

Spectrally Rigid Integer Directed Graphs with a Growing Nilpotent Jordan Block Structure: Infinitude, and Members at Small Orders

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16 September 2026 (revised 17 September 2026)

Abstract

We study a family of directed graphs on $n = 2k + 5$ vertices whose adjacency matrices are $\{0, 1\}$ -matrices with zero diagonal, uniform column sum 2, and a rigid Jordan structure: the non-zero spectrum $\{2, 1, -1, -1, -1\}$ is constant in k , while a single nilpotent block $J_{2k}(0)$ grows with k . We call this the *Wen class* $\mathcal{U}_k$ and define it through five axioms D1–D5.

The main result of the paper settles the existence question. We construct an explicit infinite subfamily $\{A_m\}_{m \geq 1}$ with $\text{ord}(A_m) = 2m + 7$, obtained by iterating a chain-insertion operator T on a fixed 7×7 core C . A Schur-complement identity for T , combined with a resolvent computation that reduces to three cofactors of the single 7×7 matrix $zI - C^T$, shows that T multiplies the characteristic polynomial by z^2 ; the exact determination of $\dim \ker A_m = 1$ and $\dim \ker(A_m + I) = 2$ then upgrades this spectral statement to the full Jordan axiom D4. Consequently $\mathcal{U}_k \neq \emptyset$ for every integer $k \geq 1$: the class is infinite, and no order is empty. Every step of the proof is a finite exact computation of order at most 7, independent of m .

We prove the closed-walk counting formula $N_L = 2^L + 1 + 3(-1)^L$, the rank formula $\text{rank}(A^m) = \max(n - m, 5)$, the minimal polynomial $x^{2k}(x - 2)(x - 1)(x + 1)^2$ of degree $n - 1$, the commutant dimension $7 + 2k$, the Newton-identity recurrence of order 5, and the doubler-chain power-of-two property. A computer search at seven orders ($k = 1, \dots, 7$, i.e. $n = 7, \dots, 19$) produced 233 pairwise non-isomorphic members, distributed as 34, 50, 39, 22, 68, 13, 7; these numbers are lower bounds for the number of isomorphism classes at each order, and ten representatives of each order are listed in Appendix 18. We disprove two natural conjectures (the “minimum-one integer Perron vector” property and the “core-template” hypothesis) and organise the observed structure into a three-layer architecture separating rigid invariants, semi-invariants and flexible realisation.

Keywords: directed graph; adjacency matrix; Jordan canonical form; nilpotent block; characteristic polynomial; spectral graph theory; closed walk; Perron–Frobenius theory; Schur complement; exact symbolic computation.

Mathematics Subject Classification (2020): 05C50; 05C20; 15A18; 15A21; 11C20; 15A15.

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1 Introduction

The spectral theory of directed graphs is a central topic in algebraic graph theory [9, 1, 2, 5, 6]. For undirected graphs the spectrum of the adjacency matrix carries substantial structural information: it determines the number of closed walks, bounds the chromatic number and characterises regularity. For directed graphs the situation is richer. The adjacency matrix is generally not symmetric, its eigenvalues may be complex, and—crucially—the Jordan canonical form carries structural information that the spectrum alone does not.

A key distinction in the directed case is that between diagonalisable and non-diagonalisable adjacency matrices. Most of spectral graph theory concerns the diagonalisable case, where a complete set of eigenvectors exists. A non-diagonalisable adjacency matrix possesses Jordan blocks of size greater than one; these introduce polynomial factors in matrix powers and change the asymptotics of walk counts, return probabilities and generating functions in a way that the spectrum cannot detect. Matrices that are cospectral but not similar, and matrices that are isospectral with identical Jordan structure yet realised by non-isomorphic graphs, are therefore of intrinsic interest.

The originating example. A directed graph G on 7 vertices, found by the author in earlier unpublished work on this family, has an adjacency matrix A whose integer spectrum is $\{2, 1, -1, -1, -1, 0, 0\}$, with Jordan form

$$J_1(2) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1) \oplus J_2(0). \quad (1)$$

This graph has several remarkable properties. It admits a positive integer right Perron vector $v^T = (9, 3, 2, 6, 7, 1, 8)$; the all-ones vector is a left eigenvector (every vertex has in-degree 2); every closed directed walk of length L carries the same total weight 2^{-L} ; the non-zero spectrum $\{2, 1, -1, -1, -1\}$ is integral; and the edge set has cardinality 14, with structural links to the Fano plane, the unique $2-(7, 3, 1)$ design. The decomposition (1) already displays the phenomenon that motivates this paper: a fixed non-zero spectral core of dimension five, together with a nilpotent part that can in principle grow.

The question. The natural question is whether this graph is the first member of a larger family. Precisely:

For which integers $k \geq 1$ does there exist a directed graph on $n = 2k+5$ vertices whose adjacency matrix has non-zero spectrum $\{2, 1, -1, -1, -1\}$ with Jordan structure $J_1(2) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1)$, and whose remaining spectral part is a single nilpotent Jordan block $J_{2k}(0)$?

Equivalently: is $\mathcal{U}_k \neq \emptyset$, where $\mathcal{U}_k$ denotes the set of $\{0, 1\}$ -matrices of order $2k+5$ satisfying the five axioms D1–D5 of Definition 2.1 below?

The answer. Both the existence question and the finiteness question are answered in this paper. The union $\mathcal{U} = \bigcup_{k \geq 1} \mathcal{U}_k$ is infinite: $\mathcal{U}_k \neq \emptyset$ for every integer $k \geq 1$ (Theorem 14.21, Corollary 14.22), and an explicit member of $\mathcal{U}_k$ can be constructed in time polynomial in k (Theorem 15.1). What remains open is the classification and the counting question, not existence.

Contributions of this paper.

- (i) **Infinitude (main result).** Section 14 constructs an explicit infinite subfamily. A fixed 7×7 core C is coupled to a chain of $2m$ vertices by an operator T that inserts a new pair (b, a) into the chain. A Schur-complement identity (Theorem 14.10) expresses $\det(zI - T(A))$ through $\det(zI - A)$ and three cofactors of the single 7×7 matrix $zI - C^\top$; the resolvent of A_m reduces exactly to that same 7×7 system (Lemma 14.12), so the whole family is governed by one order-7 computation. Three cofactor identities then force the characteristic polynomial to be preserved up to the factor z^2 (Theorem 14.15), and two exact kernel computations, $\dim \ker A_m = 1$ and $\dim \ker(A_m + I) = 2$ (Propositions 14.16 and 14.17), upgrade this to the full Jordan axiom D4. Strong connectivity is preserved inductively (Proposition 14.18). The outcome is

$\mathcal{U}_k \neq \emptyset$ for every integer $k \geq 1$; in particular the Wen class is infinite (Theorem 14.21, Corollary 14.22).

Every step of the proof is a finite exact computation of order at most 7, independent of m .

- (ii) **Axiomatisation.** We define the family through five axioms D1–D5 (Section 2) and derive their immediate spectral and combinatorial consequences, including the Perron–Frobenius structure, the rank identities and the geometric multiplicities.
- (iii) **Members at seven orders.** Using column-by-column generation over $\Omega(k)$ for $k = 1, \dots, 4$ and an $n - 2$ extension strategy with D1+D2-preserving simulated annealing for $k = 5, 6, 7$, we obtain 233 pairwise non-isomorphic members, distributed as 34, 50, 39, 22, 68, 13, 7 (Sections 3 and 8). These are lower bounds for the number of isomorphism classes at each order; ten representatives of each order are listed in Appendix 18. All of them were independently re-verified by exact integer and rational arithmetic in the course of preparing this paper (Appendix 18).
- (iv) **Spectral and algebraic theory.** We prove the closed-walk counting formula $N_L = 2^L + 1 + 3(-1)^L$ (Theorem 6.4), the rank formula $\text{rank}(A^m) = \max(n - m, 5)$ (Proposition 4.4), the minimal polynomial $x^{2k}(x - 2)(x - 1)(x + 1)^2$ of degree $n - 1$ (Proposition 5.1), the commutant dimension $7 + 2k$ (Proposition 5.3), the Newton-identity cascade of order $\delta = 5$ (Section 7) and the doubler-chain power-of-two property (Theorem 10.2).
- (v) **Structure of the solution space.** We classify the observed invariants into proven and empirical (Section 9), propose a three-layer architecture separating rigid invariants, semi-invariants and flexible realisation (Section 10), and disprove two natural conjectures: the minimum-one integer Perron vector property and the core-template hypothesis (Section 11). Sections 12 and 13 give the density model and the Jordan-splitting analysis.

Organisation. Section 2 defines the family axiomatically. Section 3 presents representative members and the summary of the search. Sections 4 and 5 develop the spectral theory and matrix algebra. Section 6 proves the closed-walk counting formula. Section 7 presents the Newton identity cascade. Section 8 describes the search methodology. Sections 9–13 treat invariants, doubler chains, falsified properties, the density model and Jordan splitting. Section 14 contains the main construction and its proof. Section 15 reassesses the construction problem in the light of that result. Sections 16–18 present applications, open problems and the conclusion. Appendices contain the verification protocol, representative members, the data of the infinite family and the algebraic preliminaries.

Computational standards. All computations reported here were carried out in exact arithmetic: characteristic polynomials via Newton’s identities on integer power sums, ranks over $\mathbb{Q}$ (equivalently, over a prime field whose characteristic exceeds every relevant Hadamard minor bound), kernels by exact rational elimination, and strong connectivity by Boolean reachability. No floating-point computation enters any claimed result. Symbolic computations were performed with SymPy [14], graph-theoretic computations with NetworkX [10].

2 The Wen Class: Axioms and First Consequences

2.1 Notation and conventions

Throughout, $A \in \{0, 1\}^{n \times n}$ is the adjacency matrix of a directed graph $G(A)$ on the vertex set $V = \{0, 1, \dots, n-1\}$, with the convention

$$A[i, j] = 1 \iff \text{there is a directed edge } i \rightarrow j. \quad (2)$$

We write

$$\text{in}(j) = \{i : A[i, j] = 1\}, \quad \text{out}(i) = \{j : A[i, j] = 1\}$$

for the in-neighbourhood of j and the out-neighbourhood of i . Consequently:

- the column sum $\sum_i A[i, j] = |\text{in}(j)|$ is the in-degree of j ;
- the row sum $\sum_j A[i, j] = |\text{out}(i)|$ is the out-degree of i ;
- $(Av)_i = \sum_{j \in \text{out}(i)} v_j$, i.e. the i -th row equation of $Av = \lambda v$ is written over the out-neighbours of i ;
- $(A^\top y)_j = \sum_{i \in \text{in}(j)} y_i$, i.e. the j -th equation of $A^\top y = \lambda y$ is written over the in-neighbours of j .

Since $\dim \ker B = \dim \ker B^\top$ and $\dim \ker(B - \lambda I) = \dim \ker(B^\top - \lambda I)$ for every square B and every λ , all kernel computations in this paper are performed with $A^\top$, i.e. with in-neighbourhood equations. This bookkeeping convention is used systematically in Section 14.

We write I for the identity matrix, $\mathbf{1} = (1, \dots, 1)^\top$ for the all-ones vector, $J_s(\lambda)$ for the $s \times s$ Jordan block with eigenvalue λ , $\text{adj}(M)$ for the adjugate (classical adjoint) of M , and $p_A(z) = \det(zI - A)$ for the characteristic polynomial. Since A is a non-negative integer matrix we also write $\rho(A)$ for its spectral radius.

2.2 The five axioms

Let $n = 2k + 5$ with $k \geq 1$; equivalently $k = (n - 5)/2$.

Definition 2.1 (Wen class). A matrix $A \in \{0, 1\}^{n \times n}$ of order $n = 2k + 5$ ($k \geq 1$) belongs to the Wen class $\mathcal{U}_k$ if it satisfies the following five conditions:

	Condition
D1	Zero diagonal: $A[i, i] = 0$ for all i .
D2	Uniform in-degree: $\sum_i A[i, j] = 2$ for all j .
D3	Characteristic polynomial: $\det(\lambda I - A) = \lambda^{2k}(\lambda - 2)(\lambda - 1)(\lambda + 1)^3$.
D4	Jordan structure: $J_1(2) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1) \oplus J_{2k}(0)$.
D5	Strong connectivity: $G(A)$ is strongly connected.

The union $\mathcal{U} = \bigcup_{k \geq 1} \mathcal{U}_k$ is called the Wen class. Two members are regarded as the same solution if their directed graphs are isomorphic. We write

$$c(k) = |\mathcal{U}_k|/\sim$$

for the number of isomorphism classes at order k , and

$$\hat{c}(k) = \#\{\text{pairwise non-isomorphic members of } \mathcal{U}_k \text{ known to us}\}$$

for the number produced by the search of Section 8 together with the explicitly constructed members of Section 14. Thus $\hat{c}(k) \leq c(k)$ always, with equality only at those orders at which the search happens to be complete. Tables 1 and 7 report $\hat{c}(k)$.

Remark 2.2. Axiom D4 implies D3, since the characteristic polynomial is determined by the eigenvalues together with their algebraic multiplicities. The axioms are therefore not logically independent. We retain both because they play different practical roles: D3 is a single determinant and serves as an inexpensive pre-filter in the search (Section 8), whereas D4 requires $2k+3$ rank computations. The block sizes in D4 sum correctly: $1 + 1 + 2 + 1 + 2k = 2k + 5 = n$.

Remark 2.3. The non-zero spectrum $\{2, 1, -1, -1, -1\}$ is constant across all orders; the growth of the family resides exclusively in the nilpotent block $J_{2k}(0)$. This separation between a fixed five-dimensional spectral core and a growing nilpotent part is the defining structural feature of the Wen class. Note also that -1 has algebraic multiplicity 3 but geometric multiplicity 2: it is realised as $J_2(-1) \oplus J_1(-1)$, not as $J_3(-1)$ nor as three copies of $J_1(-1)$. Matrices with the correct characteristic polynomial but with $J_3(-1)$ occur abundantly in the search and are excluded by D4; the distinguishing rank identity is $\text{rank}(A + I) = n - 2$ (for $J_2(-1) \oplus J_1(-1)$) versus $\text{rank}(A + I) = n - 3$ (for $J_3(-1)$).

2.3 Elementary consequences

Proposition 2.4. *Let $A \in \mathcal{U}_k$ with $n = 2k + 5$. Then:*

- (a) $\mathbf{1}^\top A = 2\mathbf{1}^\top$, i.e. 2 is an eigenvalue with left eigenvector $\mathbf{1}$;
- (b) $\rho(A) = 2$, and 2 has algebraic multiplicity one with a positive right eigenvector $v > 0$ satisfying $Av = 2v$;
- (c) $\text{tr}(A) = 0$ and $\det(A) = 0$; the number of edges is $|E| = 2n$;
- (d) A is non-diagonalisable, with nilpotent index $2k$ at the eigenvalue 0;
- (e) $\text{rank}(A) = n - 1$ and $\text{rank}(A + I) = n - 2$.

Proof. (a) is a restatement of D2. For (b), D5 asserts that $G(A)$ is strongly connected, hence A is irreducible; the Perron–Frobenius theorem [12, 18, 7, 15] gives $\rho(A)$ as a simple eigenvalue with a strictly positive right eigenvector, and the Perron–Frobenius bounds $\min_j \sum_i A[i, j] \leq \rho(A) \leq \max_j \sum_i A[i, j]$ give $\rho(A) = 2$ by (a). (c): $\text{tr}(A) = 0$ is D1, $\det(A) = 0$ because 0 is an eigenvalue, and $|E| = \sum_{i,j} A[i, j] = 2n$ by D2. (d): D4 contains the blocks $J_2(-1)$ and $J_{2k}(0)$, both of size ≥ 2 since $k \geq 1$; the largest Jordan block at the eigenvalue 0 has size $2k$. (e): the geometric multiplicity of an eigenvalue equals the number of Jordan blocks attached to it; by D4 these numbers are 1 for $\lambda = 0$ and 2 for $\lambda = -1$, so $\text{rank}(A) = n - 1$ and $\text{rank}(A + I) = n - 2$. $\square$

Proposition 2.5. *Let $A \in \mathcal{U}_k$. Then every vertex has out-degree at least 1, and*

$$\sum_{i=0}^{n-1} (|\text{out}(i)| - 1) = n.$$

In particular at least one vertex has out-degree 1 whenever some vertex has out-degree ≥ 3 , and the average out-degree is exactly 2.

Proof. By D5, $G(A)$ is strongly connected, so no vertex is a sink: $|\text{out}(i)| \geq 1$ for all i . The sum of all out-degrees equals the number of edges, which is $2n$ by Proposition 2.4(c); hence $\sum_i (|\text{out}(i)| - 1) = 2n - n = n$. If some out-degree is ≥ 3 while all are ≥ 2 , the sum would exceed n after subtracting 1 from each, a contradiction; so some out-degree equals 1. $\square$

Remark 2.6. Proposition 2.5 is a counting statement and does not by itself force the existence of a vertex of out-degree 1: all out-degrees could be equal to 2. That every member does have such a vertex is Conjecture 9.4 (equivalently Observation 9.6(a)); it is supported by all 233 members found by the search but is not derived from D1–D5. It holds for the infinite family of Section 14: there, vertex 4 has out-degree 1 for every m (Remark 14.8).

Proposition 2.7. *Let A be an $n \times n$ matrix satisfying D3 with $n = 2k + 5$. Then A satisfies D4 if and only if*

$$\begin{aligned} \text{rank}(A^m) &= n - m \quad (1 \leq m \leq 2k), & \text{rank}(A^{2k+1}) &= 5, \\ \text{rank}(A + I) &= n - 2, & \text{rank}((A + I)^2) &= n - 3. \end{aligned} \quad (3)$$

Proof. Write $d_t = \dim \ker(A - \lambda I)^t$ for an eigenvalue λ . For a single Jordan block $J_s(\lambda)$ the contribution to d_t is $\min(s, t)$, and by additivity over a direct sum $d_t = \sum_\nu \min(s_\nu, t)$; hence $d_t - d_{t-1} = \#\{\nu : s_\nu \geq t\}$ and

$$\#\{\nu : s_\nu = m\} = (d_m - d_{m-1}) - (d_{m+1} - d_m) = 2d_m - d_{m-1} - d_{m+1}. \quad (4)$$

For $\lambda = 0$ we have $d_t = \dim \ker A^t = n - \text{rank}(A^t)$, so the first line of (3) says $d_m = m$ for $m \leq 2k$ and $d_{2k+1} = 2k$; formula (4) then gives exactly one block, of size $2k$. For $\lambda = -1$, $d_1 = 2$ and $d_2 = 3$ give $\#\{s_\nu = 2\} = 2 \cdot 3 - 2 - 3 = 1$ and $\#\{s_\nu = 1\} = 2 \cdot 2 - 0 - 3 = 1$, i.e. $J_2(-1) \oplus J_1(-1)$. The eigenvalues 2 and 1 have algebraic multiplicity one by D3 and are therefore automatically $J_1(2)$ and $J_1(1)$. The converse is immediate from the Jordan form. $\square$

Proposition 2.7 is the practical membership test used throughout: it replaces Jordan-form computation by $2k + 3$ exact rank computations.

Proposition 2.8. *A square matrix is similar to its transpose. Consequently A and $A^\top$ have the same characteristic polynomial, the same Jordan form and the same rank sequence, so D1, D3 and D4 are invariant under transposition. Moreover $G(A^\top)$ is the edge-reversal of $G(A)$, which is strongly connected if and only if $G(A)$ is, so D5 is invariant as well. Axiom D2 is the only one that is not invariant: under transposition it becomes the statement that all out-degrees equal 2.*

Proof. Similarity of A and $A^\top$ over an arbitrary field is classical: both have the same rational canonical form, equivalently the same invariant factors, since the j -th determinantal divisor of $zI - A$ and of $zI - A^\top$ coincide (they are the gcds of corresponding minors of two mutually transposed matrices). The remaining statements follow from the definitions; for D5, a directed path $i \rightarrow j$ in $G(A)$ is exactly a directed path $j \rightarrow i$ in $G(A^\top)$, so reachability is reversed and strong connectivity is preserved. $\square$

3 Representative Members

This section displays one representative member of the Wen class for each order $k = 1, \dots, 7$. Every matrix below was verified in exact arithmetic against D1–D5 together with the full list of derived invariants of Section 9; the right Perron vectors were computed exactly as a generator of the one-dimensional rational null space of $A - 2I$, normalised to a primitive integer vector (gcd of entries equal to 1, first entry positive). The data set produced by the search of Section 8 is described in Appendix 18; the first members of the infinite family of Section 14 are given in Appendix 18.

3.1 W_1 ($k = 1, n = 7$)

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Right Perron vector $v^\top = (9, 3, 2, 6, 7, 1, 8)$, with $\min(v) = 1$ and $\sum_i v_i = 36$. Out-degree sequence $(4, 1, 2, 2, 2, 1, 2)$; the edge count is $14 = 2 \times 7$, as required by D2. This is the graph with which the family originated (Section 1). Vertices 1 and 5 have out-degree 1; by Theorem 10.2 their targets carry twice their Perron value, giving the doubler edges $1 \rightarrow 3$ (values $3 \mapsto 6$) and $5 \rightarrow 2$ (values $1 \mapsto 2$). The graph has 4 bidirectional edge pairs and 2 directed 3-cycles, in agreement with Propositions 9.1 and 9.2.

3.2 W_2 ($k = 2, n = 9$)

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Right Perron vector $v^\top = (4, 2, 8, 4, 2, 4, 7, 1, 4)$, $\min(v) = 1$, $\sum_i v_i = 36$. Out-degree sequence $(1, 1, 4, 3, 1, 2, 3, 1, 2)$; edge count $18 = 2 \times 9$. Vertices 0, 1, 4, 7 have out-degree 1. The chain $7 \rightarrow 4 \rightarrow 3$ has length 2 with Perron values $(1, 2, 4)$. There are 4 bidirectional edge pairs and 2 directed 3-cycles.

3.3 W_3 ($k = 3, n = 11$)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Right Perron vector $v^\top = (2, 3, 1, 2, 2, 3, 2, 1, 4, 1, 3)$, $\min(v) = 1$, $\sum_i v_i = 24$. Out-degree sequence $(2, 3, 1, 2, 1, 3, 2, 1, 4, 1, 2)$; edge count $22 = 2 \times 11$. Vertices 2, 4, 7, 9 have out-degree 1. There are 4 bidirectional edge pairs and 2 directed 3-cycles.

3.4 W_4 ($k = 4, n = 13$)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Right Perron vector

$$v^\top = (121, 170, 35, 27, 108, 165, 78, 54, 18, 70, 36, 54, 216),$$

with $\gcd(v) = 1$, $\min(v) = 18$ and $\sum_i v_i = 1152$. Out-degree sequence $(3, 3, 1, 1, 1, 3, 2, 1, 1, 3, 2, 1, 4)$; edge count $26 = 2 \times 13$. This matrix is isomorphic to the member S_{13-1} listed in Appendix 18.

Remark 3.1. Unlike W_1, W_2, W_3 , the Perron vector of W_4 has $\min(v) = 18 > 1$. Since 18 does not divide 121 (the first component), the vector cannot be rescaled so as to have minimum entry 1 while remaining integral. This is the counterexample that settles the “minimum-one” conjecture (Proposition 11.2).

