnegative even  $\phi \in C_c^\infty(B_\varepsilon)$  and replace a mode  $(k, d)$  by  $\Pi(\zeta)d\phi(\zeta - k)$ , adding its reflected packet to impose reality. For small fixed  $\varepsilon$ , the six supports are disjoint when  $K$  is large. Their  $L^2$  normalization factor tends to a positive constant. Near  $q$ , only permutations of the three positive packet centers contribute. Inner pair frequencies have size comparable to  $K$ , whereas the outer derivative in

$$B(V_1, u) + B(u, V_1) = -\mathbb{P} \operatorname{div}(V_1 \otimes u + u \otimes V_1)$$

is the bounded output frequency. The inner multiplier is of order  $K$ . Dividing the complete cubic multiplier by  $K$  gives smooth convergence in fixed packet coordinates near these centers. By (7.5), its second component stays positive after  $\varepsilon$  is chosen small enough. Integrating the nonnegative packet profiles gives a nonzero limiting output in a fixed ball near  $q$ . Thus its  $L^2$  norm grows at least linearly in  $K$ . Every fixed Sobolev weight is bounded above and below on that ball, which proves (7.4).  $\square$

The failure is not caused by measuring the output at progressively finer scales. Arbitrarily high input frequencies generate an unbounded response at a fixed output scale. Passing to a weaker fixed Sobolev topology cannot remove it. This rules out the proposed induction from energy to all weak temporal rows. It does not rule out bounds that use the higher spatial information of the initial datum or the correlations of its evolution, and it constructs no singular Euler history.

## 8 Pressure and material separation

Circulation determines how vorticity is carried by deformation. Pressure determines how that deformation accelerates. Write  $A = \nabla u$ ,  $S = (A + A^T)/2$ , and  $P = \nabla^2 p$ , reserving  $\mathbb{P}$  for the Leray projector. The material derivative is  $D_t = \partial_t + u \cdot \nabla$ . Differentiating Euler gives

$$D_t A = -A^2 - P, \quad F_{tt} = -(P \circ X)F, \quad D_t^2 \omega = -P\omega. \quad (8.1)$$

The corresponding material vorticity equation is classical; see Gibbon–Holm–Kerr–Roulstone [7] and the pressure continuation arguments in [4, 5].

In the second formula the  $A^2F$  produced by differentiating  $F_t = (A \circ X)F$  cancels the corresponding term in  $D_t A$ . The vorticity identity then follows from Cauchy's formula. Pressure acts on an actual particle separation, with its direction determined by the evolving velocity.

For a fixed unit vector  $e$  and material direction  $z = Fe$ , write  $r = |z|$ ,  $q = z/r$ , and  $\sigma = q^T(S \circ X)q$ . Invertibility makes  $r > 0$  on strict intervals. The separation equation yields

$$(\log r)' = \sigma, \quad (\log r)'' = |q'|^2 - \sigma^2 - q^T(P \circ X)q. \quad (8.2)$$

The negative square comes from differentiating the logarithm. In the equation for the length itself,  $r''/r = |q'|^2 - q^T(P \circ X)q$ , that square is absent. Twice integrating gives the exact relation

$$\begin{aligned} & \int_0^t (t-s)(|q'(s)|^2 - q(s)^T P(X(a, s), s)q(s)) \, ds \\ &= \log |F(a, t)e| - t e^T S(a, 0)e + \int_0^t (t-s)\sigma(s)^2 \, ds. \end{aligned} \quad (8.3)$$

An upper bound on the left by a fixed constant plus the final square integral is exactly a uniform upper bound on  $\log |Fe|$ , up to the bounded initial-strain term. It reformulates deformation control. It does not derive it from the sign of a term in (8.2).

## 9 Continuation from material pressure

The separation equation (8.1) gives useful continuation conditions even without an unconditional deformation bound. Its second temporal order lets us integrate pressure twice before comparing growth. We first use an absolute pressure bound, then retain directional sign, and finally cancellation across time in a matrix primitive.

**Theorem 9.1** (Inverse-square pressure criterion). *Set  $h(t) = \|\nabla^2 p(t)\|_{L^\infty, \text{op}}$ . If*

$$\limsup_{t \uparrow T} (T-t)^2 h(t) < 2, \quad (9.1)$$

*the Euler solution extends smoothly through  $T$ , and every temporal velocity and pressure row has a strong  $H^\infty$  endpoint.*

*Proof.* Choose  $t_0 < T$  and  $c < 2$  such that  $(T-t)^2 h(t) \leq c$  on the final interval. Start particle labels at  $t_0$ . Twice integrating  $z_{tt} = -(P \circ X)z$  for  $z = \omega \circ X$ , taking norms, and taking the supremum gives

$$W(t) \leq A_0 + B_0(t - t_0) + \int_{t_0}^t (t-s)h(s)W(s) \, ds, \quad (9.2)$$

where  $W(t) = \|\omega(t)\|_\infty$ ,  $A_0 = W(t_0)$ , and  $B_0 = \|D_t \omega(t_0)\|_\infty$ . For  $b < T$ , integrate once more and use nonnegative Fubini:

$$\begin{aligned} I_b &:= \int_{t_0}^b W(t) \, dt \leq A_0(b - t_0) + \frac{1}{2}B_0(b - t_0)^2 \\ &\quad + \frac{1}{2} \int_{t_0}^b (b-s)^2 h(s)W(s) \, ds \\ &\leq A_0(T - t_0) + \frac{1}{2}B_0(T - t_0)^2 + \frac{c}{2}I_b. \end{aligned}$$

Absorb the last term and let  $b \uparrow T$ :

$$\int_{t_0}^T \|\omega(t)\|_\infty dt \leq \frac{A_0(T - t_0) + B_0(T - t_0)^2/2}{1 - c/2} < \infty. \quad (9.3)$$

The Beale–Kato–Majda criterion [2] gives continuation. Persistence of spatial regularity and the full row recurrences supply the terminal intersection.  $\square$

Chae–Constantin [4, Theorem 1.1] derive an absolute pressure-Hessian continuation criterion from the material vorticity equation, including a Type-I test with their stated coefficient below 1. The integration of the Volterra inequality above gives the operator-norm coefficient 2 in (9.1). The scalar comparison in Theorem 9.1 stops at  $c = 2$ :  $y(t) = (T - t)^{-1}$  satisfies  $y'' = 2(T - t)^{-2}y$  and has divergent time integral. This checks the comparison boundary, not the existence of an Euler flow attaining it.

Their later work [5] retains directional signs. Set  $\xi = \omega/|\omega|$ ,  $\zeta = S\xi/|S\xi|$ ,  $\alpha = \xi^T S\xi$ , and  $\rho = \xi^T P\xi$ , extending the directions by zero where their denominators vanish. With  $[a]_- = \max(-a, 0)$ , put

$$b_1(t) = \|[\zeta \cdot P\xi]_-(t)\|_\infty.$$

The material identities

$$D_t|\omega| \leq |S\omega|, \quad D_t|S\omega| = -\zeta \cdot P\xi |\omega|$$

hold almost everywhere, with the norm derivatives interpreted through their absolutely continuous representatives. They give (9.2) with  $h = b_1$  and  $B_0 = \|S(t_0)\omega(t_0)\|_\infty$ . This is the Volterra inequality used in their proof. The absorption above yields the directional extension

**Corollary 9.2** (Directional coefficient two). *If*

$$\limsup_{t \uparrow T} (T - t)^2 \|[\zeta \cdot P\xi]_-(t)\|_\infty < 2, \quad (9.4)$$

*then the Euler history extends smoothly through  $T$ .*

The earlier operator condition implies this one because  $[\zeta \cdot P\xi]_- \leq \|P\|_{\text{op}}$ . The coefficient 2 here is a deduction from the stated inequality; the printed Euler Type-I coefficient in [5] is 1. Its separate coefficient 2 for two-dimensional Boussinesq concerns a different equation.

For comparison, that paper also uses

$$b_2(t) = \| [|S\xi|^2 - 2\alpha^2 - \rho]_+(t) \|_\infty$$

and the sufficient condition

$$\int_{t_0}^T \exp \left( \int_{t_0}^t (t - s) b_i(s) ds \right) dt < \infty, \quad i = 1 \text{ or } 2. \quad (9.5)$$

For  $b_2$ , twice integrating  $D_t^2 \log |\omega| = |S\xi|^2 - 2\alpha^2 - \rho$  retains the initial derivative term  $\alpha(t_0)(t - t_0)$ . Its finite exponential prefactor does not alter (9.5). The Volterra absorption for  $|\omega|$  does not transfer the coefficient 2 to this logarithmic criterion. These established signed tests give the appropriate comparison for the matrix construction below: its distinct operation is integration of the full pressure matrix before taking a norm.

A second direct consequence of (9.2) is the material weighted-action condition

$$\sup_a \int_{t_0}^T (T - s) \|P(X(a, s), s)\|_{\text{op}} ds < \infty. \quad (9.6)$$

Before taking a spatial supremum, replace  $t - s$  by  $T - s$  in the particlewise Volterra inequality. Gronwall bounds  $|z(a, t)|$  by its finite initial affine bound times the exponential of this weighted integral. Uniformity in  $a$  gives bounded vorticity and continuation. This criterion permits different pressure histories on different particles, but takes the absolute matrix norm before time integration.

### 9.1 A signed pressure primitive

Absolute values can discard cancellation along a material path. To retain it, put  $r_0 = T - t_0$ ,  $r = T - t$ , and  $\tau = \log(r_0/r)$ . Then  $\partial_\tau = r\partial_t$ , so the vorticity separation equation becomes

$$z_{\tau\tau} + z_\tau + Kz = 0, \quad K(a, \tau) = r^2 P(X(a, t), t). \quad (9.7)$$

Define a symmetric matrix primitive along that path:

$$J(a, \tau) = \int_0^\tau K(a, \sigma) d\sigma = \int_{t_0}^t (T - s) P(X(a, s), s) ds. \quad (9.8)$$

The matrix integral precedes the norm. Pressure matrices at different times are allowed to fail to commute.

Introduce  $v = z_\tau + Jz$  and  $y = z + v$  within this calculation. The derivative  $J_\tau z$  cancels the pressure force in (9.7), leaving

$$\begin{aligned} y_\tau &= -J^2 y + (J + J^2)v, \\ v_\tau &= (J - J^2)y + (J^2 - I)v. \end{aligned} \quad (9.9)$$

For  $\mathcal{E} = |y|^2 + |v|^2$ , symmetry of  $J$  cancels the two mixed  $J^2$  terms:

$$\frac{1}{2}\mathcal{E}_\tau = -|Jy - v|^2 + |Jv|^2. \quad (9.10)$$

This is an exact coupled identity, with a positive energy and a retained negative square. It does not assert monotonicity.

The symmetric block matrix of the quadratic form on the right is

$$\begin{pmatrix} -J^2 & J \\ J & J^2 - I \end{pmatrix}.$$

For an eigenvalue  $q$  of  $J$ , its corresponding two-dimensional block has trace  $-1$ , determinant  $-q^4$ , and largest eigenvalue  $(\sqrt{1 + 4q^4} - 1)/2$ . Set

$$\gamma(d) = \frac{\sqrt{1 + 4d^4} - 1}{2}. \quad (9.11)$$

Instantaneous diagonalization of this quadratic form gives  $\mathcal{E}_\tau/2 \leq \gamma(\|J\|_{\text{op}})\mathcal{E}$ . No time-dependent eigenvector change is made in the differential equation. In particular  $\gamma(d) \leq d^4$ ; estimating the separate equations before combining them would lose this small-primitive gain.

**Theorem 9.3** (Signed material pressure criterion). *Let  $\mathcal{B} = \{a : \omega(a, t_0) \neq 0\}$ , with particle labels started at  $t_0$ . Suppose, for some  $d < 2^{1/4}$ ,*

$$\sup_{a \in \mathcal{B}} \sup_{t_0 \leq t < T} \left\| \int_{t_0}^t (T - s) P(X(a, s), s) ds \right\|_{\text{op}} \leq d. \quad (9.12)$$

*Then the same Euler solution extends smoothly through  $T$ .*

*Proof.* If  $\mathcal{B}$  is empty, finite-energy divergence-free and curl-free velocity is zero. Otherwise,  $J(a, 0) = 0$  gives

$$v(a, 0) = r_0 A(a, t_0) \omega(a, t_0), \quad y(a, 0) = \omega(a, t_0) + v(a, 0).$$

The strict-time Sobolev regularity bounds these vectors uniformly. Let  $C_0 = \sqrt{2} \sup_a \mathcal{E}(a, 0)^{1/2}$ . Integrating (9.10) gives  $\|\omega(t)\|_\infty \leq C_0 e^{\gamma(d)\tau}$ . Labels with zero initial vorticity retain zero vorticity. Since  $dt = r_0 e^{-\tau} d\tau$ ,

$$\int_{t_0}^T \|\omega(t)\|_\infty dt \leq \frac{r_0 C_0}{1 - \gamma(d)} < \infty, \quad (9.13)$$

because  $\gamma(d) < 1$  exactly when  $d < 2^{1/4}$ . The estimates first hold on strict prefixes with constants independent of their endpoint. Monotone convergence and BKM complete the passage.  $\square$

Uniform Cauchy convergence of the weighted signed primitives along all vortical trajectories is also sufficient: choose a late reference time so their differences satisfy (9.12). A more flexible condition follows by retaining the growth rate in time. If

$$\mathcal{A}(\tau) = \sup_{a \in \mathcal{B}} \int_0^\tau \gamma(\|J(a, \sigma)\|_{\text{op}}) d\sigma$$

and

$$\int_0^\infty \exp[-\tau + \mathcal{A}(\tau)] d\tau < \infty, \quad (9.14)$$

the identical proof gives continuation. This permits intermittent larger primitive values when their accumulated effect remains integrable on the physical clock. For example,  $\mathcal{A}(\tau) \leq C + c\tau$  with  $c < 1$  suffices. Also,

$$\sup_{a \in \mathcal{B}} \int_0^\infty \|J(a, \tau)\|_{\text{op}}^4 d\tau < \infty$$

gives uniformly bounded vorticity, since  $\gamma(d) \leq d^4$ .

Energy restricts how much of a late pressure impulse can come from far away. For a fixed distance  $R > 0$ , define the far part of the Hessian by integrating  $\partial_i \partial_j \partial_k \partial_l G(x - y) u_k(y) u_l(y)$  over  $|x - y| > R$ , where  $G = 1/(4\pi|x|)$ . The kernel has size at most  $C|x - y|^{-5}$ , hence

$$\|P_{\text{far}}(x, t)\|_{\text{op}} \leq CN^2 R^{-5}, \quad \left\| \int_{t_0}^t (T - s) P_{\text{far}}(X(a, s), s) ds \right\|_{\text{op}} \leq CN^2 R^{-5} (T - t_0)^2. \quad (9.15)$$

If smooth continuation fails, the supremum in (9.12) cannot lie strictly below  $2^{1/4}$  on any final interval. For fixed  $R$ , the far contribution in (9.15) tends to zero as that interval shortens. A necessary late impulse must then involve the nearby pressure contribution along a vortical trajectory. This is a necessary condition for a hypothetical failure, not a bound that excludes it.

