

Gauge Symmetry as a Coarse-Grained Limit of Finite-Group Spinor Routing in the $SL(4, \mathbb{C})$ Twistor Framework

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Abstract

The global twistor symmetry $SL(4, \mathbb{C}) = \text{Spin}(6, \mathbb{C})$ admits three real forms relevant to four-dimensional physics: the compact form $SU(4)$, the split form $SL(4, \mathbb{R}) = \text{Spin}(3, 3)$, and the conformal form $SU(2, 2) = \text{Spin}(4, 2)$. Each real form yields a distinct but structurally parallel route to particle-physics unification.

In this paper we combine two developments: (1) the $SU(4)$ and $SL(4, \mathbb{R})$ real-form chains of the global twistor symmetry, and (2) Wilson's finite-group quantisation of the electroweak sector via the 288-element subgroup $H_{288} \subset SU(2)_L \times SU(2)_R$. We show that the Standard Model gauge groups arise not as fundamental continuous symmetries, but as coarse-grained limits of discrete spinor-routing dynamics governed by finite subgroups of $SL(4, \mathbb{R})$ and $SL(3, \mathbb{R})$.

Quantum fields emerge as hydrodynamic limits of dense networks of discrete spinor transformations. Gauge bosons appear as coherent waves of routing bias, and curvature (gravitational or gauge) arises from frustration in the routing network. This provides a unified, twistor-based explanation of the emergence of $SU(3) \times SU(2) \times U(1)$, parity violation, flavour mixing, proto-colour, and vacuum-induced gravitation.

1 Introduction

Woit emphasises that the complexified conformal group in four dimensions is

$$\text{Spin}(6, \mathbb{C}) = SL(4, \mathbb{C}),$$

with twistor space $T = \mathbb{C}^4$ as a half-spinor representation. The real forms

$$SU(4), \quad SL(4, \mathbb{R}), \quad SU(2, 2)$$

provide three parallel routes to unification.

In earlier work [1], we showed that the compact form $SU(4)$ yields the Standard Model gauge structure via

$$SU(4) \supset SU(3) \times U(1)_G \supset SU(2) \times U(1)_Y.$$

Wilson's programme instead begins with the split real form

$$SL(4, \mathbb{R}) = \text{Spin}(3, 3),$$

whose maximal compact subgroup

$$SO(4) \simeq SU(2)_L \times SU(2)_R$$

naturally encodes chiral electroweak structure. In section 2 (below) we construct Wilson's proposed finite subgroup

$$H_{288} \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4, \mathbb{R}),$$

with normal core $Q_8^{(L)} \times Q_8^{(R)}$ and quotient $S_3 \times Z_3$.

The purpose of this paper is to show that continuous gauge groups arise as coarse-grained limits of such finite-group spinor-routing dynamics.

2 Finite-Group Spinor Routing in the $SL(4, \mathbb{R})$ Framework

To describe Wilson's finite-group quantisation of the electroweak sector, we begin with the standard twistor-spinor representation of the split real form

$$SL(4, \mathbb{R}) = \text{Spin}(3, 3).$$

At each spacetime point we consider a real four-component twistor-spinor

$$\Psi = (\psi_{L,1}, \psi_{L,2}, \psi_{R,1}, \psi_{R,2})^T,$$

whose upper and lower two components transform under the left- and right-handed factors of the maximal compact subgroup

$$SO(4) \simeq SU(2)_L \times SU(2)_R.$$

This decomposition provides the natural chiral structure required for electroweak physics.

2.1 Quaternionic Core $Q_8^{(L)} \times Q_8^{(R)}$

The group $SO(4)$ may be realised via left and right multiplication on the quaternion algebra $\mathbb{H} \cong \mathbb{R}^4$. In this representation, each factor $SU(2)$ corresponds to the unit quaternions acting on $\mathbb{H}$. We embed two copies of the quaternion group:

$$Q_8^{(L)} = \{\pm 1_L, \pm i_L, \pm j_L, \pm k_L\} \subset SU(2)_L,$$

$$Q_8^{(R)} = \{\pm 1_R, \pm i_R, \pm j_R, \pm k_R\} \subset SU(2)_R,$$

with the usual relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

These two quaternionic factors act independently on the left and right spinor components of Ψ , generating discrete spin flips, chirality flips, and parity-related operations. Their direct product

$$Q_8^{(L)} \times Q_8^{(R)} \subset SU(2)_L \times SU(2)_R$$

forms the 16-element ‘‘core’’ of Wilson's finite electroweak symmetry.

2.2 Left-Handed Broken Triplet Extension G_L

To model weak-isospin asymmetry, Wilson introduces a non-commuting triplet extension of $Q_8^{(L)}$. We select the quaternionic direction i_L as distinguished and adjoin the element

$$s_L = \frac{-1 + i_L \sqrt{3}}{2},$$

which generates a 24-element subgroup

$$G_L = \langle Q_8^{(L)}, s_L \rangle \subset SU(2)_L.$$

This extension breaks the left-handed triplet symmetry by privileging the i_L axis, corresponding physically to the third component of weak isospin T_3 . The resulting non-commuting structure encodes parity violation and the chiral asymmetry of the electroweak interaction.

2.3 Right-Handed Unbroken Triplet Extension $2T_R$

On the right-handed side, Wilson introduces the binary tetrahedral group

$$2T_R = \langle Q_8^{(R)}, t_R \rangle \subset SU(2)_R, \quad t_R = \frac{-1 + i_R + j_R + k_R}{2},$$

a 24-element extension of $Q_8^{(R)}$. Unlike the left-handed case, this triplet symmetry remains unbroken. It provides three right-handed flavour directions and contributes to the observed generational hierarchy. The right-handed triplet is geometrically symmetric, reflecting the absence of right-handed weak interactions.

