A Unit-Based Framework for Quantity Dimensions and Base-Quantity Selection

Seungtae Kim
Department of Life Sciences, Graduate School of Arts and Sciences, The University of Tokyo, Tokyo, Japan
E-mail 1: steven-kim@g.ecc.u-tokyo.ac.jp
E-mail 2: kstt0924@gmail.com

Abstract

The International System of Quantities (ISQ) relies on conventional choice to determine its base quantities, without providing explicit mathematical and physical criteria for their selection. This paper proposes an alternative framework in which unit relations are used to derive quantity dimensions before base quantities are selected. A broader class of unit-explicit equations is also introduced for relations in which specified unit components are retained explicitly when they cannot be reduced to pure numbers without loss of physically relevant information.

Candidate base-quantity sets are first evaluated for algebraic admissibility, requiring mutual dimensional independence and dimensional completeness for the system under consideration. Because algebraically admissible sets need not be unique, a second, physically motivated criterion is introduced to identify preferred quantities on the basis of established higher-order dimensional occurrences associated with distinct physical roles. Applying this two-stage procedure to an SI-oriented set retains mass, length, time, electric current, thermodynamic temperature, and amount of substance. Plane angle is additionally selected as a preferred base quantity through the second-order angular structure of solid angle. By contrast, luminous intensity is classified as a derived quantity with the same unit-based dimensional structure as radiant intensity while remaining a distinct quantity kind. Finally, logarithmic level values conventionally expressed using the neper and decibel are reformulated as numerical index mappings whose outputs are pure numbers and whose arguments retain the relevant field- or power-quantity context. The proposed framework provides a common unit-based route from unit relations to quantity dimensions and base-quantity selection, with angular, photometric, logarithmic, and unit-explicit formulations serving as applications of the same construction.

1. Introduction

According to the International Vocabulary of Metrology (VIM), a base quantity is defined as a "quantity in a conventionally chosen subset of a given system of quantities, where no subset quantity can be expressed in terms of the others"[1]. A conceptually equivalent definition is adopted in ISO 80000-1:2022 published by the International Organization for Standardization (ISO), which provides general information and definitions concerning quantities, systems of quantities, units, and coherent unit systems, particularly the International System of Quantities (ISQ)[2]. The phrase "conventionally chosen" makes explicit that base-quantity selection contains a conventional element. This raises the question of whether additional mathematical and physical criteria can be introduced to constrain or guide the selection of base quantities beyond algebraic independence alone. For six of the current base quantities—mass, length, time, electric current, thermodynamic temperature, and amount of substance—their dimensional treatment does not present the principal problems considered in this study.

Luminous intensity, however, despite its close relationship with radiant intensity in radiometry, is treated as an independent base quantity and hence as an independent base dimension[2,3]. The current SI indeed retains the candela as the base unit of luminous intensity. This treatment contrasts with the relationship between absorbed dose $D_{\mathrm{T,R}}$, expressed in the unit gray (Gy), and equivalent dose H_{T}, expressed in the unit sievert (Sv), in radiation protection[4,5]. Specifically, H_{T} is obtained by weighting $D_{\mathrm{T,R}}$ for each radiation type R with the dimension-one radiation weighting factor w_{R}, whose pure numerical value accounts for differences in biological effectiveness without introducing an additional unit component. In the notation adopted here, retaining the respective units explicitly gives the numerical value equation as:

$$\frac{H_{\mathrm{T}}}{\mathrm{Sv}} = \sum_{\mathrm{R}} w_{\mathrm{R}} \frac{D_{\mathrm{T,R}}}{\mathrm{Gy}}$$

A structurally analogous relationship exists between luminous intensity I_{v} and radiant intensity I_{e}. Because the sensitivity of the human visual system to radiation varies with wavelength, the BIPM/CIE relation for photopic luminous intensity is[6,7]

$$I_{\mathrm{v}} = K_{\mathrm{m}} \int_{0\ \mathrm{m}}^{\infty\ \mathrm{m}} I_{\mathrm{e},\lambda}(\lambda) V(\lambda)\, \mathrm{d}\lambda$$

where $I_{\mathrm{e},\lambda}(\lambda) = dI_{\mathrm{e}}/\mathrm{d}\lambda$ is the spectral radiant intensity, $V(\lambda)$ is the spectral luminous efficiency function for photopic vision, and K_{m} is the maximum luminous efficacy for photopic vision. The latter is related to the SI defining constant K_{cd} by[6,7]