The distinction between absolute pressure and its primitive can be checked in explicit symmetric matrix equations. This check does not prescribe an Euler pressure. Take  $T = 1$ ,  $t_0 = 0$ , and

$$D_1 = \text{diag}(1, -1, 0), \quad D_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For  $\varepsilon > 0$  and  $\lambda > 0$ , set

$$J(\tau) = \varepsilon \{ \sin(\lambda\tau) D_1 + (1 - \cos(\lambda\tau)) D_2 \}, \\ K(\tau) = J_\tau = \varepsilon \lambda \{ \cos(\lambda\tau) D_1 + \sin(\lambda\tau) D_2 \}.$$

Then  $\|J(\tau)\|_{\text{op}} = 2\varepsilon|\sin(\lambda\tau/2)|$  and  $\|K(\tau)\|_{\text{op}} = \varepsilon\lambda$ . The matrices at different times need not commute. For  $\varepsilon = 1/4$ ,  $\lambda = 12$ , the auxiliary pressure  $P_{\text{aux}}(t) = (1-t)^{-2}K(-\log(1-t))$  has inverse-square coefficient 3 and infinite absolute weighted action, while its primitive has norm at most  $1/2$ . The energy identity (9.10) makes every solution of the associated Jacobi equation integrable in physical time. Conversely,  $K = \frac{1}{2}D_1$  has absolute coefficient  $1/2$ , while  $J = \frac{1}{2}\tau D_1$  is unbounded. These examples separate the two sufficient conditions at the level of symmetric matrix equations. They do not compare all directional Euler criteria or exhibit a fluid realizing either pressure history.

For an actual Euler history, the primitive has further structure. Define  $L = r(A \circ X)$ ,  $\Sigma = \text{sym } L$ , and  $w = r(\omega \circ X)$ . The gradient Riccati equation gives

$$\begin{aligned} L_\tau &= -L - L^2 - K, \\ J(\tau) &= L(0) - L(\tau) - \int_0^\tau (L + L^2) d\sigma, \\ \text{tr } J(\tau) &= \int_0^\tau \left( \frac{1}{2}|w|^2 - |\Sigma|_F^2 \right) d\sigma, \quad w_\tau = (\Sigma - I)w. \end{aligned} \tag{9.16}$$

Here  $\text{tr } L = 0$ , and the skew part of  $L$  has squared trace  $-|w|^2/2$ . The pressure matrix itself is not trace-free. Even its trace primitive compares accumulated rotation and strain with no fixed sign. The trace-free part retains additional matrix information. Thus the Euler equations identify what must be controlled in the signed criterion; these identities do not yet bound that quantity from the initial data.

## 10 How the velocity generates pressure curvature

The continuation criteria follow particles through pressure curvature. To derive that control from the initial datum, pressure has to be reconstructed from the velocity as it changes. Its Poisson equation fixes the Hessian at one instant; its material evolution identifies the interactions that change it next.

Pressure changes the matrix acceleration without reducing its  $L^2$  magnitude. Set  $B = A^2$  and  $\theta = \text{tr } B$ , so  $P = \nabla^2 \mathcal{N}\theta$ . The second Newton-tensor identity for a divergence-free Jacobian gives  $\partial_i B_{ij} = \frac{1}{2}\partial_j \theta$ . At a nonzero Fourier frequency, with  $e = \zeta/|\zeta|$ , this implies

$$\widehat{B} : (e \otimes e) = \frac{1}{2}\widehat{\theta}, \quad \widehat{P} = -\widehat{\theta} e \otimes e.$$

Thus  $B + P$  is the reflection of  $B$  across the orthogonal complement of the rank-one direction  $e \otimes e$ . Plancherel gives

$$\|D_t A\|_2 = \|A^2\|_2, \quad \langle D(B + P), E(B + P) \rangle = \langle DB, EB \rangle, \tag{10.1}$$

for constant-coefficient mixed derivatives  $D, E$  whenever these fields are in  $L^2$ . The reflection is a fixed Fourier multiplier, so it commutes with those derivatives. Pressure changes the direction of the matrix response without reducing this quadratic magnitude. The identity does not control  $A$  by  $A^2$ , since a large shear can have a small matrix square.

The reflection also separates material growth from change of direction. Put  $Z = A \circ X$ ,  $Y = \|A\|_2 > 0$ , and  $\Theta = Z/Y$ . Orthogonality of the derivative of a unit vector in the matrix-valued  $L^2$  space gives

$$\frac{\|A^2\|_2^2}{\|A\|_2^2} = |\partial_t \log \|A\|_2|^2 + \|\partial_t \Theta\|_2^2. \tag{10.2}$$

Thus  $\mathcal{L}_{[s,t]} = \int_s^t \|A^2\|_2 / \|A\|_2 \, dr$  is an exact length for radial and angular material variation, with

$$\left| \log \frac{\|A(t)\|_2}{\|A(s)\|_2} \right| \leq \mathcal{L}_{[s,t]} \leq \int_s^t \|A(r)\|_{\infty, F} \, dr.$$

Energy-preserving Euler dilation leaves this length invariant on a rescaled interval. A normalized fixed annulus carrying a uniform positive amount of energy at physical scales tending to infinity forces  $\|A\|_2 \rightarrow \infty$  along those events and hence infinite length. There is no energy-only finite-horizon bound for the length: a compact steady field  $V$  dilated by  $K$  has length  $TK^{5/2} \|(\nabla V)^2\|_2 / \|\nabla V\|_2$  over  $[0, T]$ . The quotient is positive. Otherwise its pressure would be constant and  $(V \cdot \nabla)V = 0$ , so any nonzero trajectory would be an entire straight line of constant velocity inside compact support. The length is an exact geometric identity; it supplies no finite terminal budget by itself.

### 10.1 The local contribution in pressure transport

Pressure is nonlocal, so its material evolution also contains a transport commutator. The contribution at the singular point matters. Let  $\mathcal{H} = \nabla^2 \mathcal{N}$ , and decompose

$$\mathcal{H}f = \mathcal{K}f - \frac{1}{3}fI, \quad K_{ij}(y) = \frac{3n_i n_j - \delta_{ij}}{4\pi|y|^3}, \quad n = y/|y|, \quad (10.3)$$

where  $\mathcal{K}$  uses spherical principal value. Initially take smooth rapidly decaying functions; the identities extend at strict classical times by Sobolev approximation. Writing  $\delta_y u(x) = u(x) - u(x - y)$ , the locally integrable difference formula is

$$[u \cdot \nabla, \mathcal{K}]f(x) = \int K(y) \delta_y u(x) \cdot \nabla f(x - y) \, dy.$$

Integrating by parts outside  $|y| = \varepsilon$  produces an inner boundary term. Since  $\delta_y u = Ay + O(|y|^2)$ , its limit is

$$\frac{f(x)}{4\pi} \int_{S^2} (3n_i n_j - \delta_{ij})(n^T A n) \, d\sigma(n) = \frac{2}{5} S_{ij}(x) f(x).$$

The equality uses the second and fourth spherical moments and  $\text{tr } A = 0$ . Consequently the ordinary spherical-principal-value formula is

$$\begin{aligned} [u \cdot \nabla, \mathcal{K}]f(x) &= \text{p.v.} \int [\nabla K(y) \cdot \delta_y u(x)] f(x - y) \, dy \\ &\quad + \frac{2}{5} S(x) f(x). \end{aligned} \quad (10.4)$$

A distributional derivative can encode this local contribution inside the kernel. A bulk spherical principal value written as in (10.4) requires it explicitly.

Since  $D_t \theta = -2 \text{tr}(A^3) - 2S : P$ , the deviatoric Hessian  $P^\circ = \mathcal{K}\theta$  satisfies

$$D_t P^\circ = -2\mathcal{K}[\text{tr}(A^3) + S : P] + \mathcal{C}_0(u, \theta) + \frac{2}{5}\theta S, \quad (10.5)$$

where  $\mathcal{C}_0$  is the bulk integral in (10.4). Dropping its local contribution would alter the pressure evolution even for smooth admissible data.

The full Hessian has a complementary form in which two derivative terms cancel before any estimate. Material differentiation of its Poisson equation gives

$$-\Delta D_t p = -2 \text{tr}(A^3) - 4S : P - (\Delta u) \cdot \nabla p.$$

Commuting  $D_t$  through the Hessian then gives

$$D_t P + A^T P + P A = -2\mathcal{H}[\text{tr}(A^3) + 2S : P] - \sum_k [\mathcal{H}, \partial_k p] \Delta u_k. \quad (10.6)$$

Here  $\mathcal{H}\Delta u_k = -\nabla^2 u_k$  cancels the direct product  $(\nabla^2 u_k)\partial_k p$ . The remaining commutator contains the pressure-gradient difference

$$[\mathcal{H}_{ij}, \partial_k p] \Delta u_k(x) = \int K_{ij}(y) [\partial_k p(x-y) - \partial_k p(x)] \Delta u_k(x-y) dy.$$

This difference makes the local integral absolutely convergent.

Let  $\mathcal{R}_P$  denote the right side of (10.6). Pulling the Hessian back as a bilinear form,  $\mathcal{P} = F^T(P \circ X)F$ , cancels the two matrix transport terms:

$$\partial_t \mathcal{P} = F^T(\mathcal{R}_P \circ X)F, \quad \partial_t^2(F^T F) = 2F_t^T F_t - 2\mathcal{P}. \quad (10.7)$$

The pressure curvature and the particle metric evolve together. Their coupling contains signed local and nonlocal terms; neither equation has supplied a positive bound for the other.

## 10.2 A quadratic correction leaves a local cubic term

The local term in (10.5) suggests combining pressure with a quadratic velocity-gradient expression. This can be tested inside the complete equation. Set

$$R = (\text{sym } A^2)^\circ, \quad B_c = P^\circ + cR, \quad c \in \mathbb{R}.$$

For a trace-free three-dimensional matrix, Cayley–Hamilton gives  $2(\text{sym } A^3)^\circ = \theta S$ . Differentiating  $R$  with  $D_t A = -A^2 - P$  gives

$$D_t R = -\theta S - (SP + PS)^\circ.$$

Substitute  $P = B_c - cR - \theta I/3$  into this identity and (10.5). With  $\{M, L\} = ML + LM$ , the result is

$$D_t B_c = -2\mathcal{K}[\text{tr}(A^3) + S : B_c - cS : R] + \mathcal{C}_0(u, \theta) - c\{S, B_c\}^\circ + c^2\{S, R\}^\circ + \left(\frac{2}{5} - \frac{c}{3}\right)\theta S. \quad (10.8)$$

At a point where  $A = S$ , a second use of Cayley–Hamilton gives  $\{S, R\}^\circ = \theta S/3$ . The explicit local cubic coefficient in (10.8) is then

$$\frac{2}{5} - \frac{c}{3} + \frac{c^2}{3} = \frac{1}{3}\left(c - \frac{1}{2}\right)^2 + \frac{19}{60} > 0. \quad (10.9)$$

No fixed real coefficient in this quadratic correction removes the term. This calculation rejects that algebraic cancellation. The other terms remain in (10.8); it gives no sign inequality for their sum.

Both the contact term and the symmetric-gradient test occur on admissible Euler initial data. Let  $B = \text{diag}(1, 2, -3)$  and

$$u_0(x) = \text{curl} \left[ -\frac{1}{3} e^{-|x|^2} x \times (Bx) \right]. \quad (10.10)$$

This is a solenoidal Schwartz field, with  $u_0(0) = 0$ ,  $A(0) = S(0) = B$ , and  $\theta(0) = 14$ . Using its actual quadratic pressure source, the two integrals in (10.4) give

$$[u_0 \cdot \nabla, \mathcal{K}]\theta(0) = \frac{5584}{1215}B, \quad \mathcal{C}_0(u_0, \theta)(0) = -\frac{244}{243}B. \quad (10.11)$$

These values follow by expanding the polynomial factors and using  $\int_{S^2} n_1^{2a} n_2^{2b} n_3^{2c} d\sigma = 2\Gamma(a + 1/2)\Gamma(b + 1/2)\Gamma(c + 1/2)/\Gamma(a + b + c + 3/2)$ , followed by Gaussian radial integration. Their difference is  $28B/5 = (2/5)\theta(0)S(0)$ . Omitting the local contribution changes even the sign of the first diagonal entry of the commutator.

### 10.3 Pressure evolution at every finite temporal order

The corrected equation also differentiates through the complete Euler jet. Put  $A_n = \nabla V_n$ ,  $P_n = \nabla^2 Q_n$ , and  $S_n = \text{sym } A_n$ . Its scalar source rows are

$$\theta_n = \sum_{a=0}^n \binom{n}{a} \text{tr}(A_a A_{n-a}).$$

For example, differentiation of the contact commutator gives

$$\partial_t^n [u \cdot \nabla, \mathcal{K}] \theta = \sum_{a=0}^n \binom{n}{a} [V_a \cdot \nabla, \mathcal{K}] \theta_{n-a},$$

whose local contribution is  $(2/5) \sum_{a=0}^n \binom{n}{a} S_a \theta_{n-a}$ . Differentiating (10.6) gives the full relation

$$\begin{aligned} P_{n+1} &+ \sum_{a=0}^n \binom{n}{a} V_a \cdot \nabla P_{n-a} + \sum_{a=0}^n \binom{n}{a} (A_a^T P_{n-a} + P_{n-a} A_a) \\ &= -2\mathcal{H} \left[ \sum_{a+b+c=n} \frac{n!}{a!b!c!} \text{tr}(A_a A_b A_c) \right] \\ &\quad - 4\mathcal{H} \left[ \sum_{a=0}^n \binom{n}{a} S_a : P_{n-a} \right] - \sum_{a=0}^n \binom{n}{a} \sum_k [\mathcal{H}, \partial_k Q_a] \Delta (V_{n-a})_k. \end{aligned} \tag{10.12}$$

Every pressure row is generated by the velocity rows. Formula (10.12) retains that relation when spatial, temporal, and material descriptions are compared. It is an evolution identity at strict times; convergence of its terms at  $T$  still requires the terminal information identified earlier.

## 11 Energy normalization and changing material shape

The pressure equations follow the deformation in physical coordinates. A different approach uses Euler's two scaling parameters to hold energy and peak vorticity fixed while the fluid changes. This gives a useful compactness problem, but it changes the coordinates of the problem rather than constructing the whole-interval spatial intersection. The physical scale must be recovered at the end.

### 11.1 Vorticity records and the information that survives rescaling

Put  $E = N^2 > 0$  and  $\Omega(t) = \|\omega(t)\|_\infty$ . If  $\Omega$  becomes unbounded at a finite time, choose increasing record times  $t_j \uparrow T$ , with  $\Omega_j = \Omega(t_j) \rightarrow \infty$ , and points  $x_j$  where the vorticity reaches its maximum. Such points exist at strict times since the vorticity is continuous and vanishes at spatial infinity. Define

$$\begin{aligned} K_j &= (\Omega_j^2/E)^{1/5}, \quad a_j = K_j/\Omega_j, \\ U_j(y, s) &= a_j u(x_j + K_j^{-1} y, t_j + \Omega_j^{-1} s), \\ \Pi_j(y, s) &= a_j^2 p(x_j + K_j^{-1} y, t_j + \Omega_j^{-1} s). \end{aligned} \tag{11.1}$$

Direct substitution gives Euler for  $(U_j, \Pi_j)$ , and a change of variables gives

$$\|U_j(s)\|_2 = 1, \quad \|\operatorname{curl} U_j(s)\|_\infty \leq 1 \quad (-\Omega_j t_j \leq s \leq 0), \quad |\operatorname{curl} U_j(0, 0)| = 1. \quad (11.2)$$

The backward intervals exhaust every finite subinterval of  $(-\infty, 0]$ .

