

A 2-Adic Correction to Sun's Proof of the Irrationality of Catalan's Constant

Chun Wing Lai

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Abstract

Z.-W. Sun has recently announced a determinant-based proof that Catalan's constant

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.915965594177219015$$

is irrational. We verify Sun's framework and identify one genuine gap in the final quadratic ledger: the factorial

$$F_B = \prod_{r=0}^{2B-1} r!$$

contributes a 2-adic term

$$v_2(F_B) \log 2 = (2 \log 2 + o(1))B^2,$$

which is not absorbed by the odd-prime positive-part bridge. When this term is included, the final B^2 coefficient becomes positive, and the contradiction $0 < |N_B| < 1$ cannot be derived. We propose a minimal correction: replace the monomial reference columns i^r in Sun's determinant by binomial coefficient columns $\binom{i}{r}$. The finite-difference transform then diagonalizes the reference block with unit pivots, and F_B disappears entirely from the determinant formula. All odd-prime constants—the odd small-prime constant c_{odd} , the middle-prime integral Λ_{mid} , and the large-prime improvement $83/2400$ —are unchanged. The final coefficient becomes

$$\frac{39}{200} + c_{\text{odd}} - \Lambda_{\text{mid}} - \frac{83}{2400} \leq -0.0096624265252323507,$$

which is strictly negative. The contradiction $0 < |N_B| < 1$ for a nonzero integer N_B is then restored. We also note a harmless indexing typo in Sun's rank theorem and confirm that with the corrected indexing the proof is valid. A supplementary SageMath verification suite covers every quantitative claim of the paper.

1 Introduction

Catalan's constant is

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \approx 0.915965594177219015.$$

Whether G is irrational is a long-standing open problem. In 2024, Calegari–Dimitrov–Tang [1] proved the irrationality of $L(2, \chi_{-3})$, but their adelic method did not apply to G . In a recent manuscript [2], Z.-W. Sun announced a determinant-based proof of $G \notin \mathbb{Q}$.

Sun's proof has five main stages:

1. A weighted finite-difference residual matrix $\mathcal{R}$ is shown to have full column rank by a polynomial-defect argument.
2. Three binomial Newton columns complete a nonsingular minor of $\mathcal{R}$ to a square determinant; this produces a fixed rational scalar $\hat{q}_B$ and its exact minimal integerizer $H_B^{\min}$.
3. The residual determinant is expanded by Cauchy–Binet into a Pascal factor and a Cauchy factor.
4. A prime-power positive-part bridge converts local p -adic lower bounds into a legal upper bound for the height of $\hat{q}_B$.
5. A complete asymptotic ledger at $S/B = 1/20$ yields a strict negative margin and forces $0 < |N_B| < 1$ for a nonzero integer N_B .

We verify Sun’s framework and identify one genuine gap: the factorial

$$F_B = \prod_{r=0}^{2B-1} r!$$

appears in the scalar $\hat{q}_B$ and contributes a 2-adic term

$$v_2(F_B) \log 2 = (2 \log 2 + o(1))B^2$$

that is not absorbed by the odd-prime ledger. When included, the final B^2 coefficient becomes

$$C + 2 \log 2 \approx -0.00966 + 1.386 > 0,$$

so the final margin is no longer negative and the contradiction fails.

Our correction is to replace the monomial reference columns i^r by binomial coefficient columns $\binom{i}{r}$. Because the finite-difference transform diagonalizes binomial columns with unit pivots, the factorial F_B drops out of the determinant formula. All odd-prime constants remain unchanged, and the final B^2 coefficient recovers its negative value.

Acknowledgements

The determinant framework, the rank theorem, the Pascal–Cauchy factorization, and the overall proof architecture are due to Z.-W. Sun. The author is grateful to D. Wachs for discussions on indexing in an earlier verification note, and to Sun for making his manuscript available.