$$K_{\mathrm{m}} = K_{\mathrm{cd}} \frac{V(\lambda_{\mathrm{m}})}{V(\lambda_{\mathrm{cd}})}$$

where

$$K_{\mathrm{cd}} = 683\ \mathrm{lm/W}$$

exactly, $\lambda_{\mathrm{m}} = 555\ \mathrm{nm}$ exactly, and

$$V(\lambda_{\mathrm{m}}) = 1.000\ 000\ 000\ 000\ 0$$

Thus, in standard air,

$$K_{\mathrm{m}} = (683\ \mathrm{lm/W}) \frac{1}{V(\lambda_{\mathrm{cd}})} = 683.002\ \mathrm{lm/W}$$

where λ_{cd} is the wavelength corresponding to the defining frequency 540×10^{12} Hz and is approximately $555.017\ \mathrm{nm}$ in standard air[6,7]. Thus, the relationships for equivalent dose and luminous intensity share a similar mathematical structure in that a weighting factor whose value is a pure number accounts for a biological or physiological response. An important difference, however, is that the photometric relation additionally contains the scale-setting luminous efficacy K_{m}, whose value is linked to the defining constant K_{cd}. Accordingly, the analogy alone does not determine the dimensional status of luminous intensity. Rather, it motivates examining whether the established photometric–radiometric correspondence requires luminous intensity to possess an independent dimensional structure.

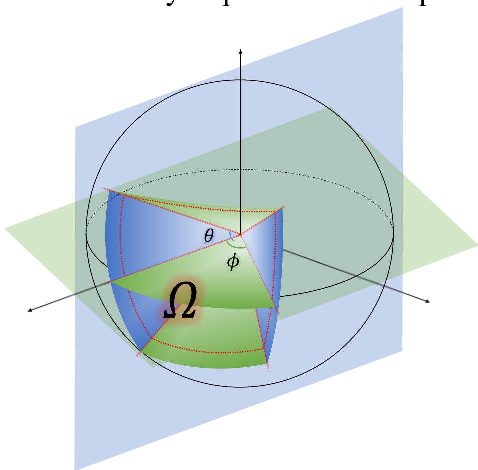

Figure 1. Geometrical relationship between plane and solid angles.
Mutually orthogonal angular coordinates/planes θ and ϕ are defined in the blue and green planes, respectively. The solid angle Ω is the three-dimensional angular region whose magnitude is obtained by double integration over the two corresponding angular coordinates, as bounded by the red dashed lines. The figure illustrates the geometrical distinction between plane angle and solid angle.

Furthermore, plane angle and solid angle possess distinct geometrical natures (Figure 1), and the current SI explicitly recognizes that angular quantities should not be identified with ordinary ratios merely because their conventional geometrical expressions involve such ratios. In the current SI Brochure, the representation of the radian as $\mathrm{m/m}$ is described as non-intrinsic and potentially misleading because plane angle is not the same kind of quantity as other length ratios; an analogous statement is made for the representation of the steradian as $\mathrm{m^2/m^2}$ because solid angle is not the same kind of quantity as other area ratios[3]. Nevertheless, for reasons of history and convention, the SI continues to treat plane and solid angles as quantities with the unit one, while the unit symbols for the radian and steradian are written explicitly where appropriate to indicate the angular nature of the quantities involved[3]. The present work accepts the distinction between angular quantities and ordinary ratios recognized in the current SI, but investigates a different dimensional consequence: whether the radian should instead retain a nontrivial angular dimension, with the steradian subsequently derived from its second-order unit structure.

A related issue arises for logarithmic quantities. For example, field and power levels are expressed as:

$$\frac{L}{\mathrm{Np}} = \ln\frac{F}{F_0}$$

$$\frac{L}{\mathrm{dB}} = 20\,\lg\frac{F}{F_0} = 10\,\lg\frac{P}{P_0}$$

where $\lg x = \log_{10} x$, and the arguments of the logarithms are ratios of quantities of the same kind that reduce algebraically to pure-numbers[8,9]. These equations are dimensionally homogeneous and explicitly relate the numerical value of the level L to its unit. Historically, the neper was adopted by the CIPM as a special name for the coherent unit one for logarithmic quantities defined using natural logarithms[8]. In the current SI Brochure, however, the neper, bel, and decibel are listed as non-SI units used to express specified logarithmic ratio quantities[3]. The issue considered here is therefore not whether these symbols constitute independent SI dimensions, but whether unit notation is required at all when a logarithmic result is represented directly as a numerical index.