The resulting velocity bounds can be proved without controlling its full gradient. For a divergence-free  $v \in L^2$  with bounded vorticity, frequency separation and the annular Biot–Savart multiplier give

$$\|P_{\leq R} v\|_\infty \leq CR^{3/2} \|v\|_2, \quad \|P_{> R} v\|_\infty \leq CR^{-1} \|\operatorname{curl} v\|_\infty.$$

Optimizing in  $R$  yields

$$\|v\|_\infty \leq C \|v\|_2^{2/5} \|\operatorname{curl} v\|_\infty^{3/5}, \quad \|v\|_{B_{\infty, \infty}^1} \leq C (\|v\|_2 + \|\operatorname{curl} v\|_\infty). \quad (11.3)$$

Here the inhomogeneous Besov bound follows by applying  $2^q \|\Delta_q v\|_\infty \leq C \|\operatorname{curl} v\|_\infty$  to each high-frequency block and using energy for the low block. These are the usual Littlewood–Paley estimates [13]. Together with the negative-order time bound from Euler, they give subsequential velocity compactness on compact space–time sets in every spatial Hölder exponent below one. For example, first choose a larger exponent below one, use compact embedding on a fixed ball, and use the negative-order time modulus to identify every time limit. Interpolation gives equicontinuity in the smaller exponent, and a diagonal extraction handles increasing balls and backward intervals.

This topology can lose the normalized vorticity. For a fixed smooth compactly supported solenoidal field  $v$ , the sequence

$$v_\varepsilon(y) = \varepsilon v(y/\varepsilon)$$

converges uniformly to zero, while its vorticity maximum is independent of  $\varepsilon$  and its squared  $L^2$  norm is  $\varepsilon^5 \|v\|_2^2$ . Adding a disjoint solenoidal field whose spatial scale tends to infinity, and adjusting its amplitude, preserves unit energy while its local velocity and vorticity tend to zero. The broad field can be placed away from the small core, with vorticity below the core maximum. Thus unit energy and a fixed vorticity maximum do not keep a nonzero vorticity profile in the velocity limit. This is a sequence of admissible spatial data; no Euler evolution realizing its successive members is asserted.

The derivative information has its own frequency width. With an inhomogeneous decomposition and  $W = \operatorname{curl} U$ , the annular Biot–Savart estimate gives

$$\|\nabla U\|_\infty \leq C \left( \|U\|_2 + \sum_{q \geq 0} \|\Delta_q W\|_\infty \right)$$

whenever the sum is finite. A bound for  $\|W\|_\infty$  controls each summand, not their sum. Increasingly many derivative scales can thus prevent this estimate from giving uniform strain control.

A dimensional core calculation shows why circulation need not prevent that loss. If a coherent core of roughly isotropic cross section has vorticity of size  $\Omega$ , circulation  $\Gamma$ , and no substantial flux cancellation, its radius obeys  $\ell \asymp (\Gamma/\Omega)^{1/2}$ . In the energy–vorticity normalization its radius is  $K\ell \asymp \Gamma^{1/2} E^{-1/5} \Omega^{-1/10}$ , and the isotropic core-energy scale is  $\Gamma^{5/2} \Omega^{-1/2}$ . A fixed circulation can accompany a core whose normalized velocity and energy vanish while its vorticity stays of order one. These are dimensional consequences of the stated core assumptions, not a construction of such a concentrating Euler history.

For consecutive doubling records,  $\Omega_{j+1} = 2\Omega_j$ , put  $\delta_j = \Omega_j(t_{j+1} - t_j)$  and  $b_j = K_j(x_{j+1} - x_j)$ . The exact relation is

$$U_{j+1}(y, s) = 2^{-3/5} U_j(b_j + 2^{-2/5} y, \delta_j + s/2), \quad T - t_J = \Omega_J^{-1} \sum_{m \geq 0} 2^{-m} \delta_{J+m}. \quad (11.4)$$

A positive lower bound on each  $\delta_j$  is compatible with a finite physical time. The material particle reaching the next record does satisfy

$$\int_{t_j}^{t_{j+1}} \xi \cdot S\xi(X(a_j, t), t) dt \geq \log 2, \quad \xi = \omega/|\omega|,$$

because its initial vorticity is at most  $\Omega_j$ . The selected particle may change between records. Summing the inequalities records stretching along recruited trajectories; it does not bound their number.

## 11.2 Continuous dilation and physical time

Let  $K(t) > 0$  be  $C^1$ , choose a material center  $x_c(t)$ , and set

$$\frac{d\tau}{dt} = E^{1/2} K^{5/2}, \quad U(y, \tau) = E^{-1/2} K^{-3/2} u(x_c + K^{-1}y, t), \quad \beta = \partial_\tau \log K.$$

With  $\Pi(y, \tau) = E^{-1} K^{-3} p(x_c + K^{-1}y, t)$  and  $U_c = U(0, \tau)$ , the chain rule gives

$$\partial_\tau U + (U - U_c) \cdot \nabla U + \beta(y \cdot \nabla U + \frac{3}{2}U) = -\nabla \Pi, \quad \operatorname{div} U = 0. \quad (11.5)$$

For  $N_m = \|\Lambda^m U\|_2^2$ , its exact homogeneous energy identity is

$$\frac{1}{2} \partial_\tau N_m + m\beta N_m = \mathcal{T}_m[U], \quad \mathcal{T}_m[U] = -\langle \Lambda^m U, \Lambda^m \mathbb{P}(U \cdot \nabla U) \rangle. \quad (11.6)$$

Indeed  $[\Lambda^m, y \cdot \nabla] = m\Lambda^m$ , and the  $3/2$  amplitude term cancels the integration-by-parts contribution from the dilation. No spatial moment hypothesis is needed for the energy identity: it also follows by differentiating the exact norm scaling  $N_m = E^{-1} K^{-2m} \|\Lambda^m u\|_2^2$ .

On an interval where  $\Omega > 0$ , choose  $K = (\Omega^2/E)^{1/5}$ . The formulas hold almost everywhere when  $\Omega$  is locally Lipschitz. Then

$$\frac{d\tau}{dt} = \Omega, \quad g = \partial_\tau \log \Omega, \quad \beta = \frac{2}{5}g, \quad \|\Lambda^m u\|_2 = E^{1/2} K^m \|\Lambda^m U\|_2. \quad (11.7)$$

An increasing  $K$  produces a positive term at every spatial height in (11.6). The physical reconstruction still multiplies the normalized norm by  $K^m$ . The term is caused by changing scale, and does not supply dissipation of the physical field.

The clock retains that distinction exactly. With  $\tau(t_0) = 0$ ,

$$t(\tau) - t_0 = \frac{1}{\Omega(t_0)} \int_0^\tau \exp\left(-\int_0^s g(r) dr\right) ds. \quad (11.8)$$

BKM requires  $\tau \rightarrow \infty$  at a finite singular time. Equation (11.8) shows that infinite normalized time can correspond to finite physical time. On an interval where one material label persistently realizes the maximum, the directional pressure law also gives

$$\partial_\tau g = |\partial_\tau \xi|^2 - 2g^2 - \rho, \quad \rho = \Omega^{-2} \xi^T (P \circ X) \xi. \quad (11.9)$$

This follows from (8.2) applied to vorticity and from  $\partial_t = \Omega \partial_\tau$ . It is a material-branch identity. A changing maximizing label does not inherit it as an ordinary scalar differential equation.

### 11.3 The anisotropy left by isotropic normalization

Volume preservation makes vorticity amplification a statement about shape. Let  $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$  be the singular values of  $F(a, t)$ , and suppose  $\omega_0(a) \neq 0$ . Cauchy's formula gives  $R = |\omega(X(a, t), t)|/|\omega_0(a)| \leq \sigma_1$ . Since  $1 = \sigma_1 \sigma_2 \sigma_3 \geq \sigma_1 \sigma_3^2$ ,

$$\|F^{-1}\|_{\text{op}} \geq R^{1/2}, \quad \kappa(F) := \|F\|_{\text{op}} \|F^{-1}\|_{\text{op}} \geq R^{3/2}. \quad (11.10)$$

At a record label,  $R \geq \Omega(t)/\|\omega_0\|_\infty$ . Dividing the amplified physical covector by the isotropic frequency scale  $K$  leaves a factor at least  $C(u_0)\Omega^{1/10}$ . This is a lower bound for the norm of the transported-covector map. It does not prove that the velocity occupies its most amplified direction, or that a particular velocity Sobolev norm grows at that rate. The latter assertion would require information about the evolving frequency amplitudes and their alignment.

Bounded shape would suffice for continuation. If  $\kappa(F) \leq L$ , then  $\sigma_3 \geq \sigma_1/L$  implies  $1 \geq \sigma_1^3/L^2$ , so  $\|F\|_{\text{op}} \leq L^{2/3}$ . Applying this to  $F^{-1}$  also gives  $\|F^{-1}\|_{\text{op}} \leq L^{2/3}$ . A uniform finite-time bound for  $\kappa(F)$  bounds vorticity by Cauchy's formula and gives BKM continuation. Conversely, a smooth extension bounds both matrices. This is an equivalent material condition, not an estimate derived from energy.

Fix two initial directions  $e_+, e_-$ , and use the notation  $z_\pm = Fe_\pm$ ,  $r_\pm = |z_\pm|$ ,  $q_\pm = z_\pm/r_\pm$ , and  $s_\pm = q_\pm^T(S \circ X)q_\pm$ . Subtracting their separation laws removes the isotropic pressure:

$$\begin{aligned} \partial_t^2 \log(r_+/r_-) &= |q'_+|^2 - |q'_-|^2 - s_+^2 + s_-^2 \\ &\quad - q_+^T(P^\circ \circ X)q_+ + q_-^T(P^\circ \circ X)q_-, \quad P^\circ = P - \frac{1}{3}(\text{tr } P)I. \end{aligned} \quad (11.11)$$

At a selected terminal strict time, choose the initial directions to be the right singular vectors for  $\sigma_1$  and  $\sigma_3$ , and hold them fixed during the integration. Twice integrating (11.11) gives  $\log \kappa(F)$ , including the initial linear term  $t(s_+(0) - s_-(0))$ . Bounds on this integral are exactly shape bounds. They require the correlation of directional rotation, strain, and anisotropic pressure through time.

### 11.4 Pressure quadrupoles and global cancellation

The pressure term in (11.11) selects a particular part of the surrounding velocity field. Set  $\theta = \text{tr}(A^2)$ . The Hessian kernel (10.3) gives, for unit vectors  $q_+, q_-$ ,

$$q_+^T P^\circ(x) q_+ - q_-^T P^\circ(x) q_- = \frac{3}{4\pi} \text{p.v.} \int_0^\infty \frac{dr}{r} \int_{S^2} ((q_+ \cdot n)^2 - (q_- \cdot n)^2) \theta(x - rn) d\sigma(n). \quad (11.12)$$

The angular factor is a degree-two spherical harmonic. It annihilates the spherical mean and the linear Taylor term. Near  $r = 0$ , the remaining angular average is  $O(r^2)$  at a strict smooth time. The large-radius integral converges through  $\theta \in L^1$ . Relative deformation thus selects the quadrupolar part of the pressure source across logarithmic radius. A bound on the scalar source's total mass does not control this signed directional integral.

Nor does integrating pressure against all strain supply a positive quantity. Two integrations by parts give

$$\int_{\mathbb{R}^3} S : P dx = \int_{\mathbb{R}^3} p \partial_i \partial_j S_{ij} dx = 0, \quad \|P^\circ\|_2^2 = \frac{2}{3} \|\theta\|_2^2. \quad (11.13)$$

The first identity uses  $\text{div } u = 0$ ; the second uses the Fourier symbol  $-e \otimes e$  of  $\mathcal{H}$  and the orthogonal scalar/deviatoric decomposition. With  $\mathbf{O} = (A - A^T)/2$  and  $\mathbf{O}^2 = (\omega \otimes \omega - |\omega|^2 I)/4$ , the strain equation and enstrophy equation then yield

$$\frac{1}{2} \frac{d}{dt} \|S\|_2^2 = - \int \text{tr}(S^3) dx - \frac{1}{4} \int \omega \cdot S \omega dx, \quad \|S\|_2^2 = \frac{1}{2} \|\omega\|_2^2.$$

Comparing their derivatives recovers the classical Betchov relation

$$\int \text{tr}(S^3) \, dx = -\frac{3}{4} \int \omega \cdot S\omega \, dx. \quad (11.14)$$

We use this established identity [14] to test a proposed pressure mechanism. Pressure has substantial size but zero unweighted global contraction with strain. A useful pressure restriction must retain the spatial, directional, or intensity selection that this global pairing removes.

The normalized high-order moments offer no automatic replacement for that selection. Use the unitary Fourier transform in the following moment identity. If  $\|U\|_2 = 1$ , then

$$L_m = \|\Lambda^m U\|_2^{1/m} = \left( \int |\zeta|^{2m} |\widehat{U}(\zeta)|^2 \, d\zeta \right)^{1/(2m)}$$

is a nondecreasing sequence of moments of a probability measure. Its extended limit is the essential supremum of  $|\zeta|$  on the Fourier support, which is infinite for unbounded support. There need not be a finite limiting spatial scale. This moment fact also gives no right to interchange the limit in  $m$  with a time derivative or to replace the signed transfer  $\mathcal{T}_m$  by one asymptotic coefficient.

## 12 Coupled transfer and material recruitment

Normalization follows concentration after it occurs. The evolution that creates the next concentration still includes the surrounding fluid. Independent rescaled packets do not supply that interaction: adding exact solutions requires the projected cross-advection to vanish. We now keep the cross terms and test whether their accumulated work can be controlled. Throughout this section  $N > 0$ . This is a separate dynamical approach to smoothness. Its conditional recurrence results do not assume that every possible singular history has the recurrence under study.

### 12.1 The exchange law and the cost of one receiver event

Let  $P_\alpha$  be a complete orthogonal Fourier partition, put  $u_\alpha = P_\alpha u$ , and define

$$E_\alpha = \frac{1}{2} \|u_\alpha\|_2^2, \quad b(a, v, w) = \int (a \cdot \nabla) v \cdot w \, dx, \quad T_{\alpha \leftarrow \gamma}^{(\beta)} = -b(u_\beta, u_\gamma, u_\alpha).$$

A finite partition includes its complementary block. At a strict time, Sobolev regularity justifies the infinite sums below. Projection of Euler and integration by parts give

$$E'_\alpha = \sum_{\beta, \gamma} T_{\alpha \leftarrow \gamma}^{(\beta)}, \quad T_{\alpha \leftarrow \gamma}^{(\beta)} = -T_{\gamma \leftarrow \alpha}^{(\beta)}. \quad (12.1)$$

Pressure pairs to zero with each solenoidal block. It remains part of the projected vector equation; zero pairing does not mean zero pressure. For weights  $w_\alpha$ , whenever the sums converge absolutely,

$$\frac{d}{dt} \sum_\alpha w_\alpha E_\alpha = \frac{1}{2} \sum_{\alpha, \beta, \gamma} (w_\alpha - w_\gamma) T_{\alpha \leftarrow \gamma}^{(\beta)}. \quad (12.2)$$

Constant weights give energy conservation. Increasing weights retain the direction of exchange between spatial scales.