2 The framework of Sun’s proof

Throughout, B is a large positive integer, $S = \lfloor B/20 \rfloor$, and $d = 2$. Set

$$N = 2B + S + 3.$$

We write

$$T_m := \sum_{r=0}^{\infty} \frac{(-1)^r}{(2m + 2r + 1)^2}, \quad u_m := \frac{T_m}{2m + 1}.$$

The tail T_m satisfies the telescoping identity

$$T_m + T_{m+1} = \frac{1}{(2m+1)^2},$$

which follows directly by summing alternating terms. Iterating this identity gives the tail recurrence

$$T_{i+j} = (-1)^j T_i + \sum_{0 \leq k < j} \frac{(-1)^{j-1-k}}{(2(i+k)+1)^2}. \quad (2.1)$$

Let

$$\Pi_i = \prod_{h=0}^{B-1} (2(i+h)+1)^2, \quad i \geq 0.$$

The residual matrix is the $(S+3) \times S$ matrix

$$\mathcal{R} = (\mathcal{R}_{a,j})_{0 \leq a \leq S+2, 1 \leq j \leq S}, \quad \mathcal{R}_{a,j} := \sum_{i=0}^{a+2B} (-1)^i \binom{a+2B}{i} \Pi_i u_{i+j}. \quad (2.2)$$

3 The rank theorem

Theorem 3.1 (Rank theorem [2, Theorem 2.1]). *For every $B > S > 0$, $\text{rank } \mathcal{R} = S$.*

Sketch following Sun. Assume for contradiction that $\lambda = (\lambda_1, \dots, \lambda_S)^T \neq 0$ lies in the right kernel of $\mathcal{R}$. Define

$$f_i = \Pi_i \sum_{j=1}^S \lambda_j u_{i+j}.$$

Then for $n = 2B, \dots, 2B + S + 2$,

$$\sum_{i=0}^n (-1)^i \binom{n}{i} f_i = 0.$$

Using the tail recurrence, write $f_i = T_i D_\lambda(i) + P_\lambda(i)$, where D_λ and P_λ are polynomials with

$$\deg D_\lambda \leq 2B - 1, \quad \deg P_\lambda \leq 2B - 3.$$

Since the n -th finite difference of a polynomial of degree $< n$ vanishes, the P_λ part drops out, leaving

$$\sum_{i=0}^n (-1)^i \binom{n}{i} T_i D_\lambda(i) = 0, \quad n = 2B, \dots, 2B + S + 2.$$

By Newton interpolation there is a polynomial A of degree $\leq 2B - 1$ with $A(i) = T_i D_\lambda(i)$ for $i = 0, \dots, 2B + S + 2$. Set

$$K(X) = (2X+3)^2 (A(X)D_\lambda(X+1) + A(X+1)D_\lambda(X)) - D_\lambda(X)D_\lambda(X+1).$$

The tail recurrence gives $K(i) = 0$ for $i = 0, \dots, 2B + S + 1$. Combining these $2B + S + 2$ integer zeros with the roots of the factor

$$G_0(X) = \prod_{j=1}^S (2X+2j+3) \prod_{S < j < B} (2X+2j+3)^2,$$

and one further zero at $X = -3/2$, gives $4B + 1$ zeros of K . Since $\deg K \leq 4B$, this forces $K \equiv 0$. Then the rational function $R = A/D_\lambda$ satisfies

$$R(X) + R(X + 1) = \frac{1}{(2X + 3)^2},$$

which is impossible: translating $z = X + 3/2$ gives $S_0(z) + S_0(z + 1) = 1/(4z^2)$, and a pole argument forces the pole of S_0 with maximal real part to be $z = 0$ and the pole with minimal real part to be $z = 1$, a contradiction. Hence $\lambda = 0$, and $\text{rank } \mathcal{R} = S$. $\square$

Remark 3.2. The proof uses $A(i) = T_{i+1}D_\lambda(i)$ implicitly, since the factor $(2X + 3)^2$ in K matches the telescoping shift. The equation $A(i) = T_iD_\lambda(i)$ in [2, §2] is a harmless typo.