Finally, the ISO 80000 series describes quantity dimensions in terms of base quantities and subsequently associates units with the corresponding quantities[1,2]. However, the numerical component of a quantity value does not by itself identify either the kind of quantity or a nontrivial dimensional class. This point is particularly clear for quantities of dimension one: their values and units may be numbers, while the quantities themselves retain information beyond the numerical value alone[1]. The present work therefore adopts a constructive, rather than ontological, unit-first ordering. Relations among unit expressions are used as the starting point for constructing dimensional subsets, while distinctions of quantity kind and physical role are retained separately through the quantity symbol, definition, and physical context.

Motivated by these examples, I examine a framework in which unit relations are first used to construct dimensional subsets, after which algebraic admissibility and physical preference are considered in base-quantity selection. The dimensional construction developed here is based on the algebraic characteristics of quantities and the principle of unit consistency, which underlies dimensional homogeneity; physical information is subsequently introduced at the stage of base-quantity preference. In this framework, conventional physical equations are also reconsidered when the omission of an explicit unit component results in the loss of physically relevant information; such relations are instead expressed as unit-explicit equations in which the required unit components are retained. Through this approach, I aim to construct a quantity and dimension system that more faithfully reflects the mathematical and physical properties of the quantities being described. Individual aspects of the problems considered here have been addressed in previous studies. In particular, independent dimensional treatments of plane angle, explicit retention of angular units in physical equations, alternative formulations of dimensional analysis, and revisions of the hierarchy of SI units have each been investigated from different perspectives[10–15]. Recent work has also reconsidered the internal organization and generative structure of the SI units, emphasizing the importance of a coherent generative framework when discussing possible revisions of the base–derived unit hierarchy[16]. The novelty claimed in the present work is therefore not the algebraic separation of quantities, dimensions, and units, nor the use of a mapping from quantities to dimensions, for which substantial precedents exist[17–20]. Nor is it claimed that the problem of distinguishing physically different quantities whose values are pure numbers is new; alternative treatments of this issue have also been proposed[21]. Rather, the distinctive feature of the present framework is the specific logical ordering in which dimensional subsets are induced from unit relations, the resulting dimensional structure is accessed through Dim, and algebraic admissibility and physical preference are subsequently considered in base-quantity selection.

2. Algebraic Characteristics of Quantities

A quantity value may be expressed as a numerical multiple of a reference unit. For example, let the mass m of an object be 3 kg:

$$m = 3\ \mathrm{kg}$$

This equation means that m is three times the reference unit, the kilogram. That is, it is equivalent to:

$$m = 3 \times \mathrm{kg}$$

where the multiplication symbol $\times$ can be considered omitted between the numerical value and the unit. Regarding notation, hereafter, in accordance with the standard notation used in quantity calculus, the numerical value and the measurement unit associated with a quantity Q are denoted as $\{Q\}$ and $[Q]$, respectively[1,2].

$$Q = \{Q\}\,[Q], \quad \{Q\} = \frac{Q}{[Q]}$$

Because $\{Q\}$ is merely the numerical component of a quantity value, it does not by itself identify the kind or physical role of the quantity represented. For a quantity having a nontrivial unit component, $[Q]$ additionally provides the reference scale relative to which its numerical value is expressed. In the present framework, when the unit component of a quantity is algebraically reducible to a pure number, the corresponding dimensional structure is classified as dimension one. This algebraic classification does not imply that all such quantities are physically identical. This distinction has precedent in both standard metrological terminology and alternative algebraic treatments. The VIM explicitly notes that quantities of dimension one convey more information than a number even though their values and measurement units are numbers[1]. Kock's algebraic formulation likewise implies that the ratio of two quantities having the same dimension is a pure quantity[18]. Johansson has proposed a different treatment in which relative quantities such as relative length and mass fraction may be unit-less while retaining a parametric dimensional descriptor[21]. The present framework does not adopt this additional parametric dimensional structure. Instead, ordinary multiplicative cancellation is retained, while distinctions among quantities of dimension one are carried by their quantity symbols, definitions, and physical roles.

Accordingly, quantities such as the Reynolds number and pH may have pure numerical values without losing their physical identity. Their symbols identify what the numerical values represent; for example $Re = 100$ and $\mathrm{pH} = 7.00$, respectively.

