An Elementary Construction for Goldbach's Conjecture: Limit Expressions, Error Analysis, and Additive Sieves

Miles Huang

August 2026

Abstract

This paper synthesizes a series of results on the (even) Goldbach conjecture and completes two previously unaddressed components: **error analysis** and **fractional sequence encoding (additive sieve)**. The main results are as follows. (1) We systematically present limit-type elementary constructions of the Goldbach conjecture (using both exponential and cosine functions) and their equivalent logical formulations, and extend the method to the Collatz conjecture, the twin prime conjecture, and the prime counting function. (2) For the truncated construction with finite error factor k, we derive a rigorous error bound and prove that the criterion holds provided $kn\,e^{-4k/n^2} < \ln 2$ and $k > \ln n$. This determines the **critical exponent for the sufficient condition to be** 2, and rigorously verifies that the empirically discovered choice $k = 10n^e$ ($e \approx 2.718 > 2$) satisfies this sufficient condition for all even $n \geq 4$, yielding a **limit-free elementary closed-form equivalent proposition** for Goldbach's conjecture. Numerical experiments show that for $n \leq 120$, the difference between the truncated construction $\gamma_k(n)$ and the true prime pair count $G(n)$ is less than 10^{-9} (see Remark 7.1). (3) We prove the closed form $A_j = j\,(d(j) - 2)$ of the "additive sieve" sequence (d being the divisor-counting function), so that $A_j = 0$ if and only if j is prime; we prove that Goldbach's conjecture is equivalent to the existence of zero positions in the symmetric superposition sequence, and give the zero-count identity $\#\{j\} = 2G(n) - [\,n/2 \in \mathbb{P}\,]$. (4) We completely resolve the **carry problem** in fractional sequence encoding: with block length $d \geq \lceil \log_{10}(4n^{3/2}) \rceil$, the encoding is carry-free and fully faithful; meanwhile we prove that for fixed d the infinite sequence inevitably carries ($\sup_j A_j = \infty$), thereby clarifying the precise scope of previous objections to the method's feasibility. (5) We provide conditional proofs for related propositions ($p_1 + p_2 + 3$ and $p_a + p_b + p_c - 1$ forms are consequences of the strong Goldbach conjecture), relate the $2p+q$ representation to Lemoine's conjecture with numerical verification up to 10^6, and correct a limit-order error in the original "irrationality test formula" (the single-limit version actually diverges as $\frac{\sqrt{\pi}}{2}k^{3/2}$).

It is emphasized: all constructions in this paper are **logical equivalents** of Goldbach's conjecture, not proofs; their value lies in recasting the conjecture into elementary analytic or combinatorial forms with precisely controllable errors.

Contents

1 Introduction

Goldbach's conjecture (specifically the even/strong form in this paper) asserts:

Conjecture 1.1 (Goldbach, 1742)**.** *Every even integer greater than 2 can be expressed as the sum of two primes.*

Since the Hardy–Littlewood circle method and Brun's sieve, the best partial result is Chen Jingrun's "1 + 2" theorem [1]; the ternary (weak) Goldbach conjecture has been completely proved by Helfgott [2].

The author's prior series of studies revolve around a simple but far-reaching idea: **formalize the "existence test" itself as an elementary function expression**. Specifically, three approaches:

1. **Limit-type elementary construction**: Using the characteristic properties of functions such as e^{-kx}, $e^{-k(x-m)^2}$, $\cos^{2k}(\pi x)$ as $k \to \infty$, encode "n decomposable into a prime pair" as the positivity of a summation (Section 3);

2. **Prime-marked symmetric stacking**: Multiply/add the prime indicator sequence on $[1, n-1]$ with its reverse pointwise; a prime pair decomposition corresponds to a 1 in the product sequence or a 0 in the sum sequence (end of Section 3 and Section 5);

3. **Fractional sequence encoding and additive sieve**: Encode integer sequences into decimal blocks of a fraction, using a "reverse sieve" (superimposing each number's proper multiples) so that prime positions become zero blocks (Section 5).

In approach 1, removing the limit depended on an error factor choice $k = 10x^e$ that was only empirically supported; approach 3 faced objections from the carry problem and was not yet rigorized. The main new work here completes these two: Section 4 provides the complete error analysis and rigorously proves the sufficiency of $k = 10n^e$ and the location of the critical exponent 2; Section 5 gives the closed form of the additive sieve, the Goldbach criterion, the zero-count identity, and the complete resolution of the carry problem (both directions). Section 6 organizes related propositions: proving two are consequences of the strong Goldbach conjecture, identifying the $2p + q$ representation as the prime-restricted case of Lemoine's conjecture, and correcting a limit-order error in the original "irrationality test." Section 7 summarizes all numerical verification.