Corollary 3.3. *There exists $A \subset \{0, \dots, S + 2\}$ with $|A| = S$ and $\det \mathcal{R}[A, J] \neq 0$ where $J = \{1, \dots, S\}$.*

4 Sun's scalar and the missing 2-adic term

Fix A as above. Define the $N \times N$ matrix $\tilde{\mathcal{A}}_B$ whose columns are

1. i^r for $0 \leq r < 2B$;
2. $\Pi_i u_{i+j}$ for $1 \leq j \leq S$;
3. $\binom{i}{2B+c_t}$ for $t = 1, 2, 3$, where $A^c = \{c_1, c_2, c_3\}$.

Divide the reference and auxiliary entries in row i by Π_i to get $\mathcal{A}_B$. Then

$$\det \tilde{\mathcal{A}}_B = \left(\prod_{i=0}^{N-1} \Pi_i \right) \det \mathcal{A}_B,$$

and, as Sun shows [2, Prop. 3.1],

$$\det \tilde{\mathcal{A}}_B = \pm F_B \det \mathcal{R}[A, J], \quad F_B := \prod_{r=0}^{2B-1} r!.$$

Hence

$$\hat{q}_B := \det \mathcal{A}_B = \pm \frac{F_B \det \mathcal{R}[A, J]}{\prod_{i=0}^{N-1} \Pi_i}. \quad (4.1)$$

Suppose $G = a/q \in \mathbb{Q}$ with $\gcd(a, q) = 1$. Define $H_B^{\min}$ as the denominator of $q^S \hat{q}_B$. Then by taking p -adic valuations,

$$v_p(H_B^{\min}) = \left[v_p \left(\prod_{i=0}^{N-1} \Pi_i \right) - v_p(F_B) - v_p(q^S \det \mathcal{R}[A, J]) \right]_+, \quad (4.2)$$

and $0 \neq q^S H_B^{\min} \hat{q}_B \in \mathbb{Z}$.

4.1 The missing 2-adic term

The positive-part bridge gives, for every odd prime p ,

$$[A_{p,B} - R_{p,B}]_+ \leq A_{p,B} - \sum_{\nu \geq 1} m_{p^\nu, B}^A,$$

where

$$A_{p,B} = v_p \left(\prod_{i=0}^{N-1} \Pi_i \right) - v_p(F_B), \quad R_{p,B} = v_p(q^S \det \mathcal{R}[A, J]).$$

Summing over odd primes,

$$\log H_B^{\min} \leq \sum_{\substack{p \text{ odd} \\ \nu \geq 1}} (a_{p^\nu, B} - m_{p^\nu, B}^A) \log p. \quad (4.3)$$

For the prime 2, Sun's Lemma 5.4 gives $[A_{2,B} - R_{2,B}]_+ = 0$.

Combining (4.1) with (4.3) yields

$$\begin{aligned} \log H_B^{\min} + \log |\hat{q}_B| &\leq \log |\det \mathcal{R}[A, J]| - \log \prod_{i=0}^{N-1} \Pi_i \\ &\quad + \sum_{\substack{p \text{ odd} \\ \nu \geq 1}} (a_{p^\nu, B} - m_{p^\nu, B}^A) \log p + \log F_B. \end{aligned} \quad (4.4)$$

Now decompose

$$\log F_B = \sum_{\substack{p \text{ odd} \\ \nu \geq 1}} \Phi_{p^\nu}(2B) \log p + v_2(F_B) \log 2. \quad (4.5)$$

Since the odd-prime part cancels the $-\Phi_{p^\nu}(2B)$ inside $a_{p^\nu, B}$, we obtain

$$\begin{aligned} \log H_B^{\min} + \log |\hat{q}_B| &\leq \log |\det \mathcal{R}[A, J]| - \log \prod_{i=0}^{N-1} \Pi_i \\ &\quad + \sum_{\substack{p \text{ odd} \\ \nu \geq 1}} (2 \sum_{i=0}^{N-1} N_{B, p^\nu}(i) - m_{p^\nu, B}^A) \log p + v_2(F_B) \log 2. \end{aligned} \quad (4.6)$$

The last term is the missing one. Since

$$v_2(F_B) = \sum_{r=0}^{2B-1} v_2(r!) = 2B^2 + O(B \log B),$$

we have $v_2(F_B) \log 2 = (2 \log 2 + o(1)) B^2$. Sun's final ledger (equation (9.3) in [2]) contains all terms except this one; adding it gives a positive leading coefficient and the contradiction fails.

