SCQ : Syracuse Conjecture Quadrature

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ABSTRACT

After circa 2300 years (*Circle Quadrature*, Archimedes, Syracuse 287 – 212 BC) the history of mathematics repeats itself in a different problem. This paper presents an original research on the Syracuse Conjecture. It proposes a theoretical development aimed at solving the conjecture through a systematic approach. One of its features suggests a process that leads to Theorem 2n+1, whose demonstration subdivides the odd numbers into seven subsets which have different behaviors applying algorithm of Collatz. It allows to replace the Collatz's cycles with the cycles of links, transforming oscillating sequences into monotone decreasing sequences. Theorem of Independence allows to manage the cycles of links to our liking to reach very high main horizons and, when we decide, go back to lower horizons. In this article it's proved that Collatz Conjecture is not fully demonstrable. In fact, if we consider the banal link $n < 2n$, there are eight cycles which connect each other in an endless of possible sequences and related links. It is a new type of *Squaring of the Circle* or *Algebraic Quadrature*, but its statement is confirmed. In other words: BIG CRUNCH (go back to 1) is always possible, but BIG BANG (to move on) has no End.

INTRODUCTION

In this paper is shown the true nature of the Syracuse conjecture, tackling it from a completely dissimilar point of view than many previous attempts. The paper proves that Syracuse conjecture, better known as Collatz conjecture, is a new type of Q*uadrature* or *Zeno's paradox Achilles and the tortoise*. The result was achieved through a procedure which allows us to manage Collatz's sequences as we wish. If we consider the banal cycle of even numbers, there are eight basic cycles which connect each other in an infinity of sequences. The unique feasible procedure for its demonstration is to prove that any numerical succession obtained by Collatz algorithm contains a value lesser than the starting number (or *generating number*). Given that the even numbers 2n go to $n < 2n$, we can consider only odd numbers. The General List of binomial inequalities until a number of steps $N(s) \leq 21$ covers *circa* 93% of $\mathbb{N}$. By Theorem of Independence we can manage the cycles of links to reach high main horizons $\Theta(m)$ ($N(s) > 21$), and when we decide go back to the lower horizons $\Theta(l) < \Theta(m)$. But we will never be able to arrive at 100% coverage of $\mathbb{N}$. It's precisely in this context that the concept of *quadrature* or *paradox*, addressed in this article, should be understood. SCQ transforms the chaos of Collatz's cycles in the descending sorting of the link cycles, which perhaps hide a type of law on the expansion of Cosmos based on the powers of 2, as prophesied by Plato in some of his writings.

Notes : This article is an exhaustive synthesized makeover of the book (110 pages plus 30 tables of links odd numbers 5 – 2999, A4 format), self-published on Amazon 2023 February by aid Mnamon Milan Italy. **[1]**
AI has been used just in Appendices, with corrections and adjustments.
The text-form is logically consistent with the proof of the result.

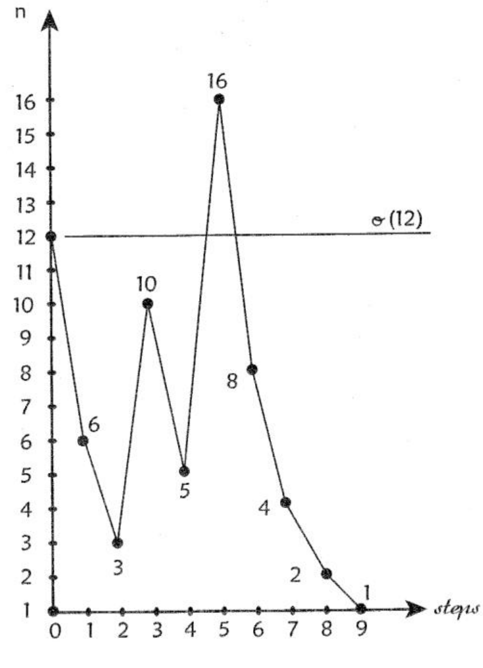
CONTENTS

1. The conjecture of Syracuse

The conjecture of Syracuse states: *If at any natural integer n, non-zero, we apply the algorithm 3n+1 if n is odd, n/2 if n is even, the sequence of the values obtained, precipitates to 1 after a finite number of steps, always in compliance with the final cycle {4; 2; 1}.*

$$\text{For every } n \in \mathbb{N} : n \neq 0 \rightarrow \begin{cases} 3n+1 \; if \; n \; is \; odd \\ \frac{n}{2} \; if \; n \; is \; even \end{cases}$$

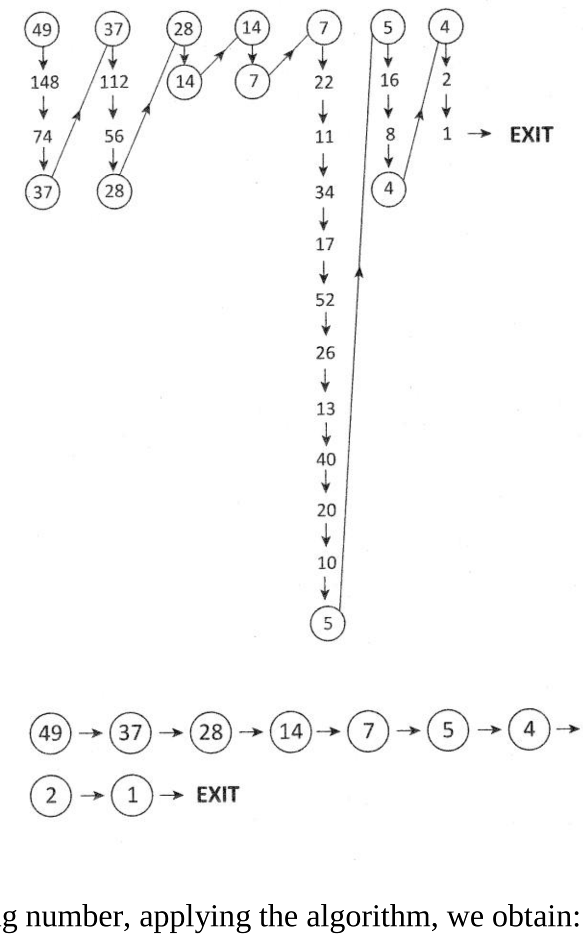
Chosen n = 12 (even), applying the algorithm of Collatz we obtain the following sequence:
n = 12 → S(12) = {12; 6; 3; 10; 5; 16; 8; 4; 2; 1} the number 12 falls down to 1 in nine steps.
The sequence generated by the number 12 is oscillating (neither increasing nor decreasing) and can be represented with a graph on a Cartesian plane having its origin in (0,1). The number 1 can be understood as the minimum level or the sea level. The steps are shown on the horizontal axis and the generating numbers on the vertical axis. The graph of fluctuation of the sequence S(12) is represented in figure. The horizontal line parallel to the abscissa axis (steps) is the horizon of the number 12 (indicated with the Greek letter θ). The number 12, during the oscillation of the sequence S(12), exceeds its horizon once, exactly in step 5, in correspondence of which reaches a peak 16.



It’s to be noted that if we choose as starting number an odd number, the second element of the sequence is always an even number, since the product of 3 for another odd number still gives an odd number, which becomes even with addition of 1. This happens for all the successors of an odd number contained in the sequence. An even number, instead, may be followed by an odd number or by one or more even numbers: n = 48 → S(48) = {48; 24; 12; 6; 3; 10; 5; 16; 8; 4; 2; 1}.
If you choose an even number power of 2, n = 2^p = 2; 4; 8; 16; 32; 64; 128; 256; 512; 1024; …, it reaches 1 after a cycle of p applications of the algorithm. A similar observation also applies to odd numbers when 3n+1 = 2^p → n = $(2^p-1)/3$: p ∈ E ⊂ ℕ; hence n = $(4^p-1)/3$. In this case the corresponding sequence S(n) contains p+2 values and it reaches 1 in p+1 steps.

If we choose the number 25 we will have the following sequence:
25 → 76 → 38 → **19** → 58 → 29 → 88 → 44 → 22 → **11** → 34 → 17 → 52 → 26 → 13 → 40 → 20 → **10** → **5** → 16 → 8 → **4** → **2** → **1**
It consists of 24 values, 23 steps. We observe that the sequence does not contain two equal values, but they are all different from each other. In short, the sequences obtained through algorithm of Collatz precipitate to 1 taking different values between them. This is obvious, because if in a succession two numbers were equal, from one of them onwards we would have the same sequence, giving rise to a vicious circle (*loop*), and this is not possible for the rules of algorithm (clearly with the exclusion of 1 which generates the final cycle).
Again with reference to the number 25, but the reasoning is valid for any other number, another *important observation* is that when the value of the sequence falls below 25, in this case 19, its succession from here on becomes perfectly the same as of this number. In this way the sequence of number 25 is linked to the succession of number 19. So, the endless sequences generated by the algorithm give rise to a kind of interaction, in which each sequence is linked to a follows one whose generating number is smaller. For 25 we have the subsequent chain of links:
25 → 19 → 11 → 10 → 5 → 4 → 2 → 1
In this way we can associate a specific procedure to the Collatz's sequences.

If we take into consideration the odd number 49, we have:



If we take 204 as generating number, applying the algorithm, we obtain:

204 → **102** < 204
102 → **51** < 102
51 → 154 → 77 → 232 → 116 → 58 → **29** < 51
29 → 88 → 44 → **22** < 29

22 → **11** < 22
11 → 34 → 17 → 52 → 26 → 13 → 40 → 20 → **10** < 11
10 → **5** < 10
5 → 16 → 8 → **4** < 5
4 → **2** → **1** final cycle (EXIT).

204 → 102 → 51 → 29 → 22 → 11 → 10 → 5 → 4 → 2 → 1
… …

Such a proceeding is applicable for every $n \in \mathbb{N} : n \neq 0$ and $n \neq 1$.

So, Collatz sequences can be imagined like a maze in which the numbered rooms are connected to each other in a descending order in accordance with the rules imposed by the algorithm, up to the final cycle 4 → 2 → 1 that's exit of the maze, which we will call *Labyrinth of Syracuse*.
SC would be proved if we could demonstrate that any sequence contains a value lesser than the starting number (generating number). If this happens it hooks to this, which in turn hooks to another lesser than it, and so on, up to precipitate necessarily to 1. Or, as they say, the main horizon of generating number descends to the lower horizons until it reaches the sea level. This can be stated with certainty for even numbers $n \in E : n = 2^p$ and for odd numbers $n \in O : n = (4^p–1)/3$. For all other numbers there is uncertainty.

1.1. Theorem 2n+1

Premise : To reduce the complexity of this theorem, its consequent logical deductions are explained in § 1.2. and § 2.1. .

Collatz's sequences are oscillating, excluding those whose generating number is power of 2. For the oscillating sequences we found that they surely fall down to 1 if a term of the sequence, generated by a generic number n, assumes a smaller value of n, i.e. below the horizon of n that we can mark with $\Theta(n)$. Marking with n_i this value ($n_i < n$; $\Theta(n_i)$ lower of $\Theta(n)$), the sequence generated by n is linked to that generated by n_i, which, in turn, is linked to that of $n_j < n_i$ ($\Theta(n_j)$ lower of $\Theta(n_i)$), which ... and so it precipitates to 1. We wonder : does this happen in all sequences? In any succession there is always an element whose value is lesser than the generator? This is true for the even starting numbers. For them, the second element of the sequence is n/2, and n/2 < n. If, instead, the generating number is odd, the next element is 3n+1, which is even and therefore divisible by 2. That is: 3n+1 → (3n+1)/2. Unless it is of the type $(3n+1)/2 = 2^p$ or $(3n+1)/2 = (4^p–1)/3$ ($n \in O$), it turns out to be (3n+1)/2 > n. But if 3n+1 were doubly even (twice divisible by 2, or divisible by 4), triply even (three times divisible by 2, or divisible by 8), then (3n+1)/4 < n and (3n+1)/8 < n they would definitely be true.

We denote the generic odd number with 2n+1.

Statement :
There are iterative cycles obtained by the algorithm of Collatz applied to odd numbers 2n+1 which contain terms a_n lesser than their generators.

Demonstration :
Applying Collatz algorithm to a generic odd number 2n+1, we will have:

2n+1 → 3(2n+1)+1 = 6n+4 → 3n+2. It can be even or odd.

a) If 3n+2 is even, then $3n+2 \rightarrow (3n+2)/2 < 2n+1$ ($3n+2 < 4n+2$, from which $n > 0$, always true).
But 3n+2 is even if $n \in E = \{2; 4; 6; 8; 10; 12; 14; 16; ... ; 2n\}$.
It follows that 2n+1 is in $\mathbf{O_1 = \{5; 9; 13; 17; 21; 25; 29; 33; \dots ; 4n+1\}}$ $[n > 0]$

b) If 3n+2 is odd, then: $3n+2 \rightarrow 3(3n+2)+1 = 9n+7$, which becomes even by virtue of the algorithm, then $9n+7 \rightarrow (9n+7)/2$, which can be even or odd.

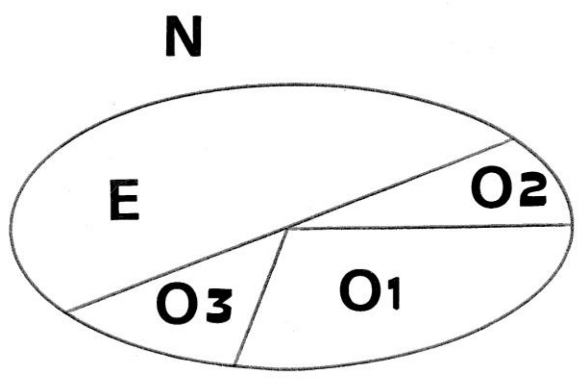
b1) If $(9n+7)/2$ is even, then $(9n+7)/2 \rightarrow (9n+7)/4 < 4n+2 = 2(2n+1) \rightarrow (9n+7)/8 < 2n+1$. And this happens if $(9n+7)/2$ is at least doubly even (divisible by 4). So: $(9n+7)/2 \rightarrow (9n+7)/4 \rightarrow (9n+7)/8 < 2n+1$. But $(9n+7)/2$ is even for $n \in O_1 = \{1; 5; 9; 13; 17; 21; 25; 29; \dots; 4n+1\}$.
It follows that 2n+1 is in a subset of $\mathbf{O_2 = \{3; 11; 19; 27; 35; 43; 51; 59; \dots; 8n+3\}}$ $[n \geq 0]$

b2) If $(9n+7)/2$ is odd, then $(9n+7)/2 \rightarrow 3((9n+7)/2)+1 = (27n+21)/2+1 = (27n+23)/2$, which being even in virtue of the algorithm: $(27n+23)/2 \rightarrow (27n+23)/4 < 8n+4 = 4(2n+1) \rightarrow (27n+23)/16 < 2n+1$. And this happens if $(27n+23)/2$ is at least triply even (divisible by 8). So: $(27n+23)/2 \rightarrow (27n+23)/4 \rightarrow (27n+23)/8 \rightarrow (27n+23)/16 < 2n+1$.
But $(27n+23)/2$ is even for $n \in O_4 = \{3; 7; 11, 15; 19; 23; 27; 31; \dots ; 4n-1; 4n+3\}$.
But $(9n+7)/2$ is odd for $n \in O_4 = \{3; 7; 11, 15; 19; 23; 27; 31; \dots ; 4n-1; 4n+3\}$.
It follows that 2n+1 is in a subset of $\mathbf{O_3 = \{7; 15; 23; 31; 39; 47; 55; 63; \dots; 8n-1; 8n+7\}}$ $[n > 0]$

If we denote with S(2n+1) any sequence obtained applying Collatz algorithm starting from a generic odd generating number 2n+1; we have identified three subsets in whose iterative cycles there are terms $a_n \in S(2n+1)$ such that $a_n < 2n+1$. They are:

$\mathbf{O_1 = \{5; 9; 13; 17; 21; 25; 29; 33; \dots ; 4n+1\}}$
$\mathbf{O_2 = \{3; 11; 19; 27; 35; 43; 51; 59; \dots; 8n+3\}}$
$\mathbf{O_3 = \{7; 15; 23; 31; 39; 47; 55; 63; \dots; 8n-1; 8n+7\}}$

$O_1 \cup O_2 \cup O_3 = \{3; 5; 7; 9; 11; 13; 15; 17; 19; 21; 23; 25; 27; 29; 31; \dots; 2n+1; 2n+3\} = O - \{1\}$, with 1 generator of the end cycle.



1.2. Addition to the theorem 2n+1

Note : In this article bold font is used for highlighting substitutions.

a) $2n+1 \in O_1$

3n+2 is even if $n \in E = \{0; 2; 4; 6; 8; \ldots; 2n\}$ $[n \geq 0]$
It is simply even if $n \in E' = \{0; 4; 8; 12; 16; \ldots; 4n\}$
n = **4n** : 2n+1 = 2(**4n**)+1 = **8n+1**; 3n+2 = 3(**4n**)+2 = 12n+2 → **6n+1** < **8n+1**
It is doubly even if $n \in E'' = \{6; 14; 22; 30; 38; \ldots; 8n+6\}$
n = **8n+6** : 2n+1 = 2(**8n+6**)+1 = **16n+13**; 3n+2 = 3(**8n+6**)+2 = 24n+20 → **12n+10** < **16n+13**
It is at least triply even if $n \in E''' = \{2; 10; 18; 26; 34, \ldots; 8n+2\}$
n = **8n+2** : 2n+1 = 2(**8n+2**)+1 = **16n+5**; 3n+2 = 3(**8n+2**)+2 = 24n+8 → **12n+4** < **16n+5**
$E' \cup E'' \cup E''' = E$ → for every $n \in E$: **(3n+2)/2 < 2n+1** $\in O_1$
n = **2n** : 2n+1 = 2(**2n**)+1 = **4n+1**; (3n+2)/2 **=** (3(**2n**)+2)/2 = **3n+1 < 4n+1** O_1

b1) $2n+1 \in O_2$

(9n+7)/2 is even if $n \in O_1 = \{1; 5; 9; 13; \ldots; 4n+1\}$ $[n \geq 0]$
It is doubly even if $n \in O_1' = \{9; 25; 41; 57; 73; \ldots; 16n+9\}$
n = **16n+9** → 2n+1 = 2(**16n+9**)+1 = **32n+19**; (9n+7)/2 = [9(**16n+9**)+7]/2 = 72n+44 → 36n+22 → **18n+11** < **32n+19**
It is at least triply even if $n \in O_1'' = \{1; 17; 33; 49; \ldots; 16n+1\}$
n = **16n+1** → 2n+1 = 2(**16n+1**)+1 = **32n+3**; (9n+7)/2 = [9(**16n+1**)+7]/2 = 72n+8 → 36n+4 → **18n+2** < **32n+3**
$O_1' \cup O_1'' = O_1''' = \{1; 9; 17; 25; \ldots; 8n+1\} \subset O_1$ → for every $n \in O_1'''$: **(9n+7)/8 < 2n+1** $\in O_2$
n = **8n+1** : 2n+1 = 2(**8n+1**)+1 = **16n+3**; (9(**8n+1**)+7)/8 **= 9n+2 < 16n+3** $O_2^* \subset O_2$

b2) $2n+1 \in O_3$

(27n+23)/2 is even if $n \in O_4 = \{3; 7; 11; 15; \ldots; 4n+3\}$ $[n \geq 0]$
It is at least triply even if $n \in O_4' = \{11; 27; 43; 59; \ldots; 16n+11\}$
n = **16n+11** → 2n+1 = 2(**16n+11**)+1 = **32n+23**; (27n+23)/2 = [27(**16n+11**)+23]/2 = 216n+160 → 108n+80 → 54n+40 → **27n+20** < **32n+23**
$O_4' \subset O_4$ → for every $n \in O_4'$: **(27n+23)/16 < 2n+1** $\in O_3$
n = **16n+11** : 2n+1 = 2(**16n+11**)+1 = **32n+23**; (27(**16n+11**)+23)/16 **= 27n+20 < 32n+23** $O_3^* \subset O_3$

1.2.1. Summary

a) $2n+1 \in O_1$ → 6n+4 → 3n+2 → (3n+2)/2 < 2n+1 : **3n+1 < 4n+1** O_1 . An odd number in O_1 assumes a value lesser than itself after three applications Collatz algorithm (3 steps). This always happens.
b1) $2n+1 \in O_2$ → 6n+4 → 3n+2 → 9n+7 → (9n+7)/2 → (9n+7)/4 → (9n+7)/8 < 2n+1 : **9n+2 < 16n+3** O_2^* . An odd number in O_2 assumes a value lesser than itself after six applications Collatz algorithm (6 steps). If this doesn't happen it's transformed into an odd number or in O_1 or in O_2 or in O_3 .
b2) $2n+1 \in O_3$ → 6n+4 → 3n+2 → 9n+7 → (9n+7)/2 → (27n+23)/2 → (27n+23)/4 → (27n+23)/8 → (27n+23)/16 < 2n+1 : **27n+20 < 32n+23** O_3^* . An odd number in O_3 assumes a value lesser than itself after eight applications Collatz algorithm (8 steps). If this doesn't happen it is transformed into an odd number or in O_1 or in O_2 or in O_3 .

1.2.2. Corollaries

Every sequence obtained by application of Collatz algorithm does not contain (never) odd numbers divisible by 3. In fact : $(2n+1) \in O$ → 3(2n+1)+1 = 6n+4 → 3n+2 can be even or odd and it's not divisible by 3 for every $n \in \mathbb{N}$. We can therefore state the following corollaries.

Corollary 1: *In the cycles of links no odd number connects to an odd number divisible by 3.*
Corollary 2: *In the cycles of links no odd number connects to an even number divisible by 3.*
Corollary 3: *If the sequences obtained by applying the algorithm of Collatz start from an odd generating number, then the odd numbers divisible by 3 are present only as generating numbers.*
Corollary 4: *If a sequence obtained by applying the algorithm of Collatz starts from an odd generating number, then in it are not present even numbers divisible by 3.*

2. Explanatory notes and Odd numbers subsets

Syracuse conjecture plan Theorem 2n+1

$\mathbf{2n+1\ O \rightarrow 6n+4\ E \rightarrow 3n+2\ O \vee E \rightarrow (3n+2)/2\ \ O_1 : 3}$ steps

$\downarrow$

$\mathbf{9n+7\ E \rightarrow (9n+7)/2\ O \vee E \rightarrow (9n+7)/4\ \ O_2 : 5}$ steps

$\downarrow$

$\mathbf{(27n+23)/2\ E \rightarrow (27n+23)/4\ \ O_3 : 6}$ steps

Notes

a) Definition: *A cycle of link is the number of steps required for to arrive from the main horizon of a generating number* (Start) *to the lower horizon of its link* (End).

We point: Θ(m) : main horizon
Θ(l) : lower horizon
N(s) : number of steps

b) Definition: *Two cycles of links,* c1, c2, *are equivalent if contain the same number of steps, i.e. :* c1 ~ c2 $\leftrightarrow$ N(s)[c1] = N(s)[c2].

c) $\mathbf{O_1 = \{5; 9; 13; 17; 21; 25; 29; 33; \ldots; 4n+1\}}$ [n > 0]
$\mathbf{O_2 = \{3; 11; 19; 27; 35; 43; 51; 59; \ldots; 8n+3\}}$ [n ≥ 0]
$\mathbf{O_3 = \{7; 15; 23; 31; 39; 47; 55; 63; \ldots; 8n-1\}}$ [n > 0]

A number n is in O_1 if n-1 is divisible by 4 .
We pose : u = (n-1)/4 ; x = 2u = (n-1)/2 $\rightarrow$ f(x) = (3x+2)/2 after N(s) = **3**
A number n is in O_2 if n-3 is divisible by 8 .
We pose : v = (n-3)/8 ; y = 4v+1 = (n-1)/2 $\rightarrow$ f(y) = (9y+7)/4 after N(s) = **5**
A number n is in O_3 if n+1 is divisible by 8 .
We pose : w = (n+1)/8 ; z = 4w-1 = (n-1)/2 $\rightarrow$ f(z) = (27z+23)/4 after N(s) = **6**

d) Important remark
We use binomials of 4 types (*e* = even number; *o* = odd number) :
1) $e \cdot n + e$ $\rightarrow$ divisible by 2
2) $o \cdot n + e$: Odd or Even. Is Odd if n = **2n+1** ; is Even if n = **2n**
3) $o \cdot n + o$: Odd or Even. Is Odd if n = **2n** ; is Even if n = **2n+1**
4) $e \cdot n + o$: it can be in O_1 or in O_2 or in O_3 :
   4a) It's in O_1 if n = **2n** or if n = **2n+1**.
   4b) It's in O_2 if n = **4n+1** or if n = **4n+3** ; if n = **4n** or if n = **4n+2**
   4c) It's in O_3 if n = **4n+3** or if n = **4n+1** ; if n = **4n+2** or if n = **4n**

e) If it's written E; O; O_1; O_2; O_3 it is an obligation; instead [E]; [O]; [O_1]; [O_2]; [O_3] it is a choice.