A ratio-form unit expression may nevertheless be retained when it provides useful scaling or contextual information. For example, a mass fraction w of dimension one may be reported as $w = 5\ \mathrm{mg/kg}$. Such a representation makes both the numerical scale and the ratio form explicit, but it does not introduce a nontrivial dimension in the present framework. The physical interpretation of the ratio remains specified by the quantity symbol w and its accompanying definition.

For the non-zero scalar quantities considered below, multiplication and division obey the usual Abelian-group properties of closure, associativity, identity, inverses, and commutativity, as in formal axiomatizations of quantity calculus[20]. Addition and subtraction are more restrictive: they are meaningful only for quantities of the same kind, and numerical values expressed in different units must first be converted to a common unit.

Importantly, additive cancellation removes the numerical magnitude without eliminating the unit reference:

$$Q - Q = 0\,[Q]$$

rather than a bare numerical zero. By contrast,

$$\frac{Q}{Q} = 1$$

so the unit component is eliminated through multiplicative cancellation. This distinction will be used below when separating dimensional structure from the particular unit selected for a concrete representation.

3. Numerical Value Equations and Unit-Explicit Equations

According to ISO 80000-1:2022, a numerical value equation is a mathematical relation between numerical quantity values, based on a given quantity equation and specified measurement units[2]. The VIM gives the same definition and denotes the numerical value of a quantity Q, when expressed in a specified measurement unit $[Q]$ of the same kind, by $Q/[Q]$[1]. The term **numerical value equation** is retained here in this conventional sense.

For the broader class of relations required in the present framework, I introduce the term **unit-explicit equation**. A unit-explicit equation is a mathematical relation in which one or more quantity expressions are explicitly divided by specified unit quantities and in which the resulting unit components are retained as part of the equation. A conventional numerical value equation is included as a special case when each unit-divided expression represents the numerical value of the corresponding quantity. More generally, however, the specified dividing unit need not be a unit of the same kind as the quantity expression being divided; in that case, the resulting expression is not itself a numerical quantity value.

This terminology does not extend or redefine the conventional meaning of a numerical value equation. Rather, it distinguishes numerical value equations from the broader unit-explicit relations used below. Because multiplication and division of scalar quantities are associative and commutative, as discussed in Section 2, the placement of an explicitly retained unit factor within a multiplicative expression does not affect the underlying algebraic relation.

The dependence of such relations on the specified unit can be demonstrated as follows.

Let us consider two different measurement units u_1 and u_2 for the same kind of quantity Q, and let q_1 and q_2 denote the corresponding numerical values. A fixed measurement unit represents a constant quantity interval: an increment of one unit represents the same physical increment irrespective of the position on the numerical scale. For the same physical interval $\varDelta Q$, let $\varDelta q_1$ and $\varDelta q_2$ denote the corresponding numerical intervals when expressed using u_1 and u_2, respectively. Then $\varDelta Q$ is expressed as:

$$\varDelta Q = \varDelta q_1\,\mathrm{u}_1 = \varDelta q_2\,\mathrm{u}_2$$

Therefore,

$$\frac{\varDelta q_1\,\mathrm{u}_1}{\varDelta q_2\,\mathrm{u}_2} = 1$$

Since u_1 and u_2 are fixed measurement units, the ratio between the corresponding numerical intervals is constant. Defining k as:

$$k = \frac{\Delta q_1}{\Delta q_2}$$

we obtain:

$$\Delta q_1 = k\Delta q_2$$

Substitution into the preceding relation gives:

$$\mathrm{u}_2 = k\mathrm{u}_1$$

Hereafter, a relation of this form, where k is a non-zero pure number, is referred to as a **unit relation** in the present framework. A unit relation concerns the algebraic structure of unit expressions and does not, in general, require the associated quantities to be of the same kind. When u_1 and u_2 are measurement units for quantities of the same kind, this relation corresponds to the conventional relation between convertible units, and k or its reciprocal represents the corresponding conversion factor[1]. In the present construction, the relevant domain consists of non-zero unit expressions admitted in the quantity system under consideration, together with expressions obtained from them by finite multiplication, division, and integer powers; these operations are sufficient for the dimensional relations considered below.