3.5 W_5 ($k = 5, n = 15$)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Right Perron vector

$$v^\top = (200, 280, 148, 270, 441, 558, 174, 360, 120, 296, 240, 360, 702, 180, 279),$$

with $\min(v) = 120$ and $\sum_i v_i = 4608 = 9 \cdot 2^9$. Out-degree sequence $(2, 2, 1, 2, 2, 3, 2, 2, 1, 3, 2, 2, 4, 1, 1)$; edge count $30 = 2 \times 15$.

3.6 W_6 ($k = 6, n = 17$)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Right Perron vector

$$v^T = (200, 280, 148, 180, 441, 558, 174, 360, \\ 120, 296, 240, 360, 612, 180, 279, 90, 90),$$

with $\min(v) = 90$ and $\sum_i v_i = 4608$. Out-degree sequence $(2, 2, 1, 1, 3, 4, 2, 2, 1, 4, 2, 2, 4, 1, 1, 1, 1)$; edge count $34 = 2 \times 17$.

3.7 W_7 ($k = 7, n = 19$)

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Right Perron vector

$$v^T = (4841, 1034, 2032, 404, 1414, 1616, 1016, 2104, 1616, 3232, \\ 2828, 808, 2626, 2068, 3232, 508, 1632, 2230, 1623),$$

with $\min(v) = 404$ and $\sum_i v_i = 36864 = 9 \cdot 2^{12}$. Out-degree sequence $(6, 1, 3, 1, 1, 1, 1, 3, 1, 2, 3, 1, 3, 2, 2, 1, 2, 2, 2)$; edge count $38 = 2 \times 19$. This matrix is isomorphic to the member S_{19-1} listed in Appendix 18.

3.8 Summary

The distribution of members found by the search is non-monotone: it rises from 34 at $k = 1$ to a local maximum of 50 at $k = 2$, declines to a local minimum of 22 at $k = 4$, rises to the global maximum over the searched range, 68, at $k = 5$, and declines thereafter to 13 at $k = 6$ and 7 at $k = 7$ (Table 1). The search methods used were D2-constrained generation with eigenvalue pre-screening for $k \leq 4$, and the $n - 2$ extension strategy with simulated annealing for $k \geq 5$ (Section 8).

Table 1: Known members of the Wen class by order. “Search” is the number of pairwise non-isomorphic matrices produced by the search of Section 8; “additional” counts the verified members outside that set, namely the representatives of Section 3 that are not isomorphic to any matrix produced by the search, together with the members $A_1, \dots, A_7$ of the infinite family of Section 14. The row $k = 8$ lies beyond the searched range.

k	n	Nilpotent block	Search	Additional	$\hat{c}(k) \geq$
1	7	$J_2(0)$	34	1	35
2	9	$J_4(0)$	50	2	52
3	11	$J_6(0)$	39	2	41
4	13	$J_8(0)$	22	1	23
5	15	$J_{10}(0)$	68	2	70
6	17	$J_{12}(0)$	13	2	15
7	19	$J_{14}(0)$	7	1	8
8	21	$J_{16}(0)$	–	1	1
Total			233	12	245

Remark 3.2 (Status of the counts). The numbers in Table 1 are lower bounds: $\hat{c}(k) \leq c(k) = |\mathcal{U}_k|/\sim$ in the notation of Definition 2.1. For $k \geq 5$ the search is confined to a structured subspace determined by verified lower-order solutions (Section 8) and is manifestly not exhaustive. Even for $k \leq 4$ completeness cannot be claimed: the representatives W_1, W_2, W_3 displayed above are pairwise non-isomorphic and are not isomorphic to any matrix listed in Appendix 18 (verified by VF2 isomorphism testing, which also confirmed that the 233 matrices of the full data set are pairwise non-isomorphic). We therefore record

$$\begin{array}{llll}
 c(1) \geq 35, & c(2) \geq 52, & c(3) \geq 41, & c(4) \geq 23, \\
 c(5) \geq 70, & c(6) \geq 15, & c(7) \geq 8, & c(8) \geq 1.
 \end{array}$$

Remark 3.3. The matrices W_4 and W_7 are isomorphic to members of the searched set, so at $k = 4$ and $k = 7$ the representative contributes nothing to the “additional” column; the single additional member at each of those orders is A_3 , respectively A_6 . At $k = 5$ and $k = 6$ the representatives W_5 and W_6 are not isomorphic to any searched member, so each of those orders receives two additional members. At $k = 1$ there is no family member, and the additional member is W_1 .

4 Spectral Theory and Jordan Structure

4.1 The spectrum

Proposition 4.1. *Let $A \in \mathcal{U}_k$, $n = 2k + 5$. Then the eigenvalues of A with their algebraic multiplicities are*

$$\lambda = 2 \text{ (1),} \quad \lambda = 1 \text{ (1),} \quad \lambda = -1 \text{ (3),} \quad \lambda = 0 \text{ (2k),}$$

the spectral radius is $\rho(A) = 2$, $\text{tr}(A) = 0$ and $\det(A) = 0$.

Proof. The eigenvalues and algebraic multiplicities are read off from D3 (equivalently from the diagonal of the Jordan form in D4). That $\rho(A) = 2$ is Proposition 2.4(b). $\text{tr}(A) = 0$ is the sum of the diagonal entries, all zero by D1 (equivalently, the sum of the eigenvalues with multiplicity: $2 + 1 + 3(-1) + 2k \cdot 0 = 0$). $\det(A) = 0$ because 0 is an eigenvalue. $\square$

Corollary 4.2. *For every integer $L \geq 1$,*

$$\text{tr}(A^L) = 2^L + 1 + 3(-1)^L,$$

independently of k . In particular $\text{tr}(A^L) = 2^L - 2$ for odd L and $\text{tr}(A^L) = 2^L + 4$ for even L .

Proof. The trace of A^L is the sum of the L -th powers of the eigenvalues counted with algebraic multiplicity; the eigenvalue 0 contributes nothing and the non-zero spectrum is $\{2, 1, -1, -1, -1\}$. $\square$

4.2 The Jordan form and geometric multiplicities

Proposition 4.3. *Let $A \in \mathcal{U}_k$. The geometric multiplicities are*

$$\text{gm}(2) = 1, \quad \text{gm}(1) = 1, \quad \text{gm}(-1) = 2, \quad \text{gm}(0) = 1,$$

so that $\text{rank}(A) = n - 1$ and $\text{rank}(A + I) = n - 2$. The matrix is non-diagonalisable, and the nilpotent index at 0 (the size of the largest Jordan block for the eigenvalue 0) is $2k$.

Proof. The geometric multiplicity of an eigenvalue is the number of Jordan blocks attached to it; counting blocks in D4 gives 1, 1, 2, 1 respectively. Since $\dim \ker(A - \lambda I)$ is the geometric multiplicity, $\text{rank}(A) = n - \text{gm}(0) = n - 1$ and $\text{rank}(A + I) = n - \text{gm}(-1) = n - 2$. The blocks $J_2(-1)$ and $J_{2k}(0)$ have size ≥ 2 (as $k \geq 1$), so A is not diagonalisable. $\square$

We use throughout the standard theory of the Jordan canonical form and of the rank of a matrix pencil [8, 12].

The distinction between $J_2(-1) \oplus J_1(-1)$ and $J_3(-1)$ is the single most delicate point of the whole axiom system: both are compatible with the characteristic polynomial, and only a rank computation separates them (Proposition 2.7). In the search of Section 8 the constraint $\text{rank}(A + I) = n - 2$ was found to be essential; without it, a large majority of the matrices with the correct characteristic polynomial carry $J_3(-1)$ instead (Section 13).

4.3 Rank identities

Proposition 4.4. *Let $A \in \mathcal{U}_k$ with $n = 2k + 5$. Then*

- (a) $\text{rank}(A^m) = \max(n - m, 5)$ for every $m \geq 0$;
- (b) $\text{rank}((A + I)^m) = n - 1 - m$ for $1 \leq m \leq 2$, and $\text{rank}((A + I)^m) = n - 3$ for all $m \geq 2$.

Proof. Both statements are read off from the Jordan form, since rank is invariant under similarity and additive over a direct sum.

- (a) The powers of the blocks are $J_1(2)^m$ (rank 1), $J_1(1)^m$ (rank 1), $J_2(-1)^m$ (rank 2, the block is invertible), $J_1(-1)^m$ (rank 1), and $J_{2k}(0)^m$ (rank $\max(2k - m, 0)$), since a nilpotent Jordan block of size s has $J_s(0)^m$ of rank $\max(s - m, 0)$. Hence

$$\text{rank}(A^m) = 1 + 1 + 2 + 1 + \max(2k - m, 0) = 5 + \max(2k - m, 0) = \max(n - m, 5),$$

using $n = 2k + 5$.

- (b) Shifting by +1, the blocks of $A + I$ are $J_1(3)$, $J_1(2)$, $J_2(0)$, $J_1(0)$ and $J_{2k}(1)$. Their m -th powers have ranks 1, 1, $\max(2 - m, 0)$, 0 and $2k$ respectively, so

$$\text{rank}((A + I)^m) = 1 + 1 + 2k + \max(2 - m, 0) = n - 3 + \max(2 - m, 0),$$

which is $n - 2$ for $m = 1$ and $n - 3$ for $m \geq 2$. $\square$

Remark 4.5. The rank stabilises at 5 rather than decaying to 0: the five-dimensional non-zero spectral core contributes a constant rank equal to the number of non-zero eigenvalues counted with multiplicity. The sequence $\text{rank}(A^m) = n - m$ for $m = 1, \dots, 2k$, followed by the plateau at 5, is a necessary and sufficient condition for the nilpotent part to be a single block $J_{2k}(0)$ rather than a direct sum of smaller blocks (Proposition 2.7).

4.4 Perron–Frobenius structure

Proposition 4.6. *Let $A \in \mathcal{U}_k$. Then:*

- (a) *the left Perron vector is $\mathbf{1}^\top$: $\mathbf{1}^\top A = 2\mathbf{1}^\top$;*
- (b) *the right Perron vector $v > 0$ with $Av = 2v$ may be chosen with entries in $\mathbb{Z}_{>0}$, and the primitive choice (gcd of entries equal to 1) is unique up to sign;*
- (c) *the Perron projection is $P_2 = v\mathbf{1}^\top/(\mathbf{1}^\top v)$; it has rank one and satisfies $P_2^2 = P_2$ and $P_2 A = A P_2 = 2P_2$.*

Proof. (a) is D2. For (b), $A - 2I$ has integer entries and its null space is one-dimensional (the algebraic multiplicity of $\lambda = 2$ is one), so the null space is defined over $\mathbb{Q}$ and contains a primitive integer vector; by the Perron–Frobenius theorem [12, 18, 7, 15] that vector may be taken positive. Uniqueness of the primitive representative is the uniqueness of a primitive generator of a rank-one lattice. For (c), compute $(v\mathbf{1}^\top)^2 = v(\mathbf{1}^\top v)\mathbf{1}^\top$, so $P_2^2 = P_2$; and $A P_2 = \frac{1}{\mathbf{1}^\top v}(Av)\mathbf{1}^\top = \frac{2}{\mathbf{1}^\top v}v\mathbf{1}^\top = 2P_2$, similarly $P_2 A = 2P_2$ using $\mathbf{1}^\top A = 2\mathbf{1}^\top$. $\square$

Remark 4.7. The integer Perron vector is far from being an invariant of the class: different members of $\mathcal{U}_k$ have Perron vectors that are unrelated by scaling or permutation (Observation 9.7). Already W_3 has $\min(v) = 1$ while W_4 has $\min(v) = 18$.

4.5 Spectral decomposition and the cyclic structure of the nilpotent part

The Jordan form induces the decomposition

$$\mathbb{R}^n = V_2 \oplus V_1 \oplus V_{-1} \oplus V_0$$

into generalised eigenspaces, of dimensions 1, 1, 3, $2k$. The spectral projections are polynomials in A obtained from the Lagrange interpolation formula for the minimal polynomial (Proposition 5.1); P_2 was made explicit in Proposition 4.6(c), and $P_0 = I - P_2 - P_1 - P_{-1}$.

Proposition 4.8. *The restriction $A|_{V_0}$ is nilpotent, and its nilpotency index is exactly $2k$:*

$$(A|_{V_0})^{2k} = 0, \quad (A|_{V_0})^{2k-1} \neq 0.$$

Consequently V_0 admits a cyclic vector x such that

$$\{x, Ax, A^2x, \dots, A^{2k-1}x\}$$

is a basis of V_0 .

Proof. By D4 the nilpotent part is the single block $J_{2k}(0)$, whose $2k$ -th power vanishes and whose $(2k - 1)$ -th power has rank one; the standard basis vector e_1 of that block is cyclic and its images under the first $2k$ powers span the block. $\square$

The existence of such a cyclic vector is equivalent to the rank condition $\text{rank}(A^m) = n - m$ for $m = 1, \dots, 2k$, which is the computational test used in practice (Proposition 2.7).

5 Matrix Algebra

5.1 The minimal polynomial

Proposition 5.1. *The minimal polynomial of $A \in \mathcal{U}_k$ is*

$$\mu_A(x) = x^{2k}(x-2)(x-1)(x+1)^2,$$

of degree $2k+4 = n-1$.

Proof. For a matrix in Jordan form, the exponent of the factor $(x-\lambda)$ in the minimal polynomial equals the size of the largest Jordan block attached to λ . By D4 these sizes are $2k$ for $\lambda = 0$, 1 for $\lambda = 2$ and $\lambda = 1$, and 2 for $\lambda = -1$ (from the block $J_2(-1)$). Hence $\mu_A = x^{2k}(x-2)(x-1)(x+1)^2$, of degree $2k+1+1+2 = 2k+4$, whereas $n = 2k+5$. The degree gap of one is exactly the extra block $J_1(-1)$: the characteristic polynomial carries $(x+1)^3$ while the minimal polynomial carries only $(x+1)^2$. $\square$

Corollary 5.2. μ_A annihilates A and is minimal with this property; equivalently

$$\begin{aligned} A^{2k}(A-2I)(A-I)(A+I)^2 &= 0, \\ A^{2k-1}(A-2I)(A-I)(A+I)^2 &\neq 0, \\ A^{2k}(A-2I)(A-I)(A+I) &\neq 0. \end{aligned}$$

In particular $\mathbb{R}^n$ is a cyclic $\mathbb{R}[x]$ -module for A if and only if $\deg \mu_A = n$, which fails here by exactly one.

The first identity of Corollary 5.2 was verified by direct integer matrix arithmetic for all members listed in this paper.

5.2 The commutant

Proposition 5.3. *For $A \in \mathcal{U}_k$ the centraliser (commutant) $\mathcal{C}(A) = \{X \in \mathbb{R}^{n \times n} : XA = AX\}$ has dimension*

$$\dim \mathcal{C}(A) = 7 + 2k = n + 2.$$

Proof. The commutant is invariant under similarity, so we may assume A is in Jordan form. The commutant of a direct sum decomposes as a direct sum over the distinct eigenvalues, and for blocks sharing an eigenvalue one has the classical formula

$$\dim \mathcal{C}\left(\bigoplus_i J_{s_i}(\lambda)\right) = \sum_{i,j} \min(s_i, s_j)$$

(see [11, Chapter 4]). Applying it block by block:

$$\lambda = 2 : \min(1, 1) = 1, \quad \lambda = 1 : 1, \quad \lambda = 0 : \min(2k, 2k) = 2k,$$

$$\lambda = -1 : \min(2, 2) + \min(2, 1) + \min(1, 2) + \min(1, 1) = 2 + 1 + 1 + 1 = 5.$$

Total: $1 + 1 + 2k + 5 = 7 + 2k$. Since $n = 2k + 5$, this is $n + 2$. $\square$

The dimension was verified computationally for $k = 1, 2, 3$ by constructing the Kronecker matrix $I \otimes A - A^\top \otimes I$ and computing its nullity ($\dim \mathcal{C}(A)$ equals the dimension of $\{X : AX = XA\}$, i.e. the nullity of that matrix), giving 9, 11, 13 respectively.

5.3 Norms, singular values and non-normality

Since A is a $\{0, 1\}$ -matrix with exactly $2n$ ones (Proposition 2.4(c)),

$$\|A\|_F^2 = \sum_{i,j} A[i, j]^2 = 2n \quad (5)$$

exactly. The spectral radius is $\rho(A) = 2$, while the operator norm satisfies $\|A\|_2 = \sigma_1(A) \geq \rho(A) = 2$; A is not normal (D4 exhibits non-trivial Jordan blocks, whereas a normal matrix is unitarily diagonalisable), so no equality is to be expected, and the eigenvalues of a non-normal matrix are in general far more sensitive to perturbations than $\|A\|_2$ alone suggests [23].

Observation 5.4 (Numerical range of the singular values). Across the 233 matrices of the searched data set the largest singular value satisfies $\sigma_1 \in [2.10, 3.05]$, the smallest is $\sigma_n = 0$ (the matrix is singular), and the number of non-zero singular values equals $\text{rank}(A) = n - 1$ for every member. These are floating-point observations; the exact statements used elsewhere in the paper are $\|A\|_F^2 = 2n$, $\sigma_n = 0$ and $\#\{i : \sigma_i \neq 0\} = n - 1$, all of which follow from D2, D3 and Proposition 4.4.

5.4 Growth of the powers induced by the Jordan structure

The powers of A decompose according to the spectral decomposition of Section 4. Write P_2, P_1, P_{-1}, P_0 for the spectral projections and set

$$M := (A + I)P_{-1},$$

the nilpotent part of A on the generalised eigenspace V_{-1} . Since $A|_{V_{-1}}$ has Jordan type $J_2(-1) \oplus J_1(-1)$, we have $M^2 = 0$ and $M \neq 0$.

Proposition 5.5. *For every $A \in \mathcal{U}_k$ and every $L \geq 1$,*

$$A^L = 2^L P_2 + P_1 + (-1)^L P_{-1} + L(-1)^{L-1} M + A^L P_0, \quad (6)$$

and $A^L P_0 = 0$ for $L \geq 2k$. In particular, for $L \geq 2k$,

$$A^L = 2^L P_2 + P_1 + (-1)^L P_{-1} + L(-1)^{L-1} M. \quad (7)$$

Proof. Since $P_2 + P_1 + P_{-1} + P_0 = I$ is a decomposition into mutually orthogonal idempotents commuting with A , we have $A^L = A^L P_2 + A^L P_1 + A^L P_{-1} + A^L P_0$. On V_2 and V_1 the operator A is scalar (2 and 1 respectively), so $A^L P_2 = 2^L P_2$ and $A^L P_1 = P_1$. On V_{-1} we write $A = -I + M$ with $M^2 = 0$, so by the binomial theorem

$$A^L P_{-1} = (-I + M)^L P_{-1} = ((-1)^L I + L(-1)^{L-1} M) P_{-1} = (-1)^L P_{-1} + L(-1)^{L-1} M,$$

the terms with M^r , $r \geq 2$, vanishing. Finally $A^L P_0 = (A|_{V_0})^L$ vanishes for $L \geq 2k$ by Proposition 4.8. $\square$

For $1 \leq L \leq 2k - 1$ the term $A^L P_0$ is non-zero: in a Jordan basis of V_0 it is the matrix with ones on the L -th superdiagonal and zeros elsewhere, i.e. the “shifting band” produced by $J_{2k}(0)^L$.

Corollary 5.6. *Let $A \in \mathcal{U}_k$ and let v be the primitive integer Perron vector. Then*

(a) $\text{tr}(A^L) = 2^L + 1 + 3(-1)^L$ for every $L \geq 1$: no polynomial prefactor in L survives in the trace;

(b) the entries of A^L grow exponentially, not polynomially: with $c_A := \max_i v_i / (\mathbf{1}^\top v) > 0$,

$$\max_{i,j} A^L[i, j] = c_A 2^L + O(L) \quad (L \rightarrow \infty);$$

(c) the only L -dependent polynomial factor in (6) is the scalar L multiplying the fixed nilpotent matrix M ; it cancels from the trace because $\text{tr}(M) = \text{tr}((A + I)P_{-1}) = 0$.

Proof. (a) follows from (7) together with $\text{tr}(P_2) = \text{tr}(P_1) = 1$, $\text{tr}(P_{-1}) = 3$ and $\text{tr}(M) = 0$, the last because M is nilpotent. For (b), apply (7): for $L \geq 2k$,

$$A^L - 2^L P_2 = P_1 + (-1)^L P_{-1} + L(-1)^{L-1} M,$$

whose entries are bounded in absolute value by $\|P_1\|_{\max} + \|P_{-1}\|_{\max} + L\|M\|_{\max} = O(L)$, where $\|\cdot\|_{\max}$ denotes the maximum absolute entry. Since $|\max_{i,j} X[i, j] - \max_{i,j} Y[i, j]| \leq \max_{i,j} |X[i, j] - Y[i, j]|$, and since $P_2 = v\mathbf{1}^\top / (\mathbf{1}^\top v)$ has $\max_{i,j} P_2[i, j] = c_A$, the claim follows. (c) is immediate: M is a fixed matrix, independent of L , and it is nilpotent, hence traceless. $\square$

Remark 5.7. Corollary 5.6(b) is the correct formulation of the growth statement. It is easy to mis-state it: the nilpotent block $J_{2k}(0)$ contributes entries of size 1 in a shifting band for $1 \leq L \leq 2k - 1$ and nothing at all for $L \geq 2k$, so the nilpotent block never produces polynomial growth of the entries; the exponential growth 2^L comes from the Perron projection, and the linear factor L comes from $J_2(-1)$. The band structure of $J_{2k}(0)^L$ is nevertheless the matrix-theoretic expression of the doubler-chain architecture of Section 10.

6 Closed-Walk Weights and Counting

6.1 Row-stochastic normalisation and the universal closed-walk weight

Let $A \in \mathcal{U}_k$ and let $v > 0$ be the right Perron vector, $Av = 2v$ (Proposition 4.6). Define the edge weight

$$w_{ij} := \frac{v_j}{2v_i} \quad \text{for each directed edge } i \rightarrow j,$$

and set $W = (w_{ij})$, $D = \text{diag}(v_0, \dots, v_{n-1})$.

Proposition 6.1 (Row-stochasticity). *W is row-stochastic: $\sum_j w_{ij} = 1$ for every i . Moreover $W = \frac{1}{2}D^{-1}AD$, so W is similar to $A/2$ and has spectral radius 1 with the same Jordan structure as A (each Jordan block $J_s(\lambda)$ becoming $J_s(\lambda/2)$).*

Proof. $\sum_j w_{ij} = \frac{1}{2v_i} \sum_j A[i, j]v_j = \frac{(Av)_i}{2v_i} = \frac{2v_i}{2v_i} = 1$. The identity $W = \frac{1}{2}D^{-1}AD$ is immediate from the definition, and similarity preserves the Jordan structure up to the scaling of eigenvalues. $\square$

Theorem 6.2 (Universal closed-walk weight). *Let $A \in \mathcal{U}_k$ and let $\gamma = (i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_L = i_0)$ be any closed directed walk of length L in $G(A)$. Then the cumulative weight*

$$\Gamma(\gamma) = \prod_{m=0}^{L-1} w_{i_m i_{m+1}}$$

equals 2^{-L} , independently of the walk.

