

Archimedean Tilings as Physical Microstructure of Compact Extra Dimensions: Finite Kaluza–Klein Spectra and an Eight-Sheet Fermion Hypothesis

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Abstract

We propose a physical-tiling hypothesis for a $5 + 1$ -dimensional Kaluza–Klein compactification: at microscopic scales the two compact dimensions are carried by finite periodic Archimedean graphs, whose vertices, edges, faces, and associated fields are physical rather than auxiliary discretization data. To accommodate all eight tilings in one four-dimensional theory, we explicitly choose a compact internal space with eight toroidal sheets. Each sheet has its own regular-polygon tiling while all sheets share the macroscopic spacetime. A fundamental quadratic action for vertex fields gives a finite graph-Laplacian mass spectrum, hence a finite KK-like tower. We prove exact formulae for its first three normalized spectral moments in terms of the vertex degree and triangle incidence. In particular, the two trivalent tilings 4.6.12 and 3.12.12 have respective third moments 54 and 52, a tiling-dependent result not contained in a generic continuum torus Fourier formula. The conventional KK dispersion arises only in a long-wavelength regime after matching an effective spectral metric. We also state a precise $2 + 3 + 3$ rule for candidate fermion sectors on the eight sheets, including the condition for a common color triplet. Distance-pair group labels, Wilson loops, gravitational volume matching, and chiral zero modes are kept at their respective logical levels. The result is a falsifiable effective model and a spectral sum rule, not a completed derivation of the Standard Model.

Keywords: Archimedean tilings; physical compactification; Kaluza–Klein spectrum; periodic graph Laplacian; fermion classification

1 Introduction

In conventional Kaluza–Klein theory, a compact continuous two-torus is attached to each point of four-dimensional spacetime, and its Fourier modes appear as a mass tower in four dimensions[1]. Dimensional deconstruction shows that higher-dimensional kinematics may also arise from finitely many internal degrees of freedom[2]. Here we adopt a specific microscopic hypothesis: the plane covering the compact two-dimensional space is tiled by regular polygons in an Archimedean pattern, and elementary degrees of freedom reside on the vertices, edges, and faces. The finite edge length ℓ and the finite number of vertices are physical parameters; they are not removed by taking $\ell \rightarrow 0$ in the definition of the theory.

This makes the role of tiling calculable. A free field on a continuous torus has a spectrum controlled by its area, shape, and boundary conditions. If a physical tiling does not change any spectral quantity beyond those data, it has no separately identifiable content. Our first goal is therefore to define a minimal internal field model and derive an exact sum rule depending on

the tiling combinatorics. Our second goal is to retain the proposed $2 + 3 + 3$ association between eight Archimedean tilings and one generation of fermion classes, while stating the additional coexistence and common-group assumptions that association requires. The eight-sheet structure is a model postulate; a geometric counting identity alone is not claimed to derive the Standard Model.

2 A physically tiled compact space

2.1 Coexistence on eight toroidal sheets

Let $\mathfrak{T}$ denote the eight vertex-transitive Euclidean tilings by regular polygons other than the three regular tilings[3]. For each $T \in \mathfrak{T}$, choose a sufficiently large sublattice Λ'_T of its translation group so that every polygon remains an ordinary embedded face in the quotient and every noncontractible closed graph walk has at least four edges. The latter requirement excludes artificial three-cycles caused by an undersized periodic quotient. Define

$$Y_T = \mathbb{R}^2 / \Lambda'_T \simeq T^2, \quad \Gamma_T = T / \Lambda'_T, \quad Y = \bigsqcup_{T \in \mathfrak{T}} Y_T. \quad (1)$$

The compact fiber Y is the disjoint union of eight distinguishable two-dimensional toroidal sheets. The total space is $M^{3,1} \times Y$, with $5 + 1$ continuous long-wavelength coordinates on each connected component. This multi-sheet choice is one minimal explicit way to make eight different regular-polygon tilings coexist as candidate four-dimensional particle sectors. It avoids the unsupported requirement that all eight retain their regular polygon geometry under one flat torus metric. Mixing between sheets must be specified by additional interactions; it does not follow from Eq. (1).