Statement of honesty. All "equivalent forms" in this paper are logically equivalent to the original conjecture, and therefore contain no substantial progress toward a proof; their significance lies in: (a) demonstrating the expressive power of elementary function systems; (b) demoting "error" from "limit" to "explicitly controllable quantity," making the construction a genuine elementary closed form; (c) translating the sieve into an additive object $A_j = j(d(j) - 2)$ with a closed form, thereby establishing a connection with the classical theory of divisor functions.

2 Notation and Preliminaries

$\mathbb{P}$ denotes the set of primes, $\mathbb{N} = \{0, 1, 2, \dots\}$. For $x \in \mathbb{R}$, write $\|x\| = \min_{m\in\mathbb{Z}} |x - m|$ for the distance from x to the nearest integer. $d(j)$ denotes the number of positive divisors of j, and $\pi(x)$ the prime counting function. For even $n > 2$, define

$$G(n) = \#\{\, p \in \mathbb{Z} :\ 2 \le p \le n/2,\ p \in \mathbb{P},\ n - p \in \mathbb{P} \,\}$$

as the Goldbach partition number of n (the number of unordered prime pairs). Conjecture 1.1 asserts $G(n) \ge 1$ for all even $n > 2$. $[\,\cdot\,]$ denotes the Iverson bracket.

3 Limit-Type Elementary Construction

3.1 Basic Construction

By definition, n satisfies Goldbach's conjecture if and only if there exists $p \in [2, n/2]$ such that both p and $n-p$ are prime. To write this existence as an analytic expression, we introduce three layers of characteristic functions.

Definition 3.1 (Zero-test)**.** $\eta : \mathbb{R}_{\geq 0} \to \{0,1\}$, $\eta(x) = 1$ if and only if $x = 0$. Its limit-type realization is

$$\eta(x) = \lim_{k\to\infty} e^{-kx}.$$

Definition 3.2 (Integer-test)**.** $\omega : \mathbb{R} \to \{0,1\}$, $\omega(x) = 1$ if and only if $x \in \mathbb{Z}$. Two limit-type realizations:

$$\omega(x) = \lim_{k\to\infty} \cos^{2k}(\pi x) \qquad \text{or (with input restrictions)} \qquad \omega(x) = \lim_{k\to\infty} \sum_{m} e^{-k(x-m)^2}.$$

The cosine version requires no restriction on the input domain and is used throughout this paper.

Definition 3.3 (Primality test)**.** For integer $x \geq 2$, let $\psi(x,m) = \lim_{k\to\infty} \cos^{2k}\big(\pi x/m\big)$,

$$\varrho(x) = \sum_{m=2}^{x-1} \psi(x,m) = \#\{\, m : \ 2 \leq m \leq x-1,\ m \mid x \,\}.$$

Clearly $\varrho(x) = 0 \iff x \in \mathbb{P}$ (for $x = 2, 3$ the sum is empty, defined as 0).

Definition 3.4 (Goldbach counting function)**.** For even $n > 2$,

$$\gamma(n) = \sum_{p=2}^{n/2} \eta\big(\varrho(p) + \varrho(n-p)\big).$$

Theorem 3.5 (Limit-type equivalence)**.** $\gamma(n) = G(n)$. *Therefore Goldbach's conjecture is equivalent to: for all even $n > 2$, $\gamma(n) > 0$; equivalently, $\eta(\gamma(n)) = 0$.*

Proof. $\varrho(p) + \varrho(n-p) = 0$ if and only if $p, n-p \in \mathbb{P}$, in which case η takes value 1, otherwise 0. Term by term gives the result. □

3.2 Prime-marked Symmetric Stacking

Let $P = (P_2, \ldots, P_{n-2})$, $P_j = [\, j \in \mathbb{P}\,]$, and its reverse $P_j^{\mathrm{rev}} = P_{n-j}$. Then n satisfies Goldbach's conjecture if and only if there exists j such that $P_j\, P_j^{\mathrm{rev}} = 1$. Rewriting with ϱ and using the trick that "a non-negative integer sequence has a zero" $\iff$ "its product is zero," we obtain the product criterion:

$$\prod_{j=2}^{n-2} \Big(\big(1 - \eta(\varrho(j))\big) + \big(1 - \eta(\varrho(n-j))\big) \Big) = 0 \iff G(n) \geq 1. \tag{1}$$

This criterion is the continuous prototype of the additive sieve in Section 5: the symmetric superposition sequence $A_j + A_{n-j}$ there is precisely a weighted discretization of $\varrho(j) + \varrho(n-j)$.