Proof. Substituting the definition,

$$\Gamma(\gamma) = \prod_{m=0}^{L-1} \frac{v_{i_{m+1}}}{2v_{i_m}} = \frac{1}{2^L} \cdot \frac{\prod_{m=0}^{L-1} v_{i_{m+1}}}{\prod_{m=0}^{L-1} v_{i_m}}.$$

Since $i_L = i_0$, the index multisets $\{i_0, i_1, \dots, i_{L-1}\}$ and $\{i_1, i_2, \dots, i_L\}$ coincide. Hence numerator and denominator are products of the same multiset of v -entries and their ratio is 1, giving $\Gamma(\gamma) = 2^{-L}$. $\square$

Remark 6.3 (Scope of Theorem 6.2). The proof uses exactly two ingredients: (i) the Perron equation $Av = 2v$, which follows from D2 and D5; and (ii) the in-out balance of closed walks, a combinatorial tautology. It uses neither D3 nor D4. The same argument shows that for any strongly connected directed graph with uniform in-degree equal to the spectral radius ρ , every closed walk of length L carries weight ρ^{-L} . The universal weight 2^{-L} is therefore not a distinguishing feature of the Wen class. What distinguishes it is the Jordan structure: the fixed non-zero spectrum together with the growing nilpotent block.

6.2 The closed-walk counting formula

The number of closed directed walks of length L in $G(A)$ is $N_L = \text{tr}(A^L)$, since $(A^L)[i, i]$ counts the closed walks of length L based at i .

Theorem 6.4 (Closed-walk count). *For every $A \in \mathcal{U}_k$ and every $L \geq 1$,*

$$N_L = \text{tr}(A^L) = 2^L + 1 + 3(-1)^L,$$

which is independent of k and of the particular member of $\mathcal{U}_k$.

Proof. This is Corollary 4.2, restated combinatorially: $\text{tr}(A^L)$ is the sum of the L -th powers of the eigenvalues with algebraic multiplicity; the eigenvalue 0 (multiplicity $2k$) contributes nothing, and the non-zero spectrum $\{2, 1, -1, -1, -1\}$ is fixed by D3. $\square$

Table 2: Closed-walk counts $N_L = \text{tr}(A^L)$ for $L = 1, \dots, 10$. The formula of Theorem 6.4 was checked against the exact integer traces of all 245 known members; the value is identical across all orders.

L	1	2	3	4	5	6	7	8	9	10
$2^L + 1 + 3(-1)^L$	0	8	6	20	30	68	126	260	510	1028

Remark 6.5. $N_1 = 0$ is D1 (no loops). $N_2 = 8$: a closed walk of length 2 is exactly a traverse of a bidirectional edge pair forth and back, and each such pair contributes two such walks, so $N_2 = 2 \times \#\{\text{bidirectional pairs}\}$; the observed value 8 is consistent with the proven count of 4 bidirectional pairs (Proposition 9.1). Equivalently $N_L = 2^L - 2$ for odd L and $N_L = 2^L + 4$ for even L .

Corollary 6.6 (Loop-model partition function). *For the row-stochastic chain W of Proposition 6.1, the total weight of all closed walks of length L is*

$$Z_L = \sum_{\gamma \text{ closed}, |\gamma|=L} \Gamma(\gamma) = N_L 2^{-L} = (2^L + 1 + 3(-1)^L) 2^{-L} = 1 + 2^{-L} + 3(-2)^{-L},$$

independent of k and of the member.

7 The Newton Identity Cascade

7.1 Characteristic polynomial versus power sums

Write $p_L = \text{tr}(A^L)$ for the L -th power sum of the eigenvalues and

$$\det(\lambda I - A) = \lambda^n - e_1 \lambda^{n-1} + e_2 \lambda^{n-2} - \dots + (-1)^n e_n$$

for the characteristic polynomial, so that e_j is the j -th elementary symmetric polynomial of the eigenvalues. Newton's identities state that for $m \geq 1$,

$$p_m - e_1 p_{m-1} + e_2 p_{m-2} - \dots + (-1)^{m-1} e_{m-1} p_1 + (-1)^m m e_m = 0, \quad (8)$$

with the convention that the terms involving e_j are absent for $j > n$. Over a field of characteristic zero the system (8) is triangular and invertible, so the datum $(p_1, \dots, p_n)$ determines $(e_1, \dots, e_n)$ and conversely.

Proposition 7.1 (Power-sum formulation of D3). *Let A be an $n \times n$ matrix with $n = 2k + 5$. Then A satisfies D3 if and only if*

$$\mathrm{tr}(A^L) = 2^L + 1 + 3(-1)^L \quad \text{for } L = 1, 2, \dots, n.$$

Proof. Both conditions determine the characteristic polynomial. D3 prescribes $\det(\lambda I - A) = \lambda^{2k}(\lambda - 2)(\lambda - 1)(\lambda + 1)^3$ exactly; by Newton's identities that prescription is equivalent to the prescription of the n power sums $p_1, \dots, p_n$, and for the stated polynomial these power sums are $p_L = 2^L + 1 + 3(-1)^L$ (the zero eigenvalue contributing nothing). $\square$

This equivalence is the basis of the fast pre-filter used in the search (Section 8): n integer traces are far cheaper than one symbolic determinant.

7.2 The recurrence of order $\delta = 5$

Expanding the non-zero spectral factor,

$$q(\lambda) = (\lambda - 2)(\lambda - 1)(\lambda + 1)^3 = \lambda^5 - 4\lambda^3 - 2\lambda^2 + 3\lambda + 2, \quad (9)$$

and hence

$$\det(\lambda I - A) = \lambda^{2k} q(\lambda) = \lambda^n - 4\lambda^{n-2} - 2\lambda^{n-3} + 3\lambda^{n-4} + 2\lambda^{n-5}.$$

Comparing coefficients gives

$$e_1 = 0, \quad e_2 = -4, \quad e_3 = 2, \quad e_4 = 3, \quad e_5 = -2, \quad e_j = 0 \quad (6 \leq j \leq n). \quad (10)$$

Proposition 7.2 (Recurrence of order five). *For every $A \in \mathcal{U}_k$ the power sums satisfy, for all $L \geq 6$,*

$$p_L = 4p_{L-2} + 2p_{L-3} - 3p_{L-4} - 2p_{L-5}. \quad (11)$$

Proof. In (8) with $m = L \geq 6$ all terms e_j with $j \geq 6$ vanish by (10), and the term $(-1)^L e_L$ vanishes as well. Solving for p_L gives $p_L = e_1 p_{L-1} - e_2 p_{L-2} + e_3 p_{L-3} - e_4 p_{L-4} + e_5 p_{L-5}$, and substituting (10) yields (11). $\square$

Corollary 7.3 (Five seeds determine everything). *Given the structural vanishing $e_j = 0$ for $j \geq 6$, the first five power sums $p_1, \dots, p_5$ determine p_L for every $L \geq 6$ through (11), and hence (by Newton's identities) determine the whole characteristic polynomial. Of these five seeds, $p_1 = 0$ is automatic from D1; the remaining four conditions are*

$$p_2 = 8, \quad p_3 = 6, \quad p_4 = 20, \quad p_5 = 30.$$

The recurrence (11) was verified exactly for $L = 6, \dots, 15$ on all members listed in this paper.

7.3 The cascade: a computational observation

Definition 7.4. The *fundamental supporting degree* of the Wen class is

$$\delta := \deg q(\lambda) = 5,$$

the degree of the non-zero spectral factor of the characteristic polynomial.

The quantity δ governs three distinct aspects.

- (a) **Number of seed conditions (proven).** By Corollary 7.3 exactly $\delta - 1 = 4$ trace conditions have to be imposed, since $p_1 = 0$ is free (it is D1).
- (b) **Cascade depth (observed).** Monte Carlo experiments (20 000 random D1+D2 matrices per order) show that once the seed $(p_2, \dots, p_5)$ is planted, the subsequent trace conditions $p_6, p_7, \dots$ become progressively more likely to hold: the conditional probability $\Pr(p_L \text{ correct} \mid p_1, \dots, p_{L-1} \text{ correct})$ exceeds the marginal probability by factors ranging from about 1.5 to about 22. The enhancement decays with L and is no longer detectable beyond roughly $L \approx \delta + 5$, where the conditions become approximately independent. This cascade is an empirical finding and is not proved.
- (c) **Critical ratio (heuristic).** The ratio $S(k) = \delta/(2k) = 5/(2k)$ compares the dimension of the spectral core with the size of the nilpotent block. Solutions are abundant when $S > 1$ ($k \leq 2$) and become scarcer when $S < 1$ ($k \geq 3$). This is a heuristic reading of the data and carries no explanatory force; in particular it does *not* predict the global maximum of the searched counts, which occurs at $k = 5$ (Table 1).

Remark 7.5. The cascade explains why a search strategy that plants the seed $(p_2, \dots, p_5)$ and then optimises the rank sequence is far more efficient than uniform sampling: the seed is a strong necessary condition for D3 that is cheap to evaluate, and conditioning on it raises the hit rate for the remaining conditions by more than an order of magnitude at small L .

8 Search Methodology

This section describes how the 233 matrices of the data set underlying Appendix 18 were produced. Two regimes are used: D2-constrained column-by-column generation with eigenvalue pre-screening for $k \leq 4$, and an $n - 2$ extension strategy with D1+D2-preserving simulated annealing for $k \geq 5$. We emphasise from the outset that neither regime is exhaustive; the resulting counts are lower bounds for $c(k)$ (Remark 3.2).

8.1 The search space

Let

$$\Omega(k) = \{A \in \{0, 1\}^{n \times n} : A \text{ satisfies D1 and D2}\}, \quad n = 2k + 5.$$

Each column of A must contain exactly two ones, chosen among the $n - 1$ off-diagonal positions, and the n columns are independent; hence

$$|\Omega(k)| = \binom{n-1}{2}^n. \tag{12}$$

Table 3: Size of the D1+D2 search space. The counts of members found are reported in Table 7 together with the resulting densities.

k	n	$\binom{n-1}{2}$	$ \Omega(k) $	$\log_{10} \Omega(k) $
1	7	15	1.71×10^8	8.23
2	9	28	1.06×10^{13}	13.02
3	11	45	1.53×10^{18}	18.19
4	13	66	4.51×10^{23}	23.65
5	15	91	2.43×10^{29}	29.39
6	17	120	2.22×10^{35}	35.35
7	19	153	3.23×10^{41}	41.51

The solution density $\rho(k) = c(k)/|\Omega(k)|$ decays by roughly five to six orders of magnitude per unit increase of k (Table 7). A quadratic fit $\log_{10} \rho \approx -0.178k^2 - 4.233k - 2.244$ reproduces the seven points with $R^2 \approx 0.9998$, but with only seven data points—all of which are lower bounds—the fit cannot distinguish a genuinely quadratic decline from a mildly super-linear one. We do not use this fit for any extrapolation.

8.2 Column-by-column generation for $k \leq 4$

For $k = 1, \dots, 4$ the search space is generated column by column: for each column, one of the $\binom{n-1}{2}$ admissible pairs of rows is selected. Candidates are filtered by

- (i) the Newton seed of Corollary 7.3, i.e. $\text{tr}(A^L) = 2^L + 1 + 3(-1)^L$ for $L = 2, 3, 4, 5$ (a necessary condition for D3, cheap to evaluate in integer arithmetic);
- (ii) the full characteristic polynomial, computed from the power sums by Newton’s identities (D3);
- (iii) the rank sequence of Proposition 2.7 (D4);
- (iv) strong connectivity by breadth-first search (D5).

Survivors are reduced modulo graph isomorphism. This is feasible up to $n = 13$ and becomes impractical beyond, because of the super-exponential growth (12).

Remark 8.1 (On completeness at small orders). Although the generation at $k \leq 4$ is systematic in the sense that it enumerates the whole of $\Omega(k)$ in principle, in practice the eigenvalue pre-screening of (i) is applied as soon as a partial assignment permits, and the search was terminated when the yield became negligible. We therefore do *not* claim that the counts at $k \leq 4$ are exact. This is not a technicality: the representatives W_1, W_2, W_3 of Section 3 were found by a different route and are not isomorphic to any matrix in the searched data set, which shows that the generation missed solutions even at $k = 1, 2, 3$. All counts reported in this paper are lower bounds.

8.3 The $n - 2$ extension strategy

For $k \geq 5$ the combinatorial space is too large for systematic generation. Given a verified solution A of order $n - 2$, a candidate of order n is built by embedding A as the upper-left block and adjoining two new vertices, indexed $n - 2$ and $n - 1$. Adding edges to and from the new vertices would violate D2, so a displacement mechanism is used: to add an edge from a vertex v into column j , one of the two existing in-edges of column j is first removed, so that the column sum stays equal to 2. Three initialisation variants supply diverse starting points:

- (A) each new vertex receives two random out-edges, inserted by displacement;
- (B) a doubler chain $n - 1 \rightarrow n - 2 \rightarrow \text{core}$, i.e. the new vertex $n - 1$ points to $n - 2$, which points into the embedded block;
- (C) the same chain with the order reversed, $n - 2 \rightarrow n - 1 \rightarrow \text{core}$.

Variants (B) and (C) implement the doubler-chain heuristic of Section 10, in which the Perron value is intended to propagate along the chain with power-of-two scaling.

8.4 D1+D2-preserving simulated annealing

After initialisation the candidate is refined by simulated annealing. The move set consists of 2×2 column swaps that preserve both D1 and D2: choose two columns j_1, j_2 and two rows i_1, i_2 with

$$A[i_1, j_1] = A[i_2, j_2] = 1, \quad A[i_1, j_2] = A[i_2, j_1] = 0,$$

and swap the two ones to the anti-diagonal positions. Both column sums and the zero diagonal are preserved, so the walk stays inside $\Omega(k)$.

The energy function is

$E = w_{\text{rank}} \cdot (\text{rank-sequence deviation}) + w_{\text{Newt}} \cdot (\text{Newton seed deviation}) + w_{\text{SCC}} \cdot (\text{SCC penalty})$, where the rank-sequence deviation is $\sum_{m=1}^{2k} |\text{rank}(A^m) - (n - m)|$ plus $|\text{rank}(A + I) - (n - 2)|$, the Newton term measures the mismatch of $\text{tr}(A^L)$ with $2^L + 1 + 3(-1)^L$ for small L , and the SCC penalty charges candidates whose digraph has more than one strongly connected component. Cooling is geometric, $T_{i+1} = \alpha T_i$, with $T_0 = 8$ and $T_f = 0.003$. Whenever $E = 0$ the candidate is submitted to exact verification of D3, D4 and D5; ranks are computed by exact integer elimination, never by floating point.

8.5 Isomorphism reduction

Isomorphism is tested with the VF2 algorithm [4] as implemented in NetworkX [10]. A union-find structure keeps the classes: each new solution is compared against all previously found solutions of the same order. The 233 matrices of the full data set were confirmed pairwise non-isomorphic by this test in the course of the present work, and the five additional representatives of Section 3 were confirmed non-isomorphic to all of them (with the two exceptions recorded in Remark 3.3).

8.6 Search effort and results

Table 4: Search effort and results for the extension regime $k = 5, 6, 7$.

k	n	bases used	annealing runs	solutions found	verified
5	15	22	198	68	68/68
6	17	17 (of 68)	68	13	13/13
7	19	13	52	7	7/7
Total			318	88	88/88

Remark 8.2. The success rate per annealing run is $68/198 \approx 34\%$ at $k = 5$, $13/68 \approx 19\%$ at $k = 6$, and $7/52 \approx 13\%$ at $k = 7$. These rates reflect the difficulty of the search inside a structured subspace, not the density of solutions in $\Omega(k)$: at $k = 7$ the uniform-sampling density is $\approx 2 \times 10^{-41}$, so structured search outperforms uniform sampling by some forty orders of magnitude at that order.

8.7 Computational cost

Each energy evaluation costs $O(2k \cdot n^3)$ operations for the rank sequence ($2k+1$ exact eliminations on $n \times n$ integer matrices). At $n = 19$ this is about $15 \times 19^3 \approx 10^5$ operations per evaluation, and a run performs on the order of 10^6 evaluations, i.e. $\approx 10^{11}$ operations per run; the 52 runs at $k = 7$ required $\approx 5 \times 10^{12}$ operations in total. Extrapolating by the factor $(21/19)^3 \times (16/14) \approx 1.6$, a run at $k = 8$ ($n = 21$) is feasible but increasingly expensive. Isomorphism checking costs $O(N^2 n^2)$ pairwise comparisons for N solutions at one order, which is manageable at present scales but would become a bottleneck if the counts grew substantially.

9 Invariants

We classify the quantities attached to members of $\mathcal{U}_k$ into *proven invariants* (derived from D1–D5), *empirical invariants* (constant across all known members but not derived from the axioms) and *variant quantities*. We begin with two combinatorial invariants that turn out to be provable from D1 and D3 alone.

9.1 Two combinatorial invariants derived from the trace formula

Proposition 9.1 (Bidirectional edge pairs). *Let $A \in \mathcal{U}_k$. Then the number of unordered pairs $\{i, j\}$ with $A[i, j] = A[j, i] = 1$ (bidirectional edge pairs) is exactly 4, independently of k .*

Proof. We have

$$\text{tr}(A^2) = \sum_{i,j} A[i, j]A[j, i].$$

The term $A[i, j]A[j, i]$ equals 1 precisely when both $i \rightarrow j$ and $j \rightarrow i$ are edges, and it occurs twice in the double sum (once as (i, j) and once as (j, i)); diagonal terms vanish by D1. Hence $\text{tr}(A^2) = 2 \#\{\text{bidirectional pairs}\}$. By Theorem 6.4, $\text{tr}(A^2) = 2^2 + 1 + 3 = 8$, so the number of bidirectional pairs is $8/2 = 4$. $\square$

Proposition 9.2 (Directed 3-cycles). *Let $A \in \mathcal{U}_k$. Then the number of directed 3-cycles (cyclic triples of pairwise distinct vertices $i \rightarrow j \rightarrow \ell \rightarrow i$, counted up to cyclic rotation of the triple) is exactly 2, independently of k .*

Proof. $\text{tr}(A^3) = \sum_{i,j,\ell} A[i, j]A[j, \ell]A[\ell, i]$ counts the closed directed walks of length 3, i.e. the triples (i_0, i_1, i_2) with $i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow i_0$. By D1 there are no loops, so $i_0 \neq i_1$, $i_1 \neq i_2$ and $i_2 \neq i_0$; hence the three vertices are pairwise distinct and every closed walk of length 3 is a directed 3-cycle. Conversely, each directed 3-cycle yields exactly three such closed walks, one for each choice of starting vertex. Therefore $\text{tr}(A^3) = 3 \#\{\text{directed 3-cycles}\}$, and by Theorem 6.4, $\text{tr}(A^3) = 2^3 + 1 + 3(-1) = 6$; the count is $6/3 = 2$. $\square$

Both statements were confirmed by direct enumeration of the edge sets of all 233 matrices of the searched data set and of the five additional representatives.

9.2 Out-degree dichotomy

Proposition 9.3. *Let $A \in \mathcal{U}_k$. Then exactly one of the following holds:*

- (a) *some vertex has out-degree 1;*
- (b) *every vertex has out-degree exactly 2 (equivalently, A is a 2-regular digraph: in-degree = out-degree = 2 at every vertex).*

Proof. By D5 every vertex has out-degree ≥ 1 , and by Proposition 2.4(c) the out-degrees sum to $2n$. If no vertex has out-degree 1, then all out-degrees are ≥ 2 , and a sum of n integers ≥ 2 equal to $2n$ forces all of them to equal 2. Conversely, if all out-degrees equal 2 then (a) fails. The parenthetical reformulation uses D2, which gives in-degree 2 everywhere. $\square$

Proposition 9.3 reduces the remaining combinatorial question to a clean dichotomy:

Conjecture 9.4 (Doubler-chain existence). *Every member of $\mathcal{U}_k$ possesses a vertex of out-degree 1, and hence (by Theorem 10.2) at least one doubler chain. Equivalently, by Proposition 9.3: no member of $\mathcal{U}_k$ has all out-degrees equal to 2.*

All 233 matrices of the searched data set and the five additional representatives have minimum out-degree 1; no member with constant out-degree 2 has been found. The infinite family of Section 14 satisfies the conjecture (Remark 14.8).

9.3 Proven and empirical invariants

Proposition 9.5 (Proven invariants). *Every $A \in \mathcal{U}_k$ satisfies:*

- (a) *characteristic polynomial $\lambda^{2k}(\lambda - 2)(\lambda - 1)(\lambda + 1)^3$;*
- (b) *Jordan form $J_1(2) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1) \oplus J_{2k}(0)$;*
- (c) *left Perron vector $\mathbf{1}$;*
- (d) *closed-walk count $N_L = 2^L + 1 + 3(-1)^L$;*
- (e) *spectral radius $\rho = 2$;*
- (f) $\text{rank}(A) = n - 1, \text{rank}(A + I) = n - 2$;
- (g) $\text{rank}(A^m) = \max(n - m, 5)$;
- (h) *minimal polynomial $x^{2k}(x - 2)(x - 1)(x + 1)^2$ of degree $n - 1$;*
- (i) *commutant dimension $7 + 2k$;*
- (j) *exactly 4 bidirectional edge pairs;*
- (k) *exactly 2 directed 3-cycles;*
- (l) $\|A\|_F^2 = 2n$;
- (m) $\sigma_{\min} = 0$ and exactly $n - 1$ non-zero singular values.

Proof. (a),(b) are D3, D4; (c) is D2; (d) is Theorem 6.4; (e) is Proposition 2.4(b); (f),(g) are Proposition 4.4; (h) is Proposition 5.1; (i) is Proposition 5.3; (j) is Proposition 9.1; (k) is Proposition 9.2; (l) is (5); (m) follows from $\text{rank}(A) = n - 1$. $\square$

Observation 9.6 (Empirical invariants). The following hold for all 233 matrices of the searched data set but are not derived from D1–D5:

- (a) minimum out-degree = 1 (equivalently Conjecture 9.4);
- (b) $\sum_i v_i \equiv 0 \pmod{3}$ for 230 of the 233 members (98.7%), and $\sum_i v_i \equiv 0 \pmod{9}$ for 206 of them (88.4%), where v is the primitive integer Perron vector;
- (c) the largest invariant factor of $\text{SNF}(A - 2I)$ is divisible by 3;
- (d) more precisely, $(\sum_i v_i) \cdot \prod_{d_i > 1} d_i = 4^k \cdot 27$, where $d_1, \dots, d_n$ run over the invariant factors of $\text{SNF}(A - 2I)$ (Section 12).

Observation 9.7 (Variant quantities). The following vary across members and are not invariants:

- (a) the right Perron vector, even after scaling and permutation;
- (b) the out-degree sequence;
- (c) the partition of $2k$ into doubler-chain lengths;
- (d) the minimum entry of the Perron vector (1 for W_1, W_2, W_3 , 18 for W_4 , 120 for W_5 , 90 for W_6 , 404 for W_7);
- (e) the specific adjacency pattern among the non-chain vertices;
- (f) the Smith normal form of $A - 2I$.

Table 5: Invariants of the Wen class. “Axiom” entries are the defining conditions; “proven” entries are derived in this paper; “empirical” entries hold for all known members without being derived from D1–D5.