To obtain the general relation between the numerical values themselves, let $Q_{\mathrm{rf},1}$, $Q_{\mathrm{rf},2}$ denote the reference quantity values assigned on respective scales to the same physical reference state. These reference quantity values need not be equal because the two scales may have different origins. Accordingly,

$$\Delta q_1 = q_1 - \frac{Q_{\mathrm{rf},1}}{\mathrm{u}_1}$$
$$\Delta q_2 = q_2 - \frac{Q_{\mathrm{rf},2}}{\mathrm{u}_2}$$

Substituting these relations into $\Delta q_1 = k\Delta q_2$ yields:

$$q_1 = kq_2 + l$$

where

$$l = \frac{Q_{\mathrm{rf},1}}{\mathrm{u}_1} - k\frac{Q_{\mathrm{rf},2}}{\mathrm{u}_2}$$

Thus, the general relation between the numerical values of two scales having constant unit intervals is an **affine relation**. The coefficient k represents the ratio of the unit intervals, whereas l represents the difference between the scale origins. Any nonlinear transformation would cause the ratio of corresponding intervals to depend on the position on the scale, contradicting the constancy of the unit interval. This result is consistent with the classical treatment of interval scales, whose admissible transformations are affine[22]. A representative example of $l \neq 0$ is the conversion formula between Celsius temperature t and thermodynamic temperature T:

$$\frac{T}{\mathrm{K}} = \frac{t}{{}^{\circ}\mathrm{C}} + 273.15$$

In this case, $k = 1$, because the degree Celsius and kelvin have the same magnitude for temperature intervals, whereas $l = 273.15$ results from the different origins of the two scales. Thus, the additive term concerns the relation between the scale origins and does not alter the unit relation for temperature intervals.

Because k is uniquely determined by u_1 and u_2, including a quantity Q divided by a specified unit in an equation is equivalent to involving the unit conversion factor k or $1/k$; that is,

$$\{Q\}_{\mathrm{u}_2} = \frac{Q}{\mathrm{u}_2} = k\frac{Q}{\mathrm{u}_1} = k\{Q\}_{\mathrm{u}_1}$$
$$\{Q\}_{\mathrm{u}_1} = \frac{Q}{\mathrm{u}_1} = \frac{1}{k}\frac{Q}{\mathrm{u}_2} = \frac{1}{k}\{Q\}_{\mathrm{u}_2}$$

Therefore, explicit division by a specified measurement unit makes the mathematical form of the relation dependent on the choice of that unit.

Consequently, when explicit division by a specified unit introduces a unit component that cannot be algebraically eliminated or consolidated without loss of physically relevant information, the resulting relation is classified here as a unit-explicit equation. Conventional numerical value equations constitute the special case in which the relevant unit-divided quantities are numerical quantity values. By contrast, a unit-explicit equation may also contain unit-divided expressions that are not numerical values in the conventional metrological sense. The purpose of retaining such unit components is therefore not to modify the conventional definition of a numerical value equation, but to preserve unit information that would otherwise be suppressed in the mathematical form of the relation. In the present framework, this distinction becomes particularly important for equations involving angular units. Accordingly, unit-explicit equations expressed using different representative units are alternative unit-dependent forms of the same underlying quantity relation, related through the unit transformations derived above.

4. Dimensional Subsets Defined by Unit Relations

As discussed in the preceding section, unit relations characterize the algebraic structure of unit expressions rather than the physical kind of the associated quantities. Unit expressions connected through a unit relation therefore share the same dimensional characteristics even when the quantities to which they are assigned are of different kinds. This is consistent with the conventional distinction between dimension and kind of quantity: quantities of the same kind necessarily have the same dimension, whereas the converse does not generally hold[1].

A unit relation of the form:

$$\mathrm{u}_2 = k\mathrm{u}_1$$

where $k \in \mathbb{R} \setminus \{0\}$ is a non-zero pure number, is reflexive, symmetric, and transitive. It therefore partitions unit expressions into mutually disjoint subsets. Accordingly, two unit expressions belong to the same subset if and only if they are connected by such a unit relation. If no such relation exists within the unit-relation system under consideration, they belong to distinct subsets; identifying the two subsets would require an additional unit relation of this form. Within the present unit-first construction, these subsets are taken to constitute quantity dimensions.

The multiplicative structure of these subsets is independent of the particular unit representatives chosen. If $\mathrm{u}_1 = k_1\mathrm{u}_1'$, $\mathrm{u}_2 = k_2\mathrm{u}_2'$, where k_1 and k_2 are non-zero pure numbers, then

$$\mathrm{u}_1\mathrm{u}_2 = (k_1k_2)\mathrm{u}_1'u_2'$$

Since k_1k_2 is again a non-zero pure number, replacing either representative by another unit in the same subset does not change the subset associated with their product. The same argument applies to division and inversion. Thus, multiplication and inversion of unit expressions induce well-defined multiplicative operations on the corresponding dimensional subsets.