3.3 Generalizations

The same family of test functions can formalize many other objects (proofs are term-by-term verifications, omitted):

$$\pi(x) = \sum_{n=2}^{x} \eta(\varrho(n)) \qquad \text{(prime coun}$$

(2)

$$f_2(x) = \sum_{n=2}^{x} \eta\big(\varrho(n) + \varrho(n+2)\big) \to \infty \iff \text{twin prime conjecture holds} \tag{3}$$

$$f_1(x) = \omega\Big(\frac{x}{2}\Big) \cdot \frac{x}{2} + \omega\Big(\frac{x-1}{2}\Big) \cdot (3x+1), \quad \forall x\, \exists m : \ f_1^{(m)}(x) = 1 \iff \text{Collatz conjecture holds} \tag{4}$$

The Collatz formulation above encodes the single-step map; a fully rigorous equivalence requires an additional stopping-time predicate (e.g. $\eta(f_1^{(m)}(x)-1)$) to capture the orbit existence quantifier. We present it here as an illustrative direction rather than a strict equivalence. The correction to the original "irrationality test formula" is discussed in Section 6.3.

4 Error Analysis: From Limit Construction to Closed Form

This section addresses the first previously unexpanded component. Truncating all limits in Definitions 3.1–3.4 at a finite parameter $k > 0$:

$$\psi_k(x,m) = \cos^{2k}\Big(\frac{\pi x}{m}\Big), \qquad \varrho_k(x) = \sum_{m=2}^{x-1} \psi_k(x,m), \qquad \gamma_k(n) = \sum_{p=2}^{n/2} \exp\Big(-k\big(\varrho_k(p)+\varrho_k(n-p)\big)\Big).$$

(the exponent $2k$ need not be integral; if one insists on elementarity, take $2\lceil k \rceil$, and all estimates differ only by an absorbable constant.) The question: how large must k (as a function of n) be to ensure that $\gamma_k(n)$ faithfully reflects $G(n)$?

4.1 Basic Lemmas

Lemma 4.1 (Cosine concentration inequality)**.** *For all $\theta \in \mathbb{R}$,*

$$\cos^2(\pi\theta) \le e^{-4\|\theta\|^2}.$$

Proof. By periodicity and symmetry, assume $\theta = t \in [0, \frac{1}{2}]$, so $\|\theta\| = t$. Then $\cos^2(\pi t) = 1 - \sin^2(\pi t) \le e^{-\sin^2(\pi t)}$ (using $1-u \le e^{-u}$), and the chord-arc inequality $\sin(\pi t) \ge 2t$ on $[0, \frac{1}{2}]$ gives $\sin^2(\pi t) \ge 4t^2$. □

Lemma 4.2 (Error bound for primality test)**.** *Let $x \ge 2$ be an integer.*

1. *If $x \in \mathbb{P}$, then $0 \le \varrho_k(x) \le x\, e^{-4k/x^2}$;*
2. *If $x \notin \mathbb{P}$ (and $x \ge 4$), then $\varrho_k(x) \ge 1$.*

Proof. (1) For $2 \le m \le x-1$ with $m \nmid x$, the distance from x/m to the nearest integer is at least $1/m \ge 1/(x-1) > 1/x$. By Lemma 4.1, $\psi_k(x,m) \le e^{-4k\|x/m\|^2} \le e^{-4k/x^2}$, and the sum is at most $(x-2)e^{-4k/x^2}$. (2) When x is composite, there exists $m \mid x$ with $2 \le m \le x-1$, giving $\psi_k(x,m) = \cos^{2k}(\pi \cdot \text{integer}) = 1$, while all other terms are non-negative. □

4.2 Main Theorem and Critical Exponent

Theorem 4.3 (Finite-k criterion)**.** *Let $n \geq 4$ be even and $k > 0$ satisfy*

$$\textit{(i)}\quad kn\,e^{-4k/n^2} < \ln 2, \qquad \textit{(ii)}\quad k > \ln n. \tag{5}$$

Then

$$G(n)\,\exp\!\left(-kn\,e^{-4k/n^2}\right) \;\leq\; \gamma_k(n) \;\leq\; G(n) + \frac{n}{2}\,e^{-k},$$

and in particular

$$\gamma_k(n) > \tfrac{1}{2} \iff G(n) \geq 1.$$

Proof. Split the summands of $\gamma_k(n)$ into two classes.

True terms (both p and $n-p$ are prime, totalling $G(n)$ terms): By Lemma 4.2(1), $\varrho_k(p) + \varrho_k(n-p) \leq p\,e^{-4k/p^2} + (n-p)e^{-4k/(n-p)^2}$. Since e^{-4k/x^2} is monotonically increasing in x for $x > 0$, and $p, n-p \leq n$, we have $e^{-4k/p^2} \leq e^{-4k/n^2}$ and $e^{-4k/(n-p)^2} \leq e^{-4k/n^2}$, giving $\varrho_k(p) + \varrho_k(n-p) \leq (p + (n-p))\,e^{-4k/n^2} = n\,e^{-4k/n^2}$, so each term is $\geq \exp(-kn\,e^{-4k/n^2})$, and obviously ≤ 1.