5 The correction: binomial reference columns

5.1 The corrected matrix

Define the $N \times N$ matrix $\tilde{\mathcal{A}}'_B$ whose columns are

1. $\binom{i}{r}$ for $0 \leq r < 2B$;
2. $\Pi_i u_{i+j}$ for $1 \leq j \leq S$;
3. $\binom{i}{2B+c_t}$ for $t = 1, 2, 3$.

The only change from Sun's matrix is the reference block: monomial columns i^r are replaced by binomial columns $\binom{i}{r}$.

5.2 Action of the finite-difference transform

Define the lower-triangular operator

$$(Tf)_n = \sum_{i=0}^n (-1)^i \binom{n}{i} f_i, \quad 0 \leq n < N.$$

Its determinant is ± 1 . Using the identity

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \binom{i}{r} = (-1)^n \delta_{n,r}, \quad (5.1)$$

each reference column becomes a signed unit vector with pivot $(-1)^r$ at row r , and each auxiliary column becomes a signed unit vector at row $2B + c_t$. The target columns pass to the residual matrix $\mathcal{R}$:

$$(T(\Pi u_{+j}))_{2B+a} = \mathcal{R}_{a,j}.$$

After applying T , the matrix $T\tilde{\mathcal{A}}'_B$ has the block structure

$$\begin{pmatrix} \Lambda & X \\ 0 & \mathcal{R} \\ 0 & U \end{pmatrix},$$

where Λ is diagonal with entries $(-1)^r$, X is a $2B \times S$ matrix, $\mathcal{R}$ is the residual matrix, and U is the $(S+3) \times 3$ matrix whose only nonzero entries are $(-1)^{2B+c_t}$ at rows $2B + c_t$.

5.3 The corrected determinant formula

Expanding along the three auxiliary columns, and using $\det T = \pm 1$ and $\det \Lambda = \pm 1$, we obtain

$$\det \tilde{\mathcal{A}}'_B = \pm \det \mathcal{R}[A, J].$$

If $\mathcal{A}'_B$ is obtained from $\tilde{\mathcal{A}}'_B$ by dividing the reference and auxiliary entries in row i by Π_i , then

$$\tilde{\mathcal{A}}'_B = \text{diag}(\Pi_0, \dots, \Pi_{N-1}) \mathcal{A}'_B,$$

and hence

$$\hat{q}'_B := \det \mathcal{A}'_B = \pm \frac{\det \mathcal{R}[A, J]}{\prod_{i=0}^{N-1} \Pi_i}. \quad (5.2)$$

No factorial F_B appears.

5.4 Relation to Sun's scalar

The change of basis from binomial columns $\binom{i}{r}$ to monomial columns i^r is given by the Stirling numbers of the second kind:

$$i^r = \sum_{k=0}^r S(r, k) k! \binom{i}{k}.$$

The change-of-basis matrix $[S(r, k) k!]_{0 \leq r, k < 2B}$ is lower triangular with diagonal entries $k!$, so its determinant is $\prod_{k=0}^{2B-1} k! = F_B$. Therefore

$$\hat{q}_B = F_B \hat{q}'_B. \quad (5.3)$$

This identifies F_B exactly as the determinant of the change of basis from binomial to monomial reference columns.

5.5 The p -adic ledger without F_B

For $G = a/q \in \mathbb{Q}$ with $\gcd(a, q) = 1$, define $H_B'^{\min}$ as the denominator of $q^S \hat{q}'_B$. From (5.2), for every prime p ,

$$v_p(H_B'^{\min}) = \left[v_p \left(\prod_{i=0}^{N-1} \Pi_i \right) - v_p(\det \mathcal{R}[A, J]) - S v_p(q) \right]_+. \quad (5.4)$$

Compared to Sun's formula (4.2), the term $-v_p(F_B)$ is absent. In particular, the large 2-adic contribution $v_2(F_B) \sim 2B^2$ no longer appears.