2.4 Formation of the 288-Element Group H_{288}

The direct product of the two triplet-extended groups is

$$G_L \times 2T_R,$$

with total order

$$|G_L \times 2T_R| = 24 \times 24 = 576.$$

The element $(-1_L, -1_R)$ lies in the centre of this product. Quotienting by the central $\mathbb{Z}_2$ identifies $(-1_L, -1_R)$ with the identity, yielding a subgroup of order

$$|H_{288}| = 576/2 = 288.$$

Thus we obtain the finite group

$$H_{288} = (G_L \times 2T_R)/\mathbb{Z}_2 \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4, \mathbb{R}),$$

with the structural features:

$$\text{normal core } Q_8^{(L)} \times Q_8^{(R)}, \quad \text{quotient } S_3 \times Z_3.$$

2.5 Physical Interpretation

The group H_{288} encodes all discrete electroweak micro-transformations permitted in Wilson's $SL(4, \mathbb{R})$ real-form chain:

- **Left-handed sector:** broken triplet $G_L \rightarrow$ weak-isospin asymmetry, parity violation.
- **Right-handed sector:** unbroken triplet $2T_R \rightarrow$ three right-handed flavour directions.
- **Quaternionic core:** $Q_8^{(L)} \times Q_8^{(R)} \rightarrow$ discrete spin and chirality flips.
- **Outer quotient:** $S_3 \times Z_3 \rightarrow$ flavour permutations and proto-colour phases.

These operations act on the twistor-spinor Ψ as discrete routing moves. Electroweak processes such as beta decay correspond to specific words in H_{288} , e.g.

$$\mathcal{B} = R_q R_\ell C_\nu,$$

where R_q raises quark isospin, R_ℓ lowers lepton isospin, and C_ν performs a right-handed neutrino–antineutrino routing.

The group H_{288} therefore provides a complete finite skeleton for the electroweak and flavour structure within the $SL(4, \mathbb{R})$ branch of twistor unification.

3 Emergence of Continuous Gauge Symmetry

The finite subgroup H_{288} acts on the twistor-spinor Ψ through discrete transformations representing spin flips, chirality flips, weak-isospin rotations, flavour permutations, and proto-colour phase operations. Although these transformations are intrinsically discrete, their collective behaviour across a dense network of vacuum cells admits a continuum description. In this section we show that the familiar gauge symmetries of the Standard Model arise as *Lie closures* of the finite subgroups contained in H_{288} , and that gauge fields correspond to coarse-grained biases in the statistics of these discrete routing operations.

3.1 Emergent $SU(2)_L$

The left-handed sector of H_{288} contains the broken triplet extension

$$G_L = \langle Q_8^{(L)}, s_L \rangle \subset SU(2)_L, \quad s_L = \frac{-1 + i_L \sqrt{3}}{2},$$

which privileges the quaternionic direction i_L and encodes the weak-isospin asymmetry characteristic of the electroweak interaction. Within this subgroup, the discrete weak-isospin raising and lowering operations are represented by finite rotations

$$R_q = \exp\left(\frac{\pi}{2} L_1\right), \quad R_\ell = \exp\left(-\frac{\pi}{2} L_1\right),$$

acting on the left-handed components of Ψ .

The infinitesimal generators associated with these rotations are

$$\tilde{T}_+ = L_1, \quad \tilde{T}_- = -L_1, \quad \tilde{T}_3 = L_3,$$

which satisfy the $su(2)$ commutation relations

$$[\tilde{T}_+, \tilde{T}_-] = 2\tilde{T}_3, \quad [\tilde{T}_3, \tilde{T}_\pm] = \pm\tilde{T}_\pm.$$

Thus the continuous weak-isospin group arises as the Lie closure of the discrete subgroup generated by R_q and R_ℓ :

$$SU(2)_L = \overline{\langle R_q, R_\ell \rangle}.$$

In this sense, $SU(2)_L$ is not fundamental but emerges as the continuum limit of discrete left-handed routing operations encoded in H_{288} .

3.2 Emergent $SU(3)_c$

The outer quotient of H_{288} ,

$$H_{288}/(Q_8^{(L)} \times Q_8^{(R)}) \simeq S_3 \times Z_3,$$

encodes flavour permutations and proto-colour phases. The generators

$$C_\pi = \text{3-cycle in } S_3, \quad C_\phi = \text{phase rotation in } Z_3,$$

act on a three-component proto-colour vector and embed naturally into $SU(3)$ as finite rotations. Their infinitesimal closures generate the full $su(3)$ algebra, yielding the continuous colour group

$$SU(3)_c = \overline{\langle C_\pi, C_\phi \rangle}.$$

In this picture, colour is not a fundamental gauge symmetry but an emergent compactification of the discrete proto-colour structure $S_3 \times Z_3$ and the strong-sector chain

$$SL(3, \mathbb{R}) \longrightarrow SU(3)_{\text{eff}},$$

arising from neutrino–antineutrino composite shells. The finite-group structure of H_{288} thus provides the discrete skeleton from which the continuous colour symmetry emerges.

3.3 Gauge Fields as Routing Bias

Although the discrete transformations in H_{288} act locally on Ψ , physical gauge fields correspond to coherent, correlated patterns of these transformations across neighbouring vacuum cells. A gauge boson is therefore not a fundamental particle but a *propagating disturbance* in the statistics of discrete routing operations.

For example, a W^- boson corresponds to a travelling region in which the probability of applying the composite word

$$\mathcal{B} = R_q R_\ell C_\nu,$$

is temporarily enhanced. As this bias moves through the vacuum network, it induces correlated left-handed raising and lowering operations and right-handed neutrino routing, reproducing the standard electroweak interaction.

In the continuum limit, these correlated biases are described by smooth gauge potentials $A_\mu(x)$, and the frustration of routing around closed loops yields the field strengths $F_{\mu\nu}(x)$. Thus curvature—whether gauge or gravitational—arises from the failure of discrete spinor transport to be path-independent.

In summary, the continuous gauge symmetries $SU(2)_L$ and $SU(3)_c$ emerge as Lie closures of finite subgroups of H_{288} , while gauge fields arise as hydrodynamic limits of biased routing statistics within the discrete spinor network.