At a snapshot whose nonzero packets lie in separated annuli  $[cK_j, CK_j]$ , with  $cK_{j+1} > 2CK_j$ , every interacting triad contains its largest packet index at least twice. Set  $J_{h\ell} = b(u_h, u_\ell, u_h)$ ,  $h > \ell$ . Then

$$E'_m = \sum_{h>m} J_{hm} - \sum_{\ell<m} J_{m\ell}, \quad |J_{h\ell}| \leq \|\nabla u_\ell\|_\infty \|u_h\|_2^2.$$

For  $M > 5/2$ , Bernstein and Cauchy–Schwarz bound the total absolute current by

$$C\|u\|_2^2 \left( \sum_j K_j^{2M} \|u_j\|_2^2 \right)^{1/2} \left( \sum_j K_j^{5-2M} \right)^{1/2}.$$

Euler can fill the spectral gaps after this snapshot. The continuing equation is (12.1), with all interactions retained.

For an orthogonal scalar Fourier receiver projection  $P_R$  supported in  $|\zeta| \leq CK$ , write  $w = P_R u$ ,  $B(f, g) = -\mathbb{P}(f \cdot \nabla g)$ , and

$$F_R^\times = P_R(B(u, u) - B(w, w)).$$

The receiver evolves by  $w_t = P_R B(w, w) + F_R^\times$ . Compare it on  $[a, b]$  with  $z_t = P_R B(z, z)$ ,  $z(a) = w(a)$ . For solenoidal  $f, g$ , the bounded output-frequency estimate

$$\|P_R B(f, g)\|_2 \leq CK^{5/2} \|f\|_2 \|g\|_2$$

follows by bounding the product transform by  $\|f\|_2 \|g\|_2$  and integrating the output derivative over the ball. Gronwall gives

$$\|w(b) - z(b)\|_2 \leq \exp \left( CK^{5/2} \int_a^b (\|w\|_2 + \|z\|_2) dt \right) \int_a^b \|F_R^\times\|_2 dt. \quad (12.3)$$

If the norms in the exponent are at most  $C_0 \sqrt{e}$ , the event lasts at most  $\gamma e^{-1/2} K^{-5/2}$ , and the final difference is at least  $\eta \sqrt{e}$ , the integrated cross force is at least  $\eta e^{-CC_0 \gamma} \sqrt{e}$ .

The energy relation is sharper and needs no restriction on duration:

$$\frac{1}{2} \frac{d}{dt} \|w\|_2^2 = \langle F_R^\times, w \rangle, \quad E_R(b) - E_R(a) = \int_a^b \langle F_R^\times, w \rangle dt. \quad (12.4)$$

Every positive receiver gain requires at least that much positive cross work. An antisymmetric exchange law does not bound the accumulated positive work: one amount of energy can pass through many receivers. Conservation balances gains against donor losses rather than paying each gain from a separate portion of initial energy.

Actual Euler data exhibit this exchange while the block pressure pairings vanish. Take centers  $p = (K, 0, 0)$ ,  $q = (0, 1, 0)$ ,  $r = (-K, -1, 0)$ , with polarizations  $e_2, e_3, ie_3$ , and add conjugate modes. Let the low block contain  $q$  and the high block contain  $p, r$ . The two resonances in  $b(u_h, u_\ell, u_h)$  each have coefficient  $-1$ ; the swapped high orientation has zero central coefficient. Narrow nonnegative Fourier bumps and Leray projection preserve the strict negative sign and give real Schwartz data. Local Euler evolution then transfers energy upward on a short interval, with zero Reynolds defect on that classical segment. Energy-preserving dilation makes the initial current arbitrarily large. It produces different initial data, not a concatenated singular history.

## 12.2 A receiver as a changing material detector

For a smooth scalar Fourier receiver  $R$ , put  $A_R = R^*R$ ,  $L_t = u \cdot \nabla$ , and  $E_R = \langle u, A_R u \rangle$ . The pressure pairing vanishes because  $A_R u$  is solenoidal, and skew-adjointness of  $L_t$  gives

$$E'_R = \langle u, [L_t, A_R] u \rangle.$$

Let  $C_t f = f \circ X(t)$ ,  $v = C_t u$ , and  $\mathcal{A}_t = C_t A_R C_t^{-1}$ . Volume preservation makes  $C_t$  unitary. Then

$$\dot{\mathcal{A}}_t = C_t [L_t, A_R] C_t^{-1}, \quad E'_R = \langle v, \dot{\mathcal{A}}_t v \rangle. \quad (12.5)$$

The derivative of  $v$  contributes zero here, since  $\langle \dot{v}, \mathcal{A}_t v \rangle = -\langle \nabla p, A_R u \rangle = 0$ . For a convolution kernel  $K$ , the material detector acts by

$$(\mathcal{A}_t f)(a) = \int K(X(a, t) - X(b, t)) f(b) db.$$

Its derivative replaces  $K$  by  $(v(a) - v(b)) \cdot \nabla K$ . Receiver work is the alignment of a velocity correlation with this changing separation kernel.

For  $K_\mu(z) = \mu^3 K(\mu z)$ , the kernel moment  $M_K = \int |z| |\nabla K(z)| dz$  is scale invariant. The commutator kernel and Schur's test give

$$\|[L_t, A_{R, \mu}]\|_{2 \rightarrow 2} \leq M_K \|\nabla u\|_\infty, \quad |E'_R| \leq N^2 M_K \|\nabla u\|_\infty.$$

For radial scalar kernels, the contraction along the separation vector removes the skew gradient, replacing  $\|\nabla u\|_\infty$  by  $\|S\|_{\infty, \text{op}}$ . A receiver gain at least  $\eta > 0$  on an event interval  $I_j$  thus requires

$$\int_{I_j} \|\nabla u\|_\infty dt \geq \frac{\eta}{N^2 M_K}, \quad \int_{I_j} \|\dot{\mathcal{A}}_t\|_{2 \rightarrow 2} dt \geq \frac{\eta}{N^2}. \quad (12.6)$$

For a uniform receiver family, disjoint events add these costs. The result is a deformation-clock requirement, not a finite initial reserve. Appendix D supplies an exact periodic Euler history with indefinitely repeated receiver gains and conserved material circulations.

## 12.3 What recurrence with a fixed energy amount would imply

Use an inhomogeneous dyadic partition with one low block and shell indices  $q \geq 0$ . Write  $e_q = \|P_q u\|_2^2/2$ ,  $\mathcal{E} = N^2/2$ , and

$$J_{pq} = -\langle (u \cdot \nabla) u_p, u_q \rangle = -J_{qp}.$$

Then  $e'_q = \sum_p J_{pq}$ . Define positive rates  $k_{pq} = J_{pq}^+/e_p$  when  $e_p > 0$ , and zero otherwise. The current vanishes when either corresponding block vanishes, so

$$e'_q = \sum_p (k_{pq} e_p - k_{qp} e_q). \quad (12.7)$$

On each strict interval,  $\Gamma(t) = \sum_{p,q} J_{pq}^+$  is integrable: the estimate  $|J_{pq}| \leq \|u\|_\infty \|\nabla u_p\|_2 \|u_q\|_2$  and weighted shell summation give  $\Gamma \leq C_s \|u\|_{H^s}^3$ ,  $s > 3/2$ . Finite-state jump constructions, followed by truncation, give a path law with marginals  $\mathbb{P}(Q_t = q) = e_q(t)/\mathcal{E}$  and expected jump count  $\mathcal{E}^{-1} \int \Gamma$ . To pass to the countable space, retain an outside state in each finite truncation; the expected jump bound makes arbitrarily many jumps unlikely. The probability of a visit outside  $\{0, \dots, L\}$  is at most the initial tail mass divided by  $\mathcal{E}$ , plus

$$\mathcal{E}^{-1} \int \sum_{p \leq L, q > L} J_{pq}^+ dt.$$

Both terms tend to zero as  $L \rightarrow \infty$ , the second by dominated convergence under the integrable total current. Conditional concatenation constructs the law on every strict interval up to  $T$ . The fixed rates specify the nonexplosive jump law. Other decompositions or path representations of the marginals need not agree with it.

**Proposition 12.1** (Consequence of fixed-mass ultraviolet recurrence). *Suppose  $t_j \uparrow T$  and shell windows  $W_j$  escape to infinite frequency, with  $\sum_{q \in W_j} e_q(t_j) \geq \delta > 0$ . Then*

$$\int_0^T \Gamma(t) dt = \infty, \quad \int_0^T \|\partial_t^n u(t)\|_2 dt = \infty \quad \text{for every } n \geq 1. \quad (12.8)$$

*Every positive spatial Sobolev norm of velocity is unbounded along the stated receiver times.*

*Proof.* The limsup of the path events  $Q_{t_j} \in W_j$  has probability at least  $\delta/\mathcal{E}$ , by decreasing tail unions. Each such path makes infinitely many jumps. For the stronger variation statement, finite  $\int \sum_q |e'_q|$  would make  $(e_q(t))_q$  Cauchy in  $\ell^1$ , whose terminal limit has vanishing tails. This contradicts the receiver hypothesis. Orthogonality gives

$$\sum_q |e'_q| \leq \left( \sum_q \|u_q\|_2^2 \right)^{1/2} \left( \sum_q \|P_q \partial_t u\|_2^2 \right)^{1/2} = N \|\partial_t u\|_2.$$

If any higher temporal row were integrable in  $L^2$  up to  $T$ , successive integrations from a strict reference time would bound all lower rows, including  $\partial_t u$ , a contradiction. The spatial claim follows by weighting the retained energy above the escaping frequency. No monotone divergence or common sequence for all temporal rows is asserted.  $\square$

A fixed energy amount at escaping frequencies also forces an energy defect. If orthogonal high-frequency projections  $Q_j$  satisfy  $\|Q_j u(t_j)\|_2^2 \geq d > 0$ , then  $Q_j u(t_j) \rightharpoonup 0$  and weak lower semicontinuity of the complementary blocks gives  $\|u_*\|_2^2 \leq N^2 - d$ . With spatial tightness, this defect is retained in the stress measure. Without tightness, spatial escape must also be included. Repeated activations do not produce independent, additive defect masses.

## 12.4 Fragmentation and what absolute shell action counts

The fixed-mass hypothesis is substantial. An explicit scalar model shows why energy conservation and scale-local currents do not force it. This model is not an Euler solution. Choose  $b > 1$ ,  $0 < \alpha < 5$ ,  $r = b^{-\alpha}$ ,  $K_j = K_0 b^j$ , and  $m_j = \mathcal{E} r^j$ . During event  $j$ , move mass  $m_{j+1}$  from shell  $j$  to  $j+1$ , leave  $(1-r)m_j$  behind, and use a smooth increasing transition with flat ends over a duration

$$\tau_j = c m_j^{-1/2} K_j^{-5/2}.$$

The total duration is finite because  $\alpha < 5$ . Total mass is  $\mathcal{E}$ , each strict-time vector has finite support, and its nearest-neighbor current is bounded by a constant times  $m_j^{3/2} K_j^{5/2}$ . The total unweighted positive action is  $\mathcal{E} r / (1-r)$ , while the terminal deposits satisfy

$$e_q^* = (1-r) \mathcal{E} r^q, \quad \sum_q K_q^{2M} e_q^* < \infty \iff 2M < \alpha. \quad (12.9)$$

No positive mass reaches infinite shell index. The recurrence proposition does not cover this pattern, and the scalar laws have not excluded it from a proposed comparison argument.

For increasing weights  $w_M(q) \asymp 2^{2Mq}$ , put  $H_M = 2 \sum_q w_M(q) e_q$ . Let  $U_M, D_M$  be the upward and downward weighted positive currents, and  $C_M = U_M + D_M$ . Exactly,

$$\frac{1}{2} H'_M = U_M - D_M, \quad \int_s^t U_M = \frac{1}{2} \int_s^t C_M + \frac{1}{4} (H_M(t) - H_M(s)). \quad (12.10)$$

Finite upward action at every height would bound the Sobolev norms. The action also counts repeated cycles. With sharp shell boundaries, that cost can exceed the smooth Sobolev energy flux. To see the scale of the defect, take triad centers

$$r = (2, 1, 0), \quad k = (R - 1, 0, 0), \quad \ell = -(R + 1, 1, 0), \quad R = 2^q,$$

with polarizations  $(1, -2, 0)/\sqrt{5}$ ,  $e_2$ , and  $(1, -(R + 1), 0)/\sqrt{(R + 1)^2 + 1}$ . The advecting coefficient is  $-(R + O(1))/\sqrt{5}$ . Choose the last packet's imaginary phase for upward transfer and give the high packets amplitude  $R^{-M}$ ,  $M \geq 1$ . Fixed-width smooth Fourier bumps yield bounded energy,  $H^M$ , and Lipschitz norm, while the two high packets straddle the sharp dyadic boundary. Their current is of order  $R^{1-2M}$  and the weight jump is of order  $R^{2M}$ , so  $U_M \geq cR$ . Local Euler existence realizes the initial test. This concerns the chosen sharp-shell absolute action. Differentiating the equivalence  $H_M \asymp \|u\|_{H^M}^2$  would incorrectly transfer it to the smooth Sobolev flux.

Removing all circulation of currents has a different cost. For integrated currents  $K_{pq} = \int_s^t J_{pq}$ , the divergence is the endpoint energy difference. A functional invariant under adding every divergence-free graph current factors through that difference. On a finite graph, a linear cycle-annihilating weight has the form  $\phi_q - \phi_p$ : fix a base vertex, integrate the edge weights along a path, and use zero cycle sums to prove path independence. On the shell line, signed integration before absolute value gives

$$\mathcal{E}W_1(\mu_s, \mu_t) = \sum_{Q \geq 1} \left| \int_s^t \Pi_Q(r) dr \right|, \quad \mu_t(q) = e_q(t)/\mathcal{E}, \quad \Pi_Q = \sum_{q \geq Q, p < Q} J_{pq}. \quad (12.11)$$

The formula is the one-dimensional transport identity obtained by summing signed mass changes beyond each edge. In (12.9), the terminal distance from the initial shell is  $r/(1 - r)$ , despite failure of higher moments. Increasing the edge costs to detect those moments restores the corresponding weighted endpoint demand. It does not independently bound it.

## 12.5 From a localized receiver to material surfaces

Frequency transfer alone does not specify which particles participate. A spatially localized receiver supplies more information. The next lemma uses circulation of the full velocity, so Kelvin's theorem can carry its conclusion back to the initial fluid.

**Lemma 12.2** (A parallel stack detected by annular energy). *Let  $F$  be real, smooth, solenoidal, with  $\|F\|_2^2 \leq E_0$ . Let  $P_b$  be componentwise multipliers whose symbols are uniformly smooth and supported in a fixed compact annulus away from zero. Suppose*

$$\int_{B_{R_0}(y)} |P_b F|^2 dx \geq \kappa > 0.$$

*There are uniform constants  $R_1, m, \gamma > 0$ ,  $\Gamma < \infty$ , an orientation, a sign  $\sigma$ , and a measurable family of parallel rectangles  $D_z$ , with side lengths in  $[1, 2]$ , such that*

$$|Z| \geq m, \quad D_z \subset B_{R_1}(y), \quad \gamma \leq \sigma \oint_{\partial D_z} F \cdot dx \leq \Gamma \quad (z \in Z).$$

The parameter  $z$  is physical distance in the normal direction; rectangles with different parameters are disjoint.