2.1. O_1 ; O_2 ; O_3

O_1

4n+1 O_1 : x = 2n → f(x) = **3n+1 < 4n+1** O_1 : N(s) = **3**

O_2

8n+3 O_2 : y = 4n+1 → f(y) = 9n+4 [E] : n = **2n**

8n+3 O_2 → 8(**2n**)+3 = **16n+3** O_2*
9n+4 → 9(**2n**)+4 = 18n+4 → **9n+2 < 16n+3** O_2* [n ≥ 0] = {3; 19; …} : N(s) = **6**

For generating odd numbers O_2 – O_2* = **16n+11** O_{2a} = {11; 27; 43; …} [n ≥ 0] we have:

16n+11 O_{2a} : y = 8n+5 → f(y) = 18n+13 $O_1 \vee O_2 \vee O_3$: N(s) = 5 steps

O_3

8n-1 O_3 : z = 4n-1 → f(z) = 27n-1 [E] : n = **2n+1**

8n-1 O_3 → 8(**2n+1**)-1 = **16n+7** O_3
27n-1 → 27(**2n+1**)-1 = 54n+26 → 27n+13 [E] : n = **2n+1**

16n+7 O_3 → 16(**2n+1**)+7 = **32n+23** O_3*
27n+13 → 27(**2n+1**)+13 = 54n+40 → **27n+20 < 32n+23** O_3* [n ≥ 0] = {23; 55; …} : N(s) = **8**

For generating odd numbers $O_3 – O_3^*$ of type:

32n-1 O_{3a} = {31; 63; 95; …} [n > 0]
32n+7 O_{3b} = {7; 39; 71; …} [n ≥ 0]
32n+15 O_{3c} = {15; 47; 79; …} [n ≥ 0]
we have:
32n-1 O_{3a} : z = 16n-1 → f(z) = 108n-1 $O_2 \vee O_3$: N(s) = 6 steps
32n+7 O_{3b} : z = 16n+3 → f(z) = 108n+26 → 54n+13 $O_1 \vee O_2 \vee O_3$: N(s) = 7 steps
32n+15 O_{3c} : z = 16n+7 → f(z) = 108n+53 O_1 : x = 54n+26 → f(x) = 81n+40 E ∨ O : N(s) = 9 steps

Keep in mind : 32n-9 O_3* [n > 0] ∼ 32n+23 O_3* ; 32n-1 O_{3a} ∼ 32n+31 O_{3a} [n ≥ 0].

Summarizing :

$O = O_1 \cup O_2 \cup O_3$
$O_2 = O_2^* \cup O_{2a}$
$O_3 = O_3^* \cup O_{3a} \cup O_{3b} \cup O_{3c}$

The seven odd numbers subsets

O_1 : $n \in O_1 = \{n \in \mathbb{N} \mid n = 4k + 1, k \in \mathbb{N}, k > 0\}$; $\{n \in 2N+1 \mid n \equiv 1 \pmod{4}\}$

O_2^* : $n \in O_2^* = \{n \in \mathbb{N} \mid n = 16k + 3, k \in \mathbb{N}, k \geq 0\}$; $\{n \in 2N+1 \mid n \equiv 3 \pmod{16}\}$
O_{2a} : $n \in O_{2a} = \{n \in \mathbb{N} \mid n = 16k + 11, k \in \mathbb{N}, k \geq 0\}$; $\{n \in 2N+1 \mid n \equiv 11 \pmod{16}\}$

$\mathbf{O_3^*}$: $n \in O_3^* = \{n \in \mathbb{N} \mid n = 32k + 23, k \in \mathbb{N}, k \geq 0\}$; $\{n \in 2N+1 \mid n \equiv 23 \pmod{32}\}$
$\mathbf{O_{3a}}$: $n \in O_{3a} = \{n \in \mathbb{N} \mid n = 32k + 31, k \in \mathbb{N}, k \geq 0\}$; $\{n \in 2N+1 \mid n \equiv 31 \pmod{32}\}$
$\mathbf{O_{3b}}$: $n \in O_{3b} = \{n \in \mathbb{N} \mid n = 32k + 7, k \in \mathbb{N}, k \geq 0\}$; $\{n \in 2N+1 \mid n \equiv 7 \pmod{32}\}$
$\mathbf{O_{3c}}$: $n \in O_{3c} = \{n \in \mathbb{N} \mid n = 32k + 15, k \in \mathbb{N}, k \geq 0\}$; $\{n \in 2N+1 \mid n \equiv 15 \pmod{32}\}$

2.2. Further analysis O_2

In this paragraph and in the following § 2.3. we apply notes c), d) § 2.

$\mathbf{O_{2a}}$

16n+11 O_{2a} : y = 8n+5 → f(y) = 18n+13 $O_1 \vee O_2 \vee O_3$: N(s) = 5

Chain of connections :
$\mathbf{O_{2a}}$ → [O_1] → **27n+10 < 32n+11** O_{2a} : N(s) = 8

Check :
32n+11 O_{2a} : y = 16n+5 → f(y) = 36n+13 O_1 : x = 18n+6 → f(x) = **27n+10 < 32n+11** O_{2a} = {11; 43; …} : N(s) = 8

Chain of connections :
$\mathbf{O_{2a}}$ → [O_2] → [E] → **81n+38 < 128n+59** O_{2a} : N(s) = 11

Check :
128n+59 O_{2a} : y = 64n+29 → f(y) = 144n+67 O_2* : y = 72n+33 → f(y) = 162n+76 →
81n+38 < 128n+59 O_{2a} = {59; 187; …} : N(s) = 11

Chain of connections. Complete procedure :
$\mathbf{O_{2a}}$ → [O_2] → [O_1] → **243n+118 < 256n +123** O_{2a} : N(s) = 13

16n+11 O_{2a} : y = 8n+5 → f(y) = 18n+13 [O_2] : n = **4n+3**

16n+11 O_{2a} → 16(**4n+3**)+11 **= 64n+59** O_{2a}
18n+13 → 18(**4n+3**)+13 = 72n+67 O_2 : y = 36n+33 → f(y) = 81n+76 [O] : n = **2n+1**

64n+59 O_{2a} → 64(**2n+1**)+59 = **128n+123** O_{2a}
f(y) = 81n+76 → 81(**2n+1**)+76 = 162n+157 [O_1] : n = **2n**

128n+123 O_{2a} → 128(**2n**)+123 = **256n+123** O_{2a}
162n+157 → 162(**2n**)+157 = 324n+157 O_1 : x = 162n+78 → f(x) = **243n+118 < 256n+123** O_{2a}

Check :
256n+123 O_{2a} : y = 128n+61 → f(y) = 288n+139 O_{2a} : y = 144n+69 → f(y) = 324n+157 O_1 :
x = 162n+78 → f(x) = **243n+118 < 256n +123** O_{2a} = {123; 379; …}: N(s) = 13

Chain of connections :
$\mathbf{O_{2a}}$ → [O_2] → [O_2] → [E] → **729n+362 < 1024n+507** : N(s) = 16

Check :
1024n+507 O_{2a} : y = 512n+253 → f(y) = 1152n+571 O_{2a} : y = 576n+285 →
f(y) = 1296n+643 O_2* : y = 648n+321 → f(y) = 1458n+728 →

$\mathbf{729n+362 < 1024n+507}$ = {507; 1531; …} : N(s) = 16

Chain of connections :
$\mathbf{O_{2a}} \to [O_2] \to [O_3] \to [E] \to [E] \to [E] \to \mathbf{2187n+955 < 4096n+1787}$ O_{2a} : N(s) = 19

Check :
$\mathbf{4096n+1787}$ O_{2a} : y = 2048n+893 → f(y) = 4608n+2011 O_{2a} : y = 2304+1005 →
f(y) = 5184n+2263 O_3* : z = 2592n+1131 → f(z) = 17496n+7640 → 8748n+3820 →
4374n+1910→ $\mathbf{2187n+955 < 4096n+1787}$ O_{2a} = {1787; …} : N(s) = 19

Chain of connections :
$\mathbf{O_{2a}} \to [O_3] \to [O_3] \to [E] \to [E] \to [E] \to [E] \to \mathbf{6561n+1868 < 8192n+2331}$ O_{2a} : N(s) = 21

Check :
$\mathbf{8192n+2331}$ O_{2a} : y = 4096n+1165 → f(y) = 9216n+2623 O_{3a} : z = 4608n+1311 →
f(z) = 31104n+8855 O_3* : z = 15552n+4427 → f(z) = 104976n+29888 →
52488n+14944 → 26244n+7472 → 13122n+3736 →
$\mathbf{6561n+1868 < 8192n+2331}$ O_{2a} = {2331; …} : N(s) = 21
… …

2.3. Further analysis O_3

$\mathbf{O_{3a}}$

$\mathbf{32n+31}$ O_{3a} : z = 16n+15 → f(z) = 108n+107 $O_2 \vee O_3$: N(s) = 6

Chain of connections :
$\mathbf{O_{3a}} \to [O_2] \to [E] \to [E] \to \mathbf{243n+91 < 256+95}$ O_{3a} : N(s) = 13

Check :
$\mathbf{256n+95}$ O_{3a} : z = 128n+47 → f(z) = 864n+323 O_2* : y = 432n+161 →
f(y) = 972n+364 → 486n+182 → $\mathbf{243n+91 < 256n+95}$ O_{3a} = {95; 351; …} : N(s) = 13

Chain of connections. Complete procedure :
$\mathbf{O_{3a}} \to [O_2] \to [O_1] \to [E] \to [E] \to \mathbf{729n+205 < 1024n+287}$ O_{3a} : N(s) = 16

$\mathbf{32n+31}$ O_{3a} : z = 16n+15 → f(z) = 108n+107 [O_2] : n = **2n**

$\mathbf{32n+31}$ O_{3a} → 32(**2n**)+31 = $\mathbf{64n+31}$ O_{3a}
108n+107 → 108(**2n**)+107 = 216+107 O_2 : y = 108+53 → f(y) = 243n+121 [O] : n = **2n**

$\mathbf{64n+31}$ O_{3a} → 64(**2n**)+31 = $\mathbf{128n+31}$ O_{3a}
243n+121 → 243(**2n**)+121 = 486n+121 [O_1] : n = **2n**

$\mathbf{128n+31}$ O_{3a} → 128(**2n**)+31 = $\mathbf{256n+31}$ O_{3a}
486n+121 → 486(**2n**)+121 = 972n+121 O_1 : x = 486n+60 →
f(x) = 729n+91 [E] : n = **2n+1**

$\mathbf{256n+31}$ O_{3a} → 256(**2n+1**)+31 = $\mathbf{512n+287}$ O_{3a}
729n+91 → 729(**2n+1**)+91 = 1458n+820 → 729n+410 [E] : n = **2n**

512n+287 O_{3a} → 512(**2n**)+287 = **1024n+287** O_{3a}
729n+410 → 729(**2n)**+410 = 1458n+410 → **729n+205 < 1024n+287** O_{3a}

Check :
1024n+287 O_{3a} : z = 512n+143 → f(z) = 3456n+971 O_{2a} : y = 1728n+485 →
f(y) = 3888n+1093 O_1 : x = 1944n+546 → f(x) = 2916n+820 → 1458n+410 →
729n+205 < 1024n+287 O_{3a} = {287; 1311; …} : N(s) = 16

Chain of connections :
$\mathbf{O_{3a}}$ → [O_3] → [O] → [E] → [E] → [E] → [E] → **2187n+205 < 4096n+383** O_{3a} : N(s) = 19

Check :
4096n+383 O_{3a} : z = 2048n+191 → f(z) = 13824n+1295 O_{3c} : z = 6912n+647 →
f(z) = 46656n+4373 O_1 : x = 23328n+2186 → f(x) = 34992n+3280 → 17496n+1640 →
8748n+820 → 4374n+410 → **2187n+205 < 4096n+383** O_{3a} = {383; …} : N(s) = 19

Chain of connections. Complete procedure :
$\mathbf{O_{3a}}$ → [O_3] → [O_2] → [E] → [E] → [E] → [E] → **6561n+205 < 8192n+255** O_{3a} : N(s) = 21

32n+31 O_{3a} : z = 16n+15 → f(z) = 108n+107 [O_3] : n = **2n+1**

32n+31 O_{3a} → 32(**2n+1**)+31 = **64n+63** O_{3a}
108n+107 → 108(**2n+1**)+107 = 216n+215 O_3 : z = 108n+107 →
f(z) = 729n+728 [O] : n = **2n+1**

64n+63 O_{3a} → 64(**2n+1**)+63 = **128n+127** O_{3a}
729n+728 → 729(**2n+1**)+728 = 1458n+1457 [O_2] : n = **4n+1**

128n+127 O_{3a} → 128(**4n+1)**+127 = **512n+255** O_{3a}
1458n+1457 → 1458(**4n+1**)+1457 = 5832n+2915 O_2 : y = 2916n+1457 →
f(y) = 6561n+3280 [E] : n = **2n**

512n+255 O_{3a} → 512(**2n**)+255 = **1024n+255** O_{3a}
6561n+3280 → 6561(**2n**)+3280 = 13122n+3280 → 6561n+1640 [E] : n = **2n**

1024n+255 O_{3a} → 1024(**2n**)+255 = **2048n+255** O_{3a}
6561n+1640 → 6561(**2n**)+1640 = 13122n+1640 → 6561n+820 [E] : n = **2n**

2048n+255 O_{3a} → 2048(**2n**)+255 = **4096n+255** O_{3a}
6561n+820 → 6561(**2n**)+820 = 13122n+820 → 6561n+410 [E] : n = **2n**

4096n+255 O_{3a} → 4096(**2n**)+255 = **8192n+255** O_{3a}
6561n+410 → 6561(**2n**)+410 = 13122n+410 → **6561n+205 < 8192n+255** O_{3a}

Check:
8192n+255 O_{3a} : z = 4096n+127 → f(z) = 27648n+863 O_{3a} : z = 13824n+431 →
f(z) = 93312n+2915 O_2* : y = 46656n+1457 → f(y) = 104976n+3280 →
52488n+1640 → 26244n+820 → 13122n+410 →
6561n+205 < 8192n+255 O_{3a} = {255; …} : N(s) = 21
… …

O_{3b}

32n+7 O_{3b} : z = 16n+3 → f(z) = 108n+26 → 54n+13 $O_1 \vee O_2 \vee O_3$: N(s) = 7

Chain of connections :
O_{3b} → E → [O_1] → [E] → **81n+5 < 128n+7** O_{3b} : N(s) =11

Check :
128n+7 O_{3b} : z = 64n+3 → f(z) = 432n+26 → 216n+13 O_1 : x = 108n+6 →
f(x) = 162n+10 → **81n+5 < 128n+7** O_{3b} = {7; 135; …} : N(s) = 11

Chain of connections :
O_{3b} → E → [O_2] → [E] → **243n+38 < 256n+39** O_{3b} : N(s) = 13

Check :
256n+39 O_{3b} : z = 128n+19 → f(z) = 864n+134 → 432n+67 O_2* : y = 216n+33 →
f(y) = 486n+76 → **243n+38 < 256n+39** O_{3b} = {39; 295; …} : N(s) = 13

Chain of connections :
O_{3b} → E → [O_2] → [O_1] → [E] → **729n+302 < 1024n+423** O_{3b} : N(s) = 16

Check :
1024n+423 O_{3b} : z = 512n+211 → f(z) = 3456n+1430 → 1728n+715 O_{2a} : y = 864n+357 →
f(y) = 1944n+805 O_1 : x = 972n+402 → f(x) = 1458n+604 → **729n+302 < 1024n+423** O_{3b} =
{423; 1447; …} : N(s) = 16

Chain of connections . Complete procedure :
O_{3b} → E → [O_3] → [O_1] → [E] → [E] → [E] → **2187n+329** < **4096n+615** O_{3b} : N(s) = 19

32n+7 O_{3b} : z = 16n+3 → f(z) = 108n+26 → 54n+13 [O_3] : n = **4n+3**

32n+7 O_{3b} → 32(**4n+3**)+7 = **128n+103** O_{3b}
54n+13 → 54(**4n+3**)+13 = 216n+175 O_3 : z = 108n+87 → f(z) = 729n+593 [O] : n = **2n**

32n+7 O_{3b} → 32(**2n**)+7 = **256n+103** O_{3b}
729n+593 → 729(**2n**)+593 = 1458n+593 [O_1] : n = **2n**

256n+103 O_{3b} → 256(**2n**)+103 **= 512n+103** O_{3b}
1458n+593 → 1458(**2n**)+593 = 2916n+593 O_1 : x = 1458n+296 →
f(x) = 2187n+445 [E]: n = **2n+1**

512n+103 O_{3b} → 512(**2n+1**)+103 = **1024n+615** O_{3b}
2187n+445 → 2187(**2n+1**)+445 = 4374n+2632 → 2187n+1316 [E] : n = **2n**

1024n+615 O_{3b} → 1024(**2n**)+615 = **2048n+615** O_{3b}
2187n+1316 → 2187(**2n**)+1316 = 4374n+2632 → 2187n+658 [E] : n = **2n**

2048n+615 O_{3b} → 2048(**2n**)+615 = **4096n+615** O_{3b}
2187n+658 → 2187(**2n**)+658 = 4374n+658 → **2187n+329** < **4096n+615** O_{3b}

Check :
4096n+615 O_{3b} : z = 2048n+307 → f(z) = 13824n+2078 → 6912n+1039 O_{3c} : z = 3456n+519 → f(z) = 23328n+3509 O_1 : x = 11664n+1754 → f(x) = 17496n+2632 → 8748n+1316 → 4374n+658 → **2187n+329** < **4096n+615** O_{3b} = {615; …} : N(s) = 19

Chain of connections :
O_{3b} → E → [O_2] → [O_3] → [E] → [E] → [E] → **6561n+955 < 8192n+1191** O_{3b} : N(s) = 21

Check :
8192n+1191 O_{3b} : z = 4096n+595 → f(z) = 27648n+4022 →
13824n+2011 O_{2a} : y = 6912n+1005 → f(y) = 15552n+2263 O_3* : z = 7776n+1131 →
f(z) = 52488n+7640 → 26244n+3820 → 13122n+1910 →
6561n+955 < 8192n+1191 O_{3b} = {1191; …} : N(s) = 21
… …

O_{3c}

32n+15 O_{3c} : z = 16n+7 → f(z) = 108n+53 O_1 : x = 54n+26 → f(x) = 81n+40 E ∨ O : N(s) = 9

Chain of connections :
O_{3c} → O_1 → [E] → [E] → **81n+10** < **128n+15** O_{3c} : N(s) = 11

Check :
128n+15 O_{3c} : z = 64n+7 → f(z) = 432n+53 O_1 : x = 216n+26 → f(x) = 324n+40 →
162n+20 → **81n+10** < **128n+15** O_{3c} = {15; 143; …} : N(s) = 11

Chain of connections :
O_{3c} → O_1 → [E] → [O_1] → **243n+76** < **256n+79** O_{3c} : N(s) = 13

Check :
256n+79 O_{3c} : z = 128n+39 → f(z) = 864n+269 O_1 : x = 432n+134 →
f(x) = 648n+202 → 324n+101 O_1 : x = 162n+50 →
f(x) = **243n+76** < **256n+79** O_{3c} = {79; 335; …} : N(s) = 13

Chain of connections :
O_{3c} → O_1 → [O_2] → [E] → [E] → **729n+262** < **1024n+367** O_{3c} : N(s) = 16

Check :
1024n+367 O_{3c} : z = 512n+183 → f(z) = 3456n+1241 O_1 : x = 1728n+620 →
f(x) = 2592n+931 O_2* : y = 1296n+465 → f(y) = 2916n+1048 → 1458n+524 →
729n+262 < **1024n+367** O_{3c} = {367; 1391; …} : N(s) = 16

Chain of connections :
O_{3c} → O_1 → [E] → [O_2] → [O_1] → [E] → **2187n+248 < 4096n+463** O_{3c} : N(s) = 19

Check :
4096n+463 O_{3c} : z = 2048n+231 → f(z) = 13824n+1565 O_1 : x = 6912n+782 →
f(x) = 10368n+1174 → 5184n+587 O_{2a} : y = 2592n+293 →
f(y) = 5832n+661 O_1 : x = 2916n+330 → f(x) = 4374n+496 →
2187n+248 < 4096n+463 O_{3c} = {463; …} : N(s) = 19

Chain of connections. Complete procedure :
$\mathbf{O_{3c}} \rightarrow \mathbf{O_1} \rightarrow [O_1] \rightarrow [O_3] \rightarrow [E] \rightarrow [E] \rightarrow [E] \rightarrow$ **6561n+2089** < **8192n+2607** O_{3c} : N(s) = 21

32n+15 O_{3c} : z = 16n+7 → f(z) = 108n+53 O_1 : x = 54n+26 → f(x) = 81n+40 [O] : n = **2n+1**

32n+15 O_{3c} → 32(**2n+1**)+15 = **64n+47** O_{3c}
81n+40 → 81(**2n+1**)+40 = 162n+121 [O_1] : n = **2n**

64n+47 O_{3c} → 64(**2n**)+47 = **128n+47** O_{3c}
162n+121 → 162(**2n**)+121 = 324n+121 O_1: x = 162n+60 → f(x) = 243n+91 [O] : n = **2n**

128n+47 O_{3c} → 128(**2n**)+47 = **256n+47** O_{3c}
243n+91 → 243(**2n**)+91 = 486n+91 [O_3] : n = **4n+2**

256n+47 O_{3c} → 256(**4n+2**)+47 = **1024n+559** O_{3c}
486n+91 → 486(**4n+2**)+91 = 1944n+1063 O_3 : z = 972n+531 → f(z) = 6561n+3590 [E] : n = **2n**

1024n+559 O_{3c} → 1024(**2n**)+559 = **2048n+559** O_{3c}
6561n+3590 → 6561(**2n**)+3590 = 13122n+3590 → 6561n+1795 [E] : n = **2n+1**

2048n+559 O_{3c} → 2048(**2n+1**)+559 = **4096n+2607** O_{3c}
6561n+1795 → 6561(**2n+1**)+1795 = 13122n+8356 → 6561n+4178 [E] : n = **2n**

4096n+2607 O_{3c} → 4096(**2n**)+2607 = **8192n+2607** O_{3c}
6561n+4178 → 6561(**2n**)+4178 → 13122n+4178 → **6561n+2089 < 8192n+2607** O_{3c}

Check :
8192n+2607 O_{3c} : z = 4096n+1303 → f(z) = 27648n+8801 O_1 : x = 13824n+ 4400 →
f(x) = 20736n+6601 O_1 : x = 10368n+3300 → f(x) = 15552n+4951 O_3* : z = 7776n+2475 →
f(z) = 52488n+16712 → 26244n+8356 → 13122n+4178 →
6561n+2089 < **8192n+2607** O_{3c} = {2607; …} : N(s) = 21
… …
… …

By the procedure previously illustrated, appropriately choosing the connections between the eight basic cycles (even numbers, plus the seven odd numbers subsets), applying the formulas obtained from Theorem 2n+1, using Theorem of Independence (§ 3.2.); we arrive to the following General List binomial inequalities until N(s) ≤ 21.

