Doctoral Dissertation Manuscript

Four-Index and Four-Sector Extensions of General Relativity

Curvature Completion, Reciprocal Geometry,
Tensor Quantum Dynamics and an $SU(m,n)$ Embedding

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Dissertation status and submission note

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Abstract

This dissertation develops a connected programme for extending the tensor organization and observer geometry of general relativity. Its first component is a four-index completion of the Einstein field equation. The Ricci-built contribution and the trace-free Weyl contribution are retained at the same tensor rank, while contraction recovers the ordinary two-index Einstein equation. A rank-four source is decomposed into a lift of the stress–energy tensor and an independent trace-free tensor with Weyl algebraic symmetries. In the classical formulation this independence is organizational rather than an enlargement of the metric solution space: the contracted Einstein equation fixes the Ricci sector and the remaining uncontracted equation fixes the trace-free source consistently with the Weyl tensor.

The second component is a four-sector completion of the orientation of a Lorentzian clock–ruler plane. The four connected non-null domains share the same null boundary and are labelled by the phase cycle $1, \mathrm{i}, -1, -\mathrm{i}$. Geometry-sector and particle-sector transformations are kept distinct. Ordinary Lorentz motion occurs inside a sector; the construction does not describe material particles accelerated through the light cone. In curved spacetime a tensor of proper scales and a solution-dependent critical tensor select an active eigendirection. Neighbouring charts use reciprocal normalization of that active scale. Schwarzschild, Kerr, Lemaître–Tolman–Bondi and spatially flat FLRW examples test curvature and geodesic distance in these charts. In the regular branches analysed, reciprocal centres are curvature-decaying asymptotic boundaries rather than finite-parameter passages.

These two components are then used to construct a tensorial quantum framework. Complete conserved rank-four configurations are weighted by complex coefficients only after their classical source content has been fixed. A rank-eight operator generates linear clock evolution, and amplitudes and probabilities are obtained by global tensor contractions. A reciprocal Kerr–Newman chart is used in a hydrogen model whose common Maxwell field retains the electron–proton cross-term before the four-index lift. The central channel produces a relativistic Coulomb spectrum and compatible Laguerre radial modes. The Ricci-flat specialization identifies a free graviton with a Weyl mode, yielding null propagation, two transverse helicities, explicit plane-wave and spherical resolutions, reciprocal transport and closed four-sector Weyl histories.

The perturbative ultraviolet problem is addressed after standard background-field quantization, gauge fixing, ghost integration, dimensional regularization and recursive subtraction. Metric variation of the remaining local pole functional gives a conserved rank-two response. Internal sector maps cancel on contracted graviton indices, while the two free covariant indices retain the character $(-1)^k$. Within the proposed complete-orbit projector, the invariant local ultraviolet source therefore vanishes order by order. This statement applies to the local pole response; tree-level Einstein dynamics and finite local and nonlocal loop terms are retained.

Finally, the programme is embedded in a pseudo-Kähler spacetime with $SU(m,n)$ symmetry. A real Lorentzian observer is represented by an admissible antiholomorphic involution. Realification and the Gauss equation show that projection of complex Bochner curvature to observer Weyl curvature requires explicit extrinsic and Ricci-compatibility conditions. The resulting synthesis connects the four-index field equation, four-sector observer atlas, null

perspective, tensor quantum carrier, worked gravitational and atomic sectors, and ultraviolet projector while making their logical status and outstanding consistency tests explicit.

Central thesis and research questions

Central thesis. Einsteinian curvature can be represented by a divergence-free rank-four field equation that preserves the ordinary two-index classical solutions while retaining the trace-free curvature sector explicitly. When this completion is combined with a four-sector orientation atlas, the same tensor hierarchy can organize reciprocal regular charts, null and helicity-two radiative states, a tensorial spectral construction and a four-sector projection of local ultraviolet pole sources. The entire hierarchy admits a proposed $SU(m, n)$-symmetric pseudo-Kähler embedding whose observable sector is a real Lorentzian projection subject to stated compatibility conditions.

The dissertation addresses five questions. First, can the full Riemann decomposition be paired with a rank-four source so that contraction recovers Einstein's equation and the uncontracted equation remains divergence free? Second, can one invariant light cone support four observer-orientation sectors without introducing physical faster-than-light particle motion? Third, can reciprocal active-scale charts remove curvature-divergent extrapolations in explicit general-relativistic geometries while retaining the original field equations? Fourth, can the complete rank-four carrier support a linear tensor quantum construction reproducing worked graviton and hydrogen sectors? Fifth, can a complete four-sector projector annihilate the local ultraviolet pole response after conventional perturbative renormalization steps have been performed?

Logical status of the principal claims

Algebraic reformulation.

The real four-index equation and its contraction are presented as a completion of tensor bookkeeping, not as an additional classical field equation. Its equivalence claim can be checked directly by contraction and curvature decomposition.

Geometric model prescription.

The four-sector and reciprocal-scale construction adds an atlas and a rule for selecting active charts. Its regularity claims are conditional on the selected chart, metric domain, curvature invariants and geodesic analysis.

Physical hypotheses.

The null perspective, coexistence postulate and interpretation of complete tensor states are hypotheses. They require operational predictions that distinguish them from standard formulations.

Worked results within the framework.

The hydrogen spectrum, graviton modes, reciprocal black-hole/cosmological geometries and ultraviolet projection are deductions within the stated assumptions. They are not independent experimental confirmations of those assumptions.

Unification proposal.

The $SU(m,n)$ construction is a candidate common geometry. The observer projection is valid only when the real-form, extrinsic-curvature and Ricci-compatibility conditions stated in the relevant chapter hold.

Notation and conventions

Greek indices denote spacetime components unless a chapter explicitly introduces complex holomorphic and antiholomorphic indices. The number of real spacetime dimensions is denoted by n in the four-index derivations and is set to four in the principal physical applications. The gravitational coupling is $\kappa = 8\pi G/c^4$ unless a perturbative chapter uses $\kappa_g^2 = 32\pi G$. The Weyl tensor is $C_{\alpha\mu\beta\nu}$, the trace-free source is $\hat{T}_{\alpha\mu\beta\nu}$, and the source sign follows the convention explicitly stated in each incorporated manuscript. Developmental appendices preserve their original conventions; they are not silently rewritten into a false uniformity.

The sector label belongs to $\mathbb{Z}_4$. Geometry and particle labels are distinguished by n_g and s_X, with total readout $N_X = n_g + s_X \pmod 4$. The phase i^n records orientation in the extended coframe; it is not an assertion that measured coordinates or physical proper time are imaginary. Repeated indices are summed, the metric signature follows the local source chapter, and all changes of convention are declared where they occur.

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Chapter 1

Research Programme and Dissertation Architecture

1.1 Motivation

The Einstein equation relates a rank-two curvature contraction to a rank-two material source. This is sufficient to determine the metric dynamics of general relativity, but it hides the trace-free part of the Riemann tensor when the equation is read only after contraction. The first aim of this research programme is to retain the Ricci and Weyl sectors together in one rank-four equation while preserving the established rank-two theory under contraction.

The second aim begins from the invariant null set. A Lorentzian time–space plane has four connected non-null orientation domains separated by the same two null directions. The proposed four-sector atlas completes these orientations and treats the active clock direction as chart data. The extension is expressly separated from superluminal material motion: a massive particle remains timelike inside its own sector, while the sector label records which non-null orientation supplies the local clock.

The third aim is constructive. The rank-four source is used as the carrier of a tensor quantum model; reciprocal charts are tested on black-hole, inhomogeneous and cosmological metrics; the Ricci-flat trace-free sector supplies a graviton model; and the sector orbit is applied to the local ultraviolet response obtained after conventional perturbative subtraction. A pseudo-Kähler $SU(m,n)$ geometry is then proposed as the common complex arena from which a real Lorentzian observer projection is obtained.

1.2 Method

The work is analytical. It uses curvature decomposition, differential identities, tensor contraction, generalized eigenvalue problems for proper scales, exact coordinate and coframe transformations, curvature invariants, geodesic and affine-parameter tests, stationary spectral constructions, null-tetrad decompositions, and background-field effective-action methods. Worked examples are used as internal consistency tests. They do not replace external empirical validation.

This dissertation is organized as a thesis by integrated manuscripts. Every scientific section and displayed formula from the supplied LaTeX sources is retained in a corresponding chapter or archival appendix. Bridge chapters add a common thesis statement, a hierarchy of claims, a reconstructed regular four-index FRW example, an integrated discussion and an explicit record of AI assistance. Repetition is reduced in the synthesis but retained inside source-derived chapters when needed for a self-contained derivation.

1.3 Order of argument

Chapter 2 establishes the rank-four formulation. Chapter 3 develops the four-sector Lorentz atlas and reciprocal normalization. Chapter 4 isolates the null-state interpretation. Chapters 5 and 6 construct the tensor quantum and free-graviton sectors. Chapter 7 applies the complete sector orbit to local ultraviolet pole responses. Chapter 8 gives the pseudo-Kähler $SU(m,n)$ embedding. Chapter 9 supplies the additional regular FRW reduction. Chapter 10 evaluates the combined claims and states the unresolved tests. The appendices preserve early formulations and thereby document the development of notation and interpretation.

Part I

Four-Index and Four-Sector Foundations

Chapter 2

Four-Index Completion of the Einstein Field Equation

Source and status. This chapter incorporates the complete scientific body of *Four Index Formulation of Einstein Field Equations in Real and Complex Spacetime*, v9 (2026). Its role in the dissertation is: Core rank-four construction in real and complex spacetime. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

In this work I present a four index formulation of Einstein field equations [24]. The purpose of the construction is to keep the Ricci-built and trace-free curvature sectors explicit at the same tensor rank and to represent the complete curvature information without changing the classical content of Einstein gravity. The source side contains the ordinary stress–energy contribution lifted to rank four together with an independent trace-free rank-four field having the algebraic symmetries and algebraic degrees of freedom of the Weyl tensor. This field is treated as an independent variable when the rank-four equations are solved. The paper shows that these tensorial degrees of freedom do not introduce new field equations, new classical information, or new metric solutions relative to the ordinary two index Einstein theory. The two index equations determine the Ricci-built rank-four sector, while the Bianchi identities, local conservation, and the complete four index equation fix the trace-free sector consistently with the Weyl curvature. The same tensorial construction is also developed for complex Kähler spacetime [25] [26].

2.1 Einstein field equations

2.1.1 Einstein tensor

Einstein tensor is basis of General Relativity [24], from it arises left side of field equations that connects matter field represented by stress momentum tensor [24] with geometry part. That geometry part is just Einstein tensor that comes from Bianachi identities [24]. It can be written as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.1}$$

It's most important property is that is divergence free [24] so that:

$$\nabla^{\mu}G_{\mu\nu} = 0 \tag{2.2}$$

It connects to left side of equations where stress momentum tensor [24] is present.

2.1.2 Stress momentum tensor

Matter sources in General Relativity [24] are all coming from right side of equation that is stress momentum tensor [24] $T_{\mu\nu}$. Units of stress momentum tensor are energy density so from it follows that on left side of equation I have units of curvature that is length to power minus two, thats why if field equations [24] I need to use Einstein constant that is equal to:

$$\kappa = \frac{8\pi G}{c^4} \tag{2.3}$$

Another crucial property of stress momentum tensor is that as on left side I have divergence free tensor , stress momentum tensor is divergence free meaning that matter fields are conserved locally. It means that divergence of stress momentum tensor is zero [24]:

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{2.4}$$

2.1.3 Field equation

That finally leads to field equation that I will call in this work a two index field equation, that is just ordinary Einstein field equation [24]:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.5}$$

The idea behind this work is to write the same gravitational content at rank four so that the part constructed from the Ricci tensor and the trace-free Weyl part remain explicit in one equation. On the source side, the tensor constructed from $T_{\mu\nu}$ is supplemented by an independent trace-free field $\hat{T}_{\alpha\mu\beta\nu}$. This field is not defined as the Weyl tensor divided by the Einstein constant. It is introduced with the algebraic symmetries of the Weyl tensor and is treated as an independent rank-four unknown when the equations are solved. The construction will show that the ordinary two index Einstein equation determines the Ricci-built rank-four sector and that the remaining trace-free part of the complete equation then fixes $\hat{T}_{\alpha\mu\beta\nu}$. Therefore the four index form contains the same classical information and the same metric solutions as the two index Einstein equation, while preserving the trace-free curvature information explicitly.

2.1.4 Cosmological constant

Two index equations used today have additional term [24], that term is cosmological constant or energy of empty space Λ it modifies field equations by adding term with it to field equations [24]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.6}$$

In vacuum it will lead to:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{2.7}$$

That solutions will be either de Sitter or anti-de Sitter [24] spacetime depending on sign of cosmological constant.

2.2 Four index Einstein tensor

2.2.1 Ricci part

Riemann tensor [24] can be decomposed into Ricci and Weyl parts:

$$
\begin{aligned}
R_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} &+ \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right) \\
&- \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R \qquad (2.8)
\end{aligned}
$$

This decomposition gives after contraction exactly Ricci tensor as part with Weyl vanishes when contracted as it's trace free for all indexes combinations. So isolating Ricci part but adding part that will after contraction reduce exactly to one half of Ricci scalar times metric tensor I will arrive at:

$$
\begin{aligned}
&\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right) \\
&- \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R - \frac{1}{2(n-1)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R \qquad (2.9)
\end{aligned}
$$

I can sum terms with Ricci scalar and will arrive at final expression for first component of four index Einstein tensor:

$$
\begin{aligned}
\tilde{R}_{\alpha\mu\beta\nu} = &\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right) \\
&- \frac{n}{2(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R \qquad (2.10)
\end{aligned}
$$

2.2.2 Divergence of Ricci part

From it follows that I can write divergence of this component with respect to first index where I will use Bianachi identities for same indexes of Ricci tensor and divergence index so index α to make equations simpler:

$$
\begin{aligned}
\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = &\frac{1}{n-2}\left(\nabla^{\alpha}R_{\alpha\beta}g_{\mu\nu} - \nabla^{\alpha}R_{\alpha\nu}g_{\beta\mu} + \nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right) \\
&- \frac{n}{2(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = &\frac{1}{n-2}\left(\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right) \\
- \frac{n}{2(n-1)(n-2)}&\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R + \frac{1}{2(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R \qquad (2.12)
\end{aligned}
$$

$$
\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right) - \frac{1}{2(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R \qquad (2.13)
$$

Taking out common denominator I will arrive at final expression:

$$\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left((\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}) - \frac{1}{2\,(n-1)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R\right) \tag{2.14}$$

2.2.3 Weyl part

Weyl [24] part divergence is equal to:

$$\nabla^{\alpha}C_{\alpha\mu\beta\nu} = \frac{n-3}{n-2}\left((\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}) - \frac{1}{2\,(n-1)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R\right) \tag{2.15}$$

This means that they differ only by a constant $n-3$, from it follows to create a field equation I need first to take that constant and divide Weyl tensor by it. Then whole tensor has to be divergence free, from it follows that I will take Ricci part and subtract Weyl dived by that constant:

$$\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} \tag{2.16}$$

2.2.4 Four index Einstein tensor

Last expression is just four index Einstein tensor, it all properties of being divergence free and it's contractions lead to two index equations. This tensor can be written:

$$G_{\alpha\mu\beta\nu} = \tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} \tag{2.17}$$

2.3 Four index energy tensor

2.3.1 Matter part

I can use same form as in Riemann decomposition into matter part of total energy tensor. This tensor will reduce to stress momentum tensor so I can write it:

$$\begin{aligned}\tilde{T}_{\alpha\mu\beta\nu} = &\frac{1}{n-2}\left(T_{\alpha\beta}g_{\mu\nu} - T_{\alpha\nu}g_{\beta\mu} + T_{\mu\nu}g_{\alpha\beta} - T_{\beta\mu}g_{\alpha\nu}\right)\\ &-\frac{1}{(n-1)\,(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T\end{aligned} \tag{2.18}$$

Now key property is to calculate divergence of this tensor with respect to first index as before, I can use property that stress momentum tensor is divergence free:

$$\nabla^{\alpha}\tilde{T}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}\right) - \frac{1}{(n-1)\,(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T \tag{2.19}$$

Taking out common denominator:

$$\nabla^{\alpha}\tilde{T}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left((\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}) - \frac{1}{(n-1)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T\right) \tag{2.20}$$

That is exactly propagation equation [24] from two index equations, where it does differ by same constant.

2.3.2 Trace-free energy part

The second part of the four index source is an independent tensor field $\hat{T}_{\alpha\mu\beta\nu}$. It is introduced independently of $C_{\alpha\mu\beta\nu}$ and has the same algebraic symmetries as the Weyl tensor:

$$\hat{T}_{\alpha\mu\beta\nu} = -\hat{T}_{\mu\alpha\beta\nu}, \qquad \hat{T}_{\alpha\mu\beta\nu} = -\hat{T}_{\alpha\mu\nu\beta}, \tag{2.21}$$

$$\hat{T}_{\alpha\mu\beta\nu} = \hat{T}_{\beta\nu\alpha\mu}, \qquad \hat{T}_{\alpha[\mu\beta\nu]} = 0, \tag{2.22}$$

$$g^{\alpha\beta}\hat{T}_{\alpha\mu\beta\nu} = 0. \tag{2.23}$$

Consequently it has the same independent algebraic components as a Weyl tensor. In n dimensions their number is

$$N_{\hat{T}} = \frac{n(n+1)(n+2)(n-3)}{12}, \tag{2.24}$$

which gives ten independent algebraic components in four dimensional spacetime. These are the tensorial degrees of freedom of $\hat{T}_{\alpha\mu\beta\nu}$ used when solving the rank-four field equation. They are not additional physical degrees of freedom beyond the two index Einstein theory, because the complete equations determine them from the same Einstein solution, as shown in Sec. 2.4.7.

Its divergence is fixed by the matter sector with the same dimensional factor that relates the Ricci-built tensor to the Weyl divergence:

$$\nabla^{\alpha}\hat{T}_{\alpha\mu\beta\nu} = \frac{n-3}{n-2}\left((\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}) - \frac{1}{n-1}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T \right). \tag{2.25}$$

This equation is an equation for the independent field $\hat{T}_{\alpha\mu\beta\nu}$; it is not a definition in terms of curvature. Together with the Weyl symmetries it specifies how the trace-free source sector enters the four index system. The complete field equation then removes any additional divergence-free contribution and gives the unique trace-free sector compatible with the two index Einstein equations.

2.3.3 Four index energy tensor

Putting this all in one equation and one tensor I will use same logic from before so I arrive at total energy tensor:

$$T_{\alpha\mu\beta\nu} = \tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu} \tag{2.26}$$

This tensor is divergence free together with the geometric side. It contains the lifted stress–energy part and a trace-free part that is paired with the Weyl sector of curvature.

2.4 Four index field equations

2.4.1 Equation

Combining the geometric and source tensors gives the four index field equation:

$$G_{\alpha\mu\beta\nu} = \kappa T_{\alpha\mu\beta\nu}. \tag{2.27}$$

The equation is divergence free:

$$\nabla^{\alpha} G_{\alpha\mu\beta\nu} = \kappa \nabla^{\alpha} T_{\alpha\mu\beta\nu} = 0. \tag{2.28}$$

Its contraction gives the ordinary two index Einstein equation [24]:

$$g^{\alpha\beta} G_{\alpha\mu\beta\nu} = \kappa g^{\alpha\beta} T_{\alpha\mu\beta\nu}, \tag{2.29}$$

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.30}$$

The rank-four equation contains two complementary algebraic sectors. The Ricci-built part gives

$$\tilde{R}_{\alpha\mu\beta\nu} = \kappa \tilde{T}_{\alpha\mu\beta\nu}, \tag{2.31}$$

and the trace-free part gives

$$C_{\alpha\mu\beta\nu} = \kappa \hat{T}_{\alpha\mu\beta\nu}. \tag{2.32}$$

The second equation is not the definition of $\hat{T}_{\alpha\mu\beta\nu}$. The field is introduced independently in Sec. 2.3.2 and is one of the rank-four unknowns. The equality is obtained when the complete field equation is solved. Sec. 2.4.7 shows explicitly that the two index Einstein equation fixes the entire Ricci-built rank-four sector and therefore leaves no additional classical information for the independent trace-free source field beyond the Weyl sector of the same Einstein solution.

2.4.2 Adding cosmological constant

I can add back to equations part with cosmological constant [24] it will be term with metric tensors and scalar, I can write that term:

$$\frac{1}{n-1} \left(g_{\mu\nu} g_{\alpha\beta} - g_{\beta\mu} g_{\alpha\nu} \right) \Lambda \tag{2.33}$$

So final field equations are:

$$\boxed{G_{\alpha\mu\beta\nu} + \frac{1}{n-1} \left(g_{\mu\nu} g_{\alpha\beta} - g_{\beta\mu} g_{\alpha\nu} \right) \Lambda = \kappa T_{\alpha\mu\beta\nu}} \tag{2.34}$$

If cosmological constant [24] is just a fixed number it's divergence vanishes but if not, it has to fulfill this equations:

$$\left(g_{\mu\nu} \partial_{\beta} - g_{\beta\mu} \partial_{\nu} \right) \Lambda = 0 \tag{2.35}$$

And from two index equations it needs to vanish when contracted so it leads to:

$$\partial_{\nu} \Lambda = 0 \tag{2.36}$$

It shows that it has to be a constant number that does not change with any spatial or a temporal direction. Otherwise equations will be not divergence free in both four index and two index [24]. After contraction it will just lead to standard form two index equation [24]:

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu} \tag{2.37}$$

2.4.3 Action formulation of equations

I can formulate field equations in as action [28] , derivation was already done in this paper [28], where there is same structure of field equations used. After adding tensors I will arrive at starting with gravity field:

$$S_G = \frac{1}{2\kappa c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(G_{\alpha\mu\beta\nu} + \lambda_{\alpha\mu\beta\nu}\right) d^D\mathbf{x} \tag{2.38}$$

Where those tensors are defined as before so:

$$S_G = \frac{1}{2\kappa c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) d^D\mathbf{x} \tag{2.39}$$

Now matter field part behaves same way but without term with Einstein constant [28]:

$$S_M = -\frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} T_{\alpha\mu\beta\nu} d^D\mathbf{x} \tag{2.40}$$

That can be expanded:

$$S_M = -\frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right) d^D\mathbf{x} \tag{2.41}$$

Adding them both into one integral [28]:

$$S = \frac{1}{2\kappa c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(G_{\alpha\mu\beta\nu} + \lambda_{\alpha\mu\beta\nu}\right) d^D\mathbf{x} - \frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} T_{\alpha\mu\beta\nu} d^D\mathbf{x} \tag{2.42}$$

$$S = \frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left[\frac{1}{2\kappa}\left(G_{\alpha\mu\beta\nu} + \lambda_{\alpha\mu\beta\nu}\right) - T_{\alpha\mu\beta\nu}\right] d^D\mathbf{x} \tag{2.43}$$

That can be written in whole expanded form as:

$$\begin{aligned} S = \frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \Bigg[\frac{1}{2\kappa}\left(\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) \\ - \left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right)\Bigg] d^D\mathbf{x} \end{aligned} \tag{2.44}$$

Since both $C_{\alpha\mu\beta\nu}$ and $\hat{T}_{\alpha\mu\beta\nu}$ are trace free, their double contractions with the metric vanish. The trace-free source field nevertheless remains an independent rank-four field in the four index formulation. The field equations obtained from the complete construction determine this sector without introducing additional classical dynamics relative to the two index Einstein equations.

2.4.4 Meaning of four index field equation

The four index equation is a different tensorial organization of the Einstein field equation, not an additional gravitational equation placed beside it. Its geometric side retains both parts of the curvature decomposition at rank four. The tensor $\tilde{R}_{\alpha\mu\beta\nu}$ contains the Ricci information that survives contraction, while $C_{\alpha\mu\beta\nu}$ contains the trace-free part that disappears under contraction. The source side is organized in the same way. The tensor $\tilde{T}_{\alpha\mu\beta\nu}$ is constructed completely from the ordinary stress–energy tensor, while $\hat{T}_{\alpha\mu\beta\nu}$ is an independent trace-free field with Weyl algebraic symmetries.

The word independent refers to the role of $\hat{T}_{\alpha\mu\beta\nu}$ in solving the four index system. Its components are not replaced by curvature before the equations are solved. They are treated as independent rank-four field components subject to the Weyl symmetries, trace-free condition, divergence equation, and the complete field equation. In four spacetime dimensions this gives the same ten independent algebraic components as the Weyl tensor. These components are therefore genuine unknown components of the rank-four formulation during the solution procedure.

This independence does not create a second gravitational dynamics. Contracting the four index equation removes both trace-free tensors and gives exactly the ordinary two index Einstein equation. The trace-reversed two index equation then fixes the Ricci tensor and Ricci scalar, so it fixes the complete tensor $\tilde{R}_{\alpha\mu\beta\nu}$. Since $\tilde{T}_{\alpha\mu\beta\nu}$ is already fixed by $T_{\mu\nu}$, the Ricci-built part of the four index equation contains exactly the same information as the two index equation. After this part is satisfied, the remaining part of the same rank-four equation is the trace-free relation between $C_{\alpha\mu\beta\nu}$ and $\hat{T}_{\alpha\mu\beta\nu}$.

Therefore $\hat{T}_{\alpha\mu\beta\nu}$ is an independent field variable but it does not introduce new field equations or new classical information relative to the two index Einstein solution. Its algebraic degrees of freedom are required to write and solve the uncontracted rank-four equation, while the equations force those degrees of freedom to reproduce the Weyl sector of the same spacetime. The four index formulation consequently keeps information that is hidden by contraction without enlarging the physical solution content of General Relativity.

The complete equation including the cosmological constant is

$$G_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda = \kappa T_{\alpha\mu\beta\nu}, \tag{2.45}$$

$$\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right). \tag{2.46}$$

The relative sign and dimensional factors follow from the divergence-free construction of the two rank-four tensors. The explicit tensor proof that the two index equation determines the rank-four sectors is given in Sec. 2.4.7.

2.4.5 Trace-reversed field equation

I can arrive at trace reversed field equations, first I take field equation:

$$\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right) \tag{2.47}$$

Expand left side and move cosmological constant part to right side of equations:

$$\begin{aligned}&\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right)\\ -&\frac{n}{2\left(n-1\right)\left(n-2\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R - \frac{1}{n-3}C_{\alpha\mu\beta\nu}\end{aligned} \tag{2.48}$$

Compare it with Riemann tensor expression:

$$R_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right) - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R \tag{2.49}$$

I can express now it in terms of Riemann tensor:

$$G_{\alpha\mu\beta\nu} = R_{\alpha\mu\beta\nu} - \frac{1}{2(n-1)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R - \left(1 + \frac{1}{n-3}\right)C_{\alpha\mu\beta\nu} \tag{2.50}$$

Use the two index equation [24] together with the trace-free field equation obtained from the complete four index equation, and move all terms other than the Riemann tensor to the right side:

$$R_{\alpha\mu\beta\nu} = \kappa\left(T_{\alpha\mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T + \left(1 + \frac{1}{n-3}\right)\hat{T}_{\alpha\mu\beta\nu}\right) \tag{2.51}$$

Expand components of total energy tensor:

$$R_{\alpha\mu\beta\nu} = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T\right) + \kappa\left(1 + \frac{1}{n-3}\right)\hat{T}_{\alpha\mu\beta\nu} \tag{2.52}$$

First collect the components of the trace-free source tensor:

$$R_{\alpha\mu\beta\nu} = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} + \hat{T}_{\alpha\mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T\right) \tag{2.53}$$

Then matter energy tensor and will arrive at final expression and add cosmological constant again:

$$R_{\alpha\mu\beta\nu} = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} + \hat{T}_{\alpha\mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T + \frac{2}{\kappa(n-1)(n-2)}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) \tag{2.54}$$

2.4.6 Geodesic deviation and geodesic equation

From fact that I can write Riemann tensor in terms of energy tensors I can go one step forward and write whole equation as a geodesic deviation equation [29] , where $\mathbf{V}$ is vector tangent to geodesic curve [29]:

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\xi^{\alpha} = R^{\alpha}_{\ \mu\beta\nu}V^{\mu}V^{\beta}\xi^{\nu} \tag{2.55}$$

Now I can switch right side of equation into energy instead of curvature parts:

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\xi^{\alpha} = \kappa\left(\tilde{T}^{\alpha}_{\ \mu\beta\nu} + \hat{T}^{\alpha}_{\ \mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\alpha}_{\beta} - g_{\beta\mu}\delta^{\alpha}_{\nu}\right)T + \frac{2}{\kappa(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\alpha}_{\beta} - g_{\beta\mu}\delta^{\alpha}_{\nu}\right)\Lambda\right)V^{\mu}V^{\beta}\xi^{\nu} \tag{2.56}$$

This gives a direct geometric interpretation of the rank-four formulation because the Riemann tensor itself controls geodesic deviation [30]. Geodesic deviation describes the relative acceleration of neighboring geodesics [29], including their focusing or defocusing. Rewriting the Riemann tensor in terms of the four index source makes both the Ricci-built and Weyl contributions visible in this equation, but it does not by itself change the singularity structure of General Relativity, because the full four index field equation is classically equivalent to the two index Einstein equation. The geodesic equation [29] can be written as:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{2.57}$$

The geodesic equation describes the motion of individual freely falling trajectories, while geodesic deviation contains the full local tidal information encoded by the Riemann tensor [29] [30]. In the four index notation the same tidal information can be separated into its matter-related Ricci sector and its trace-free Weyl sector without losing either part by contraction. This is an interpretive advantage of the notation, not an additional classical force law.

2.4.7 Field equations as connection with two index equations

The relation between the two index Einstein equation and the four index equation can be shown directly while keeping $\hat{T}_{\alpha\mu\beta\nu}$ as an independent field. The starting point is the ordinary two index Einstein equation [24], here written without the cosmological term:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}. \tag{2.58}$$

Contracting the ordinary Einstein equation gives its trace,

$$R = -\frac{2\kappa}{n-2}T, \tag{2.59}$$

and substituting this trace back into the same two index Einstein equation gives the trace-reversed Einstein equation,

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{n-2}g_{\mu\nu}T\right). \tag{2.60}$$

Thus the ordinary two index Einstein equation, together with its trace-reversed form, fixes every Ricci tensor and Ricci scalar term entering the Ricci-built part of the four index equation. Substituting these two index relations directly into the complete rank-four field equation gives

$$\begin{aligned}\frac{1}{n-2}\Bigg[&\kappa\left(T_{\alpha\beta} - \frac{1}{n-2}g_{\alpha\beta}T\right)g_{\mu\nu} - \kappa\left(T_{\alpha\nu} - \frac{1}{n-2}g_{\alpha\nu}T\right)g_{\beta\mu}\\ &+\kappa\left(T_{\mu\nu} - \frac{1}{n-2}g_{\mu\nu}T\right)g_{\alpha\beta} - \kappa\left(T_{\beta\mu} - \frac{1}{n-2}g_{\beta\mu}T\right)g_{\alpha\nu}\Bigg]\\ &-\frac{n}{2(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)\left(-\frac{2\kappa}{n-2}T\right) - \frac{1}{n-3}C_{\alpha\mu\beta\nu}\end{aligned}$$

$$= \kappa \left[\frac{1}{n-2} \left(T_{\alpha\beta} g_{\mu\nu} - T_{\alpha\nu} g_{\beta\mu} + T_{\mu\nu} g_{\alpha\beta} - T_{\beta\mu} g_{\alpha\nu} \right) \right.$$
$$\left. - \frac{1}{(n-1)(n-2)} \left(g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\beta\mu} \right) T - \frac{1}{n-3} \hat{T}_{\alpha\mu\beta\nu} \right]. \tag{2.61}$$

After collecting the terms proportional to T, the complete matter-built rank-four combination on the left is exactly the same as the matter-built combination on the right. It therefore cancels inside the full four index equation as a direct consequence of the ordinary two index Einstein equation and the trace-reversed Einstein equation. The remaining trace-free equation is

$$C_{\alpha\mu\beta\nu} = \kappa \hat{T}_{\alpha\mu\beta\nu}. \tag{2.62}$$

This equality is not used to define $\hat{T}_{\alpha\mu\beta\nu}$. The field enters the four index equations independently, with the Weyl algebraic symmetries and its own tensor components used when solving the rank-four system. The two index Einstein equation already fixes the complete Ricci-built part of that system. Once the two index equation is satisfied, the remaining trace-free part of the same four index equation fixes the independent field to the Weyl sector. Consequently the rank-four field supplies the tensorial components required to solve the uncontracted equation, but it adds no new field equation and no new classical information relative to the complete two index Einstein solution.

The same connection is visible at the differential level. Starting from the differential Bianchi identity written as the divergence of the Weyl tensor,

$$\nabla^\alpha C_{\alpha\mu\beta\nu} = \frac{n-3}{n-2} \left[\left(\nabla_\beta R_{\mu\nu} - \nabla_\nu R_{\beta\mu} \right) - \frac{1}{2(n-1)} \left(g_{\mu\nu} \nabla_\beta - g_{\beta\mu} \nabla_\nu \right) R \right], \tag{2.63}$$

we again use the ordinary two index Einstein equation. Substituting its trace-reversed form and its trace directly into the Bianchi identity gives

$$\begin{aligned} \nabla^\alpha C_{\alpha\mu\beta\nu} &= \frac{n-3}{n-2} \Bigg\{ \kappa \left[\nabla_\beta T_{\mu\nu} - \nabla_\nu T_{\beta\mu} - \frac{1}{n-2} \left(g_{\mu\nu} \nabla_\beta - g_{\beta\mu} \nabla_\nu \right) T \right] \\ &\qquad - \frac{1}{2(n-1)} \left(g_{\mu\nu} \nabla_\beta - g_{\beta\mu} \nabla_\nu \right) \left(- \frac{2\kappa}{n-2} T \right) \Bigg\} \\ &= \kappa \frac{n-3}{n-2} \left[\left(\nabla_\beta T_{\mu\nu} - \nabla_\nu T_{\beta\mu} \right) - \frac{1}{n-1} \left(g_{\mu\nu} \nabla_\beta - g_{\beta\mu} \nabla_\nu \right) T \right] \\ &= \kappa \nabla^\alpha \hat{T}_{\alpha\mu\beta\nu}. \end{aligned} \tag{2.64}$$

The differential equation of the independent trace-free source field is therefore the Weyl divergence equation after the ordinary two index Einstein equation has been used. The Ricci tensor and Ricci scalar appearing in the Bianchi identity are fixed by the two index Einstein equation, and the same Ricci information fixes the differential behaviour of $\hat{T}_{\alpha\mu\beta\nu}$. Thus the four index propagation law does not add a second independent dynamics to the Einstein system; it is the uncontracted trace-free form of the same information.

The divergence equation alone permits a divergence-free homogeneous tensor with Weyl symmetries. To show explicitly why this does not generate an additional solution of the complete four index field equation, write

$$\hat{T}_{\alpha\mu\beta\nu} = \frac{1}{\kappa} C_{\alpha\mu\beta\nu} + H_{\alpha\mu\beta\nu}, \qquad \nabla^{\alpha} H_{\alpha\mu\beta\nu} = 0. \tag{2.65}$$

Such a tensor leaves the divergence relation unchanged. The complete rank-four equation, after its Ricci-built part has been fixed by the two index Einstein equation as shown above, requires

$$C_{\alpha\mu\beta\nu} = \kappa \hat{T}_{\alpha\mu\beta\nu}. \tag{2.66}$$

Substitution of the possible homogeneous contribution gives

$$C_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} + \kappa H_{\alpha\mu\beta\nu}, \tag{2.67}$$

and therefore

$$H_{\alpha\mu\beta\nu} = 0. \tag{2.68}$$

The independent field $\hat{T}_{\alpha\mu\beta\nu}$ therefore retains the Weyl tensorial degrees of freedom required in the rank-four solution procedure, while the two index Einstein equation and the complete four index equation together ensure that those degrees of freedom contain no additional classical information beyond Einstein gravity.

For a constant cosmological term the same argument is unchanged in structure. The cosmological contribution is proportional to the metric and belongs entirely to the Ricci-built sector, while its covariant derivative vanishes. The remaining trace-free relation is therefore again $C_{\alpha\mu\beta\nu} = \kappa \hat{T}_{\alpha\mu\beta\nu}$.

2.5 Extensions to complex spacetime

2.5.1 Flat complex spacetime and SU(1,n) symmetry

Motivation behind complex spacetime is mostly here a more general symmetry that is fulfilled with flat spacetime. That symmetry will be in general case with one complex time dimension and n spatial dimensions [27] $SU(1,n)$ symmetry. Most important case is four dimensional complex spacetime that fulfills $SU(1,3)$ symmetry. This symmetry is of quarks in standard model [27]. I will write a flat spacetime metric in four dimensional case as:

$$ds^2 = \eta_{\mu\bar{\nu}} dz^{\mu} d\bar{z}^{\nu} \tag{2.69}$$

That can be expanded:

$$ds^2 = dz^0 d\bar{z}^0 - dz^1 d\bar{z}^1 - dz^2 d\bar{z}^2 - dz^3 d\bar{z}^3 \tag{2.70}$$

Where I did use mostly minus metric signature [24], this transforms under $SU(1,3)$ [27] transformations that I will denote U. This means that metric is invariant under those transformations:

$$ds^2 = ds'^2 \tag{2.71}$$

$$\eta_{\mu\bar{\nu}} dz^{\mu} d\bar{z}^{\nu} = \eta_{\mu'\bar{\nu}'} U^{\mu'}_{\alpha} dz^{\alpha} U^{\bar{\nu}'}_{\bar{\beta}} d\bar{z}^{\beta} \tag{2.72}$$

2.5.2 Einstein tensor

In complex Kähler geometry [25] Einstein tensor will be defined as:

$$G_{\mu\bar{\nu}} = R_{\mu\bar{\nu}} - g_{\mu\bar{\nu}}R \tag{2.73}$$

It is divergence free as normal Einstein tensor [25]:

$$\nabla^{\mu}G_{\mu\bar{\nu}} = 0 \tag{2.74}$$

I will pair it with a complex energy tensor that is defined as divergence free:

$$\nabla^{\mu}T_{\mu\bar{\nu}} = 0 \tag{2.75}$$

Now from it I can write a field equation for a complex spacetime [25]:

$$G_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} \tag{2.76}$$

2.5.3 Four index Einstein tensor

From using equations [26] [25] in complex geometry I can write a four index Einstein tensor as:

$$G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[R_{\alpha\bar{\beta}\gamma\bar{\delta}} - \left(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}\right) + g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}R\right] \tag{2.77}$$

Now I can check its four contractions:

$$g^{\alpha\bar{\beta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[R_{\gamma\bar{\delta}} - (nR_{\gamma\bar{\delta}} + Rg_{\gamma\bar{\delta}}) + nRg_{\gamma\bar{\delta}}\right] = R_{\gamma\bar{\delta}} - Rg_{\gamma\bar{\delta}} \tag{2.78}$$

$$g^{\gamma\bar{\beta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{n-1}\left(R_{\alpha\bar{\delta}} - Rg_{\alpha\bar{\delta}}\ \right) \tag{2.79}$$

$$g^{\alpha\bar{\delta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{n-1}\left(R_{\gamma\bar{\beta}} - Rg_{\gamma\bar{\beta}}\ \right) \tag{2.80}$$

$$g^{\gamma\bar{\delta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[R_{\alpha\bar{\beta}} - (nR_{\alpha\bar{\beta}} + Rg_{\alpha\bar{\beta}}) + nRg_{\alpha\bar{\beta}}\right] = R_{\alpha\bar{\beta}} - Rg_{\alpha\bar{\beta}} \tag{2.81}$$

Two of contractions will always give constants that can't be removed so this tensor is not fully reduced to Einstein tensor but still it is proportional to it. Only change is in dimensions two where those are equal to Einstein tensor as two minus one gives one. From fact that it is proportional to it it will be still divergence free.

2.5.4 Decomposition of Riemann tensor

I can decompose Riemann tensor in complex spacetime as [25] [26]:

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\delta}} = B_{\alpha\bar{\beta}\gamma\bar{\delta}} &+ \frac{1}{n+2}\left(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + R_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + R_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right) \\ &- \frac{1}{(n+1)(n+2)}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)R \end{aligned} \tag{2.82}$$

Now I can plug it into definition of four index Einstein tensor:

$$\begin{aligned} G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\Big[B_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{n+1}{n+2}\left(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}\right) + \frac{1}{n+2}\left(R_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + R_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right) \\ + \left(\frac{n^2+3n+1}{(n+1)(n+2)}g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} - \frac{1}{(n+1)(n+2)}g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)R\Big] \end{aligned} \tag{2.83}$$

This tensor is divergence free [26]:

$$\nabla^\alpha G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[\nabla_{\bar{\delta}}R_{\gamma\bar{\beta}} - \left((\nabla_{\bar{\beta}}R)g_{\gamma\bar{\delta}} + \nabla_{\bar{\beta}}R_{\gamma\bar{\delta}}\right) + (\nabla_{\bar{\beta}}R)g_{\gamma\bar{\delta}}\right] = 0 \tag{2.84}$$

2.5.5 Four index energy tensor

The same algebraic organization can be used in complex spacetime. The trace-free source $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is introduced as an independent rank-four field with the algebraic symmetries of the trace-free Bochner sector [25], while the remaining terms are constructed from $T_{\mu\bar{\nu}}$. As in the real case, independence means that its tensor components are variables of the rank-four equation during the solution procedure; it is not defined by the Bochner tensor before the equation is solved. The final result is:

$$\begin{aligned} T_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\Big[\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{n+1}{n+2}\left(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}\right) + \frac{1}{n+2}\left(T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right) \\ + \frac{1}{(1-n)(n+1)(n+2)}\left[-(n^2+n+1)g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + (2n+1)g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right]T\Big] \end{aligned} \tag{2.85}$$

This gives final field equations in complex spacetime:

$$\boxed{G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa T_{\alpha\bar{\beta}\gamma\bar{\delta}}} \tag{2.86}$$

The contractions reproduce the complex two index equations, and those equations determine the Ricci-built rank-four sector. The remaining trace-free part of the complete rank-four equation then fixes the independent field $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ to carry the same information as the Bochner curvature. Thus the complex four index formulation has the same relation to its two index equation as the real construction: the trace-free field is independent while solving the tensor system, but it adds no new classical equations or information to the complete two index solution. Both the four index tensor and its contractions are divergence free:

$$\nabla^\alpha G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa\nabla^\alpha T_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0 \tag{2.87}$$

2.5.6 Kähler geometry geodesic equation and its deviation

In Kähler geometry [25] geodesic equation takes form of two equations:

$$\frac{d^2z^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dz^\alpha}{ds}\frac{dz^\beta}{ds} = 0 \tag{2.88}$$

$$\frac{d^2z^{\bar{\mu}}}{ds^2} + \Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}}\frac{dz^{\bar{\alpha}}}{ds}\frac{dz^{\bar{\beta}}}{ds} = 0 \tag{2.89}$$

Geodesic deviation takes form of:

$$\frac{D^2V^\alpha}{ds^2} + R^\alpha_{\ \beta\delta\bar{\gamma}}T^\beta V^\delta T^{\bar{\gamma}} + R^\alpha_{\ \beta\bar{\delta}\gamma}T^\beta V^{\bar{\delta}}T^\gamma = 0 \tag{2.90}$$

2.5.7 Trace reversed field equations

I can use definition of Riemann tensor to change it into trace reversed field equations. I will start by writing normal two index [24] [25] trace reversed field equations:

$$g^{\mu\bar{\nu}}\left(R_{\mu\bar{\nu}} - g_{\mu\bar{\nu}}R\right) = \kappa g^{\mu\bar{\nu}}T_{\mu\bar{\nu}} \tag{2.91}$$

$$\left(1-n\right)R = \kappa T \tag{2.92}$$

$$R = \frac{\kappa}{1-n}T \tag{2.93}$$

$$R_{\mu\bar{\nu}} = \kappa\left(T_{\mu\bar{\nu}} + \frac{1}{1-n}g_{\mu\bar{\nu}}T\right) \tag{2.94}$$

Then I can use the decomposition of the Riemann tensor and replace its Ricci-built part by the corresponding stress–energy expression. The trace-free field $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ remains an independent variable of the rank-four equation, and the complete equation fixes it to the Bochner sector in the same way that $\hat{T}_{\alpha\mu\beta\nu}$ is fixed to the Weyl sector in real spacetime:

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa\Bigg(&\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{n+2}\left(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right) \\ &+\frac{2n+1}{\left(1-n\right)\left(n+2\right)\left(n+1\right)}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)T\Bigg) \end{aligned} \tag{2.95}$$

From it follows that I can write geodesic deviation in terms of only energy tensor parts:

$$\frac{D^2V^\alpha}{ds^2} + \mathcal{T}^\alpha_{\ \beta\delta\bar{\gamma}}T^\beta V^\delta T^{\bar{\gamma}} + \mathcal{T}^\alpha_{\ \beta\bar{\delta}\gamma}T^\beta V^{\bar{\delta}}T^\gamma = 0 \tag{2.96}$$

$$\begin{aligned} \mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = &\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{n+2}\left(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right) \\ &+\frac{2n+1}{\left(1-n\right)\left(n+2\right)\left(n+1\right)}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)T \end{aligned} \tag{2.97}$$

$$\mathcal{T}^\alpha_{\ \beta\delta\bar{\gamma}} = g^{\alpha\bar{\rho}}\mathcal{T}_{\beta\bar{\rho}\delta\bar{\gamma}} \tag{2.98}$$

$$\mathcal{T}^\alpha_{\ \beta\bar{\delta}\gamma} = g^{\alpha\bar{\rho}}\mathcal{T}_{\beta\bar{\delta}\gamma\bar{\rho}} \tag{2.99}$$

This only lacks cosmological constant to make equations complete.

2.5.8 Adding cosmological constant

Let me start with two index field equations [25] and add cosmological constant to them:

$$R_{\mu\bar{\nu}} - g_{\mu\bar{\nu}}R + g_{\mu\bar{\nu}}\Lambda = \kappa T_{\mu\bar{\nu}} \tag{2.100}$$

Now solve for Ricci scalar with contractions of field equations:

$$\left(1-n\right)R + n\Lambda = \kappa T \tag{2.101}$$

$$\left(1-n\right)R = \kappa T - n\Lambda \tag{2.102}$$

$$R = \frac{\kappa}{1-n}T - \frac{n}{1-n}\Lambda \tag{2.103}$$

$$R_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} + g_{\mu\bar{\nu}}\left(\frac{\kappa}{1-n}T - \frac{n}{1-n}\Lambda\right) - g_{\mu\bar{\nu}}\Lambda \tag{2.104}$$

$$R_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} + g_{\mu\bar{\nu}}\left(\frac{\kappa}{1-n}T - \left(\frac{n}{1-n}+1\right)\Lambda\right) \tag{2.105}$$

$$R_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} + g_{\mu\bar{\nu}}\left(\frac{\kappa}{1-n}T - \frac{1}{1-n}\Lambda\right) \tag{2.106}$$

This will lead plugging it into part with Ricci scalar of trace reversed field equations:

$$\left(-\frac{2}{(1-n)(n+2)} + \frac{n}{(1-n)(n+1)(n+2)}\right)\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)\Lambda \tag{2.107}$$

$$-\frac{1}{(1-n)(n+1)}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)\Lambda \tag{2.108}$$

So whole trace reversed field equations take final form:

$$\boxed{R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{(1-n)(n+1)}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)\Lambda} \tag{2.109}$$

2.6 Summary

In this work I presented a four index formulation of Einstein field equations [24] and its complex spacetime analog [25]. The construction keeps the complete rank-four curvature structure explicit instead of contracting it immediately to the Ricci tensor. On the geometric side this separates the Ricci-built and trace-free curvature sectors. On the source side the ordinary stress–energy tensor is lifted to rank four and supplemented by an independent trace-free field $\hat{T}_{\alpha\mu\beta\nu}$ with the algebraic symmetries and algebraic degrees of freedom of the Weyl tensor.

The independence of $\hat{T}_{\alpha\mu\beta\nu}$ is a property of the formulation and of the solution procedure. Its components are independent unknown components of the rank-four source and satisfy the Weyl symmetries, trace-free condition, divergence equation, and complete field equation. In four dimensional spacetime it therefore has the same ten independent algebraic components as the Weyl tensor. It is not introduced by defining it as curvature.

At the same time, the field does not add new classical information to Einstein gravity. Contraction of the four index equation gives exactly the ordinary two index Einstein equation. Its trace-reversed form determines the Ricci tensor and Ricci scalar and therefore determines the entire Ricci-built rank-four sector. Once this sector is substituted back into the complete four index equation, the remaining trace-free equation fixes the independent source field to the Weyl sector. The differential form gives the same result: the two index Einstein equation inserted into the Bianchi identity produces exactly the divergence equation of the trace-free source field. A possible additional divergence-free tensor with Weyl symmetries is removed by the complete rank-four equation.

The four index equations are therefore the same classical Einstein dynamics written without discarding the trace-free rank-four information by contraction. The field $\hat{T}_{\alpha\mu\beta\nu}$ supplies the tensorial degrees of freedom required to represent and solve the trace-free source sector, but it introduces no additional field equations, no additional classical information, and no additional metric solutions relative to the complete two index Einstein theory. The complex Kähler construction has the same structure, with the trace-free Bochner sector replacing the Weyl sector.

Chapter 3

Four-Sector Lorentz Geometry

Source and status. This chapter incorporates the complete scientific body of *Extension of Lorentz Spacetime Symmetry to Four Sectors of Flat and Curved Spacetime*, v1.13 (15 August 2026). Its role in the dissertation is: Flat and curved four-sector atlas with Schwarzschild, LTB, Kerr and FLRW examples. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

This paper develops a four-sector extension of Lorentz spacetime symmetry on a four-dimensional Lorentzian manifold with an invariant null set. Ordinary Lorentz transformations act inside a chosen causal orientation, whereas the sector extension completes the orientation of a clock–ruler plane through the four non-null domains separated by the common lightlike directions. Every active observer retains one real time axis, a three-dimensional rest space and the same locally measured light speed c. The sector label therefore records which non-null direction carries the clock; it does not represent ordinary matter accelerated through the light boundary.

The flat-space construction is written with a geometry coframe and a separate particle transformation. This keeps the active geometry sector, the particle's relative sector and its ordinary within-sector Lorentz motion as distinct data. The same structure is then extended to curved spacetime on the observer's rest bundle. A positive spatial tensor $L^{\mu}{}_{\nu}$ encodes proper deformation scales, a solution-dependent tensor $G^{\mu}{}_{\nu}$ fixes their critical values, and their generalized eigenproblem selects the clock–ruler plane that reaches a null chart boundary. The neighbouring chart uses the reciprocal eigenvalue of that same proper scale while retaining the metric, matter fields and Einstein equations of the general-relativistic solution.

Schwarzschild, Lemaître–Tolman–Bondi, Kerr and spatially flat FLRW geometries provide explicit tests. The Schwarzschild and Kerr reciprocal black-hole domains replace the zero-radius extrapolation by increasing areal radius with decaying curvature. LTB demonstrates why longitudinal shell separation and transverse areal collapse must be treated as distinct tensor modes; explicit reciprocal dust solutions are obtained for both a shell-crossing profile and a homogeneous shell-focusing profile. For the regular FLRW fluids analysed here, the reciprocal scale grows while density and Riemann curvature decay. The corresponding proper-time and affine parameter calculations show that the regular reciprocal centres reached in the analysed black-hole, homogeneous LTB and FLRW branches are asymptotic boundaries rather than finite-parameter passages.

3.1 Physical starting point: one invariant light cone

Special relativity begins with the empirical statement that every inertial observer measures the same vacuum light speed c [13]. The statement is not only about a number obtained by dividing distance by time. It says that all observers agree on the null boundary separating timelike and spacelike directions. Without that common boundary they would disagree on the causal type of the same displacement, and there would be no single causal spacetime shared by them.

The construction takes this geometrical content as its physical starting point. On a four-dimensional Lorentzian manifold with a local null set, the four sectors are defined by following the complete orientation of a clock–ruler plane around the common null directions.

Physical principle 3.1 (Observer-independent local light speed)**.** *Every local observer measures the same null speed c, independently of the observer's worldline and of which admissible sector orientation supplies the local clock.*

3.1.1 The light cone precedes a numerical velocity

A light cone is a property of the Lorentzian geometry before any observer assigns coordinate slopes to it. An observer chooses a unit timelike clock axis and three orthogonal ruler axes; only after this choice is the cone read numerically as propagation at c. A different observer uses tilted axes, but an ordinary Lorentz transformation preserves the same null directions. In a spacetime diagram the clock and the corresponding ruler tilt together. The cone is not deformed by the observer's motion.

This order prevents a common misinterpretation. A tangent may have an infinite or apparently superluminal slope when divided by the time coordinate of a sector whose clock it does not use. That quotient is not automatically a locally measured speed. Physical velocity is defined only after the tangent has been resolved in its active real clock–ruler frame.

An ordinary Lorentz boost explores one connected orientation domain. As its clock approaches a null ray, the normalisation of a unit timelike vector diverges because a null vector cannot carry a massive observer's proper time. No finite boost crosses that boundary. The four-sector extension therefore does not permit ordinary matter to pass through c. It asks instead how the neighbouring orientation domain is described when the old clock itself has reached the edge of its valid chart.

3.1.2 Why one Lorentz plane has four sectors

An ordinary observer has one time axis and three spatial axes. A side observer has exactly the same local structure: one time axis and three spatial axes. There are not several times attached to one side particle. The difference is only the orientation of its frame relative to ours. Its single time axis points along one direction that our observer calls spatial. If that direction is denoted by n^μ, then n^μ is the side observer's clock direction, while our old clock direction becomes one of the side observer's spatial directions. The two directions orthogonal to this clock–ruler plane remain spatial.

The direction of the side clock must not be confused with the direction of the particle's spatial velocity. A side particle is not forced to travel only along n^μ. Just as an ordinary particle can move in any direction of our three-dimensional rest space while its time axis remains our time axis, a side particle can move in any direction of its own three-dimensional rest space while its single time axis remains the side-clock axis. Its complete material trajectory lies inside one side Lorentz cone. That cone looks rotated by a quarter turn in our inherited diagram, but for the side observer it is an ordinary future cone with the usual local light boundary and ordinary subluminal motion inside it.

Different possible side observers may have their one clock axis oriented along different directions of our three-space. The unit possibilities form the sphere S^2, or equivalently all nonzero directions of the old $\mathbb{R}^3$ before normalisation. Thus S^2 describes the possible orientations of one time axis across the family of side frames. It does not describe many simultaneous time axes of one particle. For a particular particle the realised direction follows from its actual tangent and Lorentz transformation.

Choose the ordinary clock direction and one realised side-clock direction n^μ. Their two null lines divide the corresponding oriented Lorentz plane into four connected non-null domains. Their centres are the future clock $+t$, the first side clock $+n^\mu$, the opposite clock $-t$, and the reflected side clock $-n^\mu$:

$$+t \longrightarrow +n^\mu \longrightarrow -t \longrightarrow -n^\mu \longrightarrow +t.$$

Each domain contains its own ordinary Lorentz boosts. Such boosts change the spatial motion inside one cone but do not create a new sector and do not carry a massive observer across the null boundary. The fourfold cycle therefore completes the orientation atlas; it is not by itself a journey completed by one worldline.

3.1.3 Geometry orientation and particle orientation are different data

The same fourfold structure has two uses which must not be conflated. The geometry has an active sector n_g, telling us which orientation and which regular chart currently represent the common field. A particle X has a relative sector s_X, telling us which extended Lorentz transformation acts on the local geometric coframe. Its ordinary Lorentz tensor $\Lambda_{(X)}{}^{\mu}{}_{\nu}$ then describes motion inside that fixed particle sector, including the boost direction and the orientation of its spatial triad. No additional unit vector on the side-direction sphere is required as independent particle data.

These operations occur in a definite order. The geometry first supplies its sector coframe. The particle then applies its own sector phase and ordinary Lorentz motion to the components read from that coframe. The sum $N_X = n_g + s_X \pmod 4$ appears only after those two independent operations have been composed. A geometry-chart transition changes n_g and hence the total phase N_X, but it changes neither s_X nor the particle's local boost. Conversely, ordinary acceleration changes $\Lambda_{(X)}$ but not the two sector labels.

This separation allows ordinary, side-oriented and oppositely oriented particle transformations to be represented in the same flat or curved calculation while retaining the common null set.

3.1.4 Why curvature and geodesic distance must be tested

In flat spacetime the four sectors concern orientation alone. Curvature adds a second problem: the proper separations between neighbouring worldlines can focus, stretch or rotate. A sector continuation must therefore identify which physical scale reaches the null chart boundary and whether the neighbouring chart remains a regular representation of the same gravitational field.

This requirement cannot be replaced by a tetrad choice. A freely falling frame can remove the connection at one event, but it cannot remove a nonzero Riemann tensor. If invariant curvature diverges, the relative acceleration of nearby geodesics becomes unbounded and no regular neighbourhood remains in which the universal cone structure can be transported. A different particle sector also cannot cure this failure: it only changes the particle's local reading of the common tensor.

The curved construction below consequently has three tests. Proper scale tensors determine the direction whose inherited chart is failing and select the reciprocal scale of the neighbouring sector. Curvature invariants are then evaluated on that active domain. Finally, timelike proper time and null affine parameter determine whether the reciprocal centre is crossed, reached asymptotically, or not reached at all. Finite curvature is not assumed from the phase i^{n_g}, and finite curvature alone does not establish geodesic completeness. These separate tests decide whether a Schwarzschild, LTB, Kerr or FLRW continuation succeeds.

3.2 Four-sector Lorentz kinematics in flat spacetime

The sectors must first be defined in flat spacetime. Otherwise their meaning would depend on a particular gravitational solution and regularisation would be built into the notation. In inertial coordinates the common starting point is

$$\mathrm{d}s^2_{(0)} = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2. \tag{3.1}$$

Light follows the common null set defined by $\mathrm{d}s^2_{(0)} = 0$.

3.2.1 Notation and tensor conventions

Greek indices $\mu, \nu, \rho, \sigma, \ldots$ denote spacetime indices in both abstract tensor equations and coordinate-component expressions. Lower-case Latin indices $i, j, k, \ldots$ denote purely spatial directions or principal spatial modes. The antisymmetric pair on a metric-derived orientation angle is spacetime-valued: $\varpi_{\rho\sigma} = -\varpi_{\sigma\rho}$ with $\rho, \sigma = 0, \ldots, n-1$. It may therefore label a time–space plane as well as a spatial plane. Upper-case Latin indices $A, B, C, \ldots$ label an orthonormal Lorentz coframe with the orthonormal-frame metric has signature $(-+++)$; its spatial slots may be labelled by i, j, k. Labels in parentheses enumerate principal spatial modes but are not additional tensor indices. Sector labels take values in $\mathbb{Z}_4$ and are never summed.

The symbol x denotes one selected direction in the old spatial $\mathbb{R}^3$. It can later be represented by a spherical radial ray r, but the side sector contains every possible spatial ray rather than one preferred axis.

Let

$$\vartheta^A = \xi^A{}_\mu\,\mathrm{d}x^\mu, \qquad g^{(0)}_{\mu\nu} = \eta_{AB}\xi^A{}_\mu\xi^B{}_\nu. \tag{3.2}$$

be a real reference coframe and its metric. The coefficients $\xi^A{}_\mu(x)$ may already contain the local deformation of a curved solution; the sector construction will not multiply them by an additional physical strain.

Three pieces of information must remain distinct. The active geometry sector is $n_g(x)$. A particle X carries a sector s_X relative to that geometry. Its ordinary Lorentz transformation $\Lambda_{(X)}{}^\mu{}_\nu$ describes continuous motion inside the fixed particle sector, including the directional information of the trajectory relative to the reference coframe. A separate side-time direction is therefore not added to the particle state. Only after the geometry and particle operations are composed do we use the total label $N_X = n_g + s_X \pmod 4$ relative to the reference coframe.

The geometric variables are the metric $g_{\mu\nu}$, its null set and the curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$ together with their sector-frame representatives. Particle transformations modify tetrad-component readouts, whereas geometric flatness is determined by the Riemann tensor. Particle rest is therefore a separate kinematic statement.

3.2.2 The ordinary Lorentz transformation and the tilt of the axes

Before extending the set of orientations, consider what an ordinary boost does inside an already fixed sector. Let u^μ be the unit clock of the reference observer,

$$g_{\mu\nu}u^\mu u^\nu = -1, \qquad h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu, \tag{3.3}$$

and let n^μ be the unit direction of the particle trajectory in the old rest space,

$$u_\mu n^\mu = 0, \qquad n_\mu n^\mu = 1. \tag{3.4}$$

The vector n^μ is not an additional clock label. It is the direction already contained in the particle tangent and in the directional part of its Lorentz transformation relative to u^μ.

The active time–space plane and its hyperbolic exchange tensor are

$$P^\mu{}_\nu = -u^\mu u_\nu + n^\mu n_\nu, \qquad B^\mu{}_\nu = u^\mu n_\nu - n^\mu u_\nu, \qquad B^\mu{}_\rho B^\rho{}_\nu = P^\mu{}_\nu. \tag{3.5}$$

For $\gamma_X = (1-\beta_X^2)^{-1/2}$, the ordinary boost in the selected clock–ruler plane is the mixed tensor

$$\Lambda_{(X)}{}^\mu{}_\nu = \delta^\mu{}_\nu + (\gamma_X - 1)P^\mu{}_\nu + \gamma_X\beta_X B^\mu{}_\nu, \qquad |\beta_X| < 1. \tag{3.6}$$

It acts on the clock and the selected ruler as

$$u_X^\mu = \Lambda_{(X)}{}^\mu{}_\nu u^\nu = \gamma_X(u^\mu + \beta_X n^\mu), \qquad n_X^\mu = \Lambda_{(X)}{}^\mu{}_\nu n^\nu = \gamma_X(n^\mu + \beta_X u^\mu), \tag{3.7}$$

and leaves the two-dimensional screen orthogonal to u^μ and n^μ unchanged. Equation (3.6) displays the boost part. A general within-sector frame may additionally contain an ordinary spatial triad rotation; throughout the within-sector Lorentz factor belongs to the proper-orthochronous component $SO^+(1,3)$. Such a spatial rotation does not change the particle's state of rest. The tensor formula is four-dimensional; a 2×2 array is only its component display in the selected plane and is not used as a second notation.

The transformation preserves the metric and the null set:

$$g_{\rho\sigma}\Lambda_{(X)}{}^\rho{}_\mu\Lambda_{(X)}{}^\sigma{}_\nu = g_{\mu\nu}. \tag{3.8}$$

At $\beta_X = 0$ the transformed and reference axes coincide. As $|\beta_X|$ approaches unity, the clock and its paired ruler approach one of the common null directions, while γ_X diverges because a null vector cannot be normalised as the clock of a massive observer. No finite ordinary boost crosses that boundary.

A side observer should now be read in exactly the same physical way as an ordinary observer. It has one clock axis, not a sphere of simultaneous clock axes. Relative to the ordinary reference frame, that single clock points along one direction of the ordinary spatial rest space. Across the family of all possible side observers this direction may be any

$$n^\mu \in S^2 \subset u^\perp. \tag{3.9}$$

Before normalisation the same family is represented by the nonzero vectors of the old $\mathbb{R}^3$. For one particular particle the realised direction is not chosen as extra data: it is already fixed by the particle tangent and by the directional content of $\Lambda_{(X)}{}^\mu{}_\nu$.

