A Unit-Based Framework for Quantity Dimensions and Base-Quantity Assessment

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Abstract

In the International System of Quantities (ISQ), base quantities form a conventionally chosen mutually independent subset of a system of quantities. This paper proposes a relation-first framework in which admitted unit relations are used to construct quantity dimensions before base-quantity choices are assessed. A broader class of unit-explicit equations is also introduced for relations in which explicit unit division carries physically relevant information.

Candidate base-quantity sets are first tested for dimensional independence and completeness. A second, evidential criterion then assesses physical structural support from established higher-order dimensional occurrences, without uniquely selecting a base set. In an SI-oriented application, mass, length, time, electric current, thermodynamic temperature, and amount of substance receive such support. Relative to the specified admitted relation system, an independent angular dimension is required for completeness; plane angle is adopted as its representative, and spherical geometry gives $\mathrm{sr} = \mathrm{rad}^2$. Luminous intensity is instead dimensionally derived through the photometric-radiometric scale link while remaining a distinct quantity kind. Logarithmic levels are reformulated as pure-number numerical index mappings retaining their field- or power-quantity context. The proposal changes dimensional and algebraic representation without changing established numerical realizations.

1. Introduction

According to the International Vocabulary of Metrology (VIM), a base quantity is defined as a "quantity in a conventionally chosen subset of a given system of quantities, where no subset quantity can be expressed in terms of the others"[1]. A conceptually equivalent definition is adopted in ISO 80000-1:2022 published by the International Organization for Standardization (ISO), which provides general information and definitions concerning quantities, systems of quantities, units, and coherent unit systems, particularly the International System of Quantities (ISQ)[2]. The phrase "conventionally chosen" motivates the present work. A system of quantities is characterized by mathematical relations among quantities, and these relations exhibit definite algebraic behavior in their quantity and unit expressions. The present work therefore takes these algebraic characteristics and independently admitted unit relations, rather than an a priori conventional choice of base quantities, as the starting point for constructing dimensional structure. Although some freedom necessarily remains in choosing particular quantities to represent an established dimensional structure, the dimensional structure itself and the identification of admissible base-quantity candidates should not depend on such a prior choice. For six of the current base quantities—mass, length, time, electric current, thermodynamic temperature, and amount of substance—their dimensional treatment does not present the principal problems considered in this study.

Luminous intensity provides another case for re-examination because its SI definition involves an explicitly conventional scale-setting element. Although luminous intensity is treated in the current SI as a base quantity with the candela as its base unit[2,3], its definition is linked to the defining constant $\mathrm{K_{cd}} = 683\ \mathrm{lm/W}$. The SI Brochure characterizes $\mathrm{K_{cd}}$ as a technical constant associated with a special application and states that, in principle, it can be chosen freely, for example to incorporate conventional physiological or other weighting factors[3]. The established photometric framework further states that $\mathrm{K_{cd}}$ links the photometric units to the corresponding radiometric units[4]. This raises the question of whether such a conventionally selectable scale link should by itself generate an independent dimensional component. The general dimensional treatment of such scale-setting relations is developed in Section 4 and applied to luminous intensity in Section 5.2.

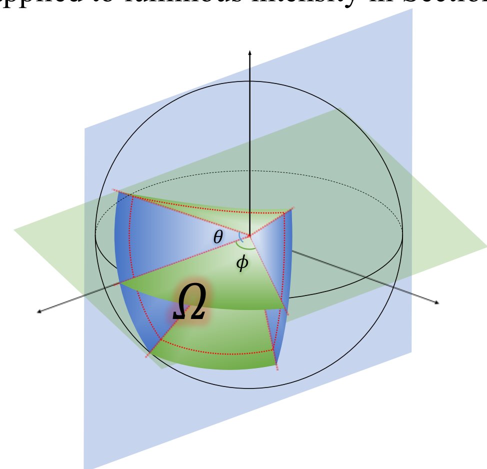

Figure 1. Geometrical relationship between plane and solid angles.
Mutually orthogonal angular coordinates θ and ϕ are defined in the vertical and horizontal coordinate planes, respectively, shown in blue and green. The solid angle Ω is the measure of the three-dimensional angular region whose magnitude is obtained by integration with respect to θ and ϕ with the appropriate spherical-coordinate geometrical weighting over the region bounded by the red dashed lines. The figure illustrates the geometrical distinction between plane angle and solid angle.

Plane angle and solid angle possess distinct geometrical natures (Figure 1), and the current SI explicitly recognizes that angular quantities should not be identified with ordinary ratios merely because their conventional geometrical expressions involve such ratios. In the current SI Brochure, the representation of the radian as $\mathrm{m/m}$ is described as non-intrinsic and potentially misleading because plane angle is not the same kind of quantity as other length ratios; an analogous statement is made for the representation of the steradian as $\mathrm{m^2/m^2}$ because solid angle is not the same kind of quantity as other area ratios[3,5]. Nevertheless, for reasons of history and convention, the SI continues to treat plane and solid angles as quantities with the unit one, while the unit symbols for the radian and steradian are written explicitly where appropriate to indicate the angular nature of the quantities involved[3]. The present work accepts the distinction between angular quantities and ordinary ratios recognized in the current SI, but investigates a different dimensional consequence: whether the radian should instead retain a nontrivial angular dimension, with the steradian subsequently derived from its second-order unit structure.

A related representational issue arises for logarithmic ratio quantities. The logarithmic operations used in their definitions yield pure numerical results. However, when a corresponding logarithmic quantity L is expressed using a unit such as the neper or decibel, the defining relation establishes only that L and the associated unit have the same dimensional structure; it does not by itself identify that structure with dimension one. Such an identification requires an additional unit convention, such as the conventional identification of the neper with the number one. Section 6 therefore examines whether a separate logarithmic unit need to be retained when the logarithmic result is represented directly as a numerical index.

Accordingly, the present work first constructs dimensional subsets from the algebraic characteristics of quantities and admitted relations among unit expressions, and then assesses candidate base-quantity sets. Criterion 1 addresses algebraic admissibility, while Criterion 2 introduces additional physical structural support among admissible alternatives. Unit-explicit equations are used where suppression of a unit component would remove physically relevant information.

Individual components of the present discussion have substantial precedent. The possibility of assigning plane angle an independent dimension and treating the radian as an independent unit has been examined by Quincey and Brown[6], Leonard[7], Quincey[8], and Mohr *et al.*[9]. Quincey, Mohr, and Phillips[10] showed specifically that the use of a length ratio in the measurement of an angle does not imply that the physical quantity angle is inherently a length ratio or dimensionless; a closely related distinction between an angle and its numerical value in radians was also emphasized by Kalinin[11].

Several consequences of a dimensional treatment of angle are likewise already established in the literature. Leonard[7] derived a second-order angular dimension for solid angle and the coherent relation $\mathrm{sr} = \mathrm{rad}^2$; Quincey and Brown[6] and Quincey[8] examined modifications to familiar equations when angle is treated independently; Quincey[12] considered explicitly the distinction between ordinary frequency and angular frequency without redefining the hertz; and Mohr *et al.*[9] developed unit-independent treatments of trigonometric and exponential functions for dimensional angles. These individual consequences are therefore not claimed here as the principal novelty of the present work.

Likewise, the algebraic separation of quantities, dimensions, and units, the use of mappings from quantities to dimensions, and the existence of alternative admissible sets of base quantities have substantial precedents[13–16]. Recent work has also reconsidered the organization and generative structure of the SI itself[17–19]. The distinctive contribution proposed here is instead the logical ordering of these elements within a common construction: an explicitly specified system of admissible relations among unit expressions is first used to construct dimensional subsets; the resulting structure is then accessed through the dimension-extraction function Dim; and algebraic admissibility and physical structural support are considered only subsequently in base-quantity assessment. Plane angle, luminous intensity, logarithmic quantities, and unit-explicit physical equations are treated as applications or consequences of this same relation-first construction rather than as independent starting assumptions.

2. Algebraic Characteristics of Quantities

A quantity value may be expressed as a numerical multiple of a reference unit. For example, let the mass m of an object be $3\ \mathrm{kg}$:

$$m = 3\ \mathrm{kg}$$

This equation means that m is three times the reference unit, the kilogram. That is, it is equivalent to:

$$m = 3 \times \mathrm{kg}$$

where the multiplication symbol $\times$ can be considered omitted between the numerical value and the unit. Regarding notation, hereafter, in accordance with the standard notation used in quantity calculus, the numerical value and the measurement unit associated with a quantity Q are denoted as $\{Q\}$ and $[Q]$, respectively[1,2].

$$Q = \{Q\}\,[Q], \quad \{Q\} = \frac{Q}{[Q]}$$

Because $\{Q\}$ is merely the numerical component of a quantity value, it does not by itself identify the kind or physical role of the quantity represented. For a quantity having a nontrivial unit component, $[Q]$ additionally provides the reference scale relative to which its numerical value is expressed. The dimensional classification introduced below depends on the algebraic reducibility of unit components. For the non-zero scalar quantities considered below, multiplication and division obey the usual Abelian-group properties of closure, associativity, identity, inverses, and commutativity, as in formal axiomatizations of quantity calculus[16]. Addition and subtraction are more restrictive: they are meaningful only for quantities of the same kind, and numerical values expressed in different units must first be converted to a common unit.

Importantly, additive cancellation removes the numerical magnitude while allowing the unit reference to remain explicit:

$$Q - Q = 0\,[Q]$$

rather than a bare numerical zero. By contrast,

$$\frac{Q}{Q} = 1$$

so the unit component is eliminated through multiplicative cancellation. This algebraic distinction will be used in Section 4 in constructing dimensional structure and in distinguishing it from the concrete unit representation of a quantity value.

3. Numerical Value Equations and Unit-Explicit Equations

According to ISO 80000-1:2022 and the VIM, a quantity equation is a “mathematical relation between quantities in a given system of quantities, independent of measurement units”, whereas a numerical value equation is a “mathematical relation between numerical quantity values, based on a given quantity equation and specified measurement units[1,2]”. Thus, an operational distinction between the two is their dependence on the units used for numerical representation: a quantity equation is invariant under a change of measurement units, whereas a numerical value equation is formulated relative to specified units and may accordingly change form when those units are changed.

In the present framework, I introduce the term **unit-explicit equation** for a broader class of unit-dependent relations. A unit-explicit equation is a mathematical relation containing one or more quantity expressions explicitly divided by specified units, with the resulting unit components retained in the equation. When each dividing unit is a measurement unit of the same kind as the corresponding quantity, the unit-divided expression is its numerical value and the relation reduces to a conventional numerical value equation. Otherwise, the unit-divided expression need not itself be a numerical quantity value. The term unit-explicit equation is introduced only to denote this broader class and does not modify the conventional definition of a numerical value equation.

The dependence of such relations on the specified unit can be demonstrated as follows.