False terms (at least one of p, $n-p$ is composite, at most $n/2$ terms): By Lemma 4.2(2), $\varrho_k(p) + \varrho_k(n-p) \geq 1$, so each term is $\leq e^{-k}$.

Adding the two classes gives the squeeze. Under condition (i), each true term exceeds $e^{-\ln 2} = \frac{1}{2}$, so $G(n) \geq 1 \Rightarrow \gamma_k(n) > \frac{1}{2}$; under condition (ii), the total false terms are $\leq \frac{n}{2}e^{-k} < \frac{1}{2}$, so $G(n) = 0 \Rightarrow \gamma_k(n) < \frac{1}{2}$. □

Corollary 4.4 (Critical exponent for sufficient condition)**.** *Take $k = c\,n^\alpha$ ($c > 0$ constant).*

1. *If $\alpha > 2$, then condition (5) holds for all sufficiently large even n;*
2. *If $\alpha \leq 2$, then condition (i) fails for all sufficiently large n: when $\alpha < 2$, $cn^{\alpha+1}e^{-4cn^{\alpha-2}} \sim cn^{\alpha+1} \to \infty$; when $\alpha = 2$ this quantity is $c\,e^{-4c}\,n^3 \to \infty$.*

Proof. (1) $\ln\!\left(kn\,e^{-4k/n^2}\right) = \ln(c) + (\alpha+1)\ln n - 4c\,n^{\alpha-2} \to -\infty$, since power growth dominates logarithmic;(ii) is immediate.(2) Direct computation as above. □

Remark 4.5. Part (2) of the corollary only shows the *sufficient condition of this paper* fails when $\alpha \leq 2$; numerical experiments (Table 2) show this sufficient condition fails extensively in $[4, 10^5]$ for $\alpha = 2$ and $\alpha = 2.2$, while holding entirely for $\alpha = e$, consistent with the corollary. The true critical behavior of the error can also be seen from a rough count: for prime x, there are roughly $O(x\delta)$ values of m with $\|x/m\| \leq \delta$, giving $\varrho_k(x) \approx x\int_0^{1/2} e^{-4kt^2}\,dt = \Theta\!\left(x/\sqrt{k}\right)$; to suppress this to $o(1/k)$, one similarly needs k exceeding a power of x^2. Thus the exponent 2 is not merely a byproduct of estimation technique, but the genuine threshold of the construction itself.

4.3 Rigorous Verification of the Empirical Formula $k = 10n^e$

The original research discovered through extensive numerical experiments that $k = 10x^e$ ($e = 2.71828\ldots$ is Euler's number) "seems to drive the error to extremely small values." We now prove it indeed falls within the safe region of Corollary 4.4, and *for all even numbers* (not merely sufficiently large ones) satisfies condition (5).

Proposition 4.6. *For all even $n \geq 4$, $k(n) = 10n^e$ satisfies condition (5).*

Proof. Condition (ii) is immediate. For condition (i), let

$$g(n) = -\ln\!\left(kn\,e^{-4k/n^2}\right) = 40\,n^{e-2} - (e+1)\ln n - \ln 20.$$

Differentiating: $g'(n) = 40(e-2)n^{e-3} - (e+1)/n = n^{-1}\big(40(e-2)n^{e-2} - (e+1)\big)$. Since $e-2 > 0.718$, for $n \geq 1$ we have $40(e-2)n^{e-2} \geq 40(e-2) > 28 > e+1$, so g is monotonically increasing on $[1,\infty)$. Moreover $g(4) = 40 \cdot 4^{0.71828\ldots} - (e+1)\ln 4 - \ln 20 \approx 40 \times 2.7028 - 3.7183 \times 1.3863 - 2.9957 \approx 108.11 - 5.155 - 2.996 \approx 99.96 > \ln \frac{1}{\ln 2} \approx 1.04$, so for all $n \geq 4$, $kn\, e^{-4k/n^2} < e^{-g(4)} < \ln 2$. □

Theorem 4.7 (Limit-free elementary closed-form equivalent proposition)**.** *Goldbach's conjecture holds if and only if: for all even $n > 2$,*

$$\sum_{p=2}^{n/2} \exp\left(-10n^e \sum_{u\in\{p,\,n-p\}} \sum_{m=2}^{u-1} \Big(\cos\frac{\pi u}{m}\Big)^{20n^e}\right) \;>\; \frac{1}{2}. \tag{6}$$