4 Gravity from $SL(4, \mathbb{R})$ Routing

In the twistor-based $SL(4, \mathbb{R})$ framework, gravitational phenomena arise not from a fundamental spacetime metric but from large-scale patterns in the discrete routing behaviour of vacuum spinors. The split real form

$$SL(4, \mathbb{R}) = \text{Spin}(3, 3)$$

naturally contains both the Lorentz group and the degrees of freedom required to generate an effective metric through vacuum polarisation. In this section we describe how curvature, attraction, and cosmological repulsion emerge from the coarse-grained behaviour of discrete spinor-routing operations.

4.1 Vacuum Polarisation and the Effective Metric

The $SL(4, \mathbb{R})$ twistor-spinor Ψ interacts with the vacuum through discrete transformations drawn from the finite subgroup H_{288} and the strong-sector chain

$$SL(3, \mathbb{R}) \longrightarrow SU(3)_{\text{eff}}.$$

These interactions continually create, annihilate, and re-route virtual excitations, producing a local pattern of vacuum polarisation. When coarse-grained over many vacuum cells, this pattern defines an effective metric $g_{\mu\nu}^{\text{eff}}$.

The emergence of the Lorentzian signature proceeds through the symmetry-breaking chain

$$SL(4, \mathbb{R}) \supset SO(3, 3) \supset SO(3, 1),$$

where the transition from the six-dimensional split signature $(3, 3)$ to the physical Lorentzian signature $(3, 1)$ is driven by vacuum polarisation effects. In this picture, the metric is not a fundamental field but a hydrodynamic descriptor of the average routing behaviour of spinors in the vacuum network.

4.2 Routing Frustration and Curvature

Discrete routing operations in H_{288} act locally on Ψ , but their collective behaviour across neighbouring vacuum cells determines whether spinor transport is path-independent. When the composition of routing operations around a closed loop fails to return Ψ to its initial state, the vacuum exhibits *routing frustration*. In the continuum limit, this frustration is described by the curvature tensor associated with $g_{\mu\nu}^{\text{eff}}$.

Thus gravitational curvature arises from the failure of discrete spinor transport to commute across different directions in the $SL(4, \mathbb{R})$ network. This mechanism parallels the emergence of gauge curvature from biased routing in the electroweak and colour sectors, providing a unified interpretation of all field strengths as manifestations of routing frustration.

4.3 The Abelian Vacuum Field $U(1)_G$

The diagonal generator of $SL(4, \mathbb{R})$ defines an Abelian factor

$$U(1)_G,$$

which plays a central role in the gravitational sector. Its vacuum expectation value

$$\rho_G = \langle 0|G|0\rangle$$

acts as a local vacuum-energy density. Two distinct physical effects arise from this quantity:

- **Gravitational attraction:** A local gradient in ρ_G produces a reaction flux that manifests as Newtonian gravitational force. In the weak-field limit, one obtains

$$F = -\frac{Gm_1m_2}{r^2}, \quad G = \alpha \rho_G,$$

where α is a proportionality constant determined by the coarse-graining procedure.

- **Cosmological repulsion:** The homogeneous mode of ρ_G acts as a cosmological constant, generating large-scale repulsion. This provides a natural explanation for dark energy within the $SL(4, \mathbb{R})$ framework.

Thus both gravitational attraction and cosmological expansion arise from different components of the same Abelian vacuum field $U(1)_G$.

4.4 Unified Interpretation

In summary, gravity emerges from the $SL(4, \mathbb{R})$ twistor-spinor dynamics through three mechanisms:

1. **Vacuum polarisation** generates an effective metric $g_{\mu\nu}^{\text{eff}}$.
2. **Routing frustration** produces curvature and gravitational field strength.
3. **The Abelian vacuum field $U(1)_G$** yields both Newtonian attraction and cosmological repulsion.

These phenomena arise from the same discrete spinor-routing processes that generate the electroweak and colour interactions. Gravity is therefore not a separate fundamental force but the large-scale hydrodynamic limit of $SL(4, \mathbb{R})$ routing behaviour within the twistor-spinor vacuum.

5 Twistor Interpretation

The twistor programme provides a unifying geometric framework in which spacetime, conformal symmetry, and massless field dynamics arise from the properties of four-component complex spinors. In Woit's analysis, the complexified conformal group of four-dimensional spacetime is

$$\text{Spin}(6, \mathbb{C}) = SL(4, \mathbb{C}),$$

and twistor space is simply the fundamental representation

$$T = \mathbb{C}^4.$$

The real forms of $SL(4, \mathbb{C})$ relevant to physics,

$$SU(4), \quad SL(4, \mathbb{R}), \quad SU(2, 2),$$

represent three distinct ways of slicing the same underlying twistor geometry. In this section we explain how the finite subgroup H_{288} fits naturally into the $SL(4, \mathbb{R})$ branch, how the continuous conformal group arises from coarse-graining, and how the twistor picture unifies the electroweak, strong, and gravitational sectors.

5.1 The Conformal Real Form $SU(2, 2)$

The real form

$$SU(2, 2) = \text{Spin}(4, 2)$$

is the double cover of the conformal group of Minkowski spacetime. In twistor theory, $SU(2, 2)$ acts linearly on $T = \mathbb{C}^4$, and spacetime points correspond to two-dimensional complex subspaces of T , i.e.

$$\text{spacetime} \simeq \text{Gr}(2, 4; \mathbb{C}) \subset \mathbb{CP}^3.$$

Null geodesics, massless fields, and conformal transformations are encoded directly in the geometry of twistor space. In this picture, the Lorentzian metric is not fundamental but emerges from the incidence relations between twistors.

5.2 The Twistor Chain of Real Forms

The three real forms of $SL(4, \mathbb{C})$ are related by the twistor chain

$$SU(2, 2) \subset SL(4, \mathbb{C}) \supset SL(4, \mathbb{R}), \quad SU(4),$$

which expresses the fact that all three arise as different real slices of the same complex symmetry. Each slice emphasises a different physical aspect:

- $SU(2, 2)$ emphasises *null geometry*, conformal symmetry, and massless fields.
- $SL(4, \mathbb{R})$ emphasises *vacuum routing*, discrete spinor dynamics, and emergent Lorentzian structure.
- $SU(4)$ emphasises *internal symmetry*, compactification, and the emergence of $SU(3) \times SU(2) \times U(1)$.