Let us consider two different measurement units u_1 and u_2 for the same kind of quantity Q, and let q_1 and q_2 denote the corresponding numerical values. A fixed measurement unit represents a constant quantity interval: an increment of one unit represents the same physical increment irrespective of the position on the numerical scale. For the same physical interval ΔQ, let Δq_1 and Δq_2 denote the corresponding numerical intervals when expressed using u_1 and u_2, respectively. Then ΔQ is expressed as:

$$\Delta Q = \Delta q_1 \, \mathrm{u}_1 = \Delta q_2 \, \mathrm{u}_2$$

Therefore,

$$\frac{\Delta q_1 \, \mathrm{u}_1}{\Delta q_2 \, \mathrm{u}_2} = 1$$

Since u_1 and u_2 are fixed measurement units, the ratio between the corresponding numerical intervals is constant. Defining

$$k \equiv \frac{\Delta q_1}{\Delta q_2} \in \mathbb{R} \setminus \{0\}$$

the preceding relation gives directly

$$\mathrm{u}_2 = k \mathrm{u}_1$$

A relation of this form between measurement units of quantities of the same kind is referred to here as an **ordinary unit relation**. Depending on the direction of conversion, k or $1/k$ is the conversion factor between the units[1].

To obtain the general relation between the numerical values themselves, let $Q_{\mathrm{rf},1}$, $Q_{\mathrm{rf},2}$ denote the reference quantity values assigned on respective scales to the same physical reference state. These reference quantity values need not be equal because the two scales may have different origins. Accordingly,

$$\Delta q_1 = q_1 - \frac{Q_{\mathrm{rf},1}}{\mathrm{u}_1}$$
$$\Delta q_2 = q_2 - \frac{Q_{\mathrm{rf},2}}{\mathrm{u}_2}$$

Substituting these relations into $\Delta q_1 = k \Delta q_2$ yields:

$$q_1 = k q_2 + l$$

where

$$l = \frac{Q_{\mathrm{rf},1}}{\mathrm{u}_1} - k\frac{Q_{\mathrm{rf},2}}{\mathrm{u}_2}$$

Thus, the general relation between the numerical values of two scales having constant unit intervals is an affine relation. The coefficient k represents the ratio of the unit intervals, whereas l represents the difference between the scale origins. Any nonlinear transformation would cause the ratio of corresponding intervals to depend on the position on the scale, contradicting the constancy of the unit interval. This result is consistent with the classical treatment of interval scales, whose admissible transformations are affine[20]. A representative example of $l \neq 0$ is the conversion formula between Celsius temperature t and thermodynamic temperature T:

$$\frac{T}{\mathrm{K}} = \frac{t}{{}^{\circ}\mathrm{C}} + 273.15$$

In this case, $k = 1$, because the degree Celsius and kelvin have the same magnitude for temperature intervals, whereas $l = 273.15$ results from the different origins of the two scales. Thus, the additive term concerns the relation between the scale origins and does not alter the unit relation for temperature intervals.

Because k is uniquely determined by u_1 and u_2, including a quantity Q' divided by a specified unit in an equation is equivalent to involving the unit conversion factor k or $1/k$; that is,

$$\frac{Q'}{\mathrm{u}_2} = k\frac{Q'}{\mathrm{u}_1}$$
$$\frac{Q'}{\mathrm{u}_1} = \frac{1}{k}\frac{Q'}{\mathrm{u}_2}$$

Therefore, explicit division by a specified measurement unit makes the mathematical form of the relation dependent on the choice of that unit. In the unit-explicit equations defined above, such unit components are retained when their elimination or consolidation would remove physically relevant information. The same underlying quantity relation may therefore have different unit-explicit forms when different units are used, and these forms are related by the corresponding unit-conversion relations.

4. Dimensional Subsets Defined by Unit Relations

As established in Section 3, measurement units of quantities of the same kind are related by ordinary unit relations of the form $\mathrm{u}_2 = k\mathrm{u}_1$, where k is a non-zero pure number. The present section uses admitted relations among unit expressions as inputs from which dimensional subsets are constructed.

Let $\mathcal{R}_0$ denote the set of unit relations admitted as inputs to the dimensional construction. To avoid circularity, relations included in $\mathcal{R}_0$ must be established independently of the dimensional classification that the construction is intended to determine. Conversion relations between units independently recognized as units of the same quantity kind may therefore be admitted without presupposing the dimension subsequently assigned to that kind. By contrast, an identification of a unit with the pure-number unit one is not admitted merely because the associated quantity is conventionally classified as having dimension one, since using that identification to establish the same dimensional classification would presuppose the result of the construction.

Relations between quantities of different kinds require separate consideration. A proportionality between quantities of different kinds is not, by itself, taken to establish dimensional equivalence. The SI Brochure describes the nature of the SI defining constants as ranging from fundamental constants of nature to technical constants. A fundamental constant is generally constrained through its relations to other constants in the equations of physics, whereas a technical constant such as K_{cd} concerns a special application and may, in principle, be chosen freely, for example to incorporate conventional physiological or other weighting factors[3]. The present construction therefore distinguishes proportionality relations constrained by physics from scale-setting relations whose normalization is conventionally selectable.

For a cross-kind scale-setting relation, let

$$k = \frac{\mathrm{u}_2}{\mathrm{u}_1}$$

Unlike in an ordinary same-kind unit relation, the dimensional status of k is not assumed a priori. When the associated technical scale-setting constant can be written as:

$$K = ak \quad (a \in \mathbb{R} \setminus \{0\})$$

where a is a pure number and K has the conventionally selectable technical role described above, k is admitted in $\mathcal{R}_0$ as a **conventional scale link**. The unit expressions u_1 and u_2 are then placed in the same dimensional subset, while the distinction between the associated quantity kinds is retained. This implements the principle that an independently selectable convention should not, by itself, generate an independent dimensional component. If a freely selectable cross-kind scale link were sufficient to generate an independent dimensional component, arbitrarily many such components could be introduced merely by defining new quantity scales and associated units and assigning conventional scale-setting constants that relate them to existing quantities, without introducing any new physical structure. Such proliferation would make the number of independent dimensional components depend on arbitrary scale conventions rather than on additional physical or algebraic structure, contrary to the purpose of the present construction. The present framework therefore adopts as an admission principle that conventional freedom in scale definition alone is not sufficient grounds for dimensional independence.

The generating system $\mathcal{R}_0$ may therefore contain two classes of admitted relations: ordinary unit relations between convertible units of the same quantity kind, and conventional scale-link relations satisfying the condition above. The admitted relations generate an equivalence relation on the unit expressions considered in the construction, with reflexivity, symmetry, and transitivity following from identity, inversion, and composition, respectively. The corresponding equivalence classes constitute the dimensional subsets. Relations obtained from the admitted relations by ordinary multiplication, division, and inversion are included in this construction. The symbol $\mathcal{R}_0$ is introduced only to make explicit which relations are used as inputs to the construction; no additional abstract algebraic structure is assumed.

The resulting dimensional structure is therefore relative to the specified relation system $\mathcal{R}_0$. An additional admitted relation may identify dimensional subsets that remain distinct under $\mathcal{R}_0$. Thus, the dimensional classification may change when the set of admitted relations is enlarged.

The multiplicative structure of these dimensional subsets is independent of the particular unit representatives chosen. If $\mathrm{u}_1 = k_1\mathrm{u}_1'$, $\mathrm{u}_2 = k_2\mathrm{u}_2'$, where k_1 and k_2 are non-zero pure numbers, then

$$\mathrm{u}_1\mathrm{u}_2 = (k_1k_2)\mathrm{u}_1'u_2'$$

Since k_1k_2 is again a non-zero pure number, replacing either representative by another unit in the same subset does not change the subset associated with their product. The same argument applies to division and inversion. Thus, multiplication and inversion of unit expressions induce well-defined multiplicative operations on the corresponding dimensional subsets.

Algebraic treatments of quantities, dimensions, and units have substantial precedent[13–15]. These works provide useful formal foundations for the multiplicative treatment of dimensional structure, but the stronger algebraic structures developed in those approaches are not required here. For present purposes, only the multiplicative behavior of the dimensional subsets induced by the admitted unit relations is required.

Within the present framework, when the unit component of a quantity is algebraically reducible to a pure number, the corresponding dimensional structure is classified as dimension one. This algebraic classification does not imply that all such quantities are physically identical. This distinction has precedent in both standard metrological terminology and alternative algebraic treatments. The VIM explicitly notes that quantities of dimension one convey more information than a number even though their values and measurement units are numbers[1]. Kock's algebraic formulation likewise implies that the ratio of two quantities having the same dimension is a pure quantity[14]. Johansson has proposed a different treatment in which relative quantities such as relative length and mass fraction may be unit-less while retaining a parametric dimensional descriptor[21]. The present framework does not adopt this additional parametric dimensional structure. Instead, ordinary multiplicative cancellation is retained, while distinctions among quantities of dimension one are carried by their quantity symbols, definitions, and physical roles.

Accordingly, a pure numerical value does not by itself erase the physical identity of a quantity of dimension one; its quantity symbol or designation specifies what the number represents. In some established usages this identifying role is visible even typographically, as in "pH 7.00" or "Mach 2", where the designation directly accompanies the numerical value without being used as a measurement unit. A ratio-form unit expression may nevertheless be retained when it provides useful scaling or contextual information. For example, a mass fraction w of dimension one may be reported as $w = 5\ \mathrm{mg/kg}$. Such a representation makes both the numerical scale and the ratio form explicit, but it does not introduce a nontrivial dimension in the present framework. The physical interpretation of the ratio remains specified by the quantity symbol w and its accompanying definition.

Numerical quantity values are pure numbers and are therefore assigned dimension one within the present framework. A normalized unit vector likewise has dimension one because its numerator and denominator possess the same dimensional structure.

Let $\mathrm{Dim}\,Q$ denote the dimensional subset associated with the unit component of a quantity Q. The same notation is used for a unit expression u, with $\mathrm{Dim}\,\mathrm{u}$ denoting the dimensional subset containing u. Equality under Dim denotes equality of unit-based dimensional structure and not identity of quantity kind. Because multiplication of unit components induces multiplication of the corresponding dimensional subsets:

$$\mathrm{Dim}(Q_1Q_2) = \mathrm{Dim}\,Q_1\,\mathrm{Dim}\,Q_2$$

and consequently

$$\mathrm{Dim}(Q^n) = (\mathrm{Dim}\, Q)^n$$

for integer n. For positive scalar quantities, the same rule extends to rational powers whenever the corresponding quantity power is defined.

Such multiplicative behavior is not itself new. Raposo explicitly introduced a surjective projection

$$\mathrm{dim} : \mathcal{Q} \to \mathcal{D}$$

from a space of quantities to a group of dimensions and required this map to be a homomorphism under multiplication[13]. The function Dim introduced here adopts the same basic multiplicative behavior. Its distinctive role lies instead in the origin of its target: the dimensional subsets have already been constructed from unit relations rather than postulated beforehand as the base structure of quantity calculus.

Additive compatibility is more restrictive than dimensional compatibility. Quantities that are added or subtracted must be of the same kind, whereas equality under Dim expresses only equality of their unit-based dimensional structure. Consequently,

$$\mathrm{Dim}(Q_1) = \mathrm{Dim}(Q_2)$$

does not by itself imply that Q_1 and Q_2 may be added.

This distinction is also recognized in Raposo's algebraic formulation: quantities of the same kind belong to the same dimensional fiber, whereas the converse need not hold, and the algebraic structure alone does not distinguish this additional quantity-kind information[13]. For a physically meaningful equation, addition and subtraction must therefore respect quantity kind, while all terms connected by an equality must remain dimensionally compatible, as required by dimensional homogeneity.