2.4. General List binomial inequalities N(s) ≤ 21

N(s) = **3**

3n+1 < 4n+1 O_1 = $\mathbf{3n+1 < 2^2 \cdot n+1}$ O_1

N(s) = **6**

9n+2 < 16n+3 O_2* = $\mathbf{3^2 \cdot n+2 < 2^4 \cdot n+3}$ O_2*

N(s) = **8**

$27n+10 < 32n+11\ O_{2a} = \mathbf{3^3 \cdot n+10 < 2^5 \cdot n+11}\ O_{2a}$
$27n+20 < 32n+23\ O_3{}^* = \mathbf{3^3 \cdot n+20 < 2^5 \cdot n+23}\ O_3{}^*$

N(s) = **11**

$81n+5 < 128n+7\ O_{3b} = \mathbf{3^4 \cdot n+5 < 2^7 \cdot n+7}\ O_{3b}$
$81n+10 < 128n+15\ O_{3c} = \mathbf{3^4 \cdot n+10 < 2^7 \cdot n+15}\ O_{3c}$
$81n+38 < 128n+59\ O_{2a} = \mathbf{3^4 \cdot n+38 < 2^7 \cdot n+59}\ O_{2a}$

N(s) = **13**

$243n+38 < 256n+39\ O_{3b} = \mathbf{3^5 \cdot n+38 < 2^8 \cdot n+39}\ O_{3b}$
$243n+76 < 256n+79\ O_{3c} = \mathbf{3^5 \cdot n+76 < 2^8 \cdot n+79}\ O_{3c}$
$243n+91 < 256n+95\ O_{3a} = \mathbf{3^5 \cdot n+91 < 2^8 \cdot n+95}\ O_{3a}$
$243n+118 < 256n+123\ O_{2a} = \mathbf{3^5 \cdot n+118 < 2^8 \cdot n+123}\ O_{2a}$
$243n+167 < 256n+175\ O_{3c} = \mathbf{3^5 \cdot n+167 < 2^8 \cdot n+175}\ O_{3c}$
$243n+190 < 256n+199\ O_{3b} = \mathbf{3^5 \cdot n+190 < 2^8 \cdot n+199}\ O_{3b}$
$243n+209 < 256n+219\ O_{2a} = \mathbf{3^5 \cdot n+209 < 2^8 \cdot n+219}\ O_{2a}$

N(s) = **16**

$729n+205 < 1024n+287\ O_{3a} = \mathbf{3^6 \cdot n+205 < 2^{10} \cdot n+287}\ O_{3a}$
$729n+248 < 1024n+347\ O_{2a} = \mathbf{3^6 \cdot n+248 < 2^{10} \cdot n+347}\ O_{2a}$
$729n+262 < 1024n+367\ O_{3c} = \mathbf{3^6 \cdot n+262 < 2^{10} \cdot n+367}\ O_{3c}$
$729n+302 < 1024n+423\ O_{3b} = \mathbf{3^6 \cdot n+302 < 2^{10} \cdot n+423}\ O_{3b}$
$729n+362 < 1024n+507\ O_{2a} = \mathbf{3^6 \cdot n+362 < 2^{10} \cdot n+507}\ O_{2a}$
$729n+410 < 1024n+575\ O_{3a} = \mathbf{3^6 \cdot n+410 < 2^{10} \cdot n+575}\ O_{3a}$
$729n+416 < 1024n+583\ O_{3b} = \mathbf{3^6 \cdot n+416 < 2^{10} \cdot n+583}\ O_{3b}$
$729n+524 < 1024n+735\ O_{3a} = \mathbf{3^6 \cdot n+524 < 2^{10} \cdot n+735}\ O_{3a}$
$729n+581 < 1024n+815\ O_{3c} = \mathbf{3^6 \cdot n+581 < 2^{10} \cdot n+815}\ O_{3c}$
$729n+658 < 1024n+923\ O_{2a} = \mathbf{3^6 \cdot n+658 < 2^{10} \cdot n+923}\ O_{2a}$
$729n+695 < 1024n+975\ O_{3c} = \mathbf{3^6 \cdot n+695 < 2^{10} \cdot n+975}\ O_{3c}$
$729n+712 < 1024n+999\ O_{3b} = \mathbf{3^6 \cdot n+712 < 2^{10} \cdot n+999}\ O_{3b}$

N(s) = **19**

$2187n+124 < 4096n+231\ O_{3b} = \mathbf{3^7 \cdot n+124 < 2^{12} \cdot n+231}\ O_{3b}$
$2187n+205 < 4096n+383\ O_{3a} = \mathbf{3^7 \cdot n+205 < 2^{12} \cdot n+383}\ O_{3a}$
$2187n+248 < 4096n+463\ O_{3c} = \mathbf{3^7 \cdot n+248 < 2^{12} \cdot n+463}\ O_{3c}$
$2187n+329 < 4096n+615\ O_{3b} = \mathbf{3^7 \cdot n+329 < 2^{12} \cdot n+615}\ O_{3b}$
$2187n+470 < 4096n+879\ O_{3c} = \mathbf{3^7 \cdot n+470 < 2^{12} \cdot n+879}\ O_{3c}$
$2187n+500 < 4096n+935\ O_{3b} = \mathbf{3^7 \cdot n+500 < 2^{12} \cdot n+935}\ O_{3b}$
$2187n+545 < 4096n+1019\ O_{2a} = \mathbf{3^7 \cdot n+545 < 2^{12} \cdot n+1019}\ O_{2a}$
$2187n+581 < 4096n+1087\ O_{3a} = \mathbf{3^7 \cdot n+581 < 2^{12} \cdot n+1087}\ O_{3a}$
$2187n+658 < 4096n+1231\ O_{3c} = \mathbf{3^7 \cdot n+658 < 2^{12} \cdot n+1231}\ O_{3c}$
$2187n+767 < 4096n+1435\ O_{2a} = \mathbf{3^7 \cdot n+767 < 2^{12} \cdot n+1435}\ O_{2a}$
$2187n+880 < 4096n+1647\ O_{3c} = \mathbf{3^7 \cdot n+880 < 2^{12} \cdot n+1647}\ O_{3c}$
$2187n+910 < 4096n+1703\ O_{3b} = \mathbf{3^7 \cdot n+910 < 2^{12} \cdot n+1703}\ O_{3b}$
$2187n+955 < 4096n+1787\ O_{2a} = \mathbf{3^7 \cdot n+955 < 2^{12} \cdot n+1787}\ O_{2a}$

$2187n+974 < 4096n+1823$ O_{3a} = $\mathbf{3^7 \cdot n+974 < 2^{12} \cdot n+1823}$ O_{3a}
$2187n+991 < 4096n+1855$ O_{3a} = $\mathbf{3^7 \cdot n+991 < 2^{12} \cdot n+1855}$ O_{3a}
$2187n+1085 < 4096n+2031$ O_{3c} = $\mathbf{3^7 \cdot n+1085 < 2^{12} \cdot n+2031}$ O_{3c}
$2187n+1177 < 4096n+2203$ O_{2a} = $\mathbf{3^7 \cdot n+1177 < 2^{12} \cdot n+2203}$ O_{2a}
$2187n+1196 < 4096n+2239$ O_{3a} = $\mathbf{3^7 \cdot n+1196 < 2^{12} \cdot n+2239}$ O_{3a}
$2187n+1256 < 4096n+2351$ O_{3c} = $\mathbf{3^7 \cdot n+1256 < 2^{12} \cdot n+2351}$ O_{3c}
$2187n+1382 < 4096n+2587$ O_{2a} = $\mathbf{3^7 \cdot n+1382 < 2^{12} \cdot n+2587}$ O_{2a}
$2187n+1384 < 4096n+2591$ O_{3a} = $\mathbf{3^7 \cdot n+1384 < 2^{12} \cdot n+2591}$ O_{3a}
$2187n+1553 < 4096n+2907$ O_{2a} = $\mathbf{3^7 \cdot n+1553 < 2^{12} \cdot n+2907}$ O_{2a}
$2187n+1589 < 4096n+2975$ O_{3a} = $\mathbf{3^7 \cdot n+1589 < 2^{12} \cdot n+2975}$ O_{3a}
$2187n+1666 < 4096n+3119$ O_{3c} = $\mathbf{3^7 \cdot n+1666 < 2^{12} \cdot n+3119}$ O_{3c}
$2187n+1679 < 4096n+3143$ O_{3b} = $\mathbf{3^7 \cdot n+1679 < 2^{12} \cdot n+3143}$ O_{3b}
$2187n+1760 < 4096n+3295$ O_{3a} = $\mathbf{3^7 \cdot n+1760 < 2^{12} \cdot n+3295}$ O_{3a}
$2187n+1901 < 4096n+3559$ O_{3b} = $\mathbf{3^7 \cdot n+1901 < 2^{12} \cdot n+3559}$ O_{3b}
$2187n+1963 < 4096n+3675$ O_{2a} = $\mathbf{3^7 \cdot n+1963 < 2^{12} \cdot n+3675}$ O_{2a}
$2187n+2089 < 4096n+3911$ O_{3b} = $\mathbf{3^7 \cdot n+2089 < 2^{12} \cdot n+3911}$ O_{3b}
$2187n+2170 < 4096n+4063$ O_{3a} = $\mathbf{3^7 \cdot n+2170 < 2^{12} \cdot n+4063}$ O_{3a}

N(s) = **21**

$6561n+154 < 8192n+191$ O_{3a} = $\mathbf{3^8 \cdot n+154 < 2^{13} \cdot n+191}$ O_{3a}
$6561n+167 < 8192n+207$ O_{3c} = $\mathbf{3^8 \cdot n+167 < 2^{13} \cdot n+207}$ O_{3c}
$6561n+205 < 8192n+255$ O_{3a} = $\mathbf{3^8 \cdot n+205 < 2^{13} \cdot n+255}$ O_{3a}
$6561n+244 < 8192n+303$ O_{3c} = $\mathbf{3^8 \cdot n+244 < 2^{13} \cdot n+303}$ O_{3c}
$6561n+433 < 8192n+539$ O_{2a} = $\mathbf{3^8 \cdot n+433 < 2^{13} \cdot n+539}$ O_{2a}
$6561n+436 < 8192n+543$ O_{3a} = $\mathbf{3^8 \cdot n+436 < 2^{13} \cdot n+543}$ O_{3a}
$6561n+500 < 8192n+623$ O_{3c} = $\mathbf{3^8 \cdot n+500 < 2^{13} \cdot n+623}$ O_{3c}
$6561n+545 < 8192n+679$ O_{3b} = $\mathbf{3^8 \cdot n+545 < 2^{13} \cdot n+679}$ O_{3b}
$6561n+577 < 8192n+719$ O_{3c} = $\mathbf{3^8 \cdot n+577 < 2^{13} \cdot n+719}$ O_{3c}
$6561n+641 < 8192n+799$ O_{3a} = $\mathbf{3^8 \cdot n+641 < 2^{13} \cdot n+799}$ O_{3a}
$6561n+859 < 8192n+1071$ O_{3c} = $\mathbf{3^8 \cdot n+859 < 2^{13} \cdot n+1071}$ O_{3c}
$6561n+910 < 8192n+1135$ O_{3c} = $\mathbf{3^8 \cdot n+910 < 2^{13} \cdot n+1135}$ O_{3c}
$6561n+955 < 8192n+1191$ O_{3b} = $\mathbf{3^8 \cdot n+955 < 2^{13} \cdot n+1191}$ O_{3b}
$6561n+974 < 8192n+1215$ O_{3a} = $\mathbf{3^8 \cdot n+974 < 2^{13} \cdot n+1215}$ O_{3a}
$6561n+1000 < 8192n+1247$ O_{3a} = $\mathbf{3^8 \cdot n+1000 < 2^{13} \cdot n+1247}$ O_{3a}
$6561n+1064 < 8192n+1327$ O_{3c} = $\mathbf{3^8 \cdot n+1064 < 2^{13} \cdot n+1327}$ O_{3c}
$6561n+1253 < 8192n+1563$ O_{2a} = $\mathbf{3^8 \cdot n+1253 < 2^{13} \cdot n+1563}$ O_{2a}
$6561n+1256 < 8192n+1567$ O_{3a} = $\mathbf{3^8 \cdot n+1256 < 2^{13} \cdot n+1567}$ O_{3a}
$6561n+1384 < 8192n+1727$ O_{3a} = $\mathbf{3^8 \cdot n+1384 < 2^{13} \cdot n+1727}$ O_{3a}
$6561n+1589 < 8192n+1983$ O_{3a} = $\mathbf{3^8 \cdot n+1589 < 2^{13} \cdot n+1983}$ O_{3a}
$6561n+1615 < 8192n+2015$ O_{3a} = $\mathbf{3^8 \cdot n+1615 < 2^{13} \cdot n+2015}$ O_{3a}
$6561n+1663 < 8192n+2075$ O_{2a} = $\mathbf{3^8 \cdot n+1663 < 2^{13} \cdot n+2075}$ O_{2a}
$6561n+1666 < 8192n+2079$ O_{3a} = $\mathbf{3^8 \cdot n+1666 < 2^{13} \cdot n+2079}$ O_{3a}
$6561n+1679 < 8192n+2095$ O_{3c} = $\mathbf{3^8 \cdot n+1679 < 2^{13} \cdot n+2095}$ O_{3c}
$6561n+1820 < 8192n+2271$ O_{3a} = $\mathbf{3^8 \cdot n+1820 < 2^{13} \cdot n+2271}$ O_{3a}
$6561n+1868 < 8192n+2331$ O_{2a} = $\mathbf{3^8 \cdot n+1868 < 2^{13} \cdot n+2331}$ O_{2a}
$6561n+1948 < 8192n+2431$ O_{3a} = $\mathbf{3^8 \cdot n+1948 < 2^{13} \cdot n+2431}$ O_{3a}
$6561n+2089 < 8192n+2607$ O_{3c} = $\mathbf{3^8 \cdot n+2089 < 2^{13} \cdot n+2607}$ O_{3c}
$6561n+2134 < 8192n+2663$ O_{3b} = $\mathbf{3^8 \cdot n+2134 < 2^{13} \cdot n+2663}$ O_{3b}
$6561n+2435 < 8192n+3039$ O_{3a} = $\mathbf{3^8 \cdot n+2435 < 2^{13} \cdot n+3039}$ O_{3a}
$6561n+2458 < 8192n+3067$ O_{2a} = $\mathbf{3^8 \cdot n+2458 < 2^{13} \cdot n+3067}$ O_{2a}

$6561n+2512 < 8192n+3135\ O_{3a} = \mathbf{3^8 \cdot n+2512 < 2^{13} \cdot n+3135}\ O_{3a}$

$6561n+2768 < 8192n+3455\ O_{3a} = \mathbf{3^8 \cdot n+2768 < 2^{13} \cdot n+3455}\ O_{3a}$

$6561n+2791 < 8192n+3483\ O_{2a} = \mathbf{3^8 \cdot n+2791 < 2^{13} \cdot n+3483}\ O_{2a}$

$6561n+2845 < 8192n+3551\ O_{3a} = \mathbf{3^8 \cdot n+2845 < 2^{13} \cdot n+3551}\ O_{3a}$

$6561n+2954 < 8192n+3687\ O_{3b} = \mathbf{3^8 \cdot n+2954 < 2^{13} \cdot n+3687}\ O_{3b}$

$6561n+3073 < 8192n+3835\ O_{2a} = \mathbf{3^8 \cdot n+3073 < 2^{13} \cdot n+3835}\ O_{2a}$

$6561n+3127 < 8192n+3903\ O_{3a} = \mathbf{3^8 \cdot n+3127 < 2^{13} \cdot n+3903}\ O_{3a}$

$6561n+3178 < 8192n+3967\ O_{3a} = \mathbf{3^8 \cdot n+3178 < 2^{13} \cdot n+3967}\ O_{3a}$

$6561n+3268 < 8192n+4079\ O_{3c} = \mathbf{3^8 \cdot n+3268 < 2^{13} \cdot n+4079}\ O_{3c}$

$6561n+3278 < 8192n+4091\ O_{2a} = \mathbf{3^8 \cdot n+3278 < 2^{13} \cdot n+4091}\ O_{2a}$

$6561n+3332 < 8192n+4159\ O_{3a} = \mathbf{3^8 \cdot n+3332 < 2^{13} \cdot n+4159}\ O_{3a}$

$6561n+3364 < 8192n+4199\ O_{3b} = \mathbf{3^8 \cdot n+3364 < 2^{13} \cdot n+4199}\ O_{3b}$

$6561n+3383 < 8192n+4223\ O_{3a} = \mathbf{3^8 \cdot n+3383 < 2^{13} \cdot n+4223}\ O_{3a}$

$6561n+3406 < 8192n+4251\ O_{2a} = \mathbf{3^8 \cdot n+3406 < 2^{13} \cdot n+4251}\ O_{2a}$

$6561n+3569 < 8192n+4455\ O_{3b} = \mathbf{3^8 \cdot n+3569 < 2^{13} \cdot n+4455}\ O_{3b}$

$6561n+3661 < 8192n+4507\ O_{2a} = \mathbf{3^8 \cdot n+3661 < 2^{13} \cdot n+4507}\ O_{2a}$

$6561n+3893 < 8192n+4859\ O_{2a} = \mathbf{3^8 \cdot n+3893 < 2^{13} \cdot n+4859}\ O_{2a}$

$6561n+3947 < 8192n+4927\ O_{3a} = \mathbf{3^8 \cdot n+3947 < 2^{13} \cdot n+4927}\ O_{3a}$

$6561n+3970 < 8192n+4955\ O_{2a} = \mathbf{3^8 \cdot n+3970 < 2^{13} \cdot n+4955}\ O_{2a}$

$6561n+4024 < 8192n+5023\ O_{3a} = \mathbf{3^8 \cdot n+4024 < 2^{13} \cdot n+5023}\ O_{3a}$

$6561n+4088 < 8192n+5103\ O_{3c} = \mathbf{3^8 \cdot n+4088 < 2^{13} \cdot n+5103}\ O_{3c}$

$6561n+4159 < 8192n+5191\ O_{3b} = \mathbf{3^8 \cdot n+4159 < 2^{13} \cdot n+5191}\ O_{3b}$

$6561n+4226 < 8192n+5275\ O_{2a} = \mathbf{3^8 \cdot n+4226 < 2^{13} \cdot n+5275}\ O_{2a}$

$6561n+4303 < 8192n+5371\ O_{2a} = \mathbf{3^8 \cdot n+4303 < 2^{13} \cdot n+5371}\ O_{2a}$

$6561n+4357 < 8192n+5439\ O_{3a} = \mathbf{3^8 \cdot n+4357 < 2^{13} \cdot n+5439}\ O_{3a}$

$6561n+4492 < 8192n+5607\ O_{3b} = \mathbf{3^8 \cdot n+4492 < 2^{13} \cdot n+5607}\ O_{3b}$

$6561n+4498 < 8192n+5615\ O_{3c} = \mathbf{3^8 \cdot n+4498 < 2^{13} \cdot n+5615}\ O_{3c}$

$6561n+4585 < 8192n+5723\ O_{2a} = \mathbf{3^8 \cdot n+4585 < 2^{13} \cdot n+5723}\ O_{2a}$

$6561n+4636 < 8192n+5787\ O_{2a} = \mathbf{3^8 \cdot n+4636 < 2^{13} \cdot n+5787}\ O_{2a}$

$6561n+4703 < 8192n+5871\ O_{3c} = \mathbf{3^8 \cdot n+4703 < 2^{13} \cdot n+5871}\ O_{3c}$

$6561n+4774 < 8192n+5959\ O_{3b} = \mathbf{3^8 \cdot n+4774 < 2^{13} \cdot n+5959}\ O_{3b}$

$6561n+4790 < 8192n+5979\ O_{2a} = \mathbf{3^8 \cdot n+4790 < 2^{13} \cdot n+5979}\ O_{2a}$

$6561n+4844 < 8192n+6047\ O_{3a} = \mathbf{3^8 \cdot n+4844 < 2^{13} \cdot n+6047}\ O_{3a}$

$6561n+4979 < 8192n+6215\ O_{3b} = \mathbf{3^8 \cdot n+4979 < 2^{13} \cdot n+6215}\ O_{3b}$

$6561n+5107 < 8192n+6375\ O_{3b} = \mathbf{3^8 \cdot n+5107 < 2^{13} \cdot n+6375}\ O_{3b}$

$6561n+5254 < 8192n+6559\ O_{3a} = \mathbf{3^8 \cdot n+5254 < 2^{13} \cdot n+6559}\ O_{3a}$

$6561n+5293 < 8192n+6607\ O_{3c} = \mathbf{3^8 \cdot n+5293 < 2^{13} \cdot n+6607}\ O_{3c}$

$6561n+5312 < 8192n+6631\ O_{3b} = \mathbf{3^8 \cdot n+5312 < 2^{13} \cdot n+6631}\ O_{3b}$

$6561n+5405 < 8192n+6747\ O_{2a} = \mathbf{3^8 \cdot n+5405 < 2^{13} \cdot n+6747}\ O_{2a}$

$6561n+5459 < 8192n+6815\ O_{3a} = \mathbf{3^8 \cdot n+5459 < 2^{13} \cdot n+6815}\ O_{3a}$

$6561n+5594 < 8192n+6983\ O_{3b} = \mathbf{3^8 \cdot n+5594 < 2^{13} \cdot n+6983}\ O_{3b}$

$6561n+5626 < 8192n+7023\ O_{3c} = \mathbf{3^8 \cdot n+5626 < 2^{13} \cdot n+7023}\ O_{3c}$

$6561n+5671 < 8192n+7079\ O_{3b} = \mathbf{3^8 \cdot n+5671 < 2^{13} \cdot n+7079}\ O_{3b}$

$6561n+5815 < 8192n+7259\ O_{2a} = \mathbf{3^8 \cdot n+5815 < 2^{13} \cdot n+7259}\ O_{2a}$

$6561n+5908 < 8192n+7375\ O_{3c} = \mathbf{3^8 \cdot n+5908 < 2^{13} \cdot n+7375}\ O_{3c}$

$6561n+5927 < 8192n+7399\ O_{3b} = \mathbf{3^8 \cdot n+5927 < 2^{13} \cdot n+7399}\ O_{3b}$

$6561n+6004 < 8192n+7495\ O_{3b} = \mathbf{3^8 \cdot n+6004 < 2^{13} \cdot n+7495}\ O_{3b}$

$6561n+6113 < 8192n+7631\ O_{3c} = \mathbf{3^8 \cdot n+6113 < 2^{13} \cdot n+7631}\ O_{3c}$

$6561n+6241 < 8192n+7791\ O_{3c} = \mathbf{3^8 \cdot n+6241 < 2^{13} \cdot n+7791}\ O_{3c}$

$6561n+6286 < 8192n+7847\ O_{3b} = \mathbf{3^8 \cdot n+6286 < 2^{13} \cdot n+7847}\ O_{3b}$

$6561n+6337 < 8192n+7911\ O_{3b} = \mathbf{3^8 \cdot n+6337 < 2^{13} \cdot n+7911}\ O_{3b}$

6561n+6382 < 8192n+7967 O_{3a} = $\mathbf{3^8 \cdot n+6382 < 2^{13} \cdot n+7967}$ O_{3a}
6561n+6446 < 8192n+8047 O_{3c} = $\mathbf{3^8 \cdot n+6446 < 2^{13} \cdot n+8047}$ O_{3c}
6561n+6491 < 8192n+8103 O_{3b} = $\mathbf{3^8 \cdot n+6491 < 2^{13} \cdot n+8103}$ O_{3b}

The binomial inequalities are of type :

$$\boldsymbol{3^h \cdot n + q < \ 2^k \cdot n + p}$$

- p is the specific known term of the main horizons; q is the corresponding specific known term of the lower horizons : $q < p$.
- h + k = N(s) number of steps : N(s) increment = 2 or 3
- Increase of k = 1 or 2 : 1 if N(s) increment = 2 ; 2 if N(s) increment = 3
- Increase of h always 1

Collatz sequences arrive at their respective links by unpredictable ways and without generalized rules, except those indicated above. For N(s) = 21 there are 85 chains of connections and related links. The number of links increase considerably with the increase of N(s), but, at the same time, their percentages are significantly reduced as well as the coverage of ℕ. Both → 0 (§ 2.4.1 ; § 3.6.).

Remark
To compile the above list, in some cases of the same N(s), the following method was used.
If ϴ(m1) < ϴ(m2) ($p_1 < p_2$) are two consecutive main horizons of two equivalent cycles of links, and p is a generic specific known term such that $p_1 < \ p \ < p_2$, then : $p = \{p_2 \text{ mod4} - p \in O_2^* \cup p \in O_3^* \cup p \sim$ N(s) previously calculated$\}$ (§ 3.3.3.).

2.4.1. Percentages

From the general list above we have the following percentages coverage of ℕ :

N(s) = 1 E : 2n → n … … … … … … … … … … … … … … … … … 50
N(s) = 3 O_1 : 4n+1 → 3n + 1 … … … … … … … … … … … … … … 25
N(s) = 6 O_2* : 16n+3 → 9n+2 … … … … … … … … … … … … … … 6.25
N(s) = 8 O_{2a} ; O_3* … … … … … … … … … … … … … … … … … … 6.25
N(s) = 11 : 0.78125 x 3 … … … … … … … … … … … … … … … … … 2.344
N(s) = 13 : 0.26042… x 7 … … … … … … … … … … … … … … … … 1.823
N(s) = 16 : 0.05229… x 12 … … … … … … … … … … … … … … … 0.625
N(s) = 19 : 0.01046… x 30 … … … … … … … … … … … … … … … 0.314
N(s) = 21 : 0.00349… x 85 … … … … … … … … … … … … … … … 0.296

≈ 93 %

The remaining 7% is covered by binomial inequalities of the infinite cycles of links. By apposite calculation tools it is possible to reach close by 100% of ℕ, but, as it will be proved, it's not possible to arrive 100% coverage of ℕ.