#	Invariant	Status
1	Zero diagonal (D1)	Axiom
2	Column sums = 2; edge count = $2n$ (D2)	Axiom
3	Char. poly. $\lambda^{2k}(\lambda - 2)(\lambda - 1)(\lambda + 1)^3$ (D3)	Axiom
4	$\text{rank}(A^m) = \max(n - m, 5)$, $m \leq 2k + 1$ (D4)	Axiom
5	$\text{rank}(A + I) = n - 2$, $\text{rank}((A + I)^2) = n - 3$ (D4)	Axiom
6	Strong connectivity (D5)	Axiom
7	Minimal polynomial $x^{2k}(x - 2)(x - 1)(x + 1)^2$	Proven
8	Commutant dimension $7 + 2k$	Proven
9	$N_L = \text{tr}(A^L) = 2^L + 1 + 3(-1)^L$	Proven
10	Bidirectional edge pairs = 4	Proven
11	Directed 3-cycles = 2	Proven
12	$\ A\ _F^2 = 2n$; $\sigma_{\min} = 0$; $n - 1$ non-zero singular values	Proven
13	Recurrence $p_L = 4p_{L-2} + 2p_{L-3} - 3p_{L-4} - 2p_{L-5}$ ($L \geq 6$)	Proven
14	Out-degree dichotomy (Prop. 9.3)	Proven
15	Minimum out-degree = 1	Empirical
16	$(\sum_i v_i) \prod_{d_i > 1} d_i = 4^k \cdot 27$	Empirical

9.4 Summary table

10 Doubler Chains and the Three-Layer Architecture

10.1 Doubler chains

Definition 10.1 (Doubler chain). Let $A \in \mathcal{U}_k$ with primitive integer Perron vector $v > 0$. A *doubler chain* of length m is a sequence of pairwise distinct vertices $c_0 \rightarrow c_1 \rightarrow \cdots \rightarrow c_m$ such that each c_i with $i < m$ has out-degree 1 and its sole outgoing edge points to c_{i+1} . The vertices $c_0, \dots, c_{m-1}$ are called *chain vertices* and the edges $c_i \rightarrow c_{i+1}$ *doubler edges*.

Theorem 10.2 (Power-of-two property). Let $A \in \mathcal{U}_k$ with Perron vector v , and suppose the vertex i has out-degree 1 with sole outgoing edge $i \rightarrow j$. Then

$$v_j = 2v_i.$$

Consequently, along any doubler chain $c_0 \rightarrow \cdots \rightarrow c_m$ the Perron values satisfy $v_{c_j} = 2^j v_{c_0}$ for $0 \leq j \leq m$.

Proof. The i -th row of the equation $Av = 2v$ reads

$$\sum_{\ell \in \text{out}(i)} v_\ell = 2v_i.$$

Since $\text{out}(i) = \{j\}$, this is $v_j = 2v_i$. The statement along a chain follows by induction. $\square$

Theorem 10.2 is a conditional statement: it applies to any vertex of out-degree 1. That every member of $\mathcal{U}_k$ has such a vertex is Conjecture 9.4, which is supported by all known members but is not derived from the axioms.

Corollary 10.3 (Chain arithmetic). Along a doubler chain of length m the Perron values have the form $a, 2a, 4a, \dots, 2^m a$ for some positive integer a ; in particular the values along a chain are totally ordered and each divides the next. Consequently a chain of length m forces $\min(v) \leq v_{c_0}$ and $2^m v_{c_0} \leq \max(v)$.

10.2 Doubler chains in the representatives

Table 6 lists the doubler structure of the representatives of Section 3; all values are exact.

Table 6: Doubler chains of the representatives $W_1, \dots, W_7$. The column “#” counts the vertices of out-degree 1 (the number of doubler edges); the last column compares it with $2k$.

Member	Doubler chains (Perron values in parentheses)	#/ $2k$
W_1	$1 \rightarrow 3$ (3, 6); $5 \rightarrow 2$ (1, 2)	2/2
W_2	$0 \rightarrow 2$ (4, 8); $1 \rightarrow 5$ (2, 4); $7 \rightarrow 4 \rightarrow 3$ (1, 2, 4)	4/4
W_3	$2 \rightarrow 0$ (1, 2); $4 \rightarrow 8$ (2, 4); $7 \rightarrow 6$ (1, 2); $9 \rightarrow 6$ (1, 2)	4/6
W_4	$2 \rightarrow 9$ (35, 70); $3 \rightarrow 7 \rightarrow 4 \rightarrow 12$ (27, 54, 108, 216); $8 \rightarrow 10$ (18, 36); $11 \rightarrow 4$ (54, 108)	6/8
W_5	$2 \rightarrow 9$ (148, 296); $8 \rightarrow 10$ (120, 240); $13 \rightarrow 11$ (180, 360); $14 \rightarrow 5$ (279, 558)	4/10
W_6	$2 \rightarrow 9$ (148, 296); $8 \rightarrow 10$ (120, 240); $14 \rightarrow 5$ (279, 558); $15 \rightarrow 3 \rightarrow 7$ (90, 180, 360); $16 \rightarrow 13 \rightarrow 11$ (90, 180, 360)	7/12
W_7	$1 \rightarrow 13$ (1034, 2068); $3 \rightarrow 11 \rightarrow 5 \rightarrow 9$ (404, 808, 1616, 3232); $4 \rightarrow 10$ (1414, 2828); $8 \rightarrow 14$ (1616, 3232); $15 \rightarrow 6 \rightarrow 2$ (508, 1016, 2032)	8/14

Observation 10.4. The number of out-degree-1 vertices varies non-monotonically (2, 4, 4, 6, 4, 7, 8 for $k = 1, \dots, 7$) and is strictly smaller than $2k$ for $k \geq 3$. In particular the length of the union of all doubler chains is not equal to the size $2k$ of the nilpotent Jordan block; the doubler chains account for only part of the nilpotent subspace V_0 . Identifying the remaining part of V_0 is an open problem (Problem 17.2). The partition of the number of doubler edges into chain lengths also varies among members of the same order, confirming that this is a flexible feature.

10.3 Three-layer architecture

The evidence accumulated over all known members suggests the following organisation of the structural features of the Wen class.

Layer 1: rigid invariants. Constant across all known members; either axioms or proven consequences of D1–D5: characteristic polynomial; Jordan form; left Perron vector $\mathbf{1}$; closed-walk count $N_L = 2^L + 1 + 3(-1)^L$; spectral radius $\rho = 2$; rank formula $\text{rank}(A^m) = \max(n - m, 5)$; minimal polynomial degree $n - 1$; commutant dimension $7 + 2k$; exactly 4 bidirectional edge pairs; exactly 2 directed 3-cycles; zero diagonal; column sums equal to 2.

Layer 2: semi-invariants. Governed by proven theorems but with realisation-dependent values: the power-of-two structure of Perron values along doubler chains (Theorem 10.2); the number of doubler chains (growing with k but not determined by it); the sum of the Perron entries (frequently, but not always, a multiple of a power of two times 9).

Layer 3: flexible realisation. Free to vary among members of the same order: the partition of the doubler edges into chains; the out-degree sequence; the adjacency pattern among non-chain vertices; the exact Perron vector; the minimum entry of the Perron vector; the Smith normal form of $A - 2I$.

The rigidity–flexibility duality expressed by this architecture is the structural signature of the family: the spectral data are pinned down completely, while the graph-theoretic realisation of those data admits many non-isomorphic solutions at each order. The infinite family of Section 14 sits at the rigid end of Layer 3: its members are obtained from a single fixed core by a deterministic chain-insertion rule, so the entire family is governed by one 7×7 matrix.

11 Falsified Properties

This section records two natural conjectures that the accumulated evidence suggested but that are in fact false. Both are disproved by explicit counterexamples drawn from the verified members, so the falsifications are unconditional. They are recorded because they constrain what a general construction can look like (Section 15).

11.1 The minimum-one Perron vector property

For W_1, W_2, W_3 the primitive integer Perron vector has minimum entry 1. This suggested:

Conjecture 11.1 (Minimum-one property — false). Every $A \in \mathcal{U}_k$ has a Perron eigenvector $v > 0$ with integer entries, $\gcd(v) = 1$, and $\min_i v_i = 1$.

Proposition 11.2. *Conjecture 11.1 holds for W_1, W_2, W_3 and fails for W_4 . Consequently it is false in general.*

Proof. The Perron vector of W_4 (computed exactly as a generator of the one-dimensional rational null space of $A - 2I$, cleared of denominators and divided by the gcd) is

$$v = (121, 170, 35, 27, 108, 165, 78, 54, 18, 70, 36, 54, 216)^T, \quad \gcd(v) = 1, \quad \min(v) = 18.$$

Since $\dim \ker(A - 2I) = 1$ by Proposition 4.1, every real Perron vector is a scalar multiple rv of this one. For rv to have integer entries with $\min(rv) = 1$ we need $r = 1/18$; but $121/18 \notin \mathbb{Z}$, so no such r exists. Hence W_4 violates the minimum-one property. For W_1, W_2, W_3 the primitive vectors displayed in Section 3 have minimum entry 1, so the property holds there. $\square$

Remark 11.3. The failure first occurs at $k = 4$, and the minimum entry of the representatives grows irregularly thereafter (1, 1, 1, 18, 120, 90, 404 for $k = 1, \dots, 7$). Thus the minimum is not a function of k , and it is not even monotone in k . The arithmetic of v is therefore a genuinely flexible feature (Layer 3).

Observation 11.4. Over the 233 matrices of the searched data set, exactly 22 have $\min(v) = 1$, and they occur only at $k = 1, 2, 3$:

k	1	2	3	4	5	6	7
$\#\{\min(v) = 1\}$	13	6	3	0	0	0	0
$\#\{\text{members}\}$	34	50	39	22	68	13	7
range of $\min(v)$	[1, 3]	[1, 12]	[1, 27]	[5, 60]	[5, 240]	[4, 734]	[32, 840]

Thus the minimum-one property is a low-order phenomenon: it holds for a substantial minority of members at $k \leq 3$ and for none of the 110 members found at $k \geq 4$. It also fails for every member of the infinite family of Section 14, all of which have $\min(v) = 3$ (Remark 14.26). Whether $\min(v) = 1$ holds for infinitely many k is open (Problem 17.5).

11.2 The core-template hypothesis

The most natural route to a general construction would be the following. Since the non-zero spectrum has dimension 5 and the nilpotent block has dimension $2k$, one might hope that every member is obtained by attaching doubler chains of total length $2k$ to a fixed 5-vertex core subgraph, the core carrying the entire non-zero spectrum.

Conjecture 11.5 (Core-template hypothesis — false). There exists a directed graph H on 5 vertices such that every $A \in \mathcal{U}_k$ (for every $k \geq 1$) contains an induced subgraph isomorphic to H .

Observation 11.6. Conjecture 11.5 is false: at $k = 1$ and at $k = 2$ the number of isomorphism types of 5-vertex induced subgraphs that occur in every member of the searched data set is 0. Moreover, if one defines the *core size* of a member as the number of vertices lying on no doubler chain (Definition 10.1), the core size is far from constant: across the 233 searched members it ranges over the sets

k	observed core sizes (with multiplicities)
1	{1:6, 2:2, 3:21, 4:1, 5:4}
2	{1:2, 2:10, 3:13, 4:12, 5:11, 7:2}
3	{0:1, 1:1, 2:4, 3:10, 4:8, 5:4, 6:8, 7:3}
4	{3:3, 4:4, 5:7, 6:7, 7:1}
5	{3:6, 4:11, 5:16, 6:14, 7:11, 8:7, 9:2, 10:1}
6	{4:1, 5:3, 6:2, 7:2, 9:4, 10:1}
7	{3:1, 5:1, 6:1, 7:2, 9:2}

where “ $s:c$ ” means that c members have core size s . The core size is not determined by k , and it exceeds the spectral value 5 for many members.

Remark 11.7. Observation 11.6 is a computational finding on the searched data set, not a consequence of the axioms. It rules out the simplest conceivable construction and explains why none of the obvious template-based approaches can succeed. Note that the infinite family of Section 14 is not a counterexample to the *spirit* of the hypothesis: it does have a fixed subgraph that carries all the non-zero spectrum, but that subgraph has 7 vertices, not 5, and it is not an induced subgraph carrying the spectrum by itself—the chain vertices feed back into it (the edge $a_1 \rightarrow 6$). The correct statement is that the interface between the fixed part and the growing part consists of a bounded number of edges, not that the fixed part is an induced subgraph of size 5.

12 Density Model and Non-monotonicity

Let

$$\Omega(k) = \{ A \in \{0, 1\}^{n \times n} : \text{D1 and D2 hold} \}, \quad n = 2k + 5,$$

be the space in which the search takes place, and let $\hat{c}(k)$ denote the number of pairwise non-isomorphic members of $\mathcal{U}_k$ known to us (Definition 2.1). Since each column must contain exactly two ones and the diagonal entry is forced to be zero, $|\Omega(k)| = \binom{n-1}{2}^n$. Define the *solution density*

$$\rho(k) := \frac{\hat{c}(k)}{|\Omega(k)|}.$$

Because $\hat{c}(k) \leq c(k)$, the quantity $\rho(k)$ is a lower bound for the true density $c(k)/|\Omega(k)|$.

Table 7: Solution density. $\hat{c}(k)$ is the number of pairwise non-isomorphic known members; it is a lower bound for $c(k)$ at every order (Remark 3.2). $|\Omega(k)| = \binom{n-1}{2}^n$ is exact.

k	n	$\hat{c}(k)$	$ \Omega(k) $	$\rho(k)$	$\log_{10} \rho(k)$
1	7	35	1.71×10^8	2.05×10^{-7}	-6.69
2	9	52	1.06×10^{13}	4.92×10^{-12}	-11.31
3	11	41	1.53×10^{18}	2.68×10^{-17}	-16.57
4	13	23	4.51×10^{23}	5.10×10^{-23}	-22.29
5	15	70	2.43×10^{29}	2.88×10^{-28}	-27.54
6	17	15	2.22×10^{35}	6.76×10^{-35}	-34.17
7	19	8	3.23×10^{41}	2.48×10^{-41}	-40.61
8	21	1	—	—	—

12.1 Solution density

Table 7 collects the data. The successive differences of $\log_{10} \rho$ are approximately

$$-4.6, -5.3, -5.7, -5.2, -6.7, -6.4,$$

so the decline accelerates overall but not monotonically. A quadratic fit $\log_{10} \rho \approx -0.178k^2 - 4.233k - 2.244$ reproduces the seven points with $R^2 \approx 0.9998$, but with seven data points—all of which are lower bounds—the fit cannot distinguish a quadratic decline from a mildly super-linear one. We do not use it for any extrapolation.

12.2 The non-monotonicity mechanism

The counts $\hat{c}(k)$ result from two competing forces: $|\Omega(k)|$ grows like $(n^2/2)^n$, i.e. super-exponentially in k , while $\rho(k)$ declines. The competition explains why the counts first rise: for small k the growth of the search space dominates. It does *not* explain the observed profile in detail. Over the searched range the sequence 34, 50, 39, 22, 68, 13, 7 has a local maximum at $k = 2$, a local minimum at $k = 4$, and its global maximum at $k = 5$, where 68 members were found—more than at $k = 2$. A monotone density decline would produce a unimodal profile; the secondary maximum at $k = 5$ shows that the profile is not governed by density alone. The most plausible explanation is an artefact of the search strategy: $k = 5$ is the first order at which the $n - 2$ extension strategy was applied, and that strategy explores a structured subspace with a substantially higher concentration of solutions than the uniform measure on $\Omega(k)$. This is a statement about our search, not about $c(k)$; we cannot exclude that the true $c(k)$ has a different profile.

Remark 12.1 (What infinitude does and does not say about $c(k)$). Theorem 14.21 below proves $c(k) \geq 1$ for every $k \geq 1$. It says nothing about the growth of $c(k)$: the one member per order produced by the infinite family is negligible compared with $|\Omega(k)|$, and the data of Table 7 are consistent both with $c(k) \rightarrow \infty$ and with $c(k)$ being eventually bounded. The question of the asymptotic behaviour of $c(k)$ therefore remains open (Problem 17.1).

12.3 Arithmetic of the Perron vector

Beyond the power-of-two structure along doubler chains, the integer Perron vector exhibits two further regularities that hold for almost all known members but not for all. We write $\text{SNF}(M) = \text{diag}(d_1, \dots, d_n)$ with $d_1 \mid d_2 \mid \dots \mid d_n$ for the Smith normal form of an integer matrix M [16].

Observation 12.2. Across the 233 matrices of the searched data set:

- (i) $\sum_i v_i$ is divisible by 3 for 230 of the 233 members (98.7%) and by 9 for 206 of them (88.4%);
- (ii) the largest invariant factor of $\text{SNF}(A - 2I)$ is divisible by 3 for all 233 members, taking values in $\{3, 6, 9, 12, 18, 24\}$;
- (iii) the quantity

$$\left(\sum_i v_i\right) \cdot \prod_{d_i > 1} d_i,$$

equals $4^k \cdot 27$ for every one of the 233 members, with no exception:

$$108, 432, 1728, 6912, 27648, 110592, 442368 \quad (k = 1, \dots, 7).$$

Part (iii) is a remarkably rigid identity: it ties the integer lattice structure of $\ker(A - 2I)$ to the order parameter k through a pure power of 4. It has been verified on the whole searched data set but no proof from D1–D5 is known; whether it is a theorem or a coincidence of the range $k \leq 7$ is open (Problem 17.4). Part (ii) is consistent with the algebraic multiplicity 3 of the eigenvalue -1 , but again no mechanism has been established.

13 Jordan Splitting Analysis

Axiom D4 is strictly stronger than D3, and the gap between them is the main obstacle to finding members by search. This section quantifies the gap. Throughout, “a D3-matrix” means a matrix satisfying D1, D2 and D3.

13.1 D3 versus D4: the true bottleneck

Write

$$\#\{\text{D4-matrices at order } k\} = \#\{\text{D3-matrices at order } k\} \times P(\text{D4} \mid \text{D3}).$$

Monte-Carlo sampling of 4000 D3-matrices per order at $k = 1, 2, 3$ gives the estimates of Table 8.

Table 8: Empirical $P(\text{D4} \mid \text{D3})$ and the two failure modes, from 4000 random D3-matrices per order at $k = 1, 2, 3$. The two mode columns are shares of the *D4-failures*, not of all sampled matrices; the last column is derived in the text by inclusion–exclusion.

k	$P(\text{D4} \mid \text{D3})$	zero-block failure	(-1) -block failure	$P(\text{both} \mid \text{failure})$
1	0.346	40% (splits)	80% (merges)	20%
2	0.291	64% (splits)	61% (merges)	25%
3	0.185	77% (splits)	55% (merges)	32%

$P(\text{D4} \mid \text{D3})$ declines with k , confirming that D4 is increasingly restrictive. These figures describe the uniform measure on the D3-fibre. The $n - 2$ extension strategy of Section 8 searches a structured subspace with a much higher concentration of solutions, which is why it succeeds at $k = 7$ where uniform sampling is hopeless.

13.2 Nilpotent block splitting

When D4 fails, the nilpotent block $J_{2k}(0)$ is observed to split in all sampled cases. In rank terms the dominant failure is

$$\text{rank}(A) = n - 2 \quad \text{instead of} \quad n - 1,$$

i.e. the geometric multiplicity of the eigenvalue 0 increases from 1 to 2; the dominant split type is a minimal fragmentation $J_1(0) \oplus J_{2k-1}(0)$. Table 9 summarises the observations at $k = 1, 2, 3$.

Table 9: Dominant nilpotent splitting patterns among D4-failures ($k = 1, 2, 3$ sampling).

k	required	dominant split	frequency	largest block
1	$J_2(0)$	$2 \times J_1(0)$	40%	1
2	$J_4(0)$	$J_1(0) \oplus J_3(0)$	52%	3
3	$J_6(0)$	$J_1(0) \oplus J_5(0)$	46%	5

13.3 The (-1) -eigenvalue: opposite tendency

The eigenvalue -1 fails in the opposite direction: instead of splitting into $J_2(-1) \oplus J_1(-1)$ it merges into $J_3(-1)$, so that $\dim \ker(A + I) = 1$ instead of 2. This occurs in 55%–80% of the sampled D4-failures at $k = 1, 2, 3$ (Table 8); the share *decreases* with k , while the share of nilpotent splittings *increases* with k .

The two modes are the only ways in which D4 can fail once D3 holds: the eigenvalues 2 and 1 have algebraic multiplicity 1 by D3 and are therefore automatically realised by $J_1(2)$ and $J_1(1)$. Hence, writing Z for the event “the nilpotent block splits” and M for the event “the (-1) -blocks merge”, we have $Z \cup M = \{\text{D4 fails}\}$ and therefore, by inclusion–exclusion conditional on failure,

$$P(Z \cap M \mid Z \cup M) = P(Z \mid Z \cup M) + P(M \mid Z \cup M) - 1,$$

which gives the last column of Table 8: 20%, 25%, 32%. Comparing with the value predicted by independence, $P(Z \mid Z \cup M)P(M \mid Z \cup M)$, namely 0.32, 0.39 and 0.42 respectively, we find

$$P(Z \cap M \mid Z \cup M) < P(Z \mid Z \cup M)P(M \mid Z \cup M) \quad (k = 1, 2, 3),$$

i.e. the two failure modes are anti-correlated. Informally, when the nilpotent block splits the (-1) -block tends to be correct, and vice versa, as if a fixed “Jordan complexity budget” had to be distributed among the eigenvalues. The margin is modest (the ratio of the two sides is 0.63, 0.64, 0.76 at $k = 1, 2, 3$), so the effect is real in the sample but not dramatic. Formalising it is Problem 17.6.

13.4 Single-block survival

The probability that a D3-matrix retains a single nilpotent block is recorded in Table 10, together with the baseline $1/p(2k)$ obtained by assuming that all partitions of $2k$ into Jordan block sizes are equally likely (p is the partition function, see [20]). D3 favours single blocks, and the enhancement factor grows from $1.5\times$ to $4.1\times$.

Table 10: Probability that a D3-matrix has a single nilpotent block, against the $1/p(2k)$ baseline.

k	$P(\text{single block} \mid \text{D3})$	$1/p(2k)$	enhancement
1	0.738	0.500	$1.5\times$
2	0.546	0.200	$2.7\times$
3	0.370	0.091	$4.1\times$

The three data points are fit by $P \approx C \exp(-\alpha\sqrt{2k})$ with $\alpha \approx 0.58$, a sub-exponential decline; with only three points, an exponential fit $\exp(-ck)$ fits comparably well. The distinction matters heuristically—a sub-exponential decline would let the super-exponential growth of the number of D3-matrices win, suggesting $c(k) \rightarrow \infty$, whereas an exponential decline could reverse the conclusion—but it cannot be settled with the available data. Sampling at $k \geq 4$ is prohibitively expensive, since the density there is of order 10^{-23} or smaller.

13.5 Do the invariants determine the count?

A natural question is whether the combinatorial invariants of Section 9 (4 bidirectional pairs, 2 directed 3-cycles, $\text{tr}(A^2) = 8$, $\text{tr}(A^3) = 6$) already force membership. They do not, by a wide margin: the number of D1+D2 matrices satisfying $\text{tr}(A^2) = 8$ and $\text{tr}(A^3) = 6$ exceeds the observed count by a factor of about 10^2 at $k = 1$ and about 10^{16} at $k = 4$. The invariants are necessary but far from sufficient; the binding constraint is D4, which no collection of trace identities can capture, since D4 is not a statement about the characteristic polynomial but about the rank sequence (Proposition 2.7).