The fact that this one side clock points along one of our spatial directions does not mean that the particle can move only along that direction. The side observer has its own three-dimensional rest space, orthogonal to its clock, just as we do. Its tangent can therefore be written as

$$U_X^\mu = c\gamma_X(u^\mu + v_X^\mu/c), \qquad u_\mu v_X^\mu = 0, \qquad g^{(1)}_{\mu\nu} v_X^\mu v_X^\nu < c^2, \tag{3.10}$$

where u^μ is its one future-directed time axis and v_X^μ may point in any direction of its own spatial rest space. The whole worldline remains inside the single Lorentz cone centred on u^μ. In our inherited diagram that cone is quarter-turned, but for the side observer it is simply its ordinary future cone. A spatial turn changes v_X^μ; it does not rotate the particle into another time orientation or another sector.

3.2.3 Four orientation sectors of one Lorentz plane

The ordinary boost has so far explored only the domain centred on $+t$. To see the missing orientations, keep the same two null lines and follow an oriented axis around the complete time–space plane. The null rays divide that plane into four open connected domains. Starting from $+t$, one meets a domain centred on a selected positive spatial direction, then the domain centred on $-t$, then the reflected spatial domain, and finally returns to $+t$. The complete cycle is

n	i^n	clock orientation	sector
0	1	$+t$	ordinary
1	i	$+n^\mu$	first side
2	-1	$-t$	opposite ordinary
3	$-\mathrm{i}$	$-n^\mu$	reflected side
$4 \equiv 0$	1	$+t$	closed orientation cycle

Thus the ordinary flat reference has its future cone centred on $+t$. The first side reference is the family of equally admissible flat cones centred on directions that were spatial in the ordinary projection. For a particular side trajectory, one direction $+n^\mu$ is determined by its tangent and by the directional content of $\Lambda_{(X)}$ relative to the reference coframe; it is not selected as an additional independent clock variable. The opposite sector has clock centre $-t$, and the second side sector is the reflected family $-n^\mu$.

The entries in the table are the centres of the four domains, not the only directions allowed inside them. Every sector has its own finite $|\beta_X| < 1$ Lorentz boosts, and none of those boosts creates another sector. In the first side class a complete material trajectory lies inside one cone whose clock centre is $+n^\mu$ relative to the ordinary observer. The cone appears quarter-turned in the inherited diagram, but it is an ordinary Lorentz cone for the side observer and has the same null boundary. Across all possible trajectories its centre may point along any unit direction of S^2, or equivalently along any nonzero direction of the old $\mathbb{R}^3$ before normalisation. One trajectory realises one such direction through its Lorentz and tangent data; it does not carry the whole sphere as simultaneous time axes.

The discrete label records the connected orientation component, whereas ${\Lambda_{(X)}}^{\mu}{}_{\nu}$, including its boost direction and spatial rotation, records continuous particle motion inside that component. The geometric ratio q_i introduced in Section 3.3.4 instead tracks the orientation of an active proper scale; it is not a particle velocity.

The table records the four orientation centres. What changes from one entry to the next is the direction chosen to bisect the null cone and to carry the observer's clock. In sector 0 this direction is the usual $+t$ axis. At the neighbouring null boundary that axis can no longer be normalised as timelike; sector 1 then reads one of the old spatial directions as its real clock direction. Repeating the same construction produces the reflected $-t$ and $-n^{\mu}$ orientations and finally returns to the starting frame.

The null rays are boundaries of the four descriptions, not additional sectors. At such a ray the old unit clock is undefined because its norm is zero. The neighbouring sector begins by assigning a new timelike centre on the other side of that boundary. This change concerns the frame used to read the geometry. It does not say that a massive tangent has crossed the null ray or changed from timelike to spacelike.

Only one old spatial direction supplies the clock of a given side observer at one event. If it is denoted by the unit vector n^{μ}, the old clock becomes one direction in the new rest space, n^{μ} becomes the new clock, and the two directions orthogonal to their plane remain spatial. The two-sphere parametrises all possible directions relative to the reference observer, but the direction of a particular particle is derived from its actual tangent U_X^{μ} and $\Lambda_{(X)}$ rather than appended as a new label. For inertial motion in flat spacetime it is constant because the tangent is constant; if the trajectory is physically accelerated, it changes continuously only with that trajectory while s_X remains fixed. In spherical language (θ, φ) label this family of possible directions; they are not two additional times on one worldline.

3.2.4 Geometry sector, particle transformation, and total orientation

The sector factors enter in two successive operations. First the geometry chooses its active coframe:

$$\vartheta^{A}_{(g)} = \mathrm{i}^{n_g}\vartheta^{A} = \mathrm{i}^{n_g}\xi^{A}{}_{\mu}\,\mathrm{d}x^{\mu}, \qquad g^{(n_g)}_{\mu\nu} = \eta_{AB}\vartheta^{A}_{(g)\mu}\vartheta^{B}_{(g)\nu} = \mathrm{i}^{2n_g}g^{(0)}_{\mu\nu}. \tag{3.11}$$

The phase i^{n_g} belongs to the geometry coframe, while its square appears in the sector representation of the metric. Odd sectors reverse the sign convention of the quadratic form and even sectors preserve it. Because multiplication by a nonzero constant leaves the null set unchanged, all four representations describe the same local light boundary. The full orientation still resides in the coframe: the metric alone cannot distinguish sectors 0 and 2, or sectors 1 and 3.

Only after the geometry has supplied $\vartheta^{A}_{(g)}$ does particle X apply its own extended Lorentz transformation

$${\mathcal{G}_{(X)}}^{A}{}_{B} = \mathrm{i}^{s_X}{\Lambda_{(X)}}^{A}{}_{B}, \qquad \mathrm{d}X^{A}_{(X)} = {\mathcal{G}_{(X)}}^{A}{}_{B}\vartheta^{B}_{(g)} = \mathrm{i}^{n_g+s_X}{\Lambda_{(X)}}^{A}{}_{B}\xi^{B}{}_{\mu}\,\mathrm{d}x^{\mu}. \tag{3.12}$$

This is the central composition law. The factor i^{s_X} belongs to the particle transformation, not to a second geometric coframe. The ordinary Lorentz tensor ${\Lambda_{(X)}}^{\mu}{}_{\nu}$ describes motion inside that particle sector. In three spatial dimensions its boost and rotation data also determine which member of the side-direction family is realised relative to the reference coframe; no independent n^μ_X is added to Eq. (3.12). The total label $N_X = n_g + s_X \pmod 4$ records the final phase relative to the reference coframe, but it does not replace the two independent sources of that phase.

The four relative classes are

s_X	relative orientation	terminology
0	ordinary	ordinary particle
1	first side	side particle I
2	opposite	anti-sector particle
3	reflected side	side particle II

The values $s_X = 1, 3$ denote the two side orientations, while $s_X = 2$ denotes the opposite orientation.

When the geometry advances by k charts, both n_g and the total readout N_X advance by k, while s_X and $\Lambda_{(X)}$ remain unchanged. By contrast, ordinary acceleration changes $\Lambda_{(X)}$ while leaving s_X fixed. Ordinary matter, a side particle and an opposite-sector particle therefore retain their class when the geometry changes sector.

Equation (3.12) may be used to form the local quadratic readout $\eta_{AB}\mathrm{d}X^A_{(X)}\mathrm{d}X^B_{(X)}$. The metric entering the field equations remains $g_{\mu\nu}$; the particle sector changes the frame through which its components are resolved.

3.2.5 Particle rest and geometric flatness

Particle rest is defined from the readout in Eq. (3.12), not from the value of a coordinate slope. If U^μ_X is its tangent, its local components are obtained by applying $\mathcal{G}_{(X)}{}^A{}_B\vartheta^B_{(g)\mu}$ to that tangent. The particle is at rest relative to the local geometry inside its own sector when its three spatial components vanish. In the adapted representative this is equivalently

$$U^i_{(X)} = 0 \qquad \Longleftrightarrow \qquad \beta_X = 0. \tag{3.13}$$

In the canonical adapted representative with no independent spatial triad rotation this is represented by $\Lambda_{(X)}{}^\mu{}_\nu = \delta^\mu{}_\nu$. A purely spatial rotation of the particle frame may remain without changing Eq. (3.13). For $s_X = 0$ this is ordinary rest relative to the geometric coframe. For $s_X = 1$ or 3 the clock axis is side-oriented, so an inherited ordinary chart can assign the particle an infinite or superluminal-looking slope even though its boost parameter vanishes. For $s_X = 2$ the particle transformation has the opposite orientation. There is no absolute rest common to all four readouts.

The sector coframe changes the decomposition of the tangent, not the tangent itself. Relative to the future clock $u^\mu_{(X)}$ of the particle's sector, write only

$$U^\mu_X = \gamma_X\big(c\,u^\mu_{(X)} + v^\mu_X\big), \qquad u_{(X)\mu}v^\mu_X = 0, \qquad \gamma_X > 0. \tag{3.14}$$

An ordinary spatial rotation acts on v^μ_X inside $u^\perp_{(X)}$ and leaves $u^\mu_{(X)}$ fixed. It may reverse radial, angular, or inherited-$X = ct$ motion, but it cannot reverse the positive temporal component of U^μ_X. Thus a side particle may turn around in its own space without turning around in proper time.

Geometric sector rest is a different statement. Let u_g^μ be the clock of the active geometry chart and $h_g{}^\mu{}_\nu = \delta^\mu{}_\nu + u_g^\mu u_{g\nu}$ its rest projector. A reference congruence is aligned with that chart when $U^\mu = c\,u_g^\mu$. It defines a flat sector centre only when the alignment extends through a neighbourhood without tidal deviation: Geometric sector rest is the pointwise alignment

$$U^\mu = c\,u_g^\mu \quad \Longleftrightarrow \quad h_g{}^\mu{}_\nu U^\nu = 0. \tag{3.15}$$

A flat geometric centre is the stronger neighbourhood condition

$$\nabla_\nu u_g^\mu = 0, \qquad R^\mu{}_{\nu\rho\sigma} = 0. \tag{3.16}$$

The first line is pointwise and can hold in curved spacetime. The second is a property of the curvature field. Hence a side-oriented particle can be at particle-sector rest in a curved region, and an ordinary particle can move through a flat geometric centre. Neither fact changes the Riemann tensor. Regularisation below relies on the second line, not on setting $\beta_X = 0$ for one particle.

The particle phase must be removed when its components are compared with the geometry norm. From Eq. (3.12),

$$\mathrm{i}^{-2s_X}\eta_{AB}U^A_{(X)}U^B_{(X)} = -c^2, \qquad \eta_{AB}k^A_{(X)}k^B_{(X)} = 0. \tag{3.17}$$

The first expression is the geometry norm written through the particle transformation; it is not a particle metric. The null equation needs no compensating factor because zero is unchanged. The numerical value c and the null boundary are identical in all four orientations. A sector-0 observer may assign a side particle a formal coordinate slope greater than c, but that quotient is not the velocity $\beta_X c$ appearing in its own $\Lambda_{(X)}$.

3.2.6 Cone orientation and the uncrossed null boundary

The four sector domains share their null boundary. For a particle with fixed sector class, $\Lambda_{(X)}$ tilts its clock and ruler axes inside that domain and Eq. (3.17) places light on its boundary. The boost can approach the boundary but cannot cross it. Hence ordinary matter cannot become a side particle by acceleration, a side particle cannot be slowed into ordinary matter, and an anti-sector particle cannot be converted into ordinary matter by changing observer.

This statement is local and independent of n_g. In flat geometry with $n_g = 0$, particles with $s_X = 0, 1, 2, 3$ may coexist at the same event and use the same null set. A side particle is already centred on $+n^\mu$ or $-n^\mu$ before any gravitational sector transition occurs, with the actual direction fixed by its tangent and $\Lambda_{(X)}$ relative to the reference observer. Its apparent ordinary-sector quantity $\mathrm{d}x/\mathrm{d}t$ can be infinite or formally exceed c, but this is a projection on a clock it does not use. Its measured velocity is the finite $\beta_X c$ obtained from its own sector-transformed components.

A geometry-sector transition is different. When the active field chart moves from n_g to $n_g + 1$, the geometry coframe acquires the next phase before each unchanged particle transformation acts on it. Every N_X shifts together, while s_X and $\Lambda_{(X)}$ are preserved. Thus an ordinary particle inside side-oriented black-hole geometry has a side-looking total phase $N_X = 1$, but it is still ordinary relative to the local field because $s_X = 0$.

The global speed limit is therefore the conjunction of two facts. Every ordinary Lorentz motion remains inside its own sector domain, and all four domains possess the same null set.

No physical process described here moves a particle through that set. What can change is the active geometry chart or the observer used to project the tangent; neither is a measurement of a particle outrunning light in its own sector.

This also fixes the meaning of the inherited coordinate $X = ct$. The clock–ruler exchange does not erase the real differential $\mathrm{d}X = c\,\mathrm{d}t$ and does not redirect the physical tangent. It changes the role assigned to that component by the active coframe. In a side sector X belongs to the spatial decomposition, so $\mathrm{d}t < 0$ can describe motion toward decreasing X while the particle's own proper time still increases. Neither the sign of $\mathrm{d}t$, a spatial turn in X, nor a relabelling of the axes reverses the future orientation of U_X^μ.

3.2.7 The sector continuation and the role of i^n

The extended Lorentz transformation is the mixed tensor

$$\mathcal{G}_{(n)}{}^{\mu}{}_{\nu}[\Lambda] = \mathrm{i}^n \Lambda^{\mu}{}_{\nu}, \qquad n = 0, 1, 2, 3. \tag{3.18}$$

The phase labels the orientation component and $\Lambda^{\mu}{}_{\nu}$ is the ordinary Lorentz transformation acting inside it. Composition and inversion are written without suppressing tensor indices:

$$\begin{aligned} \mathcal{G}_{(m)}{}^{\mu}{}_{\rho}[\Lambda_2]\mathcal{G}_{(n)}{}^{\rho}{}_{\nu}[\Lambda_1] &= \mathcal{G}_{(m+n)}{}^{\mu}{}_{\nu}[\Lambda_2 \circ \Lambda_1] \\ &= \mathrm{i}^{m+n}(\Lambda_2)^{\mu}{}_{\rho}(\Lambda_1)^{\rho}{}_{\nu}, \end{aligned} \tag{3.19}$$

$$(\mathcal{G}_{(n)}^{-1})^{\mu}{}_{\nu}[\Lambda] = \mathcal{G}_{(-n)}{}^{\mu}{}_{\nu}[\Lambda^{-1}]. \tag{3.20}$$

Sector addition is understood modulo four. The metric contraction is

$$g_{\rho\sigma}\mathcal{G}_{(n)}{}^{\rho}{}_{\mu}\mathcal{G}_{(n)}{}^{\sigma}{}_{\nu} = \mathrm{i}^{2n} g_{\mu\nu}. \tag{3.21}$$

Thus the quadratic sign alternates, but the null equation is unchanged. All compositions are written as contractions of explicitly indexed tensors.

For a particle, Eq. (3.18) is used with $n = s_X$ and $\Lambda = \Lambda_{(X)}$. For the geometry, the sector factor appears separately in its coframe as in Eq. (3.11). These must not be collapsed into one unexplained i^n. Only their ordered composition in Eq. (3.12) has the total phase i^{N_X}.

Once a sector is fixed, the derivation in Section 3.2.2 applies without modification: $g_{\rho\sigma}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu} = g_{\mu\nu}$ preserves the null rays and the measured relative speed remains below c. The four sectors therefore do not carry four different Lorentz laws. They are four orientation components on which the same Lorentz law acts.

The continuous orbit and the discrete operator label must also be distinguished. An oriented axis can approach a null boundary in one chart and recede from it in the next, so its geometric trajectory is continuous. The algebraic factors $1, i, -1, -i$ only label the four normalisation branches used along that trajectory; the operator i^n is not itself a continuous rotation parameter. Section 3.3.4 gives the single angle and the projective coordinates that cover this orbit without identifying them with a particle speed. This continuity belongs to the space of oriented frames. It does not assert that one timelike or null worldline completes the orbit at finite physical parameter.

3.2.8 The interval cycle and its physical meaning

For the geometry sector $n_g \in \mathbb{Z}_4$, successive multiplication by the sector phase gives $\mathrm{d}x^\mu$, $\mathrm{i}\mathrm{d}x^\mu$, $-\mathrm{d}x^\mu$, $-\mathrm{i}\mathrm{d}x^\mu$, and again $\mathrm{d}x^\mu$. The corresponding sector metric is

$$\mathrm{d}s^2_{(n_g)} = \mathrm{i}^{2n_g}\mathrm{d}s^2_{(0)} = (-1)^{n_g}\mathrm{d}s^2_{(0)}. \tag{3.22}$$

Applying a particle transformation adds the independent phase s_X, so its final local quadratic readout carries the total phase $N_X = n_g + s_X$. That readout does not define a particle-dependent metric. The four steps have a direct reading. Sector 0 uses the original differential and the clock $+t$. The first factor i selects the side normalisation and makes the old positive spatial direction the active clock. The second turn gives $-\mathrm{d}x^\mu$ and reverses both inherited axes, producing the $-t$ orientation. The third gives the reflected side clock $-n^\mu$; the fourth returns the orientation and the differential to their starting values. The phase is therefore a compact account of the complete axis cycle already listed in the orientation table in Sec. 3.2.3.

In one selected plane, $\mathrm{d}s^2_{(0)} = -c^2\mathrm{d}t^2 + \mathrm{d}x^2$, while the inherited sector-1 components give $\mathrm{d}s^2_{(1)} = c^2\mathrm{d}t^2 - \mathrm{d}x^2$. The sign change is the algebraic record that the old spatial direction occupies the active time slot and the old time direction belongs to the active spatial decomposition. The same exchange with reversed orientation occurs in sector 3.

Equation (3.22) records the sector sign in inherited components. In every active sector the line element has one clock direction and three ruler directions, while the common null set is unchanged by the factor $\mathrm{i}^{2n_g} = (-1)^{n_g}$.

3.2.9 What flat spacetime means in every sector

A cone does not make a region curved merely because it appears tilted or sideways in an inherited coordinate diagram. Flatness means that one active clock–ruler tetrad can be extended throughout a neighbourhood without path-dependent rotation and that the corresponding reference geodesics have no tidal acceleration. Equivalently, the second line of Eq. (3.15) holds there. A constant Lorentz tilt applied to every cone changes the observer, but it does not change this property and therefore does not create curvature.

For geometry sector $n_g = 0$, the flat reference congruence follows the $+t$ axes of a family of parallel cones. Geometry sector $n_g = 1$ uses the same flat construction after the time–space roles have been exchanged, so the old spatial direction identified by the active plane supplies its reference clock. Curvature vanishes in both descriptions because the coframes remain parallel throughout the neighbourhood.

Particle sectors are superposed on this geometric choice. Even when $n_g = 0$ everywhere, a particle with $s_X = 1$ can have a side-oriented transformation and be at rest in its own side frame. Relative to the reference coframe, the realised side direction is contained in the rotation and boost-direction data of $\Lambda_{(X)}$. At particle rest the boost parameter vanishes; ${\Lambda_{(X)}}^\mu{}_\nu = \delta^\mu{}_\nu$ is the canonical non-rotated representative rather than a missing direction. A sector-0 chart may draw its clock horizontally and assign it an infinite slope, but this is neither a divergent physical speed nor a change of the background geometry. Particles with all four values of s_X can therefore coexist in one flat region.

Geometry sectors 2 and 3 repeat the two flat references with opposite orientation. The reversal changes the clock orientation but not the vanishing of curvature or the null boundary.

The sign cycle in Eq. (3.22) is consistent with this statement. Multiplying an interval by

-1 leaves its zero set unchanged, so the null rays that separate the sectors are common. Each active geometry coframe places its selected clock in the $A = 0$ slot. The sign change records the exchange of inherited temporal and spatial roles.

This also clarifies why particle rest is not a curvature criterion. The condition $\beta_X = 0$ concerns one particle transformation, whereas a flat geometric centre requires a whole neighbouring congruence with no relative turning and no geodesic deviation. The latter is a property of an extended region and cannot be inferred from the sector or velocity of one body.

3.2.10 Null boundaries and continuity

Neighbouring orientation charts meet only at a null direction. Ordinary Lorentz motion remains inside the particle's chosen sector; only light lies on the common boundary. The geometry can reach that boundary and require a neighbouring chart, but this shifts n_g and N_X together rather than changing the relative label s_X.

The geometry orientation itself is nevertheless continuous. Starting at one sector centre, the clock–ruler plane tilts toward a common null ray. The old unit clock ceases to exist exactly at that ray; the neighbouring chart begins at the same oriented limit and follows the plane away from it toward its own centre. The discrete value n_g therefore records which finite chart is being used, not an impulse delivered to an object and not a discontinuity of the oriented-frame atlas.

No single projective coordinate covers the complete cycle. The explicit relation between the continuous angle, the scale ratio q_i, and the reciprocal rule is derived in Section 3.3.4. The important distinction is that q_i locates the field orientation, whereas β_X is the within-sector velocity contained in $\Lambda_{(X)}$. A side particle can exist without traversing the geometry orbit. Ordinary matter may also enter a neighbouring geometry chart without becoming a side particle, but whether it reaches that chart's centre is decided by its proper-time equation rather than by the orientation diagram.

3.2.11 From a uniform cone field to curvature

Up to this point every statement concerns one flat geometry. The cone axis may be boosted, exchanged with an inherited spatial direction, or reflected, yet the same choice can be repeated uniformly at neighbouring events. All these operations change the projection of the cone field without changing the fact that its axes can be made parallel throughout the region. Sector orientation alone therefore cannot be a source of gravity.

The passage to curved spacetime begins only when the local construction can no longer be extended uniformly. Each individual event still admits the four orientations and the same null boundary, but the active axes selected at neighbouring events need not remain parallel under transport. If their variation can be removed by one coordinate change or one Lorentz transformation over the whole neighbourhood, the geometry is still flat. If transport along different paths gives different relative orientations and neighbouring geodesics acquire tidal acceleration, the cone field is curved.

This is the precise bridge to general relativity. The sector construction supplies the possible local clock axes and the flat rest configuration against which their variation is compared. The connection describes how the axes change from event to event, and the Riemann tensor decides whether that change is merely a frame choice or genuine curvature. The next section therefore does not add gravity to four separate spaces. It projects one curved metric and one curvature tensor onto the four local orientations already derived in flat spacetime. Particles with different s_X perform different tetrad projections of that same tensor; none of them creates a separate curvature field.

The same bridge explains why regularity is required rather than merely hoped for. If the active curvature diverged, neighbouring timelike geodesics could acquire unbounded relative acceleration and no regular sector-rest neighbourhood would remain in which the universal null limit could be applied to arbitrary objects. A sector continuation is therefore physically complete only when its active metric and invariant curvature stay finite. The causal construction identifies the necessary chart; the tensor calculation in the next sections decides whether that chart actually satisfies the requirement.

3.3 Curved spacetime: active scales and sector charts

The flat orbit determines which local clock orientations must remain available. Curvature adds a new question: which proper spatial mode is being focused toward a null chart boundary? No choice of coordinate components can answer this invariantly. The construction must live on the observer's spatial rest bundle and must then be embedded back into the full tangent bundle.

This section concerns the sector n_g of the geometry. A particle's s_X does not choose the metric domain and does not alter the tensors used to regularise it. Once the active geometric coframe has been found, particle readouts are obtained from Eq. (3.12); all of them resolve the same curvature.

The order of the construction is fixed. A congruence determines its proper deformation scales; the solution determines the corresponding critical scales; a tensorial eigenproblem selects the active direction; and that direction is paired with the local clock. Only after this geometric identification is the reciprocal chart of the same GR solution selected. No operator introduced below is an extra metric deformation.

3.3.1 The rest bundle and the tensor of proper scales

Let u^μ be the unit clock field of the active geometry sector. Its local rest space and projector are

$$u^\mu u_\mu = -1, \qquad h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu, \qquad h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \qquad h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \quad (3.23)$$

All scale tensors below are endomorphisms of $u^\perp$: contraction with u^μ on either index vanishes. This makes their eigenvalues proper spatial scales rather than coordinate coefficients.

Let ξ_0^μ and ξ^μ be spatial separation vectors between neighbouring worldlines of a reference congruence. A spatial Jacobi tensor $\mathcal{D}^\mu{}_\nu$ relates them,

$$\xi^\mu = \mathcal{D}^\mu{}_\nu \xi_0^\nu, \qquad \mathcal{D}^\mu{}_\nu = h^\mu{}_\rho \mathcal{D}^\rho{}_\sigma h^\sigma{}_\nu. \quad (3.24)$$

Its positive spatial Cauchy–Green tensor is defined without reference to a component array by

$$C_{\mu\nu} = h_{\rho\sigma} \mathcal{D}^\rho{}_\mu \mathcal{D}^\sigma{}_\nu, \qquad C^\mu{}_\nu = h^{\mu\rho} C_{\rho\nu}. \quad (3.25)$$

The proper-scale tensor $L^\mu{}_\nu$ is the unique positive, $h_{\mu\nu}$-self-adjoint spatial tensor satisfying

$$L^\mu{}_\rho L^\rho{}_\nu = C^\mu{}_\nu, \qquad L^\mu{}_\nu = h^\mu{}_\rho L^\rho{}_\sigma h^\sigma{}_\nu, \qquad h_{\mu\rho} L^\rho{}_\nu = h_{\nu\rho} L^\rho{}_\mu. \tag{3.26}$$

Let $n^\mu_{(i)}$ be an orthonormal spatial eigenframe and define its rank-one spectral projectors by

$$\Pi_{(i)}{}^\mu{}_\nu = n^\mu_{(i)} n^{(i)}_\nu, \qquad \Pi_{(i)}{}^\mu{}_\rho \Pi_{(j)}{}^\rho{}_\nu = \delta_{ij} \Pi_{(i)}{}^\mu{}_\nu, \qquad \sum_i \Pi_{(i)}{}^\mu{}_\nu = h^\mu{}_\nu. \tag{3.27}$$

Then the proper scales are the scalar eigenvalues obtained by contraction,

$$L_i = n^{(i)}_\mu L^\mu{}_\nu n^\nu_{(i)}, \qquad L^\mu{}_\nu = \sum_i L_i \Pi_{(i)}{}^\mu{}_\nu, \qquad L_i > 0. \tag{3.28}$$

Rigid rotation carried by $\mathcal{D}^\mu{}_\nu$ is therefore separated from the invariant linear scales encoded by the positive tensor $L^\mu{}_\nu$.

The orientation of the clock and of the principal eigendirections is part of the metric data rather than an additional sector variable. Relative to any regular reference Lorentz frame $\{\bar{u}^\mu, \bar{n}^\mu_{(i)}\}$, write locally

$$\begin{aligned} u^\mu(\varpi) &= \mathcal{O}^\mu{}_\nu(\varpi_{\rho\sigma})\, \bar{u}^\nu, \qquad & n^\mu_{(i)}(\varpi) &= \mathcal{O}^\mu{}_\nu(\varpi_{\rho\sigma})\, \bar{n}^\nu_{(i)}, \\ \Pi_{(i)}{}^\mu{}_\nu(\varpi) &= n^\mu_{(i)}(\varpi) n^{(i)}_\nu(\varpi), \qquad & L^\mu{}_\nu &= \sum_i L_i \Pi_{(i)}{}^\mu{}_\nu(\varpi). \end{aligned} \tag{3.29}$$

Here $\varpi_{\rho\sigma} = -\varpi_{\sigma\rho}$ and $0 \le \rho < \sigma \le n-1$, so the full orientation contains $n(n-1)/2$ spacetime-plane angles. The subsets ϖ_{0i} and ϖ_{ij} respectively locate time–space and spatial planes. They are obtained from the decomposition of the original metric and may vary with position, as in Kerr or an inhomogeneous LTB model, but they do not themselves perform the four-sector turn. The sector angle introduced below acts only in the single time–principal plane selected by the active eigenvalue ratio.

3.3.2 Critical scales and the active tensor mode

The deformation tensor alone does not say where a sector chart ends. The given GR solution must also provide a positive, $h_{\mu\nu}$-self-adjoint spatial critical-scale tensor $G^\mu{}_\nu$ with the same dimensions as $L^\mu{}_\nu$,

$$G^\mu{}_\nu = h^\mu{}_\rho G^\rho{}_\sigma h^\sigma{}_\nu, \qquad h_{\mu\rho} G^\rho{}_\nu = h_{\nu\rho} G^\rho{}_\mu. \tag{3.30}$$

The active modes are defined by the covariant generalised eigenproblem

$$G^\mu{}_\nu n^\nu_{(i)} = q_i L^\mu{}_\nu n^\nu_{(i)}. \tag{3.31}$$

When $L^\mu{}_\nu$ and $G^\mu{}_\nu$ possess a common orthonormal spatial eigenframe, define

$$G_i = n^{(i)}_\mu G^\mu{}_\nu n^\nu_{(i)}, \qquad G^\mu{}_\nu = \sum_i G_i \Pi_{(i)}{}^\mu{}_\nu, \qquad q_i = \frac{G_i}{L_i}. \tag{3.32}$$

Thus every scalar scale used later is obtained by contraction of a spatial tensor with its spectral projector. The tensors themselves, rather than a choice of components, identify which ruler direction becomes critical. The next subsection pairs that tensorially selected direction with the clock.

The number q_i is a geometric scale ratio, not a material velocity. The active chart contains $q_i < 1$; $q_i = 1$ is its null boundary; and a formal value $q_i > 1$ must be represented from the neighbouring sector centre. Schwarzschild reduces this construction to one areal scale, LTB separates longitudinal and transverse modes, FLRW makes all three modes degenerate, and Kerr selects a position-dependent principal plane.

3.3.3 Covariant embedding in the active time–space plane

Let $n^\mu_{(k)}(\varpi)$ be the unit eigenvector of the mode that reaches $q_k = 1$. Its dependence on $\varpi_{\rho\sigma}$ is fixed by the decomposition of the original metric in Eq. (3.29). The projector onto the active plane generated by $u(\varpi)$ and $n_{(k)}(\varpi)$ is

$$P_{(k)}{}^{\mu}{}_{\nu}(\varpi) = -u^\mu(\varpi)u_\nu(\varpi) + n^\mu_{(k)}(\varpi)n^{(k)}_\nu(\varpi). \tag{3.33}$$

For readability the labels (k) and (ϖ) are suppressed in the next two equations. No angle $\varpi_{\rho\sigma}$ is inverted or advanced by a quarter turn; it only locates the principal plane on which the sector construction acts. Its complement $\delta^\mu{}_\nu - P^\mu{}_\nu$ is the transverse two-dimensional screen. The projector makes explicit which old ruler is paired with the clock and which two directions remain spectators.

The covariant quarter-turn generator on that plane is

$$J^\mu{}_\nu = u^\mu n_\nu + n^\mu u_\nu, \qquad J^\mu{}_\rho u^\rho = -n^\mu, \quad J^\mu{}_\rho n^\rho = u^\mu, \quad J^\mu{}_\rho J^\rho{}_\nu = -P^\mu{}_\nu. \tag{3.34}$$

It vanishes on the transverse screen. Let ϕ denote the local orientation parameter measured inside this selected plane. Its orientation tensor is

$$\mathcal{R}(\phi)^\mu{}_\nu = \delta^\mu{}_\nu - P^\mu{}_\nu + \cos\phi\, P^\mu{}_\nu + \sin\phi\, J^\mu{}_\nu. \tag{3.35}$$

At $\phi = 0$ it is the identity on the active plane. At $\phi = \pi/2$, u^μ is carried to $-n^\mu$ and n^μ to u^μ; the local clock and active ruler exchange roles while the transverse screen is unchanged. The parameter ϕ defines only the local action of the tensor. Its value in each sector chart is fixed by the single global angle introduced below.

Equations (3.33)–(3.35) are already four-dimensional tensor statements; no separate component representation is needed. The active geometric mode identifies one n^μ in the old rest space as the new clock direction. This geometric eigenvector must not be confused with an additional particle label. The remaining two-dimensional screen stays spatial, so changing its angular coordinates cannot reverse the active time direction.

The tensor $\mathcal{R}(\phi)^\mu{}_\nu$ records orientation and the selected plane. It is not an ordinary Lorentz boost and is not multiplied into a completed GR coframe to create a new metric. Such an operation would alter the Einstein tensor and manufacture an effective source even in a vacuum solution. Ordinary particle motion inside each active sector remains described by $\Lambda_{(X)}$ from Section 3.2.2.

3.3.4 One continuous angle and four finite projective charts

A single global angle

$$\Theta \in \mathbb{R}/(2\pi\mathbb{Z}) \tag{3.36}$$

follows the selected clock–ruler plane through the complete four-sector orbit. Away from a null boundary, the active geometry sector is the sector whose rest centre is nearest to the current value of Θ: Choose a real lift of Θ on each traversal of the orbit and define

$$\widehat{n}(\Theta) = \left\lfloor \frac{2\Theta}{\pi} + \frac{1}{2} \right\rfloor \in \mathbb{Z}, \qquad n_g(\Theta) = \widehat{n}(\Theta) \pmod 4. \tag{3.37}$$

The integer $\widehat{n}$ records the chosen lift, whereas n_g is only the physical $\mathbb{Z}_4$ sector label. At a null boundary either one-sided chart may be used. They describe the same limiting orientation.

The local parameter appearing in Eq. (3.35) is the value of this one angle measured from the active rest centre,

$$\phi(\Theta) = \Theta - \frac{\pi}{2}\widehat{n}(\Theta), \qquad |\phi(\Theta)| \le \frac{\pi}{4}. \tag{3.38}$$

Thus $\mathcal{R}(\Theta)^{\mu}{}_{\nu} = \mathcal{R}(\phi(\Theta))^{\mu}{}_{\nu}$. The finite projective variable of the active principal mode is

$$q_k(\Theta) = |\tan\phi(\Theta)|, \qquad 0 \le q_k(\Theta) \le 1. \tag{3.39}$$

It vanishes at every sector rest centre and reaches one at every common null boundary. After the boundary, Eq. (3.37) selects the neighbouring chart, so the same global function decreases from one back toward zero. No second angular field and no separate pair of sector functions is required.

The fields $\varpi_{\rho\sigma}(x)$ retain the orientation supplied by the original metric. They determine the principal eigenvectors and projectors, whereas Θ advances only the active time–principal plane selected by q_k. The angle is neither a rapidity nor a particle velocity. Particle motion inside a fixed sector remains described by β_X and $\Lambda_{(X)}{}^{\mu}{}_{\nu}$.

3.3.5 Reciprocal normalisation of the active solution

At $q_i = 1$ two adjacent charts meet at the same critical scale. Beyond that boundary the same mode must be measured from the rest centre of the new sector. The tensorial reciprocal rule fixes the boundary and exchanges a large old ratio with a small new one. Since $L^{\mu}{}_{\nu}$ is a positive endomorphism of $u^{\perp}$, define its restricted inverse tensor spectrally by

$$(L_{\perp}^{-1})^{\mu}{}_{\nu} = \sum_i \frac{1}{L_i}\Pi_{(i)}{}^{\mu}{}_{\nu}, \qquad (L_{\perp}^{-1})^{\mu}{}_{\rho}L^{\rho}{}_{\nu} = L^{\mu}{}_{\rho}(L_{\perp}^{-1})^{\rho}{}_{\nu} = h^{\mu}{}_{\nu}, \qquad (L_{\perp}^{-1})^{\mu}{}_{\nu}u^{\nu} = 0. \tag{3.40}$$

The reciprocal proper-scale tensor is then

$$\widetilde{L}^{\mu}{}_{\nu} = G^{\mu}{}_{\rho}(L_{\perp}^{-1})^{\rho}{}_{\sigma}G^{\sigma}{}_{\nu}. \tag{3.41}$$

Projection on a common eigendirection gives the scalar consequence $\widetilde{L}_i = G_i^2/L_i$. Applying the rule twice returns the original scale algebraically. Moreover, Eq. (3.31) shows that the same eigenvector has the reciprocal chart ratio $\widetilde{q}_i = q_i^{-1}$. Thus the regime described by an arbitrarily large old ratio becomes the zero-ratio rest centre of the neighbouring chart.

The rule acts on the variables with which a known GR solution is evaluated, not as a conformal or componentwise correction to its finished metric. Schematically,

$$g^{(n_g)}_{\mu\nu} = \mathrm{i}^{2n_g}\,\mathcal{I}^{*}_{n_g}\bar{g}_{\mu\nu}[L^{\mu}{}_{\nu}]. \tag{3.42}$$

The pullback evaluates the GR solution in the active proper scales. The constant factor

$\mathrm{i}^{2n_g} = (-1)^{n_g}$ records the sector sign of its tetrad representation and leaves the Levi–Civita connection and null set unchanged. At the critical surface $L_i = G_i$ the unsigned proper scale agrees in both charts. For a single scale $\widetilde{X} = X_h^2/X$, the Jacobian equals -1 at $X = X_h$; the radial coordinate orientation therefore reverses while invariant curvature remains continuous.

This sign reversal is the differential expression of the clock–ruler exchange. Along a smooth worldline, $\mathrm{d}\widetilde{X}/\mathrm{d}\tau = -(X_h^2/X^2)\mathrm{d}X/\mathrm{d}\tau$. The sign of one coordinate component changes at the transition, but the tangent itself is carried by the Jacobian of the chart map and remains timelike. There is no reflection of the body and no physical crossing of its null cone.

The same construction can be written directly with Θ without changing the field equations. A principal scale with critical value X_h is represented on the complete orbit by the single cyclic function

$$X(\Theta) = \frac{X_h}{q_k(\Theta)} = \frac{X_h}{\left|\tan(\Theta - \frac{\pi}{2}\widehat{n}(\Theta))\right|}. \tag{3.43}$$

It approaches infinity at every sector rest centre and equals X_h at every null boundary. The same profile repeats after a quarter orbit, $X(\Theta + \pi/2) = X(\Theta)$, while the sector orientation continues around the full cycle.

Let $x^\mu(\Theta, y)$ be the coordinate map of the known GR solution after its active scale has been replaced by $X(\Theta)$, and let $Y^M = (\Theta, y)$. The complete sector coframe and the one angular metric are

$$\vartheta^A{}_M(\Theta, y) = \mathrm{i}^{n_g(\Theta)}\bar{\vartheta}^A{}_\mu(x(\Theta, y))\,\frac{\partial x^\mu}{\partial Y^M}, \tag{3.44}$$

$$g_{MN}(\Theta, y) = \mathrm{i}^{2n_g(\Theta)}\bar{g}_{\mu\nu}(x(\Theta, y))\,\frac{\partial x^\mu}{\partial Y^M}\frac{\partial x^\nu}{\partial Y^N}. \tag{3.45}$$

All mixed terms generated by the full Jacobian are retained. The angles $\varpi_{\rho\sigma}(x)$ remain inside the metric-derived principal projectors and the original GR coframe; only the active scale is reparametrised by Θ.

The anti sectors reverse the inherited coframe orientation. Since $\widehat{n}(\Theta + \pi) = \widehat{n}(\Theta) + 2$, and therefore $n_g(\Theta + \pi) = n_g(\Theta) + 2 \pmod 4$, one has

$$\begin{aligned} \vartheta^A{}_M(\Theta + \pi, y) &= -\vartheta^A{}_M(\Theta, y), \\ \mathrm{d}x^\mu(\Theta + \pi, y) &= -\mathrm{d}x^\mu(\Theta, y), \\ g_{MN}(\Theta + \pi, y) &= g_{MN}(\Theta, y). \end{aligned} \tag{3.46}$$

The coordinate reversal comes from the coframe factor $\mathrm{i}^{n_g+2} = -\mathrm{i}^{n_g}$. The metric is unchanged after two sector turns because $\mathrm{i}^{2(n_g+2)} = \mathrm{i}^{2n_g}$.

3.3.6 Geometry coframe and particle readout

Let $\mathcal{I}_{n_g} : U_{n_g} \to M$ be the active geometry chart and $\bar{\vartheta}^A[L]$ a coframe of the same GR solution evaluated in its active scale tensor. The geometric sector coframe is

$$\vartheta^A_{(g)} = \mathrm{i}^{n_g}\,\mathcal{I}^*_{n_g}\bar{\vartheta}^A[L]. \tag{3.47}$$

The pullback selects a regular chart of the field, i^{n_g} selects its orientation component, and the coframe of the GR solution already contains the local geometric deformation. None of these operations creates another geometry.

The particle operation is applied only afterward. In Eq. (3.12), the reference coframe is then replaced by the pulled-back GR coframe in Eq. (3.47); the final phase is still i^{N_X}, with $N_X = n_g + s_X \pmod 4$. No additional formula is needed because this is exactly the ordered composition already defined in flat spacetime. The result is a transformed differential, not a second coframe field assigned to the manifold. Changing n_g first changes the geometry coframe; changing $\Lambda_{(X)}$ changes one particle's motion; and the conserved s_X fixes which extended Lorentz transformation that particle uses.

The metric still belongs only to the geometry and is the tensor already defined in Eq. (3.42). Its sign alternation is the quadratic trace of the sector orientation. It distinguishes even from odd sectors but not the two opposite orientations within either parity class; the missing information remains in $\vartheta^A_{(g)}$. No analogous $g^{(X)}_{\mu\nu}$ is introduced.

Before the orientation sign is attached, every physical or geometric field has one value on M and only chart pullbacks

$$\mathcal{F}_{(n_g)} = \mathcal{I}^*_{n_g}\mathcal{F}, \qquad \mathcal{F} \in \{g_{\mu\nu}, T_{\mu\nu}, R^{\mu}{}_{\nu\rho\sigma}, \ldots\}. \tag{3.48}$$

The metric readout $g^{(n_g)}$ is obtained from its pullback by the single orientation factor displayed in Eq. (3.42); it is not a second freely specifiable tensor.

3.3.7 Curvature and unchanged field equations

Let $\mathcal{I}$ denote an adjacent reciprocal chart map and let $H^A{}_B$ be the corresponding coframe transition. The Cartan connection has the usual inhomogeneous change-of-frame term, whereas its curvature two-form transforms tensorially:

$$\Omega_{(n_g+1)}{}^A{}_B = H^A{}_C\,\mathcal{I}^*\Omega_{(n_g)}{}^C{}_D (H^{-1})^D{}_B. \tag{3.49}$$

This is ordinary tensorial covariance, not a new sector curvature law [54, 63].

Scalar contractions transform by the same pullback. In particular,

$$K_{(n_g)} = \mathcal{I}^*_{n_g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{3.50}$$

Adjacent sectors give the same value at the same boundary event. A normal derivative may reverse sign because the reciprocal Jacobian reverses the normal orientation; this is not a jump in the curvature tensor.

Each particle resolves $R^{\mu}{}_{\nu\rho\sigma}$ through its transformation $\mathcal{G}_{(X)}$; only its tetrad components change. The field equation, timelike geodesic equation and null condition retain their standard form:

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}, \qquad U^{\nu}_X \nabla_\nu U^{\mu}_X = 0, \qquad g_{\mu\nu}k^{\mu}k^{\nu} = 0. \tag{3.51}$$

There is no sector-dependent Einstein tensor or independently adjustable stress tensor. Different s_X values change the temporal–spatial split measured by a particle, not the scalar invariants. In particular, inserting $\mathcal{R}(\alpha)^{\mu}{}_{\nu}$ into a completed coframe while keeping η_{AB} fixed would create a different metric. The orientation tensor selects the active plane; the pullback and reciprocal scale select the regular representative of the original solution.

The flat-spacetime kinematic meaning has already been fixed in Sections 3.2.2–3.2.6: a sector change re-resolves one tangent rather than replacing it. In curved spacetime the only additional statement is the tensorial chart law. On an overlap,

$$U_X^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} U_X^\nu, \qquad k^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} k^\nu, \tag{3.52}$$

while s_X and the physical future orientation remain unchanged. The active coframe decides which component is temporal. It does not redirect the worldline, and an inherited differential such as $\mathrm{d}X = c\,\mathrm{d}t$ may become spatial without becoming unreal or acquiring the power to reverse proper time.

3.3.8 Reciprocal centres as asymptotic boundaries

Two geometrically different limits must now be kept apart. The first is the common null boundary of adjacent finite charts. A horizon-regular frame can carry a causal curve through this boundary in finite proper or affine parameter. The second is the reciprocal centre approached after the neighbouring chart has become active. Smooth matching at the first boundary does not imply that the second limit is a finite event.

Curvature regularity and causal accessibility are therefore independent questions. Scalar invariants decide whether the reciprocal domain is singular; the integral of proper time decides whether a timelike curve reaches its centre, and an affine parameter provides the corresponding null test. A reciprocal centre can be regular and asymptotically flat while remaining at infinite timelike proper time and infinite null affine parameter. It is then an asymptotic boundary of the active chart rather than a finite-parameter point of the corresponding timelike or null geodesic.

The constant orientation sign does not alter this distinction. For any nonzero constant C, $\Gamma^\mu{}_{\nu\rho}[Cg] = \Gamma^\mu{}_{\nu\rho}[g]$, because the factor in the inverse metric cancels the factor in the metric derivatives. Nevertheless proper time must be normalised with the active representative $g^{(n_g)} = \mathrm{i}^{2n_g}\mathcal{I}^*_{n_g} g$. In an odd sector the old timelike normalisation cannot simply be copied from sector 0. The examples below apply this test explicitly rather than treating vanishing curvature as a proof of completeness.

3.3.9 Several active modes

When the principal directions of $L^\mu{}_\nu$ and $G^\mu{}_\nu$ commute, the construction reduces locally to independent two-planes and Eq. (3.35) can be applied mode by mode. If the principal planes rotate into one another, their generators need not commute and the orientation must be transported in path order. A global existence-and-uniqueness analysis for that general case lies outside this paper. Schwarzschild, the two LTB modes, isotropic FLRW, and the principal plane of Kerr are sufficient to display the construction without introducing another formalism that is not used in the examples.

3.4 Sector-normalised general-relativistic examples

The examples now apply the same physical test to four different geometries. First the Jacobi or principal structure identifies the proper scale that is being focused. The known general-relativistic solution then fixes the critical scale at which the inherited clock reaches its null boundary. Only at that boundary is the reciprocal chart introduced, and only after the chart has been identified are its curvature and geodesics evaluated.

This order matters. The reciprocal map is not chosen because a curvature formula happens to diverge, and none of the examples receives a modified Einstein equation. Each calculation asks whether another sector chart of the same solution replaces the singular extrapolation by a regular physical domain. Finite curvature answers only the first half of that question. Proper time for matter and affine parameter for light decide whether the regular centre is an event that can be crossed or an asymptotic end of the worldline.

The examples track the sector of the geometry. Unless stated otherwise, their reference matter remains ordinary relative to the active field. Other particle sectors may be read in the same chart, but they do not alter its curvature. This keeps three issues separate throughout the section: the orientation of the field, the sector class of a particle, and the part of the atlas that a causal trajectory can actually reach.

Every general sector metric below retains its geometry phase. Thus a metric written for an arbitrary active sector has the form $\mathrm{d}s^2(\Theta) = \mathrm{i}^{2n_g(\Theta)}\mathrm{d}\bar{s}^2(\Theta)$. Whenever a particular sector is displayed separately, its numerical sign is written directly as 1 or -1, rather than as a power of i.

3.4.1 Schwarzschild: from the ordinary exterior to side rest

Begin with the standard exterior Schwarzschild line element [57]

$$\mathrm{d}s^2_{(0)} = -f(R)c^2\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{f(R)} + R^2\mathrm{d}\Omega^2, \qquad f(R) = 1 - \frac{r_s}{R}, \quad r_s = \frac{2G_N M}{c^2}, \tag{3.53}$$

where R is the areal radius and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. The coordinate coefficients in Eq. (3.53) are singular at $R = r_s$, but the horizon itself is regular. The sector transition must therefore be performed in a horizon-regular coframe, such as an ingoing Eddington–Finkelstein frame, rather than inferred from the singular diagonal coordinates [47, 48].

Spherical symmetry reduces the sector test to the areal scale. On the observer's rest space its degenerate tensor representative may be written as $L^\mu{}_\nu = Rh^\mu{}_\nu$, while the Schwarzschild solution supplies $G^\mu{}_\nu = r_s h^\mu{}_\nu$. Hence every unit direction on the old spatial sphere has the same eigenvalue. This tensorial degeneracy means that any selected spatial ray reaches the same critical areal scale and can be paired with the clock by Eq. (3.35). The generalised eigenproblem reduces to

$$q(R) = \frac{r_s}{R}. \tag{3.54}$$

This is a geometric compactness ratio, not the measured velocity of a falling body. It is smaller than unity throughout the ordinary exterior, vanishes at the asymptotically flat rest centre, and reaches unity at the horizon. The horizon is therefore the null boundary of the ordinary chart.

The threefold degeneracy is physically important. It says that the old spatial three-space supplies a sphere of admissible side-sector clock directions; it does not create three times on one observer's worldline. A radial worldline determines a unit vector s^μ in that family through its local tangent and the spherical symmetry. The tensor $J^\mu{}_\nu = u^\mu s_\nu + s^\mu u_\nu$ then rotates the plane identified by $P^\mu{}_\nu = -u^\mu u_\nu + s^\mu s_\nu$. At the quarter-turn the old clock is read as the new ruler and the chosen old ruler as the new clock, while the transverse screen is unchanged. This statement is made in a horizon-regular frame. It does not add a term to the Schwarzschild metric and does not manufacture a second vacuum solution.

At the first horizon the geometry changes from the ordinary to the first side chart; the particle class does not change. Ordinary infalling matter therefore remains ordinary relative to the local field, even though its total orientation looks side-directed when described by the exterior coframe. A pre-existing side particle likewise retains its own class. In both cases the chart changes before the unchanged particle transformation is applied.

The reciprocal Schwarzschild domain is fixed only after the clock–ruler exchange is read geometrically.

The axis exchange must be interpreted before introducing another formula. In sector 0, t is the time direction and the old (x, y, z) space is spatial. In sector 1, every direction in that old three-space is an admissible time direction, while the old time direction becomes the spatial coordinate $X = ct$. Spherical symmetry labels the family of side-time rays by (r, θ, φ): the angles label a ray and r parametrises evolution along it. Sector 0 therefore uses $+t$ as its clock direction and treats (x, y, z) as space. Sector 1 reverses these roles at the level of the projection: the possible clock directions form the family $n^\mu \in u^\perp$ with $h_{\mu\nu}n^\mu n^\nu = 1$, while $X = ct$ is spatial. The essential point is that not only one distinguished r direction can supply the side clock. The whole old spatial $\mathbb{R}^3$ supplies the family of possible time axes. For a radial particle the actual trajectory has fixed $(\theta, \varphi) = (\theta_0, \varphi_0)$ and thereby determines one ray; no second independent clock choice is appended to the particle. Its single proper time is parametrised along that ray by r, and it does not evolve simultaneously along every member of the family.

The quantity $R(r)$ measures the areal radius of the sphere formed by all side-time directions at the same sector value r. It is not another name for the selected ray parameter. After the exchange, r labels evolution along a chosen old spatial ray, while $X = ct$ is spatial. Consequently the zero value of r is a value of the new time coordinate, not the origin of the old spatial radius. This change of roles is what makes an infinite-radius sphere at that endpoint geometrically meaningful rather than contradictory.

It remains to determine the size $R(r)$ of that sphere. Restricting Eq. (3.41) to any one of the degenerate eigenvectors gives $\widetilde{L} = r_s^2/L$. In the ordinary chart $L = r$, so the neighbouring areal scale is fixed, rather than postulated, as

$$R(r) = \frac{r_s^2}{r}, \qquad 0 < r \leq r_s, \tag{3.55}$$

which satisfies $R(r_s) = r_s$ and $\mathrm{d}R/\mathrm{d}r|_{r_s} = -1$. The first equality identifies the same horizon event in both charts; the second records their opposite radial orientations. Two operations must not be conflated. The axis exchange explains why the zero endpoint of the side coordinate is a temporal sector centre and why the old time coordinate is now spatial. The reciprocal chart relation then gives the value of the simultaneous areal scale there. Together, and only together, they imply that this coordinate endpoint is a time limit at which the sphere of side-time directions has infinite areal radius. This does not identify zero with infinity: r and R are coordinates with different roles in the two projections. It would be incorrect to read the zero value of r as the centre of ordinary spherical coordinates after the exchange.

First relabel the radial coordinate in the underlying Schwarzschild metric. Since $\mathrm{d}R = -r_s^2\mathrm{d}r/r^2$, the active sector-1 representative is

$$\begin{aligned} \mathrm{d}s^2_{(1)} &= -\mathcal{I}_1^* \mathrm{d}s^2_{(0)} \\ &= (1 - r/r_s)c^2\mathrm{d}t^2 - \frac{r_s^4\,\mathrm{d}r^2}{r^4(1 - r/r_s)} - \frac{r_s^4}{r^2}\mathrm{d}\Omega^2. \end{aligned} \tag{3.56}$$

The unsigned pullback $\mathcal{I}_1^* g^{(0)}$ remains a vacuum solution wherever the chart is regular, and multiplication by the constant sector sign preserves that connection and curvature: $G_{\mu\nu}[\mathcal{I}_1^* g^{(0)}] = \mathcal{I}_1^* G_{\mu\nu}[g^{(0)}] = 0$. In terms of $X = ct$ and the areal variable R, the selected clock–ruler plane of Eq. (3.56) is

$$\mathrm{d}s^2_{(1),\parallel} = f(R)\mathrm{d}X^2 - \frac{\mathrm{d}R^2}{f(R)}. \tag{3.57}$$

The old time X is now spatial and increasing R is the active timelike direction. Multiplication by the constant sign does not change the vacuum connection or curvature; it changes which direction is used in the proper-time normalisation.

The same construction can be written on the single global orientation angle introduced in Sec. 3.3.

For this metric the functions entering the angular representation are

$$\begin{aligned} \phi(\Theta) &= \Theta - \frac{\pi}{2}\widehat{n}(\Theta), & q(\Theta) &= |\tan\phi(\Theta)|, \\ R(\Theta) &= \frac{r_s}{q(\Theta)}, & R_{,\Theta} &= -\frac{r_s \tan\phi\, \sec^2\phi}{|\tan\phi|^3}, \\ \epsilon_R(\Theta) &= -\frac{R_{,\Theta}}{|R_{,\Theta}|}, & \mathrm{d}R &= R_{,\Theta}\mathrm{d}\Theta. \end{aligned} \tag{3.58}$$

The derivative formula is used on each open angular chart, away from its rest centre. The one-sided value of ϵ_R selects the regular ingoing or outgoing Painlevé–Gullstrand representative. After expanding the square, the complete metric is

$$\begin{aligned} \mathrm{d}s^2_S(\Theta) = \mathrm{i}^{2n_g(\Theta)}\Bigg[&-c^2\left(1 - \frac{r_s}{R(\Theta)}\right)\mathrm{d}T^2 + 2\epsilon_R(\Theta)c\sqrt{\frac{r_s}{R(\Theta)}}R_{,\Theta}\,\mathrm{d}T\mathrm{d}\Theta \\ &+ R_{,\Theta}^2\mathrm{d}\Theta^2 + R^2(\Theta)\mathrm{d}\Omega^2\Bigg]. \end{aligned} \tag{3.59}$$

The function $R(\Theta)$ equals r_s at every horizon and diverges at every sector rest centre. The metric repeats cyclically, while the opposite-sector coordinate orientation is supplied by Eq. (3.46).

The reciprocal endpoint can now be tested invariantly.

The angular coefficient in Eq. (3.56) has the invariant meaning of an area: at fixed side time, $A_{S^2} = 4\pi r_s^4/r^2$. As the side coordinate approaches zero, this area describes a sphere whose areal radius grows without bound. A curve with fixed X, θ, φ selects one point on each successive sphere; it neither travels around that sphere nor traverses its radius. The changing areal radius is a geometric field sampled along the curve.

Regularity follows from the curvature of the one Schwarzschild metric, not from the axis exchange itself. The Newman–Penrose amplitude and the Kretschmann scalar become

$$\begin{aligned} \Psi_2(R) &= -\frac{r_s}{2R^3}, & \Psi_{2(1)}(r) &= -\frac{r^3}{2r_s^5}, \\ K(R) &= \frac{48G_N^2M^2}{c^4R^6} = \frac{12r_s^2}{R^6}, & K_{(1)}(r) &= \frac{12r^6}{r_s^{10}}. \end{aligned} \tag{3.60}$$

The powers of r arise solely from substituting $R = r_s^2/r$. At the common horizon both charts give $K = 12/r_s^4$, while their normal derivatives have opposite signs because $\mathrm{d}R/\mathrm{d}r = -1$ there. At the zero endpoint of the side coordinate, both curvature quantities vanish. The reciprocal chart has therefore replaced the extrapolation toward zero areal radius by a domain in which the areal radius grows without bound and the geometry becomes flat. This proves curvature regularity, but it does not yet say whether the limiting endpoint is reached in finite proper time.

Curvature regularity does not determine causal distance. The timelike normalization must therefore be derived from the active side metric in Eq. (3.57), rather than copied from the ordinary Schwarzschild chart. Since X is cyclic, the momentum in this old time direction, now spatial, is conserved: $P = f(R)\mathrm{d}X/\mathrm{d}\tau$. Using $\mathrm{d}s^2_{(1),\parallel} = -c^2\mathrm{d}\tau^2$ then gives

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 = P^2 + c^2 f(R) = c^2\left(p^2 + 1 - \frac{r_s}{R}\right), \qquad p := \frac{P}{c}. \tag{3.61}$$

This sign is essential: P is not the conserved energy ε of a sector-0 timelike geodesic.

Let $A = 1 + p^2$. Integrating the outgoing side branch from $R_0 \geq r_s$ gives

$$\begin{aligned}\tau(R) - \tau(R_0) = \frac{1}{c}\Bigg[&\frac{\sqrt{R(AR - r_s)}}{A} \\ &+ \frac{r_s}{A^{3/2}}\sinh^{-1}\sqrt{\frac{AR - r_s}{r_s}}\Bigg]_{R_0}^{R}.\end{aligned} \tag{3.62}$$

Here $\sinh^{-1}$ denotes the inverse hyperbolic sine. The integrand is regular there for $p \neq 0$; for $p = 0$ its square-root divergence is integrable. Hence every horizon-to-finite-R interval is finite. The infinite-distance statement can be proved without an asymptotic expansion. Choose any finite point on the outward side branch and denote its areal radius and proper time by R_1 and τ_1. Define the two finite positive speeds

$$v_- := c\sqrt{1 + p^2 - \frac{r_s}{R_1}}, \qquad v_+ := c\sqrt{1 + p^2}. \tag{3.63}$$

The first is the radial growth rate at the chosen point; the second is the largest value approached as the reciprocal areal radius increases. Since r_s/R decreases monotonically on this branch, Eq. (3.61) means exactly that

$$0 < v_- \leq \frac{\mathrm{d}R}{\mathrm{d}\tau} \leq v_+ < \infty, \qquad R \geq R_1. \tag{3.64}$$

Equation (3.64) has a simple physical meaning: after the observer reaches R_1, the active areal scale keeps growing, its rate never falls to zero, and that rate never becomes infinite. Integrating over the elapsed proper time $\tau - \tau_1$ gives

$$R_1 + v_-(\tau - \tau_1) \leq R(\tau) \leq R_1 + v_+(\tau - \tau_1). \tag{3.65}$$

Thus $R(\tau)$ is trapped between two straight lines. The upper line is finite for every finite τ, so $R = \infty$ cannot be reached in finite proper time. The lower line grows without bound, so arbitrarily large finite areal radii are nevertheless reached as the proper time continues.

Using the reciprocal relation $r = r_s^2/R$ reverses the inequalities and gives

$$\frac{r_s^2}{R_1 + v_+(\tau - \tau_1)} \leq r(\tau) \leq \frac{r_s^2}{R_1 + v_-(\tau - \tau_1)}. \tag{3.66}$$

Equation (3.66) is the same statement in the side coordinate. Both bounding functions are positive at every finite proper time and both reach zero only in the limit of unbounded τ. Consequently the coordinate value $r = 0$ denotes the reciprocal asymptotic boundary, not an event encountered after a finite duration. The case $p = 0$ is included: after the finite interval from the horizon to any $R_1 > r_s$, the same bounds apply. Equation (3.62) already contains that side-rest case and no separate approximation is required.

The null calculation gives the corresponding affine-distance test.

Let $X = ct$, let λ be an affine parameter, and define the conserved quantity $\mathcal{E} = f(R)\,\mathrm{d}X/\mathrm{d}\lambda$. The radial null condition then reduces to

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda} = \pm\mathcal{E}, \qquad R(\lambda) = r_s + |\mathcal{E}|(\lambda - \lambda_h) \tag{3.67}$$

on the branch directed toward increasing R. The areal radius can therefore grow without bound only over an unlimited affine interval. Even light does not cross the zero endpoint of the side coordinate at finite affine parameter.

Orthonormal tidal components fall in proportion to the inverse cube of the areal radius. They therefore decay along every geodesic that approaches the boundary. For finite p, the Kretschmann scalar falls with the inverse sixth power of proper time and the orthonormal tidal scale with its inverse third power. Along radial light the scalar falls with the inverse sixth power of affine parameter. The side centre is consequently both asymptotically flat and geodesically complete in these radial directions.

Together the timelike and null results fix the physical status of the reciprocal boundary.

The algebraic atlas may contain a reflected side branch, but that branch is not the next finite-parameter segment of the geodesics just calculated. The zero endpoint of its side coordinate is another representation of asymptotic infinity, not a local event at which the first branch is glued to it. A worldline may cross the entrance horizon from the ordinary chart into the first side chart in finite time. It cannot continue through the asymptotic centre into the opposite-time chart at finite proper or affine parameter. In particular, ordinary infalling matter does not emerge from a white hole in the $-t$ sector. Algebraic reflection of the chart and physical transport of a worldline must not be identified.

3.4.2 LTB geometry: longitudinal and transverse sector modes

The Lemaître–Tolman–Bondi dust geometry is the simplest example in which a single spherical radius does not describe the full deformation [3, 60]. Using the mass coordinate F, write $R = R(t, F)$, $R_F = \partial R/\partial F$, and $\Gamma(F) = \sqrt{1 + 2E(F)}$. The comoving line element is

$$\mathrm{d}s_{(0)}^2 = -c^2\mathrm{d}t^2 + \frac{R_F^2}{\Gamma^2}\mathrm{d}F^2 + R^2\mathrm{d}\Omega^2. \tag{3.68}$$

The source dust is taken to have $s_X = 0$ relative to its local geometric coframe. This fixes its particle class but does not enter the directional scale tensors below. The proper separation of neighbouring shells is $\mathrm{d}\ell_\parallel = (R_F/\Gamma)\mathrm{d}F$, whereas both transverse scales are set by the areal radius R. Let $u^\mu = U^\mu/c$ be the unit dust clock field, s^μ the unit radial vector in its rest space, and $S^\mu{}_\nu = h^\mu{}_\nu - s^\mu s_\nu$ the projector on the spherical screen. Because F is used as the mass coordinate, the longitudinal eigenvalue is the proper shell-separation coefficient multiplying $\mathrm{d}F$, while the two transverse eigenvalues are areal scales. Introduce the radial and spherical spectral projectors

$$\Pi_\parallel{}^\mu{}_\nu = s^\mu s_\nu, \qquad \Pi_\perp{}^\mu{}_\nu = S^\mu{}_\nu = h^\mu{}_\nu - s^\mu s_\nu, \qquad \Pi_\parallel{}^\mu{}_\rho \Pi_\perp{}^\rho{}_\nu = 0. \tag{3.69}$$

The proper-scale and critical-scale tensors are then

$$\begin{aligned} L^\mu{}_\nu &= L_\parallel \Pi_\parallel{}^\mu{}_\nu + L_\perp \Pi_\perp{}^\mu{}_\nu, \\ G^\mu{}_\nu &= G_\parallel \Pi_\parallel{}^\mu{}_\nu + G_\perp \Pi_\perp{}^\mu{}_\nu, \end{aligned} \tag{3.70}$$

with eigenvalues obtained tensorially as

$$\begin{aligned} L_\parallel &= s_\mu L^\mu{}_\nu s^\nu = \frac{R_F}{\Gamma}, \qquad & G_\parallel &= s_\mu G^\mu{}_\nu s^\nu = 1, \\ L_\perp &= \frac{1}{2} S^\nu{}_\mu L^\mu{}_\nu = R, \qquad & G_\perp &= \frac{1}{2} S^\nu{}_\mu G^\mu{}_\nu = F. \end{aligned} \tag{3.71}$$

Hence the two invariant eigenvalue ratios are $q_\parallel = G_\parallel / L_\parallel = \Gamma / R_F$ and $q_\perp = G_\perp / L_\perp = F/R$. When the radial derivative of the areal radius vanishes, only the longitudinal shell-crossing projector becomes critical. When the areal radius itself vanishes, the degenerate transverse projector becomes critical instead. The distinction is entirely encoded by $\Pi_\parallel{}^\mu{}_\nu$ and $\Pi_\perp{}^\mu{}_\nu$, without introducing a component representation.