Proof. Immediate from Proposition 4.6 and Theorem 4.3 (when the even power of cos is interpreted as $\cos^{2\lceil 10n^e\rceil}$, the conclusion is unchanged, with estimates differing only by absorbable constant factors). □

Remark 4.8. Formula (6) consists entirely of addition, subtraction, multiplication, exponentiation, cosine, and exponentiation – containing no limits or any non-elementary components beyond ceiling – achieving the original research goal of a "continuous elementary construction" of Goldbach's conjecture. Numerically (Table 1), for $n \leq 120$ the difference $|\gamma_k(n) - G(n)|$ is already below 10^{-9}, and the criterion threshold $\frac{1}{2}$ has enormous safety margin. Of course, solving (6) analytically is as hard as the original conjecture – it is a *reformulation*, not a *weakening*.

By "closed form" we mean free of limit operations, not free of summation signs. Its value lies in replacing the implicit limit with the explicit controllable factor $k(n) = 10n^e$, making the construction a genuine finite expression – rather than a limit form that only holds theoretically as $k \to \infty$.

5 Additive Sieve and Fractional Sequence Encoding

This section addresses the second previously unexpanded component. The original research proposed considering the fraction $0.0102030405\ldots$ (each block of d digits stores j at position j), applying a "reverse sieve" – superimposing the proper-multiple sequences for each $k \geq 2$ – so that prime blocks become exactly zero; then adding the reverse gives the Goldbach criterion. This method was then questioned on grounds of carry issues. Below we first rigorize "superposition" as pure sequence arithmetic (decoupling from the decimal carrier), then precisely characterize when carries are controllable.

5.1 Closed Form of the Additive Sieve

Definition 5.1 (Additive sieve sequence)**.** Define $A_1 = 1$, and for $j \geq 2$

$$A_j = \sum_{\substack{k\geq 2,\ p\geq 2\\ kp=j}} j = j \cdot \#\{(k,p) :\ k,p \geq 2,\ kp = j\}.$$

That is: for each $k \geq 2$, superimpose the sequence "place the multiple itself at each proper multiple of k", then append 1 at position 1.

Theorem 5.2 (Closed form and primality characterization)**.** *For $j \geq 2$,*

$$A_j = j\,\big(d(j) - 2\big),$$

and $A_j = 0$ if and only if $j \in \mathbb{P}$. We define $A_1 = 1$ as a special convention for this formula at $j = 1$ ($d(1) - 2 = -1$, inconsistent with the definition).

Proof. The ordered pairs (k,p) with $kp=j$, $k,p\geq 2$ correspond bijectively to the divisors of j that are neither 1 nor j, totalling $d(j)-2$. For $j\geq 2$, $A_j=0 \iff d(j)=2 \iff j\in\mathbb{P}$; while $A_1=1\neq 0$. □

Remark 5.3. $A_j=j(d(j)-2)$ connects the "reverse sieve" to the classical theory of divisor functions: for instance, from $d(j)=O_\varepsilon(j^\varepsilon)$ we get $A_j=O_\varepsilon(j^{1+\varepsilon})$; from $\sum_{j\leq x} d(j)=x\ln x+(2\gamma_0-1)x+O(\sqrt{x})$ (Dirichlet; γ_0 the Euler constant) we obtain $\sum_{j\leq x} A_j=\frac{1}{2}x^2\ln x+O(x^2)$, etc. Numerically the closed form agrees with the definition for $j\leq 10^5$ (Section 7).

5.2 Symmetric Superposition and the Goldbach Criterion

Theorem 5.4 (Additive sieve criterion and zero counting)**.** *Let $n>2$ be even, and write $B_j=A_j+A_{n-j}$ $(2\leq j\leq n-2)$. Then*

$$G(n)\geq 1 \iff \exists\, j\in[2,n-2]:\ B_j=0,$$

and the number of zero positions satisfies the identity

$$\#\{\,j\in[2,n-2]:\ B_j=0\,\}=2\,G(n)-[\,n/2\in\mathbb{P}\,].$$

Proof. $A_j,A_{n-j}\geq 0$, so $B_j=0 \iff A_j=A_{n-j}=0 \iff j,\,n-j\in\mathbb{P}$ (Theorem 5.2). Each unordered prime pair $\{p,n-p\}$ $(p<n/2)$ contributes two zero positions $j=p$ and $j=n-p$, while the pair $p=n/2$ contributes only one. The counting identity follows. This identity has been numerically verified for all even $n\leq 5000$. □

5.3 Fractional Sequence Encoding and Resolution of the Carry Problem

We now embed the sequence (A_j) into a decimal carrier. With block length d, define

$$S(k,m)=\sum_{p=2}^{\lfloor m/k\rfloor} kp\cdot 10^{-dkp},\qquad P(n)=\sum_{k=2}^{n-1} S(k,n-1)+10^{-d}=\sum_{j=1}^{n-1} A_j\,10^{-dj},$$

and the reverse version $P_{\mathrm{rev}}(n)=\sum_{j=1}^{n-1} A_{n-j}\,10^{-dj}$, $C(n)=P(n)+P_{\mathrm{rev}}(n)$. This is precisely the original construction (where $+10^{-d}$ accounts for the correction "1 is not prime").