13.6 A constraint-counting heuristic

The spectral constraints imposed by D3–D4 may be counted as $2k + 4$ —namely the $2k$ rank conditions $\text{rank}(A^m) = n - m$ for $1 \leq m \leq 2k$, plus the four conditions pinning the non-zero spectrum—while A has $(2k + 5)^2$ entries. The constraint-to-entry ratio tends to zero, so the system becomes increasingly under-determined and one might expect solutions for all k .

This is an analogy, not a proof. Under-determined polynomial systems over $\mathbb{Z}$ need not have solutions, and the constraints here involve rank conditions and the integrality of the entries, not merely polynomial equations. The heuristic is nevertheless useful for two reasons. First, it explains why the family is hard to find by search: the constraints are not numerous enough to guide a search, yet they are rigid enough to make random candidates fail. Second, it points in the direction that Section 14 makes rigorous: the way to exploit an under-determined system is not to count constraints but to exhibit an explicit solution. We record the heuristic for its explanatory value only; no result in this paper depends on it.

14 Infinitude: An Explicit Infinite Subfamily

This section proves the main result of the paper: $\mathcal{U}_k \neq \emptyset$ for every integer $k \geq 1$. The proof is entirely constructive and completely explicit; every computation is carried out on matrices of order at most 7 (or at most 9 for the base case), so the entire argument can be re-verified by hand.

Throughout this section vertices are labelled $0, 1, \dots, n-1$ and we use the convention (2). We write $e_0, \dots, e_{n-1}$ for the standard basis of $\mathbb{R}^n$ (indexed from 0). Since $\dim \ker B = \dim \ker B^\top$ for every square B , all nullity computations are performed on the transpose, i.e. in terms of in-neighbourhoods.

14.1 Strategy

The construction has three ingredients.

1. A fixed *core* C of order 7, which is not itself a member of the class (its last column sums to 1, not to 2).
2. A *base member* A_1 of order 9 ($k = 2$) whose leading 7×7 block is C and whose last two vertices form the first link of a chain that feeds back into the core.
3. A *chain-insertion operator* T that takes a member of order n and returns a candidate of order $n + 2$, by cutting one edge $4 \rightarrow n - 2$ and splicing in two new vertices.

The operator T is designed so that:

- D1 and D2 are preserved automatically, by a purely combinatorial argument (Lemma 14.5);
- D5 is preserved automatically, provided the deleted edge starts at one of the two vertices that point into the first new vertex (Proposition 14.18);

- the characteristic polynomial is multiplied by z^2 , provided a certain resolvent identity holds (Theorem 14.10 and Corollary 14.11).

The decisive point is the third. The resolvent identity turns out to be independent of m . Because the interface between the core and the chain consists of a bounded number of edges, the identity reduces to a statement about three cofactors of the single matrix $zI - C^\top$, and those three cofactors satisfy a linear relation that makes the whole expression vanish identically. This is what makes the induction close.

14.2 The core, the base case, and the operator

Definition 14.1 (The core). Let

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The diagonal of C is zero, and the column sums of C are

$$2, 2, 2, 2, 2, 2, 1.$$

The last column sums to 1 rather than 2: inside the core, vertex 6 has the single in-neighbour 1, and its second in-neighbour is a chain vertex (the edge $a_1 \rightarrow 6$ of Definition 14.4 below). Thus C itself satisfies D1 but not D2. The in-neighbourhoods inside the core are recorded once and for all:

$$\begin{array}{c|ccccccc} j & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \text{in}_C(j) & \{1, 2\} & \{0, 6\} & \{0, 1\} & \{5, 6\} & \{3, 5\} & \{2, 3\} & \{1\} \end{array} \quad (13)$$

Definition 14.2 (The base case). Let

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Definition 14.3 (Chain-insertion operator). Let A be a $\{0, 1\}$ -matrix of order $n \geq 9$ with $\text{out}(4) = \{n - 2\}$. Put $b := n$, $a := n + 1$, and define $M = T(A)$ to be the matrix of order $n + 2$ obtained from A by

deleting the edge $(4, n - 2)$, adding the edges $(4, b)$, $(6, b)$, (b, a) , $(5, a)$, $(a, n - 2)$.

In graph terms, the chain $\cdots \rightarrow 4 \rightarrow n - 2$ is replaced by $\cdots \rightarrow 4 \rightarrow b \rightarrow a \rightarrow n - 2$, and the core vertices 6 and 5 are attached to b and a respectively.

Definition 14.4 (The family). For $t \geq 1$ set

$$b_t := 5 + 2t, \quad a_t := 6 + 2t.$$

Define

$$A_1 \text{ as in Definition 14.2,} \quad A_{m+1} := T(A_m) \quad (m \geq 1).$$

Thus A_m has order $n = 2m + 7$ and $k = m + 1$; if A has order $n = 2m + 7$ then $b_m = 2m + 5 = n - 2$ and $a_m = 2m + 6 = n - 1$, so the chain occupies exactly the last $2m$ vertices.

14.3 Structural induction: D1 and D2

Lemma 14.5. *Let A have order $n = 2m + 7$ with $m \geq 1$ and satisfy:*

(P1) $A[0 : 7, 0 : 7] = C$;

(P2) $\text{in}(a_t) = \{5, b_t\}$ for $1 \leq t \leq m$; $\text{in}(b_t) = \{6, a_{t+1}\}$ for $t < m$; $\text{in}(b_m) = \{4, 6\}$; $\text{in}(6) = \{1, a_1\}$; and the in-neighbours of the vertices $0, 1, \dots, 5$ all lie in the core, given by (13);

(P3) $\text{out}(4) = \{b_m\}$;

(P4) A satisfies D1 and D2.

Then $M = T(A)$ satisfies (P1)–(P4) with m replaced by $m + 1$.

Proof. Write $c := b_m = n - 2$ (legitimate by (P3)).

(P1). The positions at which M differs from A are exactly

$$(4, n - 2), (4, n), (6, n), (n, n + 1), (5, n + 1), (n + 1, n - 2).$$

A position (r, s) lies in the leading 7×7 block if and only if $r < 7$ and $s < 7$. Since $m \geq 1$ we have $n = 2m + 7 \geq 9$, hence $n - 2 \geq 7$, $n \geq 9$, $n + 1 \geq 10$. Inspecting the six positions: $(4, n - 2)$ has $s \geq 7$; $(4, n)$ and $(6, n)$ have $s \geq 9$; $(n, n + 1)$ has $r \geq 9$; $(5, n + 1)$ has $s \geq 10$; $(n + 1, n - 2)$ has $r \geq 10$. None lies in the leading block, so (P1) persists.

(P2). Directly from the definition of T :

$$\text{in}_M(n + 1) = \{n, 5\} = \{b_{m+1}, 5\}, \quad \text{in}_M(n) = \{4, 6\} = \text{in}_M(b_{m+1}),$$

$$\text{in}_M(n - 2) = (\text{in}_A(b_m) \setminus \{4\}) \cup \{n + 1\} = \{6, a_{m+1}\},$$

while for $t < m$ the sets $\text{in}(a_t)$, $\text{in}(b_t)$ and $\text{in}(6) = \{1, a_1\}$ are untouched, and the in-neighbours of $0, \dots, 5$ remain in the core. Since $b_{m+1} = 5 + 2(m + 1) = 2m + 7 = n$ and $a_{m+1} = 6 + 2(m + 1) = n + 1$, this is exactly (P2) at level $m + 1$.

(P3). $\text{out}_M(4) = (\{b_m\} \setminus \{b_m\}) \cup \{b_{m+1}\} = \{b_{m+1}\}$.

(P4), D1. All five new positions are off-diagonal: $4 \neq n$, $6 \neq n$, $n \neq n + 1$, $5 \neq n + 1$, and $n + 1 \neq n - 2$ (they differ by 3). Deleting an edge cannot create a diagonal entry.

(P4), D2. Only three columns change. Column $n - 2$ loses the entry from 4 and gains the entry from $n + 1$, so its sum remains 2; column n gains the entries from 4 and 6, so its sum is 2; column $n + 1$ gains the entries from n and 5, so its sum is 2. All other columns are unchanged. (Each of the five new positions has row or column index n or $n + 1$, so it was zero in A ; hence no double edges are created.) $\square$

Lemma 14.6 (Base case). A_1 satisfies (P1)–(P4) with $m = 1$; in particular A_1 satisfies D1 and D2.

Proof. The leading 7×7 block of A_1 is C by inspection, giving (P1). Reading the columns of A_1 :

$$\text{in}(8) = \{5, 7\} = \{5, b_1\}, \quad \text{in}(7) = \{4, 6\} = \{4, 6\} = \text{in}(b_1), \quad \text{in}(6) = \{1, 8\} = \{1, a_1\},$$

and the in-neighbours of $0, \dots, 5$ are those listed in (13), giving (P2). Row 4 of A_1 has a single one, in column $7 = b_1$, giving (P3). The diagonal of A_1 is zero and every column of A_1 sums to 2, giving (P4). $\square$

Corollary 14.7. *For every $m \geq 1$, A_m satisfies (P1)–(P4); in particular A_m satisfies D1 and D2. Moreover the explicit description of A_m underlying (P2) holds for all m .*

Proof. Immediate from Lemma 14.6 and induction on m using Lemma 14.5. $\square$

Remark 14.8. By (P3), vertex 4 has out-degree 1 in every A_m ; thus the infinite family always contains a vertex of out-degree 1 and hence a doubler chain, in agreement with Conjecture 9.4. Its out-neighbour is $b_m = n - 2$, the far end of the chain.

14.4 A Schur complement identity for T

Definition 14.9 (Setting for the identity). Let A be a matrix of order n , let $i, c, r_1, r_2, r_a \in \{0, \dots, n - 1\}$, and assume

(H1) $A[i, c] = 1$;

(H2) $r_1 \neq r_2$.

Put $b := n$, $a := n + 1$, $w := e_{r_1} + e_{r_2}$, $A_0 := A - e_i e_c^\top$, and

$$M = \begin{bmatrix} A_0 & w & e_{r_a} \\ 0 & 0 & 1 \\ e_c^\top & 0 & 0 \end{bmatrix}, \quad (14)$$

the block partition being $n \mid 1 \mid 1$ in both rows and columns; the visible bottom-left and bottom-right entries refer to the b - and a -rows and columns. Concretely, b receives edges from r_1 and r_2 and sends one edge to a ; a receives edges from b and r_a and sends one edge to c ; and the entry $A[i, c]$ has been changed from 1 to 0.

By (H1) and (H2), the matrix M of (14) is exactly $T(A)$ when $i = r_1 = 4$, $r_2 = 6$, $r_a = 5$, $c = n - 2$: indeed $T(A)$ sets $M[4, n - 2] = 0$, which agrees with $A_0[4, n - 2] = A[4, n - 2] - 1 = 0$, and adds the five edges listed in Definition 14.3. This is the only place where (H1) is used. Hypothesis (H2) guarantees that the two edges $r_1 \rightarrow b$ and $r_2 \rightarrow b$ are distinct, so that $w = e_{r_1} + e_{r_2}$ counts each once.

Theorem 14.10 (Schur complement identity). *In the setting of Definition 14.9, let $p_A(z) = \det(zI - A)$. Then, as an identity of polynomials,*

$$\det(zI - M) = z^2 p_A(z) + z^2 f_i(z) - H_b(z) - z H_a(z), \quad (15)$$

where

$$\begin{aligned} f_i(z) &= \text{adj}(zI - A)[c, i], & H_b(z) &= \text{adj}(zI - A)[c, r_1] + \text{adj}(zI - A)[c, r_2], \\ H_a(z) &= \text{adj}(zI - A)[c, r_a]. \end{aligned}$$

Proof. Partition $zI - M$ as $\begin{bmatrix} D & U \\ V & B \end{bmatrix}$ with the leading block of size n and the trailing block of size 2. From (14) the trailing block is

$$B = \begin{bmatrix} z & -1 \\ 0 & z \end{bmatrix}, \quad \det B = z^2 \neq 0, \quad B^{-1} = \frac{1}{z^2} \begin{bmatrix} z & 1 \\ 0 & z \end{bmatrix}.$$

Since B is invertible as a matrix over the field $\mathbb{R}(z)$, the Schur complement formula (Lemma .2) gives

$$\det(zI - M) = \det B \cdot \det(D - UB^{-1}V) = z^2 \det((zI - A_0) - UB^{-1}V),$$

where $D = zI - A_0$. Reading off the blocks,

$$U = [-w, -e_{r_a}] \in \mathbb{R}[z]^{n \times 2}, \quad V = \begin{bmatrix} 0^\top \\ -e_c^\top \end{bmatrix} \in \mathbb{R}[z]^{2 \times n},$$

because $(zI - M)[b, j] = 0$ and $(zI - M)[a, j] = -\delta_{jc}$ for the old columns $j < n$. Hence

$$UB^{-1}V = -\frac{1}{z^2} [w, e_{r_a}] \begin{bmatrix} z & 1 \\ 0 & z \end{bmatrix} \begin{bmatrix} 0^\top \\ -e_c^\top \end{bmatrix} = \frac{1}{z^2} (w + ze_{r_a}) e_c^\top.$$

Therefore

$$\begin{aligned} \det(zI - M) &= z^2 \det \left((zI - A_0) - \frac{1}{z^2} (w + ze_{r_a}) e_c^\top \right) \\ &= z^2 \det(zI - A_0) - e_c^\top \operatorname{adj}(zI - A_0) (w + ze_{r_a}), \end{aligned}$$

the last step by the matrix determinant lemma (Lemma .1), $\det(D + uv^\top) = \det D + v^\top \operatorname{adj}(D)u$, applied with $D = zI - A_0$, $u = -\frac{1}{z^2}(w + ze_{r_a})$, $v = e_c$, and then multiplied through by z^2 .

It remains to replace A_0 by A . Since $zI - A_0 = (zI - A) + e_i e_c^\top$, the same lemma gives

$$\det(zI - A_0) = \det(zI - A) + e_c^\top \operatorname{adj}(zI - A) e_i = p_A(z) + \operatorname{adj}(zI - A)[c, i]. \quad (16)$$

For the adjugate entries, fix any j . By definition

$$\operatorname{adj}(zI - A_0)[c, j] = (-1)^{c+j} \det((zI - A_0) \text{ with row } j \text{ and column } c \text{ deleted}).$$

The matrices $zI - A_0$ and $zI - A$ differ only in column c , and column c is deleted in the minor above; hence the two minors coincide and

$$\operatorname{adj}(zI - A_0)[c, j] = \operatorname{adj}(zI - A)[c, j] \quad \text{for every } j. \quad (17)$$

Substituting (16) and (17) into the previous display yields

$$\det(zI - M) = z^2 (p_A + \operatorname{adj}(zI - A)[c, i]) - \operatorname{adj}(zI - A)[c, r_1] - \operatorname{adj}(zI - A)[c, r_2] - z \operatorname{adj}(zI - A)[c, r_a],$$

which is (15). $\square$

Corollary 14.11 (Spectrum-preservation criterion). *In the setting of Definition 14.9, work over the field $\mathbb{R}(z)$ and put $R(z) := (zI - A)^{-1}$ and $\rho_j(z) := R(z)[c, j]$. Define*

$$E(z) := z^2 \rho_i - \rho_{r_1} - \rho_{r_2} - z \rho_{r_a}.$$

Then

$$\det(zI - M) - z^2 p_A(z) = p_A(z) E(z),$$

and consequently, since $p_A \neq 0$,

$$\det(zI - M) = z^2 p_A(z) \iff E(z) \equiv 0.$$

Proof. $\operatorname{adj}(zI - A) = p_A(z)(zI - A)^{-1}$, so $\operatorname{adj}(zI - A)[c, j] = p_A(z)\rho_j(z)$ for every j . Insert this into (15):

$$\det(zI - M) = z^2 p_A + z^2 p_A \rho_i - p_A \rho_{r_1} - p_A \rho_{r_2} - z p_A \rho_{r_a} = z^2 p_A + p_A (z^2 \rho_i - \rho_{r_1} - \rho_{r_2} - z \rho_{r_a}).$$

Rearranging gives the claim. $\square$

For our family we specialise to

$$i = r_1 = 4, \quad r_2 = 6, \quad r_a = 5, \quad c = b_m = n - 2, \quad (18)$$

so that, writing E_m for the corresponding function,

$$E_m(z) = (z^2 - 1)\rho_4(z) - \rho_6(z) - z\rho_5(z), \quad (19)$$

the term ρ_4 appearing twice because $i = r_1 = 4$.

14.5 The characteristic polynomial: D3 for all m

Lemma 14.12 (Resolvent equations and reduction to the core). *Fix $m \geq 1$, let $A = A_m$ and $c = b_m$. With $\rho_j = R[c, j]$ as above, the row of index c in the identity $R(z)(zI - A) = I$ gives, for every vertex k ,*

$$z\rho_k - \sum_{j \in \text{in}(k)} \rho_j = \delta_{c,k}. \quad (20)$$

Consequently, writing $\alpha := \rho_{a_1}$ and $\rho_C := (\rho_0, \dots, \rho_6)^\top$,

$$N\rho_C = \alpha e_6, \quad N := zI - C^\top. \quad (21)$$

In particular $\rho_C = \alpha g$ where $g := N^{-1}e_6$ is independent of m , and

$$E_m(z) = \frac{\alpha(z)}{\det N(z)} P(z), \quad P(z) := (z^2 - 1) \text{adj}(N)[4, 6] - \text{adj}(N)[6, 6] - z \text{adj}(N)[5, 6]. \quad (22)$$

Proof. Equation (20) is the (c, k) entry of $R(z)(zI - A) = I$:

$$\sum_j R[c, j](z\delta_{jk} - A[j, k]) = z\rho_k - \sum_{j \in \text{in}(k)} \rho_j.$$

For k in the core $\{0, \dots, 6\}$ we have $c = b_m \notin \{0, \dots, 6\}$, so the right-hand side of (20) is 0. By (P2) the only core vertex with an in-neighbour outside the core is vertex 6, whose in-neighbourhood is $\{1, a_1\}$. Hence for $k \in \{0, \dots, 6\}$

$$z\rho_k - \sum_{j \in \text{in}_C(k)} \rho_j = \begin{cases} \alpha, & k = 6, \\ 0, & k \neq 6. \end{cases}$$

Since $\sum_{j \in \text{in}_C(k)} \rho_j = \sum_j C[j, k]\rho_j = (C^\top \rho_C)_k$, this is exactly (21). (The matrix $N = zI - C^\top$ is invertible over $\mathbb{R}(z)$ because $\det N$ is the characteristic polynomial of $C^\top$, which is not identically zero.)

Finally, $\rho_C = \alpha g$ with $g = N^{-1}e_6$ and $g_j = \text{adj}(N)[j, 6]/\det N$ by Cramer's rule (Lemma .4); substituting $\rho_4 = \alpha g_4$, $\rho_5 = \alpha g_5$, $\rho_6 = \alpha g_6$ into (19) gives (22). $\square$

Proposition 14.13 (The three cofactors). *For $N = zI - C^\top$ one has*

$$\text{adj}(N)[4, 6] = z(z-1)(z+1)^2 = z^4 + z^3 - z^2 - z, \quad (23)$$

$$\text{adj}(N)[5, 6] = z(z-1)(z+1)^2 = z^4 + z^3 - z^2 - z, \quad (24)$$

$$\text{adj}(N)[6, 6] = z(z-1)(z+1)^2(z^2 - z - 1) = z^6 - 3z^4 - z^3 + 2z^2 + z. \quad (25)$$

Moreover $\det N = z(z-1)(z+1)^2(z^3 - z^2 - 2z + 1)$.

Proof. All four quantities are determinants (or signed 6×6 minors) of explicit 7×7 matrices with entries in $\{0, \pm 1, z\}$, so the computation is a finite expansion; we display the result and note that

$$\text{adj}(N)[j, 6] = (-1)^{j+6} \det(N \text{ with row 6 and column } j \text{ deleted}).$$

The displayed expansions were obtained by direct symbolic computation and are reproduced in Appendix 18. For the reader who wishes to verify them by hand: deleting row 6 and column 6 from

$$N = \begin{bmatrix} z & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & z & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & z & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & z & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & z \end{bmatrix}$$

leaves a block-triangular matrix whose determinant factors visibly, and the two off-diagonal cofactors are obtained from the same 6×6 determinants after a single column interchange. $\square$

Corollary 14.14 (The key cancellation). $P \equiv 0$. Consequently $E_m \equiv 0$ for every $m \geq 1$, and

$$\det(zI - A_{m+1}) = z^2 \det(zI - A_m) \quad \text{for every } m \geq 1. \quad (26)$$

Proof. Substituting (23)–(25) into the definition of P :

$$\begin{aligned} P(z) &= (z^2 - 1) \cdot z(z - 1)(z + 1)^2 - z(z - 1)(z + 1)^2(z^2 - z - 1) - z \cdot z(z - 1)(z + 1)^2 \\ &= z(z - 1)(z + 1)^2 \left[(z^2 - 1) - (z^2 - z - 1) - z \right] = z(z - 1)(z + 1)^2 \cdot 0 = 0, \end{aligned}$$

the bracket vanishing identically term by term: $z^2 - 1 - z^2 + z + 1 - z = 0$. By (22) this forces $E_m \equiv 0$ (as a rational function, hence as a polynomial identity after clearing the denominator $\det N$). Corollary 14.11, applied with the specialisation (18)—which is legitimate because $i = r_1 = 4 \neq 6 = r_2$ and, by (P3), $A_m[4, b_m] = 1$ —then gives $\det(zI - A_{m+1}) = z^2 \det(zI - A_m)$. $\square$

Theorem 14.15 (D3 for the family). For every $m \geq 1$,

$$\det(zI - A_m) = z^{2m+2}(z - 2)(z - 1)(z + 1)^3.$$

Since $\text{ord}(A_m) = 2m + 7$, i.e. $k = m + 1$ and $2k = 2m + 2$, this is axiom D3.

Proof. The base case is

$$\det(zI - A_1) = z^4(z - 2)(z - 1)(z + 1)^3, \quad (27)$$

a determinant of order 9 which is evaluated directly (Appendix 18). By (26) and induction,

$$\det(zI - A_m) = z^{2(m-1)} \det(zI - A_1) = z^{2m-2} \cdot z^4(z-2)(z-1)(z+1)^3 = z^{2m+2}(z-2)(z-1)(z+1)^3,$$

which is $\lambda^{2k}(\lambda - 2)(\lambda - 1)(\lambda + 1)^3$ with $2k = 2m + 2$. $\square$

14.6 $\dim \ker A_m = 1$

Proposition 14.16. $\dim \ker A_m = 1$ for every $m \geq 1$.

Proof. Compute $\ker A_m^\top$, i.e. solve $\sum_{j \in \text{in}(k)} y_j = 0$ for every vertex k . By (P2) (Corollary 14.7):

- $k = a_t$: $y_5 + y_{b_t} = 0$, so $y_{b_t} = -y_5$;
- $k = b_t$, $t < m$: $y_6 + y_{a_{t+1}} = 0$, so $y_{a_{t+1}} = -y_6$;
- $k = b_m$: $y_4 + y_6 = 0$ —a constraint on the core variables;
- $k = 6$: $y_1 + y_{a_1} = 0$, so $y_{a_1} = -y_1$ —this defines y_{a_1} and imposes no constraint on the core;
- $k \in \{0, \dots, 5\}$: $\sum_{j \in \text{in}_C(k)} y_j = 0$.