Applying Eq. (3.41) separately on these eigenspaces defines the neighbouring reciprocal variables

$$\widetilde{L}_\parallel = \frac{\Gamma}{R_F}, \qquad \mathrm{d}\widetilde{\ell}_\parallel = \frac{\Gamma}{R_F}\mathrm{d}F, \qquad \widetilde{L}_\perp = \widetilde{R} = \frac{F^2}{R}. \tag{3.72}$$

The vanishing old shell separation is thereby represented as an unbounded proper scale measured from the adjacent rest centre, while a transverse collapse is continued by the reciprocal areal scale. They are active-scale data for sector charts of the same LTB metric and matter fields, not rescalings of a completed coframe.

The transverse mode can also be embedded in the global orientation angle.

For the transverse mode the critical scale is F. Define all functions used in the angular metric by

$$\begin{gathered} \phi(\Theta) = \Theta - \frac{\pi}{2}\widehat{n}(\Theta), \qquad q_\perp(\Theta) = |\tan\phi(\Theta)|, \\ R(\Theta, F) = \frac{F}{q_\perp(\Theta)}, \qquad R(t(\Theta, F), F) = R(\Theta, F), \\ t_{,\Theta} = \frac{R_{,\Theta}(\Theta, F)}{R_{,t}(t(\Theta, F), F)}, \qquad t_{,F} = \frac{q_\perp^{-1}(\Theta) - R_F(t(\Theta, F), F)}{R_{,t}(t(\Theta, F), F)}, \\ \Gamma(F) = \sqrt{1 + 2E(F)}. \end{gathered} \tag{3.73}$$

The last two relations follow by differentiating the implicit equation for $t(\Theta, F)$. With

$\mathrm{d}t = t_{,\Theta}\mathrm{d}\Theta + t_{,F}\mathrm{d}F$, the full pullback, including its mixed term, is

$$\mathrm{d}s_L^2(\Theta) = \mathrm{i}^{2n_g(\Theta)}\Bigg[- c^2 t_{,\Theta}^2 \mathrm{d}\Theta^2 - 2c^2 t_{,\Theta} t_{,F} \mathrm{d}\Theta \mathrm{d}F \\ + \left(\frac{R_F^2(t(\Theta,F),F)}{\Gamma^2(F)} - c^2 t_{,F}^2 \right) \mathrm{d}F^2 + R^2(\Theta,F)\mathrm{d}\Omega^2 \Bigg]. \tag{3.74}$$

For the longitudinal shell-crossing mode the same global rule is applied to the proper shell scale R_F/Γ, with critical value one; the transverse angles and all remaining LTB fields keep the values supplied by the original solution.

Sector labels in this subsection are attached to complete geometric objects, such as $\mathrm{d}s^2_{(0)}$, $\mathrm{d}s^2_{(1)}$, and their curvature scalars. The coordinates and scalar functions t, τ, F, R, a, and b are not themselves sector-indexed; their physical role is fixed by the metric in which they occur.

The two regularity calculations must be read separately. At shell crossing only the longitudinal chart changes. The reciprocal separation in Eq. (3.72) replaces the vanishing distance between adjacent shells by an unbounded active distance measured from the neighbouring sector centre. Curvature and dust density must then be evaluated in that longitudinal domain rather than extrapolated through the degenerate comoving coordinate.

Shell focusing is different. There the radial labelling of neighbouring shells need not fail, but the transverse two-sphere loses its areal size. The reciprocal areal scale in Eq. (3.72) replaces that zero-radius extrapolation by a side domain with growing transverse area. Its curvature is evaluated independently of the shell-crossing mode. The tensor decomposition is essential precisely because one kind of collapse cannot be used as a scalar substitute for the other.

Before either LTB degeneracy is specialized, the causal equations can be written once for the parent geometry.

The geodesic equations can be written exactly before any profile is chosen. Define the ordinary-sector radial scale coefficient

$$A(t,F) = \frac{R_F(t,F)}{\Gamma(F)}. \tag{3.75}$$

By spherical symmetry every geodesic may be placed in the equatorial plane $\theta = \pi/2$. Let $V^\mu = \mathrm{d}x^\mu/\mathrm{d}\zeta$, where $\zeta = \tau$ for a massive particle and $\zeta = \lambda$ for an affinely parametrised null ray, and define Write the causal normalization as

$$g^{(0)}_{\mu\nu} V^\mu V^\nu = -\varepsilon c^2. \tag{3.76}$$

The parameter is $\varepsilon = 1$ for timelike motion and $\varepsilon = 0$ for null motion. The rotational Killing fields give the exact conserved angular momentum

$$\ell = R^2 \frac{\mathrm{d}\varphi}{\mathrm{d}\zeta}. \tag{3.77}$$

The tensor equation $V^\nu \nabla_\nu V^\mu = 0$ is then equivalent to

$$\begin{aligned} \frac{\mathrm{d}^2 t}{\mathrm{d}\zeta^2} + \frac{A\,\partial_t A}{c^2}\left(\frac{\mathrm{d}F}{\mathrm{d}\zeta}\right)^2 + \frac{\partial_t R}{c^2 R^3}\,\ell^2 &= 0, \\ \frac{\mathrm{d}^2 F}{\mathrm{d}\zeta^2} + 2\frac{\partial_t A}{A}\frac{\mathrm{d}t}{\mathrm{d}\zeta}\frac{\mathrm{d}F}{\mathrm{d}\zeta} + \frac{\partial_F A}{A}\left(\frac{\mathrm{d}F}{\mathrm{d}\zeta}\right)^2 - \frac{R_F}{A^2 R^3}\,\ell^2 &= 0, \\ \frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} &= \frac{\ell}{R^2}, \\ c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\zeta}\right)^2 = A^2\left(\frac{\mathrm{d}F}{\mathrm{d}\zeta}\right)^2 + \frac{\ell^2}{R^2} + \varepsilon c^2. & \end{aligned} \tag{3.78}$$

These equations contain no large-distance or near-centre approximation. They determine a causal curve in the sector-(0) metric once $R(t,F)$, $E(F)$, and initial data are supplied.

For radial timelike motion, $\ell = 0$, introduce

$$e_{\hat{0}}{}^{\mu} = \frac{1}{c}(\partial_t)^\mu, \qquad e_{\hat{1}}{}^{\mu} = \frac{1}{A}(\partial_F)^\mu, \tag{3.79}$$

and write

$$U^\mu = c\gamma\big(e_{\hat{0}}{}^{\mu} + \beta e_{\hat{1}}{}^{\mu}\big), \qquad p = \gamma\beta, \qquad \gamma = \sqrt{1+p^2}. \tag{3.80}$$

The exact first-order system is

$$\begin{aligned} \frac{\mathrm{d}t}{\mathrm{d}\tau} &= \sqrt{1+p^2}, \quad \frac{\mathrm{d}F}{\mathrm{d}\tau} = \frac{cp}{A}, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\partial_t A}{A}\,p. \end{aligned} \tag{3.81}$$

Hence

$$p(t) = p_0 \exp\left[-\int_{t_0}^{t} \frac{\partial_{t'} A(t',F(t'))}{A(t',F(t'))}\,\mathrm{d}t'\right], \qquad \tau - \tau_0 = \int_{t_0}^{t} \frac{\mathrm{d}t'}{\sqrt{1+p^2(t')}}. \tag{3.82}$$

The comoving dust curves are the exact special case $p = 0$, $\mathrm{d}F = 0$, and $\mathrm{d}\tau = \mathrm{d}t$.

For radial light, write

$$k^\mu = \omega\big(e_{\hat{0}}{}^{\mu} + \epsilon e_{\hat{1}}{}^{\mu}\big), \qquad \epsilon = \pm 1. \tag{3.83}$$

Then $k^\nu \nabla_\nu k^\mu = 0$ gives

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} &= \epsilon\frac{c}{A(t,F)}, \\ \frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\frac{\partial_t A(t,F(t))}{A(t,F(t))}\,\omega, \\ \lambda - \lambda_0 &= c\int_{t_0}^{t} \frac{\mathrm{d}t'}{\omega(t')}. \end{aligned} \tag{3.84}$$

The affine integral, rather than the coordinate slope alone, decides null completeness. A sector-(1) geodesic system must be derived from the complete metric $g^{(1)}_{\mu\nu}$; it is not obtained by replacing one coefficient in these sector-(0) equations.

The parent LTB curvature shows why the reciprocal domain must be tested with its own active metric.

Before introducing reciprocal domains it is useful to display the exact LTB curvature that the inactive chart assigns to the two degeneracies. In geometrised units, with the convention

$\dot{R}^2 = F/R$ and with F itself used as the mass coordinate, the Kretschmann scalar is

$$K_{(0)} = \frac{12F^2}{R^6} - \frac{8F}{R^5 R_F} + \frac{3}{R^4 R_F^2}. \tag{3.85}$$

Thus an ordinary shell crossing, $R_F \to 0$ while $R \to R_0 > 0$, is generically curvature divergent in that inactive description. Ordinary shell focusing, $R \to 0$, is also divergent. The sector construction does not turn either value finite by a coordinate change at the same event. It replaces the singular extrapolation by an explicitly specified reciprocal domain, whose curvature must then be computed from its own active metric.

Longitudinal shell crossing

A closed-form profile can be constructed in the marginally bound sector- (0) family. In this paragraph set $G_N = c = 1$ and use the convention $\dot{R}^2 = F/R$. The collapsing solution is

$$R(t, F) = \left(\frac{9F}{4}\right)^{1/3} [t_B(F) - t]^{2/3}. \tag{3.86}$$

Choose constants $F_c > 0$, $R_0 > 0$, $\alpha > 0$, and t_c, and define

$$R_c(F) = R_0 + \frac{\alpha}{2}(F - F_c)^2, \qquad t_B(F) = t_c + \frac{2R_c(F)^{3/2}}{3\sqrt{F}}. \tag{3.87}$$

Substitution gives the exact identities

$$R(t_c, F) = R_c(F), \qquad R_F(t_c, F) = \alpha(F - F_c), \qquad R(t_c, F_c) = R_0 > 0. \tag{3.88}$$

Thus $F = F_c$ is a simple shell crossing on the slice $t = t_c$, not a shell-focusing point. Its ordinary-sector Kretschmann scalar is

$$K_{(0)}^{(c)}(t_c, F) = \frac{12F^2}{R_c(F)^6} - \frac{8F}{\alpha(F - F_c)R_c(F)^5} + \frac{3}{\alpha^2(F - F_c)^2 R_c(F)^4}. \tag{3.89}$$

and diverges when the inactive chart is extrapolated to $F = F_c$.

To construct the neighbouring sector, use the branch $F < F_c$ and put

$$\delta = F_c - F > 0, \qquad L_\parallel = |R_F(t_c, F)| = \alpha\delta, \qquad q_\parallel = \frac{1}{\alpha\delta}. \tag{3.90}$$

The full radial clock–ruler orientation is selected tensorially. On the radial time–space plane define

$$P_r{}^\mu{}_\nu = -u^\mu u_\nu + s^\mu s_\nu, \qquad J_r{}^\mu{}_\nu = u^\mu s_\nu + s^\mu u_\nu, \qquad J_r{}^\mu{}_\rho J_r{}^\rho{}_\nu = -P_r{}^\mu{}_\nu. \tag{3.91}$$

With

$$\cos\psi = \frac{\alpha\delta}{\sqrt{1 + \alpha^2\delta^2}}, \qquad \sin\psi = \frac{1}{\sqrt{1 + \alpha^2\delta^2}}. \tag{3.92}$$

the radial orientation tensor is

$$\mathcal{R}_r(\psi)^\mu{}_\nu = \delta^\mu{}_\nu - P_r{}^\mu{}_\nu + \cos\psi\, P_r{}^\mu{}_\nu + \sin\psi\, J_r{}^\mu{}_\nu. \tag{3.93}$$

It acts only on the time–radial tensor plane and is the identity on the spherical screen. At the common chart boundary $q_{\|} = 1$, one has $\psi = \pi/4$; as $F \to F_c^-$, the tensor approaches the radial quarter-turn $\psi \to \pi/2$. Thus the selected longitudinal projector, not a coordinate component, identifies the mode paired with the side clock. The reciprocal scale is then used as initial data of an exact LTB solution, rather than as a componentwise rescaling of a finished metric.

Assume $F_c > 1/\alpha$, so that the entrance shell

$$F_h = F_c - \frac{1}{\alpha} > 0, \qquad R_h = R_c(F_h) = R_0 + \frac{1}{2\alpha} \tag{3.94}$$

is well defined and satisfies $|R_F(t_c, F_h)| = 1$. Realise the reciprocal longitudinal scale as the radial Jacobi scale of the side initial profile:

$$\begin{aligned} \frac{\mathrm{d}\widetilde{R}_h}{\mathrm{d}F} &= \frac{1}{\alpha(F_c - F)}, \\ \widetilde{R}_h(F) &= R_h + \frac{1}{\alpha}\ln\left(\frac{F_c - F_h}{F_c - F}\right) = R_h + \frac{1}{\alpha}\ln\left(\frac{1}{\alpha(F_c - F)}\right). \end{aligned} \tag{3.95}$$

At $F = F_h$, the areal radius and unsigned radial scale agree in both charts. The signed radial derivative reverses orientation, while its square and therefore the radial metric coefficient are continuous. Toward the reciprocal centre,

$$F \to F_c^- \quad \Longrightarrow \quad \widetilde{R}_h \to \infty, \qquad \widetilde{R}_{h,F} \to \infty. \tag{3.96}$$

The exact expanding marginally bound dust branch generated by these initial data is

$$\widetilde{R}(\tau, F) = \left[\widetilde{R}_h(F)^{3/2} + \frac{3}{2}\sqrt{F}\,(\tau - \tau_h)\right]^{2/3}, \qquad \tau \geq \tau_h. \tag{3.97}$$

Writing

$$Z(\tau, F) = \widetilde{R}_h(F)^{3/2} + \frac{3}{2}\sqrt{F}\,(\tau - \tau_h), \tag{3.98}$$

one obtains

$$\begin{aligned} &\widetilde{R} = Z^{2/3}, && \left(\frac{\partial \widetilde{R}}{\partial \tau}\right)^2 = \frac{F}{\widetilde{R}}, \\ &\widetilde{R}_F = Z^{-1/3}\left[\frac{\sqrt{\widetilde{R}_h}}{\alpha(F_c - F)} + \frac{\tau - \tau_h}{2\sqrt{F}}\right]. \end{aligned} \tag{3.99}$$

Consequently the complete sector-(1) metric is itself an exact LTB dust solution,

$$\mathrm{d}s^2_{(1,c)} = \mathrm{d}\tau^2 - \widetilde{R}_F^2(\tau, F)\,\mathrm{d}F^2 - \widetilde{R}^2(\tau, F)\,\mathrm{d}\Omega^2. \tag{3.100}$$

Its orthonormal Einstein tensor has the dust form

$$G^{(1)}{}_{\hat{A}\hat{B}} = \kappa\rho\, u_{\hat{A}} u_{\hat{B}}, \qquad \kappa\rho = \frac{1}{\widetilde{R}^2 \widetilde{R}_F}. \tag{3.101}$$

On the entrance slice $\tau = \tau_h$,

$$\kappa\rho(\tau_h, F) = \frac{\alpha(F_c - F)}{\widetilde{R}_h(F)^2} \longrightarrow 0 \qquad (F \to F_c^-). \tag{3.102}$$

Thus the reciprocal branch preserves the same pressureless matter type; no anisotropic pressure or radial flux is manufactured by the continuation.

The Kretschmann scalar of this exact sector metric is

$$K^{(c)}_{(1)}(\tau, F) = \frac{12F^2}{\widetilde{R}^6} - \frac{8F}{\widetilde{R}^5 \widetilde{R}_F} + \frac{3}{\widetilde{R}^4 \widetilde{R}_F^2}. \tag{3.103}$$

On $\tau = \tau_h$ it reduces to

$$K^{(c)}_{(1)}(\tau_h, F) = \frac{12F^2}{\widetilde{R}_h^6} - \frac{8\alpha F(F_c - F)}{\widetilde{R}_h^5} + \frac{3\alpha^2(F_c - F)^2}{\widetilde{R}_h^4}. \tag{3.104}$$

For every fixed finite $\tau - \tau_h$, the reciprocal centre obeys

$$\lim_{F \to F_c^-} \rho(\tau, F) = 0, \qquad \lim_{F \to F_c^-} K^{(c)}_{(1)}(\tau, F) = 0. \tag{3.105}$$

The inactive shell-crossing divergence is therefore replaced, for this exact profile, by an asymptotically flat-curvature dust domain. This establishes curvature regularity of the completed sector-(1) metric. Timelike and null accessibility of its spatial reciprocal centre remain separate geodesic questions.

Transverse shell focusing

Continue in geometrised units. The homogeneous marginally bound dust solution is an exact LTB subclass. Use a comoving shell label χ. The ordinary sector-(0) geometry is

$$\mathrm{d}s^2_{(0,f)} = -\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\chi^2 + a^2(t)\chi^2\mathrm{d}\Omega^2. \tag{3.106}$$

with

$$R(t,\chi) = a(t)\chi, \qquad a(t) = a_h \left(\frac{t_f - t}{t_f - t_h}\right)^{2/3}, \qquad t_h \le t < t_f, \tag{3.107}$$

and

$$\mathcal{F}(\chi) = \frac{4a_h^3}{9(t_f - t_h)^2}\,\chi^3, \qquad E(\chi) = 0. \tag{3.108}$$

Every noncentral shell has $R \to 0$ as $t \to t_f$; this is the transverse shell-focusing mode.

The neighbouring side sector uses the reciprocal scale

$$b(\tau) = \frac{a_h^2}{a(t(\tau))}, \qquad H_h = \frac{2}{3(t_f - t_h)}. \tag{3.109}$$

Its explicitly sector-indexed metric is

$$\mathrm{d}s^2_{(1,f)} = \mathrm{d}\tau^2 - b^2(\tau)\,\mathrm{d}\chi^2 - b^2(\tau)\chi^2\mathrm{d}\Omega^2. \tag{3.110}$$

The exact dust equation in sector (1) gives

$$b(\tau) = a_h\left[1 + \frac{3}{2}H_h(\tau - \tau_h)\right]^{2/3}, \tag{3.111}$$

$$t_f - t(\tau) = \frac{t_f - t_h}{1 + \frac{3}{2}H_h(\tau - \tau_h)}. \tag{3.112}$$

Consequently $t = t_f$, and hence $R = 0$ in the inherited collapse label, is approached only when τ is unbounded.

The curvatures of the two sector metrics can now be compared directly. For Eq. (3.106),

$$K_{(0)}^{(f)}(t) = 12\left[\left(\frac{\ddot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^4\right] = \frac{80}{27(t_f - t)^4}. \tag{3.113}$$

It diverges as $t \to t_f$. For the reciprocal metric Eq. (3.110),

$$K_{(1)}^{(f)}(\tau) = 12\left[\left(\frac{\ddot{b}}{b}\right)^2 + \left(\frac{\dot{b}}{b}\right)^4\right] = \frac{15H_h^4}{\left[1 + \frac{3}{2}H_h(\tau - \tau_h)\right]^4}. \tag{3.114}$$

Thus

$$K_{(1)}^{(f)}(\tau_h) = 15H_h^4, \qquad \lim_{\tau\to\infty} K_{(1)}^{(f)}(\tau) = 0. \tag{3.115}$$

Here the complete sector-(1) metric has been written explicitly, so the finite curvature and its decay to zero follow from that metric rather than from scale inversion alone.

This statement is not restricted to exactly comoving matter. For a radial massive geodesic with conserved comoving momentum Π, its own proper time σ satisfies

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 + \Pi^2/b^2(\tau)}} \geq \frac{1}{\sqrt{1 + \Pi^2/a_h^2}}, \tag{3.116}$$

so

$$\sigma - \sigma_h \geq \frac{\tau - \tau_h}{\sqrt{1 + \Pi^2/a_h^2}}. \tag{3.117}$$

For radial light, let $\mathcal{K} = b^2\mathrm{d}\chi/\mathrm{d}\lambda \neq 0$. The null normalisation gives $\mathrm{d}\lambda = b\mathrm{d}\tau/|\mathcal{K}|$, and the affine parameter is exactly

$$\lambda - \lambda_h = \frac{2a_h}{5H_h|\mathcal{K}|}\left\{\left[1 + \frac{3}{2}H_h(\tau - \tau_h)\right]^{5/3} - 1\right\}. \tag{3.118}$$

Both the massive proper time and the null affine parameter are therefore unbounded. This homogeneous profile supplies an exact sector-(1) shell-focusing metric, an exact finite Kretschmann scalar, and exact causal completeness. The longitudinal example above likewise supplies an exact sector-(1) LTB dust metric: its density and Kretschmann scalar both tend to zero at the reciprocal shell-crossing centre. Its full noncomoving timelike and null accessibility remains a separate geodesic problem.

The general equations Eqs. (3.78), (3.81), and (3.84) remain necessary for nonhomogeneous profiles. The examples do not turn the result into a universal LTB theorem: a different mass function, energy function, bang-time function, or active reciprocal completion must be tested using its own exact sector metric and causal integrals.

3.4.3 Kerr geometry: the principal sector plane with frame dragging

Rotation changes the active plane, but not the reciprocal construction. In this subsection only, set $G_N = c = 1$. Kerr has no globally preferred radial direction supplied by spherical symmetry. The repeated principal null directions instead select a local time–principal plane. Its unit clock u^μ, principal direction n^μ, projector, and quarter-turn generator are exactly those of Eqs. (3.33) and (3.34). Their position dependence carries the familiar frame dragging in the Kerr connection; it does not require another sector operator.

The outer horizon lies at $r_+ = M + \sqrt{M^2 - a^2}$. Anchoring the reciprocal chart there gives

$$R(r) = \frac{r_+^2}{r}, \qquad 0 < r \le r_+. \tag{3.119}$$

The cross term $g_{t\varphi}$ means that the active clock is the local horizon generator in a corotating, horizon-regular frame, not the coordinate one-form $\mathrm{d}t$ by itself [36]. The same worldline tangent is re-resolved in this corotating sector coframe: the inherited time direction belongs to its spatial decomposition, while n^μ supplies the active clock. Turning in azimuth or reversing inherited-t motion does not reverse the particle's proper time.

The principal Kerr scale admits the same global-angle representation.

Put $r_h = r_+$ and define every Θ-dependent quantity appearing in the metric by

$$\begin{gathered} \phi(\Theta) = \Theta - \frac{\pi}{2}\widehat{n}(\Theta), \qquad q_r(\Theta) = |\tan\phi(\Theta)|, \\ R(\Theta) = \frac{r_h}{q_r(\Theta)}, \qquad \mathrm{d}R = R_{,\Theta}\mathrm{d}\Theta, \qquad \epsilon_R(\Theta) = -\frac{R_{,\Theta}}{|R_{,\Theta}|}, \\ \Sigma(\Theta,\theta) = R^2(\Theta) + a^2\cos^2\theta. \end{gathered} \tag{3.120}$$

Let W be the horizon-regular null coordinate selected by the active one-sided chart. Expanding the final square gives one global metric:

$$\begin{aligned} \mathrm{d}s_K^2(\Theta) = \mathrm{i}^{2n_g(\Theta)}\Bigg[& \left(-1 + \frac{2MR}{\Sigma}\right)\mathrm{d}W^2 + 2\epsilon_R R_{,\Theta}\mathrm{d}W\mathrm{d}\Theta + \Sigma\mathrm{d}\theta^2 \\ & - 2\epsilon_R a\sin^2\theta\, R_{,\Theta}\mathrm{d}\Theta\mathrm{d}\widetilde{\varphi} - \frac{4MRa\sin^2\theta}{\Sigma}\mathrm{d}W\mathrm{d}\widetilde{\varphi} \\ & + \left((R^2+a^2)\sin^2\theta + \frac{2MRa^2\sin^4\theta}{\Sigma}\right)\mathrm{d}\widetilde{\varphi}^2\Bigg], \end{aligned} \tag{3.121}$$

where R, $R_{,\Theta}$, ϵ_R, and Σ in the displayed metric mean the functions defined immediately above. The position-dependent principal direction, and therefore the metric-derived angles $\varpi_{\rho\sigma}(x)$ contained in its projector, are carried unchanged by the pullback. Only the active principal scale is parametrised by Θ.

Curvature can be checked directly with one invariant. The Kerr Kretschmann scalar is

$$K_{(0)}(R,\theta) = 48M^2\frac{R^6 - 15a^2R^4\cos^2\theta + 15a^4R^2\cos^4\theta - a^6\cos^6\theta}{(R^2+a^2\cos^2\theta)^6}. \tag{3.122}$$

and the reciprocal substitution gives

$$K_{(1)}(r,\theta) = 48M^2r^6\frac{r_+^{12} - 15a^2r_+^8r^2\cos^2\theta + 15a^4r_+^4r^4\cos^4\theta - a^6r^6\cos^6\theta}{(r_+^4 + a^2r^2\cos^2\theta)^6}. \tag{3.123}$$

The overall r^6 is the residue of the ordinary $1/R^6$ decay after substitution. For a nonzero outer-horizon radius the denominator stays nonzero throughout the side domain, and the Kretschmann scalar vanishes at the zero endpoint of the side coordinate. The Kerr ring would require zero areal radius in the equatorial plane. The reciprocal domain instead begins at the outer horizon and extends toward unbounded areal radius, so it never samples the ring.

The non-rotating limit is immediate: $a = 0$ gives $r_+ = 2M = r_s$ and $K_{(1)} = 48M^2 r^6/r_s^{12} = 12r^6/r_s^{10}$, precisely the Schwarzschild result in Eq. (3.60).

The causal status of the reciprocal Kerr centre follows from the complete separated geodesic system.

The geodesic equations can be kept exact. Let

$$\Sigma = R^2 + a^2 \cos^2\theta, \qquad \Delta = R^2 - 2MR + a^2. \tag{3.124}$$

Let $V^\mu = \mathrm{d}x^\mu/\mathrm{d}\zeta$, and let $\xi^\mu_{(t)}$, $\xi^\mu_{(\varphi)}$ be the stationary and axial Killing fields. Define the conserved contractions

$$\mathcal{P} = -\xi^{(t)}_\mu V^\mu, \qquad \mathcal{L} = \xi^{(\varphi)}_\mu V^\mu, \tag{3.125}$$

and let $\mathcal{Q}$ be the Carter separation constant [4,63]. The symbol $\mathcal{P}$ is a conserved momentum along the inherited t direction; it is not called an energy in the active side chart because that direction is spatial there.

Since $g^{(1)}_{\mu\nu} = -\mathcal{I}_1^* g^{(0)}_{\mu\nu}$, it is convenient to write the causal normalization as

$$\varepsilon = -g^{(0)}_{\mu\nu} V^\mu V^\nu. \tag{3.126}$$

An active timelike tangent has $\varepsilon = -1$, equivalently $g^{(0)}_{\mu\nu} U^\mu U^\nu = 1$, while a null geodesic has $\varepsilon = 0$. Define also

$$\Pi(R) = \mathcal{P}(R^2 + a^2) - a\mathcal{L}, \qquad \mathcal{K} = (\mathcal{L} - a\mathcal{P})^2 + \mathcal{Q}. \tag{3.127}$$

The complete separated system is

$$\begin{aligned}
\Sigma^2 \left(\frac{\mathrm{d}R}{\mathrm{d}\zeta}\right)^2 &= \mathcal{R}_\varepsilon(R), \\
\mathcal{R}_\varepsilon(R) &= \Pi^2(R) - \Delta\left(\varepsilon R^2 + \mathcal{K}\right), \\
\Sigma^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)^2 &= \mathcal{T}_\varepsilon(\theta), \\
\mathcal{T}_\varepsilon(\theta) &= \mathcal{Q} - \cos^2\theta \left[a^2(\varepsilon - \mathcal{P}^2) + \frac{\mathcal{L}^2}{\sin^2\theta}\right], \\
\Sigma \frac{\mathrm{d}t}{\mathrm{d}\zeta} &= a\left(\mathcal{L} - a\mathcal{P}\sin^2\theta\right) + \frac{(R^2 + a^2)\Pi(R)}{\Delta}, \\
\Sigma \frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} &= \frac{\mathcal{L}}{\sin^2\theta} - a\mathcal{P} + \frac{a\Pi(R)}{\Delta}.
\end{aligned} \tag{3.128}$$

Here $\zeta = \tau$ for active timelike motion and $\zeta = \lambda$ for an affinely parametrised null geodesic. The value $\varepsilon = -1$ is the essential change from the ordinary timelike Kerr equation.

Using $R = r_+^2/r$, the exact side-coordinate equation is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\zeta}\right)^2 = \frac{r^4}{r_+^4 \Sigma^2} \mathcal{R}_\varepsilon\left(\frac{r_+^2}{r}\right). \tag{3.129}$$

A branch directed toward increasing R has the exact parameter integral

$$\zeta - \zeta_0 = \int_{R_0}^{R} \frac{\Sigma(R', \theta(R'))}{\sqrt{\mathcal{R}_\varepsilon(R')}}\, \mathrm{d}R'. \tag{3.130}$$

The divergence of this integral can be proved without an asymptotic series. Expanding the exact radial potential gives

$$\mathcal{R}_\varepsilon(R) = A_4 R^4 + A_3 R^3 + A_2 R^2 + A_1 R + A_0, \tag{3.131}$$

where

$$\begin{aligned} &A_4 = \mathcal{P}^2 - \varepsilon, && A_3 = 2M\varepsilon, \\ &A_2 = 2a\mathcal{P}(a\mathcal{P} - \mathcal{L}) - \mathcal{K} - \varepsilon a^2, && A_1 = 2M\mathcal{K}, \qquad A_0 = -a^2\mathcal{Q}. \end{aligned} \tag{3.132}$$

Choose any $R_0 > 0$ beyond which the considered branch is allowed, and set

$$C_\varepsilon(R_0) = |A_4| + \frac{|A_3|}{R_0} + \frac{|A_2|}{R_0^2} + \frac{|A_1|}{R_0^3} + \frac{|A_0|}{R_0^4}. \tag{3.133}$$

For every $R \geq R_0$,

$$0 \leq \mathcal{R}_\varepsilon(R) \leq C_\varepsilon(R_0) R^4, \qquad \Sigma \geq R^2. \tag{3.134}$$

Therefore the exact integral obeys

$$\zeta - \zeta_0 \geq \frac{R - R_0}{\sqrt{C_\varepsilon(R_0)}}. \tag{3.135}$$

Any causal branch that reaches arbitrarily large R consequently needs unlimited proper or affine parameter. A branch for which $\mathcal{R}_\varepsilon$ has an outer zero turns or remains bound and does not approach the reciprocal boundary.

A solvable equatorial family makes the same conclusion explicit.

The exact conclusion can also be seen in a solvable family. Set

$$\theta = \frac{\pi}{2}, \qquad \mathcal{Q} = 0, \qquad \mathcal{L} = a\mathcal{P}. \tag{3.136}$$

For active timelike motion, put $A = 1 + \mathcal{P}^2$ and

$$D(R) = AR^2 - 2MR + a^2. \tag{3.137}$$

Then

$$\mathcal{R}_{-1}(R) = R^2 D(R), \qquad \frac{\mathrm{d}R}{\mathrm{d}\tau} = \frac{\sqrt{D(R)}}{R} \tag{3.138}$$

on the outgoing branch, and direct integration gives

$$\begin{aligned} \tau(R) - \tau(R_0) = \Bigg[&\frac{\sqrt{D(R)}}{A} \\ &+ \frac{M}{A^{3/2}} \ln\Big(2\sqrt{A}\sqrt{D(R)} + 2AR - 2M\Big) \Bigg]_{R_0}^{R}. \end{aligned} \tag{3.139}$$

The first term is unbounded as R is unbounded, so this exact expression places the reciprocal centre at infinite proper time.

For null motion with the same angular data,

$$\mathcal{R}_0(R) = \mathcal{P}^2 R^4, \qquad \frac{\mathrm{d}R}{\mathrm{d}\lambda} = |\mathcal{P}|, \tag{3.140}$$

and hence

$$R(\lambda) = R_0 + |\mathcal{P}|(\lambda - \lambda_0), \qquad r(\lambda) = \frac{r_+^2}{R_0 + |\mathcal{P}|(\lambda - \lambda_0)}. \tag{3.141}$$

The side coordinate remains positive for every finite affine parameter and reaches zero only when λ is unbounded. These closed-form examples agree with the general exact bound in Eq. (3.135); no large-R approximation is used.

A reflected Kerr chart may represent the opposite orientation algebraically, but frame dragging, spin, and particle data are not transported through the side centre by a finite-parameter worldline. No Kerr black-hole trajectory derived here continues into a white-hole exit in the opposite-time sector.

3.4.4 Spatially flat FLRW: reciprocal scale and the meaning of $t = 0$

The FLRW example uses exactly the same sector mechanism as the black-hole examples. The only difference is the geometric quantity that is inverted. For a black hole it is the areal radius; for a homogeneous universe it is the scale factor.

In the ordinary $+t$ projection, write the spatially flat metric as

$$\mathrm{d}s^2_{(0)} = -c^2\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\chi^2 + a^2(t)\chi^2\mathrm{d}\Omega^2. \tag{3.142}$$

Here $a(t)$ is the ordinary scale factor [31]. Let a_h and ρ_h denote the finite scale factor and density at the null boundary where the ordinary projection ends and the side projection becomes active. The comoving fluid is assigned $s_X = 0$ relative to the active geometric coframe. A change of n_g therefore reorients the fluid and the field together without changing the fluid's particle class.

Homogeneity and isotropy make all three principal modes identical. With u^μ the comoving clock and $h^\mu{}_\nu$ its rest projector, the complete tensor statement is

$$L^\mu{}_\nu = a(t)h^\mu{}_\nu, \qquad G^\mu{}_\nu = a_h\, h^\mu{}_\nu. \tag{3.143}$$

All three spatial eigenvalues are therefore equal: the active scale is $a(t)$, the boundary scale is a_h, and their common ratio is $q_i = a_h/a(t)$. Every spatial direction consequently reaches the sector boundary together. This tensor statement already contains the common clock and the isotropic three-dimensional rest space; introducing a separate component representation would add no information.

Equation (3.41) gives

$$\begin{aligned} \widetilde{L}^\mu{}_\nu &= \frac{a_h^2}{a(t)}\, h^\mu{}_\nu, \\ b(\tau) &= \frac{a_h^2}{a(t(\tau))}, \qquad b(\tau_h) = a_h. \end{aligned} \tag{3.144}$$

Here τ is proper time along the comoving clock defined by the sector-(1) metric. The active representative is

$$\mathrm{d}s^2_{(1)} = c^2\mathrm{d}\tau^2 - b^2(\tau)\,\mathrm{d}\chi^2 - b^2(\tau)\chi^2\mathrm{d}\Omega^2. \tag{3.145}$$

This is the one FLRW geometry evaluated in its reciprocal domain, not an independently chosen second metric. The inherited time label t can approach a finite endpoint while the proper-time coordinate τ covers an infinite interval in the sector-(1) metric.

The isotropic scale can likewise be embedded in the global orientation angle.

All functions used in the global angular metric are

$$\begin{aligned} \phi(\Theta) &= \Theta - \frac{\pi}{2}\widehat{n}(\Theta), \qquad q_a(\Theta) = |\tan\phi(\Theta)|, \\ a(\Theta) &= \frac{a_h}{q_a(\Theta)}, \qquad H(\Theta) = H(a(\Theta)), \\ N(\Theta) &= c\left|\frac{a_{,\Theta}(\Theta)}{a(\Theta)H(\Theta)}\right|. \end{aligned} \tag{3.146}$$

Here $H(a)$ is the Hubble function of the original FLRW solution evaluated on the active branch. The complete angular geometry is

$$\mathrm{d}s^2_F(\Theta) = \mathrm{i}^{2n_g(\Theta)}\left[-N^2(\Theta)\mathrm{d}\Theta^2 + a^2(\Theta)\mathrm{d}\chi^2 + a^2(\Theta)\chi^2\mathrm{d}\Omega^2\right]. \tag{3.147}$$

The Friedmann equations and matter source remain unchanged. The global angle parametrises the single cyclic scale profile, while $n_g(\Theta)$ supplies its metric orientation.

The side-domain evolution is fixed by the same Friedmann equations and conservation law.

Let $p = w\rho c^2$ with constant w, and let ρ_h be the density at the common null boundary. Energy conservation in the active chart gives $\rho(\tau) = \rho_h[a_h/b(\tau)]^{3(1+w)}$. Substitution into the spatially flat Friedmann equation determines the side evolution; it is not an independently chosen expansion law. With $\alpha = 3(1+w)/2$, $H_h^2 = 8\pi G_N\rho_h/3$, and $b(\tau_h) = a_h$, its exact solution is

$$b(\tau) = a_h\left[1+\alpha H_h(\tau-\tau_h)\right]^{1/\alpha}, \qquad \rho(\tau) = \frac{\rho_h}{\left[1+\alpha H_h(\tau-\tau_h)\right]^2}. \tag{3.148}$$

The reciprocal scale grows without bound only over an unlimited interval of the side clock. The reciprocal centre is therefore not reached by comoving matter in finite proper time.

For spatially flat FLRW, the Friedmann and acceleration equations give

$$K_{(1)} = \frac{12}{c^4}\left[\left(\frac{\ddot{b}}{b}\right)^2 + \left(\frac{\dot{b}}{b}\right)^4\right] = \frac{64\pi^2 G_N^2}{3c^4}\left[(1+3w)^2+4\right]\rho^2. \tag{3.149}$$

If K_h denotes the same expression at the boundary value $\rho = \rho_h$, then

$$K_{(1)}(\tau) = \frac{K_h}{\left[1+\alpha H_h(\tau-\tau_h)\right]^4}. \tag{3.150}$$

The Weyl tensor of FLRW vanishes, so this decay describes the complete Riemann tensor. The density falls with the inverse square of the side proper time and the Kretschmann scalar with its inverse fourth power. The side branch is therefore asymptotically flat.

The relation between the inherited time label and the active proper time then fixes the status of the ordinary endpoint.

For the same single fluid, its ordinary power-law chart has $a(t) = a_h(t/t_h)^{1/\alpha}$ and $H_h = 1/(\alpha t_h)$. Combining this expression with $b = a_h^2/a$ and Eq. (3.148) yields

$$t(\tau) = \frac{t_h}{1 + \alpha H_h(\tau - \tau_h)}. \tag{3.151}$$

Equation (3.151) shows that the ordinary time label approaches zero only after an unlimited interval of the active proper time. The ordinary coordinate compresses an infinite side-sector evolution into the finite label $t = 0$; that label is not an event crossed by a comoving observer.

The result is not restricted to exactly comoving material. A freely moving massive particle with finite conserved comoving momentum has a Lorentz factor that approaches unity as the reciprocal scale grows. Its proper time σ therefore grows without bound with the side clock. For radial light, conformal transformation of Eq. (3.145) gives $\mathrm{d}\lambda \propto b(\tau)\mathrm{d}\tau$, where λ is affine. Since the scale grows as a positive power of proper time for the fluids considered here, the affine integral diverges. Null geodesics consequently approach the same boundary only after an unlimited affine interval. Equations of state outside the assumption $w > -1$ require their own sector and geodesic analysis and are not assigned a continuation in this work.

3.4.5 Geometric synthesis of the examples

The examples implement the same tensorial sequence in geometries with different active modes. A proper scale reaches its critical value, the neighbouring chart evaluates the same solution on the reciprocal scale, and curvature is then tested on that active domain. Schwarzschild and Kerr reduce the relevant black-hole continuation to a reciprocal radial scale, FLRW makes the three spatial modes degenerate, and LTB separates longitudinal shell separation from transverse areal collapse.

The geodesic calculations provide an independent test of the reciprocal centre. In the regular Schwarzschild, Kerr, homogeneous LTB and FLRW branches analysed here, the active proper scale grows without bound while the limiting coordinate approaches its finite endpoint. Timelike proper time and the corresponding null affine parameter are unbounded toward that endpoint. The reciprocal centre is therefore an asymptotic boundary of the active chart in these examples rather than a finite-parameter continuation point.

This distinction is separate from algebraic sector closure. The relation $\mathrm{i}^4 = 1$ closes the four orientation representatives of the clock–ruler plane, whereas accessibility of a chart boundary is fixed by the metric and its geodesic equations. The particle labels s_X and the within-sector Lorentz tensor $\Lambda_{(X)}{}^{\mu}{}_{\nu}$ remain unchanged under a pure geometry-chart transition, so the sector construction and the geodesic test remain logically distinct.

3.5 Conclusion

The four-sector construction begins with the invariant null set and extends the orientation of a clock–ruler plane through its four connected non-null domains. Ordinary Lorentz transformations act inside each domain. The sector label records the discrete orientation component, while the ordinary Lorentz tensor retains the continuous boost and spatial-frame data. Geometry and particle transformations are kept separate through the labels n_g and s_X, with $N_X = n_g + s_X \pmod 4$ appearing only after their ordered composition.

In curved spacetime the active direction is selected tensorially. The positive spatial tensor $L^{\mu}{}_{\nu}$ contains the proper deformation scales of a reference congruence, while $G^{\mu}{}_{\nu}$ supplies the solution-dependent critical scales. Their generalized eigenproblem selects the active eigendirection. The quarter-turn tensor identifies the associated clock–ruler plane, and the reciprocal operator maps the active eigenvalue to the scale used by the neighbouring chart. The active metric is obtained by evaluating the general-relativistic solution on these scales and applying the sector coframe orientation; curvature remains governed by the usual tensor transformation laws and Einstein equations.

The explicit examples test this prescription on different geometric modes. For Schwarzschild the reciprocal side chart converts the decreasing areal scale into an increasing one and the curvature decays toward the reciprocal centre. Kerr gives the corresponding result in a position-dependent principal plane while retaining the Carter geodesic system. Spatially flat FLRW makes the spatial modes degenerate; for the regular fluids analysed here, the reciprocal scale grows while density and Riemann curvature decay. LTB shows why a scalar radius is insufficient in an inhomogeneous geometry: longitudinal shell separation and transverse areal collapse are distinct spectral modes of the proper-scale tensor and require separate reciprocal constructions.

The proper-time and affine-parameter calculations complete the geometric test. In the regular Schwarzschild, Kerr, homogeneous LTB and FLRW branches considered here, the reciprocal centre lies at unbounded timelike or null parameter even though the inherited coordinate can approach a finite limiting value. Thus coordinate regularity, curvature regularity and geodesic distance remain separate conditions. Taken together, the tensor construction and the worked general-relativistic examples provide a four-sector Lorentz atlas in which sector continuation is fixed by proper geometric scales and tested by the active metric, its curvature and its geodesics.

Chapter 4

The Null Perspective Hypothesis

Source and status. This chapter incorporates the complete scientific body of *Null Perspective Hypothesis*, v0.21 (12 August 2026). Its role in the dissertation is: Covariant interpretation of a complete photon state on a selected null geodesic. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

This paper formulates the Null Perspective Hypothesis directly from the null interval. The analysis is formulated on a general Lorentzian spacetime and uses an orthonormal tetrad to expose the local relation between temporal and spatial displacement. The null equation fixes equality of their magnitudes but does not select a unique displacement: for every null direction there is an entire real one-parameter family of solutions, including the zero element. Physical initial data and the geodesic equation select one null branch, while the homogeneous freedom of the null equation remains. Along that selected history the metric proper time is identically zero, so photon proper time cannot order the admissible states into a sequence of intrinsic presents. The complete photon state is therefore the full family of admissible null states carried by the selected geodesic. Timelike observers describe the same trajectory in Lorentz-related orthonormal frames and assign nontrivial time coordinates to its events, thereby recovering the usual ordered motion without changing the photon state itself. The construction is covariant from the outset and does not require separate flat- and curved-spacetime formulations.

4.1 Thesis and logical structure

The geometric description of a photon begins with the null interval. At every regular event of a Lorentzian spacetime, the lightlike sector is defined by a vanishing metric norm. This is the same local statement in special relativity and in general relativity: curvature changes the metric and the way local frames are connected, but it does not replace the local null relation by a different one [13, 49, 56, 63].

The question considered here is not whether a photon follows a trajectory. A freely propagating photon follows a null geodesic. The question is more specific: what does the null equation itself determine about the state of the photon, and what additional structure is required before one can speak about a succession of photon states?

The central argument is a single chain. The null interval first determines a family of admissible displacements rather than one displacement. This follows from the equality between temporal and spatial magnitudes and from the homogeneous character of the null equation. The free real parameter appearing in that family is not fixed by nullness. Initial data and the geodesic equation then select the geometric branch followed by one photon, but they do not create a photon proper-time clock. The proper time on the selected null history is identically zero. Consequently, the admissible states are not ordered by photon proper time. If no further intrinsic variable is added to select one member as the current state, the complete intrinsic state is the entire admissible family carried by the selected null history. A timelike observer supplies a nontrivial temporal description afterwards.

This order of reasoning is essential. The null equation, the selection of a geodesic, the proper-time structure, and the observer description are different operations. Combining them too early hides the central fact that the null constraint is already degenerate before the observer enters the problem.

Thesis. *For a freely propagating photon, the null equation admits a complete real family of local states. The geodesic equation selects the physical null branch, while photon proper time is identically zero on its spacetime history and therefore supplies no intrinsic ordering of the admissible states. The photon is consequently represented by all admissible states of the selected null solution, whereas timelike observers assign the ordinary temporal description in Lorentz-related frames.*

The remainder of the paper follows exactly this chain. No separate model is introduced for curved spacetime; the tetrad and covariant geodesic equation are used throughout.

4.2 The null equation and the freedom of the state

4.2.1 The spacetime displacement in an orthonormal tetrad

Let $(\mathcal{M}, g_{\mu\nu})$ be a four-dimensional time-oriented Lorentzian spacetime with signature $(-,+,+,+)$. Greek indices denote spacetime components. At each regular event choose an orthonormal tetrad $e_A{}^\mu$ and dual coframe $e^A{}_\mu$, with Lorentz-frame indices $A, B, \ldots$. The metric and the displacement are related to their orthonormal components by

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu, \qquad \mathrm{d}x^\mu = e_A{}^\mu \mathrm{d}x^A, \qquad g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{AB}\mathrm{d}x^A \mathrm{d}x^B, \tag{4.1}$$

where $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1)$. This relation is the geometric foundation of the paper. It is valid at every regular event, independently of whether the metric is globally flat or curved.

For a photon, the displacement is null. Writing its tetrad components as $\mathrm{d}x^A = (\mathrm{d}x^{\hat{0}}, \mathrm{d}x^{\hat{\imath}})$, with $\hat{\imath} = 1, 2, 3$, Eq. (4.1) becomes $-(\mathrm{d}x^{\hat{0}})^2 + \delta_{\hat{\imath}\hat{\jmath}}\mathrm{d}x^{\hat{\imath}}\mathrm{d}x^{\hat{\jmath}} = 0$. Thus the magnitude of the temporal part equals the norm of the spatial part. The null condition fixes this relation between time and space, but it does not fix their common magnitude and it does not select one spatial direction. The solution is therefore a cone rather than a single displacement.

This equality concerns the spatial norm, not any individual spatial component. In a local orthonormal frame it is the geometric form of $c\,\mathrm{d}t = |\mathrm{d}\mathbf{x}|$. The next step is to write the full family allowed by this relation.

4.2.2 Admissible family and physical selection

The freedom left by the null equation can be written explicitly without introducing another state variable. Choose a unit spatial direction $n^{\hat{\imath}}$ satisfying $\delta_{\hat{\imath}\hat{\jmath}}n^{\hat{\imath}}n^{\hat{\jmath}} = 1$. The null displacement on the corresponding generator is

$$\boxed{\mathrm{d}x^{\mu} = \lambda \left(e_{\hat{0}}{}^{\mu} + n^{\hat{i}} e_{\hat{i}}{}^{\mu} \right), \qquad \lambda \in \mathbb{R}, \qquad \delta_{\hat{i}\hat{j}} n^{\hat{i}} n^{\hat{j}} = 1.} \tag{4.2}$$

Substitution into Eq. (4.1) gives a factor $\lambda^2[-1 + \delta_{\hat{i}\hat{j}} n^{\hat{i}} n^{\hat{j}}]$, which vanishes for every real value of λ. The null equation therefore fixes the relation between the temporal and spatial parts but does not select one scale on the generator. Positive and negative values give the two algebraic orientations and $\lambda = 0$ gives the apex of the cone.

Before physical data are supplied, the spatial direction is free as well: $n^{\hat{i}}$ may be any unit vector. The local null equation therefore describes the complete cone of admissible displacements. A particular photon is obtained only after its physical initial data select one propagation direction. That selection removes the freedom to choose an unrelated $n^{\hat{i}}$ at the next event, but it does not change the homogeneous character of the null equation. On the selected generator the real parameter λ remains unrestricted by nullness.

This separation is essential for the rest of the argument. The null equation determines which local displacements are admissible; the propagation equation determines which generator is followed through spacetime. The first gives a family of states, while the second chooses the physical trajectory on which that family is carried.

4.3 From the null family to one photon history

4.3.1 Geodesic selection

Let $\gamma : I \to \mathcal{M}$ be the worldline of one freely propagating photon and let s be an affine parameter. Its tangent $k^{\mu} = \mathrm{d}x^{\mu}/\mathrm{d}s$ obeys

$$g_{\mu\nu} k^{\mu} k^{\nu} = 0, \qquad k^{\nu} \nabla_{\nu} k^{\mu} = 0. \tag{4.3}$$

Initial data determine which null direction is physically realized at emission, and the second relation transports that direction along the resulting geodesic.

The geodesic equation therefore selects a geometric branch through the local null cones. It does not replace the local solution equation. At each event $p = \gamma(s)$, the selected tangent determines one generator of the local cone, and the null displacement on that generator still has the form

$$\mathrm{d}x^{\mu} = \lambda k^{\mu}(s), \qquad \lambda \in \mathbb{R}, \tag{4.4}$$

up to the arbitrary normalization chosen for k^{μ}. Equation (4.4) is a statement about the solution set of the null constraint at the selected event. It is not a claim that an arbitrary function $\lambda(s)$ defines a new affine tangent field.

This distinction is important. The dynamics selects the line in the tangent space, while the null equation admits every real representative of that line. The affine parameter used to integrate the geodesic is only a convenient label of the spacetime curve. A constant affine rescaling changes k^{μ} but leaves the geometric image $\gamma(I)$ unchanged, and the unrestricted real parameter in Eq. (4.4) makes the local state family independent of that normalization.

Metric compatibility ensures that the branch remains null. Along the geodesic,

$$\frac{\mathrm{d}}{\mathrm{d}s}(k_{\mu} k^{\mu}) = 2 k_{\mu} k^{\nu} \nabla_{\nu} k^{\mu} = 0. \tag{4.5}$$

Thus an initially null tangent remains null on the whole domain of the solution. This statement already contains the effect of curvature through the connection. No separate flat-space and curved-space constructions are required.

4.3.2 Spacetime events and local null states

The vector $\mathrm{d}x^\mu$ and the event $\gamma(s)$ are different geometric objects. The first belongs to a tangent space, while the second is a point of the manifold. The selected photon history links these two levels: every event of $\gamma(I)$ carries a selected null generator in its tangent space, and every real value of λ gives an admissible displacement on that generator.

It is therefore useful to retain both the spacetime support and the local state family. The complete selected null-state set is defined by

$$\mathcal{S}_\gamma = \left\{ \left(\gamma(s), \mathrm{d}x^\mu\right) : s \in I,\ \mathrm{d}x^\mu = \lambda k^\mu(s),\ \lambda \in \mathbb{R} \right\}. \tag{4.6}$$

Its projection onto spacetime is simply $\gamma(I)$. The definition says that the state contains every event of the selected history together with every local null displacement admitted by the null equation on the selected generator at that event.

This formulation restores the role of $\mathrm{d}x^\mu$ that would be lost if the photon state were identified only with a curve. The curve tells us where the selected solution lies in spacetime; the family of $\mathrm{d}x^\mu$ tells us which local null states solve the state equation above that curve.

For a finite process, emission and detection restrict the interval I to a finite segment. The same construction then applies to the portion of $\mathcal{S}_\gamma$ supported between the two interactions. Nothing in the definition depends on a global inertial coordinate system.

4.4 Zero proper time and absence of intrinsic ordering

The next step is temporal rather than dynamical. The null geodesic is future-directed and therefore has a causal orientation, but causal orientation is not the same as a clock carried by the photon. To determine whether the admissible states possess an intrinsic succession, the relevant quantity is the metric proper time.

For every null displacement in the family,

$$\mathrm{d}\tau^2 = -\frac{1}{c^2} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = 0. \tag{4.7}$$

Because this equation follows from the same null condition for every real λ, all local admissible states have the same proper-time increment: zero.

Choose the reference event $s_0 \in I$ with $\tau_\gamma(s_0) = 0$. Along the spacetime support $\gamma(I)$ the accumulated photon proper time is then

$$\tau_\gamma(s) = \frac{1}{c} \int_{s_0}^{s} \sqrt{-g_{\mu\nu} k^\mu k^\nu}\, \mathrm{d}\sigma = 0, \qquad \forall s \in I. \tag{4.8}$$

Thus the photon proper-time function vanishes identically on the selected trajectory. Distinct events of the geodesic remain distinct events and can still be causally ordered, but none is assigned a different photon proper-time value.

This is the second decisive step of the argument. The null equation first failed to select one λ. The proper-time structure now fails to provide an intrinsic rule that orders the states by successive photon clock readings. The affine parameter can order the mathematical representation of the oriented curve, but its normalization is arbitrary and it is not a nonzero photon proper time.

One can state the distinction formally. Define an intrinsic proper-time order by $p \prec_\tau q$ when $\tau_\gamma(p) < \tau_\gamma(q)$. Since $\tau_\gamma(s) = 0$ for every $s \in I$, no two distinct events satisfy this relation. The selected photon history therefore has causal order but no nontrivial order generated by photon proper time.

Proposition 4.1 (Absence of intrinsic proper-time selection)**.** *On a selected null geodesic, photon proper time cannot select one admissible event or one admissible null state as the unique intrinsic present.*

Proof. Equation (4.8) makes the photon proper-time function identically zero on the entire spacetime support. An intrinsic selection based on successive proper-time values would require distinct values. Since none exist, proper time cannot distinguish one member as current and another as earlier or later for the photon itself. □

The proposition does not remove the future direction of propagation. It removes only an internal temporal stratification of the selected state set. This distinction allows the argument to preserve ordinary causality while still reaching a stronger conclusion about the photon state.

4.5 Why the photon contains all admissible selected states

The two previous sections now combine. The state equation on a selected generator admits every real λ. The proper-time function is zero on the full selected history. Neither structure supplies a rule that picks one member of the admissible family as the photon-intrinsic current state.

If one nevertheless postulated that the photon occupies only one intrinsic member at a time, an additional state-selection map would have to be introduced. Such a map could not come from the null equation, because the null equation admits the entire family. It could not come from photon proper time, because photon proper time is identically zero. It would therefore be extra structure beyond the geometry used to define and propagate the photon.

The minimal state description is consequently the complete selected solution already defined in Eq. (4.6). In this precise sense the photon has all admissible selected states at once. The statement means simultaneous membership in the same complete solution set together with the absence of a nontrivial photon proper-time ordering. It does not mean that a timelike observer assigns the same coordinate time to every event of the state.

This conclusion is not an independent postulate placed on top of the earlier equations. It is the closure of the stated state model. The null equation defines a family of solutions, the dynamics selects the physical branch, and the intrinsic temporal variable does not reduce that branch to a sequence of exclusive proper-time presents. If the state is identified with the solutions admitted by the model, the complete family on the selected branch must be retained.

The distinction between the whole null cone and the whole selected state is important. A single photon does not contain every spatial direction in the local cone after its physical branch has been fixed. Initial data and geodesic propagation select the branch. Completeness refers to every admissible state on that branch: every event on its spacetime support and every real null representative on the corresponding local generator.

For any finite portion of the selected photon trajectory, the same conclusion holds without introducing additional notation. The complete state contains every event of that portion together with the admissible local null displacements carried by the selected generator. No photon proper-time value separates those members into an intrinsic sequence.

4.6 Observer description of the photon trajectory

4.6.1 Lorentz-related observers

The complete photon state has already been fixed by the null equation, the selected geodesic, and the zero proper-time result. A timelike observer does not alter that state. The observer only describes the same spacetime trajectory in a local orthonormal frame whose time axis is timelike.

Let two such local frames be related by a Lorentz transformation. Their tetrad components of the same displacement satisfy

$$\mathrm{d}x'^{A} = \Lambda^{A}{}_{B}\mathrm{d}x^{B}, \qquad \eta_{CD}\Lambda^{C}{}_{A}\Lambda^{D}{}_{B} = \eta_{AB}, \qquad \eta_{AB}\mathrm{d}x'^{A}\mathrm{d}x'^{B} = \eta_{AB}\mathrm{d}x^{A}\mathrm{d}x^{B} = 0. \tag{4.9}$$

The first relation changes the components assigned by the observer, while the second and third show that the Lorentzian metric and the null character of the photon displacement are preserved. Both observers therefore describe the same null solution and the same complete state.

The difference is temporal rather than intrinsic to the photon. Along the selected trajectory, one observer assigns the time component $x^{\hat{0}}$ and another assigns $x'^{\hat{0}}$. These quantities can distinguish different events on the trajectory even though the photon proper-time function is zero everywhere. Thus the observer supplies a nontrivial time coordinate with which the events can be read successively; the photon itself does not obtain a new proper-time ordering from this change of frame.

The causal orientation of a future-directed null trajectory is not reversed by an ordinary proper Lorentz transformation. What changes between observers are the numerical time and spatial components and the associated simultaneity surfaces. The selected null history $\gamma(I)$ and the full family of admissible displacements on it remain the same geometric state.

Figure 4.1 illustrates this distinction in a local Minkowski representation. The horizontal green lines represent constant $x^{\hat{0}}$ for one observer, while the dashed blue lines represent constant $x'^{\hat{0}}$ for another. They cut the same yellow null trajectory differently, although the trajectory itself is unchanged.

The standard local result that light propagates with speed c for every timelike observer already follows from the null condition together with Lorentz invariance and requires no additional state equation here. The role of the observer in this paper is only to provide a nontrivial temporal description of a state whose photon proper time remains zero.

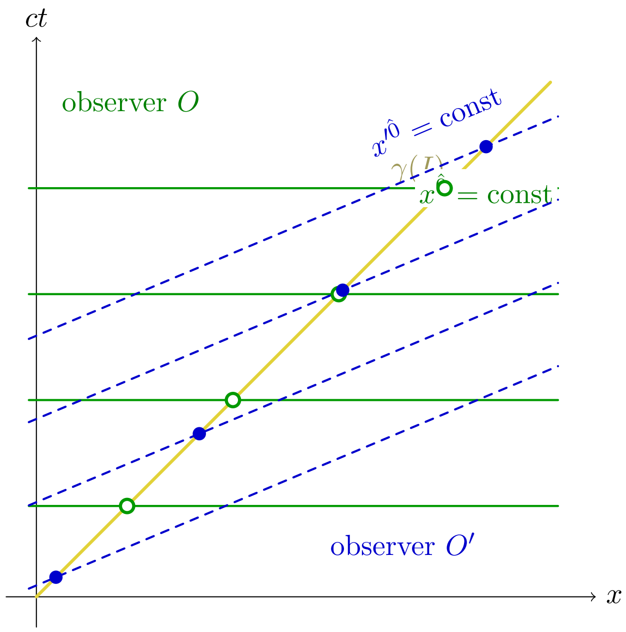

Figure 4.1: One selected photon trajectory described by two Lorentz-related observers. The yellow null history is unchanged. The green and dashed blue constant-time surfaces belong to different local orthonormal frames and assign different coordinate descriptions to the same photon state.

4.7 Complete-state result

The previous sections leave two independent freedoms that must be considered together. The geodesic equation fixes the null generator followed by the photon, but the null equation still admits every real multiple of that generator. At the same time, the metric proper time is zero on the complete trajectory. These statements can be collected without introducing any new state variable.

Proposition 4.2 (Complete-state result)**.** *Let $\gamma : I \to \mathcal{M}$ be the selected null geodesic and let $\mathrm{d}x^\mu = \lambda k^\mu(s)$ with $s \in I$ and $\lambda \in \mathbb{R}$. Every such displacement is null and has zero proper-time increment, while the accumulated photon proper time is zero at every event of $\gamma(I)$.*

Proof. For every real λ,

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \lambda^2 g_{\mu\nu}k^\mu k^\nu = 0, \qquad \mathrm{d}\tau^2 = -\frac{1}{c^2}g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 0. \tag{4.10}$$

The first equality follows from the selected tangent being null. The second is then the definition of the metric proper-time increment on that displacement. Since Eq. (4.5) keeps k^μ null along the complete geodesic, integration gives

$$\tau_\gamma(s_1) = \tau_\gamma(s_2) = 0, \qquad s_1, s_2 \in I. \tag{4.11}$$

Thus neither the null equation nor photon proper time isolates one member of the selected family. By the definition of $\mathcal{S}_\gamma$ in Eq. (4.6), all admissible displacements carried by all events of the selected trajectory remain members of the same photon state. □

Equation (4.10) concerns the local family of states, whereas Eq. (4.11) concerns the temporal relation between different events on the trajectory. Their roles are different but complementary. The first leaves the real parameter λ unrestricted; the second leaves no nontrivial photon proper-time value with which one event could be selected as the intrinsic present.

The result is independent of the orthonormal frame used to write it. The Lorentz transformation in Sec. 4.6 changes tetrad components while preserving the null scalar, and curvature is already included through the tetrad field and the covariant geodesic equation. The proposition therefore refers to the geometric null solution rather than to one coordinate representation.

4.8 Physical consequences

The mathematical result has an immediate local consequence. At any fixed event of the selected trajectory, the set of real parameters that satisfy the null state equation is

$$\{\lambda \in \mathbb{R} : \; g_{\mu\nu}(\lambda k^{\mu})(\lambda k^{\nu}) = 0\} = \mathbb{R}. \tag{4.12}$$

The null equation therefore excludes no real value of λ on the selected generator. Dynamics has selected the generator itself, but within that generator the state equation still contains the complete real family.

The temporal consequence is equally direct. Equation (4.11) means that photon proper time vanishes at every event of the selected trajectory,

$$\tau_{\gamma}(s) = 0, \qquad \forall s \in I. \tag{4.13}$$

Distinct events remain distinct spacetime events and retain the causal orientation of γ, but they are not separated by different photon proper-time values. The photon therefore has no intrinsic proper-time succession that could select one admissible event as present while excluding the others from the state.