Algebraic treatments relating quantities, dimensions, and units have substantial precedent. Kock connected physical quantities and dimensions through an explicit dimension map and treated ratios of quantities having the same dimension as pure quantities[18]. Kitano organized unit systems through transfer relations and equivalence classes within which quantity dimensions are defined[19], while Raposo developed a dimension-centered algebraic quantity calculus[17]. The present construction shares the ordinary cancellative treatment of dimensional multiplication used in these approaches but differs in its specific logical ordering: dimensional subsets are induced directly from relations among unit expressions before base-quantity selection is considered. These subsets classify unit-based dimensional structure rather than quantity kind. The present construction requires only this induced multiplicative structure and does not claim that the equivalence relation by itself establishes a free Abelian group of dimensions. Stronger algebraic structures have been adopted in previous axiomatizations—for example, a free Abelian dimension group by Raposo[17] and a multiplicative vector-space structure over the rational numbers by Kock[18] —but these additional assumptions are not required for the arguments developed here.

Within the present framework, a quantity whose unit component is reducible to a pure number is assigned dimension one. Numerical quantity values are pure numbers and are therefore assigned dimension one within the present framework, and a normalized unit vector likewise has dimension one because its numerator and denominator possess the same dimensional structure. As discussed in Section 2, this dimensional classification does not erase the quantity's physical meaning or kind.

Let $\mathrm{Dim}\,Q$ denote the dimensional subset associated with the unit component of a quantity Q. Equality under Dim denotes equality of unit-based dimensional structure and not identity of quantity kind. Because multiplication of unit components induces multiplication of the corresponding dimensional subsets:

$$\mathrm{Dim}(Q_1Q_2) = \mathrm{Dim}\,Q_1\,\mathrm{Dim}\,Q_2$$

and consequently

$$\mathrm{Dim}(Q^n) = (\mathrm{Dim}\,Q)^n$$

for integer n.

Such multiplicative behavior is not itself new. Raposo explicitly introduced a surjective projection

$$\dim : Q \to D$$

from a space of quantities to a group of dimensions and required this map to be a homomorphism under multiplication[17]. The function Dim introduced here adopts the same basic multiplicative behavior. Its distinctive role lies instead in the origin of its target: the dimensional subsets have already been constructed from unit relations rather than postulated beforehand as the base structure of quantity calculus.

Additive compatibility is more restrictive than dimensional compatibility. Quantities that are added or subtracted must be of the same kind, whereas equality under Dim expresses only equality of their unit-based dimensional structure. Consequently,

$$\mathrm{Dim}(Q_1) = \mathrm{Dim}(Q_2)$$

does not by itself imply that Q_1 and Q_2 may be added.

This distinction is also recognized in Raposo's algebraic formulation: quantities of the same kind belong to the same dimensional fiber, whereas the converse need not hold, and the algebraic structure alone does not distinguish this additional quantity-kind information[17]. For a physically meaningful equation, addition and subtraction must therefore respect quantity kind, while all terms connected by an equality must remain dimensionally compatible, as required by dimensional homogeneity.

Integration and differentiation inherit their dimensional behavior from multiplication and division. For an integrand Q and an integration variable X, the dimensional structure of the integral is:

$$\mathrm{Dim}\left(\int Q \,\mathrm{d}X\right) = \mathrm{Dim}\, Q\, \mathrm{Dim}\, X$$

whereas

$$\mathrm{Dim}\left(\frac{\mathrm{d}Q}{\mathrm{d}X}\right) = \mathrm{Dim}\, Q\, (\mathrm{Dim}\, X)^{-1}$$

These rules are consistent with the formal treatment of differential and integral calculus in algebraic quantity calculus given by Raposo, in which differentiation introduces the inverse dimension of the independent-variable quantity and integration introduces its dimension multiplicatively[23]. Detailed derivations of these standard algebraic consequences are therefore not repeated here.

As discussed in Section 2, however, dimensional structure and the concrete representation of a quantity value should be distinguished. When a quantity Q possesses a nontrivial unit component $[Q]$, elimination of its numerical magnitude does not by itself eliminate that unit component; for example, additive cancellation gives

$$Q - Q = 0\,[