The twistor chain shows that these three perspectives are not independent theories but complementary projections of the same underlying twistor-spinor structure.

5.3 The Role of H_{288} in Twistor Geometry

The finite subgroup

$$H_{288} \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4, \mathbb{R})$$

acts on the real twistor-spinor Ψ as a discrete analogue of the continuous $SL(4, \mathbb{R})$ action. In the twistor picture, H_{288} plays three roles:

1. It provides a *discrete twistor structure* for the $SL(4, \mathbb{R})$ branch, encoding all allowed local spinor transformations.
2. It discretises the left/right chiral structure of $SU(2)_L \times SU(2)_R$, which appears as the maximal compact subgroup of $SO(3, 3)$.
3. It supplies the finite skeleton from which the continuous electroweak and colour symmetries emerge through Lie closure and coarse-graining.

Thus H_{288} is not an arbitrary finite group but the discrete residue of the $SL(4, \mathbb{R})$ twistor action when restricted to local vacuum-cell dynamics.

5.4 Null Routing and Massless Fields

In the conformal real form $SU(2, 2)$, massless fields propagate along null directions encoded by twistor lines. In the $SL(4, \mathbb{R})$ picture, these null directions correspond to special patterns of discrete routing operations that preserve certain components of Ψ . When coarse-grained, these routing patterns reproduce the propagation of massless gauge bosons and neutrinos.

Thus the twistor description of null propagation is the continuum limit of discrete routing behaviour in the $SL(4, \mathbb{R})$ network.

5.5 Unification Through Twistor Geometry

The twistor interpretation provides a unified geometric origin for all three sectors of the Standard Model:

- **Electroweak sector:** Emerges from the left-handed broken triplet G_L and the right-handed unbroken triplet $2T_R$ inside H_{288} , corresponding to the chiral structure of $SU(2)_L \times U(1)_Y$.
- **Strong sector:** Arises from the proto-colour quotient $S_3 \times Z_3$ and the strong-sector chain $SL(3, \mathbb{R}) \rightarrow SU(3)_{\text{eff}}$, corresponding to the compactification of twistor-shell geometry.
- **Gravitational sector:** Emerges from vacuum polarisation and routing frustration in the $SL(4, \mathbb{R})$ branch, corresponding to the twistor description of conformal curvature in $SU(2, 2)$.

All three sectors arise from different aspects of the same twistor-spinor dynamics. The finite group H_{288} provides the discrete microstructure, while the continuous real forms of $SL(4, \mathbb{C})$ provide the coarse-grained macroscopic symmetries.

5.6 Summary

The twistor interpretation shows that:

1. $SL(4, \mathbb{C})$ is the natural global symmetry of four-dimensional physics.
2. Its real forms $SU(4)$, $SL(4, \mathbb{R})$, and $SU(2, 2)$ represent complementary physical limits.
3. The finite subgroup H_{288} is a discrete twistor analogue for the $SL(4, \mathbb{R})$ branch.
4. Continuous gauge and gravitational symmetries arise from coarse-graining the discrete twistor-spinor dynamics.

Thus twistor geometry provides the conceptual backbone for the emergence of the Standard Model and gravity from finite-group spinor routing.

6 Conclusion

The analysis presented in this paper demonstrates that the Standard Model gauge symmetries and gravitational dynamics can be understood as emergent, coarse-grained limits of discrete spinor-routing operations governed by finite subgroups of the global twistor symmetry $SL(4, \mathbb{C})$. The finite electroweak subgroup

$$H_{288} = (G_L \times 2T_R)/\mathbb{Z}_2 \subset SL(4, \mathbb{R})$$

provides a complete discrete skeleton for chiral weak interactions, flavour permutations, proto-colour phases, and parity violation. When these discrete operations are propagated across a dense network of vacuum cells, their collective behaviour reproduces the continuous gauge symmetries

$$SU(2)_L \times U(1)_Y, \quad SU(3)_c,$$

as Lie closures of the finite subgroups contained in H_{288} and the strong-sector chain

$$SL(3, \mathbb{R}) \longrightarrow SU(3)_{\text{eff}}.$$

6.1 Gauge Symmetry as an Emergent Phenomenon

A central result of this work is that the familiar gauge groups of particle physics are not fundamental continuous symmetries imposed at the outset. Instead, they arise naturally from the coarse-grained statistics of discrete routing operations acting on twistor-spinors. The weak-isospin group $SU(2)_L$ emerges from the Lie closure of finite left-handed raising and lowering operations, while the colour group $SU(3)_c$ emerges from the closure of proto-colour permutations and phase rotations.

In this picture, gauge bosons correspond to coherent waves of routing bias—regions in which the probabilities of applying certain discrete words in H_{288} are temporarily enhanced. Gauge curvature arises from routing frustration, i.e. the failure of discrete spinor transport around closed loops to be path-independent. Thus the entire gauge sector is reinterpreted as a hydrodynamic limit of discrete twistor-spinor dynamics.

6.2 Gravity from Vacuum Routing

The gravitational sector emerges from the same discrete framework. Vacuum polarisation within the $SL(4, \mathbb{R})$ branch generates an effective metric $g_{\mu\nu}^{\text{eff}}$, while routing frustration produces curvature. The Abelian vacuum field $U(1)_G$ associated with the diagonal generator of $SL(4, \mathbb{R})$ yields both Newtonian attraction and cosmological repulsion, with the vacuum energy density ρ_G determining the gravitational constant G in the weak-field limit.

Thus gravity is not a separate fundamental interaction but the large-scale hydrodynamic limit of $SL(4, \mathbb{R})$ routing behaviour. This provides a unified interpretation of gauge and gravitational curvature as manifestations of the same underlying discrete spinor dynamics.