*Proof.* Write  $C_{ik}F(x; r, s)$  for circulation around the coordinate rectangle based at  $x$ , with side lengths  $r, s \in [1, 2]$ . Set

$$A_r(z) = \int_0^r e^{i\tau z} d\tau, \quad g(z) = \int_1^2 |A_r(z)|^2 dr.$$

The function  $g$  is positive: vanishing would imply  $A_r(z) = 0$  for every  $r \in [1, 2]$ , contrary to  $\partial_r A_r(z) = e^{irz}$ . It is bounded below on compact intervals. The Fourier symbol of circulation is

$$\widehat{C_{ik}F}(\zeta; r, s) = iA_r(\zeta_i)A_s(\zeta_k)(\zeta_i \widehat{F}_k - \zeta_k \widehat{F}_i).$$

Average against the conjugate factors and divide by  $g(\zeta_i)g(\zeta_k)$ . Solenoidality then recovers the annular velocity through

$$P_b F = \sum_{i < k} \int_{[1,2]^2} K_{ik,b}^{r,s} * C_{ik} F dr ds, \quad (12.12)$$

where the Fourier symbols of the vector kernels are

$$-i m_b(\zeta) \frac{\zeta_i e_k - \zeta_k e_i}{|\zeta|^2} \frac{\overline{A_r(\zeta_i)A_s(\zeta_k)}}{g(\zeta_i)g(\zeta_k)}.$$

These kernels are uniformly Schwartz. Pair the frame with  $\phi = 1_{B_{R_0}(y)} P_b F / \|P_b F\|_{L^2(B_{R_0}(y))}$ . Its adjoint contractions  $W_{ik} = (K_{ik,b}^{r,s})^* \phi$  obey uniform bounds

$$\sum_{i < k} \int_{[1,2]^2} \|W_{ik}\|_1 dr ds \leq A_1, \quad \sup_{i,k,r,s} \|W_{ik}\|_\infty \leq A_\infty.$$

The four edges give  $\|C_{ik}F\|_2 \leq 8\sqrt{E_0}$ , hence the squared circulation norm over all parameters is at most  $L^2 = 192E_0$ . Choose a large ball so that its exterior contribution to the paired frame is at most  $\sqrt{\kappa}/4$ . This follows uniformly from the Schwartz tails and the preceding  $L^2$  bound. Circulations below  $\gamma = \sqrt{\kappa}/(4A_1)$  contribute at most another  $\sqrt{\kappa}/4$ . Cauchy–Schwarz then forces the remaining parameter set to have measure at least  $m_0 = \kappa/(4A_\infty^2 L^2)$ . Choose  $\Gamma \geq \max(2\gamma, \sqrt{2L^2/m_0})$ . Chebyshev retains measure at least  $m_0/2$  between these thresholds. One of the six orientation/sign choices retains at least  $m_0/12$ . Fubini, first in the tangential basepoint and the two side lengths, then in the normal coordinate, supplies a positive-measure parallel stack. Its rectangles remain in a uniformly enlarged ball.  $\square$

Suppose a strict-time receiver has the physical form  $u(x, t_j) = \mu_j^{3/2} F_j(\mu_j(x - x_j))$ , with the uniform hypotheses of Lemma 12.2. Scaling its surfaces gives

$$\begin{aligned} |Z_j| &\asymp \mu_j^{-1}, & |\Gamma_j(z)| &\asymp \mu_j^{1/2}, \\ \int_{Z_j} |\Gamma_j(z)| dz &\asymp \mu_j^{-1/2}, & \int_{Z_j} |\Gamma_j(z)|^2 dz &\asymp 1. \end{aligned} \quad (12.13)$$

The constants depend on the normalized receiver bounds, not on  $j$ . Pull each surface back by the actual Euler flow to  $\Sigma_{j,z}$ . Their union  $\mathcal{O}_j$  in label space has volume comparable to  $\mu_j^{-3}$ . If  $e_\ell$  is the

physical normal, the level-set coordinate has gradient  $F(a, t_j)^T e_\ell$ . Kelvin, Stokes, and coarea give

$$\begin{aligned}\Gamma_j(z) &= \int_{\Sigma_{j,z}} \omega_0 \cdot n \, dA, \\ \int_{Z_j} \int_{\Sigma_{j,z}} g \, dA \, dz &= \int_{\mathcal{O}_j} g(a) |F(a, t_j)^T e_\ell| \, da.\end{aligned}$$

Consequently, for  $1 \leq p \leq \infty$ ,

$$\int_{\mathcal{O}_j} |F^T e_\ell| \, da \geq c \mu_j^{-1/2}, \quad \|F^T e_\ell\|_{L^p(\mathcal{O}_j)} \geq c_p \mu_j^{5/2-3/p}. \quad (12.14)$$

The first inequality uses  $\|\omega_0\|_\infty$ ; Hölder gives the second. The exponent is zero at  $p = 6/5$ . These are per-event requirements on the initial fluid's deformation. Both the label set and its normal may change between events, so their lower bounds do not add into a finite initial budget.

The squared circulation has a different weight. Cauchy–Schwarz on each surface gives only

$$\int_{Z_j} \frac{|\Gamma_j(z)|^2}{|\Sigma_{j,z}|} \, dz \leq \int_{\mathcal{O}_j} |\omega_0|^2 |F^T e_\ell| \, da. \quad (12.15)$$

Volume preservation does not bound the areas in its denominator. Even simple volume-preserving geometry realizes the required exponents:  $\text{diag}(\mu^{-5/4}, \mu^{-5/4}, \mu^{5/2})$  maps a flat cylinder with horizontal area of order  $\mu^{1/2}$  and thickness of order  $\mu^{-7/2}$  to a core of diameter of order  $\mu^{-1}$ . Its normal covector factor is  $\mu^{5/2}$ . This is a kinematic example, with no velocity reconstruction imposed.

A related deformation occurs in exact affine Euler flow, with infinite energy on  $\mathbb{R}^3$ . Let  $g = s'$ ,  $\Omega = \Omega_0 e^{2s}$ , and

$$B = \begin{pmatrix} -g & -\Omega & 0 \\ \Omega & -g & 0 \\ 0 & 0 & 2g \end{pmatrix}, \quad u = Bx, \quad p = -\frac{1}{2}x^T(B' + B^2)x.$$

The relation  $\Omega' = 2g\Omega$  makes  $B' + B^2$  symmetric, so these are Euler solutions and their deformation is a horizontal rotation times  $\text{diag}(e^{-s}, e^{-s}, e^{2s})$ , with  $s(0) = 0$ . Choosing  $e^s = \mu^{5/4}$  reproduces the area scaling. This verifies compatibility with the affine equations, not with the finite-energy problem.

Finite-energy data can also carry large circulation on large surfaces. For disjoint toroidal supports centered far apart, use cylindrical coordinates on each support and take

$$u_j = a_j \chi(r - R_j) \psi(z) e_\theta, \quad a_j = \frac{\mu_j^{1/2}}{2\pi R_j}, \quad \chi(0) = \psi(0) = 1.$$

Choose  $R_j \geq 1$ , with  $\mu_j/R_j \leq 2^{-j}$ . Fixed smooth compact profiles give  $\|u_j\|_{H^m}^2 \leq C_m \mu_j/R_j$ , so their disjoint sum belongs to  $H_\sigma^\infty$ . The central circle has circulation  $\mu_j^{1/2}$ , retained on its material image during local Euler existence. Compression of those surfaces into the receivers has not been established. The example isolates the missing part: large initial surface flux and small total energy can coexist.

There is also an exact constraint on the deformation of paired directions. For fixed initial vectors  $v_0, w_0$ , put  $z = Fv_0$ ,  $w = Fw_0$ . Symmetry of pressure preserves the Wronskian

$$z' \cdot w - z \cdot w' = \omega \cdot (z \times w) = \omega_0 \cdot (v_0 \times w_0).$$

For a material area vector  $q$ , write  $n = q/|q|$ . Since  $q' = -A^T q$ , direct differentiation gives

$$\frac{(|q|)''}{|q|} = |A^T n|^2 + n^T P n + 2n^T (A^T)^2 n - (n^T S n)^2. \quad (12.16)$$

At an isotropic launch, spherical averaging and  $\text{tr } P = -\text{tr}(A^2)$  reduce this expression to  $8|S|_F^2/15$ . A surface family already deformed by the flow need not have isotropic normals or weights. Restarting an isotropic average does not establish convexity for that transported family.

## 12.6 Physical energy paths and their relation to shell paths

Local energy has its own positive transport law. Write  $e = |u|^2/2$ ,  $F_E = (e+p)u$ , and set  $b_E = F_E/e$  where  $e > 0$ , with  $b_E = 0$  elsewhere. Then

$$\partial_t e + \text{div}(b_E e) = 0, \quad \int |b_E| e \, dx = \|F_E\|_1 \leq C \|u\|_3^3. \quad (12.17)$$

The last estimate uses the Riesz-transform bound  $\|p\|_{3/2} \leq C \|u\|_3^2$ . On a strict interval its time integral is finite. Strong  $L^2$  continuity makes  $\nu_t = e(t) \, dx / \mathcal{E}$  a continuous probability curve. The superposition principle [15, Theorem 12] gives a probability law on absolutely continuous paths with marginals  $\nu_t$ , velocity  $b_E$ , and expected length

$$\mathbb{E} \int_a^b |\gamma'(t)| \, dt = \mathcal{E}^{-1} \int_a^b \|F_E(t)\|_1 \, dt. \quad (12.18)$$

These paths carry energy through the pressure flux. Their velocity differs from the material velocity; they do not carry Kelvin circulation.

A conditional terminal consequence follows. Suppose  $\nu_{t_j}(B_{r_j}(x_j)) \geq \delta > 0$ , with  $r_j \downarrow 0$  and  $t_j \uparrow T$ . If the action in (12.18) is finite up to  $T$ , almost every path has a finite endpoint and its expected remaining displacement tends to zero. Choose  $\varepsilon_j \downarrow 0$  so the probability of remaining displacement above  $\varepsilon_j$  tends to zero. The terminal law then assigns at least  $\delta - o(1)$  to  $B_{r_j+\varepsilon_j}(x_j)$ . Tightness of that probability law keeps the centers in a bounded set along a subsequence. A convergent subsequence of centers and continuity of measures from above give a terminal atom of mass at least  $\delta$ . Thus this fixed-mass spatial recurrence requires either infinite energy-path action or an atomic terminal law. Neither alternative has been excluded for a hypothetical singular Euler history.

Under a Type-I gradient bound, the atomic alternative is also excluded. In fact, Gagliardo–Nirenberg gives

$$\|F_E(t)\|_1 \leq C N^{12/5} \|\nabla u(t)\|_\infty^{3/5}.$$

If  $\sup_{t_0 < t < T} (T-t) \|\nabla u(t)\|_\infty < \infty$ , this flux is time integrable. Chae–Wolf’s terminal nonconcentration theorem [16, Theorem 1.1 and Corollary 1.2] says that the terminal energy measure of this finite-energy Euler history has no atoms. Their hypotheses hold here: the velocity is bounded in  $L^2$ , locally in time in  $W^{1,\infty}$ , and has the stated Type-I bound. Applied to the preceding atom argument, their theorem proves that fixed-positive-mass spatial recurrence in shrinking balls requires

$$\limsup_{t \uparrow T} (T-t) \|\nabla u(t)\|_\infty = \infty. \quad (12.19)$$

This excludes the Type-I branch of that recurrence. It does not prove that all possible Euler breakdowns contain a fixed-mass shrinking core.

The shell law and energy-path law can be put on a product space. Their independent product has marginals  $\rho_q(x, t) = \mu_t(q)\nu_t(x)$ , satisfying

$$\partial_t \rho_q + \operatorname{div}_x(b_E \rho_q) = \sum_p (k_{pq} \rho_p - k_{qp} \rho_q).$$

This construction proves existence of a joint probability description. Independence does not identify which spatial energy supplies a particular frequency interaction. A physical frequency correlation must come from Euler, beyond the two marginal laws.

Scaling prevents a uniform comparison of their present action costs. Under energy-preserving dilation by  $\lambda$ , the integrated current of an annular donor/receiver edge away from the low block is unchanged under dyadic shell relabeling, while  $\int \|F_E\|_1 dt$  is divided by  $\lambda$ . An exact classical segment with positive exchange on that edge supplies a fixed positive lower bound for total shell action after dilation. It thus excludes a bound of shell action by  $C(\mathcal{E})$  times physical path length over all smooth data. Giving adjacent shell edges length proportional to  $2^{-q}$  restores this scaling but puts infinite frequency at finite distance, since  $\sum_q 2^{-q} < \infty$ . This concerns that shell-line metric; it does not classify all possible couplings or all geometrically informed costs.

## 13 What the conserved structure does not determine

The pressure criteria identify quantities whose control would reach the spatial intersection. To obtain that control from the datum, one needs an additional dynamical argument. Several exact tests separate the information already present from the information still required.

### 13.1 Distant pressure can reverse the local stretching response

Pressure cannot be assigned a universal restoring sign. This can be tested with smooth compactly supported finite-energy Euler data, without using an affine flow or an infinite background velocity.

Choose  $a = Re_3$ ,  $0 < \varepsilon < R$ , and a nonzero smooth radial bump  $f$  supported in  $[0, \varepsilon)$ , smooth as a radial function on  $\mathbb{R}^3$ . For a unit vector  $m$ , set

$$u_0^{(m)}(x) = f(|x - a|) m \times (x - a), \quad C = \frac{4\pi}{3} \int_0^\varepsilon r^4 f(r)^2 dr > 0. \quad (13.1)$$

These data are divergence-free and have kinetic energy  $C$ , independently of  $m$ . All their spatial velocity derivatives vanish at the origin. Put  $\Psi(r) = -\int_r^\varepsilon s f(s)^2 ds$ , extended by zero. Direct differentiation gives

$$(u_0^{(m)} \cdot \nabla) u_0^{(m)} = -(I - mm^T) \nabla \Psi, \quad p_0 = \Psi + \partial_m^2 (G * \Psi).$$

Here the scalar functions are centered at  $a$ . Since  $\int \Psi = -C$ , the radial Newton potential outside the support is  $G * \Psi = -CG$ . Thus  $p_0(x) = -C\partial_m^2 G(x - a)$  there. At the origin,

$$\partial_{33} p_0^{(e_3)}(0) = -\frac{6C}{\pi R^5}, \quad \partial_{33} p_0^{(e_1)}(0) = \frac{3C}{\pi R^5}. \quad (13.2)$$

For a particle direction  $e_3$  starting at the origin,  $A_0 = 0$  and  $\sigma'(0) = -\partial_{33} p_0(0)$ . Equal-energy remote data with identical zero local velocity derivatives produce opposite initial stretching responses. Their local Euler evolutions exist by classical theory. The example excludes a pointwise restoring-sign argument; it does not exclude cumulative compensation involving the coupled pressure history.