Theorem 5.5 (Carry-free threshold)**.** *If*

$$d\ \geq\ \left\lceil \log_{10}\!\left(4n^{3/2}\right)\right\rceil,$$

then the j-th d-digit block $(1\leq j\leq n-1)$ of the decimal expansion of $C(n)$ equals exactly $B'_j:=A_j+A_{n-j}$ (boundary blocks $j\in\{1,n-1\}$ equal A_1+A_{n-1}), with no cross-block carries. Hence the criterion of Theorem 5.4 can be read digit by digit:

$$G(n)\geq 1 \iff C(n) \text{ has a zero block among positions } 2 \text{ through } n-2.$$

Proof. Divisor pairs $(a,j/a)$ with the smaller factor $\leq\sqrt{j}$, so $d(j)\leq 2\sqrt{j}$. For $j\geq 2$, $A_j=j(d(j)-2)\leq j(2\sqrt{j}-2)<2j^{3/2}$; for $j=1$, $A_1=1<2\cdot 1^{3/2}=2$. Thus $A_j\leq 2j^{3/2}\leq 2n^{3/2}$ for all $1\leq j\leq n-1$, and each block value $B'_j<4n^{3/2}\leq 10^d$. Since all block values $B'_j=A_j+A_{n-j}$ are non-negative integers strictly below the block capacity 10^d, digit-wise addition produces no carry, and the expansion is faithful. □

Theorem 5.6 (Impossibility of fixed block length)**.** *For any fixed d, the infinite sequence version inevitably suffers carries:* $\sup_j A_j=\infty$. *For example $A_{2^m}=2^m(m-1)\to\infty$.*

Remark 5.7 (Clarification of previous objections). The original method was dismissed outright on grounds of "carry problem and infinite operations being infeasible." Theorems 5.5 and 5.6 show this judgment was only half correct: **The infinite fractional version with fixed d is indeed infeasible** (Theorem 5.6), but**the Goldbach criterion only requires checking the finite window $[1, n-1]$ for each n**, where choosing block length $d(n) = \lceil \frac{3}{2} \log_{10} n + \log_{10} 4 \rceil = O(\log n)$ is completely rigorous (Theorem 5.5); moreover, if one abandons the decimal carrier entirely and works directly with the sequence (A_j) (Theorem 5.4), the notion of carry does not arise at all. Numerically, for $n = 10^5$ the actual required $d = 8$ while the theoretical threshold gives $d = 9$; see Table 3.

Remark 5.8 (Generating function perspective). Writing $x = 10^{-d}$, the base sequence fraction is $\sum_{j \geq 1} j\, x^j = \dfrac{x}{(1-x)^2}$, while

$$F(x) := \sum_{j \geq 2} A_j x^j = \sum_{k \geq 2} \sum_{p \geq 2} kp\, x^{kp}, \qquad \sum_{j \geq 1} d(j)\, x^j = \sum_{k \geq 1} \frac{x^k}{1 - x^k} \text{ (Lambert series)},$$

so $F(x) = x \dfrac{d}{dx} \left[\sum_{k \geq 2} \dfrac{x^{2k}}{1 - x^k} \right]$ is a Lambert-series derivative combination. Fractional encoding is merely the evaluation of the generating function at $x = 10^{-d}$; the carry problem is precisely the issue of "evaluation destroying coefficient readability." This perspective shows the method shares roots with the classical generating-function framework for additive problems (e.g., the square of $\sum_p z^p$ in the circle method): the Goldbach criterion is equivalent to the coefficient of z^n in $\left(\sum_{p \in \mathbb{P}} z^p \right)^2$ being positive, while (A_j) provides its "complementary indicator" version.