We take Γ_T , rather than the smooth Y_T , to carry the microscopic physical degrees of freedom. The latter records compact topology, embedding scale, and long-wavelength geometry; the finite graph and its physical edge length $\ell_T > 0$ define microscopic propagation. Write N_T for the number of vertices, V_T for the area, and q_T for the vertex degree. With regular polygonal faces, these parameters cannot all be varied independently.

2.2 The eight physical graphs

Figure 1 displays local windows in the planar covers of the eight sheets. Each torus is obtained by a compatible periodic identification. The windows serve only as illustrations; the physical Γ_T are finite periodic graphs without boundary.

3 Fundamental graph action and a finite KK-like spectrum

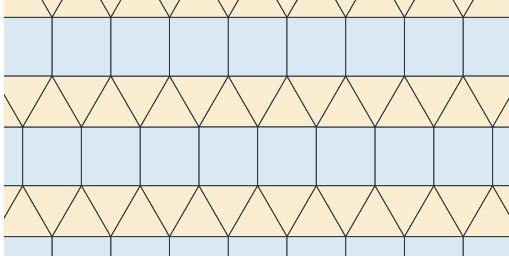
3.1 Minimal vertex-field model

Place a real scalar $\phi_{T,v}(x)$ at each vertex v of each sheet, with $x \in M^{3,1}$. On a flat four-dimensional background define the fundamental quadratic action

$$S_\phi = \frac{1}{2} \sum_{T \in \mathfrak{T}} \int d^4x \left[\sum_v \partial_\mu \phi_{T,v} \partial^\mu \phi_{T,v} - m_{0,T}^2 \sum_v \phi_{T,v}^2 - \frac{\kappa_T}{\ell_T^2} \sum_{\{u,v\} \in E_T} (\phi_{T,u} - \phi_{T,v})^2 \right], \quad \kappa_T > 0. \quad (2)$$

