

On the Conceptual Limitations of the Theoretical Substantiation of the Derivatives of Inverse Trigonometric and Hyperbolic Functions

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Abstract

This paper addresses the conceptual limitations in classical mathematical analysis regarding the differentiation of inverse trigonometric and hyperbolic functions. The traditional approach, which relies heavily on the inverse function theorem and complex virtual transformations, is reconsidered. The author introduces a novel theoretical framework based on a system of real companion functions and defines the concept of a **"transition coefficient"**. This approach establishes direct, invariant relations between trigonometric, hyperbolic, and algebraic function systems within the real domain. The practical application of the proposed apparatus is demonstrated through specific integration examples, showcasing a significant simplification of mathematical operations and analytical structures.

Keywords: New mathematical analysis, transition coefficient, companion functions, inverse trigonometric functions, hyperbolic functions, integration methods.

1 Introduction and Theoretical Framework

In classical analysis, the derivatives of inverse trigonometric and hyperbolic functions are linked to the theorem on the derivative of an inverse function, while the specific determination of such derivatives relies on abstract substitutions and transformations. What classical mathematical analysis defines as the derivatives of inverse trigonometric and hyperbolic functions is, in reality, a quotient of the derivatives of functions of different natures and the reduction of this quotient to a single system of functions. In the new mathematical analysis, the concept of a **"transition coefficient"** is introduced to explicitly define this quotient.

First, let us construct the theoretical framework — a system of real companion functions:

$$\cos bx = \frac{1}{\operatorname{ch} b\phi} = \frac{1}{by + 1}, \quad \sin bx = \operatorname{th} b\phi = \frac{\sqrt{by(by + 2)}}{by + 1},$$

$$\operatorname{tg} bx = \operatorname{sh} b\phi = \sqrt{by(by + 2)}, \quad by = \frac{1}{\cos bx} - 1 = \operatorname{ch} b\phi - 1.$$

Where $b\phi = \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{bx}{2} \right)$, and $bx = 2 \operatorname{arctg}(e^{b\phi}) - \frac{\pi}{2}$.

All interdependencies arise from a right-angled triangle where one of the legs equals 1, the hypotenuse is $\frac{1}{\cos bx} = \operatorname{ch} b\phi = (1 + by)$, and the other leg equals $\operatorname{tg} bx = \operatorname{sh} b\phi = \sqrt{by(by + 2)}$.

2 Transition Coefficients Notation

The transition coefficients for a real system of functions are the quotients of the derivatives of the companion functions and their reduction to a single system of functions. Since mathematical analysis lacks notation for algebraic functions, the following notation is proposed:

1. **Algebraic trigonometric sine:** $\text{siA}(by) = \frac{\sqrt{by(by+2)}}{by+1}$
2. **Algebraic trigonometric cosine:** $\text{coA}(by) = \frac{1}{by+1}$
3. **Algebraic trigonometric tangent:** $\text{tA}(by) = \sqrt{by(by+2)}$
4. **Algebraic hyperbolic tangent:** $\text{t}\Gamma(by) = \frac{\sqrt{by(by+2)}}{by+1}$
5. **Algebraic hyperbolic cosine:** $\text{c}\Gamma(by) = by+1$
6. **Algebraic hyperbolic sine:** $\text{s}\Gamma(by) = \sqrt{by(by+2)}$

This automatically requires a modification in the notation for finding transition coefficients, specifying which functions interact (i.e., which one transforms into another). Since the domain in real function systems is precisely defined, let:

- x — represent the trigonometric system,
- ϕ — represent the hyperbolic system,
- y — represent the algebraic system.

Therefore, the transition coefficient is denoted, for instance, as $k_{x\phi}$ — the transition coefficient from the trigonometric to the hyperbolic system of functions.