6 Related Propositions: Proofs, Positioning, and Corrections

6.1 Two Conditional Propositions

Proposition 6.1. *Assuming Conjecture 1.1 (the even/strong Goldbach conjecture) holds, then:*

1. *Every odd $n > 5$ (not necessarily prime) can be written as $n = p_1 + p_2 + 3$ ($p_1, p_2 \in \mathbb{P}$);*
2. *Every odd $n \geq 7$ can be written as $n = p_a + p_b + p_c - 1$ ($p_a, p_b, p_c \in \mathbb{P}$).*

Proof. (1) $n-3$ is an even integer > 2, decomposable directly. (2) $n+1$ is even and ≥ 8; taking $p_a = 2$, we need $n - 1 = p_b + p_c$, and $n - 1$ is even and ≥ 6, decomposable directly. □

The above proofs show these are direct consequences of the strong Goldbach conjecture (for all odd numbers), with zero counterexamples numerically up to $n \leq 10^5$. Note that this is a consequence of the even (strong) conjecture applied to $n-3$, not of the weak ternary Goldbach theorem.

6.2 $2p + q$ Representation and Lemoine's Conjecture

The original research conjectured: every prime $p_a > 10$ can be written as $p_a = 2p_b + p_c$ ($p_b, p_c \in \mathbb{P}$), attempting to derive this from the weak Goldbach conjecture by "setting $p_1 = p_2$." It should be noted: the weak Goldbach conjecture guarantees only the *existence of some* triple-prime decomposition, not one with *two equal terms*, so this derivation has a gap and the proposition should be regarded as an independent conjecture. In fact it is precisely the restriction of **Lemoine's conjecture** (1895: every odd $n > 5$ can be written as $p + 2q$, $p, q \in \mathbb{P}$) to prime arguments n, still open. We extend numerical verification to 10^6:

Proposition 6.2 (Numerical verification)**.** *For all primes $10 < p_a \leq 10^6$, the representation $p_a = 2p_b + p_c$ exists (all 78 490 primes verified, zero counterexamples).*

6.3 Correction to the Irrationality Test Formula

The original research gave the test formula

$$\lim_{k\to\infty} \sum_{p=0}^{k} \sum_{q=1}^{k} e^{-k(p/q-x)^2} = 0 \quad \overset{?}{\iff} \quad x \notin \mathbb{Q}. \tag{7}$$

This formula merges two limits – "grid scale" and "peak sharpness" – into a single parameter k, causing the left-hand side to diverge for *every* real $x > 0$: fixing denominator $q \le k$, the spacing of fractions p/q near x is $1/q$, and summing over p gives contribution $\approx q\sqrt{\pi/k}$, then summing over q yields

$$\sum_{p,q} e^{-k(p/q-x)^2} \;\approx\; \frac{\sqrt{\pi}}{2}\, k^{3/2} \;\to\; \infty.$$

Numerical experiments fully confirm this: for $x = \sqrt{2} - 1$, at $k = 1600$ the ratio $S_k/k^{3/2} = 0.8868$, differing from $\frac{\sqrt{\pi}}{2} = 0.8862$ by less than 10^{-3} (Table 4).

The correction separates the two limits and lets the peak sharpness lead. Using $x \in \mathbb{Q} \iff \exists q \ge 1 :\ qx \in \mathbb{Z}$ and the ω of Definition 3.2, we obtain

Proposition 6.3 (Corrected irrationality test)**.**

$$x \notin \mathbb{Q} \iff \lim_{m\to\infty} \sum_{q=1}^{m} \lim_{k\to\infty} \cos^{2k}(\pi q x) = 0.$$

(When x is irrational, each inner term is 0, so the outer sum is identically 0; when $x = a/b$, terms with q a multiple of b are all 1, and the outer sum diverges.) The limit order is not interchangeable – this is precisely the error in formula (7).

6.4 On Local Fitting of the Prime Number Theorem

The original research fitted $\pi(x) \sim x/\log_m x$ with $m = \frac{x}{x-10}(e + 0.383419670541)$ successfully on $[11, 10^4]$ with better accuracy than $x/\ln x$, but failed at 2×10^8. Retrospectively this was inevitable: any fitting that absorbs the lower-order terms of $\mathrm{Li}(x) - x/\ln x$ into a constant can only dominate in a finite window, since $\pi(x) = x/\ln x + x/\ln^2 x + 2!\,x/\ln^3 x + \ldots$ drifts slowly across scales. This is recorded here as a methodological lesson on "finite practice cannot extrapolate to infinite theory."

7 Numerical Verification Summary

All experimental code is available in the accompanying scripts (Python 3.11, exact integer arithmetic for encoding verification, log-domain floating point for γ_k).

Table 1: Truncated construction $\gamma_k(n)$ ($k = 10n^e$) versus true partition count $G(n)$

n	$\gamma_k(n)$	$G(n)$
30	3.0000000000	3
100	6.0000000000	6
120	12.0000000000	12

For all even $4 \le n \le 120$:
$\max_n |\gamma_k(n) - G(n)| < 10^{-9}$.
Sufficient condition (5) verified
individually on $[4, 10^5]$ with zero exceptions.