(a) Elongated triangular tiling

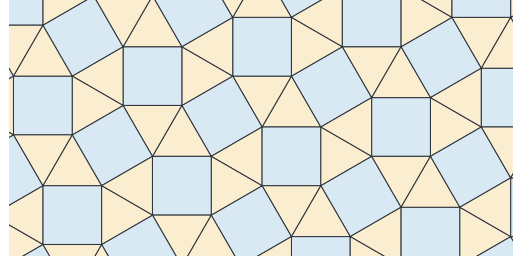
3.3.3.4.4



Candidate class: lepton

(b) Snub square tiling

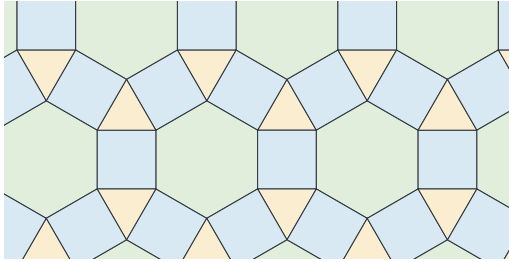
3.3.4.3.4



Candidate class: lepton

(c) Rhombitrihexagonal tiling

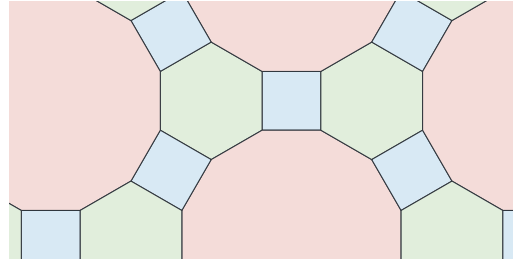
3.4.6.4



Candidate class: up-type

(d) Truncated trihexagonal tiling

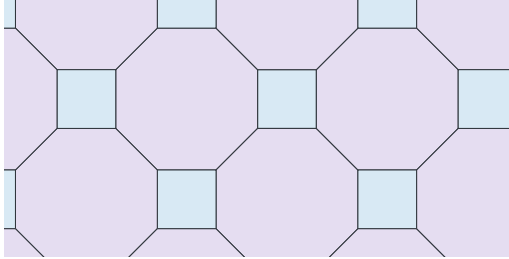
4.6.12



Candidate class: up-type

(e) Truncated square tiling

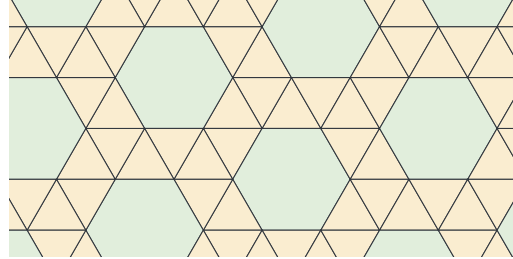
4.8.8



Candidate class: up-type

(f) Snub hexagonal tiling

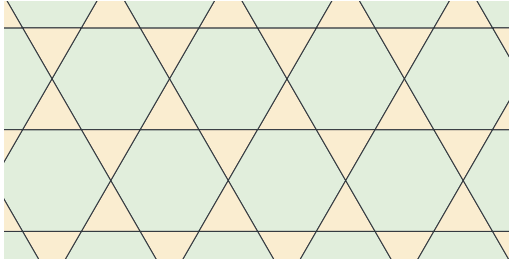
3.3.3.3.6



Candidate class: down-type

(g) Trihexagonal tiling

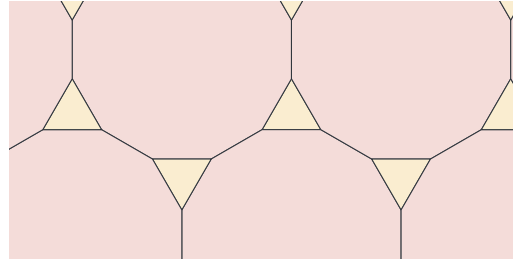
3.6.3.6



Candidate class: down-type

(h) Truncated hexagonal tiling

3.12.12



Candidate class: down-type

Figure 1: The eight Archimedean tilings and the conjectural $2 + 3 + 3$ fermion classes. Colors distinguish polygon sizes only. Each panel is a local window in the cover of a physical toroidal sheet.

The length ℓ_T is a physical microscopic scale. Equation (2) already defines a four-dimensional field theory with finitely many internal degrees of freedom. Let A_T be the adjacency matrix and $L_T = q_T I - A_T$ the combinatorial Laplacian. If $L_T \eta_{T,j} = \lambda_{T,j} \eta_{T,j}$, the expansion $\phi_{T,v} = \sum_j \varphi_{T,j} \eta_{T,j}(v)$ gives

$$M_{T,j}^2 = m_{0,T}^2 + \frac{\kappa_T}{\ell_T^2} \lambda_{T,j}, \quad j = 0, \dots, N_T - 1. \quad (3)$$

Each Γ_T is connected, so the constant zero mode $\lambda_{T,0} = 0$ is unique. The adjacency spectrum lies in $[-q_T, q_T]$, whence $0 \leq \lambda_{T,j} \leq 2q_T$. There are only N_T internal modes on a physical sheet, and

$$0 \leq M_{T,j}^2 - m_{0,T}^2 \leq \frac{2\kappa_T q_T}{\ell_T^2}. \quad (4)$$

The tower is finite. An infinite Fourier tower is, at most, a low-energy language for this microscopic model; it is not its exact spectrum.