### 13.2 Rescaled steady fields do not form a concentrating history

Euler admits nonzero compactly supported smooth steady velocities  $U$  [8]. At any fixed scale, their rescalings are legitimate steady solutions. They show immediately that all temporal rows above row zero can vanish while the spatial derivatives of row zero are nontrivial. Temporal differentiation alone has no inverse spatial-smoothing operation.

Now suppose a classical Euler evolution had the exact form

$$u(x, t) = b(t)U(\lambda(t)(x - c(t))), \quad b(t), \lambda(t) > 0. \quad (13.3)$$

Let  $\Omega = \text{curl } U$ . It is nonzero: a compactly supported divergence-free and curl-free velocity would vanish. The set  $\{\Omega \neq 0\}$  has finite positive volume. Cauchy's formula and invertibility of  $F$  transport the nonzero-vorticity set exactly, so its volume is constant along a classical Euler evolution. In (13.3), that volume is  $\lambda(t)^{-3}|\{\Omega \neq 0\}|$ . Hence  $\lambda$  is constant. Energy then makes  $b$  constant. The scale orbit cannot be turned into a concentrating solution by prescribing a shorter and shorter clock.

This obstruction depends on the fixed compact profile in (13.3). It does not exclude an evolving profile with overlap, exchange, and changing material support geometry. Those changes are exactly what a coupled concentration mechanism would have to produce.

### 13.3 Independent packets leave cross interaction unpaid

If smooth divergence-free fields  $u_j$  individually solve Euler, their sum solves it only if the projected cross interaction vanishes:

$$\mathbb{P} \sum_{j \neq k} (u_j \cdot \nabla) u_k = 0. \quad (13.4)$$

The pressure projection is part of this condition. Support separation at one time does not by itself establish it on a later interval.

The clock of an isolated rescaled packet also constrains an independent assembly. If its  $L^2$  mass is  $e$  and inverse length is  $K$ , amplitude is proportional to  $e^{1/2}K^{3/2}$ , and the nonlinear clock is proportional to

$$e^{-1/2}K^{-5/2}. \quad (13.5)$$

For disjoint copies of a fixed profile at scales  $K_j \rightarrow \infty$ , an  $H^\infty$  datum requires  $\sum_j e_j K_j^{2m} < \infty$  for every integer  $m$ , up to the fixed profile constants. In particular,  $e_j^{1/2} K_j^{5/2} \rightarrow 0$ . The independent high-frequency packet clocks become long. A sequence of increasingly fast stages would need energy transfer or a change of profile; it is not produced by placing an all-height family of independent packets in the initial datum.

### 13.4 Finite temporal smoothness need not give a Taylor radius

The distinction between material and Eulerian time can be seen in an exact periodic example. On  $\mathbb{T}^3$ , take

$$u(x, y, z, t) = (f(y - ct), c, 0), \quad c \neq 0, \quad (13.6)$$

with constant pressure and a smooth periodic function whose nonzero Fourier coefficients have magnitude  $e^{-\sqrt{|k|}}$ . Transport cancels  $\partial_t u$ , so this is a global smooth Euler solution. Every finite mixed Sobolev derivative exists. Yet  $\partial_t^n u_1 = (-c)^n f^{(n)}(y - ct)$ , and choosing frequencies near  $|k| = 4n^2$  makes  $\|\partial_t^n u_1\|_2/n!$  grow faster than any fixed exponential. The Eulerian time Taylor radius is zero.

Its material map is instead  $X(a, t) = (a_1 + tf(a_2), a_2 + ct, a_3)$ , which is entire in time. This example is periodic and has a nonzero mean flow; it is not a whole-space finite-energy counterexample. Its precise role is to prevent an inference from all finite temporal derivatives to a convergent Eulerian time series. The spatial intersection theorem requires no such analyticity.

### 13.5 A fixed positive sum can restrict the admissible datum

One might try to replace the spatial intersection by a single positive sum  $\sum_m a_m \|\Lambda^m u\|_2^2$ , with fixed  $a_m > 0$ . This introduces a function space smaller than  $H^\infty$ . For any such weights, choose disjoint Fourier annuli with radii  $K_j \rightarrow \infty$ , solenoidal unit  $L^2$  packets there, and packet energies  $e_j = K_j^{-2j}$ . Choose the radii so large that  $a_{2j} K_j^{2j} \geq 1$  and  $K_j \geq 2^j$ . Then  $\sum_j e_j K_j^{2m} < \infty$  for every fixed  $m$ , while the  $2j$ -th weighted coordinate receives a contribution at least one from packet  $j$ . The resulting real solenoidal field is in  $H^\infty$ , but the fixed positive sum diverges.

Allowing weights to depend on the datum can avoid this example, but an evolution estimate for those weights is a further assertion. The intersection allows each finite height its own bound. It contains no uniform growth law in the height index.

### 13.6 Large material distortion inside a smooth steady flow

An Eulerian norm and a material shape measure can behave very differently without singularity. Use a compact steady pair  $(U, P)$  from Gavrilov's pressure localization [8], and a regular patch inside a cutoff plateau, where  $U \cdot \nabla P = 0$ . For a smooth scalar  $f$ , supported in the permitted pressure interval, put

$$V = f(P)U, \quad \Pi = Q(P), \quad Q' = f^2.$$

Then  $\operatorname{div} V = 0$  and  $(V \cdot \nabla)V = -f(P)^2 \nabla P = -\nabla \Pi$ . Choose a regular point  $a_*$  with  $U(a_*) \neq 0$ ,  $\nabla P(a_*) \neq 0$ , and  $f(P(a_*)) = 0$ ,  $f'(P(a_*)) \neq 0$ . This is a stationary particle of  $V$ , with

$$N_* := \nabla V(a_*) = f'(P(a_*))U(a_*) \otimes \nabla P(a_*), \quad N_*^2 = 0, \quad \nabla^2 \Pi(a_*) = 0.$$

Its deformation is  $F(a_*, t) = I + tN_*$ . If  $\eta = t|f'(P(a_*))||U(a_*)||\nabla P(a_*)|$ , the singular values are

$$s_\pm = \frac{\sqrt{\eta^2 + 4} \pm \eta}{2}, \quad 1.$$

Thus  $\kappa(F)$  grows quadratically in time, while every Eulerian Sobolev norm is constant. The pressure Hessian vanishes on this carrier; the pressure work  $V \cdot \nabla \Pi$  vanishes everywhere.

This behavior has a positive-volume version. If  $\phi_t$  is the seed flow, then  $\psi_t(a) = \phi_{f(P(a))t}(a)$  is the modulated flow and

$$D\psi_t = D\phi_{f(P)t}[I + tf'(P)U \otimes \nabla P]. \quad (13.7)$$

In a fixed regular patch, bands of pressure width  $\delta$  around the simple zero of  $f$  have volume comparable to  $\delta$  and energy comparable to  $\delta^3$ . If  $t\delta \leq c$ , the outer factor and its inverse stay bounded, while the inner shear grows with  $t$ . Choosing  $\delta = c/t$  gives smaller bands at later times, with volume of order  $t^{-1}$  and energy of order  $t^{-3}$ . It does not describe energy loss from one fixed band. Exact scaling can move a prescribed large distortion into a short observation interval across different smooth data. This example defeats a coercive comparison between unweighted deformation and Eulerian energy; it does not produce finite-time breakdown of the steady flow.

### 13.7 Conserved integrals do not locate a receiver

Translations expose another information loss. For quantities that exist, let  $\mathcal{H} = \int u \cdot \omega$  be helicity,  $I = \frac{1}{2} \int x \times \omega$  impulse, and  $M = -\frac{1}{2} \int |x|^2 \omega$  angular impulse. Under  $u^{b,\lambda}(x) = bu(\lambda x)$ , their scaling, together with energy and material circulation, is

|        |                   |                   |                 |                 |                 |
|--------|-------------------|-------------------|-----------------|-----------------|-----------------|
|        | $\ u\ _2^2$       | $\mathcal{H}$     | $I$             | $M$             | $\Gamma$        |
| factor | $b^2\lambda^{-3}$ | $b^2\lambda^{-2}$ | $b\lambda^{-3}$ | $b\lambda^{-4}$ | $b\lambda^{-1}$ |

The weighted integrals require additional decay; they are not available for every datum in the paper's class. Even where available, they need not pin the location of an energy-bearing component. Let  $B, W$  be compact steady fields and form  $B + W(\cdot - y) + W(\cdot + y)$  with disjoint supports. Their pressure can be taken constant outside the respective supports, so the sum is again an exact steady Euler field. Energy and helicity add. Translation leaves impulse invariant and changes angular impulse by a linear expression in the translation and impulse; the paired translations cancel. The first signed energy moment also has its two translation terms cancel. These quantities stay fixed as  $|y| \rightarrow \infty$ , while a positive amount of energy leaves every fixed ball. This is a family of initial data, not escape through time for one datum.

A positive weighted energy moment would control tails, but its finiteness and propagation are extra hypotheses. Even the energy barycenter is not a conserved particle center: when the integrals are justified, the local energy equation gives

$$\frac{d}{dt} \int x \frac{|u|^2}{2} dx = \int (\frac{1}{2}|u|^2 + p)u dx.$$

The pressure flux participates in its motion.

### 13.8 What a scalar pressure coordinate retains

At zero frequency the pressure trace contains a kinetic tensor. With the unnormalized Fourier transform used for this calculation, let  $K_u = \int u \otimes u dx$ . Then

$$\widehat{p}(rn) \longrightarrow -n^T K_u n \quad (r \downarrow 0).$$

A weighted angular average retains only  $-A_\sigma : K_u$ , where  $A_\sigma = \int_{S^2} n \otimes n d\sigma(n)$ . Only angular degrees zero and two occur. The uniform spherical average is  $-\|u\|_2^2/3$ ; polarization gives an energy Gram pairing. Translation and energy-preserving dilation leave this zero-frequency tensor unchanged. It cannot locate a receiver or record its scale.

A nonzero pressure coordinate retains phase, but pays for that sensitivity. For fixed  $k \neq 0$ ,

$$\begin{aligned} \widehat{p}(k) &= -\widehat{k_i k_j} \widehat{u_i u_j}(k), & |\widehat{p}(k)| &\leq \|u\|_2^2, \\ \widehat{p}_{u(\cdot-y)}(k) &= e^{-ik \cdot y} \widehat{p}_u(k), & \widehat{p}_{u_\lambda}(k) &= \widehat{p}_u(k/\lambda). \end{aligned} \tag{13.8}$$

A fixed real phase can be reversed by translation while positive receiver work is unchanged. A finite linear combination of nonzero Fourier coordinates has zero translation average, so it cannot uniformly charge a strictly positive translation-invariant event. Moduli discard that phase but remain insensitive to the center. Multiplying by a moving phase  $e^{ik \cdot c(t)}$  adds the derivative  $i(k \cdot c')e^{ik \cdot c}\widehat{p}(k)$ ; selecting the center creates a new term rather than a conserved charge.

There is a corresponding scaling constraint. Suppose normalized profiles are related by an exact evolution followed by an energy-unitary dilation, and physical scales satisfy  $K_{j+1} = b_j K_j$ . A Fourier

label transported without reset has  $k_{j+1} = b_j^{-1}k_j$ , so  $K_j k_j$  is a fixed physical frequency. As  $K_j \rightarrow \infty$ , the normalized label approaches zero. It does not keep sampling a fresh normalized annulus.

The limitation also appears for bounded quadratic memories. If  $\|F_j\|_2^2 = E$ ,  $\|B_j\| \leq C$ , then  $J_j = \langle F_j, B_j F_j \rangle$  satisfies  $|J_j| \leq CE$ . A proved per-event increase  $J_{j+1} - J_j \geq \gamma\eta > 0$  would allow at most  $2CE/(\gamma\eta)$  events. Boundedness alone does not establish that increase. For a unitary dilation  $S$ , the universal identity

$$\langle S^{-1}G, CS^{-1}G \rangle - \langle F, BF \rangle = \langle G, AG \rangle - \langle F, AF \rangle \quad \text{for every } F, G$$

forces  $B = A$ ,  $C = S^{-1}AS$ , for self-adjoint operators. Set each vector in turn to zero and polarize to prove the assertion. Thus a quadratic observable transported through dilation carries its old receiver with it. A freshly chosen receiver needs a new increment law. This test concerns bounded quadratic functionals and identities for independent input pairs; it is not a classification of all functionals on Euler histories.

There is an exact finite-coordinate version of this test. Given finitely many nonzero frequencies  $k_1, \dots, k_m$ , take a compact steady pair  $(V, p_V)$ . Choose translations  $y_j$  with  $k_j \cdot y_j$  an odd multiple of  $\pi$ , so large that all  $2^m$  translates with centers  $\sum_j \epsilon_j y_j$ ,  $\epsilon_j \in \{0, 1\}$ , have disjoint supports. This is possible inductively on each unbounded phase plane. Their sum is steady. Its pressure transform contains the factor  $\prod_j (1 + e^{-ik \cdot y_j})$ , and hence vanishes at every selected frequency. Multiplication by one amplitude fixes the energy. Thus finitely many pressure coordinates can all vanish in a nonzero exact steady field.

Spatial pressure virials make the cost of detecting location explicit. When the pairings are justified,

$$\langle p, -\Delta\Phi \rangle = \int u_i u_j \partial_{ij} \Phi \, dx.$$

For a translated compact steady packet this samples  $\nabla^2\Phi$  near its center. Hessians tending to zero miss an escaping packet; Hessians tending to a constant matrix read a bounded kinetic-tensor contraction. Bounded oscillatory Hessians can produce different limits along translation subsequences, as  $\Phi(x) = \cos(k \cdot x)$  shows. Growing Hessians can detect location but require weighted moments outside the energy hypothesis. For a compact steady pair, testing with quadratic functions gives  $\int V \otimes V = -(\int p_V)I$ , so  $\int p_V = -\|V\|_2^2/3$ . The symmetric translated pair has

$$\int |x|^2 [p_V(x - y) + p_V(x + y)] \, dx = 2 \int |x|^2 p_V(x) \, dx - \frac{2}{3} |y|^2 \|V\|_2^2.$$

This detects escape through a spatial moment that grows at fixed energy. Taking a compact cutoff limit before a packet escapes and taking it after escape give different results; no uniform exchange of those limits is supplied by the virial identity.

## 14 Global continuation and the remaining spatial condition

The spatial intersection concerns one fluid throughout its evolution. Theorem 2.1 shows that whole-interval spatial membership of its velocity generates every temporal and pressure row. The finite recurrence and energy interpolation perform that generation; the next temporal row supplies a strong trace, and Sobolev embedding gives all mixed derivatives continuously on the closed slab. A single restart of the same history then excludes a finite maximal time under (2.12). The proof requires neither summability across derivative orders nor an infinite total of successive local lifespans.

Proposition 2.4 constructs the complete intersection from arbitrary smooth data on one common local interval. Its persistence estimate depends on the evolving velocity gradient. It therefore does

not yet supply the whole-interval membership needed at a candidate finite maximal time. The missing conclusion is stated precisely in (2.12).