We find the transition coefficients for the real function system:

1. $k_{x\phi} = [\arcsin bx(\text{th } b\phi)]' = [\arccos bx(\frac{1}{\text{ch } b\phi})]' = [\text{arctg } bx(\text{sh } b\phi)]' = \frac{1}{\text{ch } b\phi}$.
This is how these differentiation operations are now denoted. As can be seen, the transition coefficient does not depend on which specific companion functions interact.
2. $k_{\phi x} = [\text{Arth } b\phi(\sin bx)]' = \frac{1}{\cos bx}$
3. $k_{xy} = [\text{arctg } bx(\sqrt{by(by+2)})]' = \frac{1}{(by+1) \cdot \sqrt{by(by+2)}}$
4. $k_{\phi y} = [\text{Arch } b\phi(by+1)]' = \frac{1}{\sqrt{by(by+2)}}$
5. $k_{yx} = [\text{arccoA } by(\cos bx)]' = \frac{\sin bx}{(\cos bx)^2}$
6. $k_{y\phi} = [\text{Arc}\Gamma by(\text{ch } b\phi)]' = \text{sh } b\phi$
7. $k_{xx} = [\arcsin bx(\sin bx)]' = 1$
8. $k_{yy} = [\text{arccoA } by(\frac{1}{by+1})]' = 1$
9. $k_{\phi\phi} = [\text{Arch } b\phi(\text{ch } b\phi)]' = 1$

In standard integration or differentiation, the coefficients $k_{xx} = k_{\phi\phi} = k_{yy} = 1$ remain hidden. The new notation for the integral is $\int f(x) \cdot k_{jx} \cdot dx$, and for the derivative it is $[f(x)]'/k_{jx}$, where j represents x, ϕ, y .

The relationship between the transition coefficients is as follows:

$$k_{x\phi} \cdot k_{\phi y} \cdot k_{yx} = 1$$

Now, using the transition coefficients and the concept of companion functions, we can express any integral in trigonometric, hyperbolic, or algebraic form.

3 Practical Application and Examples

Example 1 (Hyperbolic Form):

$$\begin{aligned} \int \frac{1 + (\sin x)^2}{(\cos x)^3} \cdot dx &= \int \frac{1 + (\operatorname{th} \phi)^2}{\left(\frac{1}{\operatorname{ch} \phi}\right)^3} \cdot k_{x\phi} \cdot d\phi = \int \frac{1 + (\operatorname{th} \phi)^2}{\left(\frac{1}{\operatorname{ch} \phi}\right)^3} \cdot \left(\frac{1}{\operatorname{ch} \phi}\right) \cdot d\phi = \\ &= \int ((\operatorname{sh} \phi)^2 + (\operatorname{ch} \phi)^2) \cdot d\phi = \int \operatorname{ch} 2\phi \cdot d\phi. \end{aligned}$$

Example 2 (Algebraic Form): The same integral in algebraic form:

$$\begin{aligned} \int \frac{1 + (\sin x)^2}{(\cos x)^3} \cdot dx &= \int \frac{1 + \left(\frac{\sqrt{y(y+2)}}{y+1}\right)^2}{\left(\frac{1}{y+1}\right)^3} \cdot k_{xy} \cdot dy = \int \frac{2(y+1)^2 - 1}{\sqrt{y(y+2)}} \cdot dy = \\ &= \int (2(y+1)^2 - 1) \cdot k_{\phi y} \cdot dy = \int (2y^* + 1) \cdot k_{\phi y}^* \cdot dy^* = \int \frac{2y^* + 1}{\sqrt{2y^*(2y^* + 2)}} \cdot dy^* = \\ &= \frac{1}{2} \sqrt{2y^*(2y^* + 2)} = \frac{1}{2} \cdot \operatorname{sh} 2\phi. \end{aligned}$$

Since $2(y+1)^2 - 1 = \operatorname{ch} 2\phi$ ([1], p. 117), and $k_{\phi y} = \frac{1}{\sqrt{y(y+2)}}$, when the metric changes to $\operatorname{ch} 2\phi = (2y^* + 1)$, the transition coefficient automatically transforms into $k_{\phi y}^* = \frac{1}{\sqrt{2y^*(2y^* + 2)}}$.

4 Conclusion

The new extended theoretical substantiation of finding the derivatives of inverse trigonometric and hyperbolic functions as transition coefficients between function systems significantly simplifies the mathematical apparatus and the methods of integration.

References

- [1] Boltyan, V.M. (2004). *Theoretical Foundations of the New Mathematical Analysis*. Dnipropetrovsk: Porohy. — 117 p.