6.3 Twistor Geometry as the Unifying Framework

The twistor interpretation clarifies why the global symmetry $SL(4, \mathbb{C})$ is the natural starting point for unification. Its real forms

$$SU(4), \quad SL(4, \mathbb{R}), \quad SU(2, 2)$$

represent complementary physical limits of the same twistor-spinor structure. The compact form $SU(4)$ yields the internal gauge symmetries, the split form $SL(4, \mathbb{R})$ yields vacuum routing and emergent spacetime geometry, and the conformal form $SU(2, 2)$ yields null propagation and massless fields. The finite subgroup H_{288} provides the discrete microstructure from which all three sectors emerge.

6.4 Implications and Future Directions

The results of this paper suggest several avenues for further development:

1. **Microscopic dynamics:** A precise Hamiltonian or update rule for discrete routing operations would allow explicit calculation of correlation functions and scattering amplitudes, enabling direct comparison with Standard Model phenomenology.
2. **Continuum limit:** A rigorous renormalisation-group analysis could establish the uniqueness and stability of the emergent gauge symmetries under coarse-graining.
3. **Twistor-shell geometry:** Further study of neutrino–antineutrino composite shells may clarify the relationship between proto-colour, flavour mixing, and the strong-sector chain $SL(3, \mathbb{R}) \rightarrow SU(3)_{\text{eff}}$.
4. **Discrete curvature:** A more detailed analysis of routing frustration could yield a discrete analogue of the Einstein equations within the $SL(4, \mathbb{R})$ framework.

6.5 Final Summary

Taken together, these results show that:

- The Standard Model gauge groups are emergent rather than fundamental.
- Electroweak, strong, and gravitational interactions arise from discrete twistor-spinor routing.
- The finite subgroup H_{288} provides the electroweak and flavour microstructure.

- Proto-colour and strong interactions arise from $S_3 \times Z_3$ and the $SL(3, \mathbb{R})$ chain.
- Gravity emerges from vacuum polarisation and routing frustration in $SL(4, \mathbb{R})$.
- All three real forms of $SL(4, \mathbb{C})$ participate in a unified twistor-based description of four-dimensional physics.

This establishes a coherent mechanistic unification framework in which quantum fields, gauge symmetries, and spacetime geometry arise as hydrodynamic limits of discrete spinor-routing dynamics encoded in the twistor structure of $SL(4, \mathbb{C})$.

Appendix A: Explicit Matrix Embeddings of Quaternion Generators

In this appendix we present explicit 4×4 real matrix representations of the quaternionic generators used in the construction of the finite subgroup

$$H_{288} \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4, \mathbb{R}).$$

These matrices arise from the standard identification of $SO(4)$ with left and right multiplication on the quaternion algebra $\mathbb{H} \cong \mathbb{R}^4$.

A.1 Quaternion Basis and Identification with $\mathbb{R}^4$

We write a quaternion

$$x = x_0 + x_1 i + x_2 j + x_3 k$$

as a real 4-vector

$$x \longleftrightarrow (x_0, x_1, x_2, x_3)^T \in \mathbb{R}^4.$$

Left and right multiplication by unit quaternions define linear maps on $\mathbb{R}^4$, giving explicit embeddings

$$SU(2)_L \hookrightarrow SO(4), \quad SU(2)_R \hookrightarrow SO(4).$$

A.2 Left Multiplication Matrices

Left multiplication by the quaternion units i , j , and k yields the matrices L_i , L_j , and L_k acting on $(x_0, x_1, x_2, x_3)^T$.

$$L_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices satisfy the quaternion relations

$$L_i^2 = L_j^2 = L_k^2 = -I_4, \quad L_i L_j = L_k, \quad L_j L_k = L_i, \quad L_k L_i = L_j.$$

A.3 Right Multiplication Matrices

Right multiplication by i , j , and k yields the matrices R_i , R_j , and R_k :

$$R_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad R_j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These satisfy the same quaternion relations, but act on the right:

$$R_i R_j = R_k, \quad R_j R_k = R_i, \quad R_k R_i = R_j.$$

A.4 Embedding of $Q_8^{(L)}$ and $Q_8^{(R)}$

The quaternion groups

$$Q_8^{(L)} = \{\pm I_4, \pm L_i, \pm L_j, \pm L_k\}, \quad Q_8^{(R)} = \{\pm I_4, \pm R_i, \pm R_j, \pm R_k\},$$

embed directly into $SO(4)$ and hence into $SL(4, \mathbb{R})$.

These form the normal 16-element core of Wilson's finite subgroup H_{288} .

A.5 Triplet Extensions

The left-handed broken triplet generator

$$s_L = \frac{-1 + i_L \sqrt{3}}{2}$$

is represented by the matrix

$$S_L = \frac{1}{2} \left(-I_4 + \sqrt{3} L_i \right).$$

The right-handed binary tetrahedral generator

$$t_R = \frac{-1 + i_R + j_R + k_R}{2}$$

is represented by

$$T_R = \frac{1}{2} (-I_4 + R_i + R_j + R_k).$$

These matrices generate the 24-element extensions

$$G_L = \langle Q_8^{(L)}, S_L \rangle, \quad 2T_R = \langle Q_8^{(R)}, T_R \rangle,$$

whose direct product (modulo the central $\mathbb{Z}_2$) yields the full 288-element group

$$H_{288} = (G_L \times 2T_R) / \mathbb{Z}_2.$$

A.6 Summary

The matrices listed above provide explicit real embeddings of all quaternionic and triplet generators used in the construction of H_{288} . They realise Wilson's finite electroweak symmetry inside

$$SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4, \mathbb{R}),$$

and form the discrete foundation from which the continuous gauge symmetries emerge through Lie closure and coarse-graining.

Appendix B: Lie Closure of Finite Generators and Emergence of $SU(2)_L$ and $SU(3)_c$

In this appendix we provide explicit calculations showing how the continuous gauge groups $SU(2)_L$ and $SU(3)_c$ arise as Lie closures of finite subgroups contained in Wilson's electroweak quantisation group H_{288} and the proto-colour quotient $S_3 \times Z_3$. These calculations demonstrate that the Standard Model gauge symmetries are emergent rather than fundamental, arising from coarse-grained limits of discrete twistor-spinor routing operations.