Integration and differentiation inherit their dimensional behavior from multiplication and division. For an integrand Q and an integration variable X, the dimensional structure of the integral is:

$$\mathrm{Dim}\left(\int Q\, \mathrm{d}X\right) = \mathrm{Dim}\, Q\, \mathrm{Dim}\, X$$

whereas

$$\mathrm{Dim}\left(\frac{\mathrm{d}Q}{\mathrm{d}X}\right) = \mathrm{Dim}\, Q\, (\mathrm{Dim}\, X)^{-1}$$

These rules are consistent with the formal treatment of differential and integral calculus in algebraic quantity calculus given by Raposo, in which differentiation introduces the inverse dimension of the independent-variable quantity and integration introduces its dimension multiplicatively[22]. Detailed derivations of these standard algebraic consequences are therefore not repeated here.

The dimensional structure associated with a quantity should, however, be distinguished from the concrete representation of its value. As established algebraically in Section 2, when a quantity Q possesses a nontrivial unit component $[Q]$, elimination of its numerical magnitude does not by itself eliminate that unit component; for example,

$$Q - Q = 0\,[Q]$$

rather than a bare numerical zero. More generally, a concrete value of such a quantity is represented relative to a specified unit, whereas $\operatorname{Dim} Q$ identifies only the dimensional subset to which that unit belongs. The dimensional subset therefore does not determine which particular unit is to be used as its representative.

The choice of a representative unit is consequently a separate metrological question. Kitano has noted that the number and choice of base units are, in principle, partly arbitrary and that the resulting multiplicity of possible unit systems creates a strong need for standardization[15]. The present framework likewise separates the dimensional structure itself from the particular units chosen to represent it: dimensional subsets are established independently of whether, for example, one member of a given subset rather than another is adopted as the internationally agreed reference unit.

Once the dimensional subsets have been established, it therefore becomes necessary to distinguish two subsequent questions: which physical quantities constitute appropriate candidates for a base-quantity set, and which representative units should be internationally adopted for the quantities ultimately chosen. The former question is considered first in Section 5, where algebraic admissibility and physical structural support are treated separately. Representative units of the selected base quantities may then be designated as base units.

5. Algebraic Admissibility and Physical Structural Support in Base-Quantity Assessment

The dimensional subsets derived in the preceding section do not by themselves uniquely determine which physical quantities should be chosen as base quantities. It is therefore useful to distinguish the **algebraic admissibility** of a candidate set from the assessment of physical structural support for individual quantities within algebraically admissible alternatives.

Criterion 1: Algebraic admissibility. A candidate set of quantities for use as base quantities is regarded as algebraically admissible, within the specified system of quantities, when two conditions are satisfied. First, the dimensions associated with its members must be mutually independent: a product of rational powers of the member dimensions can equal the dimension-one element only when all corresponding exponents are zero. Second, the set must be dimensionally complete: the dimension of every quantity considered within the system must be expressible as a product of rational powers of the dimensions associated with members of the candidate set.

These conditions concern only algebraic independence and completeness. They do not require that an admissible set be unique, nor do they privilege one physical quantity over another when alternative quantities can serve as equivalent dimensional coordinates. For example, electric current and electric charge are related through time, so either may be used in an algebraically admissible representation provided that the other is then treated as derived. The choice between such alternative admissible representations is distinct from the construction of the dimensional structure itself.

A quantity of dimension one cannot provide an additional independent dimensional component and is therefore excluded automatically by the independence condition. Different algebraically admissible candidate sets may represent the same dimensional structure.

Criterion 2: Physical structural support. Because algebraic admissibility concerns only the unit-based dimensional information represented by Dim, it does not distinguish algebraically admissible alternatives according to their quantity kind or physical role. Among such alternatives, a quantity is regarded as having physical structural support for base-quantity status when its associated dimension exhibits an established higher-order occurrence, with an exponent n satisfying $|n| > 1$, in a physical relation, geometrical construction, or metrological model in which the relevant physical roles of the quantities are specified independently of the algebraic rearrangement under consideration. Here, n refers to the occurrence associated with the specified physical role of the candidate quantity, rather than to the net exponent obtained after combining the dimensional contributions of all factors in the relation.

The higher-order occurrence must be intrinsic to the specified physical, geometrical, or metrological role rather than created solely by algebraic manipulation. Multiplying an equation by an additional quantity, taking powers, or rearranging it only to generate a higher exponent therefore does not constitute structural support, nor does an ordinary change of measurement unit. The threshold $|n| > 1$ is adopted as a minimal indicator of nonlinear or repeated dimensional participation; a first-order occurrence alone is ubiquitous and provides little structural information.

Criterion 2 is therefore evidential rather than selection-determinative, and it is not invariant under every change of algebraically admissible coordinates. It assesses a physical candidate quantity in a specified admissible representation, rather than the dimensional coordinate itself independently of the quantity chosen to represent it. Consequently, alternative admissible quantities may both receive support, as in the case of electric current and electric charge, without contradiction. Criterion 2 cannot render a dimensionally dependent quantity independent, is neither necessary nor sufficient for uniquely determining a base-quantity set, and does not rank alternatives when more than one candidate is supported. Its purpose is to constrain the conventional element in base-quantity choice by additional physical evidence, not to eliminate that conventional element altogether.

The following examples show the application of Criterion 2 to individual quantities within the present SI-oriented set. They are not intended to derive this set from Criterion 2 or to establish it as unique; rather, they illustrate how independently established physical or geometrical roles can provide additional structural support for quantities in an already admissible set. Unless otherwise stated, structural support below is assessed relative to this SI-oriented choice.

Mass: Universal gravitation between two bodies contains the product of two masses,

$$F = G\frac{m_1 m_2}{r^2}$$

so that the mass dimension occurs to the second power in a physical interaction that is distinct from the gravitational field produced by a single source mass. The established bilinear role of the two source masses in this interaction therefore satisfies Criterion 2 for mass.

Length: Mutually orthogonal lengths induce the concepts of area and volume:

$$S \propto L^2$$
$$V \propto L^3$$

These quantities are physically and geometrically distinct from length itself. The established geometrical roles of area and volume therefore satisfy Criterion 2 for length.

Time: It serves as the quantity that differentiates, for example, velocity from acceleration and momentum from force. In particular,

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

so that the inverse time dimension occurs at first and second order, respectively, in physically distinct roles. The independently established second-order occurrence in acceleration therefore provides physical structural support for time under Criterion 2, while velocity provides the corresponding first-order contrast.

Electric current: Electric current is retained here as the electromagnetic coordinate of the SI-oriented set. This does not imply that it is algebraically unique; alternative electromagnetic quantities, such as electric charge, may be used in alternative admissible sets. Moreover, established physical relations contain higher powers of electric current; for example, Joule heating is expressed as:

$$P = I^2R$$

The quadratic dependence of Joule heating on current therefore provides physical structural support for electric current under Criterion 2. Electric charge may likewise satisfy Criterion 2 in an alternative algebraically admissible set, for example through its established second-order occurrence in electrostatic interactions. Criterion 2 therefore does not uniquely rank current above charge.

Thermodynamic temperature: According to the Stefan-Boltzmann law, radiant exitance is proportional to the fourth power of thermodynamic temperature:

$$M = \sigma T^4$$

Thus, the established fourth-order dependence in the Stefan-Boltzmann law satisfies Criterion 2 for thermodynamic temperature.

Amount of substance: A second-order occurrence arises, for example, in the thermodynamic Hessian:

$$\frac{\partial \mu_i}{\partial n_j} = \frac{\partial^2 G}{\partial n_j \partial n_i}$$

whose dimensional structure contains the inverse square of the amount-of-substance dimension. Because chemical-potential derivatives and the associated second derivatives of the Gibbs energy have established roles in thermodynamic response and phase stability[23,24], this inverse second-order amount-of-substance structure is not introduced solely by algebraic manipulation. It therefore satisfies Criterion 2 for amount of substance.

The corresponding SI base units are the kilogram, metre, second, ampere, kelvin, and mole, and the dimensional symbols used here are M, L, T, I, Θ, and N, respectively.

Plane angle and luminous intensity provide two contrasting applications of these criteria and are considered separately in Sections 5.1 and 5.2.

5.1. Plane Angle as a Representative Base Quantity for the Angular Dimension

As noted in Section 1, the current SI distinguishes physical plane angle from ordinary length ratios while treating plane angle as a quantity with the unit one for reasons of history and convention[3]. The present framework examines whether this distinction should instead be reflected in a nontrivial angular dimension. This dimensional status is not assumed here; it is determined from the admitted unit relations using the dimensional-subset construction developed in Section 4, before Criterion 1 is applied.

A complete revolution is assigned the angular value 2π rad. For a circular sector, the relationship among the arc length l, the radius r, and the central angle θ is derived from the proportion:

$$l : 2\pi r = \theta : 2\pi \text{ rad}$$

Therefore:

$$l = r\theta/\text{rad} \iff \frac{\theta}{\text{rad}} = \frac{l}{r}$$

These relations are unit-explicit equations in the sense defined in Section 3 because the expression involving the physical angle explicitly retains division by the specified angular unit. In this particular case, θ/rad is also the numerical value of the physical plane angle when the radian is selected as its measurement unit[6,7,10,11].

The numerical value of a physical plane angle depends on the angular unit chosen as its reference. The same physical angle may, for example, be expressed in radians, degrees, gon, or turns. These angular units are interconnected through unit relations[25,26]:

$$1^\circ = \frac{\pi}{180} \text{ rad}$$
$$1 \text{ gon} = \frac{\pi}{200} \text{ rad}$$

and, denoting one turn by tr here:

$$1 \text{ tr} = 2\pi \text{ rad}$$

Consequently, the numerical values of the same physical angle satisfy

$$\frac{\theta}{\text{rad}} = \frac{\pi}{180}\frac{\theta}{{}^\circ} = \frac{\pi}{200}\frac{\theta}{\text{gon}} = 2\pi\frac{\theta}{\text{tr}}$$

Thus, the ratio l/r in the preceding equation corresponds specifically to the numerical value of θ when the radian is selected as the reference angular unit, rather than to the physical plane angle itself[7,11].

To simplify unit-explicit expressions involving the radian, I introduce a general underbar notation in which the explicitly written denominator rad is omitted and its division is represented instead by an underbar:

$$Q/\mathrm{rad} = \frac{Q}{\mathrm{rad}} \to \frac{Q}{} = \underline{Q}$$

An underbar therefore denotes explicit division of the indicated quantity expression by one radian. It does not, in general, denote a numerical quantity value. When Q is itself a plane-angle quantity, however, the radian is a measurement unit of the same kind, and the underbarred quantity is also its numerical value when expressed in radians. Thus,

$$\underline{\theta} = \frac{\theta}{\mathrm{rad}} = \frac{l}{r}$$

and

$$\mathrm{Dim}\,\underline{\theta} = 1$$

By contrast, an expression such as:

$$\underline{r} = \frac{r}{\mathrm{rad}}$$

is a unit-divided quantity expression rather than the numerical value of r. Because multiplication and division of scalar quantities are associative and commutative, the explicitly retained radian factor may be placed at different positions within the same multiplicative relation. Accordingly,

$$l = \frac{r\theta}{\mathrm{rad}} = r\underline{\theta} = \underline{r}\theta$$

These forms are algebraically equivalent unit-explicit representations.