Hence all $2m$ chain variables are determined uniquely by the seven core variables $y_0, \dots, y_6$, and the core variables satisfy the 7×7 system $R_0 y = 0$ with

$$R_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{array}{l} k = 0 : y_1 + y_2 = 0 \\ k = 1 : y_0 + y_6 = 0 \\ k = 2 : y_0 + y_1 = 0 \\ k = 3 : y_5 + y_6 = 0 \\ k = 4 : y_3 + y_5 = 0 \\ k = 5 : y_2 + y_3 = 0 \\ k = b_m : y_4 + y_6 = 0 \end{array}$$

The matrix R_0 does not depend on m . We claim $\text{rank } R_0 = 6$.

$\text{rank } R_0 \geq 6$. The minor on rows $\{0, 1, 2, 3, 4, 6\}$ and columns $\{0, 1, 2, 3, 4, 5\}$ is

$$\det \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = 1 \neq 0.$$

$\text{rank } R_0 \leq 6$. The vector $y^* = (-1, 1, -1, 1, -1, -1, 1)^\top$ is nonzero and satisfies $R_0 y^* = 0$: indeed $y_1^* + y_2^* = 1 - 1 = 0$, $y_0^* + y_6^* = -1 + 1 = 0$, $y_0^* + y_1^* = -1 + 1 = 0$, $y_5^* + y_6^* = -1 + 1 = 0$, $y_3^* + y_5^* = 1 - 1 = 0$, $y_2^* + y_3^* = -1 + 1 = 0$, $y_4^* + y_6^* = -1 + 1 = 0$.

Thus $\dim \ker R_0 = 7 - 6 = 1$. Since the chain variables are uniquely determined by the core variables, $\dim \ker A_m^\top = 1$, and therefore $\dim \ker A_m = 1$. $\square$

14.7 $\dim \ker(A_m + I) = 2$

Proposition 14.17. $\dim \ker(A_m + I) = 2$ for every $m \geq 1$.

Proof. Solve $(A_m^\top + I)y = 0$, i.e. $y_k + \sum_{j \in \text{in}(k)} y_j = 0$ for every vertex k . Write $\beta_t := y_{b_t}$ and $\alpha_t := y_{a_t}$. By (P2):

- $k = b_m$: $\beta_m + y_4 + y_6 = 0$, so $\beta_m = -y_4 - y_6$;
- $k = a_t$: $\alpha_t + y_5 + \beta_t = 0$, so $\alpha_t = -y_5 - \beta_t$;
- $k = b_t$, $t < m$: $\beta_t + y_6 + \alpha_{t+1} = 0$, so $\beta_t = -y_6 - \alpha_{t+1}$.

Eliminating α_{t+1} from the last two relations gives

$$\beta_t = -y_6 - (-y_5 - \beta_{t+1}) = -y_6 + y_5 + \beta_{t+1}, \quad \text{i.e.} \quad \beta_{t+1} = \beta_t + (y_6 - y_5),$$

whence, by induction on t from $t = m$ downwards,

$$\beta_t = -y_4 - y_6 - (m - t)(y_6 - y_5), \quad t = 1, \dots, m,$$

and in particular $\alpha_1 = -y_5 - \beta_1 = -y_5 + y_4 + y_6 + (m - 1)(y_6 - y_5)$. The equation at $k = 6$ (whose in-neighbourhood is $\{1, a_1\}$) therefore reads

$$y_6 + y_1 + \alpha_1 = 0 \iff y_1 + y_4 + 2y_6 - y_5 + L_0(y_6 - y_5) = 0, \quad L_0 := m - 1.$$

For $k \in \{0, \dots, 5\}$ the in-neighbours lie in the core, so the equations do not involve m . Hence the $2m$ chain variables are again uniquely determined by the core variables, and the core variables satisfy the 7×7 system $S(L_0)y = 0$ with

$$S(L_0) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -L_0 - 1 & L_0 + 2 \end{bmatrix} \begin{array}{l} k = 0: y_0 + y_1 + y_2 = 0 \\ k = 1: y_0 + y_1 + y_6 = 0 \\ k = 2: y_0 + y_1 + y_2 = 0 \\ k = 3: y_3 + y_5 + y_6 = 0 \\ k = 4: y_3 + y_4 + y_5 = 0 \\ k = 5: y_2 + y_3 + y_5 = 0 \\ k = 6: y_1 + y_4 - (L_0 + 1)y_5 + (L_0 + 2)y_6 = 0 \end{array}$$

Observe that

$$S(L_0) = S(0) + L_0 e_6(e_6 - e_5)^\top,$$

a rank-one perturbation of $S(0)$. By Lemma .3, every minor of $S(L_0)$ is an affine function of L_0 : a minor that omits row 6 or omits both column 5 and column 6 is constant in L_0 , and any other minor is affine in L_0 . We claim $\text{rank } S(L_0) = 5$ for every integer $L_0 \geq 0$.

$\text{rank } S(L_0) \leq 5$. It suffices to show that all $\binom{7}{2} = 49$ minors of order 6 vanish at two distinct values of L_0 , since each is affine in L_0 . Direct computation gives $\text{rank } S(0) = \text{rank } S(1) = 5$; hence all 6×6 minors vanish at $L_0 = 0$ and at $L_0 = 1$, and therefore vanish identically.

$\text{rank } S(L_0) \geq 5$. The minor on rows $\{0, 1, 3, 4, 6\}$ and columns $\{0, 1, 2, 3, 4\}$ does not involve column 5 or column 6, hence is independent of L_0 :

$$\det \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$$

(expand along the third row, whose only nonzero entry is the 1 in column 4, then expand the resulting 4×4 determinant along its first column).

Therefore $\dim \ker(A_m^T + I) = 7 - 5 = 2$ for every $m \geq 1$, and hence $\dim \ker(A_m + I) = 2$. $\square$

14.8 Strong connectivity: D5

Proposition 14.18. *Let A be strongly connected and let the setting of Definition 14.9 hold with $i \in \{r_1, r_2\}$. Then $M = T(A)$ is strongly connected. Consequently A_m satisfies D5 for every $m \geq 1$.*

Proof. Let A_0 be A with the entry $A[i, c]$ set to 0. Deleting an outgoing edge changes no in-neighbourhood and does not change the out-neighbourhood of c ; it only removes one out-edge of i .

Every old vertex reaches i in A_0 . Fix a vertex v and a path $v \rightarrow^* \dots \rightarrow i$ in A (one exists by strong connectivity). If that path does not use the edge $i \rightarrow c$ it survives in A_0 . If it does, truncate the path at its first visit to i ; the truncated path ends at i and cannot contain the edge $i \rightarrow c$, since the edge $i \rightarrow c$ starts at i and no vertex before the first visit to i equals i .

c reaches every old vertex in A_0 . Fix w and a path $c \rightarrow^* w$ in A . If it does not use $i \rightarrow c$ it survives. Otherwise truncate at the last entry into c and keep the suffix, which starts at c and contains no edge $i \rightarrow c$ (such an edge ends at c , and c occurs only as the first vertex of the suffix).

Now in M , for any old vertices v, w ,

$$v \rightarrow^* i \rightarrow b \rightarrow a \rightarrow c \rightarrow^* w,$$

where the edge $i \rightarrow b$ exists because $i \in \{r_1, r_2\}$, and the edges $b \rightarrow a$, $a \rightarrow c$ exist by construction. Finally

$$b \rightarrow a \rightarrow c \rightarrow^* i \rightarrow b, \quad a \rightarrow c \rightarrow^* i \rightarrow b \rightarrow a,$$

so b and a are mutually reachable with every old vertex. Hence M is strongly connected.

For the family, $i = r_1 = 4 \in \{r_1, r_2\}$, so the hypothesis holds at every step. The base case A_1 is strongly connected: all off-diagonal entries of $(I + A_1)^8$ are strictly positive, which is equivalent to strong connectivity for a graph of order 9 (paths of length at most $n - 1 = 8$). Induction on m gives D5 for all A_m . $\square$

14.9 From the spectrum and the two nullities to D4

Lemma 14.19 (Jordan blocks from the rank sequence). *Let λ_0 be an eigenvalue of a matrix A of order n , and put $d_t := \dim \ker(A - \lambda_0 I)^t = n - \text{rank}(A - \lambda_0 I)^t$, $d_0 = 0$. Then the number of Jordan blocks $J_s(\lambda_0)$ with $s = m$ equals*

$$2d_m - d_{m-1} - d_{m+1}.$$

Proof. A single block $J_s(\lambda_0)$ contributes $\min(s, t)$ to d_t , and d_t is additive over blocks, so $d_t - d_{t-1} = \#\{s_\nu \geq t\}$. Hence

$$\#\{s_\nu = m\} = \#\{s_\nu \geq m\} - \#\{s_\nu \geq m+1\} = (d_m - d_{m-1}) - (d_{m+1} - d_m). \quad \square$$

Theorem 14.20 (D3 plus the two nullities implies D4). *Let A have order $n = 2k + 5$, satisfy D3, and satisfy $\dim \ker A = 1$ and $\dim \ker(A + I) = 2$. Then the Jordan form of A is*

$$J_1(2) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1) \oplus J_{2k}(0).$$

Proof. By D3 the algebraic multiplicities are 1 for $\lambda = 2$, 1 for $\lambda = 1$, 3 for $\lambda = -1$ and $2k$ for $\lambda = 0$.

$\lambda = 2$ and $\lambda = 1$. Algebraic multiplicity 1 forces a single block $J_1(2)$, resp. $J_1(1)$.

$\lambda = 0$. The algebraic multiplicity is $2k$ and the geometric multiplicity is $d_1 = \dim \ker A = 1$; by Lemma 14.19 there is therefore exactly one block. A single block of total size $2k$ is $J_{2k}(0)$, equivalently $d_t = \min(t, 2k)$ for every t .

$\lambda = -1$. Algebraic multiplicity 3 and geometric multiplicity $d_1 = \dim \ker(A + I) = 2$, so there are exactly two blocks, of sizes summing to 3 and each at least 1; hence the sizes are $\{2, 1\}$, i.e. $J_2(-1) \oplus J_1(-1)$. (This is consistent: $d_2 = \min(2, 2) + \min(1, 2) = 3$.)

The total size is $1 + 1 + 3 + 2k = 2k + 5 = n$, as required. $\square$

14.10 The main theorem

Theorem 14.21 (Main result). *For every integer $m \geq 1$,*

$$A_m \in \mathcal{U}_{m+1}, \quad \text{ord}(A_m) = 2m + 7.$$

Proof. $\text{ord}(A_m) = 2m + 7$ by Definition 14.4, and $2(m + 1) + 5 = 2m + 7$, so the order is the one prescribed for $\mathcal{U}_{m+1}$. Axiom D1 and D2 hold by Corollary 14.7; D3 by Theorem 14.15; D5 by Proposition 14.18; and D4 by Theorem 14.20 applied to A_m , whose hypotheses are Proposition 14.16 ($\dim \ker A_m = 1$) and Proposition 14.17 ($\dim \ker(A_m + I) = 2$). Hence $A_m \in \mathcal{U}_{m+1}$. $\square$

Corollary 14.22 (Infinitude). *$\mathcal{U}_k \neq \emptyset$ for every integer $k \geq 1$. Consequently the Wen class $\mathcal{U} = \bigcup_{k \geq 1} \mathcal{U}_k$ is infinite, and the set of orders for which it is nonempty is infinite.*

Proof. Given $k \geq 1$, take $m = k - 1 \geq 0$. For $k \geq 2$ this gives $A_{k-1} \in \mathcal{U}_k$ by Theorem 14.21. For $k = 1$, $\mathcal{U}_1 \neq \emptyset$ because the member W_1 of Section 3 belongs to it. Since the members A_m have pairwise distinct orders $2m + 7$, the class contains infinitely many pairwise non-isomorphic members. $\square$

Remark 14.23 (The case $k = 1$). The family starts at $m = 1$, i.e. at $k = 2$, because the operator T requires an existing edge $4 \rightarrow n - 2$ to cut; at $n = 7$ there is no separate chain to cut. The class $\mathcal{U}_1$ is nonempty nevertheless, being represented by W_1 and by 34 further pairwise non-isomorphic members found by the search (Section 3).

14.11 The family versus the search

Observation 14.24. For $m = 1, \dots, 6$ the member A_m (of order $2m + 7$, i.e. $k = m + 1$) is not isomorphic to any of the 233 matrices of the searched data set of the same order, nor to any of the representatives $W_1, \dots, W_7$; this was established by VF2 isomorphism testing. The member A_7 has order 21, i.e. $k = 8$, and is thus the first known member beyond the searched range. Combining this with Section 3, the number of known pairwise non-isomorphic members is

$$233 + 5 \text{ (the representatives } W_1, W_2, W_3, W_5, W_6) + 7 \text{ (the family } A_1, \dots, A_7) = 245.$$

Remark 14.25 (Independent numerical corroboration). The conclusions of Theorem 14.21 were verified independently by exact computation for $m = 1, \dots, 60$ (orders $n = 9, 11, \dots, 127$): D1, D2, D5, the closed-walk formula $N_L = 2^L + 1 + 3(-1)^L$ for $L \leq 10$, and the two nullities $\dim \ker A_m = 1$, $\dim \ker(A_m + I) = 2$ were all confirmed, with no exception. By Theorem 14.20 the two nullities upgrade D3 to D4, so all 60 matrices satisfy D1–D5. In addition D3 itself was checked through the n exact traces $\text{tr}(A_m^L) = 2^L + 1 + 3(-1)^L$, $L \leq n$, for $m = 1, \dots, 20$ (orders up to $n = 47$); for larger m that check becomes expensive and is in any case superseded by Theorem 14.15. Full details are in Appendices 18 and 18. This is a consistency check on the symbolic argument, not a substitute for it.

Remark 14.26 (The Perron vector of A_m). For $m = 1, \dots, 7$ the primitive integer Perron vector of A_m was computed exactly; it satisfies

$$\min(v) = 3, \quad \sum_i v_i = 144 \cdot 4^{m-1} = 144 \cdot 4^{k-2}.$$

This agrees with Observation 12.2(iii): the identity $(\sum_i v_i) \prod_{d_i > 1} d_i = 4^k \cdot 27$ here reduces to $\prod_{d_i > 1} d_i = 3$. Direct computation of the Smith normal form confirms

$$\text{SNF}(A_m - 2I) = \text{diag}(\underbrace{1, \dots, 1}_{n-2}, 3, 0) \quad (m = 1, \dots, 4),$$

i.e. exactly one nontrivial invariant factor, equal to 3. The Perron data of the family are therefore as rigid as its spectrum.

Remark 14.27 (Scope of the construction). The family $\{A_m\}$ is a single infinite subfamily. It does not exhaust $\mathcal{U}_k$ at any order (for $2 \leq k \leq 7$ the searched data set contains many members outside it, by Observation 14.24), and it says nothing about the asymptotic growth of $c(k)$. What it does settle is the existence question: no order is empty.

15 The Construction Problem, Revisited

The construction problem asks: given k , produce a member of $\mathcal{U}_k$ by an explicit formula or a polynomial-time algorithm, rather than by stochastic search. Section 14 gives a complete affirmative answer to the existence version of this problem. This section records what is now known, summarises the search technology that produced the 233 matrices of the searched data set, and isolates the parts of the construction problem that remain open.

15.1 What is solved, and what is not

Theorem 15.1 (Explicit construction). *There is an algorithm that, on input $k \geq 1$, outputs a member of $\mathcal{U}_k$ in time polynomial in k .*

Proof. For $k = 1$ output W_1 (Section 3), a fixed 7×7 matrix. For $k \geq 2$ set $m = k - 1$ and construct A_m by starting from A_1 and applying the operator T exactly $m - 1$ times. Each application of T consists of changing six fixed entries and extending the matrix by two rows and two columns, so the total work is $O(k)$ entry operations on a matrix of order $O(k)$, i.e. $O(k^2)$ time and space. Correctness is Theorem 14.21. $\square$

The following refinement of the problem, however, is still open.

Open Problem 15.2 (Surjective construction). Produce, for each k , an explicit description of all of $\mathcal{U}_k$ —that is, a parametrisation of the isomorphism classes of members—or at least a decision procedure that recognises membership in $\mathcal{U}_k$ faster than by verifying D1–D5 directly.

The obstruction is the flexibility of Layer 3 (Section 10): at $k = 5$ there are at least 70 pairwise non-isomorphic known members, and the invariant data that can be read off from the spectrum are far from determining the isomorphism class. Nothing in the construction of Section 14 helps here: the family $\{A_m\}$ is a single path through the solution space, and Observation 14.24 shows it misses every member found by the search.

15.2 Why template-based constructions fail

A natural approach—suggested by the observation that the non-zero spectrum has dimension 5 while the nilpotent block has dimension $2k$ —was the core-template hypothesis of Conjecture 11.5: identify a fixed 5-vertex core carrying the non-zero spectrum and attach doubler chains of total length $2k$. That hypothesis is false (Observation 11.6). A second approach, the doubler-chain method, is only a partial construction: Theorem 10.2 determines the Perron values along out-degree-1 edges but says nothing about the connections among non-chain vertices, and Observation 10.4 shows that the doubler chains account for only part of the nilpotent subspace V_0 .

What makes the construction of Section 14 work is the opposite of a template: the fixed part is *not* an induced subgraph carrying the whole non-zero spectrum by itself. Rather, the chain feeds back into the core through a single edge $a_1 \rightarrow 6$, and it is precisely this feedback that makes the resolvent equations (20) close on the 7×7 matrix $N = zI - C^\top$. The lesson is that the core and the chain must be treated as a single coupled system; decoupling them destroys the algebraic mechanism.

15.3 Computational cost of the search

With the construction available, the search is no longer needed to establish existence; it remains the only tool for estimating $c(k)$. For completeness we record its cost. Each energy evaluation costs $O(2k \cdot n^3)$ arithmetic operations ($2k$ rank computations on $n \times n$ matrices); at $n = 19$ this is roughly 10^5 operations per evaluation, and about 10^6 evaluations per run, giving $\approx 10^{11}$ operations per run and $\approx 5 \times 10^{12}$ for the 52 runs at $k = 7$. The pairwise isomorphism screening (VF2) costs $O(\hat{c}(k)^2 n^2)$, which is manageable at the current scales (70 known solutions at $k = 5$) but would become prohibitive if the counts grew substantially. The move set, energy function and cooling schedule are described in Section 8 and are not repeated here.

15.4 Generalisations of the construction

The ingredients of Section 14 suggest a general recipe, which we record as a programme rather than a theorem.

Open Problem 15.3 (Classification of chain-insertion operators). Classify the pairs (C, T) consisting of a core matrix C and an insertion rule T for which the argument of Section 14 goes through, i.e. for which the associated resolvent function $E(z)$ vanishes identically. Equivalently: for which 7×7 (or $r \times r$) cores does the identity

$$(z^2 - 1) \operatorname{adj}(zI - C^\top)[i, r] - \operatorname{adj}(zI - C^\top)[r, r] - z \operatorname{adj}(zI - C^\top)[r_a, r] \equiv 0$$

hold for suitable indices i, r_a, r ? Each such pair produces an infinite subfamily, and a complete classification would give a much sharper lower bound on $c(k)$.

The cancellation of Corollary 14.14 is a single linear relation among three cofactors of one 7×7 matrix; it is not a generic phenomenon, but neither is it obviously isolated. A systematic search over small cores is a natural next step.

16 Applications and Connections

The Wen class sits at the intersection of spectral graph theory, matrix analysis and combinatorial design. This section sketches the connections; none of the results of the paper depends on them. The infinitude theorem strengthens several of these connections, because it makes available an explicit member at every order rather than only at orders $k \leq 7$.

16.1 Exactly solvable loop models

In a loop model on a directed graph, each closed directed walk of length L is assigned a weight. For a member of the Wen class the normalised matrix $W = \frac{1}{2}D^{-1}AD$ of Section 6 is row-stochastic, and by Theorem 6.2 every closed walk of length L carries the same weight 2^{-L} , independently of its combinatorial structure. The cycle-expansion partition function is therefore exactly computable:

$$Z_L = \sum_{\text{closed walks of length } L} 2^{-L} = N_L 2^{-L} = \frac{2^L + 1 + 3(-1)^L}{2^L} = 1 + 2^{-L} + 3(-2)^{-L}.$$

Summing against a fugacity t gives, for $L \geq 1$,

$$\sum_{L \geq 1} Z_L t^L = \frac{t}{1-t} + \frac{t/2}{1-t/2} + \frac{-3t/2}{1+t/2},$$

a rational function of t whose denominator is $\prod_{\lambda \neq 0} (1 - \lambda t/2)$ over the non-zero eigenvalues $\{2, 1, -1\}$, independently of k . With Theorem 14.21 this construction is available at every order $k \geq 1$. What the nilpotent block contributes is not a polynomial prefactor in L : by Proposition 5.5 the nilpotent part $A^L P_0$ vanishes for $L \geq 2k$, and the only L -dependent polynomial factor in the whole decomposition is the scalar L attached to the $J_2(-1)$ block, which is traceless and hence invisible in Z_L .

16.2 Non-Hermitian exceptional points

An exceptional point of order m is a degeneracy at which the Jordan block has size m [13, 17, 22]. The block $J_{2k}(0)$ in every member of $\mathcal{U}_k$ is therefore an exceptional point of order $2k$ at the eigenvalue 0. Theorem 14.21 provides an explicit finite-dimensional $\{0, 1\}$ model with an exceptional point of order $2k$ for every $k \geq 1$, on $2k+5$ vertices and with only $2(2k+5)$ nonzero entries. The sparsity is noteworthy: the number of edges is linear in the order of the exceptional point, so these are among the sparsest known combinatorial models of high-order exceptional points.

16.3 Jordan-cospectral non-isomorphic digraphs

All known members—the 233 matrices of the searched data set, the representatives $W_1, \dots, W_7$ and the family $\{A_m\}$ —are *Jordan-cospectral*: they have identical Jordan canonical forms, hence identical characteristic and minimal polynomials. No polynomial $p(A)$ can distinguish two non-isomorphic members, since $p(A)$ is determined by the values of p and its derivatives on the spectrum. The family therefore supplies, at each order $n = 2k+5$, a set of pairwise non-isomorphic, Jordan-cospectral strongly connected digraphs whose size is at least $\hat{c}(k)$: namely 35, 52, 41, 23, 70, 15, 8 for $k = 1, \dots, 7$ and at least 1 for every $k \geq 8$. Thirty-five pairwise non-isomorphic digraphs on seven vertices with identical Jordan structure is a striking instance of the phenomenon. Infinitude implies that this supply never runs out.

16.4 Markov chains with uniform return probabilities

The row-stochastic matrix $W = \frac{1}{2}D^{-1}AD$ defines a Markov chain on the vertex set [19, 15]. By Theorem 6.2, in this chain every closed path of length L through a fixed vertex has exactly the same probability 2^{-L} , whatever route it takes—a strong uniformity property, unusual for a chain that is not a random walk on a regular undirected graph. Because the non-zero spectrum of W equals that of $\frac{1}{2}A$, namely $\{1, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}$ together with $2k$ zeros, the mixing behaviour is governed entirely by the five-dimensional non-zero part; by Proposition 5.5 the growing nilpotent part contributes only a transient of length at most $2k$.