3.2 Domain of validity of the continuum KK formula

The Bloch operator $L_T(k)$ of the periodic covering graph has a simple acoustic zero eigenvalue at $k = 0$; all other bands are separated from zero there. For $|k|\ell_T \ll 1$, analytic perturbation theory and the symmetry $\lambda(k) = \lambda(-k)$ yield

$$\lambda_{T,0}(k) = \ell_T^2 k_a C_T^{ab} k_b + O(\ell_T^4 |k|^4), \quad (5)$$

where the positive-definite effective tensor C_T is determined by the graph connectivity and embedding. On a large periodic quotient, k takes reciprocal-torus values. Equations (3) and (5) imply for the low band

$$M_{T,\mathbf{n}}^2 = m_{0,T}^2 + 4\pi^2 \kappa_T h_{\text{eff},T}^{ab} n_a n_b + O(\kappa_T \ell_T^2 / R_T^4), \quad (6)$$

where R_T is a period scale and $h_{\text{eff},T}^{ab}$ expresses C_T in unit-period coordinates. Only when this effective inverse metric matches a proposed smooth-torus h_T^{ab} does Eq. (6) become the conventional $4\pi^2 h_T^{ab} n_a n_b$ formula. Matching is a low-energy test, not a definition. The higher bands and bound (4) retain the physical finite-tiling information.

The effective tensor can itself be computed by a finite cell problem. Choose a translation cell with r_T vertices and representatives E_T^{cell} of the unoriented edges, each counted once. Let d_e be the displacement of edge e in the planar cover and $\alpha(e), \beta(e)$ its endpoint types within the cell. Then

$$k_a C_T^{ab} k_b = \frac{1}{r_T \ell_T^2} \min_{\theta \in \mathbb{R}^{r_T}} \sum_{e \in E_T^{\text{cell}}} [k \cdot d_e + \theta_{\beta(e)} - \theta_{\alpha(e)}]^2. \quad (7)$$

This follows by expanding the Rayleigh quotient of the acoustic Bloch eigenvalue to second order around $k = 0$; θ allows periodic phase relaxation among sites in a cell. Thus whether the conventional KK relation emerges is a finite-dimensional calculation for each physical tiling.

4 An exact tiling-dependent spectral sum rule

Let $m_3(T)$ be the number of triangular faces incident on each vertex. Define dimensionless mass increments $X_{T,j} = \ell_T^2 (M_{T,j}^2 - m_{0,T}^2) / \kappa_T = \lambda_{T,j}$, and normalized moments $\mu_r(T) = N_T^{-1} \sum_j X_{T,j}^r$.

Theorem 4.1 (First three moments of the finite internal spectrum). *For a sufficiently large finite periodic toroidal quotient of any of the eight Archimedean tilings,*

$$\mu_1 = q_T, \quad \mu_2 = q_T^2 + q_T, \quad \mu_3 = q_T^3 + 3q_T^2 - 2m_3(T). \quad (8)$$

These identities do not depend on the torus area, the number of vertices in the periodic quotient, or a Wilson coupling. They are exact mass-spectrum sum rules of the physical graph model.

Proof. Since $L_T = q_T I - A_T$, while $\text{Tr } A_T = 0$ and $\text{Tr } A_T^2 = N_T q_T$ for a simple q_T -regular graph, the first two formulae follow immediately. Every three-step closed walk runs around a triangle, and every triangle generates six oriented closed walks. A three-edge cycle in the regular-polygon tiling is precisely a triangular face. If F_3 is the number of triangular faces, vertex–face incidence counting gives $3F_3 = m_3(T)N_T$, so $\text{Tr } A_T^3 = 6F_3 = 2m_3(T)N_T$. Expanding $\text{Tr}(q_T I - A_T)^3/N_T$ proves the last formula. $\square$

Table 1: Low normalized dimensionless spectral moments of the eight physical sheets. Agreement through the third moment does not imply isospectrality.