2.a. Appendices
2.a.1. Discrete Growth Law. Analysis of Steps Increments

Empirical data from SCQ show that the number of steps required to establish a *link* does not increase regularly, but through discrete jumps of 2 or 3. An increment of 2 steps occurs when the

principal horizons $\Theta(m) = 2^k \cdot n+p$ has an increment δ(k) = 1; while 3 steps occurs when δ(k) = 2. This alternation suggests that the structure of the seven subsets is governed by an *arithmetic resonance logic*, following a discrete growth law. This law is repeated within the binomial inequalities observed in the connecting link of the number 27 (§ 3.3.1.). These jumps are not arbitrary events, but linear compositions of elementary increments of 2 or 3. This proves that the *arithmetical dynamics* of SCQ remains consistent even in its most chaotic phases; it simply requires a more extensive "chain" of elementary steps to trigger a resolution link.
In summary : Δ[N(s)] = 2 ∨ 3 . If *D*[N(s)] is the density of N(s), then lim(n→∞)*D*[N(s)] = ∞ .

2.a.2. Quantitative Analysis of binomial inequalities (the "Links Density")

From the analysis of the General List of Binomial Inequalities N(s) ≤ 21 (§ 2.4.), we can observe that up to a number of steps N(s) = 6, the system is rigid (only one binomial inequality relating to the subsets E; O_1; O_2*). From N(s) = 8 to N(s) = 13, the number of links increase slightly (from 2 to 7). From N(s) = 13 to N(s) = 16 (3 steps) result a links density δ(l) = 5 (from 7 to 12 links). From N(s) = 16 to N(s) = 19 (3 steps), there is a jump δ(l) = 18 (from 12 to 30 links) with an growth factor χ = 2.5. From N(s) = 19 to N(s) = 21 (2 steps) there is a jump δ(l) = 55 (from 30 to 85 links) with growth factor χ = 2.83. If the progression, as is logically predictable, maintains the speed acquired by the combinations of the eight basic subsets (E + 7 odd), from N(s) = 21 to N(s) = 24 (3 steps) we can estimate, with an acceptable approximation, a links increment Δ(l) ≈ 350-400, with a jump δ(l) ≅ 300 and a growth factor χ ≈ 3.5. We can conclude that the ratio δ(l)/*D*[N(s)] tends to diverge, i.e. : lim(N(s)→∞)(δ(l))/(*D*[N(s)]) = ∞ given that δ(l) is an exponential function whereas *D*[N(s)] is a linear steps function. Thus, while 93% of ℕ is *tamed* by a finite number of chains of connections, the extension to 100% requires an amount of binomial inequalities that grows exponentially, making the complete proof of SC (CC) an impossible computational challenge. Since there is no theoretical limit to the combinations of elementary steps, SCQ is configured as a problem with open complexity; supported by an infinite links structure, making a “closed” and definitive proof an asymptotic objective, similar to *squaring the circle*.

2.a.3. Dominance in subsets

In links cycles, subset $\mathbf{O_1}$ [class (mod4)] has twice the density of $\mathbf{O_2}$ [class (mod16)] and twice the density of $\mathbf{O_3}$ [class (mod32)]. It’s the most frequent point of convergence. It can be considered the *primary attractor of transition* toward which the other subsets, sooner or later, fall. $\mathbf{O_1}$ is a *preferential conduit* for the BIG CRUNCH (fall to 1; § 3.4.).
In $\mathbf{O_2}$***** a number becomes less than itself after 5 steps plus a link in **E**, but could remains in **E** two times, or more. In $\mathbf{O_3}$***** a number becomes less than itself after 6 steps plus two links in **E**, but could continues to be in **E** more than two times.
$\mathbf{O_{2a}}$; $\mathbf{O_{3a}}$; $\mathbf{O_{3b}}$; $\mathbf{O_{3c}}$; tend to generate growths or increments.

3. SC Quadrature

3.1. Theorem of duplication

Statement:
If a binomial inequality is valid for the horizon $2^{\alpha} \cdot n+p$ then it also applies to the horizon $2 \cdot 2^{\alpha} \cdot n+p$, i.e.: $2^{\alpha+1} \cdot n+p$.

Demonstration:
O_1 : $\mathbf{2^{\alpha+1} \cdot n+1}$ O_1 : $x = 2^{\alpha} \cdot n \rightarrow f(x) =$ $\mathbf{3 \cdot 2^{\alpha-1} \cdot n+1} < \mathbf{2^{\alpha+1} \cdot n+1}$ O_1
$\alpha = 1 \rightarrow$ **3n+1** < **4n+1** O_1 principal inequality

In O_2 we take into consideration just O_{2a} (O_2* is perfectly similar)
O_{2a} : $\mathbf{2^{\alpha+1}\cdot n+11}$ O_{2a} : y = $2^{\alpha}\cdot n+5 \rightarrow f(y) = 3^2\cdot 2^{\alpha-2}\cdot n+13$ O_1 : x = $3^2\cdot 2^{\alpha-3}\cdot n+6 \rightarrow$
f(x) = $\mathbf{3^3\cdot 2^{\alpha-4}\cdot n+10} < \mathbf{2^{\alpha+1}\cdot n+11}$ O_{2a}
$\alpha = 4 \rightarrow$ **27n+10 < 32n+11** O_{2a} principal inequality

In O_3 we take into consideration just O_{3c} (O_3*, O_{3a} and O_{3b} are perfectly similar).

O_{3c} : $\mathbf{2^{\alpha+1}\cdot n+15}$ O_{3c} : z = $2^{\alpha}\cdot n+7 \rightarrow f(z) = 3^3\cdot 2^{\alpha-2}\cdot n+53$ O_1 : x = $3^3\cdot 2^{\alpha-3}\cdot n+26 \rightarrow$
f(x) = $3^4\cdot 2^{\alpha-4}\cdot n+40 \rightarrow 3^4\cdot 2^{\alpha-5}\cdot n+20 \rightarrow \mathbf{3^4\cdot 2^{\alpha-6}\cdot n+10} < \mathbf{2^{\alpha+1}\cdot n+15}$ O_{3c}
$\alpha = 6 \rightarrow$ **81n+10 < 128n+15** O_{3c} principal inequality

3.1.1. Corollary

If a binomial inequality is valid for the horizon $2^{\alpha}\cdot n+p$ then it's valid for the horizon $2^{\alpha+i}\cdot n+p$ for every $i \in \mathbb{N}$

We use the principle of induction:
(1) i = 1 $\rightarrow 2^{\alpha+1}\cdot n+p$ is true for Theorem of duplication
(2) Supposed true $2^{\alpha+i}\cdot n+p$
(3) $2^{\alpha+i+1}\cdot n+p = 2\cdot(2^{\alpha+i})\cdot n+p \sim 2^{\alpha+i}\cdot n+p$ (1) $\rightarrow 2^{\alpha+i}\cdot n+p$ is true (2) $\rightarrow$ QED

3.2. Theorem of Independence

Statement:
In every horizon $2^{\alpha}\cdot n+p$ the specific known term p of the principal binomial inequality is independent of the parametric term from an appropriate exponent α onwards.

Demonstration:
$\mathbf{O_1}$: 4k+1 O_1 : k = {1; 2; 3; …}
$\mathbf{2^{\alpha}\cdot n+4k+1}$ O_1 : x = $2^{\alpha-1}\cdot n+2k \rightarrow$ f(x) = $\mathbf{3\cdot 2^{\alpha-2}\cdot n+3k+1} < \mathbf{2^{\alpha}\cdot n+4k+1}$ O_1 : [$\alpha \geq 2$]

$\alpha = 2 \rightarrow 3n+3k+1 < 4n+4k+1 \rightarrow 3(n + k)+1 < 4(n + k)+1$ O_1
n = 0 $\rightarrow$ **3k+1 < 4k+1** O_1 principal inequality : N(s) = **3**

$\mathbf{O_2}$***** : 16k+3 O_2* : k = {0; 1; 2; …}
$\mathbf{2^{\alpha}\cdot n+16k+3}$ O_2* : y = $2^{\alpha-1}\cdot n+8k+1 \rightarrow f(y) = 3^2\cdot 2^{\alpha-3}\cdot n+18k+4 \rightarrow$
$\mathbf{3^2\cdot 2^{\alpha-4}\cdot n+9k+2} < \mathbf{2^{\alpha}\cdot n+16k+3}$ O_2* : [$\alpha \geq 4$]

$\alpha = 4 \rightarrow 9n+9k+2 < 16n+16k+3 \rightarrow 9(n + k)+2 < 16(n + k)+3$ O_2*
n = 0 $\rightarrow$ **9k+2 < 16k+3** O_2* principal inequality : N(s) = **6**

$\mathbf{O_3}$***** : 32k-9 ~ 32k+23 O_3* : k = {0; 1; 2; …}
$\mathbf{2^{\alpha}\cdot n+32k+23}$ O_3* : z = $2^{\alpha-1}\cdot n+16k+11 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+80 \rightarrow 3^3\cdot 2^{\alpha-4}\cdot n+54k+40 \rightarrow$
$\mathbf{3^3\cdot 2^{\alpha-5}\cdot n+27k+20} < \mathbf{2^{\alpha}\cdot n+32k+23}$ O_3* : [$\alpha \geq 5$]

$\alpha = 5 \rightarrow 27n+27k+20 < 32n+32k+23 = 27(n + k)+20 < 32(n + k)+23$ O_3*
n = 0 $\rightarrow$ **27k+20 < 32k+23** O_3* principal inequality : N(s) = **8**

If 2k+1 $\in O_{2a} \vee O_{3a} \vee O_{3b} \vee O_{3c}$ we prove the Theorem of Independence for the most favorable cycles with the fewest number of steps. As it will be proved it's applicable to all other cycles.

$\mathbf{O_{2a}}$: 16k+11 O_{2a} : k = {0; 1; 2; …}

$\mathbf{2^{\alpha}\cdot n+16k+11}$ O_{2a} : y = $2^{\alpha-1}\cdot$n+8k+5 → f(y) = $3^2\cdot 2^{\alpha-3}\cdot$n+18k +13 [$O_1$] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+16k+11}$ → $2^{\alpha}\cdot$n+16(**2k**)+11 = $\mathbf{2^{\alpha}\cdot n+32k+11}$ O_{2a}
$3^2\cdot 2^{\alpha-3}\cdot$n+18k+13 → $3^2\cdot 2^{\alpha-3}\cdot$n+18(**2k**)+13 = $3^2\cdot 2^{\alpha-3}\cdot$n+36k+13 O_1 : x = $3^2\cdot 2^{\alpha-4}\cdot$n+18k+6 →
f(x) = $\mathbf{3^3\cdot 2^{\alpha-5}\cdot n+27k+10 < 2^{\alpha}\cdot n+32k+11}$ O_{2a} : [α ≥ 5]

α = 5 → 27n+27k+10 < 32n+32k+11 = 27(n + k)+10 < 32(n + k)+11 O_{2a}
n = 0 → **27k+10 < 32k+11** O_{2a} principal inequality : N(s) = **8**
n = k = 0 → **10 < 11** O_{2a} known numbers

$\mathbf{O_{3a}}$: 32k-1 ~ 32k+31 O_{3a} : k = {0; 1; 2; …}

$\mathbf{2^{\alpha}\cdot n+32k+31}$ O_{3a} : z = $2^{\alpha-1}\cdot$n+16k+15 → f(z) = $3^3\cdot 2^{\alpha-3}\cdot$n+108k+107 [$O_2$] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+32k+31}$ → $2^{\alpha}\cdot$n+32(**2k**)+31 = $\mathbf{2^{\alpha}\cdot n+64k+31}$ O_{3a}
$3^3\cdot 2^{\alpha-3}\cdot$n+108k+107 → $3^3\cdot 2^{\alpha-3}\cdot$n+108(**2k**)+107 = $3^3\cdot 2^{\alpha-3}\cdot$n+216k+107 O_2 :
y = $3^3\cdot 2^{\alpha-4}\cdot$n+108k+53 → f(y) = $3^5\cdot 2^{\alpha-6}\cdot$n+243k+121 [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+64k+31}$ → $2^{\alpha}\cdot$n+64(**2k+1**)+31 → $\mathbf{2^{\alpha}\cdot n+128k+95}$ O_{3a}
$3^5\cdot 2^{\alpha-6}\cdot$n+243k+121 → $3^5\cdot 2^{\alpha-6}\cdot$n+243(**2k+1**)+121 → $3^5\cdot 2^{\alpha-6}\cdot$n+486k+364 →
$3^5\cdot 2^{\alpha-7}\cdot$n+243k+182 [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+128k+95}$ → $2^{\alpha}\cdot$n+128(**2k**)+95 = $\mathbf{2^{\alpha}\cdot n+256k+95}$ O_{3a}
$3^5\cdot 2^{\alpha-7}\cdot$n+243k+182 → $3^5\cdot 2^{\alpha-7}\cdot$n+243(**2k**)+182 → $3^5\cdot 2^{\alpha-7}\cdot$n+486k+182 →
$\mathbf{3^5\cdot 2^{\alpha-8}\cdot n+243k+91}$ < $\mathbf{2^{\alpha}\cdot n+256k+95}$ O_{3a} : [α ≥ 8]

α = 8 → 243n+243k+91 < 256n+256k+95 = 243(n + k)+91 < 256(n + k)+95 O_{3a}
n = 0 → **243k+91 < 256k+95** O_{3a} principal inequality : N(s) = **13**
n = k = 0 → **91** O_{2a} **<** **95** O_{3a} known numbers

$\mathbf{O_{3b}}$: 32k+7 O_{3b} : k = {0; 1; 2; …}

$\mathbf{2^{\alpha}\cdot n+32k+7}$ O_{3b} : z = $2^{\alpha-1}\cdot$n+16k+3 → f(z) = $3^3\cdot 2^{\alpha-3}\cdot$n+108k+26 →
$3^3\cdot 2^{\alpha-4}\cdot$n+54k+13 [$O_1$] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+32k+7}$ → $2^{\alpha}\cdot$n+32(**2k**)+7 = $\mathbf{2^{\alpha}\cdot n+64k+7}$ O_{3b}
$3^3\cdot 2^{\alpha-4}\cdot$n+54k+13 → $3^3\cdot 2^{\alpha-4}\cdot$n+54(**2k**)+13 = $3^3\cdot 2^{\alpha-4}\cdot$n+108k+13 O_1 :
x = $3^3\cdot 2^{\alpha-5}\cdot$n+54k+6 → f(x) = $3^4\cdot 2^{\alpha-6}\cdot$n+81k+10 [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+64k+7}$ → $2^{\alpha}\cdot$n+64(**2k**)+7 = $\mathbf{2^{\alpha}\cdot n+128k+7}$ O_{3b}
$3^4\cdot 2^{\alpha-6}\cdot$n+81k+10 → $3^4\cdot 2^{\alpha-6}\cdot$n+81(**2k**)+10 = $3^4\cdot 2^{\alpha-6}\cdot$n+162k+10 →
$\mathbf{3^4\cdot 2^{\alpha-7}\cdot n+81k+5}$ < $\mathbf{2^{\alpha}\cdot n+128k+7}$ O_{3b} : [α ≥ 7]

α = 7 → 81n+81k+5 < 128n+128k+7 = 81(n + k)+5 < 128(n + k)+7 O_{3b}
n = 0 → **81k+5 < 128k+7** O_{3b} principal inequality : N(s) = **11**
n = k = 0 → **5** O_1 **< 7** O_{3b} known numbers

$\mathbf{O_{3c}}$: 32k+15 O_{3c} : k = {0; 1; 2; …}

$\mathbf{2^\alpha \cdot n+32k+15}$ O_{3c} : $z = 2^{\alpha-1}\cdot n+16k+7 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+53$ O_1 :
$x = 3^3\cdot 2^{\alpha-4}\cdot n+54k+26 \rightarrow f(x) = 3^4\cdot 2^{\alpha-5}\cdot n+81k+40$ [E] : $k = \mathbf{2k}$

$\mathbf{2^\alpha \cdot n+32k+15} \rightarrow 2^\alpha\cdot n+32(\mathbf{2k})+15 = \mathbf{2^\alpha \cdot n+64k+15}$ O_{3c}
$3^4\cdot 2^{\alpha-5}\cdot n+81k+40 \rightarrow 3^4\cdot 2^{\alpha-5}\cdot n+81(\mathbf{2k})+40 = 3^4\cdot 2^{\alpha-5}\cdot n+162k+40 \rightarrow$
$3^4\cdot 2^{\alpha-6}\cdot n+81k+20$ [E] : $k = \mathbf{2k}$

$\mathbf{2^\alpha \cdot n+64k+15} \rightarrow 2^\alpha\cdot n+64(\mathbf{2k})+15 = \mathbf{2^\alpha \cdot n+128k+15}$ O_{3c}
$3^4\cdot 2^{\alpha-6}\cdot n+81k+20 \rightarrow 3^4\cdot 2^{\alpha-6}\cdot n+81(\mathbf{2k})+20 = 3^4\cdot 2^{\alpha-6}\cdot n+162k+20 \rightarrow$
$\mathbf{3^4\cdot 2^{\alpha-7}\cdot n+81k+10} < \mathbf{2^\alpha \cdot n+128k+15}$ O_{3c} : $[\alpha \geq 7]$

$\alpha = 7 \rightarrow 81n+81k+10 < 128n+128k+15 = 81(n + k)+10 < 128(n + k)+15$ O_{3c}
$n = 0 \rightarrow \mathbf{81k+10 < 128k+15}$ O_{3c} principal inequality : $N(s) = \mathbf{11}$
$n = k = 0 \rightarrow \mathbf{10 < 15}$ O_{3c} known numbers

We take $16k+11$ O_{2a} into consideration and we apply a cycle of link managed by us

$\mathbf{2^\alpha \cdot n+16k+11}$ O_{2a} : $y = 2^{\alpha-1}\cdot n+8k+5 \rightarrow f(y) = 3^2\cdot 2^{\alpha-3}\cdot n+18k+13$ [O_3] : $k = \mathbf{4k+1}$

$\mathbf{2^\alpha \cdot n+16k+11}$ $O_{2a} \rightarrow 2^\alpha\cdot n+16(\mathbf{4k+1})+11 = \mathbf{2^\alpha \cdot n+64k+27}$ O_{2a}
$3^2\cdot 2^{\alpha-3}\cdot n+18k+13 \rightarrow 3^2\cdot 2^{\alpha-3}\cdot n+18(\mathbf{4k+1})+13 = 3^2\cdot 2^{\alpha-3}\cdot n+72k+31$ O_3 :
$z = 3^2\cdot 2^{\alpha-4}\cdot n+36k+15 \rightarrow f(z) = 3^5\cdot 2^{\alpha-6}\cdot n+243k+107$ [E] : $k = \mathbf{2k+1}$

$\mathbf{2^\alpha \cdot n+64k+27}$ $O_{2a} \rightarrow 2^\alpha\cdot n+64(\mathbf{2k+1})+27 = \mathbf{2^\alpha \cdot n+128k+91}$ O_{2a}
$3^5\cdot 2^{\alpha-6}\cdot n+243k+107 \rightarrow 3^5\cdot 2^{\alpha-6}\cdot n+243(\mathbf{2k+1})+107 = 3^5\cdot 2^{\alpha-6}\cdot n+486k+350 \rightarrow$
$3^5\cdot 2^{\alpha-7}\cdot n+243k+175$ [O] : $k = \mathbf{2k}$

$\mathbf{2^\alpha \cdot n+128k+91}$ $O_{2a} \rightarrow 2^\alpha\cdot n+128(\mathbf{2k})+91 = \mathbf{2^\alpha \cdot n+256k+91}$ O_{2a}
$3^5\cdot 2^{\alpha-7}\cdot n+243k+175 \rightarrow 3^5\cdot 2^{\alpha-7}\cdot n+243(\mathbf{2k})+175 = 3^5\cdot 2^{\alpha-7}\cdot n+486k+175$ [O_3] : $k = \mathbf{4k}$

$\mathbf{2^\alpha \cdot n+256k+91}$ $O_{2a} \rightarrow 2^\alpha\cdot n+256(\mathbf{4k})+91 = \mathbf{2^\alpha \cdot n+1024k+91}$ O_{2a}
$3^5\cdot 2^{\alpha-7}\cdot n+486k+175 \rightarrow 3^5\cdot 2^{\alpha-7}\cdot n+486(\mathbf{4k})+175 = 3^5\cdot 2^{\alpha-7}\cdot n+1944k+175$ O_3 :
$z = 3^5\cdot 2^{\alpha-8}\cdot n+972k+87 \rightarrow f(z) = 3^8\cdot 2^{\alpha-10}\cdot n+6561k+593$ [E] : $k = \mathbf{2k+1}$

$\mathbf{2^\alpha \cdot n+1024k+91}$ $O_{2a} \rightarrow 2^\alpha\cdot n+1024(\mathbf{2k+1})+91 = \mathbf{2^\alpha \cdot n+2048k+1115}$ O_{2a}
$3^8\cdot 2^{\alpha-10}\cdot n+6561k+593 \rightarrow 3^8\cdot 2^{\alpha-10}\cdot n+6561(\mathbf{2k+1})+593 = 3^8\cdot 2^{\alpha-10}\cdot n+13122k+7154 \rightarrow$
$3^8\cdot 2^{\alpha-11}\cdot n+6561k+3577$ [O] : $k = \mathbf{2k}$

$\mathbf{2^\alpha \cdot n+2048k+1115}$ $O_{2a} \rightarrow 2^\alpha\cdot n+2048(\mathbf{2k})+1115 = \mathbf{2^\alpha \cdot n+4096k+1115}$ O_{2a}
$3^8\cdot 2^{\alpha-11}\cdot n+6561k+3577 \rightarrow 3^8\cdot 2^{\alpha-11}\cdot n+6561(\mathbf{2k})+3577 = 3^8\cdot 2^{\alpha-11}\cdot n+13122k+3577$ [O_1] : $k = \mathbf{2k}$

$\mathbf{2^\alpha \cdot n+4096k+1115}$ $O_{2a} \rightarrow 2^\alpha\cdot n+40968(\mathbf{2k})+1115 = \mathbf{2^\alpha \cdot n+8192k+1115}$ O_{2a}
$3^8\cdot 2^{\alpha-11}\cdot n+13122k+3577 \rightarrow 3^8\cdot 2^{\alpha-11}\cdot n+13122(\mathbf{2k})+3577 = 3^8\cdot 2^{\alpha-11}\cdot n+26244k+3577$ O_1 :
$x = 3^8\cdot 2^{\alpha-12}\cdot n+13122k+1788 \rightarrow f(x) = 3^9\cdot 2^{\alpha-13}\cdot n+19683k+2683$ [E] : $k = \mathbf{2k+1}$

$\mathbf{2^\alpha \cdot n+8192k+1115}$ $O_{2a} \rightarrow 2^\alpha\cdot n+8192(\mathbf{2k+1})+1115 = \mathbf{2^\alpha \cdot n+16384k+9307}$ O_{2a}
$3^9\cdot 2^{\alpha-13}\cdot n+19683k+2683 \rightarrow 3^9\cdot 2^{\alpha-13}\cdot n+19683(\mathbf{2k+1})+2683 = 3^9\cdot 2^{\alpha-13}\cdot n+39366k+22366 \rightarrow$
$3^9\cdot 2^{\alpha-14}\cdot n+19683k+11183$ [E] : $k = \mathbf{2k+1}$

$\mathbf{2^\alpha \cdot n+16384k+9307}$ $O_{2a} \rightarrow 2^\alpha\cdot n+16384(\mathbf{2k+1})+9307 = \mathbf{2^\alpha \cdot n+32768k+25691}$ O_{2a}
$3^9\cdot 2^{\alpha-14}\cdot n+19683k+11183 \rightarrow 3^9\cdot 2^{\alpha-14}\cdot n+19683(\mathbf{2k+1})+11183 = 3^9\cdot 2^{\alpha-14}\cdot n+39366k+30866 \rightarrow$

$3^9 \cdot 2^{\alpha-15} \cdot n+19683k+15433 < 2^{\alpha} \cdot n+32768k+25691$ O_{2a} : [$\alpha \geq 15$]

$\alpha = 15 \rightarrow 3^9 \cdot n+3^9 \cdot k+15433 < 2^{15} \cdot n+2^{15} \cdot k+25691$ O_{2a} = $3^9 \cdot (n + k)+15433 < 2^{15} \cdot (n + k)+25691$ O_{2a}
$n = 0 \rightarrow$ $\mathbf{3^9 \cdot k+15433 < 2^{15} \cdot k+25691}$ $\mathbf{O_{2a}}$ principal inequality : N(s) = **24**
$n = k = 0 \rightarrow$ **15433** O_1 < **25691** O_{2a} known numbers

Check link known numbers :
25691 O_{2a} : y = 12845 → f(y) = 28903 O_{3b} : z = 14451 → f(z) = 97550 → 48775 O_{3b} :
z = 24387 → f(z) = 164618 → 82309 O_1 : x = 41154 → f(x) = 61732 → 30866 →
15433 O_1 < **25691** O_{2a} : N(s) = **24**

Chain of connections :
$O_{2a} \rightarrow O_{3b} \rightarrow E \rightarrow O_{3b} \rightarrow E \rightarrow O_1 \rightarrow E \rightarrow E \rightarrow$ **15433** O_1 < **25691** O_{2a}
… …

Note : *The verification on the known numbers p and q suffices to establish the inequality for the entire parametric family*.