16.5 Combinatorial design theory

Each vertex has in-degree exactly 2 (D2), so the edge set is a 2-regular in-design on n points; the combinatorial invariants of Section 9—exactly 4 bidirectional pairs and exactly 2 directed 3-cycles—are additional design-like constraints that are now known to be consequences of the spectrum (Propositions 9.1 and 9.2), not independent axioms. The member W_1 is related to the Fano plane, the unique 2-(7, 3, 1) design [21]; whether the higher-order members admit analogous descriptions as incidence structures is open (Problem 17.8).

The power-of-two structure along doubler chains (Theorem 10.2) imposes a hierarchical ordering on the Perron entries, reminiscent of the role of powers of two in Hadamard matrices and difference sets [21]. The Smith normal form provides a second arithmetic link: Observation 12.2(iii) ties $\text{SNF}(A - 2I)$ to $4^k \cdot 27$, and Remark 14.26 shows that the members of the infinite family all realise the minimal case $\text{SNF}(A_m - 2I) = \text{diag}(1, \dots, 1, 3, 0)$.

16.6 Integer matrix theory

Finally, the class is a testbed for questions about integer matrices with prescribed spectral data [3, 16]. The constraints D1–D5 mix integrality, non-negativity, a degree condition, a characteristic polynomial and a Jordan type, and the resulting feasibility problem is nontrivial precisely because the Jordan type is not determined by the characteristic polynomial. Section 13 quantifies this gap empirically; the infinite family shows that the gap can be bridged for all k by an explicit algebraic mechanism rather than by search.

17 Open Problems

The existence question is settled affirmatively by Theorem 14.21: $\mathcal{U}_k \neq \emptyset$ for every $k \geq 1$, and members at $k \geq 8$ are produced explicitly by applying T to A_1 . The following problems remain open.

Open Problem 17.1 (Asymptotics of the solution count). Determine whether $c(k)$, the number of pairwise non-isomorphic members of $\mathcal{U}_k$, tends to infinity, tends to zero, or stabilises. The known lower bounds are 35, 52, 41, 23, 70, 15, 8, 1 for $k = 1, \dots, 8$; for no order is the value known to be exact (Remark 3.2). The constraint-counting heuristic of Section 13 and the density model of Section 12 point in opposite directions, and neither is rigorous.

Open Problem 17.2 (The nilpotent subspace). Identify the part of the $2k$ -dimensional nilpotent subspace $V_0 = \ker A^{2k}$ that is not accounted for by doubler chains. By Observation 10.4 the number of doubler edges is 2, 4, 4, 6, 4, 7, 8 for $k = 1, \dots, 7$, strictly less than $2k$ for $k \geq 3$; a characterisation of the complement would presumably also explain the rank sequence $\text{rank}(A^m) = n - m$ without invoking Jordan theory.

Open Problem 17.3 (Existence of doubler chains). Is the existence of a vertex of out-degree 1 (Conjecture 9.4) a consequence of D1–D5? By Proposition 9.3 every member either has such a

vertex or has all out-degrees equal to 2; no member of the second type is known. If the answer is affirmative, what determines the partition of the doubler edges into chains?

Open Problem 17.4 (Smith normal form). Prove the identity of Observation 12.2(iii),

$$\left(\sum_i v_i\right) \prod_{d_i > 1} d_i = 4^k \cdot 27, \quad \text{SNF}(A - 2I) = \text{diag}(d_1, \dots, d_n),$$

from D1–D5, or exhibit a counterexample at some $k \geq 8$. Determine also the exact conditions under which $9 \mid \sum_i v_i$; this holds for 206 of the 233 searched members but fails for the remaining 27.

Open Problem 17.5 (Minimum of the Perron vector). Does $\min(v) = 1$ hold for infinitely many k ? Among the searched members it occurs 22 times, all at $k \leq 3$ (Observation 11.4), and the infinite family of Section 14 has $\min(v) = 3$ for every m . Is there a member with $\min(v) = 1$ at arbitrarily large k ?

Open Problem 17.6 (Anti-correlation of Jordan failures). Formalise the anti-correlation observed in Section 13: when the nilpotent block splits, the (-1) -block tends to be correct, and conversely. Is there a “Jordan complexity budget”—a rigorous inequality constraining the total off-diagonal Jordan content of a D3-matrix?

Open Problem 17.7 (Exact asymptotics of $P(\text{D4} \mid \text{D3})$). Is $P(\text{D4} \mid \text{D3})$ of the form $\exp(-c\sqrt{k})$, $\exp(-ck)$, or $\exp(-ck^2)$? Only three data points ($k = 1, 2, 3$) are available, and sampling at $k \geq 4$ is prohibitively expensive because the density falls below 10^{-23} . A rigorous lower bound of the form $P(\text{single block} \mid \text{D3}) \geq c/p(2k)$ would give an independent, non-constructive proof of infinitude.

Open Problem 17.8 (Finite geometry). The member W_1 is related to the Fano plane, the unique $2-(7, 3, 1)$ design. Do the members at higher orders admit descriptions in terms of known incidence structures, and does the core C of Definition 14.1 arise from a design-theoretic construction?

Open Problem 17.9 (Surjective construction). See Problem 15.2: parametrise all of $\mathcal{U}_k$, or decide membership faster than by verifying D1–D5.

Open Problem 17.10 (Classification of insertion operators). See Problem 15.3: classify the core/operator pairs (C, T) for which the resolvent identity of Corollary 14.11 holds identically. Each such pair yields an infinite subfamily and a better lower bound on $c(k)$.

18 Conclusion

We have studied the Wen class $\mathcal{U}_k$: $\{0, 1\}$ -matrices of order $n = 2k + 5$ with zero diagonal, uniform column sum 2, characteristic polynomial $\lambda^{2k}(\lambda - 2)(\lambda - 1)(\lambda + 1)^3$, Jordan form $J_1(2) \oplus J_1(1) \oplus J_2(-1) \oplus J_1(-1) \oplus J_{2k}(0)$, and strong connectivity. The defining feature is the coexistence of a five-dimensional non-zero spectral core $\{2, 1, -1, -1, -1\}$, constant across all orders, with a single nilpotent block $J_{2k}(0)$ that grows with k .

Structure. The spectrum forces a large amount of structure: the closed-walk count $N_L = 2^L + 1 + 3(-1)^L$ (Theorem 6.4), the rank formula $\text{rank}(A^m) = \max(n - m, 5)$ (Proposition 4.4), the minimal polynomial $x^{2k}(x - 2)(x - 1)(x + 1)^2$ of degree $n - 1$ (Proposition 5.1), the commutant dimension $7 + 2k$ (Proposition 5.3), the Newton recurrence of order $\delta = 5$ (Proposition 7.2), the power-of-two law along doubler chains (Theorem 10.2), and the exact counts of 4 bidirectional edge pairs and 2 directed 3-cycles (Propositions 9.1 and 9.2). The universal closed-walk weight 2^{-L} is shared by every strongly connected digraph whose in-degree equals the spectral radius, and is therefore not a distinguishing feature; the distinguishing feature is the Jordan structure.

Search. Seven orders ($k = 1, \dots, 7$, $n = 7, \dots, 19$) have been searched and the results independently re-verified in exact arithmetic: 233 pairwise non-isomorphic members, with order distribution 34, 50, 39, 22, 68, 13, 7. These figures are lower bounds, not exact counts (Remark 3.2). Adding the five additional representatives and the seven members $A_1, \dots, A_7$ of the infinite family gives 245 known pairwise non-isomorphic members in total.

Infinitude. The main result is Theorem 14.21: the explicit family $\{A_m\}_{m \geq 1}$ with $\text{ord}(A_m) = 2m + 7$, obtained by iterating the chain-insertion operator T on the fixed 7×7 core C , satisfies $A_m \in \mathcal{U}_{m+1}$ for every $m \geq 1$. The proof splits into a combinatorial part (D1, D2, D5 are preserved by T) and an algebraic part: a Schur complement identity expresses $\det(zI - T(A)) - z^2 \det(zI - A)$ in terms of three cofactors of $zI - A$, and a resolvent computation reduces those cofactors to three cofactors of the single 7×7 matrix $zI - C^\top$, where they satisfy a linear relation that makes the whole expression vanish identically. Exact determination of $\dim \ker A_m = 1$ and $\dim \ker(A_m + I) = 2$ then upgrades the characteristic-polynomial statement to the full Jordan axiom D4. Consequently $\mathcal{U}_k \neq \emptyset$ for every $k \geq 1$: the class is infinite, and the construction is polynomial-time in k (Theorem 15.1).

Rigidity and flexibility. The three-layer architecture of Section 10 organises the empirical picture: Layer 1 (rigid invariants) is constant across all members and mostly proven; Layer 2 (semi-invariants) is governed by theorems but flexibly realised; Layer 3 (flexible realisation) varies freely at each order. Two natural conjectures—the minimum-one Perron vector property and the core-template hypothesis—are false (Section 11), and both falsifications constrain what a surjective construction could look like.

What remains. The existence question is closed; the counting question is not. Whether $c(k)$ tends to infinity, to zero, or stabilises is open, and the available lower bounds (35, 52, 41, 23, 70, 15, 8, 1 for $k = 1, \dots, 8$) are inconclusive, being contaminated by search difficulty rather than reflecting the true counts. The structure of the non-chain part of the nilpotent subspace, the Smith normal form identity, the arithmetic of $\min(v)$, and the classification of viable insertion operators are the most accessible of the remaining problems.

The broader significance of the class is that it demonstrates, constructively and at all orders, that a prescribed spectral rigidity—a fixed non-zero spectrum together with a nilpotent Jordan block of arbitrarily large size—can be realised by sparse integer directed graphs satisfying an elementary degree condition. The mechanism that makes this possible is the coupling of a fixed core to a growing chain through a bounded interface: it is what turns an infinite family of high-dimensional algebraic conditions into a single identity among cofactors of a 7×7 matrix.

Appendix A Computational Verification Protocol

All computational claims in this paper were obtained in exact arithmetic. This appendix records the protocol so that the results can be reproduced independently. No floating-point computation enters any proof; floating-point was used only for the numerical sanity checks explicitly flagged as such (e.g. Observation 5.4).

A.1 Data sets

Three data sets were used.

- (i) *The searched set:* 233 matrices of orders $n = 7, 9, \dots, 19$, with the order distribution 34, 50, 39, 22, 68, 13, 7 for $k = 1, \dots, 7$. Ten representatives of each order (all seven at $k = 7$) are listed in Appendix 18. They are pairwise non-isomorphic (verified by VF2 testing of all $\sum_k \binom{\hat{c}^{(k)}}{2} = 5135$ pairs within each order).

- (ii) *The representatives $W_1, \dots, W_7$ of Section 3, each verified independently and each non-isomorphic to the searched members of the same order, except for W_4 and W_7 which coincide with searched members (Remark 3.3).*
- (iii) *The infinite family $A_1, \dots, A_{60}$ of Section 14, of orders $n = 9, 11, \dots, 127$.*

A.2 Characteristic polynomials by Newton's identities

Rather than computing $\det(\lambda I - A)$ by expansion, we compute the power sums $p_L = \text{tr}(A^L)$ for $L = 1, \dots, n$ by repeated integer matrix multiplication, and then recover the coefficients $c_1, \dots, c_n$ of $\chi_A(\lambda) = \lambda^n + c_1\lambda^{n-1} + \dots + c_n$ from Newton's identities (Section 7)

$$k c_k = \sum_{i=1}^k (-1)^{i-1} c_{k-i} p_i, \quad c_0 = 1.$$

All operations are over $\mathbb{Z}$; the integers involved have at most a few hundred digits at $n = 19$, so no overflow or rounding occurs. The recovered polynomial is then compared with the target $\lambda^{2k}(\lambda - 2)(\lambda - 1)(\lambda + 1)^3$.

A.3 Ranks

Ranks and nullities were computed by exact integer (fraction-free) Gaussian elimination over $\mathbb{Q}$, with no modular reduction and no floating point; this is what the symbolic engine used here (SymPy [14]) performs natively.

For the reader who prefers modular arithmetic, the standard justification is the Hadamard bound: if B is an integer $n \times n$ matrix with $H = \max_{i,j} |B_{ij}|$, then every minor of B of order $r \leq n$ is bounded in absolute value by $(\sqrt{r} H)^r$; hence if $(\sqrt{r} H)^r < p$ then reduction modulo p cannot annihilate any nonzero minor of order r , and

$$\text{rank}_{\mathbb{F}_p}(B \bmod p) = \text{rank}_{\mathbb{Q}}(B) \quad (28)$$

for any $\text{rank} \leq r$. Two remarks on the use of (28) are in order.

First, only *two* ranks are needed to certify D4, in view of Theorem 14.20: namely $\text{rank}(A) = n - 1$ and $\text{rank}(A + I) = n - 2$. Both A and $A + I$ are $\{0, 1\}$ -matrices—the diagonal of A is zero by D1, so $A + I$ has diagonal entries equal to 1—hence $H = 1$ in both cases and the bound is $r^{r/2}$ with $r = n - 1$, respectively $r = n - 2$. For the 233 searched members ($n \leq 19$) this is at most $18^9 < 2^{38}$; for the family up to $n = 127$ it is at most $126^{63} < 2^{440}$. Both are comfortably below the Mersenne prime $p = 2^{521} - 1$, so (28) applies.

Secondly, the bound is *not* adequate for the higher powers A^m at large n : the entries of A^m grow like 2^m (Corollary 5.6), so H , and with it the bound $(\sqrt{r} H)^r$, grows rapidly. Accordingly the full rank sequence $\text{rank}(A^m) = \max(n - m, 5)$ was verified by exact integer arithmetic for the 233 searched members and for $A_1, \dots, A_7$; it agreed with the theoretical values in every case. By Theorem 14.20 that sequence is in any case a consequence of D3 together with the two nullities, so it is a consistency check rather than an independent hypothesis, and no claim in this paper rests on a modular certification of it.

The nullities of Propositions 14.16 and 14.17 are obtained as $\dim \ker A = n - \text{rank}(A)$ and $\dim \ker(A + I) = n - \text{rank}(A + I)$.

A.4 Strong connectivity

Strong connectivity is decided by breadth-first search from each vertex in both the graph and its reversal, i.e. by testing whether the reachability matrix $(I + A)^{n-1}$ (computed with Boolean arithmetic, or over $\mathbb{Z}$ with positivity tested entrywise) is positive. For the base case A_1 of the infinite family we verified that every off-diagonal entry of $(I + A_1)^8$ is strictly positive.

A.5 Isomorphism

Isomorphism of directed graphs is decided by the VF2 algorithm [4], as implemented in NetworkX [10]. As a sanity check, a randomly permuted copy of a member is always reported as isomorphic to it, and the 233 searched members are pairwise non-isomorphic. The VF2 results quoted in Observation 14.24—that $A_1, \dots, A_6$ are not isomorphic to any searched member of the same order nor to any of $W_1, \dots, W_7$ —were obtained with the same implementation.

A.6 Smith normal form

The Smith normal form of $A - 2I$ was computed by the standard integer algorithm (unimodular row and column operations with gcd-based pivoting) over $\mathbb{Z}$, yielding invariant factors $d_1 \mid d_2 \mid \dots \mid d_n$ [16]. Because $\text{rank}(A - 2I) = n - 1$, exactly one invariant factor is zero. The identity $(\sum_i v_i) \prod_{d_i > 1} d_i = 4^k \cdot 27$ of Observation 12.2(iii) was checked for all 233 searched members, with no exception.

A.7 Summary of results

Check	Scope	Failures
D1, D2 (zero diagonal, column sums)	233 searched	0
D3 (characteristic polynomial)	233 searched	0
Closed-walk formula $N_L = 2^L + 1 + 3(-1)^L$, $L \leq 10$	233 searched	0
Rank sequence $\text{rank}(A^m) = \max(n - m, 5)$, $m \leq 2k + 1$	233 searched	0
$\text{rank}(A + I) = n - 2$	233 searched	0
D5 (strong connectivity)	233 searched	0
Pairwise non-isomorphism	233 searched	0
D1, D2, D5 and walk formula ($L \leq 10$)	$A_1, \dots, A_{60}$	0
$\dim \ker A_m = 1$, $\dim \ker(A_m + I) = 2$	$m = 1, \dots, 60$	0
D3 by n exact traces	$m = 1, \dots, 20$	0

The three cofactor identities of Proposition 14.13, the vanishing of P , the base characteristic polynomial (27) and the Schur complement identity (15) (tested on $A_1 \rightarrow A_2$) were additionally verified by symbolic computation over $\mathbb{Z}[z]$ with SymPy [14]; see Appendix 18.

Appendix B Representative Members of the Search

The complete data set produced by the search of Section 8 comprises 233 pairwise non-isomorphic matrices, distributed over the orders $k = 1, \dots, 7$ as 34, 50, 39, 22, 68, 13, 7. Reproducing all 233 matrices here would add some fifty pages without adding information; we therefore list ten representative members of each order (all seven at $k = 7$), selected so as to span the structural range observed at that order: the first and last member produced by the search, the members with the smallest and the largest value of $\min(v)$, the members with the fewest and the most vertices of out-degree 1, and evenly spaced members in between. Table 11 lists the invariants of the selected members; the full list of 233 matrices, together with the verification scripts of Appendix 18, is available from the author.

Every matrix below is displayed as n rows of n entries with the convention $A[i, j] = 1$ iff $i \rightarrow j$. All of them satisfy D1–D5; this was re-verified in exact arithmetic in the course of preparing this paper.

$k = 1, n = 7$. **S7-1** (first member of the order). $\min(v) = 1, \sum_i v_i = 36$.
out-degree sequence $(3, 1, 3, 2, 1, 2, 2)$.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

S7-34 (last member of the order). $\min(v) = 3, \sum_i v_i = 36$.
out-degree sequence $(2, 3, 2, 2, 1, 2, 2)$.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S7-4 (largest $\min(v)$). $\min(v) = 3, \sum_i v_i = 36$.
out-degree sequence $(2, 1, 1, 2, 2, 2, 4)$.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

S7-13 (fewest out-degree-1 vertices). $\min(v) = 3, \sum_i v_i = 36$.
out-degree sequence $(3, 2, 2, 1, 2, 2, 2)$.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S7-3 (most out-degree-1 vertices). $\min(v) = 1, \sum_i v_i = 36$.
out-degree sequence $(4, 1, 2, 3, 1, 2, 1)$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

S7-7 (evenly spaced). $\min(v) = 1$, $\sum_i v_i = 12$.
out-degree sequence $(3, 3, 1, 2, 2, 1, 2)$.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S7-12 (evenly spaced). $\min(v) = 3$, $\sum_i v_i = 36$.
out-degree sequence $(2, 2, 2, 1, 1, 3, 3)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

S7-17 (evenly spaced). $\min(v) = 3$, $\sum_i v_i = 36$.
out-degree sequence $(3, 1, 3, 2, 2, 1, 2)$.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S7-23 (evenly spaced). $\min(v) = 1$, $\sum_i v_i = 36$.
out-degree sequence $(2, 2, 2, 1, 3, 3, 1)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S7-29 (evenly spaced). $\min(v) = 2$, $\sum_i v_i = 36$.
out-degree sequence $(1, 2, 1, 2, 3, 3, 2)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$k = 2$, $n = 9$. **S9-1** (first member of the order). $\min(v) = 5$, $\sum_i v_i = 144$.
out-degree sequence $(2, 2, 1, 1, 2, 5, 2, 1, 2)$.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

S9-50 (last member of the order). $\min(v) = 4$, $\sum_i v_i = 72$.
out-degree sequence $(1, 2, 2, 2, 2, 2, 3, 2, 2)$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

S9-3 (smallest $\min(v)$). $\min(v) = 1$, $\sum_i v_i = 144$.
out-degree sequence $(1, 2, 1, 1, 4, 1, 3, 4, 1)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S9-37 (largest $\min(v)$). $\min(v) = 12$, $\sum_i v_i = 144$.
out-degree sequence $(2, 1, 2, 3, 2, 2, 2, 2, 2)$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S9-9 (evenly spaced). $\min(v) = 6$, $\sum_i v_i = 144$.
out-degree sequence $(1, 1, 5, 2, 1, 2, 2, 2, 2)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S9-17 (evenly spaced). $\min(v) = 3$, $\sum_i v_i = 144$.
out-degree sequence $(5, 2, 1, 1, 1, 2, 3, 1, 2)$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S9-25 (evenly spaced). $\min(v) = 7$, $\sum_i v_i = 144$.
out-degree sequence $(1, 2, 1, 2, 2, 2, 5, 1, 2)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

S9-34 (evenly spaced). $\min(v) = 4$, $\sum_i v_i = 144$.
out-degree sequence $(2, 3, 1, 1, 3, 2, 3, 1, 2)$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

S9-42 (evenly spaced). $\min(v) = 7$, $\sum_i v_i = 144$.
out-degree sequence $(2, 2, 1, 1, 3, 2, 3, 3, 1)$.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S9-2 (evenly spaced). $\min(v) = 6$, $\sum_i v_i = 144$.
out-degree sequence $(3, 3, 3, 2, 1, 2, 1, 1, 2)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$k = 3$, $n = 11$. **S11-1** (first member of the order). $\min(v) = 3$, $\sum_i v_i = 192$.
out-degree sequence $(2, 2, 2, 3, 1, 3, 1, 2, 2, 1, 3)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S11-39 (last member of the order). $\min(v) = 3$, $\sum_i v_i = 576$.

out-degree sequence $(1, 2, 1, 2, 1, 5, 3, 3, 1, 2, 1)$.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S11-14 (smallest $\min(v)$). $\min(v) = 1$, $\sum_i v_i = 576$.

out-degree sequence $(5, 1, 4, 1, 2, 2, 1, 1, 2, 1, 2)$.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S11-5 (largest $\min(v)$). $\min(v) = 27$, $\sum_i v_i = 576$.

out-degree sequence $(2, 1, 2, 2, 1, 2, 3, 3, 2, 2, 2)$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S11-20 (most out-degree-1 vertices). $\min(v) = 12$, $\sum_i v_i = 576$.

out-degree sequence $(1, 1, 1, 4, 1, 3, 1, 6, 1, 1, 2)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

S11-7 (evenly spaced). $\min(v) = 5$, $\sum_i v_i = 576$.

out-degree sequence $(1, 2, 1, 1, 2, 2, 1, 2, 1, 8, 1)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S11-26 (evenly spaced). $\min(v) = 18$, $\sum_i v_i = 576$.

out-degree sequence $(3, 2, 2, 2, 2, 1, 1, 3, 3, 2, 1)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S11-33 (evenly spaced). $\min(v) = 15$, $\sum_i v_i = 576$.

out-degree sequence $(1, 3, 1, 1, 2, 1, 1, 2, 3, 3, 4)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S11-2 (evenly spaced). $\min(v) = 24$, $\sum_i v_i = 576$.

out-degree sequence $(2, 1, 2, 4, 1, 2, 2, 2, 3, 2, 1)$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S11-3 (evenly spaced). $\min(v) = 9$, $\sum_i v_i = 576$.

out-degree sequence $(1, 1, 5, 3, 1, 3, 1, 2, 1, 2, 2)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$k = 4, n = 13$. **S13-1** (first member of the order). $\min(v) = 18, \sum_i v_i = 1152$.
out-degree sequence $(3, 3, 1, 1, 1, 3, 2, 1, 1, 3, 2, 1, 4)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S13-22 (last member of the order). $\min(v) = 13, \sum_i v_i = 768$.
out-degree sequence $(1, 1, 2, 3, 1, 2, 1, 2, 4, 2, 4, 2, 1)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

S13-17 (smallest $\min(v)$). $\min(v) = 5, \sum_i v_i = 256$.
out-degree sequence $(3, 1, 2, 2, 2, 4, 1, 3, 1, 4, 1, 1, 1)$.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

S13-9 (largest $\min(v)$). $\min(v) = 60, \sum_i v_i = 2304$.

out-degree sequence $(1, 2, 4, 1, 2, 1, 2, 4, 2, 2, 2, 1, 2)$.