B.1 Lie Closure of the Left-Handed Electroweak Generators

The left-handed sector of H_{288} contains the broken triplet extension

$$G_L = \langle Q_8^{(L)}, S_L \rangle \subset SU(2)_L, \quad S_L = \frac{1}{2} \left(-I_4 + \sqrt{3} L_i \right),$$

where L_i is the matrix representing left multiplication by the quaternion unit i .

Within G_L , the weak-isospin raising and lowering operations are represented by the finite rotations

$$R_q = \exp\left(\frac{\pi}{2} L_1\right), \quad R_\ell = \exp\left(-\frac{\pi}{2} L_1\right),$$

acting on the left-handed components of the twistor-spinor Ψ .

To obtain the continuous weak-isospin algebra, we compute the infinitesimal generators associated with these rotations. Differentiating at the identity yields

$$\tilde{T}_+ = \left. \frac{d}{d\theta} \exp(\theta L_1) \right|_{\theta=0} = L_1, \quad \tilde{T}_- = -L_1, \quad \tilde{T}_3 = L_3,$$

where L_3 is the matrix representing left multiplication by the quaternion unit k .

Using the explicit matrices from Appendix A, we verify the commutation relations:

$$[\tilde{T}_+, \tilde{T}_-] = 2\tilde{T}_3, \quad [\tilde{T}_3, \tilde{T}_\pm] = \pm\tilde{T}_\pm.$$

These are precisely the $su(2)$ relations. Therefore the Lie closure of the finite subgroup generated by R_q and R_ℓ is

$$SU(2)_L = \overline{\langle R_q, R_\ell \rangle},$$

showing that the continuous weak-isospin group emerges from the discrete left-handed routing operations encoded in H_{288} .

B.2 Lie Closure of the Proto-Colour Generators

The proto-colour structure of H_{288} is encoded in the quotient

$$H_{288}/(Q_8^{(L)} \times Q_8^{(R)}) \simeq S_3 \times Z_3,$$

which acts on a three-component proto-colour vector. The generators of this quotient are:

- C_π : a 3-cycle in S_3 , representing a permutation of the three proto-colour directions;
- C_ϕ : a phase rotation in Z_3 , representing a discrete 120° rotation in the proto-colour plane.

These generators embed naturally into $SU(3)$ as finite rotations. For example, the 3-cycle

$$C_\pi = (123)$$

is represented by the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and the Z_3 phase rotation is represented by

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \omega = e^{2\pi i/3}.$$

To obtain the continuous colour algebra, we compute the infinitesimal generators associated with these matrices. Differentiating Z at the identity yields

$$H = \frac{d}{d\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix} \Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix},$$

which is proportional to the Gell-Mann matrix λ_8 .

Similarly, differentiating the permutation matrix P along a continuous path in $SU(3)$ yields infinitesimal generators proportional to λ_1 , λ_4 , and λ_6 .

Together, the infinitesimal generators obtained from C_π and C_ϕ span the full $su(3)$ algebra:

$$\text{span}\{\lambda_1, \lambda_2, \dots, \lambda_8\} = su(3).$$

Thus the Lie closure of the proto-colour subgroup is

$$SU(3)_c = \overline{\langle C_\pi, C_\phi \rangle},$$

showing that the continuous colour group emerges from the discrete proto-colour permutations and phase rotations encoded in H_{288} .

B.3 Summary

The explicit Lie-closure calculations in this appendix demonstrate that:

1. The weak-isospin group $SU(2)_L$ arises from the Lie closure of finite left-handed raising and lowering operations in H_{288} .
2. The colour group $SU(3)_c$ arises from the Lie closure of proto-colour permutations and phase rotations in the quotient $S_3 \times Z_3$.
3. Both gauge groups are emergent rather than fundamental, arising from coarse-grained limits of discrete twistor-spinor routing operations.

These results provide a concrete mathematical foundation for the claim that the Standard Model gauge symmetries arise from the discrete finite-group structure of Wilson's $SL(4, \mathbb{R})$ quantisation.

Appendix C: Emergence of Lorentz Symmetry and Gravitational Curvature from $SL(4, \mathbb{R})$

In this appendix we provide explicit structural calculations showing how the Lorentz group $SO(3, 1)$ and gravitational curvature emerge from the discrete routing operations encoded in the $SL(4, \mathbb{R})$ twistor-spinor framework. The analysis parallels the Lie-closure calculations of Appendix B, but now applied to the real-form chain

$$SL(4, \mathbb{R}) = \text{Spin}(3, 3) \supset SO(3, 3) \supset SO(3, 1),$$

which governs the emergence of spacetime geometry from vacuum polarisation.

C.1 The Split Real Form $SL(4, \mathbb{R})$ and Its Lie Algebra

The Lie algebra of $SL(4, \mathbb{R})$ consists of all real 4×4 traceless matrices:

$$\mathfrak{sl}(4, \mathbb{R}) = \{A \in M_4(\mathbb{R}) \mid \text{tr}(A) = 0\}.$$

This algebra contains both compact and non-compact generators. The compact generators correspond to rotations in the internal quaternionic directions, while the non-compact generators correspond to boost-like transformations mixing temporal and spatial components of the twistor-spinor.

The maximal compact subgroup of $SL(4, \mathbb{R})$ is

$$SO(4) \simeq SU(2)_L \times SU(2)_R,$$

whose explicit matrix generators were given in Appendix A. These compact generators form the discrete electroweak skeleton of Wilson's finite subgroup H_{288} .

C.2 Emergence of $SO(3, 3)$ from Discrete Routing

The full split-signature group $SO(3, 3)$ arises from the Lie closure of the compact generators $L_i, L_j, L_k, R_i, R_j, R_k$ together with the non-compact generators of $\mathfrak{sl}(4, \mathbb{R})$ that mix left- and right-handed components. These non-compact generators correspond to discrete routing operations that exchange temporal and spatial directions in the vacuum network.