Equations (4.12) and (4.13) give the physical content of the hypothesis. The first retains every admissible local state on the selected null generator. The second shows explicitly that photon proper time is zero everywhere on the selected trajectory and therefore cannot choose only one event. Under these conditions, the photon is in all admissible states of its selected null trajectory at once. Here “all” refers to the complete family contained in $\mathcal{S}_{\gamma}$, while “at once” refers to the absence of a nontrivial ordering by photon proper time.

A timelike observer supplies a different temporal description. Lorentz-related frames assign different time components to the same null displacement while preserving its null norm, as shown in Sec. 4.6. Those observer times can order the events of $\gamma(I)$ even though τ_{γ} cannot. The familiar motion of the photon is therefore an observer-time description of the complete null state, not an intrinsic proper-time evolution of that state.

The same conclusion applies to any physically selected portion of $\gamma(I)$. Restricting the trajectory changes which events are under consideration but does not change Eq. (4.12) or the zero proper-time relation. Because the derivation was written on a general Lorentzian spacetime, curvature changes the shape of the trajectory and the components measured in a particular frame without changing these two defining properties.

4.9 Summary and conclusion

The Null Perspective Hypothesis follows from one continuous construction. The null interval is written for the spacetime displacement $\mathrm{d}x^\mu$ in an orthonormal tetrad. Its temporal magnitude equals the norm of its spatial part, while the common scale remains free. For a chosen null direction this freedom is represented by the real parameter λ, and every $\lambda \in \mathbb{R}$ satisfies the same null equation.

Physical initial data and the geodesic equation then select the trajectory $\gamma(I)$ followed by the photon. They select the null generator but do not select one value of λ. Metric compatibility preserves nullness along the trajectory, and every admissible displacement has $\mathrm{d}\tau = 0$. Consequently the accumulated photon proper time is zero on the whole trajectory.

The complete state and its temporal property can therefore be written using only the notation already introduced,

$$\mathcal{S}_\gamma = \left\{ \left(\gamma(s), \mathrm{d}x^\mu\right) : s \in I,\ \mathrm{d}x^\mu = \lambda k^\mu(s),\ \lambda \in \mathbb{R} \right\}, \qquad \tau_\gamma(s) = 0, \qquad \forall s \in I. \tag{4.14}$$

The first set contains all admissible states of the selected null solution; the second states explicitly that photon proper time is zero at every event supporting that set.

According to these conditions, the photon is in all admissible states of its selected null trajectory at once. The null equation does not choose one state from the real family, and photon proper time does not choose one event as an intrinsic present. Timelike observers may order the same trajectory through their Lorentz-frame time coordinates, but that ordering does not alter which states belong to the photon.

Part II

Tensor Quantum Dynamics and Perturbative Structure

Chapter 5

Four-Index and Four-Sector Quantum Gravity

Source and status. This chapter incorporates the complete scientific body of *Four-Index and Four-Sector Quantum Gravity*, v0.76 (15 August 2026). Its role in the dissertation is: Tensor carrier, rank-eight evolution, probability law and hydrogen construction. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

This paper develops a tensorial quantum framework built on a four-index completion of the Einstein field equation. Each stationary mode is represented by one real conserved four-index source tensor whose contraction recovers the ordinary stress–energy tensor, while its uncontracted part retains the trace-free curvature information associated with the Weyl sector. Quantum states are formed only after the classical source has been fixed: dimensionless complex coefficients weight complete tensor configurations, a rank-eight operator governs their linear clock evolution, and scalar amplitudes are formed only by global tensor contractions.

The four-sector Lorentz construction is used as an atlas for transporting the same geometry and tensor mode across active charts. It does not supply the spectral coefficients or create additional quantum states. This separation is used in the hydrogen application, where a reciprocal Kerr–Newman chart gives a regular representation of the charged spinning electron constituent and the complete atom is assembled from one common electromagnetic field together with the additional conserved source contributions required by the model.

The electron–proton interaction is obtained from the cross-term of the total Maxwell stress tensor before the four-index lift is performed. The resulting central relative channel yields the relativistic bound-state spectrum, while a compatible tensorial radial realization produces normalizable Laguerre modes with the same eigenvalues. The signed angular label fixes the orbital branch and parity, and the global probability law follows from the same two-copy tensor contraction used in the general theory. Ground-state and first-excited radial profiles, their moments and degeneracies, and a radiative comparison channel are obtained within this single tensor construction. The resulting framework connects the conserved four-index source, rank-eight spectral evolution, regular four-sector geometry and hydrogen observables without introducing an independent scalar wavefunction or a second measurement dynamics.

5.1 Theory

5.1.1 Introduction

The purpose of this paper is to formulate quantum evolution directly on the complete tensor source of a gravitational configuration. The starting object is therefore neither a point-particle amplitude nor a scalar field placed on a fixed background. It is the real four-index source associated with one classical configuration and the geometry that solves its field equation. A single contraction recovers the ordinary stress–energy tensor, whereas the uncontracted tensor retains the directional trace-free information carried by the Weyl sector. The rank-two geometric foundation and its covariant conservation law are the standard Einstein construction [14, 63]; the four-index completion and the sector atlas are developed in the companion formulations [40, 41].

The construction follows a fixed order. Real geometric and source tensors are specified first and their conservation is imposed at the classical level. Complex coefficients are attached only to complete conserved configurations, and scalar observables are formed only after the rank-four carrier has evolved. This ordering separates the physical source, the quantum coefficient and the final scalar readout throughout the calculation.

The quantum carrier is a covariant rank-four tensor. Its clock evolution is linear and is generated by a rank-eight tensor acting on complete tensor profiles. Normalization and measurement are global contractions over a spatial hypersurface and therefore retain the full spacetime-index structure until the last step. The four Lorentz sectors play a different role: they form an atlas of active descriptions of the same geometry. A sector transition transports the same source, normalization and spectral eigenvalue and is not an additional quantum number.

Hydrogen provides the explicit application. The charged spinning electron is represented by a regular reciprocal Kerr–Newman chart, after which the atom is built from one common electromagnetic field and the additional conserved source contributions of the model. The electron–proton Maxwell cross-term produces the central interaction before any radial equation is introduced. A relative mass shell and its closed radial action determine the stationary spectrum; a tensorial radial realization supplies normalizable spatial modes; and the general rank-eight spectral construction fixes their clock evolution and global probabilities. The radiative comparison channel is then formed by modifying the same complete real source and applying the original tensor measurement law.

Section 5.1.2 establishes the four-index field equation and its conservation law. Sections 5.1.3–5.1.5 define the sector transport, rank-four carrier, rank-eight evolution and global probability rule. Section 5.2 applies this structure to hydrogen, and Section 5.3 collects the resulting tensor and spectral consequences.

5.1.2 Four-index field equation

This subsection supplies the classical backbone of the paper. Its purpose is to identify exactly which information is preserved by the ordinary Einstein equation and which information is lost when the curvature and source are contracted to rank two. The construction therefore begins with standard curvature objects and enlarges their bookkeeping at fixed spacetime rank; it does not replace the Einstein equation or introduce a second metric theory.

Spacetime has dimension $d > 3$, Greek letters denote spacetime indices, the metric signature is $(-+\cdots+)$, and the connection is metric compatible. The constants are

$$\kappa = \frac{8\pi G}{c^4}, \qquad \nabla_\rho g_{\mu\nu} = 0. \tag{5.1}$$

The dimensional factors are kept explicitly because the Weyl divergence and the Ricci-built divergence differ by the factor $d-3$; the curvature decomposition and its differential identities follow the standard pseudo-Riemannian tensor calculus [54, 63].

Geometric trace sector. The construction begins with the part of the Riemann curvature determined by the Ricci tensor and scalar. Its coefficients are chosen so that contraction over the first and third indices produces the ordinary Einstein tensor:

$$\begin{aligned}\widetilde{R}_{\alpha\mu\beta\nu} = {} & \frac{1}{d-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right) \\ & - \frac{d}{2(d-1)(d-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R. \end{aligned} \tag{5.2}$$

Equation (5.2) contains only the Ricci-built information. The remaining trace-free curvature is the Weyl tensor, so the complete geometric object is obtained by adjoining that sector with the relative divergence normalization shown below:

$$G_{\alpha\mu\beta\nu} = \widetilde{R}_{\alpha\mu\beta\nu} - \frac{1}{d-3}C_{\alpha\mu\beta\nu}. \tag{5.3}$$

The factor $(d-3)^{-1}$ places the Ricci-built and Weyl parts on the same divergence scale. It equals unity in four spacetime dimensions but is kept in all general equations.

The decomposition has a direct interpretive role. The Ricci-built part is the sector that can be reconstructed from the contracted field equation, whereas the Weyl part contains the trace-free directional curvature that survives in vacuum and controls tidal structure. Keeping both pieces at rank four makes it possible to couple them to source terms with the same algebraic symmetries and to discuss their local exchange before any quantum interpretation is introduced.

Matter and trace-free energy sectors. The source side is built in the same order. First the ordinary rank-two stress–energy tensor is lifted to a rank-four tensor with the Riemann symmetries:

$$\begin{aligned}\widetilde{T}_{\alpha\mu\beta\nu} = {} & \frac{1}{d-2}\left(T_{\alpha\beta}g_{\mu\nu} - T_{\alpha\nu}g_{\beta\mu} + T_{\mu\nu}g_{\alpha\beta} - T_{\beta\mu}g_{\alpha\nu}\right) \\ & - \frac{1}{(d-1)(d-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T. \end{aligned} \tag{5.4}$$

The lift in Eq. (5.4) contains the part visible after rank-two contraction. To retain the energy content corresponding to the Weyl sector, it is completed by the trace-free tensor $\widehat{T}_{\alpha\mu\beta\nu}$:

$$T_{\alpha\mu\beta\nu} = \widetilde{T}_{\alpha\mu\beta\nu} - \frac{1}{d-3}\widehat{T}_{\alpha\mu\beta\nu}. \tag{5.5}$$

For a source resolved into physical components, write $q = (q_1, \ldots, q_k)$, with $q_i \in \mathcal{I}_i$, for one complete stationary configuration and let $a = 1, \ldots, N_T$ enumerate its simultaneous real contributions $T^{(q)}_{a\,\alpha\mu\beta\nu}(\mathbf{x}) \in \mathbb{R}$. The label q is not a spacetime index. The complete source of that configuration is

$$T^{(q)}_{\alpha\mu\beta\nu}(\mathbf{x}) = \sum_{a=1}^{N_T} T^{(q)}_{a\,\alpha\mu\beta\nu}(\mathbf{x}). \tag{5.6}$$

All numbered components belong simultaneously to the same source. In particular, a component introduced to close the conservation law cannot be multiplied by an independent state amplitude without destroying that closure.

The classical field equation is solved for one physical configuration q at a time. A raw unweighted sum over mutually exclusive stationary configurations is not a classical source. A statistical or semiclassical average, when required, must be introduced separately with explicit probabilities and with a stated approximation for the geometry.

Complete field equation and its contraction. Two checks determine whether the four-index extension is compatible with its rank-two origin. The first is algebraic: contraction must reproduce the ordinary Einstein equation with no residual Weyl term. The second is differential: the Ricci-built and trace-free sectors must exchange locally so that their sum is conserved. For one fixed configuration q,

$$G^{(q)}_{\alpha\mu\beta\nu} = \kappa T^{(q)}_{\alpha\mu\beta\nu}. \tag{5.7}$$

The trace-built and trace-free parts satisfy

$$\begin{aligned} \widetilde{R}^{(q)}_{\alpha\mu\beta\nu} &= \kappa \widetilde{T}^{(q)}_{\alpha\mu\beta\nu}, \\ C^{(q)}_{\alpha\mu\beta\nu} &= \kappa \widehat{T}^{(q)}_{\alpha\mu\beta\nu}. \end{aligned} \tag{5.8}$$

Contracting the same index pair gives the ordinary field equation directly,

$$g^{\alpha\beta} G^{(q)}_{\alpha\mu\beta\nu} = G^{(q)}_{\mu\nu} = \kappa T^{(q)}_{\mu\nu} = \kappa g^{\alpha\beta} T^{(q)}_{\alpha\mu\beta\nu}, \tag{5.9}$$

with $T^{(q)}_{\mu\nu} = \sum_{a=1}^{N_T} T^{(q)}_{a\,\mu\nu}$. Thus the higher-rank equation contains the ordinary Einstein equation without replacing it. Ordinary matter-visible rest-frame energy and four-momentum are extracted from the rank-two contraction. The trace-free Weyl sector is absent from that trace and remains available only in the uncontracted source, from which directional energy and momentum readouts can be formed without violating Weyl tracelessness.

Covariant conservation. Algebraic consistency alone is insufficient. A source that reproduces the Einstein tensor after contraction must also reproduce the Bianchi cancellation before contraction. The divergence calculation below therefore identifies the local exchange between the matter-visible sector and the trace-free sector. Neither part is required to be separately constant; conservation belongs to their complete four-index sum. The conservation law follows by comparing the divergences of the two geometric sectors. The contracted Bianchi identity first gives the Ricci-built divergence

$$\nabla^\alpha \widetilde{R}_{\alpha\mu\beta\nu} = \frac{1}{d-2}\left[\nabla_\beta R_{\mu\nu} - \nabla_\nu R_{\beta\mu} - \frac{1}{2(d-1)}\left(g_{\mu\nu}\nabla_\beta - g_{\beta\mu}\nabla_\nu\right) R\right]. \tag{5.10}$$

The differential identity for the Weyl tensor produces the same bracket, multiplied only by $d-3$:

$$\nabla^{\alpha} C_{\alpha\mu\beta\nu} = \frac{d-3}{d-2}\left[\nabla_{\beta} R_{\mu\nu} - \nabla_{\nu} R_{\beta\mu} - \frac{1}{2(d-1)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right) R\right]. \tag{5.11}$$

Equations (5.10) and (5.11) immediately yield

$$\nabla^{\alpha} G_{\alpha\mu\beta\nu} = 0. \tag{5.12}$$

The source side must reproduce the same cancellation. Assuming the ordinary matter conservation law $\nabla^{\alpha} T_{\alpha\beta} = 0$, the lifted trace sector obeys

$$\nabla^{\alpha} \widetilde{T}_{\alpha\mu\beta\nu} = \frac{1}{d-2}\left[\nabla_{\beta} T_{\mu\nu} - \nabla_{\nu} T_{\beta\mu} - \frac{1}{d-1}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right) T\right]. \tag{5.13}$$

The corresponding divergence of the trace-free energy sector is

$$\nabla^{\alpha} \widehat{T}_{\alpha\mu\beta\nu} = \frac{d-3}{d-2}\left[\nabla_{\beta} T_{\mu\nu} - \nabla_{\nu} T_{\beta\mu} - \frac{1}{d-1}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right) T\right]. \tag{5.14}$$

Consequently the complete source satisfies

$$\nabla^{\alpha} T^{(q)}_{\alpha\mu\beta\nu} = \sum_{a=1}^{N_T} \nabla^{\alpha} T^{(q)}_{a\,\alpha\mu\beta\nu} = 0, \tag{5.15}$$

and changes in the rank-two matter sector are balanced by the traceless four-index energy sector inside one local conservation law.

The result closes the classical layer of the construction. The object that will carry the quantum state is consequently not an arbitrary rank-four field, but the same real, conserved source that already contains both the contracted matter information and the directional energy associated with the Weyl sector.

5.1.3 Four-sector Lorentz geometry

The field equation fixes the tensor content but does not yet specify how one continuous geometry is represented when its active time and space orientations pass through null boundaries. The four-sector construction supplies that atlas. It is introduced before the quantum carrier because the carrier must be transported on a geometry whose sector transitions and active coordinates are already defined.

Sector representatives. Let $n \in \mathbb{Z}_4$ label the geometry sector and let I_n denote the corresponding chart identification. The coframe carries the full orientation phase, while the metric carries its square:

$$\vartheta^{\hat{a}}_{(n)} = \mathrm{i}^{n} I_n^{*} \vartheta^{\hat{a}}_{(0)}, \qquad g_{(n)\alpha\beta} = \mathrm{i}^{2n} I_n^{*} g_{(0)\alpha\beta}. \tag{5.16}$$

The squared phase is $(-1)^n$, so the inherited quadratic sign alternates between neighbouring sectors. Multiplication of the metric by this nonzero constant leaves the null set unchanged. A particle has its own relative sector label $s \in \mathbb{Z}_4$, and its local transformation is therefore

$$\Lambda_{(s)}{}^{\alpha}{}_{\beta} = \mathrm{i}^{s} \Lambda^{\alpha}{}_{\beta}. \tag{5.17}$$

The particle-sector label and the geometry-sector label are independent. A change of observer does not create additional physical particles, and a classical massive particle is not accelerated through the null boundary.

The phase factors should therefore be read as orientation data attached to coframes and chart identifications. They do not turn the four sectors into four independent backgrounds. Metric contractions, null boundaries, and the physical source are all evaluated within the active chart of the same underlying spacetime.

Sector-covariant source readout. The orientation sign must be tracked at the tensor rank at which the field equation is written. Before the quadratic metric sign is attached, the physical rank-two source and the rank-two Einstein tensor are ordinary chart pullbacks,

$$T^{(n)}_{\mu\nu} = I_n^* T^{(0)}_{\mu\nu}, \qquad G^{(n)}_{\mu\nu} = I_n^* G^{(0)}_{\mu\nu}. \tag{5.18}$$

For all-lower curvature tensors the same constant sign used in the metric is introduced when an index is lowered. Consequently

$$\begin{aligned} C^{(n)}_{\alpha\mu\beta\nu} &= (-1)^n I_n^* C^{(0)}_{\alpha\mu\beta\nu}, \\ G^{(n)}_{\alpha\mu\beta\nu} &= (-1)^n I_n^* G^{(0)}_{\alpha\mu\beta\nu}, \\ T^{(n)}_{\alpha\mu\beta\nu} &= (-1)^n I_n^* T^{(0)}_{\alpha\mu\beta\nu}. \end{aligned} \tag{5.19}$$

The last line is required by $G^{(n)}_{\alpha\mu\beta\nu} = \kappa T^{(n)}_{\alpha\mu\beta\nu}$ and is consistent with $C^{(n)} = \kappa \widehat{T}^{(n)}$. It does not alter any quadratic scalar contraction because the constant sign appears twice.

For a real electromagnetic two-form transported as $F^{(n)} = I_n^* F^{(0)}$, the standard Maxwell bracket evaluated with the signed metric $g_{(n)} = (-1)^n I_n^* g_{(0)}$ acquires one factor $(-1)^n$. The physical stress tensor is the rank-two pullback in Eq. (5.18), so its active-sector expression is

$$M^{(n)}_{\mu\nu} = \frac{(-1)^n}{\mu_0} \left(F^{(n)}_{\mu\rho} F^{(n)\rho}_{\nu} - \frac{1}{4} g^{(n)}_{\mu\nu} F^{(n)}_{\rho\sigma} F^{\rho\sigma}_{(n)} \right). \tag{5.20}$$

All index raising in the bracket uses the active signed metric. Thus Eq. (5.20) reduces to the ordinary Maxwell tensor in an even sector and gives $M^{(1)} = I_1^* M^{(0)}$ in the side sector, preserving the unchanged rank-two Einstein equation without assigning a second source to the same field.

The sector transformation is therefore used only to transport a fixed physical configuration between active charts. The tensor source, its contractions and the spectral data are carried by pullback with the sector sign appropriate to their index structure; no additional state variable is introduced by the chart change.

5.1.4 Fundamental indexed tensor equations

The quantum theory is defined before any stationary decomposition is chosen. Every tensor field in this subsection is a function only of the spacetime point $\mathbf{x}$. In an integral over Σ_t, the same symbol denotes the restriction of that point to the hypersurface with x^0 fixed. Discrete and continuous physical degrees of freedom occur as indices of the coefficient and source tensors; they are not additional arguments of the carrier.

The fundamental rank-eight evolution equation is

$$i\ell_P \nabla_0 \Psi_{\alpha\mu\beta\nu}(\mathbf{x}) = H^{\delta\gamma\phi\psi}_{\alpha\mu\beta\nu} \Psi_{\delta\gamma\phi\psi}(\mathbf{x}). \tag{5.21}$$

The four lower indices of $H^{\delta\gamma\phi\psi}_{\alpha\mu\beta\nu}$ identify the output tensor positions and the four upper indices identify the contracted input positions. No stationary eigenvalue or clock factor has been assumed in Eq. (5.21).

Only after the indexed evolution is specified are the complete scalar contractions formed:

$$\Psi(\mathbf{x}) = \Psi_{\alpha\mu\beta\nu}(\mathbf{x})\Psi^{\alpha\mu\beta\nu}(\mathbf{x}) = \Psi_{\delta\gamma\phi\psi}(\mathbf{x})\Psi_{\alpha\mu\beta\nu}(\mathbf{x}) g^{\delta\alpha} g^{\gamma\mu} g^{\phi\beta} g^{\psi\nu}. \tag{5.22}$$

The independent conjugate scalar is obtained by the same complete contraction of the conjugate carrier, so no second contraction rule is introduced.

Let $q_1, \ldots, q_k$ denote discrete degrees of freedom with $q_a \in \mathcal{I}_a$, and let $\xi_1, \ldots, \xi_m$ denote continuous degrees of freedom with $\xi_b \in \Gamma_b$. The single carrier containing both types is

$$\begin{aligned} \Psi_{\alpha\mu\beta\nu}(\mathbf{x}) = \ell_P^{\frac{5-d}{2}} \kappa \sum_{q_1 \in \mathcal{I}_1} \cdots \sum_{q_k \in \mathcal{I}_k} \int_{\Gamma_1} \cdots \int_{\Gamma_m} \\ \times c_{q_1 \ldots q_k \xi_1 \ldots \xi_m} T^{(q_1 \ldots q_k, \xi_1 \ldots \xi_m)}_{\alpha\mu\beta\nu}(\mathbf{x})\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_m. \end{aligned} \tag{5.23}$$

Every degree of freedom is therefore retained as an explicit index. If no continuous degree is present the integrals are omitted; if no discrete degree is present the sums are omitted. The coefficients are dimensionless complex scalars, the complete source tensors are real, and the independent conjugate carrier is obtained from Eq. (5.23) by $c_{q_1 \ldots q_k \xi_1 \ldots \xi_m} \mapsto c^*_{q_1 \ldots q_k \xi_1 \ldots \xi_m}$. No second set of source tensors is introduced.

On algebraic-curvature tensors the rank-eight operator retains the pair symmetries

$$\begin{aligned} H^{\delta\gamma\phi\psi}_{\alpha\mu\beta\nu} &= -H^{\delta\gamma\phi\psi}_{\mu\alpha\beta\nu} = -H^{\delta\gamma\phi\psi}_{\alpha\mu\nu\beta} = H^{\delta\gamma\phi\psi}_{\beta\nu\alpha\mu}, \\ H^{\delta\gamma\phi\psi}_{\alpha\mu\beta\nu} &= -H^{\gamma\delta\phi\psi}_{\alpha\mu\beta\nu} = -H^{\delta\gamma\psi\phi}_{\alpha\mu\beta\nu} = H^{\phi\psi\delta\gamma}_{\alpha\mu\beta\nu}. \end{aligned} \tag{5.24}$$

Its detailed differential or integral realization is fixed by the physical system. The stationary spectral realization used for hydrogen is introduced only in Section 5.2.2.

5.1.5 Tensor normalization and probability

The rank-eight equation determines the evolution of the carrier, whereas normalization and measurement require complete scalar contractions. Since the carrier is a spacetime field, these contractions are performed globally on a spatial hypersurface and only after all four tensor indices have been retained. The coefficients in Eq. (5.23) are dimensionless. Since $[\kappa T_{\alpha\mu\beta\nu}] = L^{-2}$, the carrier and comparison channel both have dimension $L^{-(d-1)/2}$. One contracted hypersurface overlap is therefore dimensionless in any spacetime dimension, with the required power of ℓ_P already contained in the carrier definition.

Normalization compares the prepared carrier with itself and with its independent conjugate copy. Keeping the two hypersurface variables separate gives

$$\begin{aligned} 1 = \int_{\Sigma_t \times \Sigma_t \subset \mathcal{M}^d} \sqrt{h(\mathbf{x})}\sqrt{h(\mathbf{y})} \\ \times \Psi_{\delta\gamma\phi\psi}(\mathbf{x})\Psi^{\delta\gamma\phi\psi}(\mathbf{x})\Psi_{*\alpha\mu\beta\nu}(\mathbf{y})\Psi_*^{\alpha\mu\beta\nu}(\mathbf{y})\mathrm{d}^{\,d-1}\mathbf{x}\,\mathrm{d}^{\,d-1}\mathbf{y}. \end{aligned} \tag{5.25}$$

The two integrations are part of the definition: a local modulus of one contraction is not substituted for the global two-copy quantity.

A detector channel $\Phi_{\alpha\mu\beta\nu}$ is compared with the same prepared carrier by replacing the second tensor in each ordered contraction,

$$\begin{aligned}\rho^2(\Psi,\Phi) = &\int_{\Sigma_t\times\Sigma_t\subset\mathcal{M}^d} \sqrt{h(\mathbf{x})}\sqrt{h(\mathbf{y})} \\ &\times \Psi_{\delta\gamma\phi\psi}(\mathbf{x})\Phi^{\delta\gamma\phi\psi}(\mathbf{x})\Phi_{*\alpha\mu\beta\nu}(\mathbf{y})\Psi_*^{\alpha\mu\beta\nu}(\mathbf{y})\mathrm{d}^{\,d-1}\mathbf{x}\,\mathrm{d}^{\,d-1}\mathbf{y}.\end{aligned} \tag{5.26}$$

Writing the single contracted hypersurface overlap as $\mathcal{A}(\Psi,\Phi)$, the double integral factorizes as $\rho^2(\Psi,\Phi) = \mathcal{A}(\Psi,\Phi)\mathcal{A}^*(\Psi,\Phi)$. The tensor contraction is therefore completed before the final product of amplitudes is formed, so ordered cross-terms and their phases are retained. Substituting the general carrier into Eqs. (5.25) and (5.26) produces four independent copies of every discrete or continuous mode label. No stationary ansatz or energy eigenvalue is required by this probability law; the hydrogen specialization below merely resolves the same construction in a discrete stationary basis.

5.2 Hydrogen

The hydrogen construction begins with the regular charged spinning geometry used for the electron. Kerr–Newman is therefore presented here as the first stage of the atomic calculation rather than as an independent element of the general theory.

The section proceeds from geometry to spectroscopy. It first verifies that the charged spinning constituent has a regular active continuation. It then assembles the electromagnetic, bridge, reaction, and optional material contributions into one conserved atomic source. Finally it contracts and projects that source to derive the central interaction, the radial action, the stationary spectrum, and the global tensor probabilities. No stage is treated as an independent model detached from the preceding one.

5.2.1 Regular four-sector Kerr–Newman geometry

Physical parameters and active boundary

The Kerr–Newman geometry is used here for a specific purpose: it provides a geometric representation of a constituent that simultaneously carries mass, charge, and angular momentum. It is not yet the geometry of the full atom. The boundary introduced below identifies where the ordinary active chart is replaced by the side chart while preserving the same physical constituent. A charged spinning constituent is represented by the Kerr–Newman parameters

$$r_s = \frac{2Gm}{c^2}, \qquad r_Q^2 = \frac{GQ^2}{4\pi\varepsilon_0 c^4}, \qquad a_{\sigma_e} = \frac{J_{\sigma_e}}{mc}, \qquad J_{\sigma_e} = \sigma_e\frac{\hbar}{2}, \qquad \sigma_e\in\{-1,+1\}. \tag{5.27}$$

The two values of σ label two admissible real orientations of the complete geometry. They do not represent two simultaneous copies of the particle. The positive active boundary used by the charged side chart is

$$R_h(\theta) = \frac{-r_s + \sqrt{r_s^2 + 4\left(r_Q^2 - a_{\sigma_e}^2\cos^2\theta\right)}}{2}, \tag{5.28}$$

and write $R_h'(\theta) = \mathrm{d}R_h/\mathrm{d}\theta$ for its derivative. The local side chart is the domain $0 < \rho \le R_h(\theta)$ with $|a_{\sigma_e}\cos\theta| < r_Q$.

Reciprocal radial scale

The reciprocal scale converts the direction that reaches the ring in the ordinary continuation into an asymptotic end of the side chart. Because the transition radius depends on angle, this is not a purely radial substitution. The angular contribution to its differential is required for the transformed metric to remain the exact pullback of the original geometry. The ordinary Kerr–Newman radial scale is represented in the active side chart by

$$\mathcal{R}(\rho,\theta) = \frac{R_h^2(\theta)}{\rho}. \tag{5.29}$$

Because the boundary depends on θ, its full differential must retain both terms:

$$\mathrm{d}\mathcal{R} = -\frac{R_h^2(\theta)}{\rho^2}\mathrm{d}\rho + \frac{2R_h(\theta)R_h'(\theta)}{\rho}\mathrm{d}\theta. \tag{5.30}$$

Define

$$\begin{aligned}
\Sigma_1(\rho,\theta) &= \mathcal{R}^2(\rho,\theta) + a_{\sigma_e}^2\cos^2\theta,\\
F_1(\rho,\theta) &= r_s\mathcal{R}(\rho,\theta) - r_Q^2,\\
\chi_1(\rho,\theta) &= \frac{F_1(\rho,\theta)}{\Sigma_1(\rho,\theta)},\\
P_1(\rho,\theta) &= \left[\mathcal{R}^2(\rho,\theta) + a_{\sigma_e}^2\right]\sin^2\theta + a_{\sigma_e}^2\chi_1(\rho,\theta)\sin^4\theta.
\end{aligned} \tag{5.31}$$

Exact regular metric

The full metric is displayed because regularity cannot be inferred from a radial rule alone. Every mixed term generated by the angularly dependent transformation must be retained. The determinant then provides the first local check: the large coordinate coefficients near the reciprocal endpoint must describe a nondegenerate chart rather than a new metric singularity. In the real active coordinates $x_{(1)}^\mu = (W,\rho,\theta,\widetilde{\phi})$, the exact Kerr–Newman representative with signature $(+---)$ is

$$\begin{aligned}
\mathrm{d}s_{(1)}^2 = {} & (1-\chi_1)\,\mathrm{d}W^2 + 2\epsilon\frac{R_h^2}{\rho^2}\mathrm{d}W\,\mathrm{d}\rho - \frac{4\epsilon R_h R_h'}{\rho}\mathrm{d}W\,\mathrm{d}\theta - \Sigma_1\mathrm{d}\theta^2\\
& - 2\epsilon a_{\sigma_e}\frac{R_h^2}{\rho^2}\sin^2\theta\,\mathrm{d}\rho\,\mathrm{d}\widetilde{\phi} + \frac{4\epsilon a_{\sigma_e}R_h R_h'}{\rho}\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}\widetilde{\phi}\\
& + 2a_{\sigma_e}\chi_1\sin^2\theta\,\mathrm{d}W\,\mathrm{d}\widetilde{\phi} - P_1\mathrm{d}\widetilde{\phi}^2, \qquad \epsilon^2 = 1.
\end{aligned} \tag{5.32}$$

All functions in Eq. (5.32) are evaluated at (ρ,θ). The terms $\mathrm{d}W\,\mathrm{d}\theta$ and $\mathrm{d}\theta\,\mathrm{d}\widetilde{\phi}$ are required by the angular dependence of R_h; omitting them would not give the pullback of the Kerr–Newman geometry. The absence of a $\mathrm{d}\rho^2$ term follows from the regular null form of the initial chart rather than from a degeneracy. Indeed,

$$\det g_{(1)} = -\frac{R_h^4(\theta)}{\rho^4}\Sigma_1^2(\rho,\theta)\sin^2\theta. \tag{5.33}$$

The metric is therefore nondegenerate throughout the local domain away from the ordinary axial coordinate degeneracy.

Nondegeneracy is necessary but not sufficient. A coordinate transformation can make components large or small without changing the curvature, so the physical regularity claim must be tested with invariants. The next subsection therefore evaluates the full Kretschmann scalar and follows it from the sector boundary to the reciprocal endpoint.

Kretschmann scalar and the removed ring singularity

Regularity is tested by curvature invariants rather than by individual metric components. The latter contain inverse powers of ρ because the reciprocal coordinate compactifies an asymptotic end. With $\mathcal{R} = R_h^2(\theta)/\rho$, the exact Kerr–Newman Kretschmann scalar $K = R_{\alpha\mu\beta\nu}R^{\alpha\mu\beta\nu}$ is [34]

$$\begin{aligned} K_{(1)}(\rho,\theta) = \frac{1}{\Sigma_1^6}\Big\{ & 12r_s^2\left(\mathcal{R}^6 - 15a_{\sigma_e}^2\mathcal{R}^4\cos^2\theta + 15a_{\sigma_e}^4\mathcal{R}^2\cos^4\theta - a_{\sigma_e}^6\cos^6\theta\right) \\ & - 48r_s r_Q^2\mathcal{R}\left(\mathcal{R}^4 - 10a_{\sigma_e}^2\mathcal{R}^2\cos^2\theta + 5a_{\sigma_e}^4\cos^4\theta\right) \\ & + 8\left(r_Q^2\right)^2\left(7\mathcal{R}^4 - 34a_{\sigma_e}^2\mathcal{R}^2\cos^2\theta + 7a_{\sigma_e}^4\cos^4\theta\right)\Big\}. \end{aligned} \tag{5.34}$$

This expression is the ordinary Kerr–Newman invariant evaluated on the reciprocal active radius. It is therefore an exact scalar of the metric in Eq. (5.32), not an additional ansatz.

The Boyer–Lindquist ring direction is the equatorial direction $\theta = \pi/2$. There the rotational terms disappear from the scalar and Eq. (5.34) reduces exactly to

$$K_{(1)}\left(\rho,\frac{\pi}{2}\right) = \frac{12r_s^2\rho^6}{R_0^{12}} - \frac{48r_s r_Q^2\rho^7}{R_0^{14}} + \frac{56\left(r_Q^2\right)^2\rho^8}{R_0^{16}}, \qquad R_0 = R_h\left(\frac{\pi}{2}\right). \tag{5.35}$$

Equation (5.35) gives $K_{(1)}(\rho,\pi/2) \to 0$ as $\rho \to 0$, so the reciprocal endpoint is asymptotically flat in curvature rather than a finite-curvature replacement core. At the sector boundary $\rho = R_0$ the same polynomial is finite. To check the behaviour between these two limits, set $u = \rho/R_0$ and $x = r_s/R_0$. Differentiation gives

$$\frac{\mathrm{d}K_{(1)}}{\mathrm{d}u} = \frac{8u^5}{R_0^4}\left[56(1+x)^2u^2 - 42x(1+x)u + 9x^2\right] > 0, \qquad 0 < u \le 1, \tag{5.36}$$

because the quadratic bracket has the negative discriminant $-252x^2(1+x)^2$. Since motion from the boundary toward the side endpoint corresponds to decreasing u, the equatorial curvature decreases strictly and monotonically after the transition. Near $\rho = 0$,

$$K_{(1)}\left(\rho,\frac{\pi}{2}\right) = \frac{12r_s^2}{R_0^{12}}\rho^6 + O(\rho^7). \tag{5.37}$$

For general θ, the Lorentzian Kretschmann scalar can change sign in rotating geometries, but every angular profile still tends to zero as $O(\rho^6)$. The equatorial profile is the relevant one for the ordinary Kerr–Newman ring and is everywhere positive and monotone in the active side coordinate.

For the electron parameters the boundary is finite by the same analytic expression, while the reciprocal endpoint remains curvature-flat. No numerical substitution is required for this regularity statement.

The result distinguishes this continuation from a prescription that merely cuts off the curvature at a large value. The invariant is finite at the transition, decreases throughout the equatorial side branch, and reaches an asymptotically flat end. Regularity is therefore a property of the completed active geometry, not a cancellation imposed on the scalar after it has been calculated.

Electromagnetic field and asymptotic curvature

A charged geometry must be accompanied by its electromagnetic field in the same chart. Using one active metric and one gauge potential ensures that the Maxwell source, the Ricci curvature, and the Weyl curvature refer to the same local geometry. This is essential before the constituent can be embedded in the common atomic source. A regular gauge potential on the same active chart is

$$A_{(1)} = -\frac{Q\mathcal{R}}{\Sigma_1}\left(\mathrm{d}W - a_{\sigma_e}\sin^2\theta\,\mathrm{d}\widetilde{\phi}\right), \qquad F^{(1)}_{\mu\nu} = 2\nabla_{[\mu}A^{(1)}_{\nu]}. \tag{5.38}$$

Thus the metric, Maxwell tensor, Riemann tensor and Weyl tensor are calculated in one and the same active chart. The real two-form in Eq. (5.38) is inserted into the sector-covariant Maxwell source (5.20); in particular $M^{(1)}_{\mu\nu} = I_1^* M^{(0)}_{\mu\nu}$ even though the active metric has the opposite quadratic sign. The exact result (5.34) establishes the regularity of the full Riemann curvature. Its decomposition also gives

$$C_{\alpha\mu\beta\nu}C^{\alpha\mu\beta\nu} = O(\rho^6), \qquad M_{\mu\nu}M^{\mu\nu} = O(\rho^8), \tag{5.39}$$

so the direction toward the Boyer–Lindquist ring is represented by an asymptotically flat end of the active side chart rather than by a curvature singularity.

One normalization in two active charts

The ordinary and side descriptions cover one physical state, so a change of active chart must not create a second normalization constant or double the probability. The relevant test is therefore the pullback of both the tensor contraction and the hypersurface measure. The endpoint behaviour also checks that the regular continuation does not hide a divergent normalization density. The reciprocal map has the exact spatial Jacobian

$$J_{(1)} = \left|\det\frac{\partial(\mathcal{R},\theta,\widetilde{\phi})}{\partial(\rho,\theta,\widetilde{\phi})}\right| = \frac{R_h^2(\theta)}{\rho^2}. \tag{5.40}$$

The tensor self-contraction is a scalar and the induced hypersurface measure is pulled back with Eq. (5.40). Applying the pullback to both independent hypersurface variables preserves one normalization, $\mathcal{N}_{(1)} = \mathcal{N}_{(0)} = 1$; the two chart values are not added. At the side endpoint one copy of the invariant integrand behaves as

$$\mathrm{d}\Sigma_{(1)}\left(\Psi_{\alpha\mu\beta\nu}\Psi^{\alpha\mu\beta\nu}\right) = O(\rho^2)\,\mathrm{d}\rho\,\mathrm{d}\Omega, \tag{5.41}$$

so the two-copy normalization remains finite without an arbitrary cutoff.

The constituent geometry is now ready to enter the atomic construction. The regular chart supplies the electron representation, while the next subsection adds the proton field, interaction terms, bridge, and response of the complete atom. The electron is not inserted again as an independent point source.

5.2.2 Conserved atomic source

The hydrogen calculation follows the same order as the general formalism. Four numbered rank-two contributions are assembled into one conserved source, and only then is the source lifted to the four-index tensor and evolved by the rank-eight operator.

This order prevents the atomic calculation from becoming a list of unrelated terms. Each numbered contribution has a distinct physical role, but none is interpreted as a complete atom by itself. Their sum is constrained by one conservation law, one common metric, and one Weyl sector. The state coefficients are attached only after this real atomic source has been built.

Atomic geometry, numbered source and Weyl-resolved carrier

The atomic state is defined before its quantum coefficient is introduced. A stationary configuration carries the radial label n_r, total angular label j, its projection m_j, the signed central orientation κ_D, and the two constituent orientation signs σ_e and σ_p. The allowed signed branch is

$$\mathcal{K}_{n_r j} = \begin{cases} \left\{-\left(j+\frac{1}{2}\right)\right\}, & n_r = 0, \\ \left\{-\left(j+\frac{1}{2}\right), +\left(j+\frac{1}{2}\right)\right\}, & n_r \geq 1. \end{cases} \tag{5.42}$$

Here $n_r \in \mathbb{N}_0$, $j \in \{\frac{1}{2}, \frac{3}{2}, \ldots\}$, $m_j = -j, \ldots, j$, $\kappa_D \in \mathcal{K}_{n_r j}$, and $\sigma_e, \sigma_p = \pm 1$. The orbital branch, parity and principal level are not independent labels but are fixed by

$$\ell(\kappa_D) = \begin{cases} -\kappa_D - 1, & \kappa_D < 0, \\ \kappa_D, & \kappa_D > 0, \end{cases} \qquad \pi_q = (-1)^{\ell(\kappa_D)}, \qquad N = n_r + |\kappa_D| = n_r + j + \frac{1}{2}. \tag{5.43}$$

The positive branch is absent at $n_r = 0$, because it would require $\ell = N$ rather than the bound-state condition $\ell \leq N - 1$. In the remainder of the atomic calculation the six labels are collected into $q = (n_r, j, m_j, \kappa_D, \sigma_e, \sigma_p) \in \mathcal{Q}$, which identifies one complete tensor configuration and is not a spacetime index.

A constituent is represented either by a complete geometry or by an explicit material source, never by both. The electron is represented by the regular Kerr–Newman geometry of Section 5.2.1 with $m = m_e$, $Q = -e$ and $a_{e,\sigma_e} = \sigma_e \hbar/(2 m_e c)$. If instead a constituent $x \in \{e, p\}$ is represented materially, its point contribution may be written

$$T_x^{\mu\nu}(\mathbf{y}) = m_x c \int u_x^\mu u_x^\nu \frac{\delta^{(d)}(y - z_x(\tau_x))}{\sqrt{-g(y)}} \mathrm{d}\tau_x. \tag{5.44}$$

This alternative prevents mass, charge or spin already encoded in a complete constituent geometry from being inserted again as a delta-supported source.

The atom has one common metric rather than a sum of constituent metrics. Its electromagnetic field is assembled first,

$$f_{\mu\nu} = f_{e\mu\nu} + f_{p\mu\nu} + f_{b\mu\nu}, \tag{5.45}$$

and the first source component is the sector-covariant Maxwell tensor

$$T_{1\mu\nu} = \frac{(-1)^{n_g}}{\mu_0}\left(f_{\mu\rho}f_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}f_{\rho\sigma}f^{\rho\sigma}\right). \tag{5.46}$$

Because the stress tensor is formed from the complete field, all self terms and all interaction cross-terms are present before the radial reduction. In particular, the electron–proton contribution is

$$T_{c\mu\nu} = \frac{(-1)^{n_g}}{\mu_0}\left(f_{e\mu\rho}f_{p\nu}{}^\rho + f_{p\mu\rho}f_{e\nu}{}^\rho - \frac{1}{2}g_{\mu\nu}f_{e\rho\sigma}f_p^{\rho\sigma}\right), \tag{5.47}$$

which supplies the electric Coulomb interaction together with its magnetic counterpart.

The second component describes the directed sector bridge. Choose paired null directions $k_\pm^\mu = u^\mu/c \pm e^\mu$, where $u^2 = -c^2$, $e^2 = 1$ and $u \cdot e = 0$, and write $T_2^{\mu\nu} = -\rho_+ k_+^\mu k_+^\nu - \rho_- k_-^\mu k_-^\nu$. With $\rho_2 = \rho_+ + \rho_-$ and $j_2 = \rho_+ - \rho_-$, the same tensor is

$$T_2^{\mu\nu} = -\rho_2\left(\frac{u^\mu u^\nu}{c^2} + e^\mu e^\nu\right) - \frac{2j_2}{c}u^{(\mu}e^{\nu)}, \qquad |j_2| \le \rho_2. \tag{5.48}$$

For a stationary bridge $j_2 = 0$, equivalently $\rho_+ = \rho_-$, so the net directed flux vanishes while the paired null directions retain their local stress.

The third component is the non-electromagnetic response required by local conservation. With $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu/c^2$, decompose it as

$$T_3^{\mu\nu} = \frac{\varepsilon_3}{c^2}u^\mu u^\nu + \frac{2}{c}u^{(\mu}f_3^{\nu)} + p_3 h^{\mu\nu} + \pi_3^{\mu\nu}. \tag{5.49}$$

The spatial pieces obey $u_\mu f_3^\mu = 0$, $u_\mu \pi_3^{\mu\nu} = 0$ and $h_{\mu\nu}\pi_3^{\mu\nu} = 0$. The fourth component $T_4^{\mu\nu}$ is included only for material not already encoded geometrically. Once the other contributions have been chosen, conservation fixes the response through

$$\nabla_\mu T_3^{\mu\nu} = -\nabla_\mu\left(T_1^{\mu\nu} + T_2^{\mu\nu} + T_4^{\mu\nu}\right). \tag{5.50}$$

Thus the response is not an independently weighted correction but the part required to close the complete atomic source.

For each fixed q the simultaneous rank-two source is therefore

$$T_{\mu\nu}^{(q)} = \sum_{a=1}^{4} T_{a\,\mu\nu}^{(q)}, \qquad \nabla_\mu T^{(q)\,\mu\nu} = 0. \tag{5.51}$$

Its trace is $T^{(q)} = -\varepsilon_3 + (d-1)p_3 + T_4$, with $T_4 = g_{\mu\nu}T_4^{\mu\nu}$. The general lift of Eqs. (5.4)–(5.5) is then applied once to this already assembled source. In particular,

$$C_{\alpha\mu\beta\nu}^{(q)} = \kappa\sum_{a=1}^{4}\widehat{T}_{a\,\alpha\mu\beta\nu}^{(q)}, \qquad T_{\alpha\mu\beta\nu}^{(q)} = \sum_{a=1}^{4}T_{a\,\alpha\mu\beta\nu}^{(q)}, \qquad \nabla^\alpha T_{\alpha\mu\beta\nu}^{(q)} = 0, \qquad g^{\alpha\beta}T_{\alpha\mu\beta\nu}^{(q)} = T_{\mu\nu}^{(q)}. \tag{5.52}$$

The Weyl contribution is consequently fixed by the common atomic geometry; it is not reconstructed by adding isolated constituent Weyl tensors.

Only after this classical source has been fixed are quantum coefficients attached to its complete stationary configurations. In four dimensions the hydrogen carrier is

$$\Psi_{\alpha\mu\beta\nu}(\mathbf{x}) = \sqrt{\ell_P}\,\kappa \sum_{q\in\mathcal{Q}} c_q(x^0) T^{(q)}_{\alpha\mu\beta\nu}(\mathbf{x}). \tag{5.53}$$

Its conjugate follows from $c_q \mapsto c_q^*$. No numbered component of one configuration receives an independent amplitude: interference between its physical contributions is already contained inside $T^{(q)}$, whereas interference between mutually exclusive stationary configurations appears only when the complete modes are combined in the carrier.

5.2.3 Central tensor reduction and bound-state spectrum

The central reduction uses only quantities defined by the complete tensor source and the centre-of-mass clock. Let u^α be that clock and $h^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta/c^2$ its spatial projector. The projected relative motion is encoded by the orbital bivector and its spatial dual,

$$l_{\alpha\beta} = 2h_\alpha{}^\rho h_\beta{}^\sigma x_{[\rho}p_{\sigma]}, \qquad l^\alpha = \frac{1}{2c}\epsilon^{\alpha\beta\gamma\delta}u_\beta l_{\gamma\delta}. \tag{5.54}$$

The intrinsic orientation s^α is spatial, has $h_{\alpha\beta}s^\alpha s^\beta = \hbar^2/4$, and combines with the orbital part as $j^\alpha = l^\alpha + s^\alpha$. A state carrying the quantum number j therefore has

$$h_{\alpha\beta}j^\alpha j^\beta = j(j+1)\hbar^2, \qquad k^2 = h_{\alpha\beta}j^\alpha j^\beta + \frac{\hbar^2}{4} = \left(j+\frac{1}{2}\right)^2\hbar^2. \tag{5.55}$$

The scalar k^2 is the angular invariant entering the radial mass shell; no independent scalar angular model is introduced.

Energy and spectrum extracted from the four-index source

The spectral calculation now returns to the conserved source rather than postulating a potential or a scalar wave equation. The rank-two energy tensor is first recovered by contraction, its interaction energy is identified in the Maxwell cross-term, and a central relative channel is then isolated. The stationary levels follow from the invariant mass shell and the closure of the radial action.

For a fixed stationary configuration, contraction of the complete four-index source gives the rank-two energy tensor used by the centre-of-mass observer. The contraction, local energy density and total stationary energy are

$$\begin{aligned} T^{(q)}_{\mu\nu} &= g^{\alpha\beta}T^{(q)}_{\alpha\mu\beta\nu} = \sum_{a=1}^{4} T^{(q)}_{a\,\mu\nu}, \\ \varepsilon^{(q)} &= \frac{1}{c^2}u^\mu u^\nu T^{(q)}_{\mu\nu}, \qquad \mathcal{E}_q = \int_{\Sigma_u} \varepsilon^{(q)}\mathrm{d}\Sigma. \end{aligned} \tag{5.56}$$

The bridge contribution enters this contraction with the sign already fixed by its definition, while the trace-free Weyl part is absent from the linear trace and remains encoded in the geometry and the uncontracted carrier.

The electron–proton interaction is contained in the Maxwell cross-term of the complete field. With $E_{e\,i} = -\partial_i\phi_e$ and $E_{p\,i} = -\partial_i\phi_p$, and using the proton Poisson equation, integration by parts gives

$$v(r) = \varepsilon_0 \int_{\Sigma_u} E_{e\,i} E_p{}^i \mathrm{d}^3x = -\varepsilon_0 \int_{\Sigma_u} \phi_e \, \partial_i \partial^i \phi_p \mathrm{d}^3 x$$
$$= \frac{q_e q_p}{4\pi\varepsilon_0 r} = -\frac{\alpha\hbar c}{r}. \tag{5.57}$$

Thus the Coulomb term is the integrated interaction part of the same conserved Maxwell source rather than an externally appended potential.

The bound two-body motion is resolved through the central relative channel. Let its rank-one contribution be $T_{r\,\mu\nu} = \varrho p_\mu p_\nu$ and let $\mu = m_e m_p/(m_e + m_p)$. Relative to the centre-of-mass clock, the momentum is resolved into radial and angular parts by

$$p_\mu = -\frac{\mathcal{E} - v(r)}{c^2} u_\mu + p_r n_\mu + \frac{1}{r} l_\mu,$$
$$u^\mu n_\mu = u^\mu l_\mu = n^\mu l_\mu = 0, \qquad l_\mu l^\mu = k^2. \tag{5.58}$$

The mass shell $p_\mu p^\mu = -\mu^2 c^2$, together with $k^2 = (j + \frac{1}{2})^2 \hbar^2$, then gives the radial constraint

$$p_r^2 = \frac{\mathcal{E}^2}{c^2} - \mu^2 c^2 + \frac{2\alpha\hbar\mathcal{E}}{cr} - \frac{\hbar^2}{r^2}\left[\left(j + \frac{1}{2}\right)^2 - \alpha^2\right]. \tag{5.59}$$

For a bound orbit the two positive roots of this expression delimit the radial cycle. Evaluating the closed action between them gives

$$j_r = 2\oint p_r \mathrm{d}r = 2\pi\hbar\left[\frac{\alpha\mathcal{E}}{\sqrt{\mu^2 c^4 - \mathcal{E}^2}} - \sqrt{\left(j + \frac{1}{2}\right)^2 - \alpha^2}\right]. \tag{5.60}$$

Imposing $j_r = 2\pi n_r \hbar$ therefore yields

$$\frac{\alpha\mathcal{E}}{\sqrt{\mu^2 c^4 - \mathcal{E}^2}} = n_r + \sqrt{\left(j + \frac{1}{2}\right)^2 - \alpha^2}, \tag{5.61}$$

$$\mathcal{E}_{n_r,j} = \mu c^2 \left[1 + \frac{\alpha^2}{\left(n_r + \sqrt{(j+\frac{1}{2})^2 - \alpha^2}\right)^2}\right]^{-1/2}. \tag{5.62}$$

Writing $N = n_r + j + \frac{1}{2}$, the binding energy $E^b_{n_r,j} = \mathcal{E}_{n_r,j} - \mu c^2$ has the leading limit $-\mu c^2 \alpha^2/(2N^2)$, so the same closed radial action reproduces the nonrelativistic Coulomb spectrum while retaining the relativistic dependence on n_r and j.

5.2.4 Stationary radial modes and rank-eight evolution

The closed action fixes the admissible energies, but the quantum carrier also requires a spatial tensor profile for each stationary level. For fixed j write $\gamma_j = \sqrt{(j+\frac{1}{2})^2 - \alpha^2}$. The central projection of a complete rank-four mode is governed by the quadratic radial operator

$$\mathcal{Q}_j(\mathcal{E}) = -\hbar^2 c^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \mu^2 c^4 + \frac{\hbar^2 c^2 \gamma_j(\gamma_j - 1)}{r^2} - \mathcal{E}^2 - \frac{2\alpha\hbar c}{r}\mathcal{E}, \qquad \mathcal{Q}_j(\mathcal{E}) u(r) = 0. \tag{5.63}$$

At $\mathcal{E} = \mathcal{E}_{n_r,j}$ its normalizable solution is

$$u_{n_rj}(r) = N_{n_rj}(2\lambda_{n_rj}r)^{\gamma_j} e^{-\lambda_{n_rj}r} L_{n_r}^{2\gamma_j-1}(2\lambda_{n_rj}r),$$
$$\lambda_{n_rj} = \frac{\sqrt{\mu^2c^4 - \mathcal{E}_{n_r,j}^2}}{\hbar c}, \qquad N_{n_rj} = \left[\frac{\lambda_{n_rj}n_r!}{(n_r+\gamma_j)\Gamma(n_r+2\gamma_j)}\right]^{1/2}. \tag{5.64}$$

The normalization is $\int_0^\infty |u_{n_rj}|^2 \mathrm{d}r = 1$, and termination of the Laguerre series is equivalent to the spectral condition Eq. (5.61). Thus the local radial equation and the closed action select the same energies. Because the operator is quadratic in the energy, distinct solutions at fixed j obey the weighted relation

$$\int_0^\infty u_{pj}(r)u_{qj}(r)\left(\mathcal{E}_p + \mathcal{E}_q + \frac{2\alpha\hbar c}{r}\right)\mathrm{d}r = 0. \tag{5.65}$$

This scalar function is only the radial projection of a complete tensor mode,

$$\mathbb{P}_r T^{(q)}_{\alpha\mu\beta\nu} = u_{n_rj}(r)\mathcal{Y}^{(q)}_{\alpha\mu\beta\nu}, \tag{5.66}$$

where $\mathcal{Y}^{(q)}$ retains the angular, orientation and remaining tensorial information. The spectral evolution must therefore act on $T^{(q)}_{\alpha\mu\beta\nu}$, not on u_{n_rj} alone.

For stationary modes $\nabla_0 T^{(q)}_{\alpha\mu\beta\nu} = 0$. Choose dual modes by the global biorthogonality condition

$$\ell_P\kappa^2 \int_{\Sigma_t} \sqrt{h}\, T^{\sharp(p)}_{\alpha\mu\beta\nu} T^{(q)\,\alpha\mu\beta\nu} \mathrm{d}^3\mathbf{x} = \delta_{pq}. \tag{5.67}$$

The associated rank-eight projector is then defined by its action on an arbitrary rank-four field,

$$(\Pi_q X)_{\alpha\mu\beta\nu}(\mathbf{x}) = \ell_P\kappa^2 T^{(q)}_{\alpha\mu\beta\nu}(\mathbf{x}) \int_{\Sigma_t} \sqrt{h(\mathbf{y})}\, T^{\sharp(q)\,\delta\gamma\phi\psi}(\mathbf{y}) X_{\delta\gamma\phi\psi}(\mathbf{y})\mathrm{d}^3\mathbf{y}. \tag{5.68}$$

Using these projectors, the stationary hydrogen generator is

$$\mathcal{H} = \sum_{q\in\mathcal{Q}} \frac{\mathcal{E}_{n_r,j}}{2E_P}\Pi_q. \tag{5.69}$$

The factor one half belongs to the quadratic scalar readout: each carrier factor supplies half of the physical stationary phase. Substitution into the fundamental rank-eight equation gives

$$c_q(x^0) = c_q(0)\exp\left[-\frac{\mathrm{i}\mathcal{E}_{n_r,j}x^0}{2\hbar c}\right]. \tag{5.70}$$

Inserting this coefficient into Eq. (5.53) gives the full time-dependent carrier without introducing a second state formula. A neighbouring sector transports the same complete mode and spectral operator by

$$T^{(q,1)}_{\alpha\mu\beta\nu} = -I_1^* T^{(q,0)}_{\alpha\mu\beta\nu}, \qquad \mathcal{H}^{(1)} = I_1^*\mathcal{H}^{(0)}(I_1^*)^{-1}. \tag{5.71}$$

The chart change therefore alters the active geometric representative but not the physical energy or the identity of the hydrogen state.

5.2.5 Global probabilities and resolved hydrogen states

The stationary phases determine temporal interference, whereas the preparation coefficients retain the initial amplitudes. The physical comparison of two complete modes is the ordered geometric overlap

$$\mathcal{G}_{pq} = \ell_P \kappa^2 \int_{\Sigma_t^{(q)}} \sqrt{h^{(q)}}\, T^{(p)}_{\alpha\mu\beta\nu} \left[T^{(q)}\right]^{\alpha\mu\beta\nu}_{g^{(q)}} \mathrm{d}^3\mathbf{x}. \tag{5.72}$$

The order matters because the contraction uses the geometry of the comparison mode. In a diagonal four-dimensional channel the field equation gives

$$\kappa^2 T^{(p)}_{\alpha\mu\beta\nu} T^{(p)\,\alpha\mu\beta\nu} = K_p, \qquad \mathcal{G}_{pp} = \ell_P \int_{\Sigma_t} \sqrt{h}\, K_p \mathrm{d}^3\mathbf{x}, \tag{5.73}$$

so the physical self-overlap is determined by the complete common geometry and is not replaced by the spectral biorthogonality normalization. The two-copy law assigns an ordered resolved pair the probability

$$P_{pq} = |c_p|^2 |c_q|^2 |\mathcal{G}_{pq}|^2. \tag{5.74}$$

For a coherent preparation the carrier itself remains normalized by the full quartic expression

$$1 = \sum_{p,q,r,s} c_p c_q c_r^* c_s^* \exp\left[-\frac{\mathrm{i}(\mathcal{E}_p + \mathcal{E}_q - \mathcal{E}_r - \mathcal{E}_s)t}{2\hbar}\right] \mathcal{G}_{pq} \mathcal{G}_{rs}^*. \tag{5.75}$$

A mutually exclusive set of resolved diagonal detector channels instead satisfies $\sum_q P_{qq} = 1$. These are different normalizations: the first belongs to the coherent carrier and the second to the resolved outcomes. The two half-energy carrier phases combine in a diagonal comparison to the usual Bohr factor $\exp[-\mathrm{i}(\mathcal{E}_p - \mathcal{E}_q)t/\hbar]$.

Joint energy–position resolution. The same general carrier can resolve a dimensionless radial coordinate ζ together with the discrete tensor label. If $f_q(\zeta)$ is the normalized radial amplitude of a resolved channel, the diagonal coefficient may be chosen so that

$$c_{q\zeta}^2(t)\mathcal{G}_{q\zeta} = A_q f_q(\zeta) \exp\left[-\frac{\mathrm{i}\mathcal{E}_{n_r,j}t}{\hbar}\right], \qquad p_q(\zeta) = |A_q|^2 |f_q(\zeta)|^2, \qquad \sum_q \int_0^\infty p_q(\zeta)\mathrm{d}\zeta = 1. \tag{5.76}$$

The geometric overlap remains part of the tensor coefficient normalization and is not reinterpreted as a local position density. The radial equation fixes only the central projection, $f^{(r)}_{n_r j}(\zeta) = u_{n_r j}(r)/\sqrt{\lambda_{n_r j}}$ with $\zeta = \lambda_{n_r j} r$; the complete f_q also contains the angular and signed tensor information.

Ground-state radial profile. For $n_r = 0$, $j = 1/2$ and $\kappa_D = -1$, define $\eta = \sqrt{1-\alpha^2}$ and $\lambda_{1s} = \mu c\alpha/\hbar$. The normalized central amplitude is

$$f^{(r)}_{1s}(\zeta) = \frac{(2\zeta)^\eta e^{-\zeta}}{\sqrt{\eta\Gamma(2\eta)}}, \qquad \zeta = \lambda_{1s} r. \tag{5.77}$$

Its radial probability density is $|f^{(r)}_{1s}|^2$, its mean radius is $\langle r\rangle_{1s} = (\eta + \frac{1}{2})/\lambda_{1s}$, and the limit $\eta \to 1$ gives the standard nonrelativistic hydrogenic profile. The cumulative probability follows by integrating the same normalized density and requires no additional state definition.

Complete first excited principal level. The three admissible state families at $N = 2$ are

$$\begin{aligned} q_s &= \left(1, \frac{1}{2}, m_j, -1, \sigma_e, \sigma_p\right), & \ell_s &= 0, & \pi_s &= +1, \\ q_{p_1} &= \left(1, \frac{1}{2}, m_j, +1, \sigma_e, \sigma_p\right), & \ell_{p_1} &= 1, & \pi_{p_1} &= -1, \\ q_{p_3} &= \left(0, \frac{3}{2}, m_j, -2, \sigma_e, \sigma_p\right), & \ell_{p_3} &= 1, & \pi_{p_3} &= -1. \end{aligned} \tag{5.78}$$

Their central energies are

$$\mathcal{E}_s = \mathcal{E}_{p_1} = \mu c^2 \sqrt{\frac{1+\eta}{2}}, \qquad \mathcal{E}_{p_3} = \frac{\mu c^2}{2}\sqrt{4-\alpha^2}. \tag{5.79}$$

The first two families share the same radial projection. With $\lambda_{1,1/2} = (\mu c/\hbar)\sqrt{(1-\eta)/2}$, it is

$$f^{(r)}_{1,\frac{1}{2}}(\zeta) = \frac{(2\zeta)^{\eta} e^{-\zeta}(2\eta - 2\zeta)}{\sqrt{(1+\eta)\Gamma(1+2\eta)}}. \tag{5.80}$$

This projection has one node at $\zeta_0 = \eta$ and stationary maxima at $\zeta_\pm = (2\eta + 1 \pm \sqrt{4\eta+1})/2$. Its first central moment is $(\eta+3)(2\eta+1)/[2(\eta+1)]$ and its variance is $(2\eta+1)(3\eta^2 + 2\eta+3)/[4(\eta+1)^2]$. Equality of the radial projection does not identify the complete states: $T^{(q_s)}_{\alpha\mu\beta\nu} \neq T^{(q_{p_1})}_{\alpha\mu\beta\nu}$, because their angular and signed-orientation tensors are different.

For the q_{p_3} family, let $\gamma_{3/2} = \sqrt{4-\alpha^2}$ and $\lambda_{0,3/2} = \mu c\alpha/(2\hbar)$. Its central projection is

$$f^{(r)}_{0,\frac{3}{2}}(\zeta) = \frac{(2\zeta)^{\gamma_{3/2}} e^{-\zeta}}{\sqrt{\gamma_{3/2}\Gamma(2\gamma_{3/2})}}. \tag{5.81}$$

It has no positive radial node, reaches its maximum at $\zeta = \gamma_{3/2}$, and has mean $\gamma_{3/2} + 1/2$ and variance $(2\gamma_{3/2}+1)/4$. These analytic properties characterize the radial projection without replacing the complete rank-four state.

For irreducible central tensor profiles the angular contraction vanishes when j, m_j or parity differ. The diagonal amplitude of the complete $N = 2$ level therefore reduces to

$$\mathcal{A}_2(t) = \left(c_s^2 \mathcal{G}_{ss} + c_{p_1}^2 \mathcal{G}_{p_1p_1}\right) e^{-\mathrm{i}\mathcal{E}_s t/\hbar} + c_{p_3}^2 \mathcal{G}_{p_3p_3} e^{-\mathrm{i}\mathcal{E}_{p_3} t/\hbar}. \tag{5.82}$$

Its probability is $|\mathcal{A}_2|^2$, and its nontrivial time dependence is controlled by the exact splitting

$$\Delta\mathcal{E}_2 = \mu c^2 \left[\frac{\sqrt{4-\alpha^2}}{2} - \sqrt{\frac{1+\sqrt{1-\alpha^2}}{2}}\right]. \tag{5.83}$$

The q_s and q_{p_1} families are degenerate in the central spectrum, so their relative preparation phase is stationary.