0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	1	0	0	1	0	0	1
0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	1	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0
0	1	1	1	0	0	0	0	0	0	0	1	0
0	0	0	0	0	1	0	1	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	1	0	0	0	0	1	0	0

S13-3 (fewest out-degree-1 vertices). $\min(v) = 35$, $\sum_i v_i = 1152$.
out-degree sequence $(1, 2, 4, 3, 3, 2, 2, 2, 1, 2, 1, 2, 1)$.

0	0	0	0	0	0	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	1
0	1	0	1	0	0	0	0	0	1	1	0	0
0	0	1	0	0	0	1	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	0	0	1	0	0	0	0	0	1	0
0	0	0	1	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0

S13-13 (most out-degree-1 vertices). $\min(v) = 12$, $\sum_i v_i = 576$.
out-degree sequence $(2, 1, 1, 1, 4, 4, 1, 1, 2, 1, 3, 1, 4)$.

0	1	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	1	0	0	1	0	1	0	1	0	0	0
1	0	0	0	0	0	0	1	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	0	0	1	0	0	0	0	0	0
1	0	0	1	0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	1	0	1	0	0

S13-5 (evenly spaced). $\min(v) = 24$, $\sum_i v_i = 2304$.

out-degree sequence $(2, 2, 2, 2, 2, 1, 1, 1, 4, 3, 2, 1, 3)$.

0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	0	1	0	0	0	0	0	1	0
0	0	0	1	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	1	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	1	0	1	0	0	1	1	0	0
0	0	0	0	1	0	0	0	1	0	1	0	0
0	0	1	1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	1	0	0	0

S13-8 (evenly spaced). $\min(v) = 14$, $\sum_i v_i = 2304$.

out-degree sequence $(1, 2, 2, 2, 2, 1, 3, 2, 3, 1, 3, 1, 3)$.

[illegible]

S13-11 (evenly spaced). $\min(v) = 12$, $\sum_i v_i = 576$.

out-degree sequence $(2, 2, 1, 3, 1, 4, 1, 1, 1, 1, 2, 2, 5)$.

0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	1	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0
1	0	0	0	1	0	1	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	0	0	0	1	0	0	0
1	1	0	0	0	0	0	1	0	1	0	1	0

S13-15 (evenly spaced). $\min(v) = 18$, $\sum_i v_i = 576$.

out-degree sequence $(1, 3, 4, 2, 1, 3, 2, 1, 2, 2, 2, 2, 1)$.

[illegible]

$k = 5$, $n = 15$. **S15-1** (first member of the order). $\min(v) = 60$, $\sum_i v_i = 4608$.
out-degree sequence $(3, 3, 1, 2, 1, 3, 1, 1, 1, 4, 2, 2, 4, 1, 1)$.

[illegible]

S15-68 (last member of the order). $\min(v) = 72$, $\sum_i v_i = 4608$.
out-degree sequence $(2, 1, 2, 3, 2, 3, 1, 2, 4, 1, 4, 2, 1, 1, 1)$.

0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	1	1
0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	1	0	1	1	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	1	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0

S15-61 (smallest $\min(v)$). $\min(v) = 5$, $\sum_i v_i = 1024$.
out-degree sequence (3, 1, 1, 1, 2, 2, 5, 2, 1, 2, 3, 1, 1, 3, 2).

0	0	0	0	1	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
1	0	0	0	1	1	0	0	1	0	1	0	0	0	0
0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	1	0	0

S15-35 (largest $\min(v)$). $\min(v) = 240$, $\sum_i v_i = 9216$.
out-degree sequence (3, 2, 2, 2, 2, 3, 2, 1, 1, 3, 2, 3, 2, 1, 1).

0	1	0	1	0	0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	0
0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	1	0	0	0	0	0	1
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

S15-37 (fewest out-degree-1 vertices). $\min(v) = 192$, $\sum_i v_i = 9216$.

out-degree sequence (1, 3, 2, 2, 2, 2, 3, 1, 2, 3, 1, 2, 2, 2, 2).

0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0

S15-29 (most out-degree-1 vertices). $\min(v) = 12$, $\sum_i v_i = 9216$.
out-degree sequence (2, 2, 1, 3, 1, 5, 1, 1, 1, 1, 2, 2, 6, 1, 1).

0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
1	0	0	0	1	1	0	1	1	0	0	0	0	1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0

S15-12 (evenly spaced). $\min(v) = 64$, $\sum_i v_i = 4608$.

out-degree sequence (3, 2, 2, 2, 2, 1, 1, 1, 4, 3, 1, 1, 4, 1, 2).

0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	1	1	0	0	1	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0

S15-23 (evenly spaced). $\min(v) = 192$, $\sum_i v_i = 9216$.
out-degree sequence (1, 3, 4, 2, 2, 1, 2, 4, 2, 2, 2, 1, 2, 1, 1).

0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	1	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	1	1	0	0	1	0	0	0
0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0	1	0	1	0	0	0
0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0

S15-46 (evenly spaced). $\min(v) = 40$, $\sum_i v_i = 9216$.

out-degree sequence $(2, 3, 2, 4, 2, 1, 1, 2, 3, 2, 2, 1, 3, 1, 1)$.

0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
1	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

S15-57 (evenly spaced). $\min(v) = 96$, $\sum_i v_i = 9216$.

out-degree sequence $(1, 2, 2, 3, 1, 3, 3, 4, 1, 2, 2, 2, 2, 1, 1)$.

0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0
1	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$k = 6$, $n = 17$. **S17-1** (first member of the order). $\min(v) = 4$, $\sum_i v_i = 6144$.
out-degree sequence $(6, 3, 1, 2, 1, 1, 1, 4, 1, 1, 3, 1, 3, 2, 2, 1, 1)$.

[illegible]

S17-13 (last member of the order). $\min(v) = 224$, $\sum_i v_i = 18432$.
out-degree sequence $(2, 1, 1, 1, 2, 2, 5, 3, 1, 2, 3, 2, 1, 3, 2, 1, 2)$.

[illegible]

S17-9 (largest $\min(v)$). $\min(v) = 734$, $\sum_i v_i = 36864$.

out-degree sequence $(1, 3, 2, 2, 2, 3, 2, 1, 2, 4, 1, 2, 2, 2, 2, 1, 2)$.

0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0
0	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0

S17-3 (evenly spaced). $\min(v) = 156$, $\sum_i v_i = 18432$.

out-degree sequence $(2, 3, 4, 1, 2, 1, 1, 4, 3, 2, 2, 1, 3, 2, 1, 1, 1)$.

0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	1	0	1	0	0	0	1	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	1	0
0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

S17-5 (evenly spaced). $\min(v) = 192$, $\sum_i v_i = 18432$.

out-degree sequence $(2, 2, 1, 2, 4, 4, 2, 1, 3, 1, 2, 1, 3, 1, 1, 2, 2)$.

0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
0	0	1	1	0	0	0	0	0	1	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0

S17-7 (evenly spaced). $\min(v) = 74$, $\sum_i v_i = 18432$.

out-degree sequence $(1, 3, 3, 1, 3, 2, 3, 1, 2, 2, 1, 4, 1, 2, 1, 2, 2)$.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0

S17-11 (evenly spaced). $\min(v) = 52$, $\sum_i v_i = 9216$.

out-degree sequence (3, 3, 2, 1, 2, 4, 1, 3, 1, 4, 1, 1, 1, 3, 2, 1, 1).

0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

S17-2 (evenly spaced). $\min(v) = 64$, $\sum_i v_i = 9216$.

out-degree sequence (3, 1, 2, 2, 2, 2, 3, 2, 4, 1, 3, 2, 2, 1, 1, 1, 2).

0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0
0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0

S17-4 (evenly spaced). $\min(v) = 32$, $\sum_i v_i = 18432$.

out-degree sequence $(3, 1, 1, 1, 2, 2, 2, 3, 5, 1, 4, 2, 1, 2, 2, 1, 1)$.

0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
0	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0
0	1	0	0	1	0	0	1	0	0	0	0	0	0	1	1	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0

S17-6 (evenly spaced). $\min(v) = 132$, $\sum_i v_i = 9216$.

out-degree sequence $(1, 2, 1, 2, 5, 4, 1, 2, 2, 2, 3, 1, 4, 1, 1, 1, 1)$.

[illegible]

$k = 7, n = 19$. **S19-1** (first member of the order). $\min(v) = 404, \sum_i v_i = 36864$.
out-degree sequence (6, 1, 3, 1, 1, 1, 1, 3, 1, 2, 3, 1, 3, 2, 2, 1, 2, 2, 2).

0	0	0	1	0	0	0	1	0	0	0	1	1	0	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0

S19-7 (last member of the order). $\min(v) = 54, \sum_i v_i = 36864$.
out-degree sequence (2, 1, 1, 2, 2, 2, 5, 4, 1, 2, 3, 2, 1, 3, 2, 1, 2, 1, 1).

0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1	0	0
0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0

S19-2 (smallest $\min(v)$). $\min(v) = 32, \sum_i v_i = 49152$.

out-degree sequence (2, 2, 1, 2, 4, 4, 1, 1, 1, 1, 3, 1, 4, 1, 1, 2, 2, 2, 3).

0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0

S19-4 (largest $\min(v)$). $\min(v) = 840$, $\sum_i v_i = 36864$.

out-degree sequence (2, 3, 2, 2, 2, 2, 3, 1, 3, 3, 1, 3, 2, 2, 3, 1, 1, 1, 1).

0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1
0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

S19-5 (most out-degree-1 vertices). $\min(v) = 648$, $\sum_i v_i = 73728$.

out-degree sequence (3, 2, 1, 1, 4, 3, 1, 4, 1, 4, 2, 1, 2, 1, 1, 3, 2, 1, 1).

0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	0	0	1	0	0	0	1	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

S19-3 (evenly spaced). $\min(v) = 828$, $\sum_i v_i = 73728$.

out-degree sequence (1, 2, 3, 1, 3, 2, 4, 1, 2, 2, 1, 6, 1, 2, 1, 2, 2, 1, 1).

0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0
0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	1	0	1	0	0	1	0	0	0	1	1	1
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0

S19-6 (evenly spaced). $\min(v) = 512$, $\sum_i v_i = 73728$.

out-degree sequence $(1, 2, 3, 3, 1, 4, 3, 4, 2, 1, 2, 3, 2, 1, 2, 1, 1, 1, 1)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Appendix C Explicit Data for the Infinite Family

This appendix collects all the explicit data used in Section 14, so that every step of the proof can be checked without recomputation.

C.1 The core

The core matrix C of Definition 14.1 is

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Column sums: $(2, 2, 2, 2, 2, 2, 1)$. Diagonal: zero. The in-neighbourhoods are given in (13). The characteristic polynomial of C is

$$\det(zI - C) = z(z - 1)(z + 1)^2(z^3 - z^2 - 2z + 1).$$

C.2 The base member A_1 and the first iterates

The base member A_1 of Definition 14.2 is

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

and $\det(zI - A_1) = z^4(z - 2)(z - 1)(z + 1)^3$. Its iterates are $A_2 = T(A_1)$:

$$A_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and $A_3 = T(A_2)$:

$$A_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

One checks directly that $\det(zI - A_2) = z^2 \det(zI - A_1)$ and $\det(zI - A_3) = z^2 \det(zI - A_2)$.

C.3 The matrix $N = zI - C^\top$

The matrix $N = zI - C^\top$ is

$$N = \begin{bmatrix} z & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & z & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & z & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & z & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & z \end{bmatrix}.$$

Its determinant is

$$\det N = z(z - 1)(z + 1)^2(z^3 - z^2 - 2z + 1),$$

which coincides with $\det(zI - C)$, as it must since $\det(zI - C^\top) = \det(zI - C)$.

C.4 The three cofactors

Writing $\text{adj}(N)[j, 6] = (-1)^{j+6} \det(N \text{ with row 6 and column } j \text{ deleted})$:

$$\begin{aligned}\text{adj}(N)[4, 6] &= z(z-1)(z+1)^2 &= z^4 + z^3 - z^2 - z, \\ \text{adj}(N)[5, 6] &= z(z-1)(z+1)^2 &= z^4 + z^3 - z^2 - z, \\ \text{adj}(N)[6, 6] &= z(z-1)(z+1)^2(z^2 - z - 1) &= z^6 - 3z^4 - z^3 + 2z^2 + z.\end{aligned}$$

Hence

$$\begin{aligned}P(z) &= (z^2 - 1) \text{adj}(N)[4, 6] - \text{adj}(N)[6, 6] - z \text{adj}(N)[5, 6] \\ &= z(z-1)(z+1)^2 \left[(z^2 - 1) - (z^2 - z - 1) - z \right] = 0.\end{aligned}$$

C.5 The reduced nullity systems

For $\ker A_m$ (Proposition 14.16) the core variables satisfy $R_0 y = 0$ with

$$R_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \text{rank } R_0 = 6, \quad y^* = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \in \ker R_0.$$

The nonvanishing 6×6 minor is the one on rows $\{0, 1, 2, 3, 4, 6\}$ and columns $\{0, 1, 2, 3, 4, 5\}$, equal to 1.

For $\ker(A_m + I)$ (Proposition 14.17) the core variables satisfy $S(L_0)y = 0$ with $L_0 = m - 1$ and

$$S(L_0) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -L_0 - 1 & L_0 + 2 \end{bmatrix} = S(0) + L_0 e_6(e_6 - e_5)^T,$$

with $\text{rank } S(L_0) = 5$ for every integer $L_0 \geq 0$; the nonvanishing 5×5 minor is the one on rows $\{0, 1, 3, 4, 6\}$ and columns $\{0, 1, 2, 3, 4\}$, equal to 1 and independent of L_0 .

C.6 Orders and parameters

m	1	2	3	4	5	6	7	...
$\text{ord}(A_m) = 2m + 7$	9	11	13	15	17	19	21	...
$k = m + 1$	2	3	4	5	6	7	8	...
$\#\{\text{edges}\} = 2n$	18	22	26	30	34	38	42	...
$\min(v)$	3	3	3	3	3	3	3	...
$\sum_i v_i$	144	576	2304	9216	36864	147456	589824	...

The closed forms are $\text{ord}(A_m) = 2m + 7$, $\#\{\text{edges}\} = 4m + 14$, $\min(v) = 3$ and $\sum_i v_i = 144 \cdot 4^{m-1}$ (Remark 14.26).

C.7 Independent verification

The following were recomputed exactly for the family, independently of the symbolic argument of Section 14:

- D1, D2 and D5 for $m = 1, \dots, 60$ (orders $n = 9, 11, \dots, 127$);
- $\dim \ker A_m = 1$ and $\dim \ker(A_m + I) = 2$ for $m = 1, \dots, 60$ (which, by Theorem 14.20, together with D3 gives D4);
- the closed-walk formula $N_L = 2^L + 1 + 3(-1)^L$ for $L = 1, \dots, 10$ and $m = 1, \dots, 60$;
- D3, i.e. $\text{tr}(A_m^L) = 2^L + 1 + 3(-1)^L$ for $L = 1, \dots, n$, for $m = 1, \dots, 20$ (orders up to $n = 47$). For $m > 20$ the enumeration of n exact traces becomes expensive, and D3 is in any case proved for all m by Theorem 14.15; the computed range therefore serves as a consistency check only.

No failure was found in any of these checks; see Appendix 18 for the protocol.

The verification boundary should be stated explicitly. D1, D2, D5, the two nullities and the closed-walk formula were checked for all 60 members, and D3 by exact traces for the first 20; beyond $m = 20$ the trace enumeration was *not* carried out, and the claim that D3 holds there rests on Theorem 14.15 alone. Since that theorem is a statement about the operator T and makes no reference to a particular m , the symbolic proof is strictly stronger than the finite check, and the finite range serves only as an independent guard against transcription and implementation errors in the symbolic computation.

Appendix D Algebraic Preliminaries

For completeness we record the four standard facts used repeatedly in the paper, with proofs. Nothing here is new.

Lemma .1 (Matrix determinant lemma). *Let D be an $n \times n$ matrix over a commutative ring and $u, v \in R^n$. Then*

$$\det(D + uv^\top) = \det D + v^\top \text{adj}(D)u.$$

Proof. Both sides are polynomial in the entries, so it suffices to prove the identity for invertible D over a field, where it follows from $\det(D + uv^\top) = \det D \cdot \det(I + D^{-1}uv^\top)$ and $\det(I + xy^\top) = 1 + y^\top x$ (Sylvester's determinant identity with a rank-one factor):

$$\det D (1 + v^\top D^{-1}u) = \det D + v^\top (\det D \cdot D^{-1})u = \det D + v^\top \text{adj}(D)u. \quad \square$$

Lemma .2 (Schur complement). *Let $M = \begin{bmatrix} D & U \\ V & B \end{bmatrix}$ be a block matrix over a field with B invertible. Then*

$$\det M = \det B \cdot \det(D - UB^{-1}V).$$

Proof. Multiply M on the left by the block-unitriangular matrix $\begin{bmatrix} I & -UB^{-1} \\ 0 & I \end{bmatrix}$, whose determinant is 1:

$$\begin{bmatrix} I & -UB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} D & U \\ V & B \end{bmatrix} = \begin{bmatrix} D - UB^{-1}V & 0 \\ V & B \end{bmatrix},$$

and take determinants; the determinant of a block-triangular matrix is the product of the determinants of its diagonal blocks. $\square$

Lemma .3 (Minors of a rank-one perturbation are affine). *Let S be an $n \times n$ matrix over a commutative ring, $u, v \in R^n$, and let λ be an indeterminate. Then every $r \times r$ minor of $S + \lambda uv^\top$ is a polynomial in λ of degree at most 1.*

Proof. Fix sets I, J of r rows and r columns. By the Cauchy–Binet formula (or by expanding the determinant multilinearly in the columns), the minor is a sum of terms each of which is a product of r entries, at most one of which can come from the rank-one part $\lambda uv^\top$, since a determinant is alternating and therefore vanishes when two columns are taken from the same rank-one matrix. Hence each term is constant or linear in λ . $\square$

Lemma .4 (Cramer’s rule). *Let N be an $n \times n$ matrix over a field with $\det N \neq 0$, and let $b \in R^n$. Then the solution x of $Nx = b$ is given by*

$$x_j = \frac{\det N_j}{\det N}, \quad j = 1, \dots, n,$$

where N_j is N with its j -th column replaced by b . Equivalently $x = \operatorname{adj}(N)b / \det N$, i.e. $x_j = \sum_i \operatorname{adj}(N)[j, i] b_i / \det N$.

Proof. $\operatorname{adj}(N)N = \det N \cdot I$, so $x = \operatorname{adj}(N)b / \det N$; the j -th entry of $\operatorname{adj}(N)b$ is the Laplace expansion of $\det N_j$ along the j -th column. $\square$

Application to Section 14

Lemmas .2 and .1 are the two ingredients of Theorem 14.10. Lemma .4 is used in Lemma 14.12 to pass from $N\rho_C = \alpha e_6$ to $\rho_C = \alpha \operatorname{adj}(N)e_6 / \det N$, which is what reduces the m -dependent resolvent to the m -independent vector $g = N^{-1}e_6$. Lemma .3 is used in Proposition 14.17 to conclude that the rank of $S(L_0)$ is at most 5 for all L_0 once it is checked at two distinct values of L_0 .

Table 11: The 67 representative members listed in this appendix. $\min(v)$ and $\sum_i v_i$ refer to the primitive integer right Perron vector; $d_{\min}$ and $\#\{d = 1\}$ are the minimum out-degree and the number of vertices of out-degree 1; the last two columns are the number of bidirectional edge pairs and of directed 3-cycles, both constant across all members by Propositions 9.1 and 9.2.

member	n	$\min(v)$	$\sum_i v_i$	$d_{\min}$	$\#\{d = 1\}$	bidir.	3-cycles
S7-1	7	1	36	1	2	4	2
S7-34	7	3	36	1	1	4	2
S7-4	7	3	36	1	2	4	2
S7-13	7	3	36	1	1	4	2
S7-3	7	1	36	1	3	4	2
S7-7	7	1	12	1	2	4	2
S7-12	7	3	36	1	2	4	2
S7-17	7	3	36	1	2	4	2
S7-23	7	1	36	1	2	4	2
S7-29	7	2	36	1	2	4	2
S9-1	9	5	144	1	3	4	2
S9-50	9	4	72	1	1	4	2
S9-3	9	1	144	1	5	4	2
S9-37	9	12	144	1	1	4	2
S9-9	9	6	144	1	3	4	2
S9-17	9	3	144	1	4	4	2
S9-25	9	7	144	1	3	4	2
S9-34	9	4	144	1	3	4	2
S9-42	9	7	144	1	3	4	2
S9-2	9	6	144	1	3	4	2
S11-1	11	3	192	1	3	4	2
S11-39	11	3	576	1	5	4	2
S11-14	11	1	576	1	5	4	2
S11-5	11	27	576	1	2	4	2
S11-20	11	12	576	1	7	4	2
S11-7	11	5	576	1	6	4	2
S11-26	11	18	576	1	3	4	2
S11-33	11	15	576	1	5	4	2
S11-2	11	24	576	1	3	4	2
S11-3	11	9	576	1	5	4	2
S13-1	13	18	1152	1	6	4	2
S13-22	13	13	768	1	5	4	2
S13-17	13	5	256	1	6	4	2
S13-9	13	60	2304	1	4	4	2
S13-3	13	35	1152	1	4	4	2
S13-13	13	12	576	1	7	4	2
S13-5	13	24	2304	1	4	4	2
S13-8	13	14	2304	1	4	4	2
S13-11	13	12	576	1	6	4	2
S13-15	13	18	576	1	4	4	2
S15-1	15	60	4608	1	7	4	2
S15-68	15	72	4608	1	6	4	2
S15-61	15	5	1024	1	6	4	2
S15-35	15	240	9216	1	4	4	2
S15-37	15	192	9216	1	3	4	2
S15-29	15	12	9216	1	8	4	2
S15-12	15	64	4608	1	6	4	2
S15-23	15	192	9216	1	5	4	2
S15-46	15	40	9216	1	5	4	2
S15-57	15	96	9216	1	5	4	2
S17-1	17	4	6144	1	9	4	2
S17-13	17	224	18432	1	6	4	2
S17-9	17	734	36864	1	4	4	2
S17-3	17	156	18432	1	7	4	2
S17-5	17	192	18432	1	6	4	2
S17-7	17	74	18432	1	6	4	2
S17-11	17	52	9216	1	8	4	2
S17-2	17	64	9216	1	5	4	2

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