Whatever cycle of link we decide to choose Theorem of Independence is always valid, also in the worst conditions for to reach highest horizons.

3.2.1. A cycle of link in worst conditions

In the calculus of the cycle of link in this paragraph and in the following ones (§ 3.2.2. ; § 3.2.3) we apply immediate substitutions as usual highlighted by bold font. We used the utmost accuracy in making the calculations.

$\mathbf{2^{\alpha} \cdot n+16k+11}$ O_{2a} : $y = 2^{\alpha-1} \cdot n+8k+5 \rightarrow f(y) = 3^2 \cdot 2^{\alpha-3} \cdot n+18k+13$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha} \cdot n+64k+27}$ O_{2a}
$3^2 \cdot 2^{\alpha-3} \cdot n+72k+31$ O_3 : $z = 3^2 \cdot 2^{\alpha-4} \cdot n+36k+15 \rightarrow f(z) = 3^5 \cdot 2^{\alpha-6} \cdot n+243k+107$ [O] : k = **2k**

$\mathbf{2^{\alpha} \cdot n+128k+27}$ O_{2a}
$3^5 \cdot 2^{\alpha-6} \cdot n+486k+107$ [O_3] : k = **4k+2**

$\mathbf{2^{\alpha} \cdot n+512k+283}$ O_{2a}
$3^5 \cdot 2^{\alpha-6} \cdot n+1944k+1079$ O_3 : $z = 3^5 \cdot 2^{\alpha-7} \cdot n+972k+539 \rightarrow f(z) = 3^8 \cdot 2^{\alpha-9} \cdot n+6561k+3644$ [O] : k = **2k+1**

$\mathbf{2^{\alpha} \cdot n+1024k+795}$ O_{2a}
$3^8 \cdot 2^{\alpha-9} \cdot n+13122k+10205$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha} \cdot n+4096k+1819}$ O_{2a}
$3^8 \cdot 2^{\alpha-9} \cdot n+52488k+23327$ O_3 : $z = 3^8 \cdot 2^{\alpha-10} \cdot n+26244k+11663 \rightarrow$
$f(z) = 3^{11} \cdot 2^{\alpha-12} \cdot n+177147k+78731$ [O] : k = **2k**

$\mathbf{2^{\alpha} \cdot n+8192k+1819}$ O_{2a}
$3^{11} \cdot 2^{\alpha-12} \cdot n+354294k+78731$ [O_3] : k = **4k+2**

$\mathbf{2^{\alpha} \cdot n+32768k+18203}$ O_{2a}
$3^{11} \cdot 2^{\alpha-12} \cdot n+1417176k+787319$ O_3 : $z = 3^{11} \cdot 2^{\alpha-13} \cdot n+708588k+393659 \rightarrow$
$f(z) = 3^{14} \cdot 2^{\alpha-15} \cdot n+4782969k+2657204$ [O] : k = **2k+1** … … we decide to go back, then:

$f(z) = 3^{14}\cdot2^{\alpha-15}\cdot n+4782969k+2657204$ [E] : k = **2k**

$2^\alpha\cdot n+65536k+18203$ O_{2a}
$3^{14}\cdot2^{\alpha-15}\cdot n+9565938k+2657204 \rightarrow 3^{14}\cdot2^{\alpha-16}\cdot n+4782969k+1328602$ [E] : k = **2k**

$2^\alpha\cdot n+131072k+18203$ O_{2a}
$3^{14}\cdot2^{\alpha-16}\cdot n+9565938k+1328602 \rightarrow 3^{14}\cdot2^{\alpha-17}\cdot n+4782969k+664301$ [E] : k = **2k+1**

$2^\alpha\cdot n+262144k+149275$ O_{2a}
$3^{14}\cdot2^{\alpha-17}\cdot n+9565938k+5447270 \rightarrow 3^{14}\cdot2^{\alpha-18}\cdot n+4782969k+2723635$ [E] : k = **2k+1**

$2^\alpha\cdot n+524288k+411419$ O_{2a}
$3^{14}\cdot2^{\alpha-18}\cdot n+9565938k+7506604 \rightarrow 3^{14}\cdot2^{\alpha-19}\cdot n+4782969k+3753302$ [E] : k = **2k**

$2^\alpha\cdot n+1048576k+411419$ O_{2a}
$3^{14}\cdot2^{\alpha-19}\cdot n+9565938k+3753302 \rightarrow 3^{14}\cdot2^{\alpha-20}\cdot n+4782969k+1876651$ [E] : k = **2k+1**

$2^\alpha\cdot n+2097152k+1459995$ O_{2a}
$3^{14}\cdot2^{\alpha-20}\cdot n+9565938k+6659620 \rightarrow 3^{14}\cdot2^{\alpha-21}\cdot n+4782969k+3329810$ [E] : k = **2k**

$2^\alpha\cdot n+4194304k+1459995$ O_{2a}
$3^{14}\cdot2^{\alpha-21}\cdot n+9565938k+3329810 \rightarrow 3^{14}\cdot2^{\alpha-22}\cdot n+4782969k+1664905$ [E] : k = **2k+1**

$2^\alpha\cdot n+8388608k+5654299$ O_{2a}
$3^{14}\cdot2^{\alpha-22}\cdot n+9565938k+6447874 \rightarrow$
$3^{14}\cdot2^{\alpha-23}\cdot n+4782969k+3223937 < 2^\alpha\cdot n+8388608k+5654299$ O_{2a} $[\alpha \geq 23]$

$\alpha = 23 \rightarrow 3^{14}\cdot n+4782969k+3223937 < 2^{23}\cdot n+8388608k+5654299$ O_{2a} =
$3^{14}\cdot n+3^{14}\cdot k+3223937 < 2^{23}\cdot n+2^{23}\cdot k+5654299$ O_{2a} =
$3^{14}\cdot(n + k)+3223937 < 2^{23}\cdot(n + k)+5654299$ O_{2a}
n = 0 → **$3^{14}\cdot k+3223937$** < **$2^{23}\cdot k+5654299$** O_{2a} principal inequality : N(s) = **37**
n = k = 0 → **3223937** O_1 < **5654299** O_{2a} known numbers.

Check link known numbers :
5654299 O_{2a} : y = 2827149 → f(y) = 6361087 O_{3a} : z = 3180543 → f(z) = 21468671 O_{3a} :
z = 10734335 → f(z) = 72456767 O_{3a} : z = 36228383 → f(z) = 244541591 O_3* :
z = 122270795 → f(z) = 825327872 → 412663936 → 206331968 → 103165984 →
51582992 → 25791496 → 12895748 → 6447874 → **3223937** O_1 < **5654299** O_{2a} : N(s) = **37**

Chain of connections :
$O_{2a} \rightarrow O_{3a} \rightarrow O_{3a} \rightarrow O_{3a} \rightarrow O_3^* \rightarrow$ E → E → E → E → E → E → E → E →
3223937 O_1 < **5654299** O_{2a}

3.2.2. A cycles of link high horizon and related check known numbers

$2^\alpha\cdot n+32k+31$ O_{3a} : $z = 2^{\alpha-1}\cdot n+16k+15 \rightarrow f(z) = 3^3\cdot2^{\alpha-3}\cdot n+108k+107$ [O_2] : k = **4k**

$2^\alpha\cdot n+128k+31$ O_{3a}
$3^3\cdot2^{\alpha-3}\cdot n+432k+107$ O_2 : $y = 3^3\cdot2^{\alpha-4}\cdot n+216k+53 \rightarrow f(y) = 3^5\cdot2^{\alpha-6}\cdot n+486k+121$ [O_2] : k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+512k+415}$ O_{3a}
$3^5\cdot 2^{\alpha-6}\cdot n+1944k+1579$ O_2 : $y = 3^5\cdot 2^{\alpha-7}\cdot n+972k+789 \rightarrow f(y) = 3^7\cdot 2^{\alpha-9}\cdot n+2187k+1777$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+1024k+415}$ O_{3a}
$3^7\cdot 2^{\alpha-9}\cdot n+4374k+1777$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha}\cdot n+4096k+1439}$ O_{3a}
$3^7\cdot 2^{\alpha-9}\cdot n+17496k+6151$ O_3 : $z = 3^7\cdot 2^{\alpha-10}\cdot n+8748k+3075 \rightarrow$
$f(z) = 3^{10}\cdot 2^{\alpha-12}\cdot n+59049k+20762$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+8192k+5535}$ O_{3a}
$3^{10}\cdot 2^{\alpha-12}\cdot n+118098k+79811$ [O_1] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+16384k+13727}$ O_{3a}
$3^{10}\cdot 2^{\alpha-12}\cdot n+236196+197909$ O_1 : $x = 3^{10}\cdot 2^{\alpha-13}\cdot n+118098k+98954 \rightarrow$
$f(x) = 3^{11}\cdot 2^{\alpha-14}\cdot n+177147k+148432$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+32768k+30111}$ O_{3a}
$3^{11}\cdot 2^{\alpha-14}\cdot n+354294k+325579$ [O_1] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+65536k+62879}$ O_{3a}
$3^{11}\cdot 2^{\alpha-14}\cdot n+708588k+679873$ O_1 : $x = 3^{11}\cdot 2^{\alpha-15}\cdot n+354294k+339936 \rightarrow$
$f(x) = 3^{12}\cdot 2^{\alpha-16}\cdot n+531441k+509905$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+131072k+128415}$ O_{3a}
$3^{12}\cdot 2^{\alpha-16}\cdot n+1062882k+1041346 \rightarrow 3^{12}\cdot 2^{\alpha-17}\cdot n+531441k+520673$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+262144k+128415}$ O_{3a}
$3^{12}\cdot 2^{\alpha-17}\cdot n+1062882k+520673$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+524288k+128415}$ O_{3a}
$3^{12}\cdot 2^{\alpha-17}\cdot n+2125764+520673$ O_1 : $x = 3^{12}\cdot 2^{\alpha-18}\cdot n+1062882+260336 \rightarrow$
$f(x) = 3^{13}\cdot 2^{\alpha-19}\cdot n+1594323k+390505$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+1048576k+128415}$ O_{3a}
$3^{13}\cdot 2^{\alpha-19}\cdot n+3188646k+390505$ [O_2] : k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+4194304k+3274143}$ O_{3a}
$3^{13}\cdot 2^{\alpha-19}\cdot n+12754584k+9956443$ O_2 : $y = 3^{13}\cdot 2^{\alpha-20}\cdot n+6377292k+4978221 \rightarrow$
$f(y) = 3^{15}\cdot 2^{\alpha-22}\cdot n+14348907k+11200999$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+8388608k+7468447}$ O_{3a}
$3^{15}\cdot 2^{\alpha-22}\cdot n+28697814k+25549906 \rightarrow 3^{15}\cdot 2^{\alpha-23}\cdot n+14348907k+12774953$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+16777216k+7468447}$ O_{3a}
$3^{15}\cdot 2^{\alpha-23}\cdot n+28697814k+12774953$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+33554432k+7468447}$ O_{3a}
$3^{15}\cdot 2^{\alpha-23}\cdot n+57395628k+12774953$ O_1 : $x = 3^{15}\cdot 2^{\alpha-24}\cdot n+28697814k+6387476 \rightarrow$
$f(x) = 3^{16}\cdot 2^{\alpha-25}\cdot n+43046721k+9581215$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+67108864k+7468447}$ O_{3a}
$3^{16}\cdot 2^{\alpha-25}\cdot n+86093442k+9581215$ $[O_2]$: k = **4k+2**

$\mathbf{2^{\alpha}\cdot n+268435456k+141686175}$ O_{3a}
$3^{16}\cdot 2^{\alpha-25}\cdot n+344373768k+181768099$ O_2 : y = $3^{16}\cdot 2^{\alpha-26}\cdot n+172186884k+90884049 \rightarrow$
f(y) = $3^{18}\cdot 2^{\alpha-28}\cdot n+387420489k+204489112$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+536870912k+410121631}$ O_{3a}
$3^{18}\cdot 2^{\alpha-28}\cdot n+774840978k+591909601$ $[O_3]$: k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+2147483648k+2020734367}$ O_{3a}
$3^{18}\cdot 2^{\alpha-28}\cdot n+3099363912k+2916432535$ O_3 : z = $3^{18}\cdot 2^{\alpha-29}\cdot n+1549681956k+1458216267 \rightarrow$
f(z) = $3^{21}\cdot 2^{\alpha-31}\cdot n+10460353203k+9842959808$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+4294967296k+2020734367}$ O_{3a}
$3^{21}\cdot 2^{\alpha-31}\cdot n+20920706406k+9842959808 \rightarrow 3^{21}\cdot 2^{\alpha-32}\cdot n+10460353203k+9842959808$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+8589934592k+2020734367}$ O_{3a}
$3^{21}\cdot 2^{\alpha-32}\cdot n+20920706406k+4921479904 \rightarrow 3^{21}\cdot 2^{\alpha-33}\cdot n+10460353203k+2460739952$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+17179869184k+10610668959}$ O_{3a}
$3^{21}\cdot 2^{\alpha-33}\cdot n+20920706406k+12921093155$ $[O_2]$: k = **4k**

$\mathbf{2^{\alpha}\cdot n+68719476736k+10610668959}$ O_{3a}
$3^{21}\cdot 2^{\alpha-33}\cdot n+83682825624k+12921093155$ O_2 : y = $3^{21}\cdot 2^{\alpha-34}\cdot n+41841412812k+6460546577 \rightarrow$
f(y) = $3^{23}\cdot 2^{\alpha-36}\cdot n+94143178827k+14536229800$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+137438953472k+79330145695}$ O_{3a}
$3^{23}\cdot 2^{\alpha-36}\cdot n+188286357654k+108679408627$ $[O_1]$: k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+274877906944k+216769099167}$ O_{3a}
$3^{23}\cdot 2^{\alpha-36}\cdot n+376572715308k+296965766281$ O_1 :
x = $3^{23}\cdot 2^{\alpha-37}\cdot n+188286357654k+148482883140 \rightarrow$
f(x) = $3^{24}\cdot 2^{\alpha-38}\cdot n+282429536481k+222724324711$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+549755813888k+491647006111}$ O_{3a}
$3^{24}\cdot 2^{\alpha-38}\cdot n+564859072962k+505153861192 \rightarrow$
$3^{24}\cdot 2^{\alpha-39}\cdot n+282429536481k+252576930596 < 2^{\alpha}\cdot n+549755813888k+491647006111$ O_{3a}

$\alpha = 39 \rightarrow 3^{24}\cdot n+3^{24}\cdot k+252576930596 < 2^{39}\cdot n+2^{39}\cdot k+491647006111$ O_{3a} =
$3^{24}\cdot(n + k)+252576930596 < 2^{39}\cdot(n + k)+491647006111$ O_{3a}
n = 0 → $\mathbf{3^{24}\cdot k+252576930596 < 2^{39}\cdot k+491647006111}$ O_{3a} principal inequality : N(s) = **63**
n = k = 0 → **252576930596 < 491647006111** O_{3a} known numbers

Check link known numbers :
491647006111 O_{3a} : z = 245823503055 → f(z) = 1659308645627 O_{2a} : y = 829654322813 →
f(y) =1866722226331 O_{2a} : y = 933361113165 → f(y) = 2100062504623 O_{3c} :
z = 1050031252311 → f(z) = 7087710953105 O_1 : x = 3543855476552 →
f(x) = 5315783214829 O_1 : x = 2857891607414 → f(x) = 3986837411122 → 1993418705561 O_1 :
x = 996709352780 → f(x) = 1495064029171 O_2^* : y = 747532014585 →
f(y) = 1681947032818 → 840973516409 O_1 : x = 420486758204 → f(x) = 630730137307 O_{2a} :

y = 315365068653 → f(y) = 709571404471 O_3^* : z = 354785702235 → f(z) = 2394803490092 → 1197401745046 → 598700872523 O_{2a} : y = 299350436261 → f(y) = 673538481589 O_1 :
x = 336769240794 → f(x) = 505153861192 → **252576930596 < 491647006111** O_{3a} : N(s) = **63**

Chain of connections :
$O_{3a} \to O_{2a} \to O_{2a} \to O_{3c} \to O_1 \to O_1 \to E \to O_1 \to O_2^* \to E \to O_1 \to O_{2a} \to O_3^* \to E \to E \to O_{2a} \to O_1 \to E \to E < O_{3a}$

Another cycle of link high horizon :

$\mathbf{2^{\alpha}{\cdot}n+16k+11}$ O_{2a} : $y = 2^{\alpha-1}{\cdot}n+8k+5 \to f(y) = 3^2{\cdot}2^{\alpha-3}{\cdot}n+18k+13$ [O_3] : k = **4k+1**

$\mathbf{2^{\alpha}{\cdot}n+64k+27}$ O_{2a}
$3^2{\cdot}2^{\alpha-3}{\cdot}n+72k+31$ O_3 : $z = 3^2{\cdot}2^{\alpha-4}{\cdot}n+36k+15 \to f(z) = 3^5{\cdot}2^{\alpha-6}{\cdot}n+243k+107$ [O] : k = **2k**

$\mathbf{2^{\alpha}{\cdot}n+128k+27}$ O_{2a}
$3^5{\cdot}2^{\alpha-6}{\cdot}n+486k+107$ [O_1] : k = **2k+1**

$\mathbf{2^{\alpha}{\cdot}n+256k+155}$ O_{2a}
$3^5{\cdot}2^{\alpha-6}{\cdot}n+972k+593$ O_1 : $x = 3^5{\cdot}2^{\alpha-7}{\cdot}n+486k+296 \to f(x) = 3^6{\cdot}2^{\alpha-8}{\cdot}n+729k+445$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}{\cdot}n+512k+411}$ O_{2a}
$3^6{\cdot}2^{\alpha-8}{\cdot}n+1458k+1174 \to 3^6{\cdot}2^{\alpha-9}{\cdot}n+729k+587$ [O] : k = **2k**

$\mathbf{2^{\alpha}{\cdot}n+1024k+411}$ O_{2a}
$3^6{\cdot}2^{\alpha-9}{\cdot}n+1458k+587$ [O_2] : k = **4k**

$\mathbf{2^{\alpha}{\cdot}n+4096k+411}$ O_{2a}
$3^6{\cdot}2^{\alpha-9}{\cdot}n+5832k+587$ O_2 : $y = 3^6{\cdot}2^{\alpha-10}{\cdot}n+2916k+293 \to f(y) = 3^8{\cdot}2^{\alpha-12}{\cdot}n+6561k+661$ [O] : k = **2k**

$\mathbf{2^{\alpha}{\cdot}n+8192k+411}$ O_{2a}
$3^8{\cdot}2^{\alpha-12}{\cdot}n+13122k+661$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}{\cdot}n+16384k+411}$ O_{2a}
$3^8{\cdot}2^{\alpha-12}{\cdot}n+26244k+661$ O_1 : $x = 3^8{\cdot}2^{\alpha-13}{\cdot}n+13122k+330 \to$
$f(x) = 3^9{\cdot}2^{\alpha-14}{\cdot}n+19683k+496$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}{\cdot}n+32768k+16795}$ O_{2a}
$3^9{\cdot}2^{\alpha-14}{\cdot}n+39366k+20179$ [O_2] : k = **4k**

$\mathbf{2^{\alpha}{\cdot}n+1310728k+16795}$ O_{2a}
$3^9{\cdot}2^{\alpha-14}{\cdot}n+157464k+20179$ O_2 : $y = 3^9{\cdot}2^{\alpha-15}{\cdot}n+78732k+10089 \to$
$f(y) = 3^{11}{\cdot}2^{\alpha-17}{\cdot}n+177147k+22702$ [O] : k = **2k+1**

$\mathbf{2^{\alpha}{\cdot}n+262144k+147867}$ O_{2a}
$3^{11}{\cdot}2^{\alpha-17}{\cdot}n+354294k+199849$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}{\cdot}n+524288k+147867}$ O_{2a}
$3^{11}{\cdot}2^{\alpha-17}{\cdot}n+708588k+199849$ O_1 : $x = 3^{11}{\cdot}2^{\alpha-18}{\cdot}n+354294k+99924 \to$
$f(x) = 3^{12}{\cdot}2^{\alpha-19}{\cdot}n+531441k+149887$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+1048576k+147867}$ O_{2a}
$3^{12}\cdot 2^{\alpha-19}\cdot n+1062882k+149887$ $[O_3]$: k = **4k**

$\mathbf{2^{\alpha}\cdot n+4194304k+147867}$ O_{2a}
$3^{12}\cdot 2^{\alpha-19}\cdot n+4251528k+149887$ O_3 : z = $3^{12}\cdot 2^{\alpha-20}\cdot n+2125764k+74943 \rightarrow$
f(z) = $3^{15}\cdot 2^{\alpha-22}\cdot n+14348907k+505871$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+8388608k+147867}$ O_{2a}
$3^{15}\cdot 2^{\alpha-22}\cdot n+28697814k+505871$ $[O_3]$: k = **4k**

$\mathbf{2^{\alpha}\cdot n+33554432k+147867}$ O_{2a}
$3^{15}\cdot 2^{\alpha-22}\cdot n+114791256k+505871$ O_3 : z = $3^{15}\cdot 2^{\alpha-23}\cdot n+57395628k+252935 \rightarrow$
f(z) = $3^{18}\cdot 2^{\alpha-25}\cdot n+387420489k+1707317$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+67108864k+147867}$ O_{2a}
$3^{18}\cdot 2^{\alpha-25}\cdot n+774840978k+1707317$ $[O_1]$: k = **2k**

$\mathbf{2^{\alpha}\cdot n+134217728k+147867}$ O_{2a}
$3^{18}\cdot 2^{\alpha-25}\cdot n+1549681956k+1707317$ O_1 : x = $3^{18}\cdot 2^{\alpha-26}\cdot n+774840978k+853658 \rightarrow$
f(x) = $3^{19}\cdot 2^{\alpha-27}\cdot n+1162261467k+1280488$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+268435456k+147867}$ O_{2a}
$3^{19}\cdot 2^{\alpha-27}\cdot n+2324522934k+1280488$ E $\rightarrow 3^{19}\cdot 2^{\alpha-28}\cdot n+1162261467k+640244$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+536870912k+147867}$ O_{2a}
$3^{19}\cdot 2^{\alpha-28}\cdot n+2324522934k+640244 \rightarrow 3^{19}\cdot 2^{\alpha-29}\cdot n+1162261467k+320122$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+1073741824k+147867}$ O_{2a}
$3^{19}\cdot 2^{\alpha-29}\cdot n+2324522934k+320122 \rightarrow 3^{19}\cdot 2^{\alpha-30}\cdot n+1162261467k+160061$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+2147483648k+147867}$ O_{2a}
$3^{19}\cdot 2^{\alpha-30}\cdot n+2324522934k+160061$ $[O_3]$: k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+8589934592k+6442598811}$ O_{2a}
$3^{19}\cdot 2^{\alpha-30}\cdot n+9298091736k+6973728863$ O_3 : z = $3^{19}\cdot 2^{\alpha-31}\cdot n+4649045868k+3486864431 \rightarrow$
f(z) = $3^{22}\cdot 2^{\alpha-33}\cdot n+31381059609k+23536334915$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+17179869184k+15032533403}$ O_{2a}
$3^{22}\cdot 2^{\alpha-33}\cdot n+62762119218k+54917394524 \rightarrow$
$3^{22}\cdot 2^{\alpha-34}\cdot n+31381059609k+27458697262$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+34359738368k+15032533403}$ O_{2a}
$3^{22}\cdot 2^{\alpha-34}\cdot n+62762119218k+27458697262 \rightarrow$
$3^{22}\cdot 2^{\alpha-35}\cdot n+31381059609k+13729348631 < 2^{\alpha}\cdot n+34359738368k+15032533403$ O_{2a}

$\alpha = 35 \rightarrow 3^{22}\cdot n+3^{22}\cdot k+13729348631 < 2^{35}\cdot n+2^{35}\cdot k+15032533403$ O_{2a} =
$3^{22}\cdot(n + k)+13729348631 < 2^{35}\cdot(n + k)+15032533403$ O_{2a}
$n = 0 \rightarrow$ $\mathbf{3^{22}\cdot k+13729348631 < 2^{35}\cdot k+15032533403}$ O_{2a} principal inequality : N(s) = **57**
$n = k = 0 \rightarrow$ **13729348631 < 15032533403** O_{2a} known numbers

Check link known numbers :
15032533403 O_{2a} : y = 7516266701 → f(y) = 16911600o79 O_{3c} : z = 8.455800039 →
f(z) = 57076650269 O_1 : x = 28538325134 → f(x) = 42807487702 → 21403743851 O_{2a} :
y = 10701871925 → f(y) = 24079211833 O_1 : x = 12039605916 → f(x) = 18059408875 O_{2a} :
y = 9029704437 → f(y) = 20316834985 O_1 : x = 10158417492 → f(x) = 15237626239 O_{3a} :
z = 7618813119 → f(z) = 51426988559 O_{3c} : z = 25713494279 → f(z) = 173566086389 O_1 :
x = 86783043194 → f(x) = 130174564792 → 65087282396 → 32543641198 →
16271820599 O_3^* : z = 8135910299 → f(z) = 54917394524 → 27458697262 →
13729348631 < 15032533403 O_{2a} : N(s) = **57**

Chain of connections :
$O_{2a} \to O_{3c} \to O_1 \to E \to O_{2a} \to O_1 \to O_{2a} \to O_1 \to O_{3a} \to O_{3c} \to O_1 \to E \to E \to$
$E \to O_3^* \to E \to E \to < O_{2a}$

Changing the last connection in the cycles of links high horizons exposed above, we will found two higher horizons ϴ(m) and ϴ(l) such that ϴ(l) < ϴ(m), and so on for an endless of links; it would be enough to have appropriate calculation tools. Therefore we can affirm that for every biggest horizon we are able to cover by a binomial inequality after N(s) steps, there is an upper horizon that needs a greater number of steps to become lower than itself. So it's proved that SC is not fully confirmable nor fully demonstrable. It's a sort of *Circle Quadrature* or a new type of *Algebraic Quadrature*.