A resolved energy probability is obtained by integrating the radial coordinate and summing the remaining discrete labels,

$$P(\mathcal{E}_{n_r,j}) = \sum_{m_j=-j}^{j} \sum_{\kappa_D \in \mathcal{K}_{n_r j}} \sum_{\sigma_e = \pm 1} \sum_{\sigma_p = \pm 1} \int_0^\infty p_q(\zeta)\mathrm{d}\zeta. \tag{5.84}$$

The corresponding energy expectation is the weighted sum over these levels. The retained diagonal multiplicities are $g_{n_r j} = 4(2j+1)(2-\delta_{n_r 0})$ and $g_N = 8N^2$; no separate normalization is introduced for degenerate states.

5.2.6 Radiative tensor transition

A radiative transition is treated as a second complete real tensor configuration, not by adding a new measurement postulate. Let $f^{(f)}_{\mu\nu}$ denote the final atomic electromagnetic field and $f^{(\gamma)}_{\mu\nu}$ a real radiation mode. The perturbed field $f^{(F)}_{\mu\nu} = f^{(f)}_{\mu\nu} + \epsilon_\gamma f^{(\gamma)}_{\mu\nu}$ changes the Maxwell source at first order through the cross-term

$$X^{(f\gamma)}_{\mu\nu} = \frac{(-1)^{n_g}}{\mu_0}\left(f^{(f)}_{\mu\rho} f^{(\gamma)\rho}_{\nu} + f^{(\gamma)}_{\mu\rho} f^{(f)\rho}_{\nu} - \frac{1}{2} g_{\mu\nu} f^{(f)}_{\rho\sigma} f^{(\gamma)\rho\sigma}\right), \qquad g^{\mu\nu} X^{(f\gamma)}_{\mu\nu} = 0. \tag{5.85}$$

The other source components respond according to the same local conservation law used for the stationary atom. Their combined first-order change is

$$\Delta T^{(f\gamma)}_{\mu\nu} = X^{(f\gamma)}_{\mu\nu} + \Delta T^{(f\gamma)}_{2\mu\nu} + \Delta T^{(f\gamma)}_{3\mu\nu} + \Delta T^{(f\gamma)}_{4\mu\nu}, \qquad \nabla_\mu \Delta T^{(f\gamma)\,\mu\nu} = 0. \tag{5.86}$$

The rank-four perturbation is obtained from the same lift and trace-free relation as the stationary source, with $\Delta C^{(f\gamma)}_{\alpha\mu\beta\nu} = \kappa \Delta \widehat{T}^{(f\gamma)}_{\alpha\mu\beta\nu}$. The complete comparison state is therefore

$$T^{(F)}_{\alpha\mu\beta\nu} = T^{(q_f)}_{\alpha\mu\beta\nu} + \epsilon_\gamma \Delta T^{(f\gamma)}_{\alpha\mu\beta\nu} + O(\epsilon_\gamma^2). \tag{5.87}$$

For a resolved initial mode q_i, the general comparison law of Section 5.1.5 reduces to the ordered overlap

$$\begin{aligned} \mathcal{G}_{iF} &= \ell_P \kappa^2 \int_{\Sigma_t} \sqrt{h}\, T^{(q_i)}_{\alpha\mu\beta\nu} T^{(F)\,\alpha\mu\beta\nu} \mathrm{d}^3\mathbf{x} = \mathcal{G}_{if} + \epsilon_\gamma \mathcal{M}_{fi} + O(\epsilon_\gamma^2), \\ \mathcal{M}_{fi} &= \ell_P \kappa^2 \int_{\Sigma_t} \sqrt{h}\, T^{(q_i)}_{\alpha\mu\beta\nu} \Delta T^{(f\gamma)\,\alpha\mu\beta\nu} \mathrm{d}^3\mathbf{x}. \end{aligned} \tag{5.88}$$

The transition probability is the same two-copy quantity as before, namely $|c_i|^2 |d_F|^2 |\mathcal{G}_{iF}|^2$. If the unperturbed overlap vanishes by symmetry, the leading contribution is proportional to $\epsilon_\gamma^2 |\mathcal{M}_{fi}|^2$.

The local four-index contraction also shows which physical sectors contribute to the transition amplitude,

$$\kappa^2 T^{(q_i)}_{\alpha\mu\beta\nu} \Delta T^{(f\gamma)\,\alpha\mu\beta\nu} = 2\kappa^2 T^{(q_i)}_{\mu\nu} \Delta T^{(f\gamma)\,\mu\nu} - \frac{\kappa^2}{3} T^{(q_i)} \Delta T^{(f\gamma)} + C^{(q_i)}_{\alpha\mu\beta\nu} \Delta C^{(f\gamma)\,\alpha\mu\beta\nu}. \tag{5.89}$$

Thus the same comparison retains both the rank-two response and the trace-free curvature response.

If the radiation tensor carries angular labels (L, M) and parity π_γ, a nonzero scalar overlap requires

$$|j_i - j_f| \le L \le j_i + j_f, \qquad m_{j_i} = m_{j_f} + M, \qquad \pi_i \pi_f \pi_\gamma = 1. \tag{5.90}$$

For the dipole-parity channel these conditions admit $2p_{1/2} \to 1s_{1/2}$ and $2p_{3/2} \to 1s_{1/2}$, while the corresponding $2s_{1/2} \to 1s_{1/2}$ overlap vanishes. Energy conservation fixes $\mathcal{E}_\gamma = \mathcal{E}_i - \mathcal{E}_f$ and $\omega_{if} = (\mathcal{E}_i - \mathcal{E}_f)/\hbar$. The two allowed lines from the first excited principal level are therefore obtained directly by inserting the already derived stationary energies; their separation is

$\Delta\mathcal{E}_2/\hbar$. No additional spectral formula is needed.

5.3 Conclusion

The construction developed here is a single tensor chain from a classical source to quantum observables. The four-index field equation preserves the ordinary Einstein equation under contraction while retaining the trace-free Weyl sector before contraction. A complete real source is conserved as one rank-four object, dimensionless complex coefficients weight complete configurations only after that source has been fixed, and a rank-eight operator governs the linear clock evolution of the resulting carrier. Normalization and transition probabilities are formed by global contractions of the rank-four state with an independent conjugate copy, so source tensors, quantum coefficients and scalar readouts remain distinct at every stage.

The four-sector geometry enters this chain through active chart transport. Rank-two sources are ordinary pullbacks, all-lower rank-four curvature and source tensors acquire the common sector orientation sign, and scalar contractions are preserved by the corresponding pullback. This permits a regular reciprocal Kerr–Newman chart to represent the charged spinning electron constituent without introducing an additional source for the same particle. The active metric is nondegenerate, its equatorial Kretschmann scalar is finite at the sector boundary and decreases to zero at the reciprocal endpoint, and the electromagnetic field is transported in the same chart as the geometry.

For a fixed hydrogen configuration the complete rank-two atom is assembled before the four-index lift. The total Maxwell field retains the electron–proton interaction cross-term, while the additional source contributions enforce the stated conservation law. The central interaction, relative mass shell and closed radial action yield the stationary relativistic spectrum. A compatible tensorial radial realization produces normalizable Laguerre modes with the same eigenvalues, and the signed angular label fixes the orbital branch and parity without an independent ground-state doubling. Ground-state and first-excited radial profiles, their nodes and moments, and the corresponding degeneracies follow from this stationary construction.

The spectral operator is built from complete stationary tensor modes and their duals. Its indexed evolution fixes the clock phase of each mode, while the quartic probability law retains ordered tensor overlaps and their interference. Diagonal state preparation, mixed stationary configurations, position-resolved channels and the radiative comparison configuration are all specializations of the same global contraction rather than separate measurement prescriptions. The hydrogen calculation therefore uses one conserved tensor source, one rank-four carrier, one rank-eight spectral evolution and one tensor probability law from the classical geometry to the resolved quantum observables.

Chapter 6

Free Gravitons as Weyl Modes

Source and status. This chapter incorporates the complete scientific body of *Free Gravitons as Weyl Modes in Four-Index and Four-Sector Quantum Gravity*, v0.80 (15 August 2026). Its role in the dissertation is: Ricci-flat graviton sector, helicity, transport, mode resolution and closed Weyl histories. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

The free gravitational quantum is obtained as the Ricci-flat radiative specialization of the complete four-index quantum carrier rather than as an additional field. The carrier continues to inherit the full four-index source, whose trace-built contribution vanishes in vacuum while its trace-free contribution is fixed by the Weyl tensor. The vacuum differential identities give null propagation and two transverse spin-two helicities, and the indexed rank-eight evolution fixes the energy of a monochromatic mode from its frequency. The four-sector geometry transports the same vacuum mode between active clock–ruler descriptions without changing its mass, energy, helicity, or global tensor overlap, provided the orientation sign of all-lower rank-four fields is retained. An exact reciprocal Rosen geometry is used as a vacuum background on which the transported local spectral mode is resolved with the active observer clock: inherited coordinate phase gradients can diverge while the measured frequency remains finite and unchanged, whereas the exact background tidal Weyl curvature decays toward a curvature-flat reciprocal endpoint. Conjugation of the active evolution generator preserves the transported positive-frequency spectral subspace, so the sector map itself produces no intrinsic positive–negative-frequency mixing. The spectral construction is made explicit by a null-tetrad dual for each type-N helicity mode, a biorthogonal plane-wave continuum, and a spherical continuum with the radial density absorbed into the mode normalization. The exact global self-contraction of plane-wave superpositions pairs antipodal momenta; in the natural antipodal screen convention the dual type-N tensor is precisely the antipodal partner of the right mode. Exactly monochromatic angular states therefore remain generalized spectral elements, while genuinely localized packets require nonzero radial bandwidth and must be normalized by the full global tensor law rather than by a phase-stripped angular kernel. The same type-N mode admits a closed four-sector Weyl history whose rank-four tensor returns after four signed pullbacks while the oriented branch momenta cancel over the complete orbit. A tensor resolution of a general local Weyl state and an explicit Schwarzschild exterior test show that this closed-history construction transports the gravitational Weyl content itself; the vacuum Bianchi equations then show that changes of the loop profile retain the null characteristic of the free field. The result is a constructive, covariant, and regular free-graviton sector that preserves the original source content, spacetime tensor hierarchy, and four-sector geometry.

6.1 Introduction

General relativity separates curvature into a Ricci-built part and a trace-free Weyl part. The first is fixed locally by the ordinary stress–energy tensor through the contracted Einstein equation, while the second survives in vacuum and carries the tidal and radiative information of the gravitational field. In the four-index formulation developed in Ref. [37], these two parts are retained at the same tensor rank as the complete source. This makes the vacuum radiative sector particularly direct: the theory already contains a rank-four field with the algebraic symmetries and the vacuum content required for gravitational radiation.

The quantum construction of Ref. [45] is built on the same tensor rank. Its carrier is formed from the complete four-index source, not from the Weyl contribution alone, and its fundamental evolution operator acts on the four spacetime index positions of that complete carrier. Discrete or continuous labels arise only when the carrier is resolved into a basis adapted to a physical problem. They are not extra spacetime coordinates and are not additional tensor slots of the fundamental field. The free-field question is therefore how the already defined complete carrier specializes when its rank-two stress–energy contraction vanishes and only its trace-free gravitational content remains.

The four-sector Lorentz extension of Ref. [38] adds a geometric requirement to this identification. The four sectors are not separate spacetimes and do not carry separately adjustable gravitational fields. They are active clock–ruler descriptions of one geometry. In curved spacetime, when the proper scale used by one chart reaches its common null boundary with a neighbouring sector, the same general-relativistic solution is evaluated on the reciprocal active scale and pulled back with the full Jacobian. A free gravitational quantum must therefore keep its physical properties when its Weyl field is represented in a neighbouring active sector, even when inherited coordinate components change strongly.

The present paper follows one continuous chain. The complete four-index source and the carrier built from it are retained as the primary objects. Their Ricci-flat specialization is then taken explicitly: the trace-built source vanishes, the trace-free source satisfies the Weyl relation, and the complete source consequently becomes proportional to the radiative Weyl tensor without being redefined as a new tensor. The vacuum differential identities determine the null characteristic of this mode, its spin content is read from the transverse Weyl curvature, and its energy follows from the indexed evolution acting on the spacetime phase of the same complete carrier mode. The state is then transported through the four-sector geometry with the orientation sign appropriate to all-lower rank-four fields. An exact reciprocal Rosen vacuum geometry supplies a nontrivial background test: its own tidal curvature is kept distinct from a local monochromatic spectral mode transported on that background, and the measured frequency is computed from the active observer clock rather than from a single null-coordinate derivative. The spectral part is then made constructive rather than merely formal: an explicit null-frame dual resolves the type-N null basis, the plane-wave continuum is normalized biorthogonally, the radial Jacobian is absorbed explicitly in a spherical mode basis, and the global self-contraction of general superpositions is evaluated with the full spacetime phase. This exposes the exact antipodal pairing of type-N modes and separates generalized monochromatic states from genuinely localized packets, which require radial bandwidth. The antipodal structure is then followed through the complete four-sector orbit at the level of the Weyl tensor itself: the free type-N mode forms one closed graviton history, a local transverse tensor resolution extends the construction to general Weyl data, and the Schwarzschild exterior supplies an exact nonradiative curvature test. Finally, the vacuum Bianchi system is used to distinguish closed-orbit bookkeeping from the propagation of a physical change of the Weyl profile. The final result therefore separates free propagation, sector transport, spectral resolution, physical preparation and closed-history transport without changing the fundamental field or introducing an additional graviton source.

6.2 Vacuum specialization of the complete four-index carrier

The quantum carrier is inherited from the complete four-index source of Ref. [37]; the radiative Weyl tensor is not introduced as a second gravitational field. In four dimensions the source separates into its trace-built and trace-free parts, with the latter fixed by the Weyl curvature,

$$T_{\alpha\mu\beta\nu} = \widetilde{T}_{\alpha\mu\beta\nu} - \widehat{T}_{\alpha\mu\beta\nu}, \qquad C_{\alpha\mu\beta\nu} = \kappa\widehat{T}_{\alpha\mu\beta\nu}. \tag{6.1}$$

The carrier therefore acts on complete rank-four modes rather than separately on matter and gravitational pieces,

$$\boxed{\Psi_{\alpha\mu\beta\nu}(x) = \sqrt{\ell_{\mathrm{P}}}\,\kappa \sum_q c_q T^{(q)}_{\alpha\mu\beta\nu}(x).} \tag{6.2}$$

Here the label q enumerates complete tensor modes and is not a spacetime index.

A free graviton is obtained by taking the Ricci-flat radiative specialization of this same source. The ordinary stress–energy tensor then vanishes, so its four-index lift vanishes, while the trace-free source remains. The complete mode and its carrier consequently reduce together according to

$$\widetilde{T}^{(q)}_{\alpha\mu\beta\nu} = 0, \qquad\qquad T^{(q)}_{\alpha\mu\beta\nu} = -\frac{1}{\kappa}C^{(q)}_{\alpha\mu\beta\nu}, \tag{6.3}$$

$$\Psi^{(q)}_{\alpha\mu\beta\nu} = -\sqrt{\ell_{\mathrm{P}}}\,c_q C^{(q)}_{\alpha\mu\beta\nu}. \tag{6.4}$$

Thus the Weyl tensor supplies the vacuum profile of the already defined carrier; it does not replace its source definition.

The differential content of the free mode is equally inherited from the vacuum geometry. In a Ricci-flat region the Weyl tensor is divergence free,

$$\nabla^\alpha C_{\alpha\mu\beta\nu} = 0. \tag{6.5}$$

Together with the differential Bianchi identity, this fixes the principal characteristic of a rapidly varying curvature phase. If k_μ denotes its covector, then

$$g^{\mu\nu}k_\mu k_\nu = 0. \tag{6.6}$$

The radiative carrier is therefore massless because its curvature propagates on the null characteristic of the underlying field equations, not because a separate particle dispersion law has been imposed.

The algebraic type of the curvature must be kept distinct from this propagation statement. A type-N mode has nonzero transverse tidal curvature even though its two quadratic Weyl invariants vanish,

$$C_{\alpha\mu\beta\nu}C^{\alpha\mu\beta\nu} = 0, \qquad {}^\star C_{\alpha\mu\beta\nu}C^{\alpha\mu\beta\nu} = 0. \tag{6.7}$$

This null self-contraction will later determine how exact plane-wave modes enter the biorthogonal spectral resolution and why a physical preparation must be treated globally.

6.3 Transverse helicity and indexed energy

Once the null direction has been fixed, the spin content is read directly from the Weyl curvature on the two-dimensional screen orthogonal to the active clock u^μ and propagation direction s^μ. The observer rest-space metric is $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. For an oriented orthonormal screen pair $e^\mu_{(1)}, e^\mu_{(2)}$, introduce the circular dyad

$$m^\mu = \frac{e^\mu_{(1)} + \mathrm{i}e^\mu_{(2)}}{\sqrt{2}}, \qquad \bar{m}^\mu = \frac{e^\mu_{(1)} - \mathrm{i}e^\mu_{(2)}}{\sqrt{2}}. \tag{6.8}$$

The two independent type-N curvature components are then represented by

$$C^{(+2)}_{\alpha\mu\beta\nu} \propto k_{[\alpha}m_{\mu]}k_{[\beta}m_{\nu]}, \qquad C^{(-2)}_{\alpha\mu\beta\nu} \propto k_{[\alpha}\bar{m}_{\mu]}k_{[\beta}\bar{m}_{\nu]}. \tag{6.9}$$

A rotation of the physical screen through ψ multiplies the two tensors by $e^{\pm 2\mathrm{i}\psi}$. The radiative Weyl sector therefore contains precisely the helicities $\lambda = \pm 2$, obtained from the transformation of the rank-four curvature itself.

The energy of the same mode is determined by the indexed evolution introduced in Ref. [45]. On a hypersurface it may be written as the action of a rank-eight kernel on the rank-four carrier,

$$\mathrm{i}\ell_{\mathrm{P}}\nabla_0\Psi_{\alpha\mu\beta\nu}(x) = \int_{\Sigma_t} H_{\alpha\mu\beta\nu}{}^{\delta\gamma\phi\psi}(x,y)\Psi_{\delta\gamma\phi\psi}(y)\,\mathrm{d}\Sigma_y. \tag{6.10}$$

All eight indices are spacetime tensor positions. Momentum and helicity appear only after a basis of complete vacuum modes has been chosen.

For a monochromatic positive-frequency mode with $x^0 = ct$, $k = \omega/c$, and phase $e^{-\mathrm{i}kx^0}$, the complete source mode is an eigenfunction of the same operator,

$$\int_{\Sigma_t} H_{\alpha\mu\beta\nu}{}^{\delta\gamma\phi\psi}(x,y)T^{(q)}_{\delta\gamma\phi\psi}(y)\,\mathrm{d}\Sigma_y = \ell_{\mathrm{P}}k\,T^{(q)}_{\alpha\mu\beta\nu}(x). \tag{6.11}$$

In vacuum this is equivalently an eigenproblem for the Weyl profile because of Eq. (6.3). Since $E_{\mathrm{P}}\ell_{\mathrm{P}} = \hbar c$, the dimensionless eigenvalue $\ell_{\mathrm{P}}k$ corresponds to the physical energy $\hbar\omega$. The null characteristic, transverse spin weight and indexed energy thus describe one degree of freedom,

$$\boxed{m_G = 0, \qquad \lambda = \pm 2, \qquad E_G = \hbar\omega.} \tag{6.12}$$

The equality $E_G = \hbar\omega$ is a spectral statement about the global tensor state. It does not introduce an additional local rank-two energy tensor for gravitational radiation.

6.4 Transport through the four-sector geometry

The four-sector extension changes the active clock–ruler description of one geometry. The reciprocal rule is applied to the proper scales before the known general-relativistic solution is evaluated in the neighbouring chart. In the active rest space the proper-scale tensor $L^\mu{}_\nu$ and the critical tensor $G^\mu{}_\nu$ define the reciprocal scale by

$$\widetilde{L}^\mu{}_\nu = G^\mu{}_\rho(L^{-1})^\rho{}_\sigma G^\sigma{}_\nu. \tag{6.13}$$

This operation determines the chart map; it is not an extra deformation multiplied into a finished metric.

Let $\bar{g}_{\mu\nu}$ be the parent GR metric and $\mathcal{I}_n$ the active chart map after the required reciprocal scales have been selected. The metric in sector n is the signed pullback

$$g^{(n)}_{MN}(Y) = \mathrm{i}^{2n}\bar{g}_{\mu\nu}(x_{(n)}(Y))\frac{\partial x^{\mu}_{(n)}}{\partial Y^M}\frac{\partial x^{\nu}_{(n)}}{\partial Y^N} = \mathrm{i}^{2n}(\mathcal{I}_n^*\bar{g})_{MN}. \tag{6.14}$$

The complete Jacobian is retained. Since $\mathrm{i}^{2n} = (-1)^n$ is constant inside a fixed sector, the null set and Levi–Civita connection are unchanged by that scalar sign.

The same map transports the complete vacuum source, the Weyl tensor and the rank-eight generator. The all-lower rank-four tensors acquire the orientation sign associated with the signed metric, while the operator is transported by conjugation,

$$T^{(n,q)}_{MNRS} = (-1)^n(\mathcal{I}_n^*T^{(0,q)})_{MNRS}, \qquad C^{(n,q)}_{MNRS} = (-1)^n(\mathcal{I}_n^*C^{(0,q)})_{MNRS}, \tag{6.15}$$

$$H^{(n)} = \mathcal{I}_n^*H^{(0)}(\mathcal{I}_n^*)^{-1}. \tag{6.16}$$

The scalar sign cancels between the input and output of the conjugated operator, so no additional sign belongs to $H^{(n)}$.

Applying these transformations to a free eigenmode leaves both its spectral eigenvalue and its characteristic class unchanged. If $k^{(n)}_M$ is the pulled-back phase covector, then

$$k^{(n)}_M = \frac{\partial x^{\mu}_{(n)}}{\partial Y^M}k_{\mu}, \qquad g^{MN}_{(n)}k^{(n)}_M k^{(n)}_N = 0, \qquad E^{(n)}_G = \hbar\omega. \tag{6.17}$$

Coordinate components can therefore change strongly across the reciprocal map while the physical massless shell and energy remain the same.

Helicity is preserved for the same geometric reason. A quarter-turn exchanges the active clock and the selected ruler direction while the transverse screen is transported continuously,

$$u^{\mu} \mapsto s^{\mu}, \qquad s^{\mu} \mapsto -u^{\mu}, \qquad m^{\mu} \mapsto m^{\mu}, \qquad \bar{m}^{\mu} \mapsto \bar{m}^{\mu}. \tag{6.18}$$

No complex conjugation of the screen dyad occurs, so the two helicities are not interchanged. The sector map therefore transports one massless spin-two mode through successive active descriptions rather than creating independent sector particles.

6.5 An explicit reciprocal radiative geometry

The transport rules can be tested on an exact vacuum geometry for which the inherited reciprocal coordinates become strongly distorted. Consider the Rosen-type line element

$$\mathrm{d}s^2_{(0)} = -2\,\mathrm{d}U\,\mathrm{d}V + \left(\frac{U}{U_h}\right)^{2p}\mathrm{d}x^2 + \left(\frac{U}{U_h}\right)^{2q}\mathrm{d}y^2, \qquad p(p-1)+q(q-1)=0. \tag{6.19}$$

If $a = (U/U_h)^p$ and $b = (U/U_h)^q$, the only potentially nonzero Ricci component is $R_{UU} = -a''/a - b''/b$, which vanishes by the displayed condition. The geometry is therefore exactly Ricci flat, while the transverse curvature is nonzero and is determined by $R_{UxUx} = -aa''$ and $R_{UyUy} = -bb''$.

Choose $U = U_h$ as the common null boundary. The odd sector evaluates the same solution on the reciprocal coordinate $U = U_h^2/\rho$. Applying the full Jacobian and the sector sign gives

$$\mathrm{d}s_{(1)}^2 = -\frac{2U_h^2}{\rho^2}\,\mathrm{d}\rho\,\mathrm{d}V - \left(\frac{U_h}{\rho}\right)^{2p}\mathrm{d}x^2 - \left(\frac{U_h}{\rho}\right)^{2q}\mathrm{d}y^2, \tag{6.20}$$

$$\det g_{(1)} = -\frac{U_h^4}{\rho^4}\left(\frac{U_h}{\rho}\right)^{2(p+q)}, \qquad R^{(1)}_{\mu\nu} = 0. \tag{6.21}$$

The active metric remains nondegenerate for $0 < \rho \le U_h$. A regular orthonormal null-frame component of the Weyl tensor behaves as

$$\left|C^{(1)}_{\hat{+}\hat{x}\hat{+}\hat{x}}\right| = \frac{|p(p-1)|}{U_h^4}\rho^2. \tag{6.22}$$

The reciprocal endpoint is therefore curvature flat even though individual inherited metric coefficients diverge.

This exact background should be distinguished from the local monochromatic spectral mode whose energy was derived in Sec. 6.3. On the ordinary side choose the null pair $n_{(0)} = \partial_U$, $\ell_{(0)} = \partial_V$ and the unit clock $u_{(0)} = (n_{(0)} + \ell_{(0)})/\sqrt{2}$. A positive-frequency phase normalized to the locally measured wave number $k = \omega/c$ is

$$\Phi_{(0)} = \sqrt{2}\,kU, \qquad K^\mu_{(0)} = -\sqrt{2}\,k\,\ell^\mu_{(0)}, \qquad \left|u^\mu_{(0)}K^{(0)}_\mu\right| = k. \tag{6.23}$$

The phase covector therefore raises along ∂_V; ∂_U is its conjugate null leg.

Under the reciprocal map the conjugate leg and ray direction become $n_{(1)} = -(\rho^2/U_h^2)\partial_\rho$ and $\ell_{(1)} = \partial_V$. With the odd-sector metric the active clock is

$$u_{(1)} = \frac{n_{(1)} + \ell_{(1)}}{\sqrt{2}}, \qquad u_{(1)}^2 = 1. \tag{6.24}$$

The pulled-back phase has a divergent radial coordinate derivative, but the active clock projection is unchanged,

$$\Phi_{(1)} = \sqrt{2}\,k\frac{U_h^2}{\rho}, \qquad K^{(1)}_\rho = -\sqrt{2}\,k\frac{U_h^2}{\rho^2}, \qquad \left|u^\mu_{(1)}K^{(1)}_\mu\right| = k. \tag{6.25}$$

Thus the inherited coordinate oscillation becomes rapid while the locally measured graviton frequency and energy remain finite.

The exact tidal background varies on the scale U, whereas the spectral phase varies on the scale k^{-1}. Their ratio is controlled by

$$\eta = \frac{\sqrt{2}}{kU} = \frac{\sqrt{2}\,\rho}{kU_h^2}, \qquad \lim_{\rho\to 0}\eta = 0. \tag{6.26}$$

The local spectral approximation therefore improves toward the curvature-flat reciprocal endpoint.

The endpoint is also asymptotic in affine parameter. Because V is cyclic, an affinely parametrized geodesic has constant conjugate momentum, which in the two active representations gives

$$-\frac{\mathrm{d}U}{\mathrm{d}\lambda} = q_0, \qquad -\frac{U_h^2}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}\lambda} = q_1. \tag{6.27}$$

For nonzero conserved momentum, U is affine-linear and $U \to \infty$, equivalently $\rho \to 0$, requires unbounded $|\lambda|$. The reciprocal centre is therefore an asymptotic radiative boundary rather than a finite event of the null geometry.

6.6 Positive-frequency transport across the reciprocal boundary

The active-clock calculation above shows directly that the reciprocal map preserves the locally measured frequency. The same result can be expressed on the complexified free solution space. Let $K_{(0)}$ be the signed frequency generator and $P_{\pm}^{(0)}$ its spectral projectors. Sector transport conjugates both the generator and its spectral splitting,

$$K_{(1)} = \mathcal{I}_1^* K_{(0)} (\mathcal{I}_1^*)^{-1}, \qquad P_{\pm}^{(1)} = \mathcal{I}_1^* P_{\pm}^{(0)} (\mathcal{I}_1^*)^{-1}. \tag{6.28}$$

The odd-sector all-lower Weyl representative additionally acquires the orientation sign of Eq. (6.15). Because this sign is scalar, it does not interchange the two spectral subspaces. A transported positive-frequency mode therefore satisfies

$$P_{-}^{(1)} C^{(1,+)} = 0, \qquad P_{+}^{(1)} C^{(1,+)} = C^{(1,+)}. \tag{6.29}$$

In the transported basis the corresponding Bogoliubov transformation is diagonal in helicity and momentum,

$$\alpha_{k\lambda,k'\lambda'} = \delta_{\lambda\lambda'} \delta(k - k'), \qquad \beta_{k\lambda,k'\lambda'} = 0. \tag{6.30}$$

The reciprocal chart transition therefore preserves the positive-frequency subspace and the helicity splitting of the free mode. No intrinsic graviton production is generated by the sector map itself.

6.7 Generalized null modes and physical preparations

An exact type-N plane wave is both a vacuum solution and an eigenmode of the indexed evolution, but its Lorentzian Weyl self-contraction vanishes. A spectral basis built from such modes therefore cannot be normalized by replacing the global tensor law with a positive local modulus. The construction instead separates the resolution of generalized null modes from the preparation of a normalizable physical state.

The continuum carrier keeps the complete-source form of Eq. (6.2). Introducing the dimensionless momentum label $\xi^{\hat{a}} = \ell_{\mathrm{P}} k^{\hat{a}}$, its vacuum specialization may be written as

$$\Psi_{\alpha\mu\beta\nu}(x) = \sqrt{\ell_{\mathrm{P}}}\,\kappa \sum_{\lambda=\pm 2} \int_{\mathbb{R}^3} c_\lambda(\xi) T^{(\xi,\lambda)}_{\alpha\mu\beta\nu}(x)\, \mathrm{d}^3\xi, \tag{6.31}$$

$$T^{(\xi,\lambda)}_{\alpha\mu\beta\nu} = -\frac{1}{\kappa} C^{(\xi,\lambda)}_{\alpha\mu\beta\nu}. \tag{6.32}$$

The complex basis resolves the real source space; reality of a classical configuration is recovered through the corresponding conjugate relation among spectral coefficients.

Because an exact type-N right mode is null, its coefficient is extracted by a dual rank-four basis. In the vacuum representation the biorthogonal condition is

$$\ell_{\mathrm{P}} \int_{\Sigma} C^{\sharp(\xi,\lambda)}_{\alpha\mu\beta\nu} C^{(\xi',\lambda')\,\alpha\mu\beta\nu}\, \mathrm{d}\Sigma = \delta_{\lambda\lambda'}\delta^{(3)}(\xi-\xi'). \tag{6.33}$$

Multiplication by the corresponding factors of κ gives the identical statement for the complete source. The dual tensor is therefore a basis covector, not a second gravitational excitation.

6.7.1 Explicit null-frame realization of the dual basis

The dual can be constructed on the same screen that carries helicity. For the active clock and propagation direction define the null pair

$$\ell^{\mu} = \frac{u^{\mu}+s^{\mu}}{\sqrt{2}}, \qquad n^{\mu} = \frac{u^{\mu}-s^{\mu}}{\sqrt{2}}, \qquad \ell^{\mu}n_{\mu} = -1. \tag{6.34}$$

Using the circular dyad of Eq. (6.8), form the right and dual bivectors

$$\begin{aligned} Z^{(+)}_{\alpha\mu} &= \ell_{\alpha}m_{\mu} - \ell_{\mu}m_{\alpha}, & Z^{(-)}_{\alpha\mu} &= \ell_{\alpha}\bar{m}_{\mu} - \ell_{\mu}\bar{m}_{\alpha}, \\ Z^{\sharp(+)}_{\alpha\mu} &= n_{\alpha}\bar{m}_{\mu} - n_{\mu}\bar{m}_{\alpha}, & Z^{\sharp(-)}_{\alpha\mu} &= n_{\alpha}m_{\mu} - n_{\mu}m_{\alpha}. \end{aligned} \tag{6.35}$$

The null-tetrad normalization fixes the ordered bivector contraction before the rank-four product is formed,

$$Z^{\sharp(\lambda)}_{\alpha\mu} Z^{(\lambda')\,\alpha\mu} = -2\delta_{\lambda\lambda'}. \tag{6.36}$$

Their quadratic products then give algebraic type-N Weyl representatives,

$$W^{(\lambda)}_{\alpha\mu\beta\nu} = Z^{(\lambda)}_{\alpha\mu} Z^{(\lambda)}_{\beta\nu}, \qquad W^{\sharp(\lambda)}_{\alpha\mu\beta\nu} = Z^{\sharp(\lambda)}_{\alpha\mu} Z^{\sharp(\lambda)}_{\beta\nu}, \tag{6.37}$$

$$W^{(\lambda)} \cdot W^{(\lambda)} = 0, \qquad W^{\sharp(\lambda)} \cdot W^{(\lambda')} = 4\delta_{\lambda\lambda'}. \tag{6.38}$$

The null self-contraction and the nonzero ordered dual pairing are therefore properties of the same rank-four basis, with no positive local norm introduced by hand.

6.7.2 Plane-wave continuum and normalization freedom

On an active locally flat hypersurface the generalized basis may be chosen as plane-wave profiles. Writing $\boldsymbol{\xi} = \ell_{\mathrm{P}}\mathbf{k}$ and $\hat{\mathbf{s}} = \boldsymbol{\xi}/|\xi|$, take

$$\begin{aligned} C^{(\xi,\lambda)}_{\alpha\mu\beta\nu}(x) &= A_{\lambda}(\xi) W^{(\lambda)}_{\alpha\mu\beta\nu}(\hat{\mathbf{s}}) \exp\left[-\frac{\mathrm{i}}{\ell_{\mathrm{P}}}(|\xi|x^{0} - \boldsymbol{\xi}\cdot\mathbf{x})\right], \\ C^{\sharp(\xi,\lambda)}_{\alpha\mu\beta\nu}(x) &= B_{\lambda}(\xi) W^{\sharp(\lambda)}_{\alpha\mu\beta\nu}(\hat{\mathbf{s}}) \exp\left[+\frac{\mathrm{i}}{\ell_{\mathrm{P}}}(|\xi|x^{0} - \boldsymbol{\xi}\cdot\mathbf{x})\right]. \end{aligned} \tag{6.39}$$

The ordinary Fourier delta distribution on the hypersurface, together with Eq. (6.38), fixes only the product of the two scalar normalizations,

$$A_{\lambda}(\xi)B_{\lambda}(\xi) = \frac{1}{4(2\pi)^{3}\ell_{\mathrm{P}}^{4}}. \tag{6.40}$$

Reciprocal rescaling of a right mode and its dual leaves the biorthogonal relation unchanged. This freedom is a basis convention; the rank-eight projector is invariant because it contains their ordered product,

$$(\Pi_{\xi,\lambda})_{\alpha\mu\beta\nu}{}^{\delta\gamma\phi\psi}(x,y) = \ell_{\mathrm{P}}\kappa^2 T^{(\xi,\lambda)}_{\alpha\mu\beta\nu}(x) T^{\sharp(\xi,\lambda)\,\delta\gamma\phi\psi}(y). \tag{6.41}$$

The free generator is consequently resolved on the same complete-source basis,

$$H_{\alpha\mu\beta\nu}{}^{\delta\gamma\phi\psi}(x,y) = \ell_{\mathrm{P}}\kappa^2 \sum_{\lambda=\pm 2} \int_{\mathbb{R}^3} |\xi| T^{(\xi,\lambda)}_{\alpha\mu\beta\nu}(x) T^{\sharp(\xi,\lambda)\,\delta\gamma\phi\psi}(y)\, \mathrm{d}^3\xi. \tag{6.42}$$

Thus the continuum labels organize complete rank-four modes inside the spectral integral; they never become additional tensor indices of the generator.

6.7.3 Spherical and multipole continuum

For radiation from localized systems it is useful to separate radial momentum from angular and helicity content. With $\boldsymbol{\xi} = \xi\hat{\mathbf{s}}$, the momentum measure and delta distribution are

$$\mathrm{d}^3\xi = \xi^2 \mathrm{d}\xi\, \mathrm{d}\Omega, \qquad \delta^{(3)}(\boldsymbol{\xi} - \boldsymbol{\xi}') = \xi^{-2}\delta(\xi - \xi')\delta^{(2)}(\Omega - \Omega'). \tag{6.43}$$

Choose an angular–helicity basis $\mathcal{Y}_{a\lambda}$ whose phase compensates the spin-two rotation of the screen. Its orthonormality and completeness may be written together as

$$\sum_{\lambda=\pm 2} \int_{S^2} \mathcal{Y}^*_{a\lambda} \mathcal{Y}_{b\lambda}\, \mathrm{d}\Omega = \delta_{ab}, \tag{6.44}$$

$$\sum_a \mathcal{Y}_{a\lambda}(\Omega) \mathcal{Y}^*_{a\lambda'}(\Omega') = \delta_{\lambda\lambda'}\delta^{(2)}(\Omega - \Omega'). \tag{6.45}$$

The radial modes must absorb one power of ξ from each side of the biorthogonal product. Define

$$\begin{aligned} T^{(\xi,a)}_{\alpha\mu\beta\nu} &= \xi \sum_{\lambda=\pm 2} \int_{S^2} \mathcal{Y}_{a\lambda}(\Omega) T^{(\xi\hat{\mathbf{s}},\lambda)}_{\alpha\mu\beta\nu}\, \mathrm{d}\Omega, \\ T^{\sharp(\xi,a)}_{\alpha\mu\beta\nu} &= \xi \sum_{\lambda=\pm 2} \int_{S^2} \mathcal{Y}^*_{a\lambda}(\Omega) T^{\sharp(\xi\hat{\mathbf{s}},\lambda)}_{\alpha\mu\beta\nu}\, \mathrm{d}\Omega. \end{aligned} \tag{6.46}$$

Their biorthogonal normalization is directly radial,

$$\ell_{\mathrm{P}}\kappa^2 \int_\Sigma T^{\sharp(\xi,a)}_{\alpha\mu\beta\nu} T^{(\xi',b)\,\alpha\mu\beta\nu}\, \mathrm{d}\Sigma = \delta_{ab}\delta(\xi - \xi'). \tag{6.47}$$

The same rank-eight generator can therefore be resolved on the radial basis without changing its tensor structure,

$$H_{\alpha\mu\beta\nu}{}^{\delta\gamma\phi\psi}(x,y) = \ell_{\mathrm{P}}\kappa^2 \sum_a \int_0^\infty \xi T^{(\xi,a)}_{\alpha\mu\beta\nu}(x) T^{\sharp(\xi,a)\,\delta\gamma\phi\psi}(y)\, \mathrm{d}\xi. \tag{6.48}$$

The spherical and plane descriptions are therefore two resolutions of the same free tensor subspace; the apparent radial density factor has been absorbed into the mode normalization rather than promoted to a new physical weight.

6.7.4 Monochromatic angular superpositions and localized packets

A generalized monochromatic state is not yet a localized physical preparation. The distinction follows already from ordinary Fourier support: if a square-integrable envelope obeys the Helmholtz equation, its Fourier transform is supported only on the sphere $|\mathbf{p}| = k$, which has zero three-dimensional measure,

$$(\nabla^2 + k^2)F = 0, \qquad (k^2 - |\mathbf{p}|^2)\widetilde{F}(\mathbf{p}) = 0, \qquad F \in L^2(\mathbb{R}^3). \tag{6.49}$$

A nonzero localized packet therefore requires radial bandwidth.

For a general preparation absorb the plane normalization into the spectral coefficient and denote the resulting weight by $a_\lambda(\boldsymbol{\xi})$. Its Weyl profile is

$$C^{(P)}_{\alpha\mu\beta\nu}(x) = \sum_{\lambda=\pm 2} \int_{\mathbb{R}^3} a_\lambda(\boldsymbol{\xi}) W^{(\lambda)}_{\alpha\mu\beta\nu}(\hat{\boldsymbol{\xi}}) \exp\left[-\frac{\mathrm{i}}{\ell_{\mathrm{P}}}(|\xi|x^0 - \boldsymbol{\xi}\cdot\mathbf{x})\right] \mathrm{d}^3\xi. \tag{6.50}$$

The complete source and carrier remain those of Eqs. (6.32) and (6.31); only the prepared coefficient has changed.

The global Lorentzian contraction retains the full spatial phase. Spatial integration therefore pairs opposite momenta, and with the natural antipodal screen convention the opposite type-N tensor is exactly the dual,

$$\int_{\mathbb{R}^3} \exp\left[\frac{\mathrm{i}}{\ell_{\mathrm{P}}}(\boldsymbol{\xi} + \boldsymbol{\xi}')\cdot\mathbf{x}\right] \mathrm{d}^3x = (2\pi)^3 \ell_{\mathrm{P}}^3 \delta^{(3)}(\boldsymbol{\xi} + \boldsymbol{\xi}'), \tag{6.51}$$

$$W^{(\lambda)}(-s) = W^{\sharp(\lambda)}(s), \qquad W^{(\lambda)}(s)\cdot W^{(\lambda')}(-s) = 4\delta_{\lambda\lambda'}. \tag{6.52}$$

The single global contraction of the prepared Weyl profile is then

$$\mathcal{G}_P(x^0) = 4(2\pi)^3 \ell_{\mathrm{P}}^4 \sum_{\lambda=\pm 2} \int_{\mathbb{R}^3} a_\lambda(\boldsymbol{\xi}) a_\lambda(-\boldsymbol{\xi}) e^{-2\mathrm{i}|\xi|x^0/\ell_{\mathrm{P}}} \, \mathrm{d}^3\xi. \tag{6.53}$$

The two-copy probability law uses this amplitude together with its independent conjugate. Antipodal pairing is therefore selected by the complete global contraction, not by a local positive replacement for the null self-norm.

A fixed radial shell can still be used as a generalized angular state. For the normalized axisymmetric weight with concentration parameter χ and unit axis z^μ, the weight and its antipodal contraction are

$$w_\chi(s) = \frac{\chi}{4\pi \sinh\chi} \exp(\chi h_{\mu\nu} z^\mu s^\nu), \qquad \int_{S^2} w_\chi \, \mathrm{d}\Omega = 1, \tag{6.54}$$

$$K^-_\chi = 4\int_{S^2} w_\chi(s) w_\chi(-s) \, \mathrm{d}\Omega = \frac{\chi^2}{\pi \sinh^2\chi}. \tag{6.55}$$

The exponential suppression of K^-_χ for a narrow cone is therefore a property of the antipodal global pairing. A genuinely localized state additionally requires a nonzero radial width in Eq. (6.50).

6.7.5 Sector-global prepared-state normalization

The ordered overlap used above is covariant under the signed sector pullback. For two vacuum profiles transported by the same active map,

$$\int_{\Sigma_{(n+1)}} C^{(n+1)}_{\alpha\mu\beta\nu} D^{(n+1)\,\alpha\mu\beta\nu}\,\mathrm{d}\Sigma_{(n+1)} = \int_{\Sigma_{(n)}} C^{(n)}_{\alpha\mu\beta\nu} D^{(n)\,\alpha\mu\beta\nu}\,\mathrm{d}\Sigma_{(n)}. \tag{6.56}$$

The all-lower orientation signs occur twice and the four inverse metrics used in the contraction contribute an even sign, so no sector-dependent normalization factor appears.

A preparation extending across several active charts is therefore normalized by evaluating the same contraction on chart domains with disjoint interiors,

$$\mathcal{G}_P = \ell_{\mathrm{P}}\kappa^2 \sum_n \int_{\Sigma\cap U_n} T^{(P,n)}_{\alpha\mu\beta\nu} T^{(P,n)\,\alpha\mu\beta\nu}\,\mathrm{d}\Sigma_{(n)}. \tag{6.57}$$

The reciprocal chart is used wherever the neighbouring sector becomes active, and Eq. (6.56) guarantees that the global result does not depend on the placement of a chart boundary inside an overlap.

6.8 Closed Weyl histories of the free graviton

The antipodal pairing of the spectral basis has a geometric counterpart in the four-sector orbit. The relevant object is the Weyl tensor itself: successive active descriptions are related by signed pullback, and one complete cycle returns the all-lower rank-four tensor to its initial representative. The type-N radiative case is then the closed history of one free graviton. The same tensorial construction may also be applied to a nonradiative Weyl field without changing its algebraic type.

6.8.1 Rank-four closure under the sector quarter-turn

Choose a reference active clock u^μ and a unit rest-space direction s^μ, with $u^\mu u_\mu = -1$, $s^\mu s_\mu = 1$, and $u_\mu s^\mu = 0$. For one orientation of the sector orbit, the successive clock–ruler axes are generated by $\theta_r = r\pi/2$,

$$u^\mu_r = \cos\theta_r\, u^\mu + \sin\theta_r\, s^\mu, \qquad s^\mu_r = -\sin\theta_r\, u^\mu + \cos\theta_r\, s^\mu. \tag{6.58}$$

For odd r these are inherited labels whose timelike and spacelike roles are read with the corresponding active metric. Reversing the orientation reverses the order of the representatives but not the closure.

Let I_r be the active pullback after r quarter-turns. The signed rank-four transport of Sec. 6.4 gives

$$C^{(r)}_{\alpha\mu\beta\nu} = (-1)^r I_r^* C^{(0)}_{\alpha\mu\beta\nu}, \qquad C^{(4)}_{\alpha\mu\beta\nu} = C^{(0)}_{\alpha\mu\beta\nu}. \tag{6.59}$$

Pullback preserves all Weyl symmetries and tracelessness, while the fourth power of the orientation sign is unity. Closure is therefore a statement about the uncontracted curvature tensor and does not depend on the Petrov type of the seed. The ordered global contractions are preserved at the same time by Eq. (6.56).

6.8.2 Type-N specialization and antipodal structure

For a radiative seed the sector axes define four inherited null directions. It is convenient to use unnormalized vectors for the momentum readout,

$$(\ell_0^\mu, \ell_1^\mu, \ell_2^\mu, \ell_3^\mu) = (u^\mu + s^\mu, -u^\mu + s^\mu, -u^\mu - s^\mu, u^\mu - s^\mu), \qquad \ell_{r+2}^\mu = -\ell_r^\mu. \tag{6.60}$$

Each branch is null and carries the same positive active energy $E_G = \hbar\omega$. With $p_r^\mu = (E_G/c)\ell_r^\mu$, the oriented first moment closes while the quadratic moment remains finite,

$$\sum_{r=0}^{3} p_r^\mu = 0, \qquad \sum_{r=0}^{3} p_r^\mu p_r^\nu = \frac{4E_G^2}{c^2}(u^\mu u^\nu + s^\mu s^\nu). \tag{6.61}$$

The cancellation of external four-momentum therefore does not erase the tensor content of the history.

The first two branches also show why the half-orbit bookkeeping is spacelike in the inherited frame. For positive affine weights a, b,

$$\Delta X_{AB}^\mu = a\ell_0^\mu + b\ell_1^\mu = (a-b)u^\mu + (a+b)s^\mu, \qquad \Delta X_{AB}^2 = 4ab, \tag{6.62}$$

$$Q_{AB}^\mu = p_0^\mu + p_1^\mu = \frac{2E_G}{c}s^\mu, \qquad Q_{AB}^2 = \frac{4E_G^2}{c^2} \tag{6.63}$$

for the symmetric choice $a = b$. In that case the inherited clock projections of both ΔX_{AB}^μ and Q_{AB}^μ vanish. The opposite half-orbit carries the negative momentum and completes the closed first moment; every local branch nevertheless remains null.

The rank-four curvature has the complementary parity. Normalize the null vectors by $\widetilde{\ell}_r^\mu = \ell_r^\mu/\sqrt{2}$. On the transported screen use $m^{(+2)} = m$ and $m^{(-2)} = \bar{m}$, and form the bivectors and Weyl representatives

$$Z_{\alpha\mu}^{(\lambda,r)} = \widetilde{\ell}_\alpha^{(r)} m_\mu^{(\lambda)} - \widetilde{\ell}_\mu^{(r)} m_\alpha^{(\lambda)}, \qquad W_{\alpha\mu\beta\nu}^{(\lambda,r)} = Z_{\alpha\mu}^{(\lambda,r)} Z_{\beta\nu}^{(\lambda,r)}. \tag{6.64}$$

The half-turn changes the sign of Z but not of W, so $W^{(\lambda,r+2)} = W^{(\lambda,r)}$ while $p_{r+2}^\mu = -p_r^\mu$. With the antipodal screen convention already used in Eq. (6.52), the adjacent branch is the dual spectral tensor.

More generally, let s and s' be two unit directions in the same observer rest space, let $\gamma = s_\mu s'^\mu$, and let ψ be the relative transported screen phase. The normalized null bivectors carry one power of the angular overlap, while the Weyl tensor carries its square,

$$\begin{aligned} Z^{(\lambda)}(s)\cdot Z^{(\lambda')}(s') &= \delta_{\lambda\lambda'}(1-\gamma)e^{\mathrm{i}\lambda\psi/2}, \\ W^{(\lambda)}(s)\cdot W^{(\lambda')}(s') &= \delta_{\lambda\lambda'}(1-\gamma)^2 e^{\mathrm{i}\lambda\psi}. \end{aligned} \tag{6.65}$$

Parallel type-N representatives are null under this ordered contraction, whereas the antipodal choice $\gamma = -1$ gives the dual normalization $4\delta_{\lambda\lambda'}$. This is the same antipodal structure selected independently by the global packet contraction.

6.8.3 Tensor resolution of an arbitrary local Weyl state

A general local Weyl tensor may be resolved relative to an active observer without changing its Petrov type. Its electric and magnetic parts are the spatial symmetric trace-free tensors

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \qquad B_{\mu\nu} = {}^\star C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \qquad \mathcal{Q}_{\mu\nu} = E_{\mu\nu} + \mathrm{i}B_{\mu\nu}. \tag{6.66}$$

They contain the ten independent real components of the four-dimensional Weyl tensor. For each unit spatial direction s^μ, define the screen projector $P_{\mu\nu} = h_{\mu\nu} - s_\mu s_\nu$ and the transverse trace-free projector

$$\Lambda_{\mu\nu}{}^{\rho\sigma}(s) = P_\mu{}^{(\rho}P_\nu{}^{\sigma)} - \frac{1}{2}P_{\mu\nu}P^{\rho\sigma}. \tag{6.67}$$

Rotational invariance fixes the angular average of this projector. Using the second and fourth moments of a unit vector on the observer sphere gives

$$\begin{aligned} \int_{S^2} s_\mu s_\nu \,\mathrm{d}\Omega &= \frac{4\pi}{3}h_{\mu\nu}, \\ \int_{S^2} s_\mu s_\nu s_\rho s_\sigma \,\mathrm{d}\Omega &= \frac{4\pi}{15}(h_{\mu\nu}h_{\rho\sigma} + h_{\mu\rho}h_{\nu\sigma} + h_{\mu\sigma}h_{\nu\rho}), \qquad (6.68)\\ \int_{S^2} \Lambda_{\mu\nu}{}^{\rho\sigma}(s) \,\mathrm{d}\Omega &= \frac{8\pi}{5}J_{\mu\nu}{}^{\rho\sigma}, \qquad (6.69) \end{aligned}$$

where $J_{\mu\nu}{}^{\rho\sigma} = h_{(\mu}{}^\rho h_{\nu)}{}^\sigma - h_{\mu\nu}h^{\rho\sigma}/3$ is the identity on spatial symmetric trace-free tensors. Since $\mathcal{Q}_{\mu\nu}$ already belongs to this subspace, the complete local Weyl data are reconstructed by

$$\mathcal{Q}_{\mu\nu} = \frac{5}{8\pi}\int_{S^2} \Lambda_{\mu\nu}{}^{\rho\sigma}(s)\mathcal{Q}_{\rho\sigma}\,\mathrm{d}\Omega_s. \tag{6.70}$$

This is a tight-frame resolution at one spacetime point. It does not replace a generic nonradiative field by a collection of independent global plane-wave solutions.

On each screen the projected tensor has the two helicity components already identified in Sec. 6.3,

$$\Lambda_{\mu\nu}{}^{\rho\sigma}\mathcal{Q}_{\rho\sigma} = q_+ m_\mu m_\nu + q_- \bar{m}_\mu \bar{m}_\nu, \qquad q_+ = \bar{m}^\mu \bar{m}^\nu \mathcal{Q}_{\mu\nu}, \qquad q_- = m^\mu m^\nu \mathcal{Q}_{\mu\nu}. \tag{6.71}$$

Self-adjointness and idempotence of Λ then give the corresponding norm identity,

$$E_{\mu\nu}E^{\mu\nu} + B_{\mu\nu}B^{\mu\nu} = \frac{5}{8\pi}\int_{S^2} [\Lambda(s)\mathcal{Q}]_{\mu\nu}[\Lambda(s)\overline{\mathcal{Q}}]^{\mu\nu}\,\mathrm{d}\Omega_s. \tag{6.72}$$

The same directional resolution therefore reconstructs both the tensor and its observer-resolved positive quadratic combination.

6.8.4 Antipodal electric–magnetic decomposition

The local screen data can be organized into the two opposite null directions of a closed type-N orbit. Let $\varepsilon_{\alpha\beta\gamma} = u^\rho \eta_{\rho\alpha\beta\gamma}$ be the spatial volume tensor and define its action on a transverse trace-free screen tensor by

$$(\mathcal{J}_s X)_{\mu\nu} = \varepsilon_{\alpha\beta(\mu}s^\alpha X_{\nu)}{}^\beta, \qquad \mathcal{J}_s^2 X = -X, \qquad \mathcal{J}_{-s} = -\mathcal{J}_s. \tag{6.73}$$

With the orientation fixed by the Hodge dual, a type-N mode in the $+s$ direction satisfies $B = \mathcal{J}_s E$, while the antipodal mode satisfies $B = -\mathcal{J}_s E$.

Project an arbitrary Weyl tensor with $\Lambda(s)$ and denote the two antipodal electric amplitudes by E_+ and E_-. Their sum must reproduce the projected electric data, while their signed screen rotations must reproduce the projected magnetic data,

$$E_+ + E_- = \Lambda(s)E, \qquad \mathcal{J}_s(E_+ - E_-) = \Lambda(s)B. \tag{6.74}$$

Since $\mathcal{J}_s^2 = -1$ on the transverse trace-free screen, this system has the unique solution

$$E_+ = \frac{1}{2}[\Lambda(s)E - \mathcal{J}_s\Lambda(s)B], \qquad E_- = \frac{1}{2}[\Lambda(s)E + \mathcal{J}_s\Lambda(s)B]. \tag{6.75}$$

The electric Weyl data are therefore antipodal even, while the magnetic data enter through the screen-rotated antipodal difference. For a purely electric field the two amplitudes coincide and each carries one half of the projected electric tensor.

6.8.5 Schwarzschild curvature as an exact gravitational test

The Schwarzschild exterior gives a nonradiative type-D test of the same rank-four transport. Since $R_{\mu\nu} = 0$, the complete exterior curvature is Weyl and the trace-free four-index source is C/κ. For a static orthonormal observer with outward unit radial vector n^μ, define $\mu_g = GM/(c^2R^3)$. The local curvature is purely electric,

$$E^S_{\mu\nu} = \mu_g(h_{\mu\nu} - 3n_\mu n_\nu), \qquad B^S_{\mu\nu} = 0. \tag{6.76}$$

Its radial eigenvalue is $-2\mu_g$ and each transverse eigenvalue is μ_g. These are directly measurable through geodesic deviation. In the local inertial frame of the observer,

$$\frac{D^2\xi^\mu}{\mathrm{d}\tau^2} = -c^2E^\mu{}_\nu\xi^\nu, \qquad \frac{D^2\xi_\parallel}{\mathrm{d}\tau^2} = \frac{2GM}{R^3}\xi_\parallel, \qquad \frac{D^2\xi_\perp}{\mathrm{d}\tau^2} = -\frac{GM}{R^3}\xi_\perp. \tag{6.77}$$

Thus the tensor transported by the sector orbit carries the ordinary Schwarzschild radial stretching and transverse compression.

The same eigenvalues fix the quadratic curvature without any additional construction,

$$E^S_{\mu\nu}E_S^{\mu\nu} = 6\mu_g^2, \qquad C^S_{\alpha\mu\beta\nu}C_S^{\alpha\mu\beta\nu} = 48\mu_g^2 = \frac{48G^2M^2}{c^4R^6}. \tag{6.78}$$

Because the Ricci tensor vanishes, the second expression is the Schwarzschild Kretschmann invariant. The radial trace-free source component is correspondingly $\widehat{T}^S_{unun} = -2GM/(\kappa c^2R^3)$.

The tight-frame resolution can also be checked explicitly. Let $\chi = s_\mu n^\mu$ and choose the first screen direction along the projection of n^μ orthogonal to s^μ. The projected electric tensor and its circular coefficients are

$$\Lambda_{\mu\nu}{}^{\rho\sigma}(s)E^S_{\rho\sigma} = \frac{3}{2}\mu_g(1-\chi^2)(e_{2\mu}e_{2\nu} - e_{1\mu}e_{1\nu}), \qquad q^S_+ = q^S_- = -\frac{3}{2}\mu_g(1-\chi^2). \tag{6.79}$$

The equality of the two helicity coefficients is precisely the purely electric antipodal relation derived above. Using $\int_{S^2}(1-\chi^2)^2\mathrm{d}\Omega = 32\pi/15$, Eq. (6.72) returns

$$\frac{5}{8\pi}\int_{S^2}\frac{9}{2}\mu_g^2(1-\chi^2)^2\,\mathrm{d}\Omega = 6\mu_g^2. \tag{6.80}$$

The local spin-two resolution therefore reproduces the exact type-D Weyl norm without changing the Petrov type of the Schwarzschild field.

6.8.6 Propagation of changes in a closed Weyl history

Closure describes the complete sector history, whereas a physical change of the curvature remains governed locally by the vacuum Weyl equations. In a local inertial frame the electric and magnetic tensors satisfy the transverse constraints and first-order evolution system

$$\partial^i E_{ij} = 0, \qquad \partial^i B_{ij} = 0, \tag{6.81}$$

$$\frac{1}{c}\partial_t E_{ij} = \epsilon_i{}^{kl}\partial_k B_{lj}, \qquad \frac{1}{c}\partial_t B_{ij} = -\epsilon_i{}^{kl}\partial_k E_{lj}. \tag{6.82}$$

Differentiating once in time and substituting the second first-order equation produces a double contraction of the spatial volume tensor. The contraction identity and the transverse constraint remove the mixed derivative term,

$$\begin{aligned}\frac{1}{c^2}\partial_t^2 E_{ij} &= -\epsilon_i{}^{kl}\epsilon_l{}^{mn}\partial_k\partial_m E_{nj},\\ \epsilon_i{}^{kl}\epsilon_l{}^{mn} &= \delta_i{}^m h^{kn} - \delta_i{}^n h^{km}.\end{aligned} \tag{6.83}$$

The same calculation applies to B_{ij}, giving wave equations directly on the curvature components,

$$\left(\frac{1}{c^2}\partial_t^2 - h^{km}\partial_k\partial_m\right)E_{ij} = 0, \qquad \left(\frac{1}{c^2}\partial_t^2 - h^{km}\partial_k\partial_m\right)B_{ij} = 0. \tag{6.84}$$

No metric gauge has entered this derivation.

For a local Fourier perturbation with covector k_μ, the transverse constraints and the wave equation give

$$k^i e_{ij} = 0, \qquad k^i b_{ij} = 0, \qquad \Omega^2 = c^2 h^{ab}k_a k_b, \qquad g^{\mu\nu}k_\mu k_\nu = 0. \tag{6.85}$$

Changes of the free Weyl profile therefore propagate on the same null characteristic as the underlying graviton mode. The result is obtained directly from the curvature equations and is independent of how the same perturbation might be represented by a metric potential.

On a curved Ricci-flat background the second-order equation acquires algebraic couplings to the background curvature. Writing their coefficient tensor as $\mathcal{A}_{\alpha\mu\beta\nu}{}^{\rho\sigma\lambda\tau}$, the propagated perturbation has the form

$$\nabla^\rho\nabla_\rho \delta C_{\alpha\mu\beta\nu} + \mathcal{A}_{\alpha\mu\beta\nu}{}^{\rho\sigma\lambda\tau}\delta C_{\rho\sigma\lambda\tau} = 0. \tag{6.86}$$

The second term contains no second derivatives of δC, so the principal characteristic remains null.

The same conclusion is visible in geometric optics. For a slowly modulated type-N mode $C = AWe^{\mathrm{i}S/\epsilon}$, with $k_\mu = \nabla_\mu S$, the first two orders of the vacuum differential identities give

$$k^\mu k_\mu = 0, \qquad k^\rho\nabla_\rho k^\mu = 0, \qquad k^\rho\nabla_\rho A + \frac{1}{2}A\nabla_\rho k^\rho = 0. \tag{6.87}$$

The sector pullback carries these relations into the neighbouring active chart without changing the eigenvalue, helicity or characteristic class. The spacelike half-orbit bookkeeping and the null propagation of physical changes are therefore compatible properties of the same closed free-graviton history.

6.9 Conclusion

The free gravitational quantum obtained here is the Ricci-flat radiative specialization of the complete four-index carrier. In vacuum the complete source reduces algebraically to its trace-free Weyl sector, the vacuum differential identities fix a null characteristic, the transverse screen carries the two helicities $\lambda = \pm 2$, and the rank-eight evolution assigns the energy $E_G = \hbar\omega$. Masslessness, spin and energy are therefore properties of one rank-four tensor mode rather than independent ingredients attached to a separate metric-polarization state.

The four-sector extension transports this same mode through successive active clock–ruler descriptions of one geometry. The complete source and Weyl tensor follow by signed pullback, while the rank-eight generator is transported by conjugation and retains its spectral eigenvalue. The reciprocal Rosen geometry makes the distinction between inherited coordinates and local observables explicit: an inherited coordinate phase gradient can become arbitrarily large while the active-clock frequency remains finite, and the exact tidal background approaches a curvature-flat reciprocal endpoint. The transported positive-frequency subspace and transverse helicity basis are preserved at the same time.

The spectral resolution is necessarily biorthogonal because an exact type-N Weyl mode has vanishing Lorentzian self-contraction. The explicit null-frame dual produces the nonzero ordered pairing required by the spectral projector, and the plane and spherical continua are equivalent resolutions of the same free tensor subspace. For localized preparations the complete phase-dependent contraction is retained; it pairs antipodal momenta and identifies the opposite-direction type-N tensor with the corresponding dual element without introducing another physical excitation.

This antipodal structure extends naturally to a closed four-sector history. A single type-N quantum is read through four successive null branches whose oriented momenta cancel over the complete orbit, while its rank-four Weyl content is even under the antipodal half-turn. The all-lower tensor returns to its initial form after four signed pullbacks, and the same closure applies to arbitrary Weyl data without changing their algebraic type. The Schwarzschild exterior provides an explicit type-D test: the closed transport preserves the exact tidal eigenvalues and quadratic curvature invariant, while the local spin-two screen resolution reconstructs the same Weyl tensor rather than replacing it by a radiative type-N field.