The fact that $\underline{\theta} = l/r$ has dimension one does not by itself establish that either the physical plane angle θ or the radian has dimension one[7,11]. Under the construction of Section 4, the dimensional classification is determined relative to the generating relation system $\mathcal{R}_0$. The circular-sector relation:

$$\frac{\theta}{\mathrm{rad}} = \frac{l}{r}$$

relates the radian-based numerical value of the physical angle to the dimension-one length ratio l/r, but it does not itself provide a generating unit relation between the radian and the unit one. Likewise, the angular-unit conversion relations established above connect the radian, degree, gon, and turn to one another without identifying their common dimensional subset with the dimension-one subset. This means that the conventional identification of the radian with the unit one is not used as a generator in $\mathcal{R}_0$, since the dimensional structure is to be determined prior to such a convention.

Within the generating relation system $\mathcal{R}_0$, the angular-unit relations established above place the radian, degree, gon, and turn in the same dimensional subset, while no admitted relation identifies this angular subset with the dimension-one subset. The angular dimensional subset obtained relative to $\mathcal{R}_0$ is therefore denoted by:

$$\mathrm{Dim(rad)} = \mathsf{A}, \quad \mathsf{A} \neq 1$$

The result $\mathsf{A} \neq 1$ is therefore relative to $\mathcal{R}_0$. If the conventional SI identification of the radian with the unit one is added to the admitted relation system, the angular subset merges with the dimension-one subset and the current SI treatment is recovered.

The corresponding dimension of the steradian will be obtained below by applying the dimensional-angle representation established above to the standard spherical geometry of solid angle, following the unit-to-dimension hierarchy established in Section 4. Before carrying out this geometrical construction, it is necessary to clarify the dimensional interpretation of trigonometric and inverse trigonometric functions. The sine and cosine functions are expressed by Taylor series:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

For the ordinary mathematical definitions of these functions, x is a pure numerical variable. Correspondingly, dimensional homogeneity of the power-series representation requires:

$$\mathrm{Dim}\, x = 1$$

However, this condition alone does not determine which angular unit is involved. As shown above,

$$\frac{\theta}{\mathrm{rad}} \quad \frac{\theta}{{}^\circ} \quad \frac{\theta}{\mathrm{gon}} \quad \frac{\theta}{\mathrm{tr}}$$

are all pure numbers. The ordinary mathematical sine and cosine functions conventionally used in scientific equations act on pure numerical arguments corresponding to radian-based numerical angular values[9,25]. Therefore, when x represents the numerical value of a physical plane angle θ,

$$x = \frac{\theta}{\mathrm{rad}} = \underline{\theta}$$

Accordingly, when θ explicitly denotes the dimensional physical plane angle, the corresponding trigonometric expressions are written as:

$$\sin \underline{\theta} = \sin\left(\frac{\theta}{\mathrm{rad}}\right)$$
$$\cos \underline{\theta} = \cos\left(\frac{\theta}{\mathrm{rad}}\right)$$

For example, if the same physical angle is instead represented by its numerical value in degrees,

$$\{\theta_{\circ}\} = \frac{\theta}{\circ}$$

then the ordinary sine function is evaluated as:

$$\sin\left(\frac{\pi}{180}\{\theta_{\circ}\}\right)$$

rather than as $\sin\{\theta_{\circ}\}$. Thus, the familiar Taylor coefficients apply directly to the radian-based numerical value $\underline{\theta}$, rather than to an arbitrary numerical value of angle expressed in another angular unit[9].

This conclusion is also consistent with the geometrical definition of trigonometric functions. For a circle of radius r centered at the origin and a point (a, b) on the circumference

$$a^2 + b^2 = r^2$$

and

$$\sin x = \frac{b}{r}$$
$$\cos x = \frac{a}{r}$$

Since a, b, and r possess the same dimension,

$$\mathrm{Dim}\left(\frac{b}{r}\right) = \mathrm{Dim}\left(\frac{a}{r}\right) = 1$$

The inverse trigonometric functions likewise return the radian-based numerical angular value rather than dimensional physical angle itself[7,9]. For example,

$$\arcsin\frac{b}{r} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!\cdot(2n+1)}\left(\frac{b}{r}\right)^{2n+1} = \frac{b}{r} + \frac{1}{6}\left(\frac{b}{r}\right)^3 + \frac{3}{40}\left(\frac{b}{r}\right)^5 + \cdots$$

and therefore

$$\arcsin\left(\frac{b}{r}\right) = x = \underline{\theta} = \frac{\theta}{\mathrm{rad}}$$

The corresponding physical angle is

$$\theta = \left(\arcsin\left(\frac{b}{r}\right)\right)\ \mathrm{rad}$$

Thus, ordinary trigonometric functions operate on radian-based numerical angular values, and ordinary inverse trigonometric functions return such numerical values[9]. This does not imply that the corresponding physical plane angle itself has dimension one. If the argument of a mathematical trigonometric function is not interpreted as a physical angular quantity, no explicit angular unit need be introduced.

With this distinction established, the relation between the units of plane angle and solid angle can be obtained from the complete solid angle surrounding a point. Let θ and ϕ denote the physical polar and azimuthal plane angles, and let the corresponding underbarred symbols denote their numerical values when expressed in radians, as defined above.

Using the standard spherical-coordinate geometrical measure, the complete solid angle is represented in terms of the two physical angular differentials as:

$$\Omega_{\mathrm{full}} = \int_{0\ \mathrm{rad}}^{2\pi\ \mathrm{rad}} \int_{0\ \mathrm{rad}}^{\pi\ \mathrm{rad}} \sin\underline{\theta}\, \mathrm{d}\theta\, \mathrm{d}\phi$$

Here, $\sin\underline{\theta}$ is the dimension-one geometrical weighting associated with the spherical-coordinate measure; the second-order angular structure is carried by the product $\mathrm{d}\theta\, \mathrm{d}\phi$. Because

$$\mathrm{d}\theta = \mathrm{rad}\ \mathrm{d}\underline{\theta}$$
$$\mathrm{d}\phi = \mathrm{rad}\ \mathrm{d}\underline{\phi}$$

this becomes

$$\Omega_{\mathrm{full}} = \int_{0\ \mathrm{rad}}^{2\pi\ \mathrm{rad}} \int_{0\ \mathrm{rad}}^{\pi\ \mathrm{rad}} \sin\underline{\theta}\, \mathrm{d}\theta\, \mathrm{d}\phi = 4\pi\ \mathrm{rad}^2$$

Using the equivalent complete-sphere definition of the steradian, the complete solid angle surrounding a point has the value $\Omega_{\mathrm{full}} = 4\pi\ \mathrm{sr}$. Since these expressions represent the same physical solid angle,

$$4\pi\ \mathrm{sr} = 4\pi\ \mathrm{rad}^2$$

and hence

$$\mathrm{sr} = \mathrm{rad}^2$$

Only after this unit relation has been established is the dimension-extraction function introduced in Section 4 applied:

$$\mathrm{Dim}\,\Omega = \mathrm{Dim}(\mathrm{sr}) = \mathrm{Dim}(\mathrm{rad}^2) = \mathsf{A}^2$$

Thus, within the present dimensional-angle representation, the geometrical solid-angle measure, represented in spherical coordinates, contains an intrinsic second-order occurrence of the plane-angle dimension: the dimension-one factor $\sin \underline{\theta}$ weights the product of two physical angular differentials. Because this higher-order occurrence is specified by the spherical geometry rather than generated by algebraic rearrangement, it provides physical structural support for plane angle under Criterion 2.

Previous treatments in which plane angle is assigned an independent dimension likewise obtain solid angle as a derived angular quantity with dimension A^2 and a coherent unit satisfying $\mathrm{sr} = \mathrm{rad}^2$[7,11]. This second-order angular structure also has a practical analogue in the long-standing use of the square degree as a unit of solid angle in astronomy[27,28]. These precedents are consistent with the result derived above, but are not used as premises in the present construction.

For a spherical surface area S subtending a solid angle Ω at radius r, the corresponding numerical-value relation is:

$$\frac{\Omega}{\mathrm{sr}} = \frac{S}{r^2}$$

In analogy with $\underline{\theta} = \theta/\mathrm{rad}$, the underbar square notation may be introduced:

$$\frac{\Omega}{\mathrm{sr}} = \frac{\Omega}{\mathrm{rad}^2} \to \frac{\Omega}{\underline{}_2} \to \underline{\Omega}_2$$

Accordingly,

$$S = \underline{\Omega}_2 r^2 = \Omega \underline{r}^2$$

Thus, solid angle is a geometrically distinct quantity whose unit structure is generated by a second-order product of the plane-angle unit:

$$\mathrm{rad} \quad \to \quad \mathrm{rad}^2 = \mathrm{sr}$$

In particular, the 19th meeting of the Consultative Committee for Length noted that participating laboratories generally do not realize plane-angle units through the geometrical length-ratio representation in the SI Brochure, but instead mainly use circular-closure methods[29]. Lewis and Coveney subsequently developed this practical dimensional-metrology viewpoint in greater detail[30]. These practical-realization arguments are logically independent of, but compatible with, the unit-based dimensional construction developed here.

An important practical feature of the proposed reform is that its adoption would not require a discontinuity in the numerical realization of plane angle. Existing relations among angular units, such as a complete revolution corresponding to 2π rad, remain unchanged, as do numerical calibration results obtained using established angle standards. The principal change concerns the dimensional and unit representation of angular quantities and of equations in which angular units are conventionally suppressed. Accordingly, existing calibration procedures, uncertainty evaluations, and key-comparison results need not be repeated merely because the radian is assigned a nontrivial dimension; if the proposed framework were adopted, changes would instead arise primarily in dimensional formulae, coherent derived-unit expressions, and associated metrological notation or metadata.

The present treatment differs from the current SI status of the radian, but is proposed as a candidate framework for resolving the dimensional ambiguities that have motivated continuing discussion of angle within the SI. The CCU has considered positions ranging from introducing the radian as a base unit to retaining the status quo, without reaching consensus; the subsequent task group improved the description of angle, while reconsideration of its status within the SI lay outside the scope of the task group[31]. The unit-based construction developed here is intended to contribute to this continuing discussion by providing a systematic basis on which a future revision of the SI dimensional structure and its base- and derived-unit organization could be considered.

Representative consequences are considered in Section 7, with detailed derivations for rotational kinematics and dynamics, periodic quantities, quantum-mechanical relations, and radiometric equations provided in Appendix A.

5.2. Luminous Intensity as a Derived Quantity

Luminous intensity provides a contrasting application of the framework. For photopic luminous intensity, the established photometric-radiometric correspondence through spectral weighting of the visual response[3,4,32] may be written as:

$$I_{\mathrm{v}} = \frac{K_{\mathrm{cd}}}{V(\lambda_{\mathrm{cd}})} \int_{0\,\mathrm{m}}^{\infty\,\mathrm{m}} I_{\mathrm{e},\lambda}(\lambda) V(\lambda)\,\mathrm{d}\lambda$$

where $V(\lambda)$ is the spectral luminous efficiency function for photopic vision and λ_{cd} corresponds to the defining frequency 540×10^{12} Hz[3,4,32].