Vertex configuration	q_T	$m_3(T)$	μ_1	μ_2	μ_3
3.3.3.4.4	5	3	5	30	194
3.3.4.3.4	5	3	5	30	194
3.4.6.4	4	1	4	20	110
4.6.12	3	0	3	12	54
4.8.8	3	0	3	12	54
3.3.3.3.6	5	4	5	30	192
3.6.3.6	4	2	4	20	108
3.12.12	3	1	3	12	52

The 4.6.12 and 3.12.12 graphs are both trivalent. Their first two moments agree, but their third moments differ by 2. This difference records the presence of triangular faces in the physical compactification graph. A continuum free-torus formula specified only by area and shape does not contain that combinatorial datum. The sum rule is an in-principle distinguishing spectral prediction of the model, not a claim that new visible particles have been observed. The first three moments of 4.8.8 and 4.6.12 coincide; higher moments or full-band calculations are required to distinguish them.

Equivalently, define the normalized internal heat trace $K_T(t) = N_T^{-1} \text{Tr } e^{-tL_T}$. Theorem 4.1 implies

$$K_T(t) = 1 - q_T t + \frac{q_T^2 + q_T}{2} t^2 - \frac{q_T^3 + 3q_T^2 - 2m_3(T)}{6} t^3 + O(t^4), \quad (9)$$

so that $K_{3.12.12}(t) - K_{4.6.12}(t) = t^3/3 + O(t^4)$. At finite microscopic scale this difference cannot be absorbed into the generic area–shape formula of a free continuous torus.

5 Distance pairs, group labels, and fermion sheets

5.1 Geometric input from face-boundary paths

Choose two nonadjacent edges of one regular n -gon. Let $a, b \geq 1$ count the intermediate edges in the two boundary directions, so $a + b = n - 2$, and take $a \leq b$. The number of unordered pairs of this type per face is $c_n(a, b) = n$ if $a < b$ and $n/2$ if $a = b$. If $m_n(T)$ is the number of

n -gons at each vertex, the density per vertex on a toroidal quotient is

$$\rho_{n,a,b}(T) = \frac{m_n(T)}{n} c_n(a,b) = \begin{cases} m_n(T), & a < b, \\ m_n(T)/2, & a = b. \end{cases} \quad (10)$$

This again uses the vertex–face identity $nF_n = m_n(T)N_T$. Equation (10) is purely geometric and does not depend on a dynamical choice.

We separately postulate the candidate group-label rule

$$H(1) = \text{U}(1), \quad H(k) = \text{SU}(k) \ (k \geq 2), \quad \Phi(a,b) = H(a) \times H(b). \quad (11)$$

It gives eleven candidate labels across the eight tilings. A path length does not generate a Lie group by itself; an actual gauge field requires a representation and an action. Table 3 uses the presence of an $\text{SU}(3)$ factor in a candidate label and the presence of a square face to formulate the $2 + 3 + 3$ classification.

Table 2: The eleven candidate labels of the path–group rule. Triangles have no admissible distance pair.

n	(a,b)	$\Phi(a,b)$	n	(a,b)	$\Phi(a,b)$
4	(1,1)	$\text{U}(1)^2$	6	(1,3)	$\text{U}(1) \times \text{SU}(3)$
6	(2,2)	$\text{SU}(2)^2$	8	(1,5)	$\text{U}(1) \times \text{SU}(5)$
8	(2,4)	$\text{SU}(2) \times \text{SU}(4)$	8	(3,3)	$\text{SU}(3)^2$
12	(1,9)	$\text{U}(1) \times \text{SU}(9)$	12	(2,8)	$\text{SU}(2) \times \text{SU}(8)$
12	(3,7)	$\text{SU}(3) \times \text{SU}(7)$	12	(4,6)	$\text{SU}(4) \times \text{SU}(6)$
12	(5,5)	$\text{SU}(5)^2$			

Table 3: Candidate fermion classes on the eight physical sheets. A color witness is only a geometric admissibility condition under Eq. (11).