The closed geometry does not modify the local propagation law. The electric and magnetic parts of the vacuum Weyl tensor obey a first-order Bianchi system whose second-order consequence is a curvature wave equation with null characteristic. On a curved Ricci-flat background the lower-order curvature couplings leave the principal characteristic unchanged, and the geometric-optics limit transports the mode along null geodesics. The resulting construction therefore combines a closed four-sector tensor history, biorthogonal free-graviton spectral resolution and ordinary null propagation of physical Weyl perturbations within one covariant rank-four framework.

Chapter 7

Four-Sector Projection of Local Ultraviolet Sources

Source and status. This chapter incorporates the complete scientific body of *A Four-Sector Cancellation Mechanism for Local Ultraviolet Sources in Perturbative Einstein Gravity*, v0.08 (19 August 2026). Its role in the dissertation is: Background-field quantization, two-loop benchmark and complete-orbit projector. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

Perturbative Einstein gravity is finite on shell at one loop but develops a genuine local ultraviolet divergence at two loops, and higher loop order permits increasingly higher-derivative local counterterms. This paper studies that problem inside the previously introduced four-sector Lorentz and four-index source framework. The metric is quantized by the standard background-field path integral with gauge fixing and Faddeev–Popov ghosts, with the usual dimensionally regulated loop expansion. Proper subdivergences, including evanescent contributions, are removed before the remaining local ultraviolet residue is extracted. Metric variation turns that residue into a symmetric conserved rank-two gravitational response. The particle-sector action is then written explicitly as an indexed tensor map on that response and on the graviton kernels that generate it. Internal sector maps cancel against the inverse maps carried by contracted propagator indices, whereas the two free covariant indices retain the quadratic character $\mathrm{i}^{2k} = (-1)^k$. The four representatives of one complete particle orbit therefore contain the same scalar loop coefficients and have a vanishing invariant rank-two ultraviolet component. Geometry-sector homogeneity is treated independently, with separate geometry and particle labels throughout. The two-loop cubic-curvature divergence of pure Einstein gravity supplies an explicit benchmark, including the regulated Gauss–Bonnet subdivergence structure and the full metric Euler tensor of the cubic invariant. The rank-two result extends term by term to the local residue at arbitrary perturbative loop order and is then embedded explicitly in the divergence-free four-index source decomposition. The projector acts on the loop-generated local ultraviolet response, while the tree-level Einstein term and the finite loop contribution retain their standard roles in the effective action. The result is an order-by-order cancellation of local ultraviolet gravitational sources in pure Einstein gravity with vanishing cosmological constant within the four-sector source completion defined in the underlying framework.

7.1 Introduction

Einstein gravity treated as a perturbative quantum field theory has a sharp ultraviolet hierarchy. Pure gravity is on-shell finite at one loop after the four-dimensional curvature identities are taken into account, while at two loops a genuine cubic-curvature divergence remains [32, 33, 59, 61]. Modern amplitude calculations make the same separation visible in a particularly clean way: the regulated two-loop amplitude contains a local ultraviolet pole together with finite and nonlocal renormalized contributions [1, 2].

The four-sector construction developed in Refs. [41–43] uses two independent sector labels. A geometry sector specifies the active clock–ruler chart of one spacetime, while a particle sector specifies the relative four-sector type of a physical source in that geometry. The particle labels form a closed four-step orientation cycle. For a rank-two source the sector phase is quadratic and therefore alternates in sign around that cycle. This paper applies that rank-two transformation to the local source generated by quantum graviton loops.

The calculation below uses the standard perturbative quantization. The Einstein metric is split into a background and a quantum fluctuation, the fluctuation and the Faddeev–Popov ghosts are integrated in the usual background-field functional integral, dimensional regularization is retained, and recursive subtraction is performed before any four-sector operation. This order matters. In particular, evanescent Gauss–Bonnet terms can participate in a higher-loop subdivergence even though their four-dimensional on-shell matrix elements vanish [2].

The four-sector step begins only after the regulated loop calculation has produced a local diffeomorphism-covariant pole functional. Varying this functional with respect to the metric produces a conserved rank-two local gravitational response. The main observation is tensorial: the sector map acts on the two indices of the final local response, whereas sector factors attached to internally contracted graviton indices cancel against the inverse factors carried by the propagator tensor. A complete particle-sector orbit therefore has a vanishing invariant local ultraviolet component. The finite local and nonlocal terms remain as separate contributions in the effective action.

The calculation is organized as one sequence. The background-field path integral and power counting first define the perturbative setting. Recursive forest subtraction then isolates the regulator-level local pole, whose metric Euler derivative gives a symmetric conserved rank-two source. The particle-sector map is applied to that tensor and verified directly on indexed graviton kernels before the complete-orbit projector is formed. Geometry-sector covariance and homogeneity are then treated as an independent transformation of the same local response. The Goroff–Sagnotti cubic-curvature pole provides the explicit two-loop benchmark, after which the tensor argument is extended to the complete local pole basis at arbitrary loop order. The resulting rank-two source is finally embedded in the divergence-free four-index source equation and separated from the finite loop contribution in the effective action.

7.2 Background-field quantization of Einstein gravity

We use four spacetime dimensions for the physical theory and dimensional regularization with

$$D = 4 - 2\epsilon \tag{7.1}$$

in intermediate loop calculations. Natural units are used in the perturbative sections. With the mostly-plus metric convention, the dimensionally continued Einstein–Hilbert action is written as

$$S_{\rm EH}^{(D)}[g] = -\frac{2}{\kappa_D^2}\int \mathrm{d}^D x\,\sqrt{-g}\,R, \qquad \kappa_D = \mu^\epsilon \kappa_g, \qquad \kappa_g^2 = 32\pi G. \tag{7.2}$$

Here μ is the dimensional-regularization scale and κ_g is the physical four-dimensional coupling recovered at $\epsilon \to 0$. Equation (7.2) is the pure Einstein action at $\Lambda = 0$, which is used throughout. The background-field split in the regulated theory is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa_D h_{\mu\nu}. \tag{7.3}$$

The background $\bar{g}_{\mu\nu}$ is held fixed in the functional integral, while the symmetric fluctuation $h_{\mu\nu}$ is the quantum field. Its gauge redundancy is fixed before the loop expansion is formed.

A convenient background de Donder condition is

$$F_\mu[h;\bar{g}] = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu h, \qquad h = \bar{g}^{\rho\sigma}h_{\rho\sigma}. \tag{7.4}$$

At unit gauge parameter the gauge-fixing action may be written

$$S_{\rm gf} = -\frac{1}{2}\int \mathrm{d}^D x\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu. \tag{7.5}$$

The corresponding Faddeev–Popov operator acting on a ghost vector is

$$(\Delta_{\rm gh})^\mu{}_\nu = -\delta^\mu{}_\nu\bar{\nabla}^2 - \bar{R}^\mu{}_\nu, \tag{7.6}$$

and

$$S_{\rm gh} = \int \mathrm{d}^D x\sqrt{-\bar{g}}\,\bar{c}_\mu(\Delta_{\rm gh})^\mu{}_\nu c^\nu. \tag{7.7}$$

The gauge-fixed quantum theory is therefore defined by the functional integral

$$\exp\!\left(\frac{\mathrm{i}}{\hbar}\Gamma[\bar{g}]\right) = \int \mathcal{D}h\,\mathcal{D}\bar{c}\,\mathcal{D}c\,\exp\!\left[\frac{\mathrm{i}}{\hbar}\left(S_{\rm EH}^{(D)}[\bar{g}+\kappa_D h] + S_{\rm gf} + S_{\rm gh}\right)\right], \tag{7.8}$$

with the ordinary normalization factors understood. The loop expansion of the background effective action is

$$\Gamma[\bar{g}] = S_{\rm EH}^{(D)}[\bar{g}] + \sum_{L=1}^{\infty}\hbar^L\Gamma_L[\bar{g}]. \tag{7.9}$$

Equation (7.8) is the quantization step used throughout the paper. In the local-covariant sections below, the background argument $\bar{g}_{\mu\nu}$ of the effective action is denoted simply by $g_{\mu\nu}$. The four-sector map enters later, at the level of the local tensor source extracted from the regulated effective action.

Expanding the action to second order gives a minimal second-order graviton operator after gauge fixing. In background-covariant notation it has the form

$$\Delta_h = -\bar{\nabla}^2\mathbf{1}_{S^2} + \mathcal{U}(\bar{R}_{\mu\nu\rho\sigma},\bar{R}_{\mu\nu}), \tag{7.10}$$

where $\mathbf{1}_{S^2}$ is the identity on symmetric rank-two tensors and $\mathcal{U}$ is algebraic in the background curvature. On a Ricci-flat background the physical spin-two part reduces to the Lichnerowicz operator

$$(\Delta_L h)_{\mu\nu} = -\bar{\nabla}^2 h_{\mu\nu} - 2\bar{R}_{\mu\alpha\nu\beta}h^{\alpha\beta}. \tag{7.11}$$

At one loop the two Gaussian integrations therefore give the standard determinant structure

$$\Gamma_1 = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\Delta_h - \mathrm{i}\,\mathrm{Tr}\ln\Delta_{\mathrm{gh}}, \tag{7.12}$$

up to the usual treatment of zero modes and normalization. Higher loops are generated by the cubic and higher vertices in the expansion of $S_{\mathrm{EH}}^{(D)}[\bar{g} + \kappa_D h]$.

The superficial momentum degree follows directly from this expansion. Every Einstein vertex carries two derivatives. In the regulated dimension $D = 4 - 2\epsilon$, a connected graph with L loops, I internal graviton lines and V vertices has

$$\omega_D = DL + 2V - 2I. \tag{7.13}$$

Using $L = I - V + 1$ gives

$$\omega_D = (D-2)L + 2 = 2L + 2 - 2\epsilon L. \tag{7.14}$$

The physical four-dimensional degree is therefore

$$\omega \equiv \lim_{\epsilon\to 0}\omega_D = 2L + 2. \tag{7.15}$$

Thus the four-dimensional local logarithmic structures allowed at loop order L have total derivative order $2L+2$. At one loop the local basis begins at curvature-squared order. At two loops the first genuine on-shell divergence of pure Einstein gravity is cubic in the Riemann tensor [33, 59].

Power counting therefore fixes the derivative order of the local ultraviolet structures that can occur at each loop order. The next step is to construct those local residues from the regulated graphs before any sector projection is applied.

7.3 Regulated subtraction and local ultraviolet residue

7.3.1 Recursive forest subtraction and evanescent terms

The local ultraviolet source used later in the sector construction is obtained only after the regulated subdivergences have been treated. The recursive structure is written with the usual forest operation. For every proper divergent one-particle-irreducible subgraph γ, define the regulator-level counterterm

$$C_D(\gamma) = -K_\gamma R'_D \gamma. \tag{7.16}$$

If $\mathfrak{F}'(\Gamma)$ denotes the set of forests built from mutually compatible proper divergent subgraphs, including the empty forest, then schematically

$$R'_D\Gamma = \sum_{F\in\mathfrak{F}'(\Gamma)}\left(\prod_{\gamma\in F} C_D(\gamma)\right)\Gamma/F. \tag{7.17}$$

Nested subgraphs are treated from the inside outward, while genuinely overlapping subgraphs occur in distinct compatible forest terms. After all proper subdivergences have been removed, the overall renormalized graph is

$$R_D\Gamma = (1 - K_\Gamma)R'_D\Gamma. \tag{7.18}$$

The forest operation first constructs $K_\Gamma R'_D\Gamma$ at regulator level. The role of evanescent terms in this construction is already visible at two loops. In dimensional regularization the one-loop pure-gravity calculation contains the evanescent Gauss–Bonnet density

$$E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{7.19}$$

and in the convention of Ref. [2] its pure-gravity counterterm contains

$$\Gamma^{(1)}_{\mathrm{GB}} = \frac{1}{(4\pi)^2}\frac{53}{90}\frac{1}{\epsilon}\int \mathrm{d}^D x\sqrt{-g}\,E_4. \tag{7.20}$$

The metric variation of the integrand is the Lanczos tensor,

$$\begin{aligned} L^{\mathrm{GB}}_{\mu\nu} = {} & 2RR_{\mu\nu} - 4R_{\mu\alpha}R_\nu{}^\alpha - 4R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ & + 2R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - \frac{1}{2}g_{\mu\nu}E_4. \end{aligned} \tag{7.21}$$

In exactly four dimensions the integrated Euler density is topological, and $L^{\mathrm{GB}}_{\mu\nu}$ vanishes identically for variations with the standard boundary conditions. In $D = 4 - 2\epsilon$, however, its insertions are part of the regulated subdivergence structure [2].

The relevant mixing can be seen algebraically. If an evanescent vertex has the physical expansion

$$V_{\mathrm{ev}}(D) = \epsilon\,\widehat{V} + O(\epsilon^2), \tag{7.22}$$

then insertion into another ultraviolet-divergent integration can produce

$$\frac{1}{\epsilon}\left(\epsilon\widehat{V}\right)\left(\frac{B}{\epsilon} + F + O(\epsilon)\right) = \frac{B\widehat{V}}{\epsilon} + \widehat{V}F + O(\epsilon). \tag{7.23}$$

A term that vanishes when the regulator is removed can therefore change the coefficient of the physical simple pole at the next loop order. These evanescent contributions are incorporated into the scalar coefficients of the fully subtracted local residue. The output of the regulator-level stage is therefore the local functional $KR'_D\Gamma_L$, with all proper subdivergences already organized.

7.3.2 Local pole as a conserved tensor source

The forest operation now supplies a local diffeomorphism-covariant pole functional. At loop order L define

$$\Gamma^{(L)}_{\mathrm{pole}}[g] \equiv KR'_D\Gamma_L[g]. \tag{7.24}$$

After all proper subdivergences have been subtracted, the pole coefficients are first obtained in $D = 4 - 2\epsilon$. Once the regulator-level tensor algebra has been completed, the physical residue may be reduced to a four-dimensional covariant local basis. Locality and background diffeomorphism covariance then allow the expansion

$$\Gamma^{(L)}_{\text{pole}}[g] = \sum_{r=1}^{r_{\max}} \frac{1}{\epsilon^r} \sum_A c^{(L)}_{A,r} \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{O}^{(L)}_A(g, R, \nabla R, \ldots). \tag{7.25}$$

Integrations by parts, Bianchi identities and local field redefinitions relate equivalent representatives of this basis. The tensor relations below hold for any such equivalent covariant basis.

The local rank-two response is defined by the metric Euler derivative

$$H^{(L),\text{UV}}_{\mu\nu}[g] = \frac{2}{\sqrt{-g}} \frac{\delta\Gamma^{(L)}_{\text{pole}}}{\delta g^{\mu\nu}}. \tag{7.26}$$

For an infinitesimal diffeomorphism, $\delta_\xi g^{\mu\nu} = -2\nabla^{(\mu}\xi^{\nu)}$, one has

$$\begin{aligned} 0 &= \delta_\xi \Gamma^{(L)}_{\text{pole}} \\ &= -\int \mathrm{d}^4x \sqrt{-g}\, H^{(L),\text{UV}}_{\mu\nu} \nabla^\mu \xi^\nu \\ &= \int \mathrm{d}^4x \sqrt{-g}\, \xi^\nu \nabla^\mu H^{(L),\text{UV}}_{\mu\nu}, \end{aligned} \tag{7.27}$$

up to a boundary term. Since ξ^ν is arbitrary,

$$\nabla^\mu H^{(L),\text{UV}}_{\mu\nu} = 0. \tag{7.28}$$

Thus the object on which the four-sector source rule acts is an explicitly defined, symmetric, conserved covariant rank-two tensor.

The regulated quantum calculation has therefore been reduced to the single tensor $H^{(L),\text{UV}}_{\mu\nu}$. The particle-sector construction can now be applied directly to this covariant source.

7.4 Particle-sector tensor projection

7.4.1 Particle-sector action on a covariant rank-two tensor

The particle transformation introduced in the underlying four-sector theory is

$${\mathcal{G}_{(k)}}^\mu{}_\nu = \mathrm{i}^k \Lambda^\mu{}_\nu, \qquad k \in \mathbb{Z}_4. \tag{7.29}$$

For the discrete sector calculation it is useful to separate the ordinary Lorentz part from the pure quarter-turn factor,

$${\mathcal{G}_{(k)}}^\mu{}_\nu = {\mathcal{Q}_{(k)}}^\mu{}_\rho \Lambda^\rho{}_\nu, \qquad {\mathcal{Q}_{(k)}}^\mu{}_\nu = \mathrm{i}^k \delta^\mu{}_\nu, \qquad {\mathcal{Q}^{-1}_{(k)}}{}^\mu{}_\nu = \mathrm{i}^{-k} \delta^\mu{}_\nu. \tag{7.30}$$

The ordinary Lorentz transformation and the discrete particle class are independent. The sector orbit is therefore obtained by holding the ordinary Lorentz representative fixed and applying only $\mathcal{Q}_{(k)}$.

For any covariant rank-two physical source $X_{\mu\nu}$ the pure sector map acts on both tensor indices,

$$X_{\mu\nu}^{(k)} = \mathcal{Q}_{(k)}{}^{\alpha}{}_{\mu}\mathcal{Q}_{(k)}{}^{\beta}{}_{\nu}X_{\alpha\beta}^{(0)}$$
$$= \mathrm{i}^{2k}X_{\mu\nu}^{(0)} = (-1)^{k}X_{\mu\nu}^{(0)}. \tag{7.31}$$

Equation (7.31) is the tensor representation of the particle-sector quarter turn on a covariant rank-two source and is the sector rule used below.

Applying Eq. (7.31) to the ultraviolet response gives

$$H_{\mu\nu}^{(L),\mathrm{UV}}(k) = (-1)^{k}H_{\mu\nu}^{(L),\mathrm{UV}}(0). \tag{7.32}$$

The remaining point is to verify that the completely contracted scalar loop coefficients carry no particle-sector factor. This follows directly from the indexed kernel contractions.

7.4.2 Indexed contraction check for graviton kernels

The same result can be verified directly at the level of tensor contractions. Let

$$V^{(n)}_{\mu_1\nu_1\cdots\mu_n\nu_n}(x_1,\ldots,x_n) \tag{7.33}$$

denote a graviton kernel with all graviton indices lowered using the background metric. Here $V^{(2)}$ is the gauge-fixed quadratic graviton kernel, while $V^{(n)}$ with $n \geq 3$ denotes the interaction kernels; mixed ghost–graviton kernels are classified by the number of graviton index pairs they carry. Let the graviton propagator be the inverse symmetric rank-four kernel $D^{\mu\nu\rho\sigma}(x,y)$, normalized schematically by

$$\int \mathrm{d}^{D}z\, V^{(2)}_{\mu\nu\alpha\beta}(x,z)D^{\alpha\beta\rho\sigma}(z,y) = \delta^{\rho}_{(\mu}\delta^{\sigma}_{\nu)}\,\delta^{(D)}(x-y). \tag{7.34}$$

The map $\mathcal{Q}_{(k)}$ acts on graviton index pairs. Ghost indices follow the ordinary gauge-fixed contraction rules, with the Faddeev–Popov fields retaining their gauge-auxiliary role.

Under the pure sector map each graviton index pair of an n-point kernel transforms tensorially,

$$V^{(n,k)}_{\mu_1\nu_1\cdots\mu_n\nu_n} = \prod_{j=1}^{n}\left(\mathcal{Q}_{(k)}{}^{\alpha_j}{}_{\mu_j}\mathcal{Q}_{(k)}{}^{\beta_j}{}_{\nu_j}\right)V^{(n,0)}_{\alpha_1\beta_1\cdots\alpha_n\beta_n}$$
$$= (-1)^{kn}V^{(n,0)}_{\mu_1\nu_1\cdots\mu_n\nu_n}. \tag{7.35}$$

The inverse kernel transforms with the inverse map on each of its four contravariant indices,

$$D^{\mu\nu\rho\sigma}_{(k)} = (\mathcal{Q}^{-1}_{(k)})^{\mu}{}_{\alpha}(\mathcal{Q}^{-1}_{(k)})^{\nu}{}_{\beta}(\mathcal{Q}^{-1}_{(k)})^{\rho}{}_{\gamma}(\mathcal{Q}^{-1}_{(k)})^{\sigma}{}_{\delta}D^{\alpha\beta\gamma\delta}_{(0)}$$
$$= \mathrm{i}^{-4k}D^{\mu\nu\rho\sigma}_{(0)} = D^{\mu\nu\rho\sigma}_{(0)}. \tag{7.36}$$

More explicitly, on every internal contraction the sector maps reduce to identities,

$$(\mathcal{Q}^{-1}_{(k)})^{\alpha}{}_{\rho}\mathcal{Q}_{(k)}{}^{\rho}{}_{\beta} = \delta^{\alpha}{}_{\beta}. \tag{7.37}$$

Hence every internal graviton line reduces its sector factors to Kronecker deltas.

For a connected graph let I be the number of internal graviton lines, let V_n count every interaction kernel carrying n graviton index pairs (including mixed ghost–graviton kernels), and let E be the number of external graviton rank-two factors. The purely combinatorial identity

$$\sum_n nV_n = 2I + E \tag{7.38}$$

now serves only as a check of the explicit tensor contraction. Equations (7.35)–(7.37) give

$$\prod_n (-1)^{knV_n} = (-1)^{k(2I+E)} = (-1)^{kE}. \tag{7.39}$$

Thus a completely contracted scalar loop coefficient has $E = 0$ and is sector independent,

$$c^{(L)}_{A,r}(k) = c^{(L)}_{A,r}, \tag{7.40}$$

while a single external covariant rank-two response has $E = 1$ and transforms exactly as Eq. (7.32). The tensor calculation therefore fixes both parts of the sector decomposition: the scalar coefficient is invariant and the external rank-two tensor carries the character $(-1)^k$.

7.4.3 Invariant projection of the complete particle orbit

The pure sector maps satisfy

$$\mathcal{Q}_{(r)}\mathcal{Q}_{(s)} = \mathcal{Q}_{(r+s)}, \qquad r + s \bmod 4. \tag{7.41}$$

Their action on a covariant rank-two tensor defines the group-average operator

$$(\mathcal{P}^{\mathrm{UV}}_2 X)_{\mu\nu} = \frac{1}{4}\sum_{k=0}^{3} \mathcal{Q}_{(k)}{}^{\alpha}{}_{\mu}\mathcal{Q}_{(k)}{}^{\beta}{}_{\nu} X_{\alpha\beta}. \tag{7.42}$$

Using Eq. (7.41),

$$\begin{aligned}(\mathcal{P}^{\mathrm{UV}}_2)^2 X &= \frac{1}{16}\sum_{r,s=0}^{3} \mathcal{Q}^{\otimes 2}_{(r+s)} X \\ &= \frac{1}{4}\sum_{t=0}^{3} \mathcal{Q}^{\otimes 2}_{(t)} X = \mathcal{P}^{\mathrm{UV}}_2 X,\end{aligned} \tag{7.43}$$

so this is the ordinary finite-group projector onto the invariant part of the rank-two representation. Equation (7.31) shows that the rank-two representation carries the one-dimensional character $(-1)^k$. Its invariant subspace is therefore trivial,

$$(\mathcal{P}^{\mathrm{UV}}_2 H^{(L),\mathrm{UV}})_{\mu\nu} = \frac{1}{4}\sum_{k=0}^{3}(-1)^k H^{(L),\mathrm{UV}}_{\mu\nu} = 0. \tag{7.44}$$

The factor $1/4$ is the normalization of the finite-group average over the four particle-sector representatives.

Combining the regulator-level construction with the tensor projection gives the complete order of operations,

$$\Gamma_L \xrightarrow{R'_D} R'_D\Gamma_L \xrightarrow{K} KR'_D\Gamma_L \xrightarrow{2(\sqrt{-g})^{-1}\delta/\delta g^{\mu\nu}} H^{(L),\mathrm{UV}}_{\mu\nu} \xrightarrow{\mathcal{P}^{\mathrm{UV}}_2} 0. \tag{7.45}$$

This establishes the particle-sector cancellation at a fixed active geometry sector. The geometry label can therefore be restored independently, without altering the particle-orbit proof.

7.5 Geometry-sector covariance and homogeneity

The particle projection above is defined at fixed active geometry sector. The independent geometry label is now restored in two steps: first at the level of the regulated background operators, and then through the homogeneity of the local ultraviolet response under the constant metric orientation sign.

7.5.1 Covariance of the regulated background operators

The active geometry sector is a chart representative of the same spacetime geometry. Let I_m be the corresponding active pullback and let $c_m = (-1)^m$. The background and its fluctuation are transported together,

$$\bar{g}^{(m)}_{\mu\nu} = c_m I^*_m \bar{g}_{\mu\nu}, \qquad h^{(m)}_{\mu\nu} = c_m I^*_m h_{\mu\nu}. \tag{7.46}$$

Because c_m is constant, the Levi–Civita connection is carried entirely by the pullback. The trace obeys

$$h^{(m)} = (\bar{g}^{(m)})^{\mu\nu} h^{(m)}_{\mu\nu} = I^*_m h, \tag{7.47}$$

and the gauge condition becomes

$$F^{(m)}_\mu = I^*_m F_\mu. \tag{7.48}$$

The Faddeev–Popov ghost is transported as the gauge vector field of the background-field calculation. Consequently the two quadratic operators are related to their reference representatives by conjugation,

$$\Delta^{(m)}_h = U^{(2)}_m \Delta_h (U^{(2)}_m)^{-1}, \qquad \Delta^{(m)}_{\mathrm{gh}} = U^{(1)}_m \Delta_{\mathrm{gh}} (U^{(1)}_m)^{-1}, \tag{7.49}$$

where $U^{(r)}_m$ denotes the pullback on rank-r fields together with the fixed geometry-sector orientation factor appropriate to the tensor. In particular,

$$\operatorname{Tr}\ln \Delta^{(m)}_h = \operatorname{Tr}\ln \Delta_h, \qquad \operatorname{Tr}\ln \Delta^{(m)}_{\mathrm{gh}} = \operatorname{Tr}\ln \Delta_{\mathrm{gh}}. \tag{7.50}$$

The conjugation establishes the invariance of the one-loop determinants. At higher loop order the same pullback acts tensorially on the interaction kernels and propagators, so the numerical scalar coefficients remain common to the active geometry representatives; the geometry-sector sign is carried by the covariant local operators and their metric responses, as computed next. This is the background-field counterpart of the sector pullback used for free graviton modes in Ref. [42].

7.5.2 Homogeneity of the local ultraviolet response

For the local pole basis, the constant geometry-sector sign acts according to the derivative order of the corresponding covariant operator.

Let

$$g_{\mu\nu} \longrightarrow c\, g_{\mu\nu}, \qquad c = \pm 1. \tag{7.51}$$

For a parity-even local scalar operator built from the metric, curvature and covariant derivatives with total derivative order $2N$, as in the pure-Einstein ultraviolet basis considered here, write

$$\Gamma_N[g] = \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{O}_{2N}[g]. \tag{7.52}$$

Under a constant metric rescaling the connection is unchanged, while each contracted curvature or equivalent pair of derivatives gives one inverse metric weight. Hence

$$\mathcal{O}_{2N}[cg] = c^{-N}\mathcal{O}_{2N}[g], \qquad \Gamma_N[cg] = c^{2-N}\Gamma_N[g]. \tag{7.53}$$

Define the rank-two response

$$H^{(N)}_{\mu\nu}[g] = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma_N}{\delta g^{\mu\nu}}. \tag{7.54}$$

Since $g'^{\mu\nu} = c^{-1} g^{\mu\nu}$ and $\sqrt{-g'} = c^2\sqrt{-g}$ in four dimensions, the functional chain rule applied to the homogeneity relation gives

$$H^{(N)}_{\mu\nu}[cg] = c^{1-N} H^{(N)}_{\mu\nu}[g]. \tag{7.55}$$

For a pure Einstein L-loop logarithmic ultraviolet structure, power counting gives $N = L+1$, so an active geometry sector m contributes

$$H^{(L)}_{\mu\nu}(m) = (-1)^{mL} I^*_m H^{(L)}_{\mu\nu}(0). \tag{7.56}$$

At odd loop order the geometry representatives alternate in sign, while at even loop order they are invariant under the geometry-sector sign. The ultraviolet projection is therefore evaluated at fixed active geometry sector. Combining the independent labels gives

$$H^{(L),\mathrm{UV}}_{\mu\nu}(k,m) = (-1)^k (-1)^{mL} I^*_m H^{(L),\mathrm{UV}}_{\mu\nu}(0,0), \tag{7.57}$$

but the sum over the particle orbit already vanishes for every fixed m,

$$\sum_{k=0}^{3} H^{(L),\mathrm{UV}}_{\mu\nu}(k,m) = 0. \tag{7.58}$$

The particle and geometry labels are independent.

With the particle and geometry transformations now separated explicitly, the two-loop cubic-curvature divergence can be used as a complete benchmark of the construction.

7.6 The two-loop cubic-curvature benchmark

The first genuine on-shell ultraviolet divergence of pure Einstein gravity is the Goroff–Sagnotti term. In the convention used by Ref. [2],

$$\Gamma^{(2)}_{\mathrm{div}} = -\frac{209}{1440}\left(\frac{\kappa_g}{2}\right)^2 \frac{1}{(4\pi)^4}\frac{1}{\epsilon}\int \mathrm{d}^4x \sqrt{-g}\, \mathcal{R}_3, \tag{7.59}$$

where

$$\mathcal{R}_3 = R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta}. \tag{7.60}$$

The numerical coefficient is nonzero. The four-sector operation is applied only after the regulated two-loop pole, including the required lower-loop insertions, has been assembled.

For the tensor variation write

$$\Gamma_3[g] = c_3 \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{R}_3, \tag{7.61}$$

where c_3 contains the complete regulated coefficient. For a scalar Lagrangian depending algebraically on $R_{\mu\nu\rho\sigma}$ define the Riemann derivative at fixed metric,

$$P^{\mu\nu\rho\sigma} \equiv \frac{\partial \mathcal{R}_3}{\partial R_{\mu\nu\rho\sigma}}. \tag{7.62}$$

Because $\mathcal{R}_3$ is the cyclic trace of the cubic Riemann endomorphism on bivectors, differentiation gives three identical terms,

$$P^{\mu\nu\rho\sigma} = 3R^{\mu\nu}{}_{\alpha\beta} R^{\alpha\beta\rho\sigma}. \tag{7.63}$$

The tensor $P^{\mu\nu\rho\sigma}$ inherits the algebraic symmetries of the Riemann tensor,

$$P^{\mu\nu\rho\sigma} = -P^{\nu\mu\rho\sigma} = -P^{\mu\nu\sigma\rho} = P^{\rho\sigma\mu\nu}. \tag{7.64}$$

The complete metric variation is, up to the standard boundary term,

$$\delta\Gamma_3 = c_3 \int \mathrm{d}^4x \sqrt{-g}\, E^{(3)}_{\mu\nu}\, \delta g^{\mu\nu}, \tag{7.65}$$

where the Euler tensor for a curvature Lagrangian is

$$E^{(3)}_{\mu\nu} = P_{\mu}{}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma} - 2\nabla^{\alpha}\nabla^{\beta} P_{\mu\alpha\beta\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}_3. \tag{7.66}$$

Substituting Eq. (7.63) gives the explicit cubic-curvature source,

$$\begin{aligned} E^{(3)}_{\mu\nu} &= 3R_{\mu}{}^{\alpha}{}_{\rho\sigma} R^{\rho\sigma\beta\gamma} R_{\nu\alpha\beta\gamma} \\ &\quad - 6\nabla^{\alpha}\nabla^{\beta}\left(R_{\mu\alpha}{}^{\rho\sigma} R_{\rho\sigma\beta\nu}\right) - \frac{1}{2} g_{\mu\nu} \mathcal{R}_3. \end{aligned} \tag{7.67}$$

The rank-two ultraviolet response generated by Eq. (7.61) is therefore

$$H^{R^3}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta\Gamma_3}{\delta g^{\mu\nu}} = 2c_3 E^{(3)}_{\mu\nu}. \tag{7.68}$$

The tensor is symmetric and its covariant divergence vanishes identically as the Noether identity of the diffeomorphism-invariant functional,

$$E^{(3)}_{\mu\nu} = E^{(3)}_{\nu\mu}, \qquad \nabla^{\mu} E^{(3)}_{\mu\nu} = 0, \qquad \nabla^{\mu} H^{R^3}_{\mu\nu} = 0. \tag{7.69}$$

On a Ricci-flat background, $R_{\mu\nu} = 0$, one may replace every Riemann tensor in Eqs. (7.60)–(7.67) by the Weyl tensor. The pole functional is scalar, whereas its metric Euler derivative is a covariant rank-two tensor.

The geometry-sector homogeneity derived in the previous section also gives a useful check.

The cubic invariant has $N = 3$, and the two-loop order has $L = 2$, so

$$H^{R^3}_{\mu\nu}(m) = (-1)^{2m} I^*_m H^{R^3}_{\mu\nu}(0) = I^*_m H^{R^3}_{\mu\nu}(0). \tag{7.70}$$

Thus the two-loop cubic response is even under the geometry-sector sign, exactly as Eq. (7.56) requires.

Its independent particle-sector transformation and complete-orbit projection are instead

$$\begin{aligned} H^{R^3}_{\mu\nu}(k) &= (-1)^k H^{R^3}_{\mu\nu}(0), \\ (\mathcal{P}^{\mathrm{UV}}_2 H^{R^3})_{\mu\nu} &= 0. \end{aligned} \tag{7.71}$$

The benchmark contains a nonzero ordinary two-loop coefficient, evanescent one-loop contributions in the recursive subtraction, a nonzero local rank-two response before particle-sector projection, and the finite two-loop amplitude as a separate renormalized contribution [1, 2].

7.7 Order-by-order cancellation of local ultraviolet sources

The two-loop calculation is a special case of the same tensor statement at arbitrary perturbative order. From Eq. (7.25),

$$H^{(L),\mathrm{UV}}_{\mu\nu} = \sum_{r=1}^{r_{\max}} \frac{1}{\epsilon^r} \sum_A c^{(L)}_{A,r} H^{(L)}_{A\,\mu\nu}, \tag{7.72}$$

with

$$H^{(L)}_{A\,\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{O}^{(L)}_A. \tag{7.73}$$

Every $c^{(L)}_{A,r}$ is a scalar produced by the completely contracted regulated loop calculation. Every $H^{(L)}_{A\,\mu\nu}$ is a covariant rank-two tensor. Therefore Eqs. (7.40) and (7.31) apply term by term,

$$c^{(L)}_{A,r}(k) = c^{(L)}_{A,r}, \qquad H^{(L)}_{A\,\mu\nu}(k) = (-1)^k H^{(L)}_{A\,\mu\nu}(0). \tag{7.74}$$

Linearity of the finite-group projector then gives

$$\begin{aligned} (\mathcal{P}^{\mathrm{UV}}_2 H^{(L),\mathrm{UV}})_{\mu\nu} &= \sum_{r,A} \frac{c^{(L)}_{A,r}}{\epsilon^r} (\mathcal{P}^{\mathrm{UV}}_2 H^{(L)}_A)_{\mu\nu} \\ &= 0. \end{aligned} \tag{7.75}$$

Equivalently,

$$\boxed{\mathcal{P}^{\mathrm{UV}}_2 \left[\frac{2}{\sqrt{-g}} \frac{\delta K R'_D \Gamma_L}{\delta g^{\mu\nu}} \right] = 0, \qquad L \geq 1.} \tag{7.76}$$

Equation (7.76) is an equality of covariant rank-two tensors at every fixed active geometry sector. It holds separately for every pole order $1/\epsilon^r$ and for every local operator in any equivalent covariant basis.

The geometry label m and particle label k are independent. Equations (7.57) and (7.58) give the combined transformation and the vanishing particle-orbit sum for every fixed geometry sector.

At each loop order, the regulated background-field functional integral and recursive subtraction produce the local covariant pole response on which the four-sector projector acts.

This completes the all-loop rank-two statement. The same tensor can then be placed in the four-index source architecture used by the underlying formulation.

7.8 Embedding in the four-index source equation

The rank-two ultraviolet response can be embedded explicitly in the four-index source structure used in Refs. [40, 43]. This makes the relation between the present cancellation rule and the four-index field equation tensorial rather than implicit.

Let

$$H^{(L),\mathrm{UV}} = g^{\mu\nu} H^{(L),\mathrm{UV}}_{\mu\nu} \tag{7.77}$$

be the trace of the conserved rank-two response. In four dimensions its Ricci-type rank-four lift is

$$\begin{aligned}\widetilde{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu} &= \frac{1}{2}\Big(H^{(L),\mathrm{UV}}_{\alpha\beta} g_{\mu\nu} - H^{(L),\mathrm{UV}}_{\alpha\nu} g_{\beta\mu} + H^{(L),\mathrm{UV}}_{\mu\nu} g_{\alpha\beta} - H^{(L),\mathrm{UV}}_{\beta\mu} g_{\alpha\nu}\Big) \\ &\quad - \frac{1}{6}\left(g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\beta\mu}\right) H^{(L),\mathrm{UV}}.\end{aligned} \tag{7.78}$$

Its contraction is exactly the original source,

$$g^{\alpha\beta} \widetilde{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu} = H^{(L),\mathrm{UV}}_{\mu\nu}. \tag{7.79}$$

Using $\nabla^\alpha H^{(L),\mathrm{UV}}_{\alpha\beta} = 0$ and metric compatibility gives

$$\begin{aligned}\nabla^\alpha \widetilde{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu} &= \frac{1}{2}\left(\nabla_\beta H^{(L),\mathrm{UV}}_{\mu\nu} - \nabla_\nu H^{(L),\mathrm{UV}}_{\beta\mu}\right) \\ &\quad - \frac{1}{6}\left(g_{\mu\nu}\nabla_\beta - g_{\beta\mu}\nabla_\nu\right) H^{(L),\mathrm{UV}}.\end{aligned} \tag{7.80}$$

Introduce the trace-free rank-four completion $\widehat{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu}$, the loop-generated analogue of the trace-free source field in the four-index formulation, with the Weyl algebraic symmetries,

$$\begin{aligned}&\widehat{H}_{\alpha\mu\beta\nu} = -\widehat{H}_{\mu\alpha\beta\nu} = -\widehat{H}_{\alpha\mu\nu\beta}, \\ &\widehat{H}_{\alpha\mu\beta\nu} = \widehat{H}_{\beta\nu\alpha\mu}, \qquad \widehat{H}_{\alpha[\mu\beta\nu]} = 0, \qquad g^{\alpha\beta}\widehat{H}_{\alpha\mu\beta\nu} = 0,\end{aligned} \tag{7.81}$$

where the loop and ultraviolet labels are suppressed only inside Eq. (7.81). Its divergence is fixed by the same four-index propagation law as the trace-free source sector,

$$\begin{aligned}\nabla^\alpha \widehat{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu} &= \frac{1}{2}\left(\nabla_\beta H^{(L),\mathrm{UV}}_{\mu\nu} - \nabla_\nu H^{(L),\mathrm{UV}}_{\beta\mu}\right) \\ &\quad - \frac{1}{6}\left(g_{\mu\nu}\nabla_\beta - g_{\beta\mu}\nabla_\nu\right) H^{(L),\mathrm{UV}}.\end{aligned} \tag{7.82}$$

The complete four-index ultraviolet source is then

$$H^{(4,L),\mathrm{UV}}_{\alpha\mu\beta\nu} = \widetilde{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu} - \widehat{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu}. \tag{7.83}$$

Equations (7.79)–(7.82) immediately give

$$\nabla^\alpha H^{(4,L),\mathrm{UV}}_{\alpha\mu\beta\nu} = 0, \qquad g^{\alpha\beta} H^{(4,L),\mathrm{UV}}_{\alpha\mu\beta\nu} = H^{(L),\mathrm{UV}}_{\mu\nu}. \tag{7.84}$$

Thus the loop-generated ultraviolet response fits the same divergence-free rank-four source architecture as the classical stress–energy source.

At fixed geometry sector the metric is unchanged when only the particle label k is varied. Equation (7.78) is linear in $H^{(L),\mathrm{UV}}_{\mu\nu}$, and the propagation equation (7.82) is linear in the same source. With the same covariant completion condition used in the four-index theory, which fixes the possible divergence-free homogeneous part of the trace-free sector, both rank-four pieces and their difference therefore inherit the same particle character,

$$\begin{aligned}\widetilde{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu}(k) &= (-1)^k \widetilde{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu}(0),\\ \widehat{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu}(k) &= (-1)^k \widehat{H}^{(L),\mathrm{UV}}_{\alpha\mu\beta\nu}(0),\\ H^{(4,L),\mathrm{UV}}_{\alpha\mu\beta\nu}(k) &= (-1)^k H^{(4,L),\mathrm{UV}}_{\alpha\mu\beta\nu}(0).\end{aligned} \tag{7.85}$$

The complete particle orbit therefore also vanishes at the four-index source level,

$$\frac{1}{4}\sum_{k=0}^{3} H^{(4,L),\mathrm{UV}}_{\alpha\mu\beta\nu}(k) = 0. \tag{7.86}$$

The cancellation therefore admits a divergence-free four-index lift built with the same source decomposition, and contraction of that lifted result reproduces the rank-two cancellation. With the local ultraviolet source fixed at both tensor ranks, the effective action can now be separated into its projected pole response and its finite loop contribution.

7.9 Ultraviolet projection and finite loop terms

After the regulated subdivergences have been removed, the effective contribution may be separated schematically as

$$R'_D \Gamma_L = \Gamma^{(L)}_{\mathrm{pole,local}} + \Gamma^{(L)}_{\mathrm{fin},local} + \Gamma^{(L)}_{\mathrm{fin},nonlocal}. \tag{7.87}$$

The four-sector ultraviolet projector acts on the metric response of the local pole term. Therefore

$$(\mathcal{P}_2^{\mathrm{UV}} H^{(L)}_{\mathrm{pole}})_{\mu\nu} = 0, \tag{7.88}$$

The finite part of the effective contribution is carried by

$$\Gamma^{(L)}_{\mathrm{fin}} = \Gamma^{(L)}_{\mathrm{fin},local} + \Gamma^{(L)}_{\mathrm{fin},nonlocal}. \tag{7.89}$$

This decomposition retains the finite remainders of explicit two-loop four-graviton amplitudes [1]. The tree-level Einstein action is the zeroth-order term in Eq. (7.9), while $\mathcal{P}_2^{\mathrm{UV}}$ acts on the loop-generated local pole response.

7.10 Summary

The calculation forms a single chain from perturbative quantization to sector completion. The quantum metric fluctuation and Faddeev–Popov ghosts are integrated in the background-field path integral, dimensional regularization and recursive forest subtraction isolate the local diffeomorphism-covariant pole functional, and metric variation of that functional produces the symmetric conserved rank-two response on which the four-sector source rule acts.

The particle-sector action is fixed by the indexed tensor map. On every internal graviton contraction the sector maps attached to covariant graviton indices cancel against the inverse maps carried by the propagator, while the two free covariant indices of the local response retain the quadratic character $(-1)^k$. The scalar loop coefficients, including ghost and evanescent contributions, are therefore common to the four particle representatives. The independent geometry-sector calculation gives the factor $(-1)^{mL}$, and for every fixed geometry sector the normalized four-element rank-two projector annihilates the local ultraviolet response.

The Goroff–Sagnotti cubic-curvature divergence provides the explicit two-loop benchmark. Its nonzero coefficient, the regulated Gauss–Bonnet subdivergence structure, and the complete conserved metric Euler tensor are retained before the particle-sector projection is applied. The same tensor argument extends term by term to the full local pole basis at arbitrary perturbative loop order, giving Eq. (7.76). The resulting rank-two source also admits the explicit divergence-free four-index lift constructed from its Ricci-type and trace-free parts, and both pieces inherit the same particle-sector character and the same vanishing complete-orbit sum.

The effective action therefore separates naturally into the projected local ultraviolet pole response, the tree-level Einstein contribution, and the finite local and nonlocal loop terms at their respective stages of the calculation. For pure Einstein gravity with vanishing cosmological constant, the four-sector source framework gives an order-by-order cancellation rule for the local ultraviolet gravitational response in both the rank-two and four-index formulations.

Part III

Unified Geometry and Regular Cosmology

CHAPTER 8

The SU(m,n) Unified Field Framework

Source and status. This chapter incorporates the complete scientific body of *SU(m,n) Unified Field Theory*, v0.01 (22 August 2026). Its role in the dissertation is: Pseudo-Kaehler embedding, observer real forms, tensor quantum state and worked sectors. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

This work develops one $SU(m,n)$-symmetric framework governed by four postulates: invariant locally measured light speed; observer-independent laws in inertial and non-inertial frames; coexistence of all physically admissible prepared states with a single observer record; and a fundamentally complex spacetime observed through a real Lorentzian projection. The fundamental arena is a pseudo-Kähler manifold with Hermitian metric and a complex four-index field equation whose trace-free sector is carried by Bochner curvature. An observer is defined by an admissible antiholomorphic involution. Its fixed real form inherits the measured metric, sources and state tensor. The relation between complex and real curvature is derived with realification, the Gauss equation and the real Weyl projector; in particular, Bochner curvature projects to observer Weyl curvature only when explicit extrinsic and Ricci-compatibility corrections vanish. The four-sector construction is then formulated as an orientation completion of the real observer clock–ruler plane, rather than as a second fundamental spacetime. Quantum coexistence is represented by a complex rank-four tensor carrier with full Hermitian conjugation, a hypersurface overlap and covariant rank-eight evolution along the lifted observer clock. The common structure contains flat, Schwarzschild, Kerr, Kerr–Newman, LTB, FLRW, photon, free-graviton, hydrogen and ultraviolet worked sectors. The framework yields the locally invariant null speed, massless helicity-two graviton kinematics, reciprocal regular geometries, the relativistic Coulomb spectrum and the four-sector cancellation of local ultraviolet pole sources.

8.1 Introduction

Special relativity turns the invariant speed of light into a restriction on the relation between inertial frames, and general relativity replaces that restricted covariance by local tensor laws on curved spacetime [13, 14, 63]. Quantum theory, meanwhile, assigns one state to several possible measurement outcomes. The present theory joins four-index gravity, complex curvature, four observer sectors, reciprocal geometry, tensor quantum states, null histories, gravitons and ultraviolet cancellation in one hierarchy of fundamental and observer-dependent objects.

This paper imposes that hierarchy through four physical postulates. Every observer measures the same local vacuum light speed. Physical laws retain their tensorial form for inertial and non-inertial observers. Every physically admissible configuration selected by a preparation coexists in one complete state, while a particular observer obtains one record at one event. Finally, spacetime itself is fundamentally complex and the spacetime measured by an observer is a real Lorentzian projection.

The fourth postulate fixes the order of construction. We begin with a pseudo-Kähler manifold $(\mathcal{M}, J, g, \omega)$ of complex dimension $N = m + n$ and signature (m, n), not with four copies of a real Lorentzian metric. Its general pseudo-unitary special symmetry is $SU(m, n)$. The physical Lorentzian branch is the special case $(m, n) = (1, 3)$, with ambient symmetry $SU(1, 3)$ and the nested four-sector frame actions $SU(1, 3) \times SU(1, 2) \times SU(1, 1)$.

Gravity is written at the same complex level. A Hermitian four-index geometric tensor is equated to a Hermitian four-index source, including a trace-free Bochner-type component. The equation retains curvature information that is invisible after one selected two-index contraction. The four-index equation is therefore retained as the fundamental field equation.

The central bridge of the theory is the observer projection. An observer real form is the fixed set of an admissible antiholomorphic involutive isometry, with embedding ι. The measured Lorentzian metric is the real symmetric pullback of the Hermitian metric. Curvature is not projected by merely erasing bars on indices: it is realified, evaluated on embedded tangent vectors and related to intrinsic observer curvature by the Gauss equation. Consequently the real Weyl tensor satisfies $C_{\mu\nu\rho\sigma} = (\mathsf{W}_g[\Re B])_{\mu\nu\rho\sigma} + (\Delta_{\mathrm{Ric}})_{\mu\nu\rho\sigma} + (\Delta_K)_{\mu\nu\rho\sigma}$. Only a field-compatible, totally geodesic real form reduces this to the shorter Bochner-to-Weyl relation.

The four-sector construction is introduced only after this bridge. It completes the orientations of a real clock–ruler plane around one common null set and separates a geometry-sector label from a particle-sector label. This supplies a concrete realization of invariant local c without interpreting a coordinate speed as a local measurement or allowing a massive particle to cross its null boundary dynamically. Reciprocal proper-scale maps then generate the regular continuations of the curved solutions analysed in this theory.

The third postulate is implemented by a complex tensor carrier rather than by adding mutually exclusive classical stress tensors. Each configuration solves its own field equation, and the carrier is the weighted complete state of the complex rank-four sources. Both coefficients and tensor bases are conjugated in the Hermitian overlap, and evolution is generated by a rank-eight operator along the lifted physical clock rather than a preferred coordinate derivative.

The theory uses one common convention throughout. The one-carrier phase $\exp(-\mathrm{i}E\tau/\hbar)$ is used consistently, affine rescalings of one null tangent are identified as descriptions of the same null history, and the ultraviolet sector projector acts on the complete local pole source.

The paper follows the logical sequence postulates, mathematical structure, predictions and solutions. Sections 8.2–8.8 define the fundamental complex theory, its real observer projection, the four-sector atlas and the tensor state. Section 8.9 classifies consequences by status. The subsequent worked sections collect the flat, black-hole, cosmological, null, graviton, hydrogen and ultraviolet examples.

Part IV

Fundamental postulates

8.2 Four physical postulates

Postulate 8.1 (Invariant speed of light)**.** *Every observer measures the speed of light to be constant, independently of the observer's state of motion.*

The postulate concerns a local measurement made with the observer's physical clock and rulers. It asserts one common null structure, including for an accelerated, freely falling or sector-oriented observer. It does not assert that an arbitrary coordinate quotient $\mathrm{d}x/\mathrm{d}t$ must equal c, and it does not allow a massive observer to be accelerated dynamically through a null boundary.

Postulate 8.2 (Observer-independent physical laws)**.** *The laws of physics are the same in every reference frame, whether inertial or non-inertial.*

The fundamental equations must therefore be covariant. Tensor components, the local clock and the connection coefficients may differ between observers; the law does not. Acceleration and rotation remain measurable physical data and enter through the observer congruence and its transported frame.

Postulate 8.3 (Coexistence of physical states)**.** *All physically admissible states of a system coexist simultaneously, while an observer measures one of them at any given moment.*

To make this statement compatible with Postulate 2, "simultaneously" does not mean equality of one preferred global coordinate time. It means that all configurations compatible with a preparation belong to one complete physical state. "At a given moment" refers to one event on the worldline of a particular observer. Each classical configuration remains self-consistent; coexistence is implemented in the quantum carrier and not by an unweighted sum of mutually exclusive classical stress tensors.

Postulate 8.4 (Complex spacetime)**.** *Spacetime is fundamentally complex, while every observer measures its real Lorentzian projection.*

The fundamental metric, curvature, sources and quantum carrier live on a pseudo-Hermitian complex geometry. A physical observer is associated with an admissible real form of that geometry. The four-sector construction is introduced only after this real observer projection has been defined.

Part V

Mathematical structure

8.3 Conventions and tensor hierarchy

The fundamental manifold $\mathcal{M}$ has complex dimension $N = m + n$ and pseudo-Hermitian signature (m, n). The physical case used throughout is $(m, n) = (1, 3)$, hence $N = 4$, so the ambient realification has dimension eight. An observer projection $\mathcal{N}$ has real dimension $d = 4$. Dimensional regularization is denoted separately by $D = 4 - 2\epsilon$ and occurs only in the ultraviolet calculation.

Holomorphic indices are $\alpha, \gamma, \kappa, \eta = 0, \ldots, N - 1$, while antiholomorphic indices are barred. Real Lorentzian spacetime indices are $\mu, \nu, \rho, \sigma = 0, \ldots, 3$, and orthonormal-frame indices are A, B. The labels $N_g, S_X \in \mathbb{Z}_4$ denote the geometry and particle sectors and are never summed. A label $q \in \mathcal{Q}$ enumerates complete physical configurations and is not a tensor index.

The fundamental pseudo-Hermitian metric obeys

$$g_{\alpha\bar{\beta}} = \overline{g_{\beta\bar{\alpha}}}, \qquad \nabla g = 0, \qquad \mathrm{sig}(g) = (m, n). \tag{8.1}$$

The observed real metric uses signature $(-+++)$ in even sectors; the neighbouring odd-sector representative has the opposite overall sign but the same Lorentzian null set.

Three different complex structures must not be conflated:

1. the holomorphic coordinates and pseudo-Hermitian geometry of $\mathcal{M}$;
2. the complex coefficients of the quantum carrier;
3. the four bookkeeping phases $1, \mathrm{i}, -1, -\mathrm{i}$ used by the real four-sector observer atlas.

8.4 Fundamental complex spacetime

8.4.1 Pseudo-Kähler geometry

The fundamental geometric object is

$$(\mathcal{M}, J, g_{\alpha\bar{\beta}}, \omega), \qquad \omega = \mathrm{i} g_{\alpha\bar{\beta}} \mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{z}^{\bar{\beta}}, \qquad \mathrm{d}\omega = 0. \tag{8.2}$$

The invariant interval may be written

$$\mathrm{d}s^2 = g_{\alpha\bar{\beta}}\, \mathrm{d}z^{\alpha} \mathrm{d}\bar{z}^{\bar{\beta}}. \tag{8.3}$$

The metric is pseudo-Hermitian rather than positive definite, so an admissible real form can inherit Lorentzian signature.

In flat complex spacetime,

$$\eta_{\alpha\bar{\beta}} = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{m}, \underbrace{+1, \ldots, +1}_{n}), \qquad \mathrm{d}s^2 = \eta_{\alpha\bar{\beta}} \mathrm{d}z^{\alpha} \mathrm{d}\bar{z}^{\bar{\beta}}. \tag{8.4}$$

The pseudo-unitary special group preserving this form is $SU(m, n)$. The previously used $SU(1, n)$ family is its one-time-direction special case.

8.4.2 $SU(m,n)$ symmetry and the physical nested frame

The general fundamental symmetry is $SU(m,n)$, where $N = m+n$. The Lorentzian four-dimensional case has $(m,n) = (1,3)$. Its full complex tangent frame is preserved by $SU(1,3)$. The nested clock–ruler modules of the four-sector description carry the additional independent lower-frame actions. They form the independently acting product

$$\boxed{SU(1,3) \times SU(1,2) \times SU(1,1)} \tag{8.5}$$

on the three corresponding pseudo-unitary frame modules. Thus $SU(m,n)$ is the general ambient symmetry, while Eq. (8.5) is its physical $(1,3)$ nested realization.

8.5 Complex four-index field equations

8.5.1 Curvature and Bochner decomposition

The nonzero Kähler curvature components are written $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$. Their decomposition is

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\delta}} = B_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{N+2}\Big(&R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} \\ &+R_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + R_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\Big) \\ &- \frac{1}{(N+1)(N+2)}\Big(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\Big)R. \end{aligned} \tag{8.6}$$

The tensor $B_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is the trace-free Bochner curvature. It is the complex Kähler counterpart of the information carried by the real Weyl tensor, but the two tensors are not identified before the real projection in Sec. 8.6.

8.5.2 Complex four-index Einstein tensor

The complex four-index geometric tensor is

$$\boxed{G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-N}\Big[R_{\alpha\bar{\beta}\gamma\bar{\delta}} - R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} - R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}R\Big].} \tag{8.7}$$

Its first natural contraction gives

$$g^{\alpha\bar{\beta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\delta}} - g_{\gamma\bar{\delta}}R. \tag{8.8}$$

The crossed contractions can differ by fixed dimension-dependent factors; the uncontracted equation is fundamental and is not replaced by one selected two-index contraction.

8.5.3 Complex four-index source

Let $T_{\alpha\bar{\beta}}$ be the Hermitian rank-two source, let $T = g^{\alpha\bar{\beta}}T_{\alpha\bar{\beta}}$, and let $\widehat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ have the algebraic symmetries and traces of the Bochner sector. The source used in the complex four-index equation is

$$T_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-N}\Bigg[\widehat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{N+1}{N+2}\Big(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}\Big)$$
$$+ \frac{1}{N+2}\Big(T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\Big)$$
$$+ \frac{1}{(1-N)(N+1)(N+2)}\Big(-(N^2+N+1)g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}$$
$$+ (2N+1)g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\Big)T\Bigg]. \tag{8.9}$$

The fundamental field equation is

$$\boxed{G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa T_{\alpha\bar{\beta}\gamma\bar{\delta}}.} \tag{8.10}$$

The trace-free part gives the complex Bochner-source relation

$$B_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa\widehat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} \tag{8.11}$$

with the normalization fixed by Eqs. (8.7) and (8.9). The complex Bianchi identity and source equation are required to give

$$\nabla^\alpha G_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0, \qquad \nabla^\alpha T_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0. \tag{8.12}$$

The Hermitian reality conditions are

$$\begin{aligned} \overline{T_{\alpha\bar{\beta}}} &= T_{\beta\bar{\alpha}}, \\ \overline{T_{\alpha\bar{\beta}\gamma\bar{\delta}}} &= T_{\beta\bar{\alpha}\delta\bar{\gamma}}, \\ \overline{G_{\alpha\bar{\beta}\gamma\bar{\delta}}} &= G_{\beta\bar{\alpha}\delta\bar{\gamma}}. \end{aligned} \tag{8.13}$$

8.6 Real Lorentzian observer projection

8.6.1 Observer real form

An observer does not obtain a real spacetime by deleting the imaginary parts of coordinates component by component. The projection is defined geometrically. Let

$$\sigma : \mathcal{M} \to \mathcal{M} \tag{8.14}$$

be an antiholomorphic involutive isometry,

$$\sigma^2 = 1, \qquad \sigma_* J = -J\sigma_*, \qquad \sigma^* g = g. \tag{8.15}$$

Its fixed set is the observer real form

$$\mathcal{N} = \mathrm{Fix}(\sigma), \qquad \iota : \mathcal{N} \hookrightarrow \mathcal{M}. \tag{8.16}$$

For a regular middle-dimensional fixed component, the anti-holomorphic isometry makes the real form Lagrangian and the isometry makes it totally geodesic in the standard real-form construction [5,35]. In the pseudo-Kähler setting used here these properties are imposed as admissibility conditions together with Lorentzian signature of the induced metric.

Let

$$E^{\alpha}{}_{\mu} = \frac{\partial z^{\alpha}}{\partial x^{\mu}} \tag{8.17}$$

be the holomorphic part of the tangent map. The observed metric is the real symmetric pullback

$$\boxed{g_{\mu\nu} = \mathrm{Re}\Big(g_{\alpha\bar{\beta}} E^{\alpha}{}_{\mu} \overline{E^{\beta}{}_{\nu}}\Big)\,, \qquad \mathrm{sig}(g) = (-+++).} \tag{8.18}$$

An overall conventional factor of two may be absorbed into the definition of the realification of the Hermitian metric.

The real observer map is therefore

$$\boxed{\mathcal{M} \xrightarrow{(\sigma,\iota)} (\mathcal{N}, g_{\mu\nu}).} \tag{8.19}$$

8.6.2 Curvature realification and the Gauss equation

For a complex Kähler curvature-type tensor X, let $\Re X$ denote its realification followed by evaluation on four embedded real tangent vectors. Abstractly,

$$(\Re X)_{\mu\nu\rho\sigma} := X(\mathrm{d}\iota\, e_{\mu}, \mathrm{d}\iota\, e_{\nu}, \mathrm{d}\iota\, e_{\rho}, \mathrm{d}\iota\, e_{\sigma}), \tag{8.20}$$

where on the right-hand side X denotes the unique real multilinear tensor obtained from the mixed holomorphic components and their complex conjugates. This definition retains the antisymmetries of a real curvature tensor; a raw replacement of barred indices by unbarred ones would not.

Let $K^{I}{}_{\mu\nu}$ be the second fundamental form of the observer real form, with I a normal-bundle index. The semi-Riemannian Gauss equation is

$$R_{\mu\nu\rho\sigma} = (\Re R_{\alpha\bar{\beta}\gamma\bar{\delta}})_{\mu\nu\rho\sigma} + \eta_{IJ}\left(K^{I}{}_{\mu\rho}K^{J}{}_{\nu\sigma} - K^{I}{}_{\mu\sigma}K^{J}{}_{\nu\rho}\right). \tag{8.21}$$

For the fixed set of an isometry, $K^{I}{}_{\mu\nu} = 0$; nevertheless the full formula is kept because it shows exactly what changes for a more general observer section.

8.6.3 The Bochner-to-Weyl bridge

For any real algebraic curvature tensor $X_{\mu\nu\rho\sigma}$ in dimension $d > 3$, define its Weyl projector by

$$\begin{aligned}(\mathsf{W}_g X)_{\mu\nu\rho\sigma} = X_{\mu\nu\rho\sigma} &- \frac{2}{d-2}\left(g_{\mu[\rho}X_{\sigma]\nu} - g_{\nu[\rho}X_{\sigma]\mu}\right)\\ &+ \frac{2X}{(d-1)(d-2)} g_{\mu[\rho}g_{\sigma]\nu},\end{aligned} \tag{8.22}$$

where $X_{\mu\nu} = g^{\rho\sigma} X_{\rho\mu\sigma\nu}$ and $X = g^{\mu\nu} X_{\mu\nu}$. Applying this projector to Eq. (8.21) and using the complex Bochner decomposition gives the exact bridge

$$\boxed{C_{\mu\nu\rho\sigma} = (\mathsf{W}_g[\mathfrak{R}B])_{\mu\nu\rho\sigma} + (\Delta_{\mathrm{Ric}})_{\mu\nu\rho\sigma} + (\Delta_K)_{\mu\nu\rho\sigma}.} \tag{8.23}$$

where

$$\begin{aligned}
(\Delta_{\mathrm{Ric}})_{\mu\nu\rho\sigma} &:= \left(\mathsf{W}_g[\mathfrak{R}(R^{(\mathrm{Ric})}_{\alpha\bar\beta\gamma\bar\delta})]\right)_{\mu\nu\rho\sigma}, \\
(\Delta_K)_{\mu\nu\rho\sigma} &:= \mathsf{W}_g\left[\eta_{IJ}(K^I{}_{\mu\rho}K^J{}_{\nu\sigma} - K^I{}_{\mu\sigma}K^J{}_{\nu\rho})\right]_{\mu\nu\rho\sigma}.
\end{aligned} \tag{8.24}$$

Consistency condition 8.1 (Projection compatibility). *An observer real form is field-compatible when it is totally geodesic, $K = 0$, and the realification of the complex Ricci-built sector is a real Ricci-built algebraic curvature tensor. Equivalently, $\Delta_K = \Delta_{\mathrm{Ric}} = 0$.*

On a field-compatible observer form,

$$\boxed{C_{\mu\nu\rho\sigma} = \mathsf{W}_g[\mathfrak{R}B]_{\mu\nu\rho\sigma}.} \tag{8.25}$$

Thus $B \to C$ is not assumed. It is the composition of complex realification, pullback to the observer form and the real Weyl projector. If the observer section is not field-compatible, Eq. (8.23) predicts explicit intrinsic corrections.

8.6.4 Projection of sources and field equations

The observed rank-two source is

$$T_{\mu\nu} = \mathrm{Re}\Big(T_{\alpha\bar\beta} E^\alpha{}_\mu \overline{E^\beta{}_\nu}\Big). \tag{8.26}$$

The observed rank-four source is obtained by the same realification as a curvature-type tensor, followed by the real Ricci/trace-free decomposition:

$$T_{\mu\nu\rho\sigma} = \mathsf{P}^{(4)}[T_{\alpha\bar\beta\gamma\bar\delta}]_{\mu\nu\rho\sigma}. \tag{8.27}$$

Here $\mathsf{P}^{(4)}$ is not merely component restriction; it includes $\mathfrak{R}$, the Hermitian reality condition and the algebraic projection to real Riemann symmetries.