To separate the response weighting from the subsequent photometric scale normalization, define the auxiliary weighted radiometric quantity:

$$I_{\mathrm{e},V} \equiv \frac{1}{V(\lambda_{\mathrm{cd}})} \int_{0\,\mathrm{m}}^{\infty\,\mathrm{m}} I_{\mathrm{e},\lambda}(\lambda) V(\lambda)\,\mathrm{d}\lambda$$

Since $V(\lambda)$ and $V(\lambda_{\mathrm{cd}})$ are pure-number weighting factors, and

$$\mathrm{Dim}\big(I_{\mathrm{e},\lambda}(\lambda)\,\mathrm{d}\lambda\big) = \mathrm{Dim}\,I_{\mathrm{e}}$$

the dimensional rules for multiplication and integration established in Section 4 give:

$$\mathrm{Dim}\,I_{\mathrm{e},V} = \mathrm{Dim}\,I_{\mathrm{e}}$$

This result concerns only the response-weighting operation and does not yet assume any dimensional classification for luminous intensity or for K_{cd}.

The photometric relation can now be written as:

$$I_{\mathrm{v}} = K_{\mathrm{cd}} I_{\mathrm{e},V}$$

In the current SI, $K_{\mathrm{cd}} = 683\ \mathrm{lm/W} = 683\ \mathrm{cd \cdot sr/W}$. As discussed in Section 4, the SI Brochure characterizes K_{cd} as a technical constant having the conventionally selectable scale-setting role considered there[3]. This characterization is used here to determine the admissibility of the scale link, not to assume the dimensional status of K_{cd}.

Taking

$$\mathrm{u}_2 = \mathrm{cd}$$
$$\mathrm{u}_1 = \mathrm{W/sr}$$

define the corresponding unit-ratio factor k_{cd} as:

$$k_{\mathrm{cd}} = \frac{\mathrm{u}_2}{\mathrm{u}_1} = \frac{\mathrm{cd}}{\mathrm{W/sr}} = \frac{\mathrm{cd}\cdot\mathrm{sr}}{\mathrm{W}}$$

The defining constant may then be written as:

$$K_{\mathrm{cd}} = 683k_{\mathrm{cd}}$$

Because 683 is a pure number and K_{cd} has the conventionally selectable technical role specified in Section 4, k_{cd} is admitted in $\mathcal{R}_0$ as a conventional scale link. Consequently,

$$\mathrm{Dim(cd)} = \mathrm{Dim(W/sr)}$$

This equivalence concerns unit-based dimensional structure only. Luminous intensity and radiant intensity remain different kinds of quantity, consistently with the distinction between quantity dimension and quantity kind discussed in Section 4.

Only after this dimensional equivalence has been established does it follow that:

$$\mathrm{Dim}\,k_{\mathrm{cd}} = \frac{\mathrm{Dim(cd)}}{\mathrm{Dim(W/sr)}} = 1$$

and hence

$$\mathrm{Dim}\,K_{\mathrm{cd}} = \mathrm{Dim}(683k_{\mathrm{cd}}) = 1$$

Thus, the dimension-one classification of K_{cd} is a consequence of the admitted scale-link relation rather than an assumption used to establish it.

As established in Section 5.1, $\mathrm{sr} = \mathrm{rad}^2$, $\mathrm{Dim(rad)} = \mathsf{A}$. Therefore,

$$\mathrm{Dim}(I_{\mathrm{v}}) = \mathsf{ML^2T^{-3}A^{-2}}$$

The candela consequently becomes a unit of a derived quantity, with its scale fixed through K_{cd}, rather than a base unit:

$$\mathrm{Dim(cd)} = \mathrm{Dim(W/sr)} = \mathsf{ML^2T^{-3}A^{-2}}$$

Although this dimensional classification differs from the current SI treatment of the candela as a base unit, the proposal leaves the fixed numerical photometric-radiometric correspondence defined by K_{cd} unchanged and instead assigns a different dimensional interpretation to the connected unit structures.

The status of the candela within the hierarchy of SI units has also been reconsidered recently from different perspectives. Lehman et al. proposed a three-category organization of SI units in which the candela is classified among physiologically relevant derived units[18].

Finkelstein and Whitehead distinguish the candela, kelvin, and mole conceptually from the kilogram, metre, second, and ampere, and propose assigning the former group to a separate category of "scale-spanning units" rather than treating all seven base units as playing the same structural role[19]. Their proposal differs from the present framework, which does not introduce an additional category of units but instead classifies luminous intensity as dimensionally derived through the admissible photometric-radiometric scale-link relation established above.

Consequently, luminous intensity does not provide an additional independent dimensional component under Criterion 1 and is therefore classified as a derived quantity. This conclusion does not depend on Criterion 2, which cannot override dimensional dependence established under Criterion 1.

The dimensional completeness of the resulting SI-oriented representation can now be stated explicitly. In the current ISQ, a quantity dimension may be written in the general form:

$$\mathsf{M}^{\alpha}\mathsf{L}^{\beta}\mathsf{T}^{\gamma}\mathsf{I}^{\delta}\Theta^{\epsilon}\mathsf{N}^{\zeta}\mathsf{J}^{\eta}$$

Within the present construction, the luminous-intensity dimension is not independent but satisfies

$$\mathsf{J} = \mathsf{M}\mathsf{L}^{2}\mathsf{T}^{-3}\mathsf{A}^{-2}$$

Hence any such dimension can be rewritten as:

$$\mathsf{M}^{\alpha+\eta}\mathsf{L}^{\beta+2\eta}\mathsf{T}^{\gamma-3\eta}\mathsf{I}^{\delta}\Theta^{\epsilon}\mathsf{N}^{\zeta}\mathsf{A}^{-2\eta}$$

Conversely, because the angular dimension A is not generated by M, L, T, I, Θ and N under $\mathcal{R}_0$, an angular dimensional coordinate is required for completeness. Thus, within the SI-oriented system considered here, the dimensional coordinates M, L, T, I, Θ, N and A provide a complete representation, with plane angle adopted here as the representative quantity for the angular coordinate. Under $\mathcal{R}_0$, no admitted relation expresses any of these seven dimensional coordinates as a product of powers of the others; together with the completeness established above, the corresponding SI-oriented set is therefore algebraically admissible under Criterion 1. This completeness is relative to the specified system of quantities and does not imply that the dimensional structure is permanently limited to these seven coordinates. If future empirical developments reveal an additional dimensional component independent of the present set, Criterion 1 would require an extension of the dimensional representation, with Criterion 2 available to assess physical structural support for a representative quantity of that new dimension.

6. Reformulation of Logarithmic Levels as Numerical Indices

Regarding the neper and decibel, the current metrological framework establishes a specific quantity termed the level L, expressed respectively as:

$$\frac{L}{\mathrm{Np}} = \ln\frac{F}{F_0}$$

$$\frac{L}{\mathrm{dB}} = 10\,\mathrm{lg}\frac{P}{P_0} = 20\,\mathrm{lg}\frac{F}{F_0}$$

where

$$\lg x = \log_{10} x = \frac{\ln x}{\ln 10}.$$

In the conventional definitions, F/F_0 and P/P_0 are ratios of quantities of the same kind and therefore reduce algebraically to pure numbers. The logarithmic functions consequently act on pure numerical arguments, and their numerical outputs are pure numbers and therefore have dimension one[33]. Hence,

$$\mathrm{Dim}\left(\frac{L}{\mathrm{Np}}\right) = \mathrm{Dim}\left(\frac{L}{\mathrm{dB}}\right) = 1$$

which establishes:

$$\mathrm{Dim}\, L = \mathrm{Dim}(\mathrm{Np}) = \mathrm{Dim}(\mathrm{dB})$$

but does not, from these defining equations alone, determine the common dimensional subset before a convention for the logarithmic units is supplied. Within the relation-first construction of Section 4, any further identification of this common dimensional subset, including with dimension one, must therefore enter through an additional convention rather than follow from the logarithmic relation itself.

Historically, the neper was adopted as a special name for the number one for logarithmic quantities defined using natural logarithms[34]. In the current SI Brochure, the neper, bel, and decibel are listed as non-SI units used to express specified logarithmic ratio quantities[3]. The current SI also requires that the quantity concerned and any reference value used be specified, so the conventional notation is fully capable of retaining the physical context of the logarithmic ratio. The question considered here is therefore not whether such information can be represented conventionally, but whether a separate logarithmic unit is required within the present unit-based dimensional construction when the defining logarithmic relation itself yields a pure numerical result.

An alternative treatment within the present framework is to represent logarithmic levels directly by numerical index mappings. In the proposed notation, Np and dB are retained as function labels to preserve explicit correspondence with the established neperian and decadic conventions; when conventional units are discussed, they retain their established meanings as unit symbols. The numerical index mappings are therefore written explicitly as functions, $\mathrm{Np}(\cdot)$ and $\mathrm{dB}(\cdot)$, throughout this section. To retain explicitly the type of physical ratio and the corresponding reference quantity, let F and P be field quantity and power quantity, respectively, and define

$$\mathrm{Np}_{F_0}(F) \coloneqq \ln\frac{F}{F_0}$$
$$\mathrm{Np}_{P_0}(P) \coloneqq \frac{1}{2}\ln\frac{P}{P_0}$$

Likewise, the corresponding decadic indices are defined as:

$$\mathrm{dB}_{F_0}(F) \coloneqq 20\lg\frac{F}{F_0}$$
$$\mathrm{dB}_{P_0}(P) \coloneqq 10\lg\frac{P}{P_0}$$

The defining numerical functions are the same as those conventionally used for field and power levels[33]; what changes in the present framework is the metrological role assigned to Np and dB. The reformulation instead consolidates the physical quantity, its reference quantity, and the logarithmic convention directly in the mapping notation while treating the resulting value as a pure numerical index. Thus, after the index mapping has been defined, a value may also optionally be reported compactly in a prefix form, for example

$$\mathrm{Np}(F)\ 3$$
$$\mathrm{dB}(P)\ 30$$

analogously to expressions such as $\mathrm{pH}\ 7.00$ and $\mathrm{Mach}\ 2$ discussed in Section 2. In these expressions, $\mathrm{Np}(F)$ and $\mathrm{dB}(P)$ identify the type of numerical index and are not factors multiplying the following number or symbols for measurement units. For mathematical derivations, the equivalent equation forms

$$\mathrm{Np}(F) = 3$$
$$\mathrm{dB}(P) = 30$$

are used to avoid ambiguity.

Because the arguments of the logarithms are ratios of quantities of the same kind, they reduce algebraically to pure numbers, and the resulting index values are likewise pure numbers and therefore have dimension one:

$$\mathrm{Dim}\big(\mathrm{Np}_{F_0}(F)\big) = \mathrm{Dim}\left(\mathrm{Np}_{P_0}(P)\right) = \mathrm{Dim}\left(\mathrm{dB}_{F_0}(F)\right) = \mathrm{Dim}\left(\mathrm{dB}_{P_0}(P)\right) = 1$$

This assignment applies to the outputs of the index mappings; the labels Np and dB themselves are not physical quantities and are therefore not assigned independent dimensions.