Vertex configuration	Square	$\text{SU}(3)$ label	Color-witness pair	Candidate class
3.3.3.4.4	yes	no	none	lepton sheet 1
3.3.4.3.4	yes	no	none	lepton sheet 2
3.4.6.4	yes	yes	$6 : (1,3)$	up sheet 1
4.6.12	yes	yes	$6 : (1,3)$	up sheet 2
4.8.8	yes	yes	$8 : (3,3)$	up sheet 3
3.3.3.3.6	no	yes	$6 : (1,3)$	down sheet 1
3.6.3.6	no	yes	$6 : (1,3)$	down sheet 2
3.12.12	no	yes	$12 : (3,7)$	down sheet 3

5.2 A precise rule for eight sheets in one four-dimensional theory

Place a four-dimensional Grassmann field $\psi_{T,e}(x)$ on each edge of each physical graph. Let $\mathcal{H}_T^{(1)} = \mathbb{C}^{E_T} \otimes W_T$ be its edge-state space, where W_T is an internal representation space. The full internal fermion space is

$$\mathcal{H}_{\text{int}} = \bigoplus_{T \in \mathfrak{T}} \mathcal{H}_T^{(1)}. \quad (12)$$

A physical model must specify a gauge-covariant Hermitian hopping operator or an appropriate lattice Dirac operator on each $\mathcal{H}_T^{(1)}$, together with any allowed intersheet terms. As the Γ_T differ, their mode numbers and spectra generally differ; Eq. (12) alone does not imply particle degeneracies.

Hypothesis 5.1 (Eight-sheet fermion identification). *The edge-state operator on each sheet has one dynamically selected low-energy state η_T . The two lepton sheets of Table 3 supply two lepton classes. The three up sheets and three down sheets span respective three-dimensional subspaces $\mathcal{H}_u = \text{span}\{\eta_T : T \in \mathfrak{T}_u\}$ and $\mathcal{H}_d = \text{span}\{\eta_T : T \in \mathfrak{T}_d\}$. A common four-dimensional $\text{SU}(3)_c$, if present, acts in its fundamental representation on these intersheet subspaces, and the corresponding low-energy operator commutes with that action on each subspace.*

Hypothesis 5.1 specifies how eight tilings could represent one fermion generation simultaneously, without treating eight mutually different patterns as one continuous torus. It also exposes a real obstruction: if the three effective masses in an up or down sector differ, the low-energy operator does not commute with the intersheet color action. Three sheets by themselves are not a color triplet. Color acts on the intersheet low-energy subspace; the $\text{SU}(3)$ factors in individual distance-pair labels serve only as possible geometric witnesses for its admissibility.

Electric charges, weak doublets, and four-dimensional chirality likewise do not follow from Table 3. A Standard Model generation still requires the correct $\text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y$ representations, anomaly cancellation, and a suitable lattice Dirac construction[4, 5]. In a smooth-torus long-wavelength limit with $\text{U}(1)$ magnetic flux, the two-dimensional index can be $\text{ind } \mathcal{D} = qm$; a finite physical graph must reproduce an appropriate counterpart through its actual Dirac operator. Counting three sheets does not replace an index calculation[6].

Each tiling also has a Laves combinatorial dual Γ_T^* : faces become dual vertices, edges become dual edges, and $(\Gamma_T^*)^* \cong \Gamma_T$. This makes the dual graph a possible carrier of antiparticle degrees of freedom, but combinatorial duality is not by itself charge conjugation. Such an interpretation would require an analogous edge-state space $\mathcal{H}_{T^*}^{(1)}$ on the dual graph and an antilinear map $J_T : \mathcal{H}_T^{(1)} \rightarrow \mathcal{H}_{T^*}^{(1)}$ taking a gauge representation to its complex conjugate and intertwining the low-energy dynamics and chirality structure. Dual graphs are therefore not counted as additional particle sheets among the eight.