For a field-compatible real form the following diagram commutes:

$$\begin{array}{ccc}
G_{\alpha\bar\beta\gamma\bar\delta} & = & \kappa T_{\alpha\bar\beta\gamma\bar\delta} \\
\big\downarrow \mathsf{P}^{(4)} & & \big\downarrow \mathsf{P}^{(4)} \\
G_{\mu\nu\rho\sigma} & = & \kappa T_{\mu\nu\rho\sigma}.
\end{array} \tag{8.28}$$

The lower equation is the real four-index formulation, whose contraction gives the observed two-index Einstein equation. If the projection is not field-compatible, the two correction tensors in Eq. (8.23) enter the lower equation as geometric projection sources and must not be hidden inside $\widehat{T}$. For a connection-compatible real form the projection also commutes with the covariant derivative. A general embedding instead contributes the Codazzi and extrinsic-curvature terms already encoded by the full bridge.

The fundamental complex theory is therefore not reduced to the real Einstein equation. The real equation is one observer projection of the complex four-index equation.

8.7 Four-sector geometry of the real observer projection

8.7.1 Four orientations of one clock–ruler plane

Let ϑ^A be an orthonormal coframe of the real observer projection. The two null lines in a selected clock–ruler plane divide the non-null orientations into four connected domains. Their centres form the cycle

$$+t \longrightarrow +n \longrightarrow -t \longrightarrow -n \longrightarrow +t. \tag{8.29}$$

The geometry sector $N_g \in \mathbb{Z}_4$ is represented by

$$\boxed{\vartheta^A_{(N_g)} = \mathrm{i}^{N_g} \mathcal{I}^*_{N_g} \vartheta^A, \qquad g^{(N_g)}_{\mu\nu} = (-1)^{N_g} \mathcal{I}^*_{N_g} g_{\mu\nu}.} \tag{8.30}$$

The coframe distinguishes all four orientations, whereas the metric retains only the even–odd sign. Multiplication by a nonzero constant preserves the null set and the Levi–Civita connection inside a fixed chart. The phases in Eq. (8.30) belong to the atlas of the real projection; they are not a replacement for the fundamental complex coordinates of Sec. 8.4.

A particle X has an independent relative sector $S_X \in \mathbb{Z}_4$ and an ordinary within-sector Lorentz transformation ${\Lambda_X}^A{}_B$:

$$\mathrm{d}X^A = \mathrm{i}^{S_X} {\Lambda_X}^A{}_B \vartheta^B_{(N_g)}, \qquad N_X = N_g + S_X \pmod 4. \tag{8.31}$$

A geometry-chart transition changes N_g and the total readout N_X but leaves S_X and the particle's within-sector motion unchanged. Ordinary acceleration changes Λ_X but not S_X. Geometry orientation, particle orientation and motion are therefore distinct data.

8.7.2 Invariant local light speed

Let u^μ be the unit clock of any observer in the active real metric and let k^μ be null. The local decomposition is

$$k^\mu = \frac{\omega_u}{c}(u^\mu + s^\mu), \qquad u_\mu s^\mu = 0, \qquad s_\mu s^\mu = 1, \qquad \omega_u = -c\, u_\mu k^\mu. \tag{8.32}$$

The spatial and temporal magnitudes measured by that observer are equal, so the measured speed is c. This calculation is local and does not require the observer worldline to be geodesic. Non-inertial motion changes the transport of successive frames, not the null relation at one event.

Every massive within-sector Lorentz motion has $|\beta_X| < 1$. A side observer uses one clock direction that the reference observer calls spatial, but it has an ordinary three-dimensional rest space of its own. A formally superluminal slope obtained by dividing by the wrong sector clock is not a locally measured speed.

8.7.3 Proper-scale tensor and reciprocal continuation

Let u^μ be the active geometry clock and $h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu$ its rest projector. A Jacobi map $\mathcal{D}^\mu{}_\nu$ between neighbouring worldlines defines the positive Cauchy–Green tensor and its positive square root,

$$C^\mu{}_\nu = h^{\mu\rho} h_{\lambda\kappa} \mathcal{D}^\lambda{}_\rho \mathcal{D}^\kappa{}_\nu, \qquad L^\mu{}_\rho L^\rho{}_\nu = C^\mu{}_\nu. \tag{8.33}$$

The self-adjoint spatial tensor $L^\mu{}_\nu$ contains the invariant proper scales. A given projected GR solution supplies a positive critical-scale tensor $G^\mu{}_\nu$. Active directions solve

$$G^\mu{}_\nu n^\nu_{(i)} = q_i L^\mu{}_\nu n^\nu_{(i)}. \tag{8.34}$$

The active chart has $0 \le q_i < 1$, and $q_i = 1$ is its common null boundary. On the rest bundle, define

$$\boxed{\widetilde{L}^\mu{}_\nu = G^\mu{}_\rho (L_\perp^{-1})^\rho{}_\sigma G^\sigma{}_\nu.} \tag{8.35}$$

On a common eigendirection,

$$\widetilde{L}_i = \frac{G_i^2}{L_i}, \qquad \widetilde{q}_i = q_i^{-1}. \tag{8.36}$$

The reciprocal map is involutive and fixes the boundary. The active metric is the full signed pullback of the already projected real solution,

$$g^{(N_g)} = (-1)^{N_g} \mathcal{I}^*_{N_g} \bar{g}[L]. \tag{8.37}$$

All Jacobian-generated mixed terms are retained. Equation (8.35), rather than the constant phase alone, is the geometric content of the reciprocal continuation.

Curvature regularity, timelike proper time and null affine parameter are three independent tests. A reciprocal endpoint with vanishing curvature can still lie at infinite causal parameter and then represents an asymptotic boundary, not a finite passage into the next algebraic sector.

8.7.4 Projected tensor transport

Rank-two physical sources are ordinary chart pullbacks. All-lower real rank-four curvature and source tensors acquire the orientation sign produced when an index is lowered with the signed metric:

$$\begin{aligned}
T^{(N_g)}_{\mu\nu} &= \mathcal{I}^*_{N_g} T_{\mu\nu}, \\
C^{(N_g)}_{\mu\nu\rho\sigma} &= (-1)^{N_g} \mathcal{I}^*_{N_g} C_{\mu\nu\rho\sigma}, \\
T^{(N_g)}_{\mu\nu\rho\sigma} &= (-1)^{N_g} \mathcal{I}^*_{N_g} T_{\mu\nu\rho\sigma}.
\end{aligned} \tag{8.38}$$

Scalar contractions and physical spectral values are unchanged when the metric, tensor and integration measure are transported together.

8.8 Fundamental complex quantum state

8.8.1 Complete classical configurations

Let $q \in \mathcal{Q}_{\text{prep}}$ label one complete complex configuration compatible with a preparation. Its source components are assembled first and then lifted to the complex four-index source $T^{(q)}_{\alpha\bar{\beta}\gamma\bar{\delta}}$. Each such configuration separately obeys

$$G^{(q)} = \kappa T^{(q)}, \qquad \nabla^\alpha T^{(q)}_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0. \tag{8.39}$$

Components introduced to close the conservation law belong simultaneously to one configuration and never receive independent state coefficients.

If different configurations carry different self-consistent geometries, their state space is the direct sum

$$\mathfrak{H}_{\text{prep}} = \bigoplus_{q \in \mathcal{Q}_{\text{prep}}} \mathfrak{H}_q. \tag{8.40}$$

This prevents a pointwise addition of tensors on different manifolds. A local sum is used only after a common complex background or an explicit relational identification has been supplied.

8.8.2 Complex rank-four carrier

On a common fundamental complex geometry the complete carrier is

$$\boxed{\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}(z,\bar{z}) = \ell_{\mathrm{P}}^{(5-d)/2}\kappa \sum_{q \in \mathcal{Q}_{\text{prep}}} c_q T^{(q)}_{\alpha\bar{\beta}\gamma\bar{\delta}}(z,\bar{z}), \qquad c_q \in \mathbb{C}.} \tag{8.41}$$

The projected hypersurface has real spacetime dimension d. Since G and κT have dimension L^{-2}, for $d = 4$ the carrier has dimension $L^{-3/2}$, and its Hermitian hypersurface overlap is dimensionless. In the general branch case, Eq. (8.41) is understood as the direct-sum section $\boldsymbol{\Psi} = \bigoplus_q c_q \Psi_q$.

Postulate 3 is realized by the support of this carrier: every configuration with nonzero preparation coefficient coexists in the complete state. The classical equation remains branchwise and is not sourced by the raw sum on the right-hand side of Eq. (8.41).

8.8.3 Full Hermitian conjugation

Complex conjugation acts on both the coefficients and the tensor basis:

$$\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}} \longleftrightarrow \overline{\Psi}_{\bar{\alpha}\beta\bar{\gamma}\delta}. \tag{8.42}$$

Using the pseudo-Hermitian metric, define

$$\overline{\Phi}^{\alpha\bar{\beta}\gamma\bar{\delta}} = g^{\alpha\bar{\kappa}} g^{\lambda\bar{\beta}} g^{\gamma\bar{\eta}} g^{\theta\bar{\delta}} \overline{\Phi}_{\bar{\kappa}\lambda\bar{\eta}\theta}. \tag{8.43}$$

For a projected observer hypersurface Σ, the ordered Hermitian overlap is

$$\boxed{\langle \Phi, \Psi \rangle_\Sigma = \int_\Sigma \sqrt{h}\, \overline{\Phi}^{\alpha\bar{\beta}\gamma\bar{\delta}} \Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}\, \mathrm{d}^{d-1}x.} \tag{8.44}$$

The embedding tensors that identify the observer slice are understood in the integrand. Equivalently, the complete tensors may first be projected with $\mathsf{P}^{(4)}$ and then contracted with the observed real metric.

Normalization and a resolved comparison channel are

$$\langle \Psi, \Psi \rangle = 1, \qquad P(\Phi|\Psi) = \frac{|\langle \Phi, \Psi \rangle|^2}{\sum_A |\langle \Phi_A, \Psi \rangle|^2}. \tag{8.45}$$

The sum runs over a complete mutually exclusive set of observer channels. The probability remains a global full-tensor contraction. No independent scalar wavefunction and no second measurement dynamics are introduced.

The physical channel basis is normalized with the Hermitian or biorthogonal tensor pairing. The type-N graviton dual basis below gives its explicit radiative realization.

8.8.4 Covariant rank-eight evolution

Let U^α be the holomorphic part of the lifted observer clock, with real tangent

$$U = U^\alpha \partial_\alpha + \overline{U}^{\bar\alpha} \partial_{\bar\alpha}, \qquad g_{\alpha\bar\beta} U^\alpha \overline{U}^{\bar\beta} = -1. \tag{8.46}$$

Let $\mathscr{D}_U$ be the covariant derivative of the full tensor along U, including the observer-frame connection. The fundamental evolution is

$$\boxed{\mathrm{i}\ell_{\mathrm{P}}\, \mathscr{D}_U \Psi_{\alpha\bar\beta\gamma\bar\delta} = H_{\alpha\bar\beta\gamma\bar\delta}{}^{\kappa\bar\lambda\eta\bar\theta} \Psi_{\kappa\bar\lambda\eta\bar\theta}.} \tag{8.47}$$

The operator remains rank eight. It is Hermitian or bi-Hermitian on the physical state space and transforms by conjugation under a complex frame map or real sector pullback,

$$\Psi' = U\Psi, \qquad H' = UHU^{-1}. \tag{8.48}$$

Equation (8.47) therefore does not privilege a coordinate x^0. In an inertial adapted frame it reduces to the earlier $\mathrm{i}\ell_{\mathrm{P}}\nabla_0\Psi = H\Psi$.

For a stationary eigenmode, the unified one-carrier convention is

$$H\Psi_q = \frac{E_q}{E_{\mathrm{P}}}\Psi_q, \qquad c_q(\tau) = c_q(0)e^{-\mathrm{i}E_q\tau/\hbar}, \qquad E_{\mathrm{P}}\ell_{\mathrm{P}} = \hbar c. \tag{8.49}$$

This convention is required by the Hermitian one-carrier overlap and by the free-graviton relation $E = \hbar\omega$. It fixes the phase of the carrier; quadratic observables are constructed from that phase without redefining the one-particle energy.

8.8.5 Projection of the quantum state

The tensor measured on an observer real form is

$$\boxed{\Psi_{\mu\nu\rho\sigma} = \mathsf{P}^{(4)}[\Psi_{\alpha\bar\beta\gamma\bar\delta}]_{\mu\nu\rho\sigma}.} \tag{8.50}$$

The projection acts on the full complex tensor, not only on its coefficients. The sector representatives are subsequently

$$\Psi^{(N_g)}_{\mu\nu\rho\sigma} = (-1)^{N_g}\mathcal{I}^*_{N_g}\Psi_{\mu\nu\rho\sigma}. \tag{8.51}$$

A consistent projection must preserve the physical overlap, spectral value and probability:

$$\langle\Phi_{\alpha\bar\beta\gamma\bar\delta}, \Psi_{\alpha\bar\beta\gamma\bar\delta}\rangle = \langle\Phi_{\mu\nu\rho\sigma}, \Psi_{\mu\nu\rho\sigma}\rangle, \qquad E'_q = E_q, \qquad P' = P. \tag{8.52}$$

Equation (8.52) defines an admissible observer projection of the physical state space.

Part VI

Physical predictions and mathematical consequences

8.9 Consequences of the theory

Derived result 8.1 (Universal locally measured c)**.** *Equation (8.32) implies that every admissible local observer measures the same vacuum light speed c. This includes inertial, accelerated, freely falling and sector-oriented observers.*

Derived result 8.2 (Four observer sectors)**.** *Completing the orientation of one real clock–ruler plane around its common null lines gives four non-null connected domains and the $\mathbb{Z}_4$ cycle in Eq. (8.29). This is a mathematical consequence of the chosen orientation completion of the real projection.*

Physical prediction 8.1 (No ordinary massive FTL crossing)**.** *A massive particle remains within the Lorentz cone of its fixed relative sector. Acceleration changes Λ_X, not S_X, and cannot convert an ordinary particle into a side particle by passage through c.*

Derived result 8.3 (Complex complete quantum state)**.** *Postulates 3 and 4 together require the complete carrier to combine complex four-index configurations on the fundamental pseudo-Hermitian geometry, as in Eq. (8.41). Complex coefficients on an otherwise purely real tensor basis are only the observer-projected special case.*

Physical prediction 8.2 (Single observer readout)**.** *At one event an observer records one resolved channel Φ_A with the conditional probability in Eq. (8.45), although the complete state retains all nonzero prepared configurations.*

Derived result 8.4 (Null-state limit)**.** *For a selected null history, metric proper time is identically zero. The events therefore possess no nontrivial ordering by the photon's own proper time, although timelike observers order them with their clocks. Affine rescalings of the same null ray are gauge-equivalent and do not create extra physical states.*

Derived result 8.5 (Free graviton sector)**.** *In a Ricci-flat radiative projection the complete source reduces to its trace-free curvature part. The vacuum Bianchi equations, transverse screen representation and rank-eight spectrum give*

$$m_G = 0, \qquad \lambda_G = \pm 2, \qquad E_G = \hbar\omega. \tag{8.53}$$

This is the free radiative spectrum of the projected four-index vacuum sector.

Physical prediction 8.3 (Reciprocal regularity)**.** *For the explicit Schwarzschild, Kerr, Kerr–Newman, homogeneous LTB and regular-fluid FLRW branches analysed below, the reciprocal active scale grows while the relevant curvature invariants decay. Their limiting reciprocal centres are asymptotic in the timelike and null directions that have been checked.*

Derived result 8.6 (Local ultraviolet sector cancellation)**.** *The complete equal-weight particle-sector orbit annihilates a local covariant rank-two UV pole response with sector character $(-1)^{S_X}$, as shown by the complete-orbit projector in Eq. (8.103).*

Part VII

Exact and worked solutions

8.10 Flat complex spacetime

Worked solution 8.1 (Pseudo-Hermitian vacuum). *For $N = 4$ with $(m, n) = (1, 3)$,*

$$\mathrm{d}s^2 = -\mathrm{d}z^0\mathrm{d}\bar{z}^{\bar{0}} + \mathrm{d}z^1\mathrm{d}\bar{z}^{\bar{1}} + \mathrm{d}z^2\mathrm{d}\bar{z}^{\bar{2}} + \mathrm{d}z^3\mathrm{d}\bar{z}^{\bar{3}}. \tag{8.54}$$

The full tangent frame is preserved by $SU(1,3)$. The nested observer modules carry the additional independent actions of $SU(1,2)$ and $SU(1,1)$, giving the nested product in Eq. (8.5). For the canonical real structure $\sigma(z) = \bar{z}$, the fixed form $z^\alpha = x^\alpha$ has

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2. \tag{8.55}$$

The embedding is totally geodesic, all curvatures vanish, and the complex-to-real field diagram commutes trivially.

8.11 Flat four-sector observer spacetime

Worked solution 8.2 (Sector cycle). *Starting from Eq. (8.55), the sector representatives are*

$$\mathrm{d}s^2_{(N_g)} = (-1)^{N_g}\mathrm{d}s^2_{(0)}, \qquad N_g = 0, 1, 2, 3. \tag{8.56}$$

The four clock centres are $+t, +n, -t, -n$. Each sector has one real clock, three rulers and the same null equation. The continuous motion inside one sector remains an ordinary Lorentz motion with $|\beta| < 1$. The full orientation resides in the coframe because the metric cannot distinguish sectors separated by a half-turn.

8.12 Reciprocal Schwarzschild solution

The ordinary projected metric is

$$\mathrm{d}s^2_{(0)} = -f(\mathcal{R})c^2\mathrm{d}t^2 + \frac{\mathrm{d}\mathcal{R}^2}{f(\mathcal{R})} + \mathcal{R}^2\mathrm{d}\Omega^2, \qquad f(\mathcal{R}) = 1 - \frac{r_s}{\mathcal{R}}. \tag{8.57}$$

The proper and critical scales are $L = \mathcal{R}$ and $G = r_s$. The reciprocal chart uses

$$\mathcal{R}(r) = \frac{r_s^2}{r}, \qquad 0 < r \leq r_s. \tag{8.58}$$

After the full radial pullback and odd-sector sign,

$$\mathrm{d}s^2_{(1)} = (1 - r/r_s)c^2\mathrm{d}t^2 - \frac{r_s^4}{r^4(1 - r/r_s)}\mathrm{d}r^2 - \frac{r_s^4}{r^2}\mathrm{d}\Omega^2. \tag{8.59}$$

Worked solution 8.3 (Curvature and causal distance). *The Kretschmann scalar is*

$$K_{(1)}(r) = \frac{12r^6}{r_s^{10}}, \tag{8.60}$$

so it is finite at $r = r_s$ and vanishes at $r \to 0$. With $X = ct$ spatial in the active side plane *and $P = f\,\mathrm{d}X/\mathrm{d}\tau$, a timelike geodesic satisfies*

$$\left(\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau}\right)^2 = c^2\left(1 + p^2 - \frac{r_s}{\mathcal{R}}\right), \qquad p = P/c. \tag{8.61}$$

Its growth rate is bounded above and below by finite positive constants after any $\mathcal{R}_1 > r_s$, so $\mathcal{R} = \infty$, equivalently $r = 0$, requires infinite proper time. Radial light obeys

$$\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\lambda} = \pm\mathcal{E}, \tag{8.62}$$

and reaches the same limit only at infinite affine parameter. The reciprocal centre is an asymptotically flat boundary, not a finite transition to a white hole branch.

For a field-compatible complexification of this analytic real metric, Eq. (8.25) projects the complex Bochner curvature to the Schwarzschild Weyl tensor. Existence of the local analytic complexification does not by itself establish a unique global fundamental complex manifold.

8.13 Reciprocal Kerr solution

Let $M = Gm/c^2$, $a = J/(mc)$, and

$$r_+ = M + \sqrt{M^2 - a^2}. \tag{8.63}$$

The reciprocal principal radial scale is

$$\mathcal{R}(r) = \frac{r_+^2}{r}. \tag{8.64}$$

The exact active metric is the odd signed pullback of a horizon-regular Kerr metric with the full Jacobian retained. The curvature scalar becomes the ordinary Kerr invariant evaluated at $\mathcal{R} = r_+^2/r$ and tends to zero as $O(r^6)$ at the reciprocal endpoint.

Worked solution 8.4 (Carter parameter bound)**.** *With $\Sigma = \mathcal{R}^2 + a^2\cos^2\theta$, $\Delta = \mathcal{R}^2 - 2M\mathcal{R} + a^2$, and Carter constants $(\mathcal{P}, \mathcal{L}, \mathcal{Q})$, the radial potential is*

$$\Sigma^2\left(\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\zeta}\right)^2 = \mathcal{R}_\varepsilon(\mathcal{R}) = \Pi^2 - \Delta(\varepsilon\mathcal{R}^2 + \mathcal{K}), \tag{8.65}$$

where $\varepsilon = -1$ for active side-timelike motion and $\varepsilon = 0$ for null motion. Since $0 \le \mathcal{R}_\varepsilon \le C\mathcal{R}^4$ and $\Sigma \ge \mathcal{R}^2$ on every unbounded allowed branch,

$$\zeta - \zeta_0 = \int_{\mathcal{R}_0}^{\mathcal{R}} \frac{\Sigma}{\sqrt{\mathcal{R}_\varepsilon}}\mathrm{d}\mathcal{R}' \ge \frac{\mathcal{R} - \mathcal{R}_0}{\sqrt{C}}. \tag{8.66}$$

The reciprocal endpoint is therefore at infinite proper or affine parameter for every such branch.

8.14 Reciprocal Kerr–Newman solution

For mass m, charge Q and spin orientation σ, define

$$r_s = \frac{2Gm}{c^2}, \qquad r_Q^2 = \frac{GQ^2}{4\pi\epsilon_0 c^4}, \qquad a_\sigma = \sigma\frac{\hbar}{2mc}. \tag{8.67}$$

The angular critical scale and reciprocal radius are

$$\begin{aligned} R_h(\theta) &= \frac{-r_s + \sqrt{r_s^2 + 4(r_Q^2 - a_\sigma^2\cos^2\theta)}}{2}, \\ \mathcal{R}(\rho,\theta) &= \frac{R_h^2(\theta)}{\rho}, \\ \mathrm{d}\mathcal{R} &= -\frac{R_h^2}{\rho^2}\mathrm{d}\rho + \frac{2R_hR_h'}{\rho}\mathrm{d}\theta. \end{aligned} \tag{8.68}$$

The angular term in $\mathrm{d}\mathcal{R}$ generates indispensable $\mathrm{d}W\mathrm{d}\theta$ and $\mathrm{d}\theta\mathrm{d}\widetilde{\phi}$ terms in the exact pullback metric.

Worked solution 8.5 (Nondegeneracy and removed ring direction)**.** *With $\Sigma_1 = \mathcal{R}^2 + a_\sigma^2\cos^2\theta$, the determinant of the full active metric is*

$$\det g_{(1)} = -\frac{R_h^4(\theta)}{\rho^4}\Sigma_1^2\sin^2\theta. \tag{8.69}$$

It is nonzero away from the usual polar coordinate axis. In the equatorial ring direction,

$$K_{(1)}\left(\rho,\frac{\pi}{2}\right) = \frac{12r_s^2\rho^6}{R_0^{12}} - \frac{48r_sr_Q^2\rho^7}{R_0^{14}} + \frac{56r_Q^4\rho^8}{R_0^{16}}, \qquad R_0 = R_h(\pi/2). \tag{8.70}$$

The scalar decreases toward zero as $\rho \to 0$. The Maxwell potential

$$A_{(1)} = -\frac{Q\mathcal{R}}{\Sigma_1}(\mathrm{d}W - a_\sigma\sin^2\theta\mathrm{d}\widetilde{\phi}) \tag{8.71}$$

is transported in the same active chart, so the metric, electromagnetic source, Ricci curvature and Weyl curvature refer to one constituent geometry.

8.15 LTB longitudinal and transverse modes

The projected Lemaître–Tolman–Bondi dust geometry is

$$\mathrm{d}s_{(0)}^2 = -c^2\mathrm{d}t^2 + \frac{R_F^2}{1+2E(F)}\mathrm{d}F^2 + R^2\mathrm{d}\Omega^2. \tag{8.72}$$

Let s^μ be the radial unit vector and $S^\mu{}_\nu = h^\mu{}_\nu - s^\mu s_\nu$ the spherical screen projector. The two proper scale modes and their critical scales are

$$\begin{aligned} L_\parallel &= \frac{R_F}{\sqrt{1+2E}}, & G_\parallel &= 1, & q_\parallel &= \frac{\sqrt{1+2E}}{R_F}, \\ L_\perp &= R, & G_\perp &= F, & q_\perp &= \frac{F}{R}. \end{aligned} \tag{8.73}$$

Shell crossing $R_F \to 0$ activates the longitudinal projector, whereas shell focusing $R \to 0$ activates the degenerate transverse projector. One scalar radius cannot replace these two tensor modes.

Worked solution 8.6 (Exact reciprocal shell-crossing dust)**.** *In geometrized units, take*

$$R(t,F) = \left(\frac{9F}{4}\right)^{1/3} [t_B(F) - t]^{2/3}, \qquad R_c(F) = R_0 + \frac{\alpha}{2}(F - F_c)^2, \tag{8.74}$$

with

$$t_B(F) = t_c + \frac{2R_c(F)^{3/2}}{3\sqrt{F}}. \tag{8.75}$$

Then $R_F(t_c, F) = \alpha(F - F_c)$ and $R(t_c, F_c) = R_0 > 0$. On the reciprocal branch $F < F_c$, the entrance profile is

$$\widetilde{R}_h(F) = R_h + \frac{1}{\alpha}\ln\left[\frac{1}{\alpha(F_c - F)}\right]. \tag{8.76}$$

The exact expanding side solution is

$$\widetilde{R}(\tau, F) = \left[\widetilde{R}_h(F)^{3/2} + \frac{3}{2}\sqrt{F}(\tau - \tau_h)\right]^{2/3}, \tag{8.77}$$

with metric

$$\mathrm{d}s^2_{(1)} = \mathrm{d}\tau^2 - \widetilde{R}_F^2 \mathrm{d}F^2 - \widetilde{R}^2 \mathrm{d}\Omega^2. \tag{8.78}$$

Its density and Kretschmann scalar are

$$\begin{aligned} \kappa\rho &= \frac{1}{\widetilde{R}^2 \widetilde{R}_F}, \\ K_{(1)} &= \frac{12F^2}{\widetilde{R}^6} - \frac{8F}{\widetilde{R}^5 \widetilde{R}_F} + \frac{3}{\widetilde{R}^4 \widetilde{R}_F^2}, \end{aligned} \tag{8.79}$$

and both tend to zero at $F \to F_c^-$ for every fixed finite $\tau - \tau_h$. This proves curvature regularity for this exact profile, not for every LTB shell crossing.

Worked solution 8.7 (Homogeneous reciprocal shell focusing)**.** *For the homogeneous dust collapse*

$$a(t) = a_h \left(\frac{t_f - t}{t_f - t_h}\right)^{2/3}, \qquad \mathrm{d}s^2_{(0)} = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}\chi^2 + \chi^2 \mathrm{d}\Omega^2), \tag{8.80}$$

the side scale and metric are

$$\begin{aligned} b(\tau) &= a_h \left[1 + \frac{3}{2}H_h(\tau - \tau_h)\right]^{2/3}, \\ \mathrm{d}s^2_{(1)} &= \mathrm{d}\tau^2 - b^2(\tau)(\mathrm{d}\chi^2 + \chi^2 \mathrm{d}\Omega^2). \end{aligned} \tag{8.81}$$

The active curvature is

$$K_{(1)}(\tau) = \frac{15H_h^4}{[1 + \frac{3}{2}H_h(\tau - \tau_h)]^4} \longrightarrow 0. \tag{8.82}$$

Massive proper time and radial-null affine parameter both diverge toward the reciprocal centre. This homogeneous solution is causally complete in the directions analysed.

8.16 Spatially flat FLRW reciprocal solution

For

$$\mathrm{d}s_{(0)}^2 = -c^2\mathrm{d}t^2 + a^2(t)(\mathrm{d}\chi^2 + \chi^2\mathrm{d}\Omega^2), \tag{8.83}$$

homogeneity makes all three proper-scale eigenvalues equal. With critical scale a_h,

$$b(\tau) = \frac{a_h^2}{a(t(\tau))}. \tag{8.84}$$

For $p = w\rho c^2$, $w > -1$, and $\alpha = 3(1+w)/2$, the exact active solution is

$$\begin{aligned} b(\tau) &= a_h[1 + \alpha H_h(\tau - \tau_h)]^{1/\alpha}, \\ \rho(\tau) &= \frac{\rho_h}{[1 + \alpha H_h(\tau - \tau_h)]^2}. \end{aligned} \tag{8.85}$$

The FLRW Weyl tensor vanishes and the full Riemann invariant is

$$K_{(1)}(\tau) = \frac{K_h}{[1 + \alpha H_h(\tau - \tau_h)]^4} \longrightarrow 0. \tag{8.86}$$

For the corresponding ordinary power law,

$$t(\tau) = \frac{t_h}{1 + \alpha H_h(\tau - \tau_h)}. \tag{8.87}$$

Thus the inherited label $t = 0$ compresses an infinite interval of active proper time. Finite-comoving-momentum massive particles and radial light also require unbounded proper and affine parameter. This gives the reciprocal FLRW branch for $w > -1$.

8.17 Photon and null-perspective solution

Let $\gamma : I \to \mathcal{N}$ be a selected null geodesic with tangent k^μ. The local null representatives are

$$\mathrm{d}x^\mu = \lambda k^\mu(s), \qquad g_{\mu\nu}k^\mu k^\nu = 0. \tag{8.88}$$

Initial data and the geodesic equation select the null ray and transport it; they do not create a photon rest clock. The accumulated proper time is

$$\tau_\gamma(s) = \frac{1}{c}\int_{s_0}^{s} \sqrt{-g_{\mu\nu}k^\mu k^\nu}\,\mathrm{d}s' = 0. \tag{8.89}$$

Worked solution 8.8 (Projective complete null state)**.** *The physical state supported on the selected history is*

$$\mathcal{S}_\gamma = \{(\gamma(s), [k(s)], \omega_u(s), \pi) : s \in I\}, \tag{8.90}$$

where $[k]$ is the projective future null ray, ω_u fixes its physical scale relative to an observer, and π denotes polarization or helicity. Constant positive affine rescalings are not counted as new states, the zero vector is not a propagating photon, and the past orientation is a separate branch. Distinct events remain causally ordered, but no two receive different photon proper-time values. Timelike observer slices supply the ordinary measured temporal sequence.

This solution realizes Postulate 3 in the null limit; timelike configurations are carried by the complete tensor state of Sec. 8.8.

8.18 Free graviton solution

In a projected Ricci-flat region,

$$T_{\mu\nu} = 0, \qquad T^{(q)}_{\mu\nu\rho\sigma} = -\frac{1}{\kappa} C^{(q)}_{\mu\nu\rho\sigma}, \qquad \Psi^{(q)}_{\mu\nu\rho\sigma} = -\sqrt{\ell_{\mathrm{P}}}\, c_q C^{(q)}_{\mu\nu\rho\sigma}. \tag{8.91}$$

The vacuum Bianchi equation gives $\nabla^\mu C_{\mu\nu\rho\sigma} = 0$, whose principal covector is null. On the screen transverse to the observer clock u^μ and propagation direction s^μ, define

$$m^\mu = \frac{e_1^\mu + \mathrm{i} e_2^\mu}{\sqrt{2}}. \tag{8.92}$$

The type-N tensors

$$C^{(+2)}_{\mu\nu\rho\sigma} \propto k_{[\mu} m_{\nu]} k_{[\rho} m_{\sigma]}, \qquad C^{(-2)}_{\mu\nu\rho\sigma} \propto k_{[\mu} \bar{m}_{\nu]} k_{[\rho} \bar{m}_{\sigma]} \tag{8.93}$$

acquire phases $e^{\pm 2\mathrm{i}\psi}$ under a screen rotation.

Worked solution 8.9 (Spectrum and sector transport)**.** *For a positive-frequency mode, Eq. (8.47) gives $E_G = \hbar\omega$. The projected sector transport is*

$$C^{(N_g)} = (-1)^{N_g} \mathcal{I}^*_{N_g} C^{(0)}, \qquad H^{(N_g)} = \mathcal{I}^*_{N_g} H^{(0)} (\mathcal{I}^*_{N_g})^{-1}. \tag{8.94}$$

It preserves the null characteristic, energy and helicity. Exact type-N plane waves have zero Lorentzian self-contraction, so their spectral basis is biorthogonal: the dual mode is represented by the antipodal null direction. Localized physical packets require nonzero radial bandwidth and the complete Hermitian tensor normalization.

Four signed pullbacks close the rank-four Weyl representative algebraically. The oriented momenta of the four inherited null branches can sum to zero, but changes of the physical Weyl profile still obey a local curvature wave equation with null principal characteristic. Closed sector history is not nonlocal instantaneous propagation.

8.19 Hydrogen solution

The electron constituent uses the reciprocal Kerr–Newman geometry of Sec. 8.14. The atom itself has one common field and one common geometry. Its electromagnetic tensor is assembled before the source lift,

$$F_{\mu\nu} = F^{(e)}_{\mu\nu} + F^{(p)}_{\mu\nu} + F^{(b)}_{\mu\nu}. \tag{8.95}$$

The Maxwell stress tensor consequently contains the electron–proton cross-term. Integrating its electrostatic part gives

$$V(r) = \frac{q_e q_p}{4\pi\epsilon_0 r} = -\frac{\alpha\hbar c}{r}. \tag{8.96}$$

The electromagnetic, paired-null bridge, conservation response and optional material contributions are assembled into one real source

$$T^{(q)}_{\mu\nu} = \sum_{A=1}^{4} T^{(q)}_{A\mu\nu}, \qquad \nabla_\mu T^{(q)\mu\nu} = 0. \tag{8.97}$$

It is then lifted to the complex four-index configuration and only afterward weighted by c_q in Eq. (8.41).

Worked solution 8.10 (Central spectrum)**.** *With reduced mass μ and $k^2 = (j+\frac{1}{2})^2\hbar^2$, the central mass shell gives*

$$p_r^2 = \frac{E^2}{c^2} - \mu^2c^2 + \frac{2\alpha\hbar E}{cr} - \frac{\hbar^2}{r^2}\left[(j+\tfrac{1}{2})^2 - \alpha^2\right]. \tag{8.98}$$

The closed radial action yields

$$\boxed{E_{n_rj} = \mu c^2\left[1 + \frac{\alpha^2}{\left(n_r + \sqrt{(j+\frac{1}{2})^2-\alpha^2}\right)^2}\right]^{-1/2}}. \tag{8.99}$$

The compatible radial projection is

$$u_{n_rj}(r) = N_{n_rj}(2\lambda r)^{\gamma_j} e^{-\lambda r} L_{n_r}^{2\gamma_j-1}(2\lambda r), \qquad \gamma_j = \sqrt{(j+\tfrac{1}{2})^2-\alpha^2}. \tag{8.100}$$

The complex carrier uses the full tensor mode, not this scalar radial projection alone. Hermitian overlaps resolve energy, position, angular and spin channels without introducing another state field.

The result reproduces the relativistic central Coulomb spectrum within the complete four-index tensor carrier.

8.20 Ultraviolet quantum-gravity solution

In the projected real theory, standard background-field quantization uses $D = 4 - 2\epsilon$, gauge fixing and Faddeev–Popov ghosts. Recursive forest subtraction is completed before the sector operation. At loop order L, the remaining local pole functional defines

$$H^{(L),\mathrm{UV}}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\Gamma^{(L)}_{\mathrm{pole}}}{\delta g^{\mu\nu}}, \qquad \nabla^\mu H^{(L),\mathrm{UV}}_{\mu\nu} = 0. \tag{8.101}$$

Internal sector factors cancel against inverse factors on propagator indices, whereas the two free covariant indices retain

$$H^{(L),\mathrm{UV}}_{\mu\nu}(S_X) = (-1)^{S_X} H^{(L),\mathrm{UV}}_{\mu\nu}(0). \tag{8.102}$$

The normalized complete-orbit projector therefore gives

$$\mathcal{P}_2^{\mathrm{UV}} H^{(L),\mathrm{UV}}_{\mu\nu} = \frac{1}{4}\sum_{S_X=0}^{3}(-1)^{S_X} H^{(L),\mathrm{UV}}_{\mu\nu} = 0. \tag{8.103}$$

Worked solution 8.11 (Two-loop cubic-curvature benchmark)**.** *The Goroff–Sagnotti local pole is proportional to*

$$\Gamma^{(2)}_{\mathrm{div}} \propto \frac{1}{\epsilon} \int \sqrt{-g}\, R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{ef} R^{ef}{}_{\mu\nu}\, \mathrm{d}^4x. \tag{8.104}$$

Its metric Euler tensor is symmetric, conserved and nonzero before sector projection. It has the character in Eq. (8.102), so its complete particle-orbit component vanishes. Evanescent Gauss–Bonnet subdivergences are included before this projection and therefore modify the common scalar coefficient without modifying the sector character.

The rank-two response has the real four-index lift

$$\begin{aligned}\widetilde{H}_{\mu\nu\rho\sigma} = \frac{1}{2}&(H_{\mu\rho}g_{\nu\sigma} - H_{\mu\sigma}g_{\nu\rho} + H_{\nu\sigma}g_{\mu\rho} - H_{\nu\rho}g_{\mu\sigma}) \\ &- \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})H,\end{aligned} \tag{8.105}$$

completed by a trace-free $\widehat{H}_{\mu\nu\rho\sigma}$ obeying the same divergence exchange law. Both pieces inherit $(-1)^{S_X}$, so the four-index orbit also cancels.

Thus every local UV pole source with the sector character $(-1)^{S_X}$ vanishes under the complete four-sector orbit projection, both in its rank-two response and in its four-index lift.

8.21 Conclusion

The $SU(m, n)$ Unified Field Theory has a complex pseudo-Kähler spacetime of dimension $N = m + n$, carrying Hermitian four-index geometry and sources. The physical Lorentzian branch is $(m, n) = (1, 3)$. A real spacetime is the geometry measured on an admissible observer real form, and the four-sector atlas acts after this projection.

The principal mathematical result of the theory is the explicit complex-to-real bridge. Realification, pullback and the Gauss equation give the observed curvature, while the Weyl projector separates the projected trace-free content. The concise relation between Bochner and Weyl tensors is therefore conditional on vanishing extrinsic and Ricci-projection corrections. The same observer map projects the complex source, field equation and quantum carrier, so geometry and state are not related by separate ad hoc rules.

Postulate 3 is realized by a complete tensor state whose nonzero branches coexist without being summed into one classical source. Full Hermitian conjugation, a hypersurface overlap and covariant rank-eight evolution supply one probability and phase convention. At a given event the observer resolves one channel of the complete state.

The framework yields one common local null speed, four oriented observer sectors, no ordinary massive crossing of the local light cone, and massless helicity-two free-graviton modes with $E = \hbar\omega$. Schwarzschild, Kerr, Kerr–Newman, LTB and FLRW geometries acquire reciprocal regular continuations. The photon solution has zero proper-time separation along its null history, and the hydrogen construction reproduces the relativistic central Coulomb spectrum.

Finally, the complete equal-weight particle-sector orbit annihilates local UV pole sources carrying the character $(-1)^{S_X}$. The same cancellation is preserved by the real four-index lift. Complex geometry, real observer projection, four-sector transport, complete tensor states and the collected solutions therefore constitute one common field-theoretic structure governed by the four postulates stated at the beginning.

Regular Four-Index FRW Solution from a Four-Sector Reduction

Source and reconstruction note. This chapter incorporates the supplied document *Regular Four-Index FRW Solution from a Four-Sector Reduction*. Several equation objects in that DOCX contained empty or incomplete text runs. The formulas below are reconstructed directly from the definitions that remained readable and from the four-dimensional rank-four convention used by the surrounding manuscripts. The reconstruction should be independently checked against the author's original calculation before formal submission.

9.1 Four-sector reduction

Consider four spacetime sectors labelled by $n \in \mathbb{Z}_4$,

$$ds_n^2 = a_n^2(t)\,\mathrm{i}^{2n}\eta_{\mu\nu}\,dX^\mu_{(n)}dX^\nu_{(n)}. \tag{9.1}$$

They are combined according to

$$dS^2 = ds_0^2 - ds_1^2 + ds_2^2 - ds_3^2. \tag{9.2}$$

Because $\mathrm{i}^{2n} = (-1)^n$, the explicit sector signs cancel:

$$dS^2 = \sum_{n=0}^{3} a_n^2(t)\,\eta_{\mu\nu}dX^\mu_{(n)}dX^\nu_{(n)}. \tag{9.3}$$

Under the diagonal $\mathbb{Z}_4$ projection onto one physical four-dimensional spacetime, normalized so that $a_n = 1$ reproduces the Minkowski metric, the effective line element is

$$ds^2 = A^2(t)\eta_{\mu\nu}dx^\mu dx^\nu, \qquad A^2(t) = \frac{1}{4}\sum_{n=0}^{3} a_n^2(t). \tag{9.4}$$

For the special configuration

$$a_0 = 1, \qquad a_1 = a_2 = a_3 = t, \tag{9.5}$$

one obtains

$$A^2(t) = \frac{1+3t^2}{4}, \qquad A(t) = \frac{1}{2}\sqrt{1+3t^2}, \tag{9.6}$$

and hence

$$ds^2 = \frac{1+3t^2}{4}\,\eta_{\mu\nu}dx^\mu dx^\nu. \tag{9.7}$$

Here t is conformal time.

9.2 Curvature regularity

For a spatially flat conformal-FRW metric the Kretschmann scalar can be written

$$K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 12\frac{\left(AA'' - (A')^2\right)^2 + (A')^4}{A^8}. \tag{9.8}$$

The unregularized ansatz $A = t$ therefore gives

$$K_{A=t} = \frac{24}{t^8}, \tag{9.9}$$

which is singular at $t = 0$. For the four-sector scale factor,

$$A' = \frac{3t}{2\sqrt{1+3t^2}}, \qquad A'' = \frac{3}{2(1+3t^2)^{3/2}}, \tag{9.10}$$

and

$$K(t) = \frac{1728\,(1 - 6t^2 + 18t^4)}{(1+3t^2)^6}. \tag{9.11}$$

Consequently,

$$K(0) = 1728 < \infty, \qquad A(0) = \frac{1}{2}, \tag{9.12}$$

so the metric is non-degenerate at the bounce. The geometry is conformally flat,

$$C_{\alpha\beta\gamma\delta} = 0. \tag{9.13}$$

9.3 Four-index Einstein tensor and source

In four dimensions the all-lower four-index Einstein tensor is taken in the convention

$$\begin{aligned} G_{\alpha\beta\gamma\delta} = {} & \frac{1}{2}\left(R_{\alpha\gamma}g_{\beta\delta} - R_{\alpha\delta}g_{\beta\gamma} + R_{\beta\delta}g_{\alpha\gamma} - R_{\beta\gamma}g_{\alpha\delta}\right) \\ & - \frac{1}{3}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right)R - C_{\alpha\beta\gamma\delta}. \end{aligned} \tag{9.14}$$

Isotropy reduces its independent nonzero components to two classes. For $i = 1, 2, 3$,

$$G_{0i0i} = (A')^2 = \frac{9t^2}{4(1+3t^2)}, \tag{9.15}$$

whereas for $i \neq j$,

$$G_{ijij} = (A')^2 - AA'' = \frac{3(3t^2-1)}{4(1+3t^2)}. \tag{9.16}$$

At the bounce,

$$G_{0i0i}(0) = 0, \qquad G_{ijij}(0) = -\frac{3}{4}. \tag{9.17}$$

All independent components therefore remain finite.

Using

$$G_{\alpha\beta\gamma\delta} = \kappa T_{\alpha\beta\gamma\delta}, \tag{9.18}$$

the corresponding nonzero source classes are

$$T_{0i0i} = \frac{9t^2}{4\kappa(1+3t^2)}, \tag{9.19}$$

and

$$T_{ijij} = \frac{3(3t^2-1)}{4\kappa(1+3t^2)}. \tag{9.20}$$

Because the spacetime is conformally flat, the trace-free gravitational-energy sector vanishes in the convention $C_{\alpha\beta\gamma\delta} = \kappa\hat{T}_{\alpha\beta\gamma\delta}$:

$$\hat{T}_{\alpha\beta\gamma\delta} = 0. \tag{9.21}$$

The complete source is finite through $t = 0$, and its conservation law is

$$\nabla^\alpha T_{\alpha\beta\gamma\delta} = 0. \tag{9.22}$$

The construction therefore replaces the curvature-divergent conformal ansatz $A = t$ by a non-degenerate, conformally flat four-sector reduction with finite curvature and finite rank-four source at $t = 0$. It is a bounce candidate within the stated diagonal-projection prescription; its dynamical matter interpretation and stability remain to be established.

Chapter 10

Integrated Results, Limitations and Research Programme

10.1 What is established internally

The rank-four curvature decomposition is the most direct algebraic component of the programme. In the convention of Chapter 2, contraction of the complete equation recovers the ordinary Einstein equation, while the trace-free remainder pairs the Weyl tensor with a trace-free source. The formulation makes tidal curvature visible on the source side without, by itself, changing the classical metric solution set. This distinction between an independent variable during the rank-four solution procedure and an additional physical degree of freedom is essential.

The four-sector construction is internally organized around another distinction: geometry-sector orientation, particle-sector orientation and ordinary within-sector Lorentz motion are separate data. The null boundary is common to all sectors and massive particles are not accelerated through it. Reciprocal normalization is a chart prescription selected by proper geometric scales. In the black-hole, homogeneous LTB and FLRW branches analysed in Chapter 3, curvature decay is accompanied by unbounded proper or affine parameter toward the reciprocal centre. Regularity therefore means a curvature-regular asymptotic boundary in those examples, not an automatically traversable continuation.

The tensor quantum chapters maintain a consistent hierarchy: a real classical source is fixed first; complex coefficients then weight complete configurations; rank-eight evolution acts on the carrier; and scalar probabilities are obtained only after global contraction. The free-graviton construction is the Ricci-flat trace-free specialization of that hierarchy. The ultraviolet chapter likewise begins with the conventional regulated loop calculation and applies the sector projector only to the resulting local pole response.

10.2 Results that remain conditional

The hydrogen calculation is conditional on the adopted source assembly, reciprocal Kerr–Newman representation, central reduction and tensor normalization. Agreement of a spectral formula is an important internal check, but a complete comparison must include fine structure, radiative corrections, transition strengths, scattering sectors and the standard precision observables that distinguish competing formulations.

The ultraviolet cancellation is a statement about a projected local source after subtraction. It is not yet a proof of a complete ultraviolet-finite quantum theory of gravity. A full result would require gauge-parameter independence of the projector, BRST compatibility, treatment of anomalies, unitarity and cutting rules, a careful account of finite renormalized couplings, and a demonstration that the sector sum is a physical operation rather than an imposed cancellation among redundant representatives.

The $SU(m, n)$ framework supplies a common complex geometry only under explicit observer-projection conditions. Bochner curvature does not automatically equal the Weyl tensor of an arbitrary real form. Extrinsic-curvature corrections and the projected Ricci sector must satisfy the compatibility relations derived in Chapter 8. The theory must also specify dynamically which real form represents an observer and whether different admissible involutions lead to equivalent measurable predictions.

10.3 Notation and sign consistency

The developmental sources contain more than one sign convention for the trace-free source and more than one dimension symbol. The integrated argument is invariant only when each equation is read with the definitions local to its chapter. Before journal or doctoral submission, a final normalization pass should select one global convention for the Riemann tensor, metric signature, Weyl-source sign, gravitational coupling and complex dimension, then rederive every contracted and divergence identity under that convention. The archival appendices should remain unchanged as a record of development, but the principal chapters should ultimately use the selected notation uniformly.

10.4 Independent verification priorities

The next mathematical priority is a computer-algebra verification of all rank-four contractions, divergences and sign factors in real and pseudo-Kähler geometry. The next geometric priority is to check junction regularity, maximal extension and geodesic completeness for each reciprocal metric without relying only on a selected curvature scalar. The next quantum priority is to prove positivity and basis independence of the global tensor norm and to derive operational measurement probabilities without importing an unacknowledged scalar-state structure. The next perturbative priority is to test the four-sector projector on explicit amplitudes and Ward identities beyond the local Euler response.

On the empirical side, the framework requires predictions that differ quantitatively from general relativity and standard quantum theory. Candidate domains include strong-field lensing and ringdown near reciprocal compact-object branches, cosmological signatures of the regular sector reduction, precision spectroscopy beyond the leading relativistic Coulomb spectrum, and propagation or correlation effects that distinguish the four-sector atlas without assigning material particles speeds greater than c.

10.5 Conclusion

Taken together, the supplied works form a recognizable research programme rather than a collection of unrelated conjectures. The common object is the complete tensor content hidden by contraction or by a single observer chart. The four-index equation retains Ricci and Weyl information at one rank; the four-sector atlas retains the four non-null orientations around one invariant null set; the tensor quantum model retains complete classical configurations before scalar readout; and the $SU(m, n)$ proposal retains a complex geometry before real observer projection.

The programme's strongest current result is structural coherence across these layers. Its main scientific burden is now equally clear: global convention control, independent symbolic checks, precise mathematical existence statements, and falsifiable predictions. The dissertation therefore closes not by treating every proposed mechanism as established physics, but by giving the framework a form in which each claim can be checked, challenged and developed.

Part VIII

Archival Developmental Manuscripts

Chapter A

Archival Four-Index Thesis Draft

Source and status. This chapter incorporates the complete scientific body of *Four Index Einstein Field Equations of Real and Complex Spacetime*, archival draft (2026). Its role in the dissertation is: Earlier derivational form retained for provenance and equation coverage. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

In this work I present extensions of Einstein field equations [6] into four index equations. This extensions give as natural result a energy tensor for vacuum thus for gravity field. It's all construct in spirit of two index field equations and in truth does not need any additional assumptions about field equations. Form it follows that it's natural completeness of two index equations not a true extension as it fully defines curvature tensor not only Ricci part of curvature as it happens in two index equations. Additionally I extend this frame work to complex spacetime [7] [8].

A.1 Einstein field equations

A.1.1 Einstein tensor

Einstein tensor is basis of General Relativity [6], from it arises left side of field equations that connects matter field represented by stress momentum tensor [6] with geometry part. That geometry part is just Einstein tensor that comes from Bianachi identities [6]. It can be written as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{A.1}$$

It's most important property is that is divergence free [6] so that:

$$\nabla^{\mu}G_{\mu\nu} = 0 \tag{A.2}$$

It connects to left side of equations where stress momentum tensor [6] is present.

A.1.2 Stress momentum tensor

Matter sources in General Relativity [6] are all coming from right side of equation that is stress momentum tensor [6] $T_{\mu\nu}$. Units of stress momentum tensor are energy density so from it follows that on left side of equation I have units of curvature that is length to power minus two, thats why if field equations [6] I need to use Einstein constant that is equal to:

$$\kappa = \frac{8\pi G}{c^4} \tag{A.3}$$

Another crucial property of stress momentum tensor is that as on left side I have divergence free tensor , stress momentum tensor is divergence free meaning that matter fields are conserved locally. It means that divergence of stress momentum tensor is zero [6]:

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{A.4}$$

A.1.3 Field equation

That finally leads to field equation that I will call in this work a two index field equation, that is just ordinary Einstein field equation [6]:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{A.5}$$

Idea behind this work is simple extended this equation into four index equation, adding part that is responsible for vacuum energy or more precise gravity field energy.

A.1.4 Cosmological constant

Two index equations used today have additional term [6], that term is cosmological constant or energy of empty space Λ it modifies field equations by adding term with it to field equations [6]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{A.6}$$

In vacuum it will lead to:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{A.7}$$

That solutions will be either de Sitter or anti-de Sitter [6] spacetime depending on sign of cosmological constant.

A.2 Four index Einstein tensor

A.2.1 Ricci part

Riemann tensor [6] can be decomposed into Ricci and Weyl parts:

$$\begin{aligned} R_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} &+ \frac{1}{n-2}\left(R_{\alpha\beta} g_{\mu\nu} - R_{\alpha\nu} g_{\beta\mu} + R_{\mu\nu} g_{\alpha\beta} - R_{\beta\mu} g_{\alpha\nu}\right) \\ &- \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\beta\mu}\right) R \end{aligned} \tag{A.8}$$

This decomposition gives after contraction exactly Ricci tensor as part with Weyl vanishes when contracted as it's trace free for all indexes combinations. So isolating Ricci part but adding part that will after contraction reduce exactly to one half of Ricci scalar times metric tensor I will arrive at:

$$\begin{aligned}&\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu}-R_{\alpha\nu}g_{\beta\mu}+R_{\mu\nu}g_{\alpha\beta}-R_{\beta\mu}g_{\alpha\nu}\right)\\ -&\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)R-\frac{1}{2(n-1)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)R\end{aligned}\tag{A.9}$$

I can sum terms with Ricci scalar and will arrive at final expression for first component of four index Einstein tensor:

$$\begin{aligned}\tilde{R}_{\alpha\mu\beta\nu}=&\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu}-R_{\alpha\nu}g_{\beta\mu}+R_{\mu\nu}g_{\alpha\beta}-R_{\beta\mu}g_{\alpha\nu}\right)\\ &-\frac{n}{2(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)R\end{aligned}\tag{A.10}$$

A.2.2 Divergence of Ricci part

From it follows that I can write divergence of this component with respect to first index where I will use Bianachi identities for same indexes of Ricci tensor and divergence index so index α to make equations simpler:

$$\begin{aligned}\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu}=&\frac{1}{n-2}\left(\nabla^{\alpha}R_{\alpha\beta}g_{\mu\nu}-\nabla^{\alpha}R_{\alpha\nu}g_{\beta\mu}+\nabla_{\beta}R_{\mu\nu}-\nabla_{\nu}R_{\beta\mu}\right)\\ &-\frac{n}{2(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta}-g_{\beta\mu}\nabla_{\nu}\right)R\end{aligned}\tag{A.11}$$

$$\begin{aligned}\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu}=&\frac{1}{n-2}\left(\nabla_{\beta}R_{\mu\nu}-\nabla_{\nu}R_{\beta\mu}\right)\\ -\frac{n}{2(n-1)(n-2)}&\left(g_{\mu\nu}\nabla_{\beta}-g_{\beta\mu}\nabla_{\nu}\right)R+\frac{1}{2(n-2)}\left(g_{\mu\nu}\nabla_{\beta}-g_{\beta\mu}\nabla_{\nu}\right)R\end{aligned}\tag{A.12}$$

$$\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu}=\frac{1}{n-2}\left(\nabla_{\beta}R_{\mu\nu}-\nabla_{\nu}R_{\beta\mu}\right)-\frac{1}{2(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta}-g_{\beta\mu}\nabla_{\nu}\right)R\tag{A.13}$$

Taking out common denominator I will arrive at final expression:

$$\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu}=\frac{1}{n-2}\Big(\left(\nabla_{\beta}R_{\mu\nu}-\nabla_{\nu}R_{\beta\mu}\right)-\frac{1}{2(n-1)}\left(g_{\mu\nu}\nabla_{\beta}-g_{\beta\mu}\nabla_{\nu}\right)R\Big)\tag{A.14}$$

A.2.3 Weyl part

Weyl [6] part divergence is equal to:

$$\nabla^{\alpha}C_{\alpha\mu\beta\nu}=\frac{n-3}{n-2}\Big(\left(\nabla_{\beta}R_{\mu\nu}-\nabla_{\nu}R_{\beta\mu}\right)-\frac{1}{2(n-1)}\left(g_{\mu\nu}\nabla_{\beta}-g_{\beta\mu}\nabla_{\nu}\right)R\Big)\tag{A.15}$$

This means that they differ only by a constant $n-3$, from it follows to create a field equation I need first to take that constant and divide Weyl tensor by it. Then whole tensor has to be divergence free, from it follows that I will take Ricci part and subtract Weyl dived by that constant:

$$\tilde{R}_{\alpha\mu\beta\nu}-\frac{1}{n-3}C_{\alpha\mu\beta\nu}\tag{A.16}$$

A.2.4 Four index Einstein tensor

Last expression is just four index Einstein tensor, it all properties of being divergence free and it's contractions lead to two index equations. This tensor can be written:

$$G_{\alpha\mu\beta\nu} = \tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3} C_{\alpha\mu\beta\nu} \tag{A.17}$$

A.3 Four index energy tensor

A.3.1 Matter part

I can use same form as in Riemann decomposition into matter part of total energy tensor. This tensor will reduce to stress momentum tensor so I can write it:

$$\begin{aligned}\tilde{T}_{\alpha\mu\beta\nu} = &\frac{1}{n-2}\left(T_{\alpha\beta}g_{\mu\nu} - T_{\alpha\nu}g_{\beta\mu} + T_{\mu\nu}g_{\alpha\beta} - T_{\beta\mu}g_{\alpha\nu}\right) \\ &- \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T\end{aligned} \tag{A.18}$$

Now key property is to calculate divergence of this tensor with respect to first index as before, I can use property that stress momentum tensor is divergence free:

$$\nabla^{\alpha}\tilde{T}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}\right) - \frac{1}{(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T \tag{A.19}$$

Taking out common denominator:

$$\nabla^{\alpha}\tilde{T}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\left(\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}\right) - \frac{1}{(n-1)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T\right) \tag{A.20}$$

That is exactly propagation equation [6] from two index equations, where it does differ by same constant.

A.3.2 Vacuum energy part

From previous equations I can see a logic here, vacuum energy tensor divergence will be have to be equal to same as matter part times a constant defined before:

$$\nabla^{\alpha}\hat{T}_{\alpha\mu\beta\nu} = \frac{n-3}{n-2}\left(\left(\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}\right) - \frac{1}{(n-1)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T\right) \tag{A.21}$$

This tensor has to be trace free for all indexes just as Weyl tensor. That means if it has symmetries of Riemann tensor:

$$g^{\alpha\beta}\hat{T}_{\alpha\mu\beta\nu} = 0 \tag{A.22}$$

In vacuum solutions it just equal Weyl tensor, but I need to put Einstein constant [6] in it.

A.3.3 Four index energy tensor

Putting this all in one equation and one tensor I will use same logic from before so I arrive at total energy tensor:

$$T_{\alpha\mu\beta\nu} = \tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu} \tag{A.23}$$

This tensor is divergence free as part with geometry, it has a trace part (stress momentum tensor) and trace-less part (vacuum energy tensor).

A.4 Four index field equations

A.4.1 Equation

Combining all from before I arrive at four index equation:

$$G_{\alpha\mu\beta\nu} = \kappa T_{\alpha\mu\beta\nu} \tag{A.24}$$

Equation is divergence free:

$$\nabla^{\alpha} G_{\alpha\mu\beta\nu} = \kappa \nabla^{\alpha} T_{\alpha\mu\beta\nu} = 0 \tag{A.25}$$

It's contraction leads to two index equation [6]:

$$g^{\alpha\beta} G_{\alpha\mu\beta\nu} = \kappa g^{\alpha\beta} T_{\alpha\mu\beta\nu} \tag{A.26}$$

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{A.27}$$

It has trace and trace-less parts that are equal to each other:

$$\kappa \tilde{T}_{\alpha\mu\beta\nu} = \tilde{R}_{\alpha\mu\beta\nu} \tag{A.28}$$

$$\kappa \hat{T}_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} \tag{A.29}$$

A.4.2 Adding cosmological constant

I can add back to equations part with cosmological constant [6] it will be term with metric tensors and scalar, I can write that term:

$$\frac{1}{n-1}\left(g_{\mu\nu} g_{\alpha\beta} - g_{\beta\mu} g_{\alpha\nu}\right)\Lambda \tag{A.30}$$

So final field equations are:

$$\boxed{G_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu} g_{\alpha\beta} - g_{\beta\mu} g_{\alpha\nu}\right)\Lambda = \kappa T_{\alpha\mu\beta\nu}} \tag{A.31}$$

If cosmological constant [6] is just a fixed number it's divergence vanishes but if not, it has to fulfill this equations:

$$\left(g_{\mu\nu} \partial_{\beta} - g_{\beta\mu} \partial_{\nu}\right)\Lambda = 0 \tag{A.32}$$

And from two index equations it needs to vanish when contracted so it leads to:

$$\partial_\nu \Lambda = 0 \tag{A.33}$$

It shows that it has to be a constant number that does not change with any spatial or a temporal direction. Otherwise equations will be not divergence free in both four index and two index [6]. After contraction it will just lead to standard form two index equation [6]:

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu} \tag{A.34}$$

A.4.3 Action formulation of equations

I can formulate field equations in as action [10] , derivation was already done in this paper [10], where there is same structure of field equations used. After adding tensors I will arrive at starting with gravity field:

$$S_G = \frac{1}{2\kappa c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(G_{\alpha\mu\beta\nu} + \lambda_{\alpha\mu\beta\nu}\right) d^D\mathbf{x} \tag{A.35}$$

Where those tensors are defined as before so:

$$S_G = \frac{1}{2\kappa c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) d^D\mathbf{x} \tag{A.36}$$

Now matter field part behaves same way but without term with Einstein constant [10]:

$$S_M = -\frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} T_{\alpha\mu\beta\nu} d^D\mathbf{x} \tag{A.37}$$

That can be expanded:

$$S_M = -\frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right) d^D\mathbf{x} \tag{A.38}$$

Adding them both into one integral [10]:

$$S = \frac{1}{2\kappa c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(G_{\alpha\mu\beta\nu} + \lambda_{\alpha\mu\beta\nu}\right) d^D\mathbf{x} - \frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} T_{\alpha\mu\beta\nu} d^D\mathbf{x} \tag{A.39}$$

$$S = \frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left[\frac{1}{2\kappa}\left(G_{\alpha\mu\beta\nu} + \lambda_{\alpha\mu\beta\nu}\right) - T_{\alpha\mu\beta\nu}\right] d^D\mathbf{x} \tag{A.40}$$

That can be written in whole expanded form as:

$$\begin{aligned} S = \frac{1}{c}\int \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \Bigg[\frac{1}{2\kappa}\left(\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) \\ - \left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right)\Bigg] d^D\mathbf{x} \end{aligned} \tag{A.41}$$

A.4.4 Meaning of four index field equation

Four index field equation is natural extension of two index equation, what is new in this field equation that there is new tensor that is responsible for vacuum energy. This tensor is energy of matter field in vacuum. It means that it's an extension of matter field into vacuum and it consists of physical matter. This matter is not exactly like normal matter field, it is trace-less part of total energy, but still it's matter field.

This matter field is carrying energy of matter field into the vacuum, it's simplest candidate for gravity particle. From it follows most important part, in this model gravity as particle carrying energy is not need to quantize field to arrive at it- it's done in pure classical terms. And crucial is that energy of that particle being equal to Weyl tensor is an equation to be solved not identity, that leads to two equations but what is important are not two separate equations but come from a tensor that is created when they are combined:

$$G_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda = \kappa T_{\alpha\mu\beta\nu} \tag{A.42}$$

$$\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right) \tag{A.43}$$

As this tensor is a divergence free object. It leads to simple question, why there is a minus sign in the parts with vacuum energy and Weyl tensor? Answer to that question is that otherwise two sides would not be divergence free. This gives a physical meaning to it, total tensor takes all inputs from vacuum as negative in sign. Still this is not a final story as it lacks any kind of quantum effects.