An important consequence of this reinterpretation is that the algebraic relation between the numerical index values must be distinguished from the conventional relation between the corresponding unit symbols. Under the conventional treatment as measurement units, the established relation is[3,33]:

$$1\ \mathrm{Np} = (20\lg e)\ \mathrm{dB} \approx 8.686\ \mathrm{dB}$$
$$1\ \mathrm{dB} = \frac{\ln 10}{20}\ \mathrm{Np}$$

By contrast, when Np and dB denote the numerical index mappings defined above, their values for the same underlying logarithmic ratio satisfy:

$$\mathrm{dB}(X) = (20\lg e)\mathrm{Np}(X) \approx 8.686\ \mathrm{Np}(X)$$

where X denotes either the corresponding field or power quantity with its specified reference quantity.

The same conversion factor therefore appears with the opposite orientation: the unit relation expresses one neper as $(20\lg e)$ decibels, whereas the numerical-index relation expresses the decibel index value as $(20\lg e)$ times the neper index value. This reversal of the dependent and independent quantities follows directly from the distinction between unit relations and numerical-value relations established in Section 3. Conventional unit-conversion equations must therefore be translated with care into the proposed index notation.

7. Relation to Previous Dimensional-Angle Treatments and Representative Consequences

Several of the individual angular consequences considered below have precedents in the literature[6–9,12], as reviewed in Section 1. The equation forms below are therefore presented as consistency consequences of the dimensional structure established in Sections 3-5, rather than as premises for assigning plane angle a dimension.

For example, the framework gives:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}/\mathrm{rad}, \quad \mathrm{Dim}\,\boldsymbol{\tau} = \mathsf{ML}^2\mathsf{T}^{-2}\mathsf{A}^{-1}$$
$$\mathbf{L} = \mathbf{r} \times \mathbf{p}/\mathrm{rad}, \quad \mathrm{Dim}\,\mathbf{L} = \mathsf{ML}^2\mathsf{T}^{-1}\mathsf{A}^{-1}$$

and

$$\nu = \frac{\omega}{2\pi\ \mathrm{rad}}$$

These results parallel previously proposed dimensional-angle formulations[6–8,12] and are presented here as consistency checks rather than independent arguments for the dimensional assignment of plane angle. Detailed derivations are collected in Appendix A.

A consolidated comparison of the conventional and corresponding unit-explicit equation forms is provided in supplementary table S1.

8. Conclusion

In this study, quantity dimensions were constructed from admitted unit relations before base-quantity assessment. The conventional concept of a numerical value equation was retained, while the broader concept of a unit-explicit equation was introduced for relations in which explicitly retained unit division carries information that would otherwise be suppressed. The admitted unit relations are used to construct dimensional subsets, and Dim identifies the subset associated with a quantity without, in general, identifying its quantity kind. Base-quantity assessment is then separated into algebraic admissibility under Criterion 1 and physical structural support under Criterion 2. Criterion 2 is evidential rather than selection-determinative: it provides additional physical support among algebraically admissible alternatives but does not uniquely determine or rank a base-quantity set.

Within the SI-oriented application considered here, mass, length, time, electric current, thermodynamic temperature, and amount of substance receive physical structural support under Criterion 2. The admitted angular-unit relations require an additional angular dimensional component for completeness, for which plane angle is adopted as a representative; applying this dimensional-angle representation to the geometrically specified complete-solid-angle measure further gives:

$$\mathrm{sr} = \mathrm{rad}^2$$
$$\mathrm{Dim}\,\Omega = \mathsf{A}^2$$

providing further structural support for plane angle. Luminous intensity, by contrast, is classified as dimensionally derived through the admitted photometric-radiometric scale-link relation while remaining a quantity kind distinct from radiant intensity. Logarithmic levels are reformulated as dimension-one numerical index mappings whose physical context is retained by their arguments and reference quantities. These classifications are relative to the specified system of admitted relations $\mathcal{R}_0$; in particular, adding the conventional SI identification of the radian with the unit one recovers the current SI angular classification.

The proposed changes concern dimensional interpretation and algebraic representation rather than numerical realization. Existing numerical angle values, angular-unit conversion relations, calibration results, and the numerical photometric-radiometric correspondence remain unchanged; the changes arise in the dimensional classification of angular and photometric quantities and in equations where angular unit factors are conventionally suppressed. The framework therefore provides a unit-based route from admitted relations to dimensional structure and subsequent base-quantity assessment while keeping quantity kind and physical role conceptually distinct from dimension.

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Appendix A. Detailed Consequences of the Proposed Angular Dimension

Throughout Appendix A, the underbar notation introduced in Section 5.1 is used in the general sense

$$\underline{Q} = \frac{Q}{\mathrm{rad}}$$

Thus, an underbar denotes explicit division of the indicated quantity expression by one radian. It represents a numerical quantity value only when the radian is a measurement unit of the quantity being divided; otherwise, it denotes a unit-divided quantity expression in the sense defined in Section 3.

A.1. Infinitesimal Rotations in Three-Dimensional Space

Let us consider a vector $\mathbf{r}$ that undergoes a rotation by an angle θ around an axis defined by a unit vector $\hat{\mathbf{n}}$, resulting in a new vector $\mathbf{r}'$ (Figure 2). According to Rodrigues' rotation formula, $\mathbf{r}'$ is expressed as:

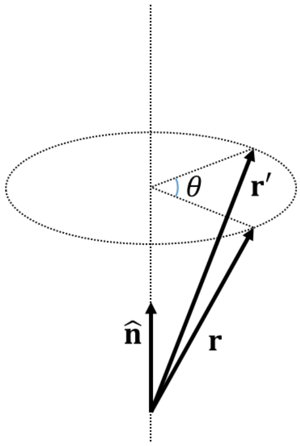

Figure 2. Rotation of $\mathbf{r}$ to $\mathbf{r}'$ around the axis.

$$
\begin{aligned}
\mathbf{r}' &= \left(\cos\underline{\theta}\right)\mathbf{r} + \left(1 - \cos\underline{\theta}\right)(\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \left(\sin\underline{\theta}\right)\hat{\mathbf{n}}\times\mathbf{r} \\
&= \mathbf{r} + \left(1 - \cos\underline{\theta}\right)\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{r}) + \left(\sin\underline{\theta}\right)\hat{\mathbf{n}}\times\mathbf{r}
\end{aligned}
$$

As $\hat{\mathbf{n}}$ and $\mathbf{r}$ are constant vectors, $\mathrm{d}\mathbf{l}/\mathrm{d}\underline{\theta}$ can be expressed by the derivative of $\mathbf{r}'$ with respect to $\underline{\theta}$ at the initial state $\underline{\theta} = 0$:

$$
\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\underline{\theta}} = \left.\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}\underline{\theta}}\right|_{\underline{\theta}=0} = \hat{\mathbf{n}}\times\mathbf{r}
$$

Therefore, the infinitesimal arc vector $\mathrm{d}\mathbf{l}$ is obtained as:

$$
\mathrm{d}\mathbf{l} = \mathrm{d}\underline{\theta}\,\hat{\mathbf{n}}\times\mathbf{r} = \mathrm{d}\theta\,\hat{\mathbf{n}}\times\underline{\mathbf{r}} = \mathrm{d}\theta\,\hat{\mathbf{n}}\times\mathbf{r}/\mathrm{rad}
$$

Here, since $\hat{\mathbf{n}}$ is a unit vector—a quantity of dimension one—in the direction of the rotation axis, it can be treated as defining the direction of the infinitesimal rotation vector. Denoting this vector as $\mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta\,\hat{\mathbf{n}}$, we obtain:

$$
\mathrm{d}\mathbf{l} = \mathrm{d}\underline{\boldsymbol{\theta}}\times\mathbf{r} = \mathrm{d}\boldsymbol{\theta}\times\underline{\mathbf{r}} = \mathrm{d}\boldsymbol{\theta}\times\mathbf{r}/\mathrm{rad}
$$

Consequently, the tangential velocity $\mathbf{v}$ is:

$$
\begin{aligned}
\mathbf{v} = \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = \frac{\mathrm{d}\underline{\boldsymbol{\theta}}}{\mathrm{d}t}\times\mathbf{r} &= \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\mathbf{r}/\mathrm{rad} \\
&= \boldsymbol{\omega}\times\mathbf{r}/\mathrm{rad}
\end{aligned}
$$

Using the underbar notation introduced in Section 5.1, the same unit-explicit relation may be written equivalently as:

$$
\mathbf{v} = \frac{\boldsymbol{\omega}\times\mathbf{r}}{\mathrm{rad}} = \underline{\boldsymbol{\omega}}\times\mathbf{r} = \boldsymbol{\omega}\times\underline{\mathbf{r}}
$$

The last two forms illustrate that, for scalar division by the radian, the explicitly retained unit factor may be associated algebraically with either multiplicative factor.

Similarly, the acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ becomes:

$$
\begin{aligned}
\mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \times \mathbf{r})/\mathrm{rad} \\
&= \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)/\mathrm{rad} \\
&= (\boldsymbol{\alpha} \times \mathbf{r})/\mathrm{rad} + \underline{\boldsymbol{\omega}} \times \left(\underline{\boldsymbol{\omega}} \times \mathbf{r}\right)
\end{aligned}
$$

Similarly, with $\underline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/\mathrm{rad}$, $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega}/\mathrm{rad}$, the acceleration may be written as:

$$
\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} + \underline{\boldsymbol{\omega}} \times \left(\underline{\boldsymbol{\omega}} \times \mathbf{r}\right)
$$

For motion in which $\boldsymbol{\omega} \cdot \mathbf{r} = \mathbf{0}\ \mathrm{m} \cdot \mathrm{rad/s}$, this reduces to:

$$
\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} - \left\|\underline{\boldsymbol{\omega}}\right\|^2 \mathbf{r}
$$

These are unit-explicit forms in the sense defined in Section 3. Similar explicit-radian formulations of rotational kinematics have been discussed previously[8,9].