In SCQ, Independence Theorem shows the *disordered* harmony of odd numbers link cycles. In fact, if we compared our choices with the chain of connections, we note :
O_3 is O_3^* if it's connected to E; O_3 is O_{3a} if it's connected to O_2 or O_3; O_3 is O_{3b} if it's connected to E just a time; O_3 is O_{3c} if it's connected to O_1 .
O_2 is O_2^* if it's connected to E; O_2 is O_{2a} if it's connected to O_3 or O_2 or O_1 .
O_1 can be connected to E or O_1 or O_2 or O_3 and then follows the rules illustrated above.

3.2.3. An example of a cycle of link with an infinite number of steps

16k+11 O_{2a}

$2^\alpha \cdot n+16k+11$ O_{2a} : $y = 2^{\alpha-1}\cdot n+8k+5 \to f(y) = 3^2\cdot 2^{\alpha-3}\cdot n+18k+13$ [O_3] : k = **4k+1**

$2^\alpha \cdot n+64k+27$ O_{2a}
$3^2\cdot 2^{\alpha-3}\cdot n+72k+31$ O_3 : $z = 3^2\cdot 2^{\alpha-4}\cdot n+36k+15 \to f(z) = 3^5\cdot 2^{\alpha-6}\cdot n+243k+107$ [O] : k = **2k**

$2^\alpha \cdot n+128k+27$ O_{2a}
$3^5\cdot 2^{\alpha-6}\cdot n+486k+107$ [O_1] : k = **2k+1**

$2^\alpha \cdot n +256k+155$ O_{2a}
$3^5\cdot 2^{\alpha-6}\cdot n+972k+593$ O_1 : $x = 3^5\cdot 2^{\alpha-7}\cdot n+486k+296 \to f(x) = 3^6\cdot 2^{\alpha-8}\cdot n+729k+445$ [O] : k = **2k**

$2^\alpha \cdot n+512k+155$ O_{2a}
$3^6\cdot 2^{\alpha-8}\cdot n+1458k+445$ [O_1] : k = **2k**

$2^\alpha \cdot n+1024k+155$ O_{2a}
$3^6\cdot 2^{\alpha-8}\cdot n+2916k+445$ O_1 : $x = 3^6\cdot 2^{\alpha-9}\cdot n+1458k+222 \to f(x) = 3^7\cdot 2^{\alpha-10}\cdot n+2187k+334$ [O] : k = **2k+1**

$2^\alpha \cdot n+2048k+1179$ O_{2a}
$3^7\cdot 2^{\alpha-10}\cdot n+4374k+2521$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+4096k+1179}$ O_{2a}
$3^7\cdot 2^{\alpha-10}\cdot n+8748k+2521$ O_1 : $x = 3^7\cdot 2^{\alpha-11}\cdot n+4374k+1260 \rightarrow$
$f(x) = 3^8\cdot 2^{\alpha-12}\cdot n+6561k+1891$ [O] : k = **2k** … … here we stop.

If we choose [O]→[O_1] twice more, the cycle collapses to the lower horizon $3^{10}\cdot n+52732 < 2^{16}\cdot n+58523$ O_{2a} : N(s) = 26; but if we choose [O]→[O_1] only once and then [O]→[O_3], the cycle rebounds to a higher horizon. In this way we could continue to go on for an infinity of steps, where O_1 is always presents, but its value doesn't become (never) lesser than that of O_{2a} . Only if we decide to go back this is possible; and this is possible if we choose [E] and not [O]. In fact:

$f(x) = 3^8\cdot 2^{\alpha-12}\cdot n+6561k+1891$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+4096k+1179}$ O_{2a} → $\mathbf{2^{\alpha}\cdot n+8192k+5275}$ O_{2a}
$3^8\cdot 2^{\alpha-12}\cdot n+6561k+1891 \rightarrow 3^8\cdot 2^{\alpha-12}\cdot n+13122k+8452 \rightarrow$
$3^8\cdot 2^{\alpha-13}\cdot n+6561k+4226 < 2^{\alpha}\cdot n+8192k+5275$ O_{2a} $[\alpha \geq 13]$

$\alpha = 13 \rightarrow 3^8\cdot n+3^8\cdot k+4226 < 2^{13}\cdot n+2^{13}\cdot k+5275$ O_{2a} $= 3^8\cdot(n + k)+4226 < 2^{13}\cdot(n + k)+5275$ O_{2a}
n = 0 → $\mathbf{3^8\cdot k+4226 < 2^{13}\cdot k+5275}$ O_{2a} principal inequality : N(s) = **21**
n = k = 0 → **4226 < 5275** O_{2a} known numbers.

Another cycle of link with an infinite number of steps managed so :

$\mathbf{O_{3b}}$ → **E** → [O_3] → [O_1] → [O_3] → [O_1] → … … → [E] → … …

$\mathbf{2^{\alpha}\cdot n+32k+7}$ O_{3b} : $z = 2^{\alpha-1}\cdot n+16k+3 \rightarrow f(z) = 3^3\cdot 2^{\alpha-3}\cdot n+108k+26 \rightarrow 3^3\cdot 2^{\alpha-4}\cdot n+54k+13$ [O_3] :
k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+128k+103}$ O_{3b}
$3^3\cdot 2^{\alpha-4}\cdot n+216k+175$ O_3 : $z = 3^3\cdot 2^{\alpha-5}\cdot n+108k+87 \rightarrow f(z) = 3^6\cdot 2^{\alpha-7}\cdot n+729k+593$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+256k+103}$ O_{3b}
$3^6\cdot 2^{\alpha-7}\cdot n+1458k+593$ [O_1] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+512k+103}$ O_{3b}
$3^6\cdot 2^{\alpha-7}\cdot n+2916k+593$ O_1 : $x = 3^6\cdot 2^{\alpha-8}\cdot n+1458k+296 \rightarrow f(x) = 3^7\cdot 2^{\alpha-9}\cdot n+2187k+445$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+1024k+103}$ O_{3b}
$3^7\cdot 2^{\alpha-9}\cdot n+4374k+445$ [O_3] : k = **4k+3**

$\mathbf{2^{\alpha}\cdot n+4096k+3175}$ O_{3b}
$3^7\cdot 2^{\alpha-9}\cdot n+17496k+13567$ O_3 : $z = 3^7\cdot 2^{\alpha-10}\cdot n+8748k+6783 \rightarrow$
$f(z) = 3^{10}\cdot 2^{\alpha-12}\cdot n+59049k+45791$ [O] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+8192k+3175}$ O_{3b}
$3^{10}\cdot 2^{\alpha-12}\cdot n+118098k+45791$ [O_1] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+16384k+11367}$ O_{3b}
$3^{10}\cdot 2^{\alpha-12}\cdot n+236196k+163889$ O_1 : $x = 3^{10}\cdot 2^{\alpha-13}\cdot n+118098k+81944 \rightarrow$
$f(x) = 3^{11}\cdot 2^{\alpha-14}\cdot n+177147k+122917$ [O] : k = **2k** … …

we could go on for a number of steps tends towards infinity, but we decide to go back, hence :

$f(x) = 3^{11}\cdot 2^{\alpha-14}\cdot n+177147k+122917$ [E] : k = **2k+1**

$\mathbf{2^{\alpha}\cdot n+16384k+11367}$ O_{3b} → $\mathbf{2^{\alpha}\cdot n+32768k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-14}\cdot n+177147k+122917 \rightarrow 3^{11}\cdot 2^{\alpha-14}\cdot n+354294k+300064 \rightarrow$
$3^{11}\cdot 2^{\alpha-15}\cdot n+177147k+150032$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+65536k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-15}\cdot n+354294k+150032 \rightarrow 3^{11}\cdot 2^{\alpha-16}\cdot n+177147k+75016$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+131072k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-16}\cdot n+354294k+75016 \rightarrow 3^{11}\cdot 2^{\alpha-17}\cdot n+177147k+37508$ [E] : k = **2k**

$\mathbf{2^{\alpha}\cdot n+262144k+27751}$ O_{3b}
$3^{11}\cdot 2^{\alpha-17}\cdot n+354294k+37508 \rightarrow 3^{11}\cdot 2^{\alpha-18}\cdot n+177147k+18754 < 2^{\alpha}\cdot n+262144k+27751$ O_{3b} $[\alpha \geq 18]$

$\alpha = 18 \rightarrow 3^{11}\cdot n+3^{11}\cdot k+18754 < 2^{18}\cdot n+2^{18}\cdot k+27751$ O_{3b} =
$3^{11}\cdot(n + k)+18754 < 2^{18}\cdot(n + k)+27751$ O_{3b}
n = 0 → $\mathbf{3^{11}\cdot k+18754 < 2^{18}\cdot k+27751}$ O_{3b} principal inequality : N(s) = **29**
n = k = 0 → **18754 < 27751** O_{3b} known numbers

Check link known numbers :
27751 O_{3b} : z = 13875 → f(z) = 93662 → 46831 O_{3c} : z = 23415 → f(z) = 158057 O_1 :
x = 79028 → f(x) = 118543 O_{3c} : z = 59271 → f(z) = 400085 O_1 : x = 200042 →
f(x) = 300064 → 150032 → 75016 → 37508 → **18754** < **27751** O_{3b} : N(s) = **29**

Two chains of connections with an infinite number of steps (We avoid to indicate the choices by [ ].

$\mathbf{O_{3a}}$ → O_3 → O_3 → E → O_3 → O_3 → O_1 → O_3 → O_3 → O_2 → E → E → E → O_3 → … → O_3 →
E → O_3 → E → O_3 → O_1 → O_2 → … → O_2 → O_3 → O_1 → O_1 → E → O_3 → … ... N(s) → ∞

$\mathbf{O_{3a}}$ → O_2 → O_2 → O_3 → O_1 → E → O_2 → O_3 → E → O_2 → O_3 → O_3 → … → O_2 → O_3 → O_3 →
O_2 → E → E → O_2 → O_3 → O_1 → O_1 → O_3 → O_2 → E → E → O_3 → O_1 → … … N(s) → ∞

There are an infinity of possible combinations of connections in the cycles of links which contain a number of steps N(s) → ∞. So, by the procedure applied above, is further confirmed that SC (CC) is a new type of *Quadrature* or *Zeno's paradox* (both understood as successive approximations of the percentage of coverage of ℕ by binomial inequalities).

As we saw Independence Theorem allows to manage the cycles of links to our liking. So, chosen an odd number $p \in O$ we can associate the horizon $2^k\cdot n+p$ to it. By formulas plan Theorem 2n+1 we can calculate $3^h\cdot n+q < 2^k\cdot n+p$ and the number of steps needed to move from the main horizon $\Theta(m) = 2^k\cdot n+p$ to its lower horizon $\Theta(l) = 3^h\cdot n+q$: N(s) = h + k .

Test of some binomial inequalities in General List by Theorem of Independence

N(s) = **8** : **27n+20 < 32n+23** O_3*
$\mathbf{2^k\cdot n+23}$ O_3* : $z = 2^{k-1}\cdot n+11 \rightarrow f(z) = 3^3\cdot 2^{k-3}\cdot n+80 \rightarrow 3^3\cdot 2^{k-4}\cdot n+40 \rightarrow$
$\mathbf{3^3\cdot 2^{k-5}\cdot n+20}$ < $\mathbf{2^k\cdot n+23}$ O_3* $[k \geq 5]$: k = 5 → **27n+20 < 32n+23** O_3*

N(s) = **11** : **81n+38 < 128n+59** O_{2a}
2^k·n+59 O_{2a} : y = 2^{k-1}·n+29 → f(y) = 3^2·2^{k-3}·n+67 O_2* : y = 3^2·2^{k-4}·n+33 →
f(y) = 3^4·2^{k-6}·n+76 → **3^4·2^{k-7}·n+38** < **2^k·n+59** O_{2a} [k ≥ 7] : k = 7 → **81n+38 < 128n+59** O_{2a}

N(s) = **13** : **243n+91 < 256n+95** O_{3a}
2^k·n+95 O_{3a} : z = 2^{k-1}·n+47 → f(z) = 3^3·2^{k-3}·n+323 O_2* : y = 3^3·2^{k-4}·n+161 →
f(y) = 3^5·2^{k-6}·n+364 → 3^5·2^{k-7}·n+182 → **3^5·2^{k-8}·n+91** < **2^k·n+95** O_{3a} [k ≥ 8] : k = 8 →
243n+91 < 256n+95 O_{3a}

N(s) = **16** : **729n+302 < 1024n+423** O_{3b}
2^k·n+423 O_{3b} : z = 2^{k-1}·n+211 → f(z) = 3^3·2^{k-3}·n+1430 → 3^3·2^{k-4}·n+715 O_{2a} :
y = 3^3·2^{k-5}·n+357 → f(y) = 3^5·2^{k-7}·n+805 O_1 : x = 3^5·2^{k-8}·n+402 → f(x) = 3^6·2^{k-9}·n+604 →
3^6·2^{k-10}·n+302 < **2^k·n+423** O_{3b} [k ≥ 10] : k = 10 → **729n+302 < 1024n+423** O_{3b}

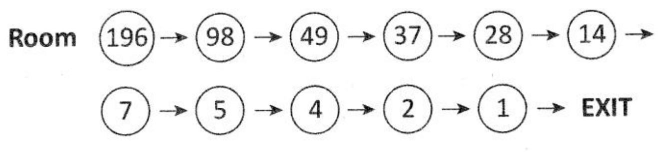
N(s) = **19** : **2187n + 205 < 4096n+383** O_{3a}
2^k·n+383 O_{3a} : z = 2^{k-1}·n + 191 → f(z) = 3^3·2^{k-3}·n + 1295 O_{3c} : z = 3^3·2^{k-4}·n + 647 →
f(z) = 3^6·2^{k-6}·n + 4373 O_1 : x = 3^6·2^{k-7}·n + 2186 → f(x) = 3^7·2^{k-8}·n + 3280 →
3^7·2^{k-9}·n + 1640 → 3^7·2^{k-10}·n + 820 → 3^7·2^{k-11}·n + 410 →
3^7·2^{k-12}·n + 205 < 2^k·n+383 O_{3a} [k ≥ 12] : k = 12 → **2187n + 205 < 4096n+383** O_{3a}

N(s) **= 21** : **6561n+205 < 8192n+255** O_{3a}
2^k·n+255 O_{3a} : z = 2^{k-1}·n+127 → f(z) = 3^3·2^{k-3}·n+863 O_{3a} : z = 3^3·2^{k-4}·n+431 →
f(z) = 3^6·2^{k-6}·n+2915 O_2* : y = 3^6·2^{k-7}·n+1457 → f(y) = 3^8·2^{k-9}·n+3280 →
3^8·2^{k-10}·n+1640 → 3^8·2^{k-11}·n+820 → 3^8·2^{k-12}·n+410 →
3^8·2^{k-13}·n+205 < 2^k·n+255 O_{3a} [k ≥ 13] : k = 13 → **6561n+205 < 8192n+255** O_{3a}

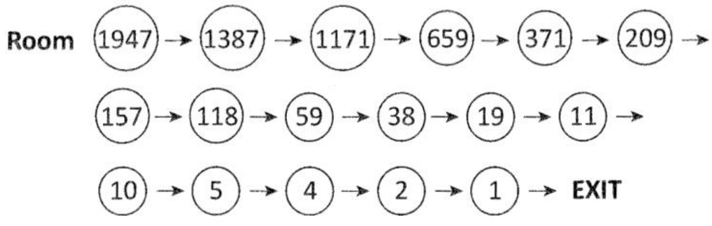
3.2.4. Labyrinth of Syracuse and Descent to the sea

In all cycles of links, the final cycle (last five elements = four steps) to exit *Labyrinth of Syracuse* is: **{10; 5; 4; 2; 1}**or **{7; 5; 4; 2; 1}**; unless in the cycle appears an even number like that of kind 2^p, or an odd number of kind (4^p-1)/3. In this case the subsequent steps are evaluable *a priori*, and the last five elements are : 16 → 8 → 4 → 2 → 1 = **{16; 8; 4; 2; 1}**.

Strategy to exit labyrinth of Syracuse as we stand in room 196 :



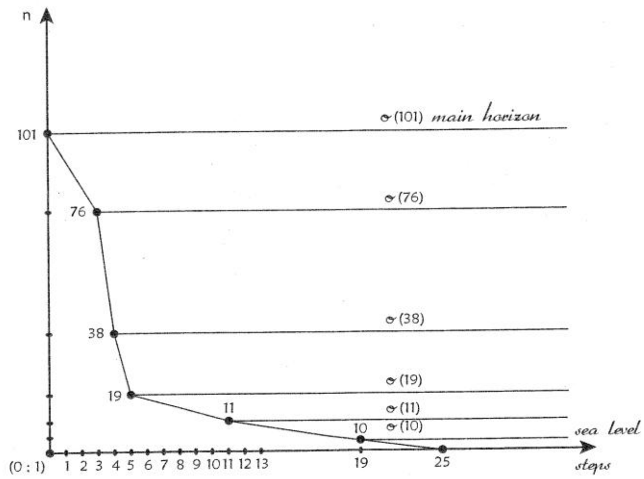
Strategy to exit labyrinth of Syracuse as we stand in room 1947 :



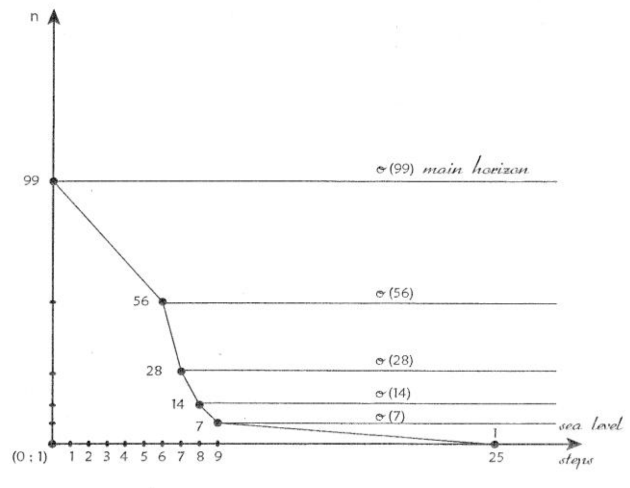
Another final cycle in cycles of links is **{12; 6; 3; 2; 1}.** This cycle is anomalous, since it appears only if the generating numbers are of type n = 3·2^p ∈ I(3) = {3; 6; 12; 24; 48; 96; 192; …; 3·2^p }. The n ∈ I(3) do not appear in any other iterative cycle. Not in those generated by even numbers, in

fact: if $n \in E$, then $n \to n/2$, if $n/2 = 3 \cdot 2^p$: $n = 2 \cdot 3 \cdot 2^p = 3 \cdot 2^{p+1} \in I(3)$. Not in those generated by odd numbers; in fact if $n \in O$ then $n \to 3n+1$, if $3n+1 = 3 \cdot 2^p$: $n = (3 \cdot 2^p - 1)/3 = 2^p - 1/3$ which is impossible. The only odd number in I(3) is 3, which generates the sequence: **3** $\to 10 \to 5 \to 16 \to 8 \to 4 \to$ **2** < 3. In the *Labyrinth of Syracuse* this anomalous cycle, up value 3, can be considered as an infinite sequence of even rooms, numbered one double of the other, aligned in a long hallway perfectly straight that is not connected to any other room of the maze. In the descent to the sea, it can be interpreted as a river that flows into the sea without having any tributaries or being affluent. For $n > 3$ all sequences generated by $n = 3 \cdot 2^p \in I(3)$ are monotone decreasing, such that $a_n = a_{n-1}/2$. If, for example, the generating number is $n = 3 \cdot 2^{10} = 3072$ we have S(3072) = {3072; 1536; 768; 384; 192; 96; 48; 24; 12; 6; 3}.
In the descent to sea level the examples could be translated into graphs. It should be borne in mind that the final cycle {10; 5; 4; 2; 1} is done in 6 steps, since $\Theta(5)$ passes to $\Theta(4)$ in three steps, while the final cycle {7; 5; 4; 2; 1} is accomplished in 16 steps, since $\Theta(7)$ passes to $\Theta(5)$ in 11 steps and, as we already said, $\Theta(5)$ passes to $\Theta(4)$ in three steps. For convenience these end cycles are represented by a single line section, one step. Of course is very difficult to realize graphs of large numbers. Here we present two examples.

Graph of the descent to the sea level from the main horizon $\Theta(m) = 101$. It arrives at sea in 6 steps instead of 25.



Graph of the descent to the sea level from the main horizon $\Theta(m) = 99$. It arrives at sea in 5 steps instead of 25



In this way we can arrive at the sea level (1) from highest heights, greatly reducing the number of steps and remaining always below the main horizon.

3.3. Long cycles of links

In this paragraph we analyze Odd generating numbers which require N(s) > 21 to become lesser than themselves.