A.4.5 Trace-reversed field equation

I can arrive at trace reversed field equations, first I take field equation:

$$\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3}C_{\alpha\mu\beta\nu} + \frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right) \tag{A.44}$$

Expand left side and move cosmological constant part to right side of equations:

$$\begin{aligned}&\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right)\\ &-\frac{n}{2\left(n-1\right)\left(n-2\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R - \frac{1}{n-3}C_{\alpha\mu\beta\nu}\end{aligned} \tag{A.45}$$

Compare it with Riemann tensor expression:

$$\begin{aligned}R_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} &+ \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right)\\ &-\frac{1}{\left(n-1\right)\left(n-2\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R\end{aligned} \tag{A.46}$$

I can express now it in terms of Riemann tensor:

$$G_{\alpha\mu\beta\nu} = R_{\alpha\mu\beta\nu} - \frac{1}{2\left(n-1\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R - \left(1 + \frac{1}{n-3}\right)C_{\alpha\mu\beta\nu} \tag{A.47}$$

Use two index equations [6] contractions and fact that Weyl tensor is equal to energy tensor of vacuum and move all that is not Riemann tensor to right side:

$$R_{\alpha\mu\beta\nu} = \kappa\left(T_{\alpha\mu\beta\nu} - \frac{1}{\left(n-1\right)\left(n-2\right)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T + \left(1 + \frac{1}{n-3}\right)\hat{T}_{\alpha\mu\beta\nu}\right) \tag{A.48}$$

Expand components of total energy tensor:

$$R_{\alpha\mu\beta\nu} = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T\right) + \kappa\left(1 + \frac{1}{n-3}\right)\hat{T}_{\alpha\mu\beta\nu} \tag{A.49}$$

Sort out components first of vacuum energy tensor:

$$R_{\alpha\mu\beta\nu} = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} + \hat{T}_{\alpha\mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T\right) \tag{A.50}$$

Then matter energy tensor and will arrive at final expression and add cosmological constant again:

$$R_{\alpha\mu\beta\nu} = \kappa\left(\tilde{T}_{\alpha\mu\beta\nu} + \hat{T}_{\alpha\mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T + \frac{2}{\kappa(n-1)(n-2)}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) \tag{A.51}$$

A.4.6 Geodesic deviation and geodesic equation

From fact that I can write Riemann tensor in terms of energy tensors I can go one step forward and write whole equation as a geodesic deviation equation [11] , where $\mathbf{V}$ is vector tangent to geodesic curve [11]:

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\xi^{\alpha} = R^{\alpha}_{\ \mu\beta\nu}V^{\mu}V^{\beta}\xi^{\nu} \tag{A.52}$$

Now I can switch right side of equation into energy instead of curvature parts:

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\xi^{\alpha} = \kappa\left(\tilde{T}^{\alpha}_{\ \mu\beta\nu} + \hat{T}^{\alpha}_{\ \mu\beta\nu} - \frac{1}{(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\alpha}_{\beta} - g_{\beta\mu}\delta^{\alpha}_{\nu}\right)T + \frac{2}{\kappa(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\alpha}_{\beta} - g_{\beta\mu}\delta^{\alpha}_{\nu}\right)\Lambda\right)V^{\mu}V^{\beta}\xi^{\nu} \tag{A.53}$$

That leads to most general interpretation of field equations in geometric way as it is interpretation of Riemann tensor itself [12]. In general geodesic deviation says how "fast" geodesic [11] accelerate with respect to each other. They can either converge or diverge. Field equations will not lead to singularities only if both side of equations stay finite, from fact that Im defining Riemann tensor that encodes whole curvature I can impose that right side with energy content stays finite in whole manifold then solve for left side of equation. This gives control over singularities that is absent in two index equation [6]. If geodesic deviation acts on global spacetime, geodesic equation [6] [11] acts locally when effects of particle or systems of particles is small compared to source of gravity field. Geodesic equation [11] can be written as:

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\ \alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 0 \tag{A.54}$$

This equations in truth works only if I don't take a field as whole but a isolated part of a field into account. Field as whole is generated from curvature itself. It leads to idea that object not only follow locally geodesic equation but field in general has obey geodesic deviation equation [11] that is more general statement of gravity field as it has whole curvature as a source. So locally geodesic equation is still valid but in truth more general statement about field is geodesic deviation equation [11] [12]. In truth field should be considered as a general object, that's why geodesic equation [11] is not whole story. Energy makes geodesic accelerate towards each other or away from each other, this happens both in case of matter field or gravity field energy. Gravity field should be thought then in geometric sense [12] as focusing or un-focusing of geodesics. As long as both side of equations stay finite this leads to well defined curvature and from it follows field.

A.4.7 Field equations as connection with two index equations

I can re-write field equations using two equations that are separated, one equation is just two index field equation [6] written in four index form and one deals with Weyl tensor [6]:

$$\tilde{R}_{\alpha\mu\beta\nu} = \kappa\tilde{T}_{\alpha\mu\beta\nu} \tag{A.55}$$

$$C_{\alpha\mu\beta\nu} = \kappa\hat{T}_{\alpha\mu\beta\nu} \tag{A.56}$$

As first equation can be reduced to two index equation [6] I can re-write field equations as:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{A.57}$$

$$C_{\alpha\mu\beta\nu} = \kappa\hat{T}_{\alpha\mu\beta\nu} \tag{A.58}$$

Where there is connection between two index equations and four index equations, this connection is covariant derivative. When I do take covariant derivative of vacuum energy tensor I need to arrive exactly at same expression as for energy momentum part:

$$\nabla^{\alpha}\hat{T}_{\alpha\mu\beta\nu} = \nabla^{\alpha}\tilde{T}_{\alpha\mu\beta\nu} \tag{A.59}$$

At first it may seem as it's just propagation equation [6] from two index equation [6] but key here is that vacuum energy tensor is equation to be solved not an identity. It means that it agrees with two index equation [6] in form of propagation equation [6]:

$$\nabla^{\alpha}C_{\alpha\mu\beta\nu} = \kappa\nabla^{\alpha}\hat{T}_{\alpha\mu\beta\nu} \tag{A.60}$$

$$\begin{gathered}\frac{n-3}{n-2}\left(\left(\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right) - \frac{1}{2\left(n-1\right)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R\right) = \\ \frac{n-3}{n-2}\kappa\left(\left(\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}\right) - \frac{1}{\left(n-1\right)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T\right)\end{gathered} \tag{A.61}$$

$$\begin{gathered}\left(\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right) - \frac{1}{2\left(n-1\right)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R = \\ \kappa\left(\left(\nabla_{\beta}T_{\mu\nu} - \nabla_{\nu}T_{\beta\mu}\right) - \frac{1}{\left(n-1\right)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)T\right)\end{gathered} \tag{A.62}$$

It means that change in energy tensor of vacuum is connected to change in matter field, or stating opposite change of vacuum energy tensor is connected to change of matter field. From it follows that if vacuum energy stays finite there is need for matter field to stay finite if I started from finite values. From it follows that each change in gravity field is connected to change in matter field and matter field change are connected to gravity field, if those changes remain finite whole equations will remain finite. It's imposing on two index equations [6] that energy momentum tensor [6] stays finite is equally imposing on vacuum energy to stay finite from it follows Weyl tensor so gravity field itself. This separation of field equations shows how on one hand it's close to two index equations [6] and where it does differ. As to create a whole field equations I need to use four index formalism. It follows from conservation equations that are written in four index formalism. So in truth I can't state that those two are equally valid and can be separated into two sets of one four index and one two index [6] equations. It means that I need to use whole field equations or trace reversed ones, that capture four index dynamics rather than using separated equations. It means that connections between those two equations two index [6] and four index does not mean that they are exactly same equations.

A.5 Extensions to complex spacetime

A.5.1 Flat complex spacetime and SU(1,n) symmetry

Motivation behind complex spacetime is mostly here a more general symmetry that is fulfilled with flat spacetime. That symmetry will be in general case with one complex time dimension and n spatial dimensions [9] $SU(1,n)$ symmetry. Most important case is four dimensional complex spacetime that fulfills $SU(1,3)$ symmetry. This symmetry is of quarks in standard model [9]. I will write a flat spacetime metric in four dimensional case as:

$$ds^2 = \eta_{\mu\bar{\nu}} dz^\mu d\bar{z}^\nu \tag{A.63}$$

That can be expanded:

$$ds^2 = dz^0 d\bar{z}^0 - dz^1 d\bar{z}^1 - dz^2 d\bar{z}^2 - dz^3 d\bar{z}^3 \tag{A.64}$$

Where I did use mostly minus metric signature [6], this transforms under $SU(1,3)$ [9] transformations that I will denote U. This means that metric is invariant under those transformations:

$$ds^2 = ds'^2 \tag{A.65}$$

$$\eta_{\mu\bar{\nu}} dz^\mu d\bar{z}^\nu = \eta_{\mu'\bar{\nu}'} U^{\mu'}_{\alpha} dz^\alpha U^{\bar{\nu}'}_{\bar{\beta}} d\bar{z}^\beta \tag{A.66}$$

A.5.2 Einstein tensor

In complex Kähler geometry [7] Einstein tensor will be defined as:

$$G_{\mu\bar{\nu}} = R_{\mu\bar{\nu}} - g_{\mu\bar{\nu}} R \tag{A.67}$$

It is divergence free as normal Einstein tensor [7]:

$$\nabla^\mu G_{\mu\bar{\nu}} = 0 \tag{A.68}$$

I will pair it with a complex energy tensor that is defined as divergence free:

$$\nabla^\mu T_{\mu\bar{\nu}} = 0 \tag{A.69}$$

Now from it I can write a field equation for a complex spacetime [7]:

$$G_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} \tag{A.70}$$

A.5.3 Four index Einstein tensor

From using equations [8] [7] in complex geometry I can write a four index Einstein tensor as:

$$G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[R_{\alpha\bar{\beta}\gamma\bar{\delta}} - \left(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}\right) + g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}R\right] \tag{A.71}$$

Now I can check its four contractions:

$$g^{\alpha\bar{\beta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[R_{\gamma\bar{\delta}} - (nR_{\gamma\bar{\delta}} + Rg_{\gamma\bar{\delta}}) + nRg_{\gamma\bar{\delta}}\right] = R_{\gamma\bar{\delta}} - Rg_{\gamma\bar{\delta}} \tag{A.72}$$

$$g^{\gamma\bar{\beta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{n-1}\left(R_{\alpha\bar{\delta}} - Rg_{\alpha\bar{\delta}}\ \right) \tag{A.73}$$

$$g^{\alpha\bar{\delta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{n-1}\left(R_{\gamma\bar{\beta}} - Rg_{\gamma\bar{\beta}}\ \right) \tag{A.74}$$

$$g^{\gamma\bar{\delta}}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[R_{\alpha\bar{\beta}} - (nR_{\alpha\bar{\beta}} + Rg_{\alpha\bar{\beta}}) + nRg_{\alpha\bar{\beta}}\right] = R_{\alpha\bar{\beta}} - Rg_{\alpha\bar{\beta}} \tag{A.75}$$

Two of contractions will always give constants that can't be removed so this tensor is not fully reduced to Einstein tensor but still it is proportional to it. Only change is in dimensions two where those are equal to Einstein tensor as two minus one gives one. From fact that it is proportional to it it will be still divergence free.

A.5.4 Decomposition of Riemann tensor

I can decompose Riemann tensor in complex spacetime as [7] [8]:

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\delta}} = B_{\alpha\bar{\beta}\gamma\bar{\delta}} &+ \frac{1}{n+2}\left(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + R_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + R_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right) \\ &- \frac{1}{(n+1)(n+2)}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)R \end{aligned} \tag{A.76}$$

Now I can plug it into definition of four index Einstein tensor:

$$\begin{aligned} G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\Bigg[&B_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{n+1}{n+2}\left(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}\right) + \frac{1}{n+2}\left(R_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + R_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right) \\ &+ \left(\frac{n^2+3n+1}{(n+1)(n+2)}g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} - \frac{1}{(n+1)(n+2)}g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)R\Bigg] \end{aligned} \tag{A.77}$$

This tensor is divergence free [8]:

$$\nabla^{\alpha}G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[\nabla_{\bar{\delta}}R_{\gamma\bar{\beta}} - \left((\nabla_{\bar{\beta}}R)g_{\gamma\bar{\delta}} + \nabla_{\bar{\beta}}R_{\gamma\bar{\delta}}\right) + (\nabla_{\bar{\beta}}R)g_{\gamma\bar{\delta}}\right] = 0 \tag{A.78}$$

A.5.5 Four index energy tensor

I can use simple trick to turn four index Einstein tensor into a energy part, I will turn Bochner [7] tensor into gravity energy tensor and group parts of Ricci tensor matching

Einstein tensor and final result is:

$$T_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-n}\left[\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{n+1}{n+2}\left(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}\right) + \frac{1}{n+2}\left(T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right)\right.$$
$$\left. + \frac{1}{(1-n)(n+1)(n+2)}\left[-(n^2+n+1)g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + (2n+1)g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right]T\right] \tag{A.79}$$

This gives final field equations in complex spacetime:

$$\boxed{G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa T_{\alpha\bar{\beta}\gamma\bar{\delta}}} \tag{A.80}$$

That always lead to Einstein field equations regardless of contractions. It means that each of four contractions give two index Einstein field equations and both four index tensor and it's contractions are divergence free:

$$\nabla^{\alpha} G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \nabla^{\alpha} T_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0 \tag{A.81}$$

A.5.6 Kähler geometry geodesic equation and its deviation

In Kähler geometry [7] geodesic equation takes form of two equations:

$$\frac{d^2 z^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dz^{\alpha}}{ds}\frac{dz^{\beta}}{ds} = 0 \tag{A.82}$$

$$\frac{d^2 z^{\bar{\mu}}}{ds^2} + \Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}}\frac{dz^{\bar{\alpha}}}{ds}\frac{dz^{\bar{\beta}}}{ds} = 0 \tag{A.83}$$

Geodesic deviation takes form of:

$$\frac{D^2 V^{\alpha}}{ds^2} + R^{\alpha}_{\beta\delta\bar{\gamma}} T^{\beta} V^{\delta} T^{\bar{\gamma}} + R^{\alpha}_{\beta\bar{\delta}\gamma} T^{\beta} V^{\bar{\delta}} T^{\gamma} = 0 \tag{A.84}$$

A.5.7 Trace reversed field equations

I can use definition of Riemann tensor to change it into trace reversed field equations. I will start by writing normal two index [6] [7] trace reversed field equations:

$$g^{\mu\bar{\nu}}\left(R_{\mu\bar{\nu}} - g_{\mu\bar{\nu}}R\right) = \kappa g^{\mu\bar{\nu}} T_{\mu\bar{\nu}} \tag{A.85}$$

$$(1-n)\,R = \kappa T \tag{A.86}$$

$$R = \frac{\kappa}{1-n} T \tag{A.87}$$

$$R_{\mu\bar{\nu}} = \kappa\left(T_{\mu\bar{\nu}} + \frac{1}{1-n} g_{\mu\bar{\nu}} T\right) \tag{A.88}$$

Then I can use definition of Riemann tensor and change Ricci tensor and scalar parts with gravity energy part:

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa\left(\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{n+2}\left(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}\right)\right.$$
$$\left. + \frac{2n+1}{(1-n)\,(n+2)\,(n+1)}\left(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\right)T\right) \tag{A.89}$$

From it follows that I can write geodesic deviation in terms of only energy tensor parts:

$$\frac{D^2 V^\alpha}{ds^2} + \mathcal{T}^{\alpha}_{\beta\delta\bar{\gamma}} T^\beta V^\delta T^{\bar{\gamma}} + \mathcal{T}^{\alpha}_{\beta\bar{\delta}\gamma} T^\beta V^{\bar{\delta}} T^\gamma = 0 \tag{A.90}$$

$$\begin{aligned}\mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{n+2}\left(T_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}} g_{\alpha\bar{\beta}} + T_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}}\right) \\ + \frac{2n+1}{(1-n)(n+2)(n+1)}\left(g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}}\right) T\end{aligned} \tag{A.91}$$

$$\mathcal{T}^{\alpha}_{\beta\delta\bar{\gamma}} = g^{\alpha\bar{\rho}} \mathcal{T}_{\beta\bar{\rho}\delta\bar{\gamma}} \tag{A.92}$$

$$\mathcal{T}^{\alpha}_{\beta\bar{\delta}\gamma} = g^{\alpha\bar{\rho}} \mathcal{T}_{\beta\bar{\delta}\gamma\bar{\rho}} \tag{A.93}$$

This only lacks cosmological constant to make equations complete.

A.5.8 Adding cosmological constant

Let me start with two index field equations [7] and add cosmological constant to them:

$$R_{\mu\bar{\nu}} - g_{\mu\bar{\nu}} R + g_{\mu\bar{\nu}} \Lambda = \kappa T_{\mu\bar{\nu}} \tag{A.94}$$

Now solve for Ricci scalar with contractions of field equations:

$$(1-n) R + n\Lambda = \kappa T \tag{A.95}$$

$$(1-n) R = \kappa T - n\Lambda \tag{A.96}$$

$$R = \frac{\kappa}{1-n} T - \frac{n}{1-n}\Lambda \tag{A.97}$$

$$R_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} + g_{\mu\bar{\nu}}\left(\frac{\kappa}{1-n} T - \frac{n}{1-n}\Lambda\right) - g_{\mu\bar{\nu}}\Lambda \tag{A.98}$$

$$R_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} + g_{\mu\bar{\nu}}\left(\frac{\kappa}{1-n} T - \left(\frac{n}{1-n} + 1\right)\Lambda\right) \tag{A.99}$$

$$R_{\mu\bar{\nu}} = \kappa T_{\mu\bar{\nu}} + g_{\mu\bar{\nu}}\left(\frac{\kappa}{1-n} T - \frac{1}{1-n}\Lambda\right) \tag{A.100}$$

This will lead plugging it into part with Ricci scalar of trace reversed field equations:

$$\left(-\frac{2}{(1-n)(n+2)} + \frac{n}{(1-n)(n+1)(n+2)}\right)\left(g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}}\right)\Lambda \tag{A.101}$$

$$-\frac{1}{(1-n)(n+1)}\left(g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}}\right)\Lambda \tag{A.102}$$

So whole trace reversed field equations take final form:

$$\boxed{R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{(1-n)(n+1)}\left(g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}}\right)\Lambda} \tag{A.103}$$

A.6 Summary

In this work I did present four index Einstein field equations [6] and it complex spacetime analog [7]. Those equations are natural extension of two index equations [6] [7] where they give energy to vacuum itself so gravity field. Equations can lead to new dynamics as they use new energy term that is connected to matter field [6] but still independent of it. It's connected by conservation laws and it's trace-less part of energy tensor that in vacuum is equal to Weyl tensor [6]. Those are natural and simplest way to write two index field equations [6] into four index formalism. Question do they differ from two index [6] equations when solved still is not showed in this paper, but in principle they should differ specially when there are two things assumed, matter field stays finite and from it follows vacuum energy stays finite that could lead to new solutions that are singularity free. This follows from fact that in conserved version of equations Weyl tensor [6] and vacuum energy tensor has a negative contribution to field equations. Key future of equations is that even it's a classical theory gravity field has a energy that it carries. In future work it should be checked is there a way to generate a solutions that are absent in two index equations or in two index equations [6] are non physical that are free of singularity problem. In vacuum and in only matter field those equations reduce to two index equations and do not bring anything new. Only interaction between matter field and energy of gravity field gives new dynamics.

CHAPTER B

Exploratory Curvature, Measurement and Orientation Constructions

Source and status. This chapter incorporates the complete scientific body of *Four Index Einstein Field Equations and Quantum Effects*, archival exploratory draft (2026). Its role in the dissertation is: Early normalized-curvature, invariant-energy, orientation and spin constructions. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

In this work I present extensions of Einstein field equations [15] into four index equations. This extensions give as natural result a energy tensor for vacuum thus for gravity field. It's all construct in spirit of two index field equations and in truth does not need any additional assumptions about field equations. Form it follows that it's natural completeness of two index equations not a true extension as it fully defines curvature tensor not only Ricci part of curvature as it happens in two index equations.

Quantum effects are divided into two parts, one is about wave function like object and measurement, next one is about spin as orientation of manifold. Wave function like object is constructed from normalized curvature invariant. That plays role of "probability" of finding object in given volume of spacetime at given interval of time. I did no present direct solutions to those equations or concrete examples where it differs from General Relativity [15].

B.1 Einstein field equations

B.1.1 Einstein tensor

Einstein tensor is basis of General Relativity [15], from it arises left side of field equations that connects matter field represented by stress momentum tensor [15] with geometry part. That geometry part is just Einstein tensor that comes from Bianachi identities [15]. It can be written as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{B.1}$$

It's most important property is that is divergence free [15] so that:

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{B.2}$$

It connects to left side of equations where stress momentum tensor [15] is present.

B.1.2 Stress momentum tensor

Matter sources in General Relativity [15] are all coming from right side of equation that is stress momentum tensor [15] $T_{\mu\nu}$. Units of stress momentum tensor are energy density so from it follows that on left side of equation I have units of curvature that is length to power minus two, thats why if field equations [15] I need to use Einstein constant that is equal to:

$$\kappa = \frac{8\pi G}{c^4} \tag{B.3}$$

Another crucial property of stress momentum tensor is that as on left side I have divergence free tensor , stress momentum tensor is divergence free meaning that matter fields are conserved locally. It means that divergence of stress momentum tensor is zero [15]:

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{B.4}$$

B.1.3 Field equation

That finally leads to field equation that I will call in this work a two index field equation, that is just ordinary Einstein field equation [15]:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{B.5}$$

Idea behind this work is simple extended this equation into four index equation, adding part that is responsible for vacuum energy or more precise gravity field energy.

B.1.4 Cosmological constant

Two index equations used today have additional term [15], that term is cosmological constant or energy of empty space Λ it modifies field equations by adding term with it to field equations [15]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{B.6}$$

In vacuum it will lead to:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{B.7}$$

That solutions will be either de Sitter or anti-de Sitter [15] spacetime depending on sign of cosmological constant.

B.2 Four index Einstein tensor

B.2.1 Ricci part

Riemann tensor [15] can be decomposed into Ricci and Weyl parts:

$$R_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right)$$
$$-\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R \tag{B.8}$$

This decomposition gives after contraction exactly Ricci tensor as part with Weyl vanishes when contracted as it's trace free for all indexes combinations. So isolating Ricci part but adding part that will after contraction reduce exactly to one half of Ricci scalar times metric tensor I will arrive at:

$$\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right)$$
$$-\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R - \frac{1}{2(n-1)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R \tag{B.9}$$

I can sum terms with Ricci scalar and will arrive at final expression for first component of four index Einstein tensor:

$$\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\mu\nu}g_{\alpha\beta} - R_{\beta\mu}g_{\alpha\nu}\right)$$
$$-\frac{n}{2(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)R \tag{B.10}$$

B.2.2 Divergence of Ricci part

From it follows that I can write divergence of this component with respect to first index where I will use Bianachi identities for same indexes of Ricci tensor and divergence index so index α to make equations simpler:

$$\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\nabla^{\alpha}R_{\alpha\beta}g_{\mu\nu} - \nabla^{\alpha}R_{\alpha\nu}g_{\beta\mu} + \nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right)$$
$$-\frac{n}{2(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R \tag{B.11}$$

$$\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right)$$
$$-\frac{n}{2(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R + \frac{1}{2(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R \tag{B.12}$$

$$\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right) - \frac{1}{2(n-1)(n-2)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R \tag{B.13}$$

Taking out common denominator I will arrive at final expression:

$$\nabla^{\alpha}\tilde{R}_{\alpha\mu\beta\nu} = \frac{1}{n-2}\left(\left(\nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\beta\mu}\right) - \frac{1}{2(n-1)}\left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right)R\right) \tag{B.14}$$

B.2.3 Weyl part

Weyl [15] part divergence is equal to:

$$\nabla^{\alpha} C_{\alpha\mu\beta\nu} = \frac{n-3}{n-2} \left(\left(\nabla_{\beta} R_{\mu\nu} - \nabla_{\nu} R_{\beta\mu} \right) - \frac{1}{2\,(n-1)} \left(g_{\mu\nu} \nabla_{\beta} - g_{\beta\mu} \nabla_{\nu} \right) R \right) \tag{B.15}$$

This means that they differ only by a constant $n-3$, from it follows to create a field equation I need first to take that constant and divide Weyl tensor by it. Then whole tensor has to be divergence free, from it follows that I will take Ricci part and subtract Weyl dived by that constant:

$$\tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3} C_{\alpha\mu\beta\nu} \tag{B.16}$$

B.2.4 Four index Einstein tensor

Last expression is just four index Einstein tensor, it all properties of being divergence free and it's contractions lead to two index equations. This tensor can be written:

$$G_{\alpha\mu\beta\nu} = \tilde{R}_{\alpha\mu\beta\nu} - \frac{1}{n-3} C_{\alpha\mu\beta\nu} \tag{B.17}$$

B.3 Four index energy tensor

B.3.1 Matter part

I can use same form as in Riemann decomposition into matter part of total energy tensor. This tensor will reduce to stress momentum tensor so I can write it:

$$\begin{aligned} \tilde{T}_{\alpha\mu\beta\nu} = {} & \frac{1}{n-2} \left(T_{\alpha\beta} g_{\mu\nu} - T_{\alpha\nu} g_{\beta\mu} + T_{\mu\nu} g_{\alpha\beta} - T_{\beta\mu} g_{\alpha\nu} \right) \\ & - \frac{1}{(n-1)\,(n-2)} \left(g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\nu} g_{\beta\mu} \right) T \end{aligned} \tag{B.18}$$

Now key property is to calculate divergence of this tensor with respect to first index as before, I can use property that stress momentum tensor is divergence free:

$$\nabla^{\alpha} \tilde{T}_{\alpha\mu\beta\nu} = \frac{1}{n-2} \left(\nabla_{\beta} T_{\mu\nu} - \nabla_{\nu} T_{\beta\mu} \right) - \frac{1}{(n-1)\,(n-2)} \left(g_{\mu\nu} \nabla_{\beta} - g_{\beta\mu} \nabla_{\nu} \right) T \tag{B.19}$$

Taking out common denominator:

$$\nabla^{\alpha} \tilde{T}_{\alpha\mu\beta\nu} = \frac{1}{n-2} \left(\left(\nabla_{\beta} T_{\mu\nu} - \nabla_{\nu} T_{\beta\mu} \right) - \frac{1}{(n-1)} \left(g_{\mu\nu} \nabla_{\beta} - g_{\beta\mu} \nabla_{\nu} \right) T \right) \tag{B.20}$$

That is exactly propagation equation [15] from two index equations, where it does differ by same constant.

B.3.2 Vacuum energy part

From previous equations I can see a logic here, vacuum energy tensor divergence will be have to be equal to same as matter part times a constant defined before:

$$\nabla^\alpha \hat{T}_{\alpha\mu\beta\nu} = \frac{n-3}{n-2}\left((\nabla_\beta T_{\mu\nu} - \nabla_\nu T_{\beta\mu}) - \frac{1}{(n-1)} (g_{\mu\nu}\nabla_\beta - g_{\beta\mu}\nabla_\nu) T \right) \tag{B.21}$$

This tensor has to be trace free for all indexes just as Weyl tensor. That means if it has symmetries of Riemann tensor:

$$g^{\alpha\beta}\hat{T}_{\alpha\mu\beta\nu} = 0 \tag{B.22}$$

In vacuum solutions it just equal Weyl tensor, but I need to put Einstein constant [15] in it.

B.3.3 Four index energy tensor

Putting this all in one equation and one tensor I will use same logic from before so I arrive at total energy tensor:

$$T_{\alpha\mu\beta\nu} = \tilde{T}_{\alpha\mu\beta\nu} - \frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu} \tag{B.23}$$

This tensor is divergence free as part with geometry, it has a trace part (stress momentum tensor) and trace-less part (vacuum energy tensor).

B.4 Four index field equations

B.4.1 Equation

Combining all from before I arrive at four index equation:

$$G_{\alpha\mu\beta\nu} = \kappa T_{\alpha\mu\beta\nu} \tag{B.24}$$

Equation is divergence free:

$$\nabla^\alpha G_{\alpha\mu\beta\nu} = \kappa \nabla^\alpha T_{\alpha\mu\beta\nu} = 0 \tag{B.25}$$

It's contraction leads to two index equation [15]:

$$g^{\alpha\beta} G_{\alpha\mu\beta\nu} = \kappa g^{\alpha\beta} T_{\alpha\mu\beta\nu} \tag{B.26}$$

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{B.27}$$

It has trace and trace-less parts that are equal to each other:

$$\kappa \tilde{T}_{\alpha\mu\beta\nu} = \tilde{R}_{\alpha\mu\beta\nu} \tag{B.28}$$

$$\kappa \hat{T}_{\alpha\mu\beta\nu} = C_{\alpha\mu\beta\nu} \tag{B.29}$$

B.4.2 Adding cosmological constant

I can add back to equations part with cosmological constant [15] it will be term with metric tensors and scalar, I can write that term:

$$\frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda \tag{B.30}$$

So final field equations are:

$$\boxed{G_{\alpha\mu\beta\nu}+\frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda=\kappa T_{\alpha\mu\beta\nu}} \tag{B.31}$$

If cosmological constant [15] is just a fixed number it's divergence vanishes but if not, it has to fulfill this equations:

$$\left(g_{\mu\nu}\partial_{\beta}-g_{\beta\mu}\partial_{\nu}\right)\Lambda=0 \tag{B.32}$$

And from two index equations it needs to vanish when contracted so it leads to:

$$\partial_{\nu}\Lambda=0 \tag{B.33}$$

It shows that it has to be a constant number that does not change with any spatial or a temporal direction. Otherwise equations will be not divergence free in both four index and two index [15]. After contraction it will just lead to standard form two index equation [15]:

$$G_{\mu\nu}+g_{\mu\nu}\Lambda=\kappa T_{\mu\nu} \tag{B.34}$$

B.4.3 Action formulation of equations

I can formulate field equations in as action [21] , derivation was already done in this paper [21], where there is same structure of field equations used. After adding tensors I will arrive at starting with gravity field:

$$S_G=\frac{1}{2\kappa c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}\left(G_{\alpha\mu\beta\nu}+\lambda_{\alpha\mu\beta\nu}\right)d^D\mathbf{x} \tag{B.35}$$

Where those tensors are defined as before so:

$$S_G=\frac{1}{2\kappa c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}\left(\tilde{R}_{\alpha\mu\beta\nu}-\frac{1}{n-3}C_{\alpha\mu\beta\nu}+\frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right)d^D\mathbf{x} \tag{B.36}$$

Now matter field part behaves same way but without term with Einstein constant [21]:

$$S_M=-\frac{1}{c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}T_{\alpha\mu\beta\nu}d^D\mathbf{x} \tag{B.37}$$

That can be expanded:

$$S_M=-\frac{1}{c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}\left(\tilde{T}_{\alpha\mu\beta\nu}-\frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right)d^D\mathbf{x} \tag{B.38}$$

Adding them both into one integral [21]:

$$S=\frac{1}{2\kappa c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}\left(G_{\alpha\mu\beta\nu}+\lambda_{\alpha\mu\beta\nu}\right)d^{D}\mathbf{x}-\frac{1}{c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}T_{\alpha\mu\beta\nu}d^{D}\mathbf{x} \tag{B.39}$$

$$S=\frac{1}{c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}\left[\frac{1}{2\kappa}\left(G_{\alpha\mu\beta\nu}+\lambda_{\alpha\mu\beta\nu}\right)-T_{\alpha\mu\beta\nu}\right]d^{D}\mathbf{x} \tag{B.40}$$

That can be written in whole expanded form as:

$$S=\frac{1}{c}\int\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}\left[\frac{1}{2\kappa}\left(\tilde{R}_{\alpha\mu\beta\nu}-\frac{1}{n-3}C_{\alpha\mu\beta\nu}+\frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right)\right.$$
$$\left.-\left(\tilde{T}_{\alpha\mu\beta\nu}-\frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right)\right]d^{D}\mathbf{x} \tag{B.41}$$

B.4.4 Meaning of four index field equation

Four index field equation is natural extension of two index equation, what is new in this field equation that there is new tensor that is responsible for vacuum energy. This tensor is energy of matter field in vacuum. It means that it's an extension of matter field into vacuum and it consists of physical matter. This matter is not exactly like normal matter field, it is trace-less part of total energy, but still it's matter field.

This matter field is carrying energy of matter field into the vacuum, it's simplest candidate for gravity particle. From it follows most important part, in this model gravity as particle carrying energy is not need to quantize field to arrive at it- it's done in pure classical terms. And crucial is that energy of that particle being equal to Weyl tensor is an equation to be solved not identity, that leads to two equations but what is important are not two separate equations but come from a tensor that is created when they are combined:

$$G_{\alpha\mu\beta\nu}+\frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda=\kappa T_{\alpha\mu\beta\nu} \tag{B.42}$$

$$\tilde{R}_{\alpha\mu\beta\nu}-\frac{1}{n-3}C_{\alpha\mu\beta\nu}+\frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda=\kappa\left(\tilde{T}_{\alpha\mu\beta\nu}-\frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right) \tag{B.43}$$

As this tensor is a divergence free object. It leads to simple question, why there is a minus sign in the parts with vacuum energy and Weyl tensor? Answer to that question is that otherwise two sides would not be divergence free. This gives a physical meaning to it, total tensor takes all inputs from vacuum as negative in sign. Still this is not a final story as it lacks any kind of quantum effects.

B.4.5 Trace-reversed field equation

I can arrive at trace reversed field equations, first I take field equation:

$$\tilde{R}_{\alpha\mu\beta\nu}-\frac{1}{n-3}C_{\alpha\mu\beta\nu}+\frac{1}{n-1}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda=\kappa\left(\tilde{T}_{\alpha\mu\beta\nu}-\frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}\right) \tag{B.44}$$

Expand left side and move cosmological constant part to right side of equations:

$$\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu}-R_{\alpha\nu}g_{\beta\mu}+R_{\mu\nu}g_{\alpha\beta}-R_{\beta\mu}g_{\alpha\nu}\right)$$

$$-\frac{n}{2(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)R-\frac{1}{n-3}C_{\alpha\mu\beta\nu} \tag{B.45}$$

Compare it with Riemann tensor expression:

$$R_{\alpha\mu\beta\nu}=C_{\alpha\mu\beta\nu}+\frac{1}{n-2}\left(R_{\alpha\beta}g_{\mu\nu}-R_{\alpha\nu}g_{\beta\mu}+R_{\mu\nu}g_{\alpha\beta}-R_{\beta\mu}g_{\alpha\nu}\right)$$
$$-\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)R \tag{B.46}$$

I can express now it in terms of Riemann tensor:

$$G_{\alpha\mu\beta\nu}=R_{\alpha\mu\beta\nu}-\frac{1}{2(n-1)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)R-\left(1+\frac{1}{n-3}\right)C_{\alpha\mu\beta\nu} \tag{B.47}$$

Use two index equations [15] contractions and fact that Weyl tensor is equal to energy tensor of vacuum and move all that is not Riemann tensor to right side:

$$R_{\alpha\mu\beta\nu}=\kappa\left(T_{\alpha\mu\beta\nu}-\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)T+\left(1+\frac{1}{n-3}\right)\hat{T}_{\alpha\mu\beta\nu}\right) \tag{B.48}$$

Expand components of total energy tensor:

$$R_{\alpha\mu\beta\nu}=\kappa\left(\tilde{T}_{\alpha\mu\beta\nu}-\frac{1}{n-3}\hat{T}_{\alpha\mu\beta\nu}-\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)T\right)$$
$$+\kappa\left(1+\frac{1}{n-3}\right)\hat{T}_{\alpha\mu\beta\nu} \tag{B.49}$$

Sort out components first of vacuum energy tensor:

$$R_{\alpha\mu\beta\nu}=\kappa\left(\tilde{T}_{\alpha\mu\beta\nu}+\hat{T}_{\alpha\mu\beta\nu}-\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)T\right) \tag{B.50}$$

Then matter energy tensor and will arrive at final expression and add cosmological constant again:

$$R_{\alpha\mu\beta\nu}=\kappa\left(\tilde{T}_{\alpha\mu\beta\nu}+\hat{T}_{\alpha\mu\beta\nu}-\frac{1}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}\right)T\right.$$
$$\left.+\frac{2}{\kappa(n-1)(n-2)}\left(g_{\mu\nu}g_{\alpha\beta}-g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) \tag{B.51}$$

B.4.6 Geodesic deviation and geodesic equation

From fact that I can write Riemann tensor in terms of energy tensors I can go one step forward and write whole equation as a geodesic deviation equation [22] , where $\mathbf{V}$ is vector tangent to geodesic curve [22]:

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\xi^{\alpha}=R^{\alpha}_{\ \mu\beta\nu}V^{\mu}V^{\beta}\xi^{\nu} \tag{B.52}$$

Now I can switch right side of equation into energy instead of curvature parts:

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\xi^{\alpha}=\kappa\left(\tilde{T}^{\alpha}_{\ \mu\beta\nu}+\hat{T}^{\alpha}_{\ \mu\beta\nu}-\frac{1}{(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\alpha}_{\beta}-g_{\beta\mu}\delta^{\alpha}_{\nu}\right)T\right.$$

$$+\frac{2}{\kappa(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\alpha}_{\beta}-g_{\beta\mu}\delta^{\alpha}_{\nu}\right)\Lambda\Big)V^{\mu}V^{\beta}\xi^{\nu} \tag{B.53}$$

That leads to most general interpretation of field equations in geometric way as it is interpretation of Riemann tensor itself [23]. In general geodesic deviation says how "fast" geodesic [22] accelerate with respect to each other. They can either converge or diverge. Field equations will not lead to singularities only if both side of equations stay finite, from fact that Im defining Riemann tensor that encodes whole curvature I can impose that right side with energy content stays finite in whole manifold then solve for left side of equation. This gives control over singularities that is absent in two index equation [15]. If geodesic deviation acts on global spacetime, geodesic equation [15] [22] acts locally when effects of particle or systems of particles is small compared to source of gravity field. Geodesic equation [22] can be written as:

$$\frac{d^2x^{\mu}}{ds^2}+\Gamma^{\mu}_{\ \alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds}=0 \tag{B.54}$$

This equations in truth works only if I don't take a field as whole but a isolated part of a field into account. Field as whole is generated from curvature itself. It leads to idea that object not only follow locally geodesic equation but field in general has obey geodesic deviation equation [22] that is more general statement of gravity field as it has whole curvature as a source. So locally geodesic equation is still valid but in truth more general statement about field is geodesic deviation equation [22] [23]. In truth field should be considered as a general object, that's why geodesic equation [22] is not whole story. Energy makes geodesic accelerate towards each other or away from each other, this happens both in case of matter field or gravity field energy. Gravity field should be thought then in geometric sense [23] as focusing or un-focusing of geodesics. As long as both side of equations stay finite this leads to well defined curvature and from it follows field.

B.4.7 Field equations as connection with two index equations

I can re-write field equations using two equations that are separated, one equation is just two index field equation [15] written in four index form and one deals with Weyl tensor [15]:

$$\tilde{R}_{\alpha\mu\beta\nu}=\kappa\tilde{T}_{\alpha\mu\beta\nu} \tag{B.55}$$

$$C_{\alpha\mu\beta\nu}=\kappa\hat{T}_{\alpha\mu\beta\nu} \tag{B.56}$$

As first equation can be reduced to two index equation [15] I can re-write field equations as:

$$G_{\mu\nu}=\kappa T_{\mu\nu} \tag{B.57}$$

$$C_{\alpha\mu\beta\nu}=\kappa\hat{T}_{\alpha\mu\beta\nu} \tag{B.58}$$

Where there is connection between two index equations and four index equations, this connection is covariant derivative. When I do take covariant derivative of vacuum energy tensor I need to arrive exactly at same expression as for energy momentum part:

$$\nabla^{\alpha}\hat{T}_{\alpha\mu\beta\nu}=\nabla^{\alpha}\tilde{T}_{\alpha\mu\beta\nu} \tag{B.59}$$

At first it may seem as it's just propagation equation [15] from two index equation [15] but key here is that vacuum energy tensor is equation to be solved not an identity. It means that it agrees with two index equation [15] in form of propagation equation [15]:

$$\nabla^{\alpha} C_{\alpha\mu\beta\nu} = \kappa \nabla^{\alpha} \hat{T}_{\alpha\mu\beta\nu} \tag{B.60}$$

$$\begin{aligned} &\frac{n-3}{n-2}\left(\left(\nabla_{\beta} R_{\mu\nu} - \nabla_{\nu} R_{\beta\mu}\right) - \frac{1}{2\,(n-1)} \left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right) R \right) = \\ &\frac{n-3}{n-2}\kappa \left(\left(\nabla_{\beta} T_{\mu\nu} - \nabla_{\nu} T_{\beta\mu}\right) - \frac{1}{(n-1)} \left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right) T \right) \end{aligned} \tag{B.61}$$

$$\begin{aligned} &\left(\nabla_{\beta} R_{\mu\nu} - \nabla_{\nu} R_{\beta\mu}\right) - \frac{1}{2\,(n-1)} \left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right) R = \\ &\kappa \left(\left(\nabla_{\beta} T_{\mu\nu} - \nabla_{\nu} T_{\beta\mu}\right) - \frac{1}{(n-1)} \left(g_{\mu\nu}\nabla_{\beta} - g_{\beta\mu}\nabla_{\nu}\right) T \right) \end{aligned} \tag{B.62}$$

It means that change in energy tensor of vacuum is connected to change in matter field, or stating opposite change of vacuum energy tensor is connected to change of matter field. From it follows that if vacuum energy stays finite there is need for matter field to stay finite if I started from finite values. From it follows that each change in gravity field is connected to change in matter field and matter field change are connected to gravity field, if those changes remain finite whole equations will remain finite. It's imposing on two index equations [15] that energy momentum tensor [15] stays finite is equally imposing on vacuum energy to stay finite from it follows Weyl tensor so gravity field itself. This separation of field equations shows how on one hand it's close to two index equations [15] and where it does differ. As to create a whole field equations I need to use four index formalism. It follows from conservation equations that are written in four index formalism. So in truth I can't state that those two are equally valid and can be separated into two sets of one four index and one two index [15] equations. It means that I need to use whole field equations or trace reversed ones, that capture four index dynamics rather than using separated equations. It means that connections between those two equations two index [15] and four index does not mean that they are exactly same equations.

B.5 Quantum effects form geometry

B.5.1 Is there matter field or mater particles?

In physics there are two notions of matter, matter as field and matter as a particle. I will postulate that matter field is more of a field than of a distinct particles that can be independent of field itself. Let me start by in general I can think of gravity as a particle (source) and it's field generated by that particle (effect in space). It does work this way for all other forces, my assumption is that matter field and effect of that matter field (gravity field) both are same thing but affecting two kinds of curvature effects. Matter field affects Ricci curvature [15] and, gravity affects Weyl curvature [15]. But both are forms of same field and both carry energy. Key question would be can one be turn into another? First in static solutions there is clear separation between gravity field and matter field. But in general non-static solutions it's possible to have this kind of separation. If Weyl curvature [15] mixes with Ricci curvature [15], I have both matter field and gravity energy in same point of spacetime acting as one field. This is key insight into understanding how quantum like effects emerge from pure geometry of spacetime. Gravity field is in truth not localized in a point, same as wave function in quantum mechanics. It's spread across all space. I will assume that this spreading of gravity field is not only hint of how to get into quantum effects only from pure geometry but possibly a hint about another path, instead of quantizing gravity, trying to build model that uses only gravity to explain quantum effects. First I will start by a simplest construction that is normalized curvature scalar.

B.5.2 Normalized curvature scalar

From trace revered field equations I can directly calculate curvature scalar [15] $K = R_{\alpha\mu\beta\nu}R^{\alpha\mu\beta\nu}$ using only energy parts. I will write it as:

$$R_{\alpha\mu\beta\nu} = \kappa \left(\hat{T}_{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(T_{\alpha\beta}g_{\mu\nu} - T_{\alpha\nu}g_{\beta\mu} + T_{\mu\nu}g_{\alpha\beta} - T_{\beta\mu}g_{\alpha\nu}\right)\right.$$
$$\left. -\frac{2}{(n-1)(n-2)}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)T + \frac{2}{(n-1)(n-2)}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\beta\mu}g_{\alpha\nu}\right)\Lambda\right) \tag{B.63}$$

$$R^{\alpha\mu\beta\nu} = \kappa \left(\hat{T}^{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(T^{\alpha\beta}g^{\mu\nu} - T^{\alpha\nu}g^{\beta\mu} + T^{\mu\nu}g^{\alpha\beta} - T^{\beta\mu}g^{\alpha\nu}\right)\right.$$
$$\left. -\frac{2}{(n-1)(n-2)}\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\beta\mu}\right)T + \frac{2}{(n-1)(n-2)}\left(g^{\mu\nu}g^{\alpha\beta} - g^{\beta\mu}g^{\alpha\nu}\right)\Lambda\right) \tag{B.64}$$

$$K(x) = R_{\alpha\mu\beta\nu}R^{\alpha\mu\beta\nu} \tag{B.65}$$

This will lead to last final expression for normalized curvature scalar:

$$\alpha = \int_{ct_A}^{ct_B} \int_{\Sigma_t} \sqrt{-g}K(x)\, d^{n-1}x dx^0 \tag{B.66}$$

$$N(x) = \frac{1}{\alpha} \int_{ct_A}^{ct_B} \int_{V^{n-1}} \sqrt{-g}K(x)\, d^{n-1}x dx^0 \tag{B.67}$$

Where key here is that this scalar represents total curvature and comes from trace reversed field equations. It plays role like wave function in quantum mechanics. With change that instead of calculating probability of finding particle in given volume there is curvature density in given volume compared to total curvature. That curvature field has simple interpretation it's density of field compared with some region of space and time. Formally it says how much curvature affects given region of spacetime compared to total curvature. Key here is that it has to be finite in order to make this integral possible so all singular solutions are discarded.

B.5.3 Non-normalized curvature scalar

From it I can create just normal integral that is not normalized, this integral will have direct meaning as if there is enough curvature in given volume of space and given time interval I can detect particle. First I will define a particle focus region where most of curvature of particle or system of particles is focused then will need for object to be measured only if flow with some region of space and given interval of time is equal to that value of curvature:

$$N_0 = \int_{ct_a}^{ct_b} \int_{V_0^{n-1}} \sqrt{-g}K(x)\, d^{n-1}x dx^0 \tag{B.68}$$

Where there is not only focused volume V_0^{n-1} but time focused time interval $\Delta t = t_b - t_a$. Now to detect particle there has to be at least that amount of curvature, it means when I do measurement in given volume and time interval particle can be only detected if this curvature in this region is greater or equal to that value:

$$N_0 \leq \int_{ct_A}^{ct_B} \int_{V^{n-1}} \sqrt{-g} K\left(x\right) d^{n-1}x dx^0 = \alpha N(x) \tag{B.69}$$

To detect particle or system of particles I need at least N_0 value of curvature. Detecting particle acts as focusing of curvature into some volume of space in given time interval. Particle will be detected only if there is enough curvature in focusing region on the detector. Focusing comes from interaction of curvature field with other objects or even with detector itself.

B.5.4 How well I can measure objects

More precise I do measure objects position less less volume goes into integral, from it follows less total curvature I do take into integral. Smaller the time interval of measurement it too makes curvature less focused. It means that If i want to measure position precise I need to take into account big interval of time, and opposite fast measurements need big amount of space im measuring. This can be expressed that If I take small volume of space need to take big interval of time to detect a particle:

$$N_0 \leq \int_{ct_1}^{ct_2} \int_{\epsilon^{n-1}} \sqrt{-g} K\left(x\right) d^{n-1}x dx^0 \tag{B.70}$$

$$\int_{ct_1}^{ct_2} \sqrt{-g} K\left(x\right) dx^0 \gg \int_{\epsilon^{n-1}} \sqrt{-g} K\left(x\right) d^{n-1}x \tag{B.71}$$

And opposite if I take small time interval I need big volume of space:

$$N_0 \leq \int_{ct_{\epsilon_1}}^{ct_{\epsilon_2}} \int_{V^{n-1}} \sqrt{-g} K\left(x\right) d^{n-1}x dx^0 \tag{B.72}$$

$$\int_{V^{n-1}} \sqrt{-g} K\left(x\right) d^{n-1}x \gg \int_{\epsilon_1}^{\epsilon_2} \sqrt{-g} K\left(x\right) dx^0 \tag{B.73}$$

If I do take a small volume of space and small time interval particle becomes hardest to detect.

B.5.5 Energy of system invariant

I can use Einstein constant to arrive at same invariant but now instead of calculating curvature I do calculate total energy density squared. This can be expressed just as:

$$\Phi\left(x\right) = \frac{1}{\kappa^2} N(x) = \frac{1}{\alpha\kappa^2} \int_{ct_A}^{ct_B} \int_{V^{n-1}} \sqrt{-g} K\left(x\right) d^{n-1}x dx^0 \tag{B.74}$$

From it follows that I can write field equations in scalar form as:

$$K\left(x\right) = R_{\alpha\mu\beta\nu} R^{\alpha\mu\beta\nu} \tag{B.75}$$

$$\Phi\left(x\right) = \left(\hat{T}^{\alpha\mu\beta\nu} + \frac{1}{n-2}\left(T^{\alpha\beta}g^{\mu\nu} - T^{\alpha\nu}g^{\beta\mu} + T^{\mu\nu}g^{\alpha\beta} - T^{\beta\mu}g^{\alpha\nu}\right)\right.$$

$$-\frac{2}{(n-1)(n-2)}\left(g^{\alpha\beta}g^{\mu\nu}-g^{\alpha\nu}g^{\beta\mu}\right)T+\frac{2}{(n-1)(n-2)}\left(g^{\mu\nu}g^{\alpha\beta}-g^{\beta\mu}g^{\alpha\nu}\right)\Lambda\Big)$$

$$\Big(\hat{T}^{\alpha\mu\beta\nu}+\frac{1}{n-2}\left(T^{\alpha\beta}g^{\mu\nu}-T^{\alpha\nu}g^{\beta\mu}+T^{\mu\nu}g^{\alpha\beta}-T^{\beta\mu}g^{\alpha\nu}\right)$$

$$-\frac{2}{(n-1)(n-2)}\left(g^{\alpha\beta}g^{\mu\nu}-g^{\alpha\nu}g^{\beta\mu}\right)T+\frac{2}{(n-1)(n-2)}\left(g^{\mu\nu}g^{\alpha\beta}-g^{\beta\mu}g^{\alpha\nu}\right)\Lambda\Big) \tag{B.76}$$

$$\Phi(x)=\frac{1}{\kappa^2}K(x) \tag{B.77}$$

That scalar form of field equations is invariant form of equations , if it's finite it meaning that both fields stay finite, spacetime has no singularities.

B.6 Spin

B.6.1 Spin as a manifold orientation

Last example is idea that spin is orientation of manifold in space. So for example for spin one half particles I need to use hyper-sphere. What I do mean by orientation of manifold in space? If i take a vector that goes form center of hyper-sphere and points to it's surface I will call it orientation vector. This vector says how this manifold is pointing in space. When manifold rotates it will rotate around this vector. In general there are infinite number of possible orientations on any manifold. When I do measurement I project orientation of manifold onto my measurement axis.

From it follows that I get two possible outcomes, orientation changes to being up or down with measurement axis. As there are only two possible solutions that align with measurement axis that are vectors one is pointing up one is point down with measurement axis. Orientation vector can't be split into smaller vectors as manifold can have only one orientation it explains why there are only two possible states of spin. I will start with simple math let me write black hole solutions [15] from it it can be easy seen that it's a solution of a hyper-sphere:

$$ds^2=-\left(1-\frac{r_s}{r}\right)c^2dt^2+\frac{dr^2}{(1-\frac{r_s}{r})}+r^2d\Omega^2 \tag{B.78}$$

Isolate part with black hole radius so skip part with time components:

$$ds^2=\frac{dr^2}{(1-\frac{r_s}{r})}+r^2d\Omega^2 \tag{B.79}$$

Write hyper-sphere metric with not a constant radius:

$$ds^2=\frac{dr^2}{\left(1-\frac{r^2}{R^2(r)}\right)}+r^2d\Omega^2 \tag{B.80}$$

Where radius of hyper-sphere is defined as $R(r) = \sqrt{\frac{r^3}{r_s}}$ same can be done with inside metric:

$$ds^2 = -\frac{1}{4}\left(3\sqrt{1-\frac{r_s}{r_g}} - \sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2 c^2 dt^2 + \frac{dr^2}{1-\frac{r^2 r_s}{r_g^3}} + r^2 d\Omega^2 \tag{B.81}$$

Where again I take only spatial part without time:

$$\frac{dr^2}{1-\frac{r^2 r_s}{r_g^3}} + r^2 d\Omega^2 \tag{B.82}$$

$$ds^2 = \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2 d\Omega^2 \tag{B.83}$$

Where in this case radius is equal to $R = \sqrt{\frac{r_g^3}{r_s}}$ so it shows clear that those solutions give exactly hyper-sphere so come from matter field with spin one half. Same can be done with de Sitter universe [15] in coordinates that I did use before:

$$ds^2 = -dT^2 + \frac{1}{H^2}\cosh^2(HT)\, d\Omega_3^2 \tag{B.84}$$

Where here whole hyper-sphere does change with time not only radial part. But again I can use another coordinates that will show that is true hyper-sphere , ones used before:

$$ds^2 = -\left(1-\frac{\Lambda r^2}{3}\right)c^2 dt^2 + \left(1-\frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{B.85}$$

Here radius is constant of hyper-sphere and is equal to:

$$R = \sqrt{\frac{3}{\Lambda}} \tag{B.86}$$

So again metric can be written as in spatial parts:

$$ds^2 = \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2 d\Omega^2 \tag{B.87}$$

That shows that two base solutions used in cosmology and black holes are in truth a hypes-sphere solutions so their use matter field with spin one half.

B.6.2 Simple projection of orientation vector model

I will start with only space components of metric as I care only about them. First denote spatial part of metric as h_{ab} and its orientation vector as s^a, that vector length squared is equal to:

$$L^2 = h_{ab} s^a s^b \tag{B.88}$$

Now I need to add measurement axis vector that will be projected onto orientation axis. This will be denoted as v^a and it's length squared is defined same way:

$$L^2 = h_{ab} v^a v^b \tag{B.89}$$

Both have to lie on same sphere or hyper-sphere and have same length equal to its radius. So I can set that:

$$h_{ab} v^a v^b = h_{ab} s^a s^b = L^2 \tag{B.90}$$

Now i create two vectors that are projections of measurement axis vector on spin state, they represent spin up and down, so two orientations projection of orientation vector can take. One state has to be rotated π radians compared to another for sphere so spin one and one has to have state 2π for spin one half. That represents spin up and down states. Total length of final vector is sum of orientation vector with its measurement axis vector normalized by total length first for sphere so spin one:

$$\psi_+ (\theta) = \frac{L + L\cos(\theta)}{2L} \tag{B.91}$$

$$\psi_- (\theta) = \frac{L + L\cos(\pi - \theta)}{2L} \tag{B.92}$$

Then for hyper-sphere so spin one half:

$$\psi_+ \left(\frac{\theta}{2}\right) = \frac{L + L\cos(\frac{\theta}{2})}{2L} \tag{B.93}$$

$$\psi_- \left(\frac{\theta}{2}\right) = \frac{L + L\cos(\pi - \frac{\theta}{2})}{2L} \tag{B.94}$$

This can be re-written using formula for angle between vectors, and lengths using only metric tensors:

$$\cos(\theta) = \frac{h_{ab} s^a v^b}{\sqrt{h_{ab} s^a s^b}\sqrt{h_{ab} v^a v^b}} = \frac{h_{ab} s^a v^b}{L^2} \tag{B.95}$$

Idea is that before measurement orientation vector is random in direction as it's unknown and there is infinite number of possible orientation so there is exactly zero probability of guessing right one, after measurement we project that vector into our measurement axis and get known state of orientation, when we measure another axis there is new projection and so on. So measurement does change orientation state from s^a to $s^{a'}$ with probability equal to $\psi_+ \left(\frac{\theta}{2}\right)$ or $\psi_- \left(\frac{\theta}{2}\right)$ for up or down state and spin one half. Where state up represents orientation vector align with measurement axis and down opposite to it.

B.6.3 Spin state when there is correlation between states

When there is a correlation between states [20] its best candidate to test will this formalism break. I will use correlation between states where angle between them is always anti-correlated [20] it means that if one particle is spin up one is spin down. Here I have two body system [20], but I can use same logic. Now If i take axis between one system and another system as angle θ and there is no correlation [20] will arrive at same equation as before but if

I assume anti-correlation I will arrive at states:

$$\psi_+(\theta) = \frac{L + L\cos(\vartheta + \pi)}{2L} \tag{B.96}$$

$$\psi_-(\theta) = \frac{L + L\cos(\vartheta)}{2L} \tag{B.97}$$

Then for hyper-sphere so spin one half:

$$\psi_+\left(\frac{\theta}{2}\right) = \frac{L + L\cos(\pi + \frac{\vartheta}{2})}{2L} \tag{B.98}$$

$$\psi_-\left(\frac{\theta}{2}\right) = \frac{L + L\cos(\frac{\vartheta}{2})}{2L} \tag{B.99}$$

As first angle is always rotated by π compared to second so I have change in angle equal to π:

$$\phi_+ = \pi + \vartheta \quad \phi_- = \vartheta \tag{B.100}$$

$$\phi_+ - \phi_- = \pi \tag{B.101}$$

I can use inverse correlation [20] but then I need to be aware that those states are now giving opposite relation:

$$\phi_+ = \vartheta \quad \phi_- = \vartheta + \pi \tag{B.102}$$

$$\phi_+ - \phi_- = -\pi \tag{B.103}$$

It means that states are reversed. Up state is now down state and down state is now up state, as I need to flip orientation (from minus sign). In case where there is no correlation angles between are random. So if one vector has angle ϑ another one has $\pi - \vartheta$, as total angle between spin state up and down is equal to π:

$$\vartheta + \pi - \vartheta = \pi \tag{B.104}$$

In case of correlated states of two systems I ask what is correlation [20] of them if I chose random axis to measure them, angle is angle between measurement axis just like in Bell theorem [20]. Now I can again use same logic and get correct answer for correlation between both anti-correlated states and not correlated states:

$$E_{\text{Correlated}}(\vartheta) = \psi_+(\vartheta) - \psi_-(\vartheta) \tag{B.105}$$

$$E_{\text{Correlated}}(\vartheta) = \frac{L + L\cos(\vartheta + \pi)}{2L} - \frac{L + L\cos(\vartheta)}{2L} = \sin^2\left(\frac{\vartheta}{2}\right) - \cos^2\left(\frac{\vartheta}{2}\right)$$

$$= -\cos(\vartheta) \tag{B.106}$$

$$E_{\text{Random}}(\vartheta) = \psi_+(\vartheta) - \psi_-(\vartheta) \tag{B.107}$$

$$E_{\text{Random}}(\vartheta) = \frac{L + L\cos(\vartheta)}{2L} - \frac{L + L\cos(\pi - \vartheta)}{2L} = \cos^2\left(\frac{\vartheta}{2}\right) - \sin^2\left(\frac{\vartheta}{2}\right)$$

$$= \cos(\vartheta) \tag{B.108}$$

That is exactly result one would expect from quantum theory [20] but here it's done only by geometric relation between orientation vectors.

Chapter C

Exploratory Notes on Motion Relative to Spacetime

Source and status. This chapter incorporates the complete scientific body of *Motion Relative to Spacetime*, archival note (2026). Its role in the dissertation is: Conceptual vector-field model for motion and reaction of spacetime. Repeated definitions are retained where they are required for a self-contained derivation; the synthesis chapters distinguish reformulations, model-dependent results, hypotheses and open claims.

Chapter abstract

I did present a model of how gravity field can arise just from pure vector fields transformations that are always relative to spacetime itself. Effects of motion is always connected to opposite reaction of spacetime. Combined those effect give rise gravity field.

C.1 Motion

C.1.1 Motion is relative but to what?

It may seem like a great success of modern physics to explain motion in terms of Minkowski spacetime [53], Einstein special relativity [51] is basis of kinematics in physics. Still it's misleading to say that we understand motion, as gravity field is seen as curvature of spacetime [53] but still there is no mechanical reason to assume that spacetime itself can act on objects. Or saying more precise there is no mechanism that explains what gravity is , but only what are effects of gravity field thus motion of objects. I will try to show that motion of objects is more fundamental that geometry of field itself. Goal of this paper is to show that gravity field is just simply a motion field, objects move in spacetime and spacetime moves in opposite way relative to objects that is crucial statement I will explore. Taking it to it's limits with Minkowski spacetime [53] and it's crucial Lorentz transformations [52] that are key to length contraction and time dilation if applied correctly. Key change is that I do not assume motion is relative to observers but to background spacetime itself that is by itself static but reacts to motion. It's big change with relativity [51] that tries to explain motion in terms of observers perspective not with things that are not frame dependent. I will assume that any observer can transform his coordinates but it does not change how it moves relative to spacetime itself. This means that Lorentz transformations need to be applied correctly. But correctly I mean that I always need to keep im my mind who is in motion and who is not. How observers will see motion of other observers is not a key physical fact but how they really move is.

C.1.2 What is spacetime really?

Minkowski idea about merging space and time into one object [53] may seem revolutionary at first but what hides behind this mathematical structure and what is it's physical meaning is key. Let me start with general idea of space itself I have a sets of coordinates that assign to each point of that space a position in spacetime change is that each position is an event, another key property is that it uses hyperbolic rotations [52] instead of normal spherical ones. Or saying more simply instead of preserving vector length by just normal space distance it adds minus sign to either space or time dimensions. So length that is preserved no longer acts as rotation on a sphere but a hyperbola. This trick makes all good stuff that come from Lorentz transformations [52] being build naturally into spacetime geometry. But problem is when we confuse mathematical tool with reality. Stating that there is a Minkowski metric as base spacetime geometry only from fact that it agrees with physical transformations of object in motion does not mean that it's build into reality itself. It's a tool that is useful in making our mathematical model consistent. Spacetime itself is a tool that explains motion, not the other way around as we can't observe spacetime but we can only observe motion.

C.1.3 Motion is always absolute

With big success or relativity [51] it may seem like motion is relative and Newton views of absolute space and time are demolish. I will give simple objection to this kind of thinking. There is no measurement that can be done relative to something, measurement needs to take into account two factors at least when we are talking about motion one frame is always base frame of reference that we say is frame we measure motion with respect too, same space so rods and clocks. And there can be other frames that are relative to that our base frame. We can for example chose earth centered frame. solar system centered frame or Milky way centered frame but to say that we can measure motion as relative is simply misleading. There always has to be a frame a lets call it a master frame whom relative measurements are taken with respect too.

Source Coverage and Equation-Retention Audit

The table records every supplied source and its role in the dissertation. The displayed-mathematics count is computed from the scientific body of each LaTeX source after front matter and references are removed. All of those bodies are incorporated without deleting displayed equations. The separate FRW document is reconstructed in Chapter 9 because several stored equation objects were incomplete.

Source version	Dissertation location and role	Math blocks
v9 (2026)	Chapter 2: Core rank-four construction in real and complex spacetime	85
v1.13 (15 August 2026)	Chapter 3: Flat and curved four-sector atlas with Schwarzschild, LTB, Kerr and FLRW examples	151
v0.21 (12 August 2026)	Chapter 4: Covariant interpretation of a complete photon state on a selected null geodesic	14
v0.76 (15 August 2026)	Chapter 5: Tensor carrier, rank-eight evolution, probability law and hydrogen construction	92
v0.80 (15 August 2026)	Chapter 6: Ricci-flat graviton sector, helicity, transport, mode resolution and closed Weyl histories	76
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archival note (2026)	Appendix C: Conceptual vector-field model for motion and reaction of spacetime	0
FRW reduction document	Chapter 9: reconstructed regular four-sector FRW solution	24

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