A.2. Torque

Consider a situation in which an external force $\mathbf{F}$ is applied to a rigid body, causing it to rotate by an infinitesimal displacement $\mathrm{d}\mathbf{l}$. The resulting infinitesimal work $\mathrm{d}W$ is expressed as:

$$
\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{l}
$$

Since $\mathrm{d}\mathbf{l} = \mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}$, this becomes:

$$
\mathrm{d}W = \mathbf{F} \cdot \left(\mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}\right)
$$

Using the scalar triple-product identity,

$$
\begin{aligned}
\mathrm{d}W &= \mathrm{d}\underline{\boldsymbol{\theta}} \cdot (\mathbf{r} \times \mathbf{F}) = (\mathbf{r} \times \mathbf{F}) \cdot \mathrm{d}\underline{\boldsymbol{\theta}} \\
&= (\mathbf{r} \times \mathbf{F}) \cdot \mathrm{d}\boldsymbol{\theta}/\mathrm{rad}
\end{aligned}
$$

Within the present framework, torque is the quantity conjugate to the physical angular displacement $\boldsymbol{\theta}$:

$$
\mathrm{d}W = \boldsymbol{\tau} \cdot \mathrm{d}\boldsymbol{\theta}
$$

Hence,

$$
\boldsymbol{\tau} = \frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}} = \frac{\mathbf{r} \times \mathbf{F}}{\mathrm{rad}} = \underline{\mathbf{r}} \times \mathbf{F} = \mathbf{r} \times \underline{\mathbf{F}}
$$

These are algebraically equivalent unit-explicit forms. The conventional expression:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

is recovered when the explicit radian factor is suppressed.
Dimensional analysis yields:

$$\begin{aligned} \mathrm{Dim}(\boldsymbol{\tau}) &= \mathrm{Dim}(\mathbf{r} \times \mathbf{F}/\mathrm{rad}) \\ &= \frac{\mathsf{L} \cdot \mathsf{MLT}^{-2}}{\mathsf{A}} \\ &= \mathsf{ML}^2\mathsf{T}^{-2}\mathsf{A}^{-1} \end{aligned}$$

Accordingly, the coherent unit of torque becomes, within the present framework:

$$\mathrm{J/rad} = \mathrm{N} \cdot \mathrm{m/rad}$$

This angle-dependent assignment for torque was previously proposed by Quincey[8]. Leonard subsequently contested this conclusion[35], to which Quincey replied[36]. In the present framework, the $\mathrm{J/rad}$ assignment follows specifically from treating torque as the quantity conjugate to the dimensional physical angle in $\mathrm{d}W = \boldsymbol{\tau} \cdot \mathrm{d}\boldsymbol{\theta}$; it is therefore a consequence of the unit-explicit formulation rather than an independent choice made to modify the conventional cross-product equation.

A.3. Angular Momentum

Since $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$, the torque becomes:

$$\boldsymbol{\tau} = \underline{\mathbf{r}} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

Because

$$\frac{\mathrm{d}\underline{\mathbf{r}}}{\mathrm{d}t} \times \mathbf{p} = \frac{\mathbf{v}}{\mathrm{rad}} \times m\mathbf{v} = \mathbf{0}\ \mathrm{kg} \cdot \mathrm{m}^2/(\mathrm{s}^2\mathrm{rad})$$

we obtain:

$$\boldsymbol{\tau} = \frac{\mathrm{d}}{\mathrm{d}t}(\underline{\mathbf{r}} \times \mathbf{p})$$

Accordingly, using the general underbar notation,

$$\mathbf{L} = \int \boldsymbol{\tau}\, \mathrm{d}t = \underline{\mathbf{r}} \times \mathbf{p} = \frac{\mathbf{r} \times \mathbf{p}}{\mathrm{rad}}$$

Thus, $\underline{\mathbf{r}}$ is not a numerical value of position but the unit-divided expression $\mathbf{r}/\mathrm{rad}$. The familiar expression $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ corresponds to the radian-suppressed form.
Dimensional analysis gives:

$$\mathrm{Dim}(\mathbf{L}) = \frac{\mathrm{Dim}(\mathbf{r}) \cdot \mathrm{Dim}(\mathbf{p})}{\mathrm{Dim}(\mathrm{rad})}$$
$$= \frac{\mathsf{L} \cdot \mathsf{MLT}^{-1}}{\mathsf{A}}$$
$$= \mathsf{ML}^2\mathsf{T}^{-1}\mathsf{A}^{-1}$$

The corresponding coherent unit is

$$\mathrm{J} \cdot \mathrm{s/rad}$$

Furthermore, substituting $\mathbf{p} = m\mathbf{v} = m\underline{\boldsymbol{\omega}} \times \mathbf{r}$ and assuming $\underline{\boldsymbol{\omega}} \cdot \mathbf{r} = 0\ \mathrm{m/s}$, we obtain:

$$\mathbf{L} = \underline{\mathbf{r}} \times \left(m\underline{\boldsymbol{\omega}} \times \mathbf{r}\right) = m\left\|\underline{\mathbf{r}}\right\|^2 \boldsymbol{\omega} = m\|\mathbf{r}\|^2\underline{\boldsymbol{\omega}}_2 = m\|\mathbf{r}\|^2\boldsymbol{\omega}/\mathrm{rad}^2$$

Dimensional analysis gives:

$$\mathrm{Dim}\left(m\left\|\underline{\mathbf{r}}\right\|^2\boldsymbol{\omega}\right) = \mathsf{ML}^2\mathsf{T}^{-1}\mathsf{A}^{-1}$$

consistent with the dimension obtained above.

The explicit-radian units of angular momentum proposed here likewise correspond to one side of the published Quincey-Leonard debate[8,35,36].

A.4. Moment of Inertia

As derived in Appendix A.1, the acceleration of a point undergoing rotational motion is:

$$\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} - \left\|\underline{\boldsymbol{\omega}}\right\|^2 \mathbf{r}$$

Substituting this into the expression for torque yields:

$$\boldsymbol{\tau} = \underline{\mathbf{r}} \times m\mathbf{a}$$
$$= m\underline{\mathbf{r}} \times \left(\underline{\boldsymbol{\alpha}} \times \mathbf{r}\right) - m\underline{\mathbf{r}} \times \left(\left\|\underline{\boldsymbol{\omega}}\right\|^2 \mathbf{r}\right)$$

The second term vanishes because $\underline{\mathbf{r}} \times \mathbf{r} = \mathbf{r} \times \mathbf{r}/\mathrm{rad} = \mathbf{0}\ \mathrm{m}^2/\mathrm{rad}$. When the rotation axis is perpendicular to $\mathbf{r}$, so that $\underline{\mathbf{r}} \cdot \underline{\boldsymbol{\alpha}} = 0\ \mathrm{m}/(\mathrm{s}^2\mathrm{rad})$, the first term reduces to:

$$\boldsymbol{\tau} = m\left\|\underline{\mathbf{r}}\right\|^2 \boldsymbol{\alpha}$$

Using the underbar notation, the rotational equation may be written as:

$$\boldsymbol{\tau} = I\boldsymbol{\alpha}$$

with

$$I = m\left\|\underline{\mathbf{r}}\right\|^2 = \frac{m\|\mathbf{r}\|^2}{\mathrm{rad}^2}$$

Thus, the underbar on $\mathbf{r}$ provides a compact representation of the two explicit radian factors required by the quadratic dependence on distance.

More generally, if $\mathbf{I}$ denotes the inertia tensor, its components I_{ij} are written as:

$$I_{ij} = \sum_{\alpha} m_{\alpha} \left[\delta_{ij} \left(\sum_{k} \underline{x_{\alpha k}}^2 \right) - \underline{x_{\alpha i}} \underline{x_{\alpha j}} \right]$$

where δ_{ij} is the Kronecker delta. Therefore,

$$\mathrm{Dim}\, I = \mathsf{ML^2A^{-2}}$$

and its coherent unit within the present framework is $\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{rad}^2$.

A.5. Periodic Quantities and Dimensional Consistency of Angular Units

Consider a system undergoing periodic motion with angular velocity ω. Since one full rotation corresponds to an angle of 2π rad, the period T is:

$$T = \frac{2\pi\ \mathrm{rad}}{\omega}$$

Accordingly, the ordinary frequency $\nu = 1/T$ is:

$$\nu = \frac{\omega}{2\pi\ \mathrm{rad}} = \frac{\underline{\omega}}{2\pi}$$

Here, $\underline{\omega} = \omega/\mathrm{rad}$ has the dimension of ordinary frequency, $\mathsf{T^{-1}}$, whereas the physical angular frequency ω has dimension $\mathsf{T^{-1}A}$.

Thus, ordinary frequency and angular frequency remain distinct quantities:

$$\mathrm{Dim}\, \nu = \mathsf{T^{-1}}$$

whereas

$$\mathrm{Dim}\, \omega = \mathsf{T^{-1}A}$$

The familiar expression $\nu = \omega/(2\pi)$ is recovered when the radian is suppressed. This distinction allows the conventional definition $1\ \mathrm{Hz} = 1\ \mathrm{s}^{-1}$ to remain unchanged[12].

Likewise, the wavenumber $\tilde{\nu}$ is:

$$\tilde{\nu} = 1/\lambda$$

and the angular wavenumber k is:

$$k = 2\pi\tilde{\nu}\ \mathrm{rad} = \frac{2\pi\ \mathrm{rad}}{\lambda}$$

Using the general underbar notation in Section 5.1

$$\underline{k} = \frac{k}{\text{rad}} = 2\pi\tilde{\nu} = \frac{2\pi}{\lambda}$$

The angular wavelength $ƛ$, defined as the reciprocal of $\underline{k}$, is therefore:

$$ƛ = \frac{1}{\underline{k}} = \frac{\lambda}{2\pi\ \text{rad}}$$

In a similar manner, the energy E of a quantum can be expressed as:

$$E = h\nu = \frac{h}{2\pi\ \text{rad}}\omega$$

Here, the Planck constant h is retained as the action-valued constant, with

$$\text{Dim}\, h = \mathsf{ML^2T^{-1}}$$

The conventional reduced Planck constant, or Dirac constant, is retained as:

$$\hbar = \frac{h}{2\pi}$$

with

$$\text{Dim}\, \hbar = \mathsf{ML^2T^{-1}}$$

Using the general underbar notation introduced in Section 5.1, its radian-divided form is:

$$\underline{\hbar} = \frac{\hbar}{\text{rad}} = \frac{h}{2\pi\ \text{rad}}$$

and therefore

$$\text{Dim}\, \underline{\hbar} = \mathsf{ML^2T^{-1}A^{-1}}$$

which is the dimensional structure of angular momentum in the present framework. The energy relation may consequently be written equivalently as:

$$E = h\nu = \underline{\hbar}\omega$$

Accordingly, the magnitude of orbital angular momentum $\mathbf{J}$ is expressed as:

$$\|\mathbf{J}\| = \sqrt{l(l+1)}\underline{\hbar}$$

where l is a numerical quantum number and therefore introduces no additional unit component. Thus, the conventional symbol $\hbar$ retains its established action dimension, whereas $\underline{\hbar}$ denotes its explicitly radian-divided form and has the dimensional structure of angular momentum. A distinction between the action and angular-momentum roles conventionally associated with $\hbar$ has previously been discussed by Quincey[8].

A.6. Planck's Law

Planck's law describes the spectral energy density of blackbody radiation at a frequency ν and thermodynamic temperature T as:

$$u_\nu(\nu, T)\,\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,\mathrm{d}\nu$$

Performing dimensional analysis on the components:

$$\begin{cases} \mathrm{Dim}(h) = \mathsf{ML^2T^{-1}} \\ \mathrm{Dim}\left(\frac{\nu}{c}\right) = \mathsf{L^{-1}} \\ \mathrm{Dim}(h\nu) = \mathrm{Dim}(k_\mathrm{B}T) = \mathsf{ML^2T^{-2}} \\ \mathrm{Dim}(\mathrm{d}\nu) = \mathsf{T^{-1}} \end{cases}$$

we obtain:

$$\mathrm{Dim}(u_\nu\,\mathrm{d}\nu) = \mathsf{ML^{-1}T^{-2}} = \frac{\mathsf{ML^2T^{-2}}}{\mathsf{L^3}}$$

which corresponds to the dimension of energy density.