The terminal constructions use energy to give information at a candidate finite endpoint. There is one terminal velocity distribution  $u_*$ , one projected-transport isometry  $W$ , and one measure-preserving particle limit  $X_*$ . Their relations retain momentum, circulation as a current, and an exact description of the energy defect. They leave the full spatial regularity unresolved. In the operator formulation the required membership is  $u_0 \in W(H_\sigma^\infty)$ ; in the material formulation, derivatives of the deformation enter the elliptic conversion from conserved circulation to physical velocity. The row-two counterexample shows why iterating an energy-only weak time estimate cannot supply this information.

The pressure analysis reaches continuation by a different route. Its material separation equation admits both an absolute comparison and a signed matrix energy identity. The latter permits cancellation between noncommuting pressure histories before a norm is taken. Its weighted primitive is constrained by the actual strain and vorticity equations, and energy makes the late contribution from fixed-distance exterior regions small. A further proof could use this near-field, directional, and temporal structure. The present identities identify that structure without proving its required bound.

The coupled transfer results give another concrete requirement. A receiver can gain energy only through cross work with the surrounding velocity. Fixed positive energy recurring at successively finer scales forces infinite shell activity and infinite integrated temporal variation. With spatial localization in shrinking cores, this recurrence also requires growth beyond the Type-I gradient bound. A localized receiver also demands substantial circulation on a parallel family of surfaces and an explicit deformation factor in the initial labels. Those statements retain participation that a scalar norm does not record. Their hypotheses still matter: fragmentation can evade fixed-mass recurrence, the participating label sets can change, and conserved flux does not bound the areas through which it passes.

These approaches need not be forced into one proof template. They meet at a physical question: what restriction does the coupled evolution impose on the correlations that would sustain concentration? The terminal intersection, signed pressure history, and material recruitment describe different places where a sufficient restriction could act. The results proved here make those places explicit and exclude several proposed shortcuts. They do not yet establish the required restriction for every smooth finite-energy initial velocity.

## A Spatial tails and the full increment equation

### A.1 A terminal trace as a Fourier-tail condition

The common low terminal trace gives a useful exact criterion at a fixed spatial height. Let  $P_{>K}$  be the sharp Fourier projection onto  $|\zeta| > K$ . For an integer  $m \geq 0$ , the two statements

$$u(t) \longrightarrow u_* \quad \text{strongly in } H^m \tag{A.1}$$

and

$$\lim_{K \rightarrow \infty} \limsup_{t \uparrow T} \|P_{>K} u(t)\|_{H^m} = 0 \tag{A.2}$$

are equivalent. In the second statement  $u_* \in H^m$  follows from the condition; it need not be added separately.

To see this, at fixed  $K$  the low-frequency difference obeys

$$\|P_{\leq K}(u(t) - u(s))\|_{H^m} \leq (1 + K^2)^{(m+3)/2} \|u(t) - u(s)\|_{H^{-3}}.$$

It tends to zero as  $s, t \uparrow T$ . The tail condition bounds the two remaining high-frequency terms and makes the full path Cauchy in  $H^m$ . Its limit is identified by the low trace. Conversely, strong convergence gives  $\limsup_{t \uparrow T} \|P_{>K}u(t)\|_{H^m} \leq \|P_{>K}u_*\|_{H^m}$ , which tends to zero.

This criterion concerns uniform tails through the terminal approach. The tail of every individual smooth snapshot tends to zero, but exchanging that spatial limit with the terminal limit is an additional assertion. At  $m = 0$ , the condition is equivalent to  $d_* = 0$ .

## A.2 Increment moments retain pressure at higher order

For a spatial displacement  $h$ , put  $\delta_h u(x) = u(x + h) - u(x)$  and define  $\delta_h p$  likewise. Subtracting Euler at the two positions gives

$$\partial_t \delta_h u + u \cdot \nabla \delta_h u + \delta_h u \cdot \nabla u(x + h) + \nabla \delta_h p = 0. \quad (\text{A.3})$$

For  $\Phi(z) = |z|^{2q}/(2q)$ ,  $q \geq 1$ , multiplication by  $D\Phi(\delta_h u)$  and spatial integration yields

$$\begin{aligned} \frac{d}{dt} \int \Phi(\delta_h u) dx &= - \int D\Phi(\delta_h u) \cdot (\delta_h u \cdot \nabla) u(x + h) dx \\ &\quad + \int \delta_h p \operatorname{div}(D\Phi(\delta_h u)) dx. \end{aligned} \quad (\text{A.4})$$

All integrals are finite at strict times by the spatial Sobolev assumptions. For  $q = 1$ ,  $\operatorname{div} \delta_h u = 0$  removes the pressure pairing. For higher moments it does not: differentiation also acts on the weight  $|\delta_h u|^{2q-2}$ . The pressure-complete hierarchy retains this term at every order.

For the quadratic moment, translation and incompressibility put the result in the exact displacement-flux form

$$\partial_t \int |\delta_h u|^2 dx = - \operatorname{div}_h \int \delta_h u |\delta_h u|^2 dx. \quad (\text{A.5})$$

For  $0 < s < 1$ , the Fourier identity

$$\|\Lambda^s u\|_2^2 = c_s \int_{\mathbb{R}^3} \frac{\|\delta_h u\|_2^2}{|h|^{3+2s}} dh \quad (\text{A.6})$$

then recovers the Sobolev energy flux by integrating (A.5). Higher difference operators supply higher orders. Full temporal differentiation distributes derivatives across every factor, as in (2.3). These representations retain spatial direction and nonlinear coupling, but integrating them exactly returns the corresponding Sobolev flux. A terminal bound does not arise from the change of representation alone.

For a smooth radial cutoff  $\chi$ , a related continuous-frequency identity is

$$\frac{d}{dt} \|\chi(\Lambda/K) \Lambda^m u\|_2^2 = -2 \langle \chi(\Lambda/K) \Lambda^m u, \chi(\Lambda/K) \Lambda^m \mathbb{P}(u \cdot \nabla u) \rangle.$$

Any integration over cutoff scale must preserve its exact weight. Two equivalent spatial norms need not have comparable signed time derivatives; norm equivalence is a bound on values, not on their rates of change.

### A.3 A conditional translation modulus through normalized stages

The quadratic increment law does give a useful conditional compactness statement. For  $w_h = u(\cdot + h) - u$ , write the difference equation with advecting velocity  $u(\cdot + h)$  and the remaining gradient at  $x$ . Its energy pairing is

$$\frac{1}{2} \partial_t \|w_h\|_2^2 + \int w_h^T S w_h \, dx = 0.$$

Let  $a(t) = \|(-S(t))_+\|_{\infty, \text{op}}$ , with the positive part taken spectrally, and  $\omega_f(r) = \sup_{|h| \leq r} \|f(\cdot + h) - f\|_2$ . Gronwall gives  $\omega_{u(t)}(r) \leq e^{\int_s^t a} \omega_{u(s)}(r)$ . For  $S_b f(x) = b^{3/2} f(bx)$  and a normalized recurrence  $F_{j+1} = S_{b_j}^{-1} \Phi_{c_j}(F_j)$ , where  $\Phi$  denotes the actual Euler evolution on that segment, set  $A_j = \int_0^{c_j} \|(-S[\Phi_t(F_j)])_+\|_{\infty} \, dt$ . Then

$$\omega_{F_N}(r) \leq r \|\nabla F_0\|_2 \exp \left( \sum_{j < N} (A_j - \log b_j) \right). \quad (\text{A.7})$$

Indeed, one stage gives  $\omega_{F_{j+1}}(r) \leq e^{A_j} \omega_{F_j}(r/b_j)$ ; iterate and use  $F_0 \in H^1$ . Translations used to recenter profiles leave the modulus unchanged.

If the cumulative excess in (A.7) is bounded above, the normalized snapshots have uniform translation continuity. Together with their energy bound and uniform spatial tightness, the Fréchet–Kolmogorov criterion gives strong  $L^2$  precompactness. This removes the snapshot stress defect. It still requires the two stated bounds; time compactness of a whole normalized history is an additional step. The formula identifies the balance between compressive strain and the compacting dilation, without estimating that balance from energy.

### A.4 No viscosity limit is implicit in the construction

If a family of Navier–Stokes solutions  $u^\nu$  is used to approximate Euler, passing to the terminal intersection requires input uniform as  $\nu \downarrow 0$  in the relevant physical coordinates. At the level of differentiated equations,

$$V_{n+1}^\nu + \mathbb{P} \sum_{j=0}^n \binom{n}{j} (V_j^\nu \cdot \nabla) V_{n-j}^\nu = \nu \Delta V_n^\nu.$$

The spatial recovery in (1.3) divides by positive powers of  $\nu$ . Existence of a finite bound for each fixed viscosity does not bound those quotients uniformly as viscosity vanishes. An inviscid passage would need an additional uniform theorem. No result in this paper assumes that passage.

## B Equivalent forms of the endpoint construction

For the fixed classical Euler history used throughout, the following conditions are equivalent:

- (i) The solution extends through  $T$  in  $C_t H_\sigma^\infty$ .
- (ii) Every temporal velocity row belongs to the common spatial intersection  $\mathcal{J}_T^w$ .
- (iii) The snapshot condition (3.5) holds at every finite spatial height.
- (iv) The low terminal velocity satisfies  $u_* \in H_\sigma^\infty$  and  $\|u_*\|_2 = N$ .
- (v) The initial datum belongs to  $W(H_\sigma^\infty)$ .

The implications from (ii), (iii), (iv), and (v) to (i) were proved in Corollary 2.3, Theorem 3.3, Lemma 3.2, and Corollary 4.2. An extension gives (ii) and (iii) by smoothness on the resulting compact interval, (iv) by energy conservation, and (v) by the terminal propagator.

This equivalence records the exact scope of the unresolved construction. It does not claim that changing among these formulations has reduced its difficulty. The pressure conditions in Section 9 are explicit sufficient hypotheses that imply all five conditions. Establishing one of them from arbitrary smooth data remains a separate theorem.

## C Pressure tests with explicit realizability limits

The material pressure criteria use a time weight or a signed primitive. Replacing them by a square-root action would be attractive because that action has the scaling of a first-order clock. Symmetric Jacobi equations show why such a replacement needs additional Euler structure. A separate static construction shows how large Hessian values can occur even among finite-energy Euler data with a fixed gradient maximum.

### C.1 Finite square-root action and unbounded Jacobi solutions

Consider  $F'' + PF = 0$  on  $[0, T]$ . There are smooth symmetric trace-free coefficients with  $\int_0^T \|P\|^{1/2} dt < \infty$  and unbounded  $F$ . To construct one, set

$$B_j = 2^{2j}, \quad w_j = 2^{-4j}, \quad \ell_j = 2^{-2j}, \quad q_j = B_j/w_j = 2^{6j}.$$

Use successive pulse windows of length  $2w_j$ , followed by free intervals of length  $\ell_j$ . In each window choose a smooth nonnegative pulse  $q \leq q_j$ , equal to  $q_j$  on a central interval of length  $w_j$ , and flat at its endpoints. Set  $q = 0$  between pulses. Their total duration is finite, and

$$\int_0^T \sqrt{q(t)} dt \leq 2 \sum_j w_j \sqrt{q_j} = 2 \sum_j 2^{-j} < \infty.$$

For  $y'' = qy$ ,  $y(0) = 1$ ,  $y'(0) = 0$ , positivity makes both  $y$  and  $y'$  nondecreasing. If  $Y_j$  is the value at the start of pulse  $j$ , its plateau increases  $y'$  by at least  $B_j Y_j$ ; the following free interval increases  $y$  by at least  $\ell_j B_j Y_j = Y_j$ . Hence  $Y_{j+1} \geq 2Y_j$ . Taking  $P = \text{diag}(-q, q, 0)$ ,  $F(0) = I$ ,  $F'(0) = 0$ , embeds  $y$  as the first diagonal entry. This example imposes no determinant condition. Nor does finite action ensure invertibility: constant  $P = \text{diag}(\pi^2, -\pi^2, 0)$  gives first entry  $\cos \pi t$ , which vanishes at  $t = 1/2$ .

The growth construction extends to every exponent  $0 < \alpha < 1$ . Use  $B_j = 2^j$ ,  $\ell_j = B_j^{-1}$ ,  $w_j = 2^{-rj}$ , with  $r > \alpha/(1 - \alpha)$ . Each stage still doubles  $y$ , while  $\sum_j q_j^\alpha w_j < \infty$ . The weighted absolute criterion in (9.6) is not contradicted: the remaining time through a pulse includes its following free interval, so each pulse contributes at least  $\ell_j q_j w_j = 1$  to  $\int_0^T (T - t)q(t) dt$ .

Volume preservation alone does not repair the square-root implication. The next construction imposes it exactly, with a weaker coefficient regularity stated explicitly.

**Proposition C.1** (A volume-preserving auxiliary Jacobi counterexample). *There are  $T < \infty$ , a continuous locally Lipschitz trace-free matrix  $A$ , and a symmetric locally bounded piecewise smooth coefficient  $P$ , such that*

$$\begin{aligned} F' &= AF, & F(0) &= I, & F'' + PF &= 0 \quad a.e., \\ \det F &= 1, & \int_0^T \|P\|^{1/2} dt &< \infty, & \sup_{t < T} (\|F\| + \|F^{-1}\|) &= \infty. \end{aligned} \tag{C.1}$$

Both matrix norms are unbounded. Here  $F \in C^1 \cap W_{\text{loc}}^{2,\infty}$ . No Euler realization or smooth-coefficient upgrade is asserted.

*Proof.* Let  $N$  have entries  $N_{12} = N_{23} = 1$  and all others zero. Then  $N^3 = 0$ ,  $\|\text{skw } N\|_F = 1$ , and  $\|\text{skw}(N - 2N^2)\|_F = \sqrt{3}$ . For  $R \in SO(3)$ , the arc

$$A(t) = rR(N - rtN^2)R^T, \quad 0 \leq t \leq 2/r,$$

satisfies  $A' + A^2 = 0$ ,  $\text{tr } A = 0$ . It costs no pressure action and increases the skew norm from  $r$  to  $\sqrt{3}r$ . At its endpoint write  $A_- = S_- + \Omega_-$ ,  $s = \|\Omega_-\|_F$ ; then  $\|S_-\|_F \leq C_0 s$ .

Reset this endpoint over a time  $\varepsilon = s^{-3}$ . On  $\|\Omega - \Omega_-\|_F < s/4$ , choose a smooth local section

$$\mathcal{C}(\Omega) = \|\Omega\|_F Q(\Omega) N Q(\Omega)^T, \quad \text{skw } \mathcal{C}(\Omega) = \Omega.$$

Identify skew matrices with axial vectors and rotate their unit directions from that of  $\Omega_-$ . This gives  $\|\mathcal{C}\|_F \leq Cs$ ,  $\|D\mathcal{C}\| \leq C$  with uniform constants. For a fixed smooth cutoff  $\chi$  from zero to one, solve

$$\begin{aligned} S(t) &= (1 - \chi(t/\varepsilon))S_- + \chi(t/\varepsilon) \text{sym } \mathcal{C}(\Omega(t)), \\ \Omega' &= -(S\Omega + \Omega S), \quad \Omega(0) = \Omega_-. \end{aligned}$$

The equation preserves skew symmetry. Within the stated neighborhood,  $\|S\| + \|\Omega\| \leq Cs$ ,  $\|\Omega'\| \leq Cs^2$ , so  $\|\Omega(t) - \Omega_-\| \leq C/s$ . For sufficiently large  $s$ , this closes the neighborhood bound and supplies the full reset. Differentiation gives  $\|S'\| \leq C(s/\varepsilon + s^2)$ . With  $A = S + \Omega$ ,  $P = -A' - A^2$ , the skew equation makes  $P$  symmetric, and  $\text{tr } A = 0$ . Its cost is

$$\int_{\text{reset}} \|P\|^{1/2} dt \leq C(\sqrt{s\varepsilon} + s\varepsilon) \leq C/s.$$

The endpoint  $A = \mathcal{C}(\Omega(\varepsilon))$  begins a new arc. Thus the successive arc scales satisfy  $r_{j+1} \geq \sqrt{3}r_j - C/r_j$ . For large  $r_1$  they grow at least geometrically, giving finite total duration and square-root action. Endpoint values of  $A$  match, so  $A$  is continuous and locally Lipschitz. Solving  $F' = AF$  gives the claimed regularity, the Jacobi equation almost everywhere, and  $\det F = 1$ .