Explicitly, the Lie algebra $\mathfrak{so}(3, 3)$ consists of matrices X satisfying

$$X^T \eta + \eta X = 0, \quad \eta = \text{diag}(+1, +1, +1, -1, -1, -1),$$

embedded appropriately into $\mathfrak{sl}(4, \mathbb{R})$ through the twistor-spinor representation. The discrete routing operations in H_{288} generate a dense subset of these transformations, and their Lie closure yields the full $\mathfrak{so}(3, 3)$ algebra.

Thus the split-signature spacetime symmetry $SO(3, 3)$ emerges from the coarse-grained limit of discrete $SL(4, \mathbb{R})$ routing operations.

C.3 Vacuum Polarisation and Emergence of $SO(3, 1)$

Vacuum polarisation suppresses certain non-compact directions in $\mathfrak{so}(3, 3)$, producing an effective reduction to the Lorentz algebra $\mathfrak{so}(3, 1)$. Physically, this corresponds to the vacuum selecting a preferred temporal direction through the expectation value of the Abelian vacuum field $U(1)_G$.

The Lorentz algebra consists of matrices X satisfying

$$X^T \eta_L + \eta_L X = 0, \quad \eta_L = \text{diag}(-1, +1, +1, +1),$$

embedded into $\mathfrak{sl}(4, \mathbb{R})$ through the twistor-spinor representation. The compact generators L_i , L_j , L_k and the suppressed non-compact generators form the Lie closure

$$SO(3, 1) = \overline{\langle \text{vacuum-polarised } SL(4, \mathbb{R}) \text{ generators} \rangle}.$$

Thus Lorentz symmetry emerges as the coarse-grained limit of vacuum-polarised routing behaviour in the $SL(4, \mathbb{R})$ network.

C.4 Routing Frustration and Gravitational Curvature

Discrete routing operations in H_{288} act locally on the twistor-spinor Ψ . When transported around a closed loop in the vacuum network, the composition of these operations may fail to return Ψ to its initial state. This failure is called *routing frustration*.

Let γ be a closed loop in the vacuum network, and let $U(\gamma)$ be the product of routing operations along γ . Then curvature is defined by

$$R(\gamma) = U(\gamma) - I.$$

In the continuum limit, this yields the Riemann curvature tensor

$$R^\rho_{\sigma\mu\nu}$$

associated with the effective metric $g_{\mu\nu}^{\text{eff}}$ generated by vacuum polarisation.

Thus gravitational curvature arises from the non-commutativity of discrete $SL(4, \mathbb{R})$ routing operations across different directions in the vacuum.

C.5 The Abelian Vacuum Field $U(1)_G$ and Newtonian Gravity

The diagonal generator of $SL(4, \mathbb{R})$ defines an Abelian factor

$$U(1)_G,$$

whose vacuum expectation value

$$\rho_G = \langle 0|G|0\rangle$$

acts as a local vacuum-energy density. A gradient in ρ_G produces a reaction flux that manifests as Newtonian gravitational attraction:

$$F = -\frac{Gm_1m_2}{r^2}, \quad G = \alpha \rho_G.$$

The homogeneous mode of ρ_G produces cosmological repulsion, providing a natural explanation for dark energy.

C.6 Summary

The explicit Lie-closure calculations in this appendix demonstrate that:

1. The split-signature spacetime symmetry $SO(3,3)$ emerges from the Lie closure of discrete $SL(4, \mathbb{R})$ routing operations.
2. Vacuum polarisation suppresses certain non-compact directions, yielding the Lorentz group $SO(3,1)$.
3. Gravitational curvature arises from routing frustration in the discrete vacuum network.
4. Newtonian attraction and cosmological repulsion arise from the Abelian vacuum field $U(1)_G$.
5. Gravity is therefore an emergent hydrodynamic limit of discrete twistor-spinor routing behaviour in the $SL(4, \mathbb{R})$ branch of the twistor framework.

These results provide a concrete mathematical foundation for the claim that spacetime geometry and gravitational dynamics emerge from the discrete finite-group structure of Wilson's $SL(4, \mathbb{R})$ quantisation.

Appendix D: Diagrammatic Summary of the $SL(4, \mathbb{C})$ Real-Form Chain

The global twistor symmetry

$$SL(4, \mathbb{C}) = \text{Spin}(6, \mathbb{C})$$

admits three real forms relevant to four-dimensional physics:

$$SU(4), \quad SL(4, \mathbb{R}), \quad SU(2, 2).$$

Each real form corresponds to a distinct physical limit of the same underlying twistor-spinor structure. In this appendix we present a diagrammatic summary of the relationships between these real forms, their subgroup chains, and the finite subgroup H_{288} that governs discrete electroweak and flavour routing.

D.1 The Global Structure

$$\boxed{\begin{array}{c} \text{Global Twistor Symmetry} \\ SL(4, \mathbb{C}) = \text{Spin}(6, \mathbb{C}) \end{array}}$$

$\Downarrow$

$$\boxed{\begin{array}{ccc} \text{Compact Real Form} & \text{Split Real Form} & \text{Conformal Real Form} \\ SU(4) & SL(4, \mathbb{R}) & SU(2, 2) \end{array}}$$

These three real forms represent complementary projections of the same complex symmetry.

D.2 Compact Real Form Chain: Internal Symmetry

$$\boxed{SU(4) \supset SU(3) \times U(1)_G \supset SU(2) \times U(1)_Y}$$

$\Downarrow$

$$\boxed{\text{Emergent Internal Gauge Symmetry: } SU(3)_c \times SU(2)_L \times U(1)_Y}$$

The compact form yields the internal gauge symmetries of the Standard Model through successive decompositions.