Spectral radiance is related to radiation energy density through integration over solid angle:

$$u_\nu = \frac{1}{c}\int B_\nu\,\mathrm{d}\Omega$$

For an isotropic blackbody field, B_ν is independent of direction, and therefore:

$$u_\nu = \frac{4\pi\ \mathrm{sr}}{c} B_\nu$$

Hence,

$$B_\nu(\nu, T)\,\mathrm{d}\nu = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\ \mathrm{sr}^{-1}\,\mathrm{d}\nu$$

Within the present framework, the explicit factor sr^{-1} is retained because the steradian is no longer reducible to the number one. Accordingly,

$$\mathrm{Dim}(B_\nu(\nu, T)\,\mathrm{d}\nu) = \mathsf{MT^{-3}A^{-2}} = \frac{\mathsf{ML^2T^{-2}}}{\mathsf{T}} \cdot \frac{1}{\mathsf{L^2}} \cdot \frac{1}{\mathsf{A^2}}$$

Integrating over all frequencies yields the total blackbody radiance $L(T)$:

$$L(T) = \int_{0\ \mathrm{Hz}}^{\infty\ \mathrm{Hz}} B_\nu(\nu, T)\ \mathrm{d}\nu$$

and therefore:

$$L = \frac{2\pi^4 {k_\mathrm{B}}^4}{15c^2h^3} T^4\ \mathrm{sr}^{-1}$$
$$= \frac{\sigma}{\pi} T^4/\mathrm{sr}$$

Thus, within the present framework, the conventional Stefan-Boltzmann expression for blackbody radiance

$$L(T) = \frac{\sigma}{\pi} T^4$$

is rewritten with the explicit solid-angle unit as:

$$L(T) = \frac{\sigma}{\pi} \frac{T^4}{\mathrm{sr}}$$

This is a unit-explicit equation in the sense defined in Section 3.

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Supplementary Table S1. Comparison of conventional equation forms with the corresponding unit-explicit forms in the proposed framework.

Notation: An underbar denotes explicit division by one radian, $\underline{Q} = Q/\mathrm{rad}$; it represents a numerical value only when Q is an angular quantity expressed in radians.

Equations	Conventional equation	Form in the present framework
	(Dimensional analysis)	
Arc length l	$l = r\theta$ (L ≠ L · A)	$l = r\theta/\mathrm{rad} = r\underline{\theta} = \underline{r}\theta$ (L = L · A · A^{-1} = L · 1 = LA^{-1} · A)
Area on spherical surface S	$S = \Omega r^2$ (L^2 ≠ A^2 · L^2)	$S = \Omega r^2/\mathrm{sr} = \underline{\Omega}_{2} r^2 = \Omega \underline{r}^2$ (L^2 = A^2 · L^2 · A^{-2} = 1 · L^2 = A^2 · L^{2}A^{-2})
Trigonometric and inverse trigonometric functions	$a = \sin\theta \Leftrightarrow \theta = \arcsin a$ (On the equation of θ, A ≠ 1)	$a = \sin\underline{\theta} \Leftrightarrow \underline{\theta} = \arcsin a$ (On the equation of θ, 1 = 1)
Period T	$T = \dfrac{2\pi}{\omega}$ (T ≠ TA^{-1})	$T = \dfrac{2\pi\ \mathrm{rad}}{\omega} = \dfrac{2\pi}{\underline{\omega}}$ (T = A · TA^{-1} = T)
Angular frequency ω	$\omega = 2\pi\nu$ (T^{-1}A ≠ T^{-1})	$\omega/\mathrm{rad} = \underline{\omega} = 2\pi\nu$ (T^{-1}A · A^{-1} = T^{-1} = T^{-1})
Angular wavenumber k	$k = 2\pi\tilde{\nu} = \dfrac{2\pi}{\lambda}$ (L^{-1}A ≠ L^{-1})	$k = 2\pi\tilde{\nu}\ \mathrm{rad} = \dfrac{2\pi\ \mathrm{rad}}{\lambda}$ (L^{-1}A = L^{-1} · A = A · L^{-1})
Angular wavelength $ƛ$	$ƛ = \dfrac{\lambda}{2\pi}$ (LA^{-1} ≠ L)	$ƛ = \dfrac{\lambda}{2\pi\ \mathrm{rad}} = \dfrac{\underline{\lambda}}{2\pi}$ (LA^{-1} = L · A^{-1})
Wave equation $u(\mathbf{x},t)$ of amplitude A	$u(\mathbf{x},t) = \dfrac{A}{r}e^{i(\omega t \pm \mathbf{k}\cdot\mathbf{x})}$ (The exponent part is A ≠ 1)	$u(\mathbf{x},t) = \dfrac{A}{r}e^{i\{\underline{\omega} t \pm \underline{\mathbf{k}}\cdot\mathbf{x}\}}$ (The exponent part is 1)
Infinitesimal displacement $\mathrm{d}\mathbf{l}$ by infinitesimal rotation	$\mathrm{d}\mathbf{l} = \mathrm{d}\boldsymbol{\theta} \times \mathbf{r}$ (L ≠ A · L)	$\mathrm{d}\mathbf{l} = \mathrm{d}\boldsymbol{\theta} \times \mathbf{r}/\mathrm{rad} = \mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}$ (L = A · L · A^{-1} = 1 · L)
Tangential velocity $\mathbf{v}$	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ (LT^{-1} ≠ T^{-1}A · L)	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}/\mathrm{rad} = \underline{\boldsymbol{\omega}} \times \mathbf{r}$ (LT^{-1} = T^{-1}A · L · A^{-1} = T^{-1} · L)
Tangential acceleration $\mathbf{a_t}$	$\mathbf{a_t} = \boldsymbol{\alpha} \times \mathbf{r}$ (LT^{-2} ≠ T^{-2}A · L)	$\mathbf{a_t} = \boldsymbol{\alpha} \times \mathbf{r}/\mathrm{rad} = \underline{\boldsymbol{\alpha}} \times \mathbf{r}$ (LT^{-2} = T^{-2}A · L · A^{-1} = T^{-2} · L)
Centripetal acceleration $\mathbf{a_c}$	$\mathbf{a_c} = \lVert\boldsymbol{\omega}\rVert^2\mathbf{r}$ (LT^{-2} ≠ T^{-2}A^2 · L)	$\mathbf{a_c} = \lVert\boldsymbol{\omega}\rVert^2\mathbf{r}/\mathrm{rad}^2 = \lVert\underline{\boldsymbol{\omega}}\rVert^2\mathbf{r}$ (LT^{-2} = T^{-2}A^2 · L · A^{-2} = T^{-2} · L)
Torque $\boldsymbol{\tau}$	$\boldsymbol{\tau} = \dfrac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}} = \mathbf{r} \times \mathbf{F}$ (ML^{2}T^{-2}A^{-1} ≠ L · MLT^{-2})	$\boldsymbol{\tau} = \dfrac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}} = \mathbf{r} \times \mathbf{F}/\mathrm{rad} = \underline{\mathbf{r}} \times \mathbf{F}$ (ML^{2}T^{-2}A^{-1} = L · MLT^{-2} · A^{-1})
Angular momentum $\mathbf{L}$	$\mathbf{L} = \int \boldsymbol{\tau}\,\mathrm{d}t = \mathbf{r} \times \mathbf{p}$ (ML^{2}T^{-1}A^{-1} ≠ L · MLT^{-1})	$\mathbf{L} = \int \boldsymbol{\tau}\,\mathrm{d}t = \mathbf{r} \times \mathbf{p}/\mathrm{rad} = \underline{\mathbf{r}} \times \mathbf{p}$ (ML^{2}T^{-1}A^{-1} = L · MLT^{-1} · A^{-1})
	$\mathbf{L} = \mathbf{r} \times (m\boldsymbol{\omega} \times \mathbf{r}) = m\lVert\mathbf{r}\rVert^2\boldsymbol{\omega}$ (ML^{2}T^{-1}A^{-1} ≠ M · L^2 · AT^{-1})	$\mathbf{L} = \underline{\mathbf{r}} \times (m\underline{\boldsymbol{\omega}} \times \mathbf{r}) = m\lVert\underline{\mathbf{r}}\rVert^2\boldsymbol{\omega}$ (ML^{2}T^{-1}A^{-1} = M · L^{2}A^{-2} · AT^{-1})
Moment of inertia I	$I = \dfrac{\lVert\boldsymbol{\tau}\rVert}{\lVert\boldsymbol{\alpha}\rVert} = \sum_i m_i\lVert\mathbf{r}_i\rVert^2$ (ML^{2}A^{-2} ≠ M · L^2)	$I = \dfrac{\lVert\boldsymbol{\tau}\rVert}{\lVert\boldsymbol{\alpha}\rVert} = \sum_i m_i\lVert\underline{\mathbf{r}_i}\rVert^2$ (ML^{2}A^{-2} = M · L^{2}A^{-2})

Stefan–Boltzmann law for blackbody radiance	$L(T) = \frac{\sigma}{\pi} T^4$ $(\mathrm{MT^{-3}A^{-2}} \neq \mathrm{MT^{-3}\Theta^{-4}} \cdot \Theta^4)$	$L(T) = \frac{\sigma}{\pi} T^4/\mathrm{sr}$ $(\mathrm{MT^{-3}A^{-2}} = \mathrm{MT^{-3}\Theta^{-4}} \cdot \Theta^4 \cdot \mathrm{A^{-2}})$
Angular momentum $\mathbf{J}$	$\lVert\mathbf{J}\rVert = \sqrt{l(l+1)}\hbar$ $(\mathrm{ML^2T^{-1}A^{-1}} \neq 1 \cdot 1 \cdot \mathrm{ML^2T^{-1}})$	$\lVert\mathbf{J}\rVert = \sqrt{l(l+1)}\underline{\hbar}$ $(\mathrm{ML^2T^{-1}A^{-1}} = 1 \cdot 1 \cdot \mathrm{ML^2T^{-1}A^{-1}})$
Logarithmic levels	$\frac{L}{\mathrm{Np}} = \ln\frac{F}{F_0}$ $(\mathrm{Dim}\, L \,/\, \mathrm{Dim}(\mathrm{Np}) = 1,$ $\mathrm{Dim}\, L$ is undetermined)	$\mathrm{Np}_{F_0}(F) = \ln\frac{F}{F_0}, \quad \mathrm{Np}_{P_0}(P) = \frac{1}{2}\ln\frac{P}{P_0}$ $\left(\mathrm{Dim}\left(\mathrm{Np}_{F_0}(F)\right) = \mathrm{Dim}\left(\mathrm{Np}_{P_0}(P)\right) = 1\right)$
	$\frac{L}{\mathrm{dB}} = 10\lg\frac{P}{P_0} = 20\lg\frac{F}{F_0}$ $(\mathrm{Dim}\, L \,/\, \mathrm{Dim}(\mathrm{dB}) = 1,$ $\mathrm{Dim}\, L$ is undetermined)	$\mathrm{dB}_{F_0}(F) = 20\lg\frac{F}{F_0}, \quad \mathrm{dB}_{P_0}(P) = 10\lg\frac{P}{P_0}$ $\left(\mathrm{Dim}\left(\mathrm{dB}_{P_0}(P)\right) = \mathrm{Dim}\left(\mathrm{dB}_{F_0}(F)\right) = 1\right)$