Table 2: Number of failing evens in $[4, 10^5]$ under sufficient condition (i) with $k = n^\alpha$

α	Failing count	Largest failing n
2.0	49 999 (all)	10^5
2.2	31 139	62 280
$e \approx 2.718$	0	—

Table 3: Carry threshold: actual required block length versus Theorem 5.5 theoretical bound. *Actual min* d is computed as $\lceil \log_{10}(\max_j B'_j) \rceil$, i.e. the minimum digit count of the largest block value.

n	$\max_j(A_j + A_{n-j})$	Actual min d	Theoretical threshold d
10^2	964	3	4
10^4	583 520	6	7
10^5	12 407 360	8	9

Table 4: Divergence of the left-hand side of (7) at $x = \sqrt{2} - 1$ ($S_k/k^{3/2} \to \sqrt{\pi}/2 \approx 0.8862$)

k	100	200	400	800	1600
$S_k/k^{3/2}$	0.8950	0.8905	0.8884	0.8873	0.8868

Remark 7.1 (Note). The numerical precision shown in Table 1 demonstrates the good behavior of the truncated construction but does not constitute a theoretical proof of Goldbach's conjecture. The individual verification of sufficient condition (5) only confirms the premises of Theorem 4.3, and does not imply $G(n) \geq 1$ for all n. The exact match at small n occurs because $k = 10n^e$ is so large that $\cos^{2k}$ decays to machine precision away from integer arguments.

Additional verification: the additive sieve closed form $A_j = j(d(j) - 2)$ and primality characterization hold individually for $j \leq 10^5$; the zero-count identity holds individually for even $n \leq 5000$; the encoding criterion (exact integer arithmetic) perfectly matches prime pair decompositions in the examples $(n, d) = (8, 2), (12, 2), (30, 3)$; zero counterexamples for $p_1 + p_2 + 3$ and $p_a + p_b + p_c - 1$ representations with $n \leq 10^5$; zero counterexamples for $2p + q$ representations with $p_a \leq 10^6$.

8 Conclusion and Outlook

This paper completes the two missing pieces of the original research plan:

1. **Error analysis** reduces the limit-type construction to a fully elementary closed-form criterion (Theorem 4.7), determines the critical growth exponent of the error factor to be 2, and provides a rigorous proof for the empirical formula $k = 10n^e$;

2. **Additive sieve** obtains the closed form $A_j = j(d(j) - 2)$, the exact Goldbach criterion and zero-count identity, and the carry problem is completely resolved in the finite window while being proved inherently unavoidable in the fixed-block-length infinite version.

Possible future directions: (a) Using $A_j = j(d(j) - 2)$ and the mean/moment theory of divisor functions to study the average behavior of zero positions in the symmetric superposition B_j (this essentially leads to Hardy–Littlewood type asymptotics, i.e., the singular series prediction for $G(n)$); (b) Making a quantitative comparison between the exponential sums in (6) and trigonometric sums in the circle method, to understand whether this "smoothed counting"

leaves any new angle of attack for analytic tools; (c) Generalizing the additive sieve to $p + 2q$ (Lemoine) and the symmetric superposition version of twin primes. The author maintains a clear-eyed assessment: every step above is a reformulation until it reaches the genuine difficulty (the bilinear/parity obstacle in prime distribution); but as the original research postscript states – the process itself is the reward.

Acknowledgments

This article was independently completed by the author without the guidance of a supervisor.

Thanks to QClaw and Claude Code (Agnes) for providing the LaTeX typesetting organization, text polishing, and supplementary evidence analysis for this article. They helped me transform scattered research notes into standardized academic texts and refined the error bound derivation in section 4. It should be made clear that all the core concepts, mathematical constructions and logical deductions of this article were independently completed by the author, and the AI tools merely served as auxiliary "digital laboratory assistants".

Finally, I would like to express my gratitude to my classmates and friends for their understanding and support during my days of exploring alone.

References

[1] J. Chen. On the representation of a large even integer as the sum of a prime and an integer containing at most two prime factors. *Science in China*, 1973(2): 111–128.

[2] H. A. Helfgott. The ternary Goldbach conjecture is true. *arXiv preprint* arXiv:1312.7748, 2013.

[3] Y. Wang. On the representation of a large even integer as the sum of a product of at most three primes and a product of at most four primes. *Acta Mathematica Sinica*, 1956(3): 500–513.

[4] É. Lemoine. L'intermédiaire des mathématiciens, 1 (1894), 179; ibid 3 (1896), 151.

[5] T. M. Apostol. *Introduction to Analytic Number Theory*. Springer, 1976.

[6] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*, 6th ed. Oxford University Press, 2008.

[7] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *MRC 56*, 1962.