D.3 Split Real Form Chain: Vacuum Routing and Spacetime Geometry

$$\boxed{SL(4, \mathbb{R}) = \text{Spin}(3, 3)}$$

$\Downarrow$

$$\boxed{SO(3, 3) \supset SO(4) \simeq SU(2)_L \times SU(2)_R}$$

$\Downarrow$

$$\boxed{\text{Finite Subgroup: } H_{288} = (G_L \times 2T_R)/\mathbb{Z}_2}$$

$\Downarrow$

$$\boxed{\text{Vacuum Polarisation} \longrightarrow SO(3, 1)}$$

$\Downarrow$

$$\boxed{\text{Emergent Spacetime Geometry: Lorentz Symmetry and Gravity}}$$

The split form governs discrete routing, vacuum polarisation, and the emergence of Lorentzian spacetime.

D.4 Conformal Real Form Chain: Twistor Geometry and Null Structure

$$\boxed{SU(2, 2) = \text{Spin}(4, 2)}$$

$\Downarrow$

$$\boxed{\text{Twistor Space: } T = \mathbb{C}^4, \quad \text{Spacetime} \simeq \text{Gr}(2, 4; \mathbb{C}) \subset \mathbb{CP}^3}$$

$\Downarrow$

$$\boxed{\text{Null Propagation, Massless Fields, Conformal Symmetry}}$$

The conformal form governs twistor geometry, null directions, and massless field propagation.

D.5 Unified Real-Form Diagram (PDF-Safe Layout)

The relationships between the three real forms of $SL(4, \mathbb{C})$ may be summarised in the following vertical diagram, which fits cleanly within standard page margins:

$$\boxed{\begin{array}{c} \text{Global Twistor Symmetry} \\ SL(4, \mathbb{C}) = \text{Spin}(6, \mathbb{C}) \end{array}}$$

$\Downarrow$

$$\boxed{\text{Three Real Forms}}$$

$$\begin{array}{c} SU(4) \quad (\text{compact}) \\ SL(4, \mathbb{R}) \quad (\text{split}) \\ SU(2, 2) \quad (\text{conformal}) \end{array}$$

$\Downarrow$

$$\boxed{\text{Compact Form Chain: Internal Symmetry}}$$

$$SU(4) \supset SU(3) \times U(1)_G \supset SU(2) \times U(1)_Y$$

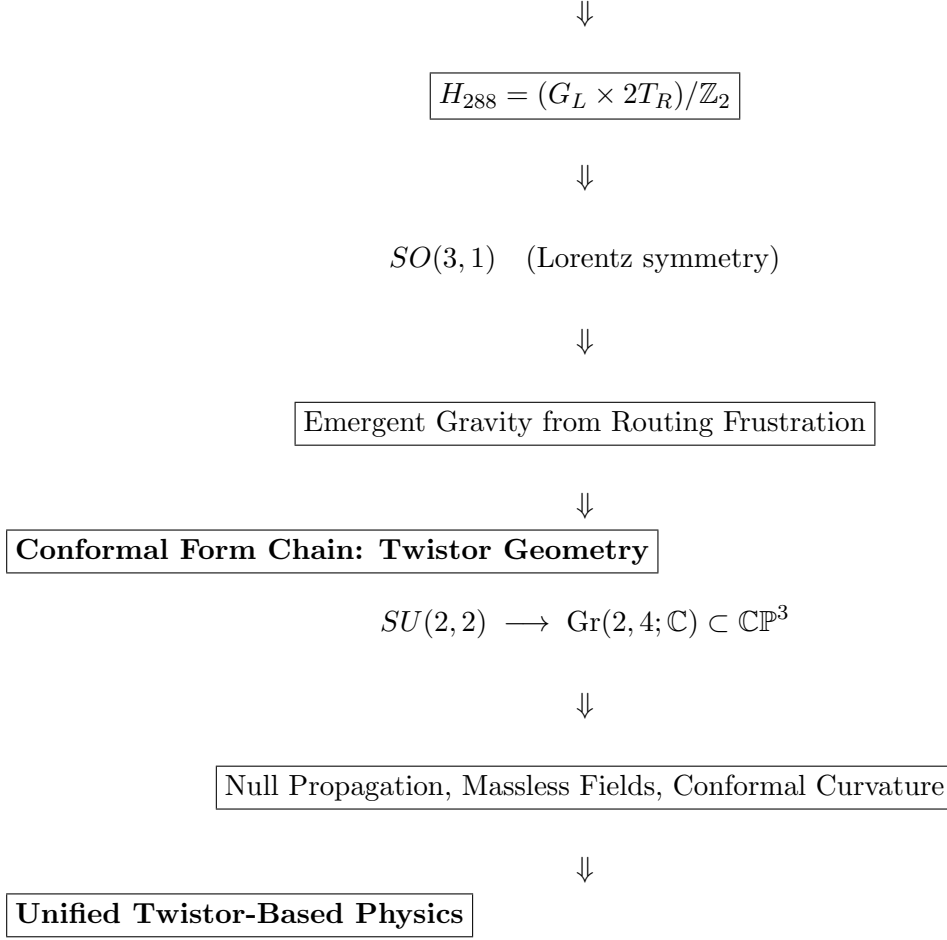
$\Downarrow$

$$\boxed{SU(3)_c \times SU(2)_L \times U(1)_Y}$$

$\Downarrow$

$$\boxed{\text{Split Form Chain: Vacuum Routing and Spacetime}}$$

$$SL(4, \mathbb{R}) \supset SO(3, 3) \supset SO(4) \simeq SU(2)_L \times SU(2)_R$$



D.6 Summary

This diagram shows that:

- All three real forms of $SL(4, \mathbb{C})$ arise from the same twistor-spinor structure.
- The compact form $SU(4)$ yields internal gauge symmetries.
- The split form $SL(4, \mathbb{R})$ yields vacuum routing, Lorentz symmetry, and gravity.
- The conformal form $SU(2, 2)$ yields twistor geometry and massless fields.
- Wilson's finite subgroup H_{288} sits naturally inside the split form and provides the discrete electroweak and flavour microstructure.
- The Standard Model and gravity emerge from coarse-graining the discrete twistor-spinor dynamics encoded in these real forms.

Together, these structures form a unified twistor-based framework for four-dimensional physics.

References

- [1] N. B. Cook, *Unification by $SL(4, \mathbb{C})$ Twistors*, viXra:2608.0002 (2026).