3.3.1. The cycle of link number 27 and its internal binomial inequalities

$\mathbf{2^k \cdot n+27}$ $O_{2a} \rightarrow 3\cdot2^k\cdot n+82 \rightarrow 3\cdot2^{k-1}\cdot n+41$ $O_1 \rightarrow 3^2\cdot2^{k-1}\cdot n+124 \rightarrow 3^2\cdot2^{k-2}\cdot n+62 \rightarrow$
$3^2\cdot2^{k-3}\cdot n+31$ $O_{3a} \rightarrow 3^3\cdot2^{k-3}\cdot n+94 \rightarrow 3^3\cdot2^{k-4}\cdot n+47$ $O_{3c} \rightarrow 3^4\cdot2^{k-4}\cdot n+142 \rightarrow 3^4\cdot2^{k-5}\cdot n+71$ $O_{3b} \rightarrow$
$3^5\cdot2^{k-5}\cdot n+214 \rightarrow 3^5\cdot2^{k-6}\cdot n+107$ $O_{2a} \rightarrow 3^6\cdot2^{k-6}\cdot n+322 \rightarrow 3^6\cdot2^{k-7}\cdot n+161$ $O_1 \rightarrow 3^7\cdot2^{k-7}\cdot n+484 \rightarrow$
$3^7\cdot2^{k-8}\cdot n+242 \rightarrow 3^7\cdot2^{k-9}\cdot n+121$ $O_1 \rightarrow 3^8\cdot2^{k-9}\cdot n+364 \rightarrow 3^8\cdot2^{k-10}\cdot n+182 \rightarrow 3^8\cdot2^{k-11}\cdot n+91$ $O_{2a} \rightarrow$
$3^9\cdot2^{k-11}\cdot n+274 \rightarrow 3^9\cdot2^{k-12}\cdot n+137$ $O_1 \rightarrow 3^{10}\cdot2^{k-12}\cdot n+412 \rightarrow 3^{10}\cdot2^{k-13}\cdot n+206 \rightarrow$
$3^{10}\cdot2^{k-14}\cdot n+103$ $O_{3b} \rightarrow 3^{11}\cdot2^{k-14}\cdot n+310 \rightarrow 3^{11}\cdot2^{k-15}\cdot n+155$ $O_{2a} \rightarrow 3^{12}\cdot2^{k-15}\cdot n+466 \rightarrow$
$3^{12}\cdot2^{k-16}\cdot n+233$ $O_1 \rightarrow 3^{13}\cdot2^{k-16}\cdot n+700 \rightarrow 3^{13}\cdot2^{k-17}\cdot n+350 \rightarrow 3^{13}\cdot2^{k-18}\cdot n+175$ $O_{3c} \rightarrow$
$3^{14}\cdot2^{k-18}\cdot n+526 \rightarrow 3^{14}\cdot2^{k-19}\cdot n+263$ $O_{3b} \rightarrow 3^{15}\cdot2^{k-19}\cdot n+790 \rightarrow 3^{15}\cdot2^{k-20}\cdot n+395$ $O_{2a} \rightarrow$
$3^{16}\cdot2^{k-20}\cdot n+1186 \rightarrow 3^{16}\cdot2^{k-21}\cdot n+593$ $O_1 \rightarrow 3^{17}\cdot2^{k-21}\cdot n+1780 \rightarrow 3^{17}\cdot2^{k-22}\cdot n+890 \rightarrow$
$3^{17}\cdot2^{k-23}\cdot n+445$ $O_1 \rightarrow 3^{18}\cdot2^{k-23}\cdot n+1336 \rightarrow 3^{18}\cdot2^{k-24}\cdot n+668 \rightarrow 3^{18}\cdot2^{k-25}\cdot n+334 \rightarrow$
$3^{18}\cdot2^{k-26}\cdot n+167$ $O_{3b} \rightarrow 3^{19}\cdot2^{k-26}\cdot n+502 \rightarrow 3^{19}\cdot2^{k-27}\cdot n+251$ $O_{2a} \rightarrow 3^{20}\cdot2^{k-27}\cdot n+754 \rightarrow$
$3^{20}\cdot2^{k-28}\cdot n+377$ $O_1 \rightarrow 3^{21}\cdot2^{k-28}\cdot n+1132 \rightarrow 3^{21}\cdot2^{k-29}\cdot n+566 \rightarrow 3^{21}\cdot2^{k-30}\cdot n+283$ $O_{2a} \rightarrow$
$3^{22}\cdot2^{k-30}\cdot n+850 \rightarrow 3^{22}\cdot2^{k-31}\cdot n+425$ $O_1 \rightarrow 3^{23}\cdot2^{k-31}\cdot n+1276 \rightarrow 3^{23}\cdot2^{k-32}\cdot n+638 \rightarrow$
$3^{23}\cdot2^{k-33}\cdot n+319$ $O_{3a} \rightarrow 3^{24}\cdot2^{k-33}\cdot n+958 \rightarrow 3^{24}\cdot2^{k-34}\cdot n+479$ $O_{3a} \rightarrow 3^{25}\cdot2^{k-34}\cdot n+1438 \rightarrow$
$3^{25}\cdot2^{k-35}\cdot n+719$ $O_{3c} \rightarrow 3^{26}\cdot2^{k-35}\cdot n+2158 \rightarrow 3^{26}\cdot2^{k-36}\cdot n+1079$ $O_3^* \rightarrow 3^{27}\cdot2^{k-36}\cdot n+3238 \rightarrow$
$3^{27}\cdot2^{k-37}\cdot n+1619$ $O_2^* \rightarrow 3^{28}\cdot2^{k-37}\cdot n+4858 \rightarrow 3^{28}\cdot2^{k-38}\cdot n+2429$ $O_1 \rightarrow 3^{29}\cdot2^{k-38}\cdot n+7288 \rightarrow$
$3^{29}\cdot2^{k-39}\cdot n+3644 \rightarrow 3^{29}\cdot2^{k-40}\cdot n+1822 \rightarrow 3^{29}\cdot2^{k-41}\cdot n+911$ $O_{3c} \rightarrow 3^{30}\cdot2^{k-41}\cdot n+2734 \rightarrow$
$3^{30}\cdot2^{k-42}\cdot n+1367$ $O_3^* \rightarrow 3^{31}\cdot2^{k-42}\cdot n+4102 \rightarrow 3^{31}\cdot2^{k-43}\cdot n+2051$ $O_2^* \rightarrow 3^{32}\cdot2^{k-43}\cdot n+6154 \rightarrow$
$3^{32}\cdot2^{k-44}\cdot n+3077$ $O_1 \rightarrow 3^{33}\cdot2^{k-44}\cdot n+9232 \rightarrow 3^{33}\cdot2^{k-45}\cdot n+4616 \rightarrow 3^{33}\cdot2^{k-46}\cdot n+2308 \rightarrow$
$3^{33}\cdot2^{k-47}\cdot n+1154 \rightarrow 3^{33}\cdot2^{k-48}\cdot n+577$ $O_1 \rightarrow 3^{34}\cdot2^{k-48}\cdot n+1732 \rightarrow 3^{34}\cdot2^{k-49}\cdot n+866 \rightarrow$
$3^{34}\cdot2^{k-50}\cdot n+433$ $O_1 \rightarrow 3^{35}\cdot2^{k-50}\cdot n+1300 \rightarrow 3^{35}\cdot2^{k-51}\cdot n+650 \rightarrow 3^{35}\cdot2^{k-52}\cdot n+325$ $O_1 \rightarrow$
$3^{36}\cdot2^{k-52}\cdot n+976 \rightarrow 3^{36}\cdot2^{k-53}\cdot n+488 \rightarrow 3^{36}\cdot2^{k-54}\cdot n+244 \rightarrow 3^{36}\cdot2^{k-55}\cdot n+122 \rightarrow$
$3^{36}\cdot2^{k-56}\cdot n+61$ $O_1 \rightarrow 3^{37}\cdot2^{k-56}\cdot n+184 \rightarrow 3^{37}\cdot2^{k-57}\cdot n+92 \rightarrow 3^{37}\cdot2^{k-58}\cdot n+46 \rightarrow$
$\mathbf{3^{37}\cdot2^{k-59}\cdot n+23} < \mathbf{2^k\cdot n+27}$ O_{2a} : k = 59 → $\mathbf{3^{37}\cdot n+23} < \mathbf{2^{59}\cdot n+27}$ O_{2a} : N(s) = **96**

Chain of transformations:
$O_{2a} \rightarrow O_1 \rightarrow O_{3a} \rightarrow O_{3c} \rightarrow O_{3b} \rightarrow O_{2a} \rightarrow O_1 \rightarrow O_1 \rightarrow O_{2a} \rightarrow O_1 \rightarrow O_{3b} \rightarrow O_{2a} \rightarrow O_1 \rightarrow O_{3c} \rightarrow$
$O_{3b} \rightarrow O_{2a} \rightarrow O_1 \rightarrow O_1 \rightarrow O_{3b} \rightarrow O_{2a} \rightarrow O_1 \rightarrow O_{2a} \rightarrow O_1 \rightarrow O_{3a} \rightarrow O_{3a} \rightarrow O_{3c} \rightarrow O_3^* \rightarrow O_2^* \rightarrow$
$O_1 \rightarrow O_{3c} \rightarrow O_3^* \rightarrow O_2^* \rightarrow O_1 \rightarrow O_1 \rightarrow O_1 \rightarrow O_1 \rightarrow O_1 \rightarrow E \rightarrow < O_{2a}$

In the chain of transformations we have considered → E only at the end. We note that O_1 appears many more times than the other subsets, in perfect agreement with Theorem 2n+1 and its addition.

Inside the cycle of link of 27 O_{2a} are present the following binomial inequalities

$3^{37}\cdot2^{k-59}\cdot n+23$ $O_3^* < 3^2\cdot2^{k-3}\cdot n+31$ $O_{3a} \rightarrow 3^{35}\cdot2^{k-59}\cdot n+23 < 2^{k-3}\cdot n+31$: k = 59 →
$\mathbf{3^{35}\cdot n+23 < 2^{56}\cdot n+31}$ O_{3a} : N(s) = **91**

$3^{37}\cdot2^{k-58}\cdot n+46 < 3^3\cdot2^{k-4}\cdot n+47$ $O_{3c} \rightarrow 3^{34}\cdot2^{k-58}\cdot n+46 < 2^{k-4}\cdot n+47$: k = 58 →
$\mathbf{3^{34}\cdot n+46 < 2^{54}\cdot n+47}$ O_{3c} : N(s) = **88**

$3^{36}\cdot2^{k-56}\cdot n+61$ O_1 < $3^4\cdot2^{k-5}\cdot n+71$ O_{3b} → $3^{32}\cdot2^{k-56}\cdot n+61 < 2^{k-5}\cdot n+71$: k = 56 →
$3^{32}\cdot n+61 < 2^{51}\cdot n+71$ O_{3b} : N(s) = **83**

$3^{36}\cdot2^{k-56}\cdot n+61$ O_1 < $3^8\cdot2^{k-11}\cdot n+91$ O_{2a} → $3^{28}\cdot2^{k-56}\cdot n+61 < 2^{k-11}\cdot n+91$: k = 56 →
$3^{28}\cdot n+61 < 2^{45}\cdot n+91$ O_{2a} : N(s) = **73**

$3^{36}\cdot2^{k-56}\cdot n+61$ O_1 < $3^{10}\cdot2^{k-14}\cdot n+103$ O_{3b} → $3^{26}\cdot2^{k-56}\cdot n+61 < 2^{k-14}\cdot n+103$: k = 56 →
$3^{26}\cdot n+61 < 2^{42}\cdot n+103$ O_{3b} : N(s) = **68**

$3^{36}\cdot2^{k-55}\cdot n+122 < 3^{11}\cdot2^{k-15}\cdot n+155$ O_{2a} → $3^{25}\cdot2^{k-55}\cdot n+122 < 2^{k-15}\cdot n+155$: k = 55 →
$3^{25}\cdot n+122 < 2^{40}\cdot n+155$ O_{2a} : N(s) = **65**

$3^{36}\cdot2^{k-55}\cdot n+122 < 3^{18}\cdot2^{k-26}\cdot n+167$ O_{3b} → $3^{18}\cdot2^{k-55}\cdot n+122 < 2^{k-26}\cdot n+167$: k = 55 →
$3^{18}\cdot n+122 < 2^{29}\cdot n+167$ O_{3b} : N(s) = **47**

$3^{36}\cdot2^{k-54}\cdot n+244 < 3^{19}\cdot2^{k-27}\cdot n+251$ O_{2a} → $3^{17}\cdot2^{k-54}\cdot n+244 < 2^{k-27}\cdot n+251$: k = 54 →
$3^{17}\cdot n+244 < 2^{27}\cdot n+251$ O_{2a} : N(s) = **44**

$3^{36}\cdot2^{k-54}\cdot n+244 < 3^{21}\cdot2^{k-30}\cdot n+283$ O_{2a} → $3^{15}\cdot2^{k-54}\cdot n+244 < 2^{k-30}\cdot n+283$: k = 54 →
$3^{15}\cdot n+244 < 2^{24}\cdot n+283$ O_{2a} : N(s) = **39**

$3^{36}\cdot2^{k-54}\cdot n+244 < 3^{23}\cdot2^{k-33}\cdot n+319$ O_{3a} → $3^{13}\cdot2^{k-54}\cdot n+244 < 2^{k-33}\cdot n+319$: k = 54 →
$3^{13}\cdot n+244 < 2^{21}\cdot n+319$ O_{3a} : N(s) = **34**

$3^{34}\cdot2^{k-50}\cdot n+433$ O_1 < $3^{24}\cdot2^{k-34}\cdot n+479$ O_{3a} → $3^{10}\cdot2^{k-50}\cdot n+433 < 2^{k-34}\cdot n+479$: k = 50 →
$3^{10}\cdot n+433 < 2^{16}\cdot n+479$ O_{3a} : N(s) = **26**

$3^{33}\cdot2^{k-48}\cdot n+577$ O_1 < $3^{25}\cdot2^{k-35}\cdot n+719$ O_{3c} → $3^{8}\cdot2^{k-48}\cdot n+577 < 2^{k-35}\cdot n+719$: k = 48 →
$3^{8}\cdot n+577 < 2^{13}\cdot n+719$ O_{3c} : N(s) = **21**

In the cycle of link number 27 we applied Collatz algorithm, because applying Independence Theorem and formulas plan Theorem 2n+1 (see § 3.3.2 below) we would have lost some of the internal binomial inequalities.

3.3.2. Applications Independence Theorem

Cycle of link number 27 by application Independence Theorem

$2^k\cdot n+27$ O_{2a} : y = $2^{k-1}\cdot n+13$ → $3^2\cdot2^{k-3}\cdot n+31$ O_{3a} : z = $3^2\cdot2^{2k-4}\cdot n+15$ → f(z) = $3^5\cdot2^{k-6}\cdot n+107$ O_{2a} :
y = $3^5\cdot2^{k-7}\cdot n+53$ → f(y) = $3^7\cdot2^{k-9}\cdot n+121$ O_1 : x = $3^7\cdot2^{k-10}\cdot n+60$ → f(x) = $3^8\cdot2^{k-11}\cdot n+91$ O_{2a} :
y = $3^8\cdot2^{k-12}\cdot n+45$ → f(y) = $3^{10}\cdot2^{k-14}\cdot n+103$ O_{3b} : z = $3^{10}\cdot2^{k-15}\cdot n+51$ → f(z) = $3^{13}\cdot2^{k-17}\cdot n+350$ →
$3^{13}\cdot2^{k-18}\cdot n+175$ O_{3c} : z = $3^{13}\cdot2^{k-19}\cdot n+87$ → f(z) = $3^{16}\cdot2^{k-21}\cdot n+593$ O_1 : x = $3^{16}\cdot2^{k-22}\cdot n+296$ →
$3^{17}\cdot2^{k-23}\cdot n+445$ O_1 : x = $3^{17}\cdot2^{k-24}\cdot n+222$ → f(x) = $3^{18}\cdot2^{k-26}\cdot n+167$ O_{3b} : z = $3^{18}\cdot2^{k-27}\cdot n+83$ →
f(z) = $3^{21}\cdot2^{k-29}\cdot n+566$ → $3^{21}\cdot2^{k-30}\cdot n+283$ O_{2a} : y = $3^{21}\cdot2^{k-31}\cdot n+141$ → f(y) = $3^{23}\cdot2^{k-33}\cdot n+319$ O_{3a} :
z = $3^{23}\cdot2^{k-34}\cdot n+159$ → f(z) = $3^{26}\cdot2^{k-36}\cdot n+1079$ O_3* : z = $3^{26}\cdot2^{k-37}\cdot n+539$ →
f(z) = $3^{29}\cdot2^{k-39}\cdot n+3644$ → $3^{29}\cdot2^{k-40}\cdot n+1822$ → $3^{29}\cdot2^{k-41}\cdot n+911$ O_{3c} : z = $3^{29}\cdot2^{k-42}\cdot n+455$ →
f(z) = $3^{32}\cdot2^{k-44}\cdot n+3077$ O_1 : x = $3^{32}\cdot2^{k-45}\cdot n+1538$ → f(x) = $3^{33}\cdot2^{k-46}\cdot n+2308$ →
$3^{33}\cdot2^{k-47}\cdot n+1154$ → $3^{33}\cdot2^{k-48}\cdot n+577$ O_1 : x = $3^{33}\cdot2^{k-49}\cdot n+288$ → f(x) = $3^{34}\cdot2^{k-50}\cdot n+433$ O_1 :
x = $3^{34}\cdot2^{k-51}\cdot n+216$ → f(x) = $3^{35}\cdot2^{k-52}\cdot n+325$ O_1 : x = $3^{35}\cdot2^{k-53}\cdot n+162$ → f(x) = $3^{36}\cdot2^{k-54}\cdot n+244$ →
$3^{36}\cdot2^{k-55}\cdot n+122$ → $3^{36}\cdot2^{k-56}\cdot n+61$ O_1 : x = $3^{36}\cdot2^{k-57}\cdot n+30$ → f(x) = $3^{37}\cdot2^{k-58}\cdot n+46$ →
$3^{37}\cdot2^{k-59}\cdot n+23$ < **$2^k\cdot n+27$** O_{2a} : k = 59 → **$3^{37}\cdot n+23 < 2^{59}\cdot n+27$** O_{2a} : N(s) = **96**

Chain of connections:
$O_{2a} \rightarrow O_{3a} \rightarrow O_{2a} \rightarrow O_1 \rightarrow O_{2a} \rightarrow O_{3b} \rightarrow E \rightarrow O_{3c} \rightarrow O_1 \rightarrow O_1 \rightarrow E \rightarrow O_{3b} \rightarrow E \rightarrow O_{2a} \rightarrow$
$O_{3a} \rightarrow O_3^* \rightarrow E \rightarrow E \rightarrow O_{3c} \rightarrow O_1 \rightarrow E \rightarrow E \rightarrow O_1 \rightarrow O_1 \rightarrow O_1 \rightarrow E \rightarrow E \rightarrow O_1 \rightarrow E \rightarrow < O_{2a}$:
N(s) = **96** has been reduced to N(s) = **29**.

Example

We apply Theorem of Independence to odd numbers in O_{2a} ; O_{3a} ; O_{3b} ; O_{3c} . We don't take into consideration O_1 ; O_2* ; O_3* because they are completely covered by their own respective binomial inequalities.

$$\text{Remembering : } \begin{cases} x = \frac{n-1}{2} \rightarrow f(x) = \frac{3x+2}{2} : N(s) = 3 \\ y = \frac{n-1}{2} \rightarrow f(y) = \frac{9y+7}{4} : N(s) = 5 \\ z = \frac{n-1}{2} \rightarrow f(z) = \frac{27z+23}{4} : N(s) = 6 \end{cases}$$

The three functions, derived from theorem 2n+1, allow longer jumps which reduce the number of steps in Collatz cycles. If *2n+1* is a generic number belonging to the odd subsets, we have :

- **f(x) = (3x+2)/2** [n (mod4)] (3 steps); (direct contraction) : applies when the number enters in the flow of O_1, significantly reducing the value towards the lower horizon.
- **f(y) = (9y+7)/4** [n (mod16)] (5 steps); (shift function). It can move to class [n (mod4)], or remain in class [n (mod16)] ($O_{2a} \vee O_2$*), or move up to class [n (mod32)].
- **f(z) = (27z+23)/4** [n (mod32)] (6 steps); (growth function or rebound function). It moves to class [n (mod16)] or [n (mod32)]).

n = **2407**
16n+11 = 38523 O_{2a}
$\mathbf{2^k \cdot n+38523}$ O_{2a} : $y = 2^{k-1}\cdot n+19261 \rightarrow f(y) = 3^2\cdot 2^{k-3}\cdot n+43339$ O_{2a} : $y = 3^2\cdot 2^{k-4}\cdot n+21669 \rightarrow$
$f(y) = 3^4\cdot 2^{k-6}\cdot n+48757$ O_1 : $x = 3^4\cdot 2^{k-7}\cdot n+24378 \rightarrow f(x) = 3^5\cdot 2^{k-8}\cdot n+36568 < 2^k\cdot n+38523$ O_{2a}
k = 8 → $\mathbf{3^5\cdot n+36568 < 2^8\cdot n+38523}$ O_{2a} : N(s) = **13** ~ 243n+118 < 256n+123 O_{2a} : n = 150

32n-1 = 77023 O_{3a}
$\mathbf{2^k\cdot n+77023}$ O_{3a} : $z = 2^{k-1}\cdot n+38511 \rightarrow f(z) = 3^3\cdot 2^{k-3}\cdot n+259955$ O_2^* : $y = 3^3\cdot 2^{k-4}\cdot n+129977 \rightarrow$
$f(y) = 3^5\cdot 2^{k-6}\cdot n+292450 \rightarrow 3^5\cdot 2^{k-7}\cdot n+146225$ O_1 : $x = 3^5\cdot 2^{k-8}\cdot n+73112 \rightarrow$
$f(x) = 3^6\cdot 2^{k-9}\cdot n+109669$ O_1 : $x = 3^6\cdot 2^{k-10}\cdot n+54834 \rightarrow f(x) = 3^7\cdot 2^{k-11}\cdot n+82252 \rightarrow$
$3^7\cdot 2^{k-12}\cdot n+41126 < 2^k\cdot n+77023$ O_{3a}
k = 12 → $\mathbf{3^7\cdot n+41126 < 2^{12}\cdot n+77023}$ O_{3a} : N(s) = **19** ~ 2187n+1760 < 4096n+3295 O_{3a} : n = 18

32n+7 = 77031 O_{3b}
$\mathbf{2^k\cdot n+77031}$ O_{3b} : $z = 2^{k-1}\cdot n+38515 \rightarrow f(z) = 3^3\cdot 2^{k-3}\cdot n+259982 \rightarrow 3^3\cdot 2^{k-4}\cdot n+129991$ O_{3b} :
$z = 3^3\cdot 2^{k-5}\cdot n+64995 \rightarrow f(z) = 3^6\cdot 2^{k-7}\cdot n+438722 \rightarrow 3^6\cdot 2^{k-8}\cdot n+219361$ O_1 : $x = 3^6\cdot 2^{k-9}\cdot n+109680 \rightarrow$
$f(x) = 3^7\cdot 2^{k-10}\cdot n+164521$ O_1 : $x = 3^7\cdot 2^{k-11}\cdot n+82260 \rightarrow f(x) = 3^8\cdot 2^{k-12}\cdot n+123391$ O_{3a} :
$z = 3^8\cdot 2^{k-13}\cdot n+61695 \rightarrow f(z) = 3^{11}\cdot 2^{k-15}\cdot n+416447$ O_{3a} : $z = 3^{11}\cdot 2^{k-16}\cdot n+208223 \rightarrow$
$f(z) = 3^{14}\cdot 2^{k-18}\cdot n+1405511$ O_{3b} : $z = 3^{14}\cdot 2^{k-19}\cdot n+702755 \rightarrow f(z) = 3^{17}\cdot 2^{k-21}\cdot n+4743602 \rightarrow$
$3^{17}\cdot 2^{k-22}\cdot n+2371801$ O_1 : $x = 3^{17}\cdot 2^{k-23}\cdot n+1185900 \rightarrow f(x) = 3^{18}\cdot 2^{k-24}\cdot n+1778851$ O_2^* :
$y = 3^{18}\cdot 2^{k-25}\cdot n+889425 \rightarrow f(y) = 3^{20}\cdot 2^{k-27}\cdot n+2001208 \rightarrow 3^{20}\cdot 2^{k-28}\cdot n+1000604 \rightarrow$