6 Gauge loops, a gravitational zero mode, and symmetries

Assign an independent link variable $U_{T,uv} \in G_T$ to every directed edge of each graph, with $U_{T,vu} = U_{T,uv}^{-1}$ and a preselected compact group G_T . Let $\rho_T : G_T \rightarrow \text{U}(d_T)$ be a unitary representation. Local transformations $U_{T,uv} \mapsto g_{T,u} U_{T,uv} g_{T,v}^{-1}$ leave traces of face holonomies $U_{T,\partial f}$ invariant. A physical internal action may include

$$S_W = \sum_T \beta_T \sum_{f \in F_T} w_{T,f} \left(1 - \frac{1}{d_T} \text{Re Tr } \rho_T(U_{T,\partial f}) \right), \quad (13)$$

where $\beta_T, w_{T,f} > 0$. Equation (13) is a Wilson energy on a finite physical graph. It does not need $\ell_T \rightarrow 0$ to be defined[7]. Coupling the sheets to one common low-energy color field requires additional embeddings of the common group into the G_T , or projections between common and sheet state spaces. Without those maps Eq. (13) describes independent sheets.

An effective term for a common four-dimensional gravitational zero mode can be written using physical vertex-area weights $\omega_{T,v} > 0$ with $\sum_v \omega_{T,v} = V_T$. If the sheets share one slowly varying four-dimensional metric $g_{\mu\nu}(x)$, take

$$S_{g,0} = \frac{M_6^4}{2} \sum_{T,v} \omega_{T,v} \int_{M^{3,1}} d^4x \sqrt{-g} R_4(g), \quad M_4^2 = M_6^4 \sum_T V_T. \quad (14)$$

This is an effective-action postulate for the diagonal gravitational zero mode, not a theorem derived from an unspecified microscopic graph gravity. It parallels $M_4^2 = M_6^4 V_2$ for a single

smooth torus, but the volumes must add when eight sheets are present. Massive gravitational modes and intersheet couplings require a further consistent dynamical definition.

A fixed physical tiling generally breaks the continuous translation symmetry of a smooth torus. Consequently, the two $U(1)$ vectors obtained from continuous torus isometries in a smooth metric reduction *cannot be assumed to remain exactly massless gauge fields* in this model. Their effective masses must be calculated in the tiling background, or a protective residual mechanism established. The non-Abelian groups of Eq. (11) are conceptually distinct from these metric modes. This distinction follows directly from treating the tiling as physical.

7 Tests and conclusions

The minimal model makes statements of different strengths. Theorem 4.1, the finite spectral bound (4), and the distance-pair density (10) are exact within the specified graph action and labeling rule. The infinite Fourier KK spectrum, the four-dimensional gravitational volume law, and the magnetic-flux index enter respectively as a long-wavelength matching condition, an effective gravitational postulate, and a condition to be realized by the fermion sector. The association of eight sheets with one generation is Hypothesis 5.1; it is not proved by the spectral-moment theorem.

There are clear ways for this hypothesis to fail. A common color symmetry may not commute with the dynamics of the three up or three down low-energy states; a concrete lattice Dirac spectrum may not supply the required chirality; or continuous translation modes and internal masses may be incompatible with the proposed low-energy reduction. The next calculation is to choose comparable ℓ_T , V_T and intersheet interactions, compute the full Bloch bands, fermion modes, and intersheet mass matrices, and couple the relevant subspaces to one common four-dimensional gauge field. That program will test whether the $2 + 3 + 3$ structure is more than a combinatorial classification and can become a physical multiplet.

Reproducibility. Scripts for generating the eight regular-polygon patches, checking their interior vertex configurations, and checking the numerical values of Eq. (8) accompany this work. The drawings are finite windows; the theorem concerns sufficiently large periodic toroidal quotients. No experimental data are used.

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