Finally, symmetry preserves  $\mathcal{W} = 2F^T \Omega F = 2r_1 \text{skw } N \neq 0$ . Consequently

$$\|F^{-1}\|_{\text{op}}^2 \geq \frac{2\|\Omega\|_F}{\|\mathcal{W}\|_F}.$$

At arc starts the right side tends to infinity. Determinant one then implies  $\|F\|_{\text{op}}^2 \geq \|F^{-1}\|_{\text{op}}$ , proving growth of both. The coefficient obeys  $\text{tr } P = -\text{tr}(A^2)$ ; it need not be trace-free.  $\square$

## C.2 A fixed-energy static Hessian test

**Proposition C.2** (Nested annular pressure contributions). *There are real solenoidal  $u_n \in C_c^\infty(B_8)$  and constants  $E_*, h > 0$ , such that*

$$\|u_n\|_2^2 = E_*, \quad \|\nabla u_n\|_{L^\infty(F)} = 1, \quad \nabla^2 p_{u_n}(0) = (n + b_n^2)h \text{diag}(1, 1, -2),$$

where  $0 < b_n < 1$ . Each field vanishes near zero. In particular, energy and the gradient maximum do not bound the pressure Hessian at a point of zero local velocity jet.

*Proof.* Let  $\phi$  be a nonzero radial smooth bump in the unit ball and  $w = (\partial_2 \phi, -\partial_1 \phi, 0)$ . Radial symmetry gives  $\int w \otimes w = m \operatorname{diag}(1, 1, 0)$ ,  $m > 0$ . Put  $y_0 = \frac{3}{2}e_3$  and  $v_\eta(x) = \eta w((x - y_0)/\eta)$ . For small  $\eta$ , its support lies in  $1 < |x| < 2$ . With  $G = 1/(4\pi|x|)$ , the pressure tensor is

$$(H_\eta)_{ab} = \int \partial_{abij} G(-x) (v_\eta)_i (v_\eta)_j \, dx.$$

The kernel is smooth on this support. Taylor expansion and change of variables give

$$\eta^{-5} H_\eta \longrightarrow -m \nabla^2 \partial_{33} G(y_0) = \frac{3m}{\pi(3/2)^5} \operatorname{diag}(1, 1, -2).$$

Axial symmetry and harmonicity near zero give exactly  $H_\eta = h_\eta \operatorname{diag}(1, 1, -2)$ , with  $h_\eta > 0$  for small fixed  $\eta$ . Normalize one such field to  $v$ , with  $\|\nabla v\|_{L^\infty(F)} = 1$ . Write  $H = \nabla^2 p_v(0) = h \operatorname{diag}(1, 1, -2)$  and  $E_v = \|v\|_2^2$ .

Let  $\rho = 1/4$ ,  $r_k = \rho^{k-1}$ , and  $U_n = \sum_{k=1}^n r_k v(\cdot/r_k)$ . The supporting annuli are disjoint. Hence

$$\|\nabla U_n\|_\infty = 1, \quad E_n = E_v \sum_{k=1}^n r_k^5 \leq \frac{E_v}{1 - \rho^5}, \quad \nabla^2 p_{U_n}(0) = nH.$$

The last equality uses disjointness of the quadratic stresses and the scaling  $p_{r_k v(\cdot/r_k)}(x) = r_k^2 p_v(x/r_k)$ . Set  $E_* = 2E_v/(1 - \rho^5)$ ,  $L = 4$ , and

$$b_n^2 = \frac{E_* - E_n}{L^5 E_v}, \quad u_n = U_n + b_n L v(\cdot/L).$$

The outer copy is supported in  $4 < |x| < 8$  and  $0 < b_n^2 \leq 2/1023 < 1$ . It fixes the energy, leaves the gradient maximum equal to one, and adds  $b_n^2 H$  to the Hessian.  $\square$

Every  $u_n$  starts a local classical finite-energy Euler solution. The construction gives no uniform higher Sobolev norms, no prescribed pressure history, and no singular evolution from a fixed datum. Its consequence is the failure of the indicated static pressure estimate.

## D Periodic receiver and conservative-defect tests

The examples in this appendix are exact Euler histories on  $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ , with normalized Haar measure. They test proposed consequences of conservation and local compactness. They are not finite-energy histories on  $\mathbb{R}^3$ .

### D.1 Repeated positive receiver gains without spending circulation

Fix  $\varepsilon > 0$  and take

$$u(x, t) = (\sin x_2, 0, \varepsilon \cos(x_1 - t \sin x_2)), \quad p = 0. \quad (\text{D.1})$$

Transport by the first component cancels the time derivative of the third, so this is a smooth global Euler solution. Its flow and pulled-back one-form are

$$\begin{aligned} X(t, a) &= (a_1 + t \sin a_2, a_2, a_3 + \varepsilon t \cos a_1), \\ X(t)^*(u \cdot dx) &= u_0 \cdot da + da \left[ \frac{t}{2} (\sin^2 a_2 + \varepsilon^2 \cos^2 a_1) \right]. \end{aligned}$$

All material circulations are constant.

Choose a smooth radial receiver  $R_K$  with symbol  $r(|k|/K)$ , where  $0 \leq r \leq 1$ ,  $r = 1$  on  $[1/2, 2]$ , and  $\text{supp } r \subset [2/5, 5/2]$ . For  $K \geq 8$  it removes the stationary shear. Let  $\mathcal{E}_v = \varepsilon^2/2$ . Define the Bessel coefficients by  $e^{-it \sin y} = \sum_n (-1)^n J_n(t) e^{iny}$ . Parseval gives

$$\frac{\|R_K u(t)\|_2^2}{\mathcal{E}_v} = \sum_n r(\sqrt{1+n^2}/K)^2 J_n(t)^2.$$

Parseval applied to this phase and its first two derivatives gives

$$\sum_n J_n^2 = 1, \quad \sum_n n^2 J_n^2 = t^2/2, \quad \sum_n n^4 J_n^2 = (3t^4 + 4t^2)/8.$$

At  $t = K$ , Paley–Zygmund for  $n^2$  and a fourth-moment tail bound yield

$$\sum_{K/2 \leq |n| \leq 3K/2} J_n(K)^2 \geq \frac{K^2}{2(3K^2 + 4)} - \frac{2(3 + 4/K^2)}{81} \geq \frac{1}{12}.$$

These modes lie in the receiver plateau. At  $t = K/32$ , every retained mode has  $|n| \geq K/3$ , and the second moment bounds its total mass by  $9/2048$ . Thus

$$\|R_K u(K)\|_2^2 - \|R_K u(K/32)\|_2^2 \geq \mathcal{E}_v/16. \quad (\text{D.2})$$

Taking  $K_j = 64^j K_0$ ,  $K_0 \geq 8$ , gives disjoint event intervals with an unbounded sum of positive gains as the horizon tends to infinity. The energy passes through successive bands; circulation is conserved throughout. For integer  $\lambda$ ,  $u(\lambda x, \lambda t)$  has unchanged torus energy and puts any finite number of these events into a prescribed positive time interval by increasing  $\lambda$ . This excludes a stage bound based only on energy and elapsed time across periodic smooth data. The infinite event sequence of the fixed original datum still takes infinite time.

## D.2 A nonzero conservative defect with a positive receiver

Positivity of receiver work does not force every unresolved stress to participate in it. Fix  $a \neq 0$  and, for integers  $n \rightarrow \infty$ , set

$$\theta(t, x, y) = \cos(x - t \sin y), \quad U_n = (\sin y, 0, \theta + a \sin(ny)), \quad p_n = 0.$$

Every component is transported by  $\partial_t + \sin y \partial_x$ . These histories are smooth for all  $t \in \mathbb{R}$ , including arbitrarily negative times, and  $\|U_n(t)\|_2^2 = 1 + a^2/2$ . Their limits on compact time intervals are

$$U_n \rightharpoonup U = (\sin y, 0, \theta), \quad U_n \otimes U_n \rightharpoonup U \otimes U + \mathcal{R}, \quad \mathcal{R} = \frac{a^2}{2} e_3 \otimes e_3. \quad (\text{D.3})$$

The positive defect has zero divergence and double divergence; pressure is zero both before and after the limit.

Take a smooth Fourier multiplier  $P$  equal to one at the four lattice points  $(\pm 1, \pm 1, 0)$  and zero at every other lattice point. It can be supported inside four small annular balls. It removes both shears in  $U_n$ , and the phase expansion gives  $\|PU_n(t)\|_2^2 = J_1(t)^2$ . Choose  $0 < \alpha < \beta < \pi/2$  small enough that  $J_1, J_1' > 0$  on this interval. Then

$$\|PU_n(\beta)\|_2^2 - \|PU_n(\alpha)\|_2^2 = J_1(\beta)^2 - J_1(\alpha)^2 > 0$$

is uniform in  $n$ . The selected receiver and its work survive in the smooth mean. The defect pairs to zero with its gradient because every field is independent of  $z$ .

A full-field circulation stack survives as well. For an  $(x, z)$  rectangle with side lengths  $r, s$ , the two horizontal edges and the oscillatory shear cancel, leaving

$$\oint_{\partial D} U_n \cdot dx = s[\theta(t, x + r, y) - \theta(t, x, y)].$$

Fix  $r = s = 3/2$  and  $x = -\pi/2 - r/2$ . The expression becomes  $2s \sin(r/2) \cos(t \sin y)$ , bounded below by  $2s \sin(r/2) \cos \beta > 0$ . Varying  $y$  supplies a positive-measure family of disjoint parallel surfaces with one-sign circulation, uniformly in  $n$ .

There is no anomalous energy loss hidden in this defect. Put  $e = |U|^2/2$ ,  $k = a^2/4$ . Direct expansion gives

$$\frac{1}{2}|U_n|^2 \rightharpoonup e + k, \quad \frac{1}{2}|U_n|^2 U_n \rightharpoonup ((e + k) \sin y, 0, (e + 3k)\theta).$$

The latter flux equals  $eU + kU + \mathcal{R}U$ . The mean obeys local energy equality, while  $\mathcal{R} : \nabla U = 0$  and  $\partial_t k + \operatorname{div}(kU + \mathcal{R}U) = 0$ . Thus the unresolved energy is conservative. More generally, if a Reynolds limit admits a cubic energy-flux limit  $\bar{\mathcal{F}}$  and a smooth mean, subtraction of the resolved energy equation gives

$$\partial_t k + \operatorname{div} \mathcal{J}_{\mathcal{R}} = -\mathcal{R} : \nabla U, \quad \mathcal{J}_{\mathcal{R}} = \bar{\mathcal{F}} - (\tfrac{1}{2}|U|^2 + p)U - \mathcal{R}U.$$

An energy bound alone does not guarantee the cubic flux limit. When it exists, positive semidefiniteness of  $\mathcal{R}$  supplies no sign for its contraction with trace-free strain. The explicit histories above retain pressure slaving, positive receiver work, and circulation together with a nonzero conservative defect. They exclude a rigidity argument based only on those local inputs. They do not realize successive resamplings of one whole-space Euler history.

## References

- [1] T. Birnie, *Global Smoothness for the Three-Dimensional Incompressible Navier–Stokes Equations*, author’s companion manuscript, 2026, 73 pp.; version consulted 8 September 2026.
- [2] J. T. Beale, T. Kato, and A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Communications in Mathematical Physics* **94** (1984), 61–66. doi:10.1007/BF01212349.
- [3] N. Besse and U. Frisch, Geometric formulation of the Cauchy invariants for incompressible Euler flow in flat and curved spaces, *Journal of Fluid Mechanics* **825** (2017), 412–478. doi:10.1017/jfm.2017.402. arXiv:1701.01592.
- [4] D. Chae and P. Constantin, On a Type I singularity condition in terms of the pressure for the Euler equations in  $\mathbb{R}^3$ , *International Mathematics Research Notices* **2022**, no. 12, 9013–9023. doi:10.1093/imrn/rnab014. arXiv:2012.11948.
- [5] D. Chae and P. Constantin, Remarks on Type I blow-up for the 3D Euler equations and the 2D Boussinesq equations, *Journal of Nonlinear Science* **31** (2021), article 77. doi:10.1007/s00332-021-09734-0. arXiv:2103.10672.

- [6] P. Constantin, An Eulerian–Lagrangian approach for incompressible fluids: local theory, *Journal of the American Mathematical Society* **14** (2001), no. 2, 263–278. doi:10.1090/S0894-0347-00-00364-7. arXiv:math/0004059.
- [7] J. D. Gibbon, D. D. Holm, R. M. Kerr, and I. Roulstone, Quaternions and particle dynamics in the Euler fluid equations, *Nonlinearity* **19** (2006), no. 8, 1969–1983. doi:10.1088/0951-7715/19/8/011.
- [8] A. V. Gavrillov, A steady Euler flow with compact support, *Geometric and Functional Analysis* **29** (2019), 190–197. doi:10.1007/s00039-019-00476-6.
- [9] T. Kato, Nonstationary flows of viscous and ideal fluids in  $\mathbb{R}^3$ , *Journal of Functional Analysis* **9** (1972), 296–305. doi:10.1016/0022-1236(72)90003-1.
- [10] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Springer-Verlag, Berlin, 1972.
- [11] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics 27, Cambridge University Press, 2002. doi:10.1017/CBO9780511613203.
- [12] V. Zheligovsky and U. Frisch, Time-analyticity of Lagrangian particle trajectories in ideal fluid flow, *Journal of Fluid Mechanics* **749** (2014), 404–430. doi:10.1017/jfm.2014.221.
- [13] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften 343, Springer, 2011. doi:10.1007/978-3-642-16830-7.
- [14] R. Betchov, An inequality concerning the production of vorticity in isotropic turbulence, *Journal of Fluid Mechanics* **1** (1956), no. 5, 497–504. doi:10.1017/S0022112056000317.
- [15] L. Ambrosio and G. Crippa, Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields, in *Transport Equations and Multi-D Hyperbolic Conservation Laws*, Lecture Notes of the Unione Matematica Italiana 5, Springer, 2008, 3–57. doi:10.1007/978-3-540-76781-7\_1. Author version, Theorem 12.
- [16] D. Chae and J. Wolf, Energy concentrations and Type I blow-up for the 3D Euler equations, *Communications in Mathematical Physics* **376** (2020), 1627–1669. doi:10.1007/s00220-019-03566-6. arXiv:1706.02020.