$3^{20}\cdot 2^{k-29}\cdot n+500302 \rightarrow 3^{20}\cdot 2^{k-30}\cdot n+250151$ O_{3b} : $z = 3^{20}\cdot 2^{k-31}\cdot n+125075 \rightarrow$
$f(z) = 3^{23}\cdot 2^{k-33}\cdot n+844262 \rightarrow 3^{23}\cdot 2^{k-34}\cdot n+422131$ O_2^* : $y = 3^{23}\cdot 2^{k-35}\cdot n+211065 \rightarrow$
$f(y) = 3^{25}\cdot 2^{k-37}\cdot n+474898 \rightarrow 3^{25}\cdot 2^{k-38}\cdot n+237449$ O_1 : $x = 3^{25}\cdot 2^{k-39}\cdot n+118724 \rightarrow$
$f(x) = 3^{26}\cdot 2^{k-40}\cdot n+178087$ O_{3b} : $z = 3^{26}\cdot 2^{k-41}\cdot n+189043 \rightarrow f(z) = 3^{29}\cdot 2^{k-43}\cdot n+601046 \rightarrow$
$3^{29}\cdot 2^{k-44}\cdot n+300523$ O_{2a} : $y = 3^{29}\cdot 2^{k-45}\cdot n+150261 \rightarrow f(y) = 3^{31}\cdot 2^{k-47}\cdot n+338089$ O_1 :
$x = 3^{31}\cdot 2^{k-48}\cdot n+169044 \rightarrow f(x) = 3^{32}\cdot 2^{k-49}\cdot n+253567$ O_{3a} : $z = 3^{32}\cdot 2^{k-50}\cdot n+126783 \rightarrow$
$f(z) = 3^{35}\cdot 2^{k-52}\cdot n+855791$ O_{3c} : $z = 3^{35}\cdot 2^{k-53}\cdot n+855791 \rightarrow f(z) = 3^{38}\cdot 2^{k-55}\cdot n+2888297$ O_1 :
$x = 3^{38}\cdot 2^{k-56}\cdot n+1444148 \rightarrow f(x) = 3^{39}\cdot 2^{k-57}\cdot n+2166223$ O_{3c} : $z = 3^{39}\cdot 2^{k-58}\cdot n+1083111 \rightarrow$
$f(z) = 3^{42}\cdot 2^{k-60}\cdot n+7311005$ O_1 : $x = 3^{42}\cdot 2^{k-61}\cdot n+3655502 \rightarrow f(x) = 3^{43}\cdot 2^{k-62}\cdot n+5483254 \rightarrow$
$3^{43}\cdot 2^{k-63}\cdot n+2741627$ O_{2a} : $y = 3^{43}\cdot 2^{k-64}\cdot n+1370813 \rightarrow f(y) = 3^{45}\cdot 2^{k-66}\cdot n+3084331$ O_{2a} :
$y = 3^{45}\cdot 2^{k-67}\cdot n+1542165 \rightarrow f(y) = 3^{47}\cdot 2^{k-69}\cdot n+3469873$ O_1 : $x = 3^{47}\cdot 2^{k-70}\cdot n+1734936 \rightarrow$
$f(x) = 3^{48}\cdot 2^{k-71}\cdot n+2602405$ O_1 : $x = 3^{48}\cdot 2^{k-72}\cdot n+1301202 \rightarrow f(x) = 3^{49}\cdot 2^{k-73}\cdot n+1951804 \rightarrow$
$3^{49}\cdot 2^{k-74}\cdot n+975902 \rightarrow 3^{49}\cdot 2^{k-75}\cdot n+487951$ O_{3b} : $z = 3^{49}\cdot 2^{k-76}\cdot n+243975 \rightarrow$
$f(z) = 3^{52}\cdot 2^{k-78}\cdot n+1646837$ O_1 : $x = 3^{52}\cdot 2^{k-79}\cdot n+823418 \rightarrow f(x) = 3^{53}\cdot 2^{k-80}\cdot n+1235128 \rightarrow$
$3^{53}\cdot 2^{k-81}\cdot n+617564 \rightarrow 3^{53}\cdot 2^{k-82}\cdot n+308782 \rightarrow 3^{53}\cdot 2^{k-83}\cdot n+154391$ O_3^* : $z = 3^{53}\cdot 2^{k-84}\cdot n+77195 \rightarrow$
$f(z) = 3^{56}\cdot 2^{k-86}\cdot n+521072 \rightarrow 3^{56}\cdot 2^{k-87}\cdot n+260536 \rightarrow 3^{56}\cdot 2^{k-88}\cdot n+130268 \rightarrow$
$3^{56}\cdot 2^{k-89}\cdot n+65134 < 2^k\cdot n+77031$ O_{3b}
$k = 89 \rightarrow$ $\mathbf{3^{56}\cdot n+65134 < 2^{89}\cdot n+77031}$ O_{3b} : N(s) = **145**

32n+15 = 77039 O_{3c}
$\mathbf{2^k\cdot n+77039}$ O_{3c} : $z = 2^{k-1}\cdot n+38519 \rightarrow f(z) = 3^3\cdot 2^{k-3}\cdot n+260009$ O_1 : $x = 3^3\cdot 2^{k-4}\cdot n+130004 \rightarrow$
$f(x) = 3^4\cdot 2^{k-5}\cdot n+195007$ O_{3a} : $z = 3^4\cdot 2^{k-6}\cdot n+97503 \rightarrow f(z) = 3^7\cdot 2^{k-8}\cdot n+658151$ O_{3b} :
$z = 3^7\cdot 2^{k-9}\cdot n+329075 \rightarrow f(z) = 3^{10}\cdot 2^{k-11}\cdot n+2221262 \rightarrow 3^{10}\cdot 2^{k-12}\cdot n+1110631$ O_{3b} :
$z = 3^{10}\cdot 2^{k-13}\cdot n+555315 \rightarrow f(z) = 3^{13}\cdot 2^{k-15}\cdot n+3748382 \rightarrow 3^{13}\cdot 2^{k-16}\cdot n+1874191$ O_{3c} :
$z = 3^{13}\cdot 2^{k-17}\cdot n+937095 \rightarrow f(z) = 3^{16}\cdot 2^{k-19}\cdot n+6325397$ O_1 : $x = 3^{16}\cdot 2^{k-20}\cdot n+3162698 \rightarrow$
$f(x) = 3^{17}\cdot 2^{k-21}\cdot n+4744048 \rightarrow 3^{17}\cdot 2^{k-22}\cdot n+2372024 \rightarrow 3^{17}\cdot 2^{k-23}\cdot n+1186012 \rightarrow$
$3^{17}\cdot 2^{k-24}\cdot n+593006 \rightarrow 3^{17}\cdot 2^{k-25}\cdot n+296503$ O_3^* : $z = 3^{17}\cdot 2^{k-26}\cdot n+148251 \rightarrow$
$f(z) = 3^{20}\cdot 2^{k-28}\cdot n+1000700 \rightarrow 3^{20}\cdot 2^{k-29}\cdot n+500350 \rightarrow 3^{20}\cdot 2^{k-30}\cdot n+250175$ O_{3a} :
$z = 3^{20}\cdot 2^{k-31}\cdot n+125087 \rightarrow f(z) = 3^{23}\cdot 2^{k-33}\cdot n+844343$ O_3^* : $z = 3^{23}\cdot 2^{k-34}\cdot n+422171 \rightarrow$
$f(z) = 3^{26}\cdot 2^{k-36}\cdot n+2849660 \rightarrow 3^{26}\cdot 2^{k-37}\cdot n+1424830 \rightarrow 3^{26}\cdot 2^{k-38}\cdot n+712415$ O_{3a} :
$z = 3^{26}\cdot 2^{k-39}\cdot n+356207 \rightarrow f(z) = 3^{29}\cdot 2^{k-41}\cdot n+2404403$ O_2^* : $y = 3^{29}\cdot 2^{k-42}\cdot n+1202201 \rightarrow$
$f(y) = 3^{31}\cdot 2^{k-44}\cdot n+2704954 \rightarrow 3^{31}\cdot 2^{k-45}\cdot n+1352477$ O_1 : $x = 3^{31}\cdot 2^{k-46}\cdot n+676238 \rightarrow$
$f(x) = 3^{32}\cdot 2^{k-47}\cdot n+1014358 \rightarrow 3^{32}\cdot 2^{k-48}\cdot n+507179$ O_{2a} : $y = 3^{32}\cdot 2^{k-49}\cdot n+253589 \rightarrow$
$f(y) = 3^{34}\cdot 2^{k-51}\cdot n+570577$ O_1 : $x = 3^{34}\cdot 2^{k-52}\cdot n+285288 \rightarrow f(x) = 3^{35}\cdot 2^{k-53}\cdot n+427933$ O_1 :
$x = 3^{35}\cdot 2^{k-54}\cdot n+213966 \rightarrow f(x) = 3^{36}\cdot 2^{k-55}\cdot n+320950 \rightarrow 3^{36}\cdot 2^{k-56}\cdot n+160475$ O_{2a} :
$y = 3^{36}\cdot 2^{k-57}\cdot n+80237 \rightarrow f(y) = 3^{38}\cdot 2^{k-59}\cdot n+180535$ O_3^* : $z = 3^{38}\cdot 2^{k-60}\cdot n+90267 \rightarrow$
$f(z) = 3^{41}\cdot 2^{k-62}\cdot n+609308 \rightarrow 3^{41}\cdot 2^{k-63}\cdot n+304654 \rightarrow 3^{41}\cdot 2^{k-64}\cdot n+152327$ O_{3b} :
$z = 3^{41}\cdot 2^{k-65}\cdot n+76163 \rightarrow f(z) = 3^{44}\cdot 2^{k-67}\cdot n+514106 \rightarrow 3^{44}\cdot 2^{k-68}\cdot n+257053$ O_1 :
$x = 3^{44}\cdot 2^{k-69}\cdot n+128256 \rightarrow f(x) = 3^{45}\cdot 2^{k-70}\cdot n+192790 \rightarrow 3^{45}\cdot 2^{k-71}\cdot n+96395$ O_{2a} :
$y = 3^{45}\cdot 2^{k-72}\cdot n+48197 \rightarrow f(y) = 3^{47}\cdot 2^{k-74}\cdot n+108445$ O_1 : $x = 3^{47}\cdot 2^{k-75}\cdot n+54222 \rightarrow$
$f(x) = 3^{48}\cdot 2^{k-76}\cdot n+81334 \rightarrow 3^{48}\cdot 2^{k-77}\cdot n+40667 < 2^k\cdot n+77039$ O_{3c}
$k = 77 \rightarrow$ $\mathbf{3^{48}\cdot n+40667}$ < $\mathbf{2^{77}\cdot n+77039}$ O_{3c} : N(s) = **125**

3.3.3. Two examples completion General List

By Theorem of Independence and formulas derived from Theorem 2n+1, we research cycles of links N(s) ≤ 21. If N(s) > 21 we don't write the corresponding cycle.

N(s) = **16** : 729n+362 < 1024n+507 O_{2a} ; 729n+410 < 1024n+575 O_{3a}

p : 507 < p < 575 = {575mod4 - $p \in O_2* \cup p \in O_3* \cup p \sim N(s) = 8 \cup p \sim N(s) = 11 \cup p \sim N(s) = 13$} = {511; 539; 543; 559}
$p = 511 \rightarrow 2^k \cdot n+511\ O_{3a} : N(s) > 21$
$p = 539 \rightarrow 2^k \cdot n+539\ O_{2a} : y = 2^{k-1} \cdot n+269 \rightarrow f(y) = 3^2 \cdot 2^{k-3} \cdot n+607\ O_{3a} : z = 3^2 \cdot 2^{k-4} \cdot n+303 \rightarrow$
$f(z) = 3^5 \cdot 2^{k-6} \cdot n+2051\ O_2* : y = 3^5 \cdot 2^{k-7} \cdot n+1025\ O_1 \rightarrow f(y) = 3^7 \cdot 2^{k-9} \cdot n+2308 \rightarrow$
$3^7 \cdot 2^{k-10} \cdot n+1154 \rightarrow \quad 3^7 \cdot 2^{k-11} \cdot n+577\ O_1 : x = 3^7 \cdot 2^{k-12} \cdot n+288 \rightarrow f(x) = 3^8 \cdot 2^{k-13} \cdot n+433 < 2^k \cdot n+539\ O_{2a}$
$k = 13 \rightarrow$ $\mathbf{3^8 \cdot n+433 < 2^{13} \cdot n+539}$ O_{2a} : to add General list N(s) = **21**
$p = 543 \rightarrow 2^k \cdot n+543\ O_{3a} : z = 2^{k-1} \cdot n+271 \rightarrow f(z) = 3^3 \cdot 2^{k-3} \cdot n+1835\ O_{2a} : y = 3^3 \cdot 2^{k-4} \cdot n+917 \rightarrow$
$f(y) = 3^5 \cdot 2^{k-6} \cdot n+2065\ O_1 : x = 3^5 \cdot 2^{k-7} \cdot n+1032 \rightarrow f(x) = 3^6 \cdot 2^{k-8} \cdot n+1549\ O_1 :$
$x = 3^6 \cdot 2^{k-9} \cdot n+774 \rightarrow f(x) = 3^7 \cdot 2^{k-10} \cdot n+1162 \rightarrow 3^7 \cdot 2^{k-11} \cdot n+581\ O_1 : x = 3^7 \cdot 2^{k-12} \cdot n+290 \rightarrow$
$f(x) = 3^8 \cdot 2^{k-13} \cdot n+436 < 2^k \cdot n+543\ O_{3a}$
$k = 13 \rightarrow$ $\mathbf{3^8 \cdot n+436 < 2^{13} \cdot n+543}$ O_{3a} : to add General list N(s) = **21**
$p = 559 \rightarrow 2^k \cdot n+559\ O_{3c} : N(s) > 21$

N(s) = **19** : $3^7 \cdot n+910 < 2^{12} \cdot n+1703\ O_{3b}$; $3^7 \cdot n+955 < 2^{12} \cdot n+1787\ O_{2a}$

p : 1703 < p < 1787 = {1787mod4 - $p \in O_2* \cup p \in O_3* \cup p \sim N(s) = 8 \cup p \sim N(s) = 11 \cup p \sim N(s) = 13 \cup p \sim N(s) = 16$} = {1727; 1743; 1775}
$p = 1727$: $\mathbf{2^k \cdot n+1727}$ $O_{3a} : z = 2^{k-1} \cdot n+863 \rightarrow f(z) = 3^3 \cdot 2^{k-3} \cdot n+5831\ O_{3b} : z = 3^3 \cdot 2^{k-4} \cdot n+2915 \rightarrow$
$f(z) = 3^6 \cdot 2^{k-6} \cdot n+19682 \rightarrow 3^6 \cdot 2^{k-7} \cdot n+9841\ O_1 : x = 3^6 \cdot 2^{k-8} \cdot n+4920 \rightarrow f(x) = 3^7 \cdot 2^{k-9} \cdot n+7381\ O_1 :$
$x = 3^7 \cdot 2^{k-10} \cdot n+3690 \rightarrow f(x) = 3^8 \cdot 2^{k-11} \cdot n+5536 \rightarrow 3^8 \cdot 2^{k-12} \cdot n+2768 \rightarrow$
$3^8 \cdot 2^{k-13} \cdot n+1384 < 2^k \cdot n+1727\ O_{3a}$
$k = 13 \rightarrow$ $\mathbf{3^8 \cdot n+1384 < 2^{13} \cdot n+1727}$ O_{3a} : to add General list N(s) = **21**
$p = 1743 \rightarrow 2^k \cdot n+1743\ O_{3c} : N(s) > 21$
$p = 1775 \rightarrow 2^k \cdot n+1775\ O_{3c} : N(s) > 21$

3.4. Big Bang and Big Crunch

"*The numbers reign on the Universe*" (Pythagoras)
"*ONE and DIADE are generators of Cosmos*" (Plato)

Maybe the cycles of links of SCQ represents a promising kind of law on expansion of Cosmos called BIG BANG. Starting from 1 we reach very high and very vast horizons. As we proved : BIG CRUNCH is always possible, but BIG BANG has no End.

BIG BANG :

$1 \rightarrow 2 \rightarrow 4 \rightarrow 16 \rightarrow 32 \rightarrow 128 \rightarrow 256 \rightarrow 1024 \rightarrow 4096 \rightarrow 8192 \rightarrow 2^{15} \rightarrow 2^{16} \rightarrow 2^{18} \rightarrow \ldots \rightarrow$
$2^{21} \rightarrow 2^{23} \rightarrow 2^{24} \rightarrow 2^{26} \rightarrow 2^{27} \rightarrow 2^{29} \rightarrow 2^{31} \rightarrow \ldots \rightarrow 2^{35} \rightarrow \ldots \rightarrow 2^{39} \rightarrow 2^{40} \rightarrow 2^{42} \rightarrow \ldots \rightarrow 2^{45} \rightarrow$
$\ldots \rightarrow 2^{51} \rightarrow \ldots \rightarrow 2^{54} \rightarrow 2^{56} \rightarrow \ldots \rightarrow 2^{59} \rightarrow \ldots \rightarrow 2^{77} \rightarrow \ldots \rightarrow 2^{89} \rightarrow \ldots \rightarrow 2^k \rightarrow \ldots \rightarrow \infty$

Where … are occupied by 2^k of the cycles of links to us still unknown. Increment of k = 1 or 2.

BIG CRUNCH :

$3^h \rightarrow 3^{h-1} \rightarrow 3^{h-2} \rightarrow 3^{h-3} \rightarrow \ldots \rightarrow 3^{37} \rightarrow 3^{36} \rightarrow 3^{35} \rightarrow 3^{34} \rightarrow 3^{33} \rightarrow \ldots \rightarrow 3^{20} \rightarrow 3^{19} \rightarrow 3^{18} \rightarrow$
$\ldots \rightarrow 3^{12} \rightarrow 3^{11} \rightarrow 3^{10} \rightarrow 3^9 \rightarrow 6561 \rightarrow 2187 \rightarrow 729 \rightarrow 243 \rightarrow 81 \rightarrow 27 \rightarrow 9 \rightarrow 3 \rightarrow 1$

Decrease of h always 1 .

3.a. Appendix : The Independence Theorem and Global Quadrature

The Independence Theorem (§ 3.2.) allows us to manage links cycles arbitrarily, theoretically reaching very high main horizons ϴ(m) and corresponding lower horizons ϴ(l) such that :

$$\mathbf{3^{\beta}\cdot(n + k) + q < 2^{\alpha}\cdot(n + k) + p \ \mid\ \beta < \alpha \ ;\ q < p}$$

Given sufficient computational tools, *Independence Theorem* allows us to create cycles with a number of steps N(s) that tends to infinity, through the systematic repetition of rebounds, or by organizing chains of connections to avoid encounter a *transition attractor* or sequences of even numbers (§ 3.2.3). This tendency towards infinity of link cycles exists as *combinatorial freedom* among the eight subsets (E + the seven odd numbers subsets), and the open infinite structure of the links library (§ 2.4.). SC (CC) can have trajectories that extend indefinitely, but these extensions are not escapes, but search for saturation. Since the links density δ(l) grows faster than the density *D*[N(s)] of the main horizon ϴ(m), the intersection with a collapse link towards the lower horizon ϴ(l) is an asymptotically certain event. Moreover, it is ensured by the framework of the seven subsets and the odd numbers *2n+1* classes membership ([4]; [16]; [32]). This is precisely the *revolution* of natural numbers in SCQ ; that is, despite the existence of infinite trajectories the validity of the *global quadrature* is proven. Collatz Conjecture is unsolvable in a closed system, but it's solved in SCQ, showing that *the return to 1* is the result of an asymptotic saturation of the phases space of links.

4. Final Proof

We have seen that SC (CC) is a new sort of *Quadrature*, and this paper proves that it's not possible to arrive 100% coverage of ℕ with the binomial inequalities, but we can claim that its initial statement is true; in fact, despite infinite paths to reach very high horizons ϴ(m) (BIG BANG), the framework of the seven subsets (Theorem 2n+1) and Independence Theorem, guarantee logical return to lower horizons ϴ(l) (BIG CRUNCH) (§ 3.a.). Hence :

For any ϴ(m) = $2^k\cdot n+p$; there is ϴ(l) = $3^h\cdot n+q$: $\begin{cases} h < k \\ q < p \end{cases}$: ϴ(l) < ϴ(m)

Since there exists $a_n \in S(2n+1) : a_n < 2n+1$ after N(s) = h + k steps.
Since there exists $a_n \in S(2n) : a_n < 2n$ after N(s) = one step : 2n → n . Being $E \cup O = \mathbb{N}$.
So every $n \in \mathbb{N}$ falls down to 1 → QED

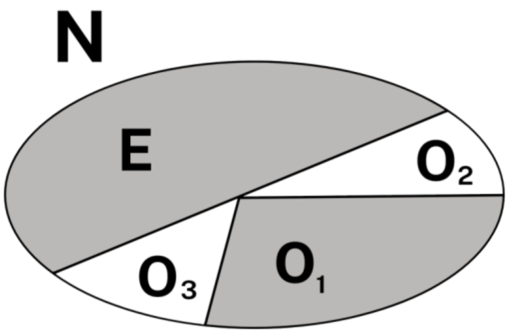
4.1. Conclusion / Finale

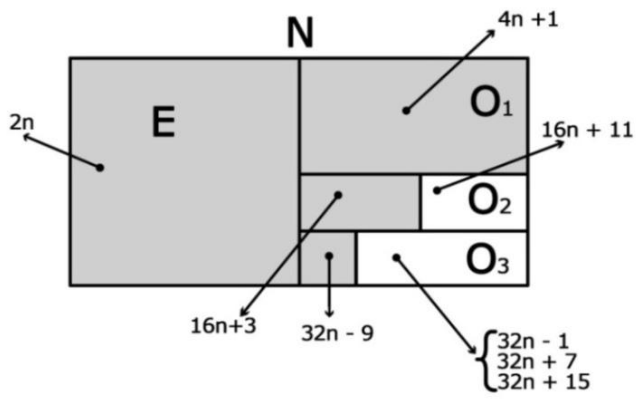
Odd generating numbers in O_1 become lesser than themselves after one application of Theorem 2n+1. So also it is for odd generating numbers in O_2* of type 16n+3. So also it is for odd generating numbers in O_3* of type 32n-9 ∼ 32n+23. For the others odd generating numbers in O_2 – O_2* of type 16n+11 O_{2a} and in O_3 - O_3* of type 32n-1 ∼ 32n+31 O_{3a} ; 32n+7 O_{3b} ; 32n+15 O_{3c} ; Theorem 2n+1 must be iterated two, three, or more times, or very many times, until the odd generating number becomes lesser than itself. In this way the blanks in the following PATTERNS they fill more and more with grey almost to complete the set ℕ. We can conclude that the starting odd numbers $O \subset \mathbb{N}$ become lesser than themselves after a number of steps (i.e. applications Collatz algorithm) from N(s) = 3 (O_1) until N(s) → ∞ .

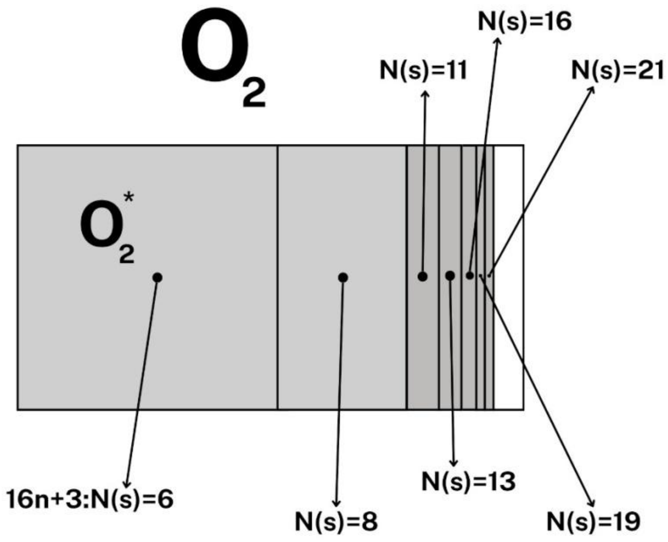
4.a. Appendix : Infinity within a Global Logic

In SCQ, infinity is not a place where numbers get lost, but a *saturation limit*. In the *Asymptotic Infinity* regardless of how large a number is or how complex its trajectory becomes, the network of links (§ 3.a.) expands faster than the trajectory itself. There is no final "boundary" (due to the Independence Theorem); however, the links density becomes so high that the probability of *not* encountering a downward connection tends to zero. In summary : a "closed" proof of the Collatz Conjecture is asymptotic because it would require mapping infinite links. Nevertheless, SCQ is fundamentally valid, because the modular structure of the seven subsets guarantees that a trajectory devoid of link to lower horizon cannot exist. In SCQ, *infinity is possible, but return to 1 is inevitable.* In SCQ, Collatz Conjecture is solved not through a lot of formulas, but by a *structural certainty*.

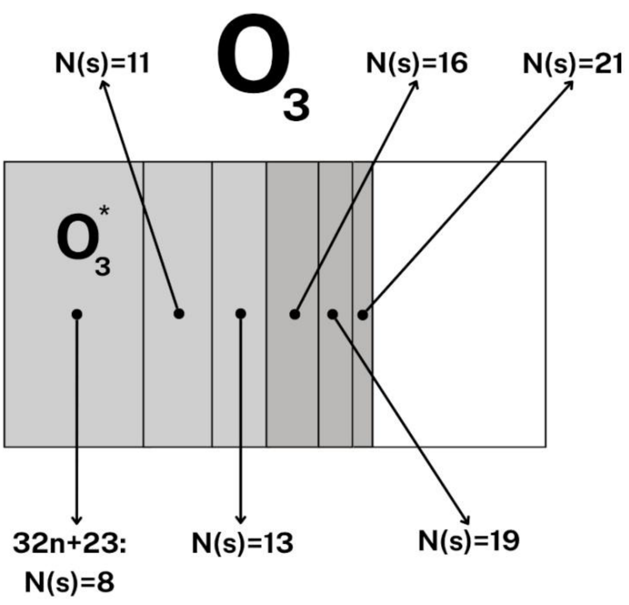
PATTERNS







O2
N(s)=16
N(s)=11
N(s)=21
O2*
16n+3:N(s)=6
N(s)=8
N(s)=13
N(s)=19



O3
N(s)=11
N(s)=16
N(s)=21
O3*
32n+23:
N(s)=8
N(s)=13
N(s)=19

References / Essential Bibliography

[1] Zucchini, R. (2023). *The logical solution Syracuse conjecture*. Amazon on demand by aid Publisher Mnamon Milan Italy

Lagarias, J.C. (1985). *The 3x+1 problem and its generalizations.* The American Mathematical Monthly, 92(1), 3-23.

Wirsching, G. J. (1998). *The Dynamical System Generated by 3n+1 Function*. Springer-Verlag

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