6 Recomputation of the constants

We now verify that all odd-prime constants in Sun's ledger are unchanged by the substitution of binomial reference columns.

6.1 The odd small-prime constant

For odd prime powers $Q \leq S$, the local layer is

$$\lambda_Q^A(I) = C_Q^A + 2 \sum_{r \bmod Q} \binom{n_{Q,r}(I)}{2} + \sum_{i \in I} (2N_{B,Q}(i) - N_{S,Q}(i) - 2\mathbf{1}_{Q \leq 2i+1} - F_{N,Q}(i)).$$

The minimum $m_{Q,B}^A$ depends only on the target and auxiliary data, which are the same in both constructions, so $m_{Q,B}^A$ is unchanged. The denominator layer in the amended construction is

$$A'_{Q,B} = 2 \sum_{i=0}^{N-1} N_{B,Q}(i),$$

whereas Sun's layer is

$$a_{Q,B} = 2 \sum_{i=0}^{N-1} N_{B,Q}(i) - \Phi_Q(2B).$$

The difference is $\Phi_Q(2B)$, which is the odd part of $\log F_B$. Since the odd part of $\log F_B$ already cancelled against the $-\Phi_Q(2B)$ term inside Sun's $a_{Q,B}$ in the original ledger, the amended construction loses nothing:

$$c'_{\text{odd}} = c_{\text{odd}}.$$

Numerically,

$$c_{\text{odd}} \leq 0.006276744728100982604597600317605548505. \quad (6.1)$$

6.2 The middle-prime integral

For $S < p < B$, only the first p -adic layer occurs. The change in the denominator layer is $A'_{p,B} = a_{p,B} + \Phi_p(2B)$ and the change in the archimedean factor is $-\log F_B$. The middle-prime part of $-\log F_B$ cancels the change $\Phi_p(2B)$, so the middle-prime contribution is unchanged:

$$\Lambda'_{\text{mid}} = \Lambda_{\text{mid}} = 0.1763558379286482956996630430101674632453 \dots \quad (6.2)$$

In fact, the exact value is the rational number

$$\Lambda_{\text{mid}} = \frac{33042423784278900654752890582560690565493664595111}{187362234062518051579626183549876762148305272280000}.$$

6.3 The large-prime refinement

For $p > B$, only the first p -adic layer occurs and the same cancellation applies. Hence

$$\Delta'_{>B} = \Delta_{>B} = \frac{83}{2400} = 0.034583333 \dots \quad (6.3)$$

6.4 The raw quadratic coefficient

Sun's raw quadratic coefficient is

$$4\rho - 2\rho^2 = 4 \cdot \frac{1}{20} - 2 \cdot \frac{1}{400} = \frac{39}{200} = 0.195, \quad (6.4)$$

with $\rho = S/B = 1/20$. This is unchanged.

7 The final estimate

Combining the four contributions,

$$\log H_B'^{\min} + \log |\hat{q}'_B| \leq \left(\frac{39}{200} + c_{\text{odd}} - \Lambda_{\text{mid}} - \frac{83}{2400} \right) B^2 + o(B^2). \quad (7.1)$$

Substituting the bounds (6.1)–(6.3),

$$\frac{39}{200} + c_{\text{odd}} - \Lambda_{\text{mid}} - \frac{83}{2400} \leq -0.0096624265252323507. \quad (7.2)$$

Thus there is $\delta_0 > 0.00966242652523235$ with

$$\log H_B'^{\min} + \log |\hat{q}'_B| \leq -\delta_0 B^2 + o(B^2). \quad (7.3)$$

8 Final contradiction

Assume for contradiction that $G = a/q \in \mathbb{Q}$ with $a, q \in \mathbb{Z}^+$ and $\gcd(a, q) = 1$. Define

$$N_B := q^S H_B'^{\min} \hat{q}'_B.$$

By (5.4), $N_B \in \mathbb{Z}$, and it is nonzero because $\hat{q}'_B \neq 0$. Hence $N_B \in \mathbb{Z} \setminus \{0\}$. Taking logarithms,

$$\log |N_B| = S \log q + \log H_B'^{\min} + \log |\hat{q}'_B|.$$

By (7.3),

$$\log |N_B| \leq S \log q - \delta_0 B^2 + o(B^2).$$

Since $S = \lfloor B/20 \rfloor = B/20 + O(1)$, we have $S \log q = O(B) = o(B^2)$, so

$$\log |N_B| \leq -\delta_0 B^2 + o(B^2) \rightarrow -\infty.$$

Thus for all sufficiently large B ,

$$0 < |N_B| < 1.$$

But N_B is a nonzero integer, so $|N_B| \geq 1$, a contradiction. Therefore $G \notin \mathbb{Q}$, and Catalan's constant is irrational.

9 Verification suite

We have implemented a SageMath verification suite covering every quantitative claim in this paper. The suite is organized into eleven independent modules:

Module	Content	Method
M1	Missing 2-adic term $v_2(F_B) \log 2$	Legendre formula
M2	Elimination of F_B via binomial columns	Symbolic rank over $\mathbb{Q}(G)$
M3	Constants $c_{\text{odd}}, \Lambda_{\text{mid}}, 83/2400$	Exact rationals
M4	Pascal–Cauchy factorization	Exact determinants
M5	Positive-part bridge, 2-adic lemma	Exact counts
M6	Structural c_{odd}	Lemma 6.6, antiderivative
M7	238-cell certificate for c_{odd}	Exact rational integration
M8	Middle-prime integral Λ_{mid}	Water-filling, numerical
M9	Archimedean bound $-4B^2 \log B$	Richardson extrapolation
M10	Mertens and Chebyshev sums	Sieve up to 5×10^6
M11	Rank theorem for $B \leq 30$	Modular arithmetic mod $2^{61} - 1$

All modules pass. The script writes a log file that is suitable for inclusion as an appendix.

<https://github.com/laichunwing/Irrationality of Catalan Constant>

A Notation index

G	Catalan’s constant $\sum_{k \geq 0} (-1)^k / (2k+1)^2$
T_m	tail $\sum_{r \geq 0} (-1)^r / (2m+2r+1)^2$
u_m	weighted tail $T_m / (2m+1)$
B	main integer parameter
S	$\lfloor B/20 \rfloor$
N	$2B+S+3$
Π_i	$\prod_{h=0}^{B-1} (2(i+h)+1)^2$
$\mathcal{R}_{a,j}$	residual matrix entries
A	selected row set, $ A =S$
F_B	$\prod_{r=0}^{2B-1} r!$ (eliminated in amended construction)
$\hat{q}_B$	Sun’s scalar, contains F_B
$\hat{q}'_B$	amended scalar, no F_B
$H_B^{\min}$	denominator of $q^S \hat{q}_B$
$H_B'^{\min}$	denominator of $q^S \hat{q}'_B$
N_B	$q^S H_B'^{\min} \hat{q}'_B$
$\Phi_Q(n)$	$\sum_{r=0}^{n-1} \lfloor r/Q \rfloor$
$N_{B,Q}(i)$	$\#\{0 \leq h \leq B-1 : Q \mid 2i+2h+1\}$
$\lambda_Q^A(I)$	local layer cost
$m_{Q,B}^A$	minimum of $\lambda_Q^A(I)$
c_{odd}	odd small-prime constant
Λ_{mid}	middle-prime integral
$\Delta_{>B}$	large-prime improvement 83/2400

B Explicit constants

Constant	Value
39/200	0.195
c_{odd}	$\leq 0.006276744728100982604597600317605548505$
Λ_{mid}	$0.1763558379286482956996630430101674632453 \dots$
83/2400	$0.0345833 \dots$
δ_0	> 0.00966242652523235
$v_2(F_B) \log 2$	$(2 \log 2 + o(1))B^2$ (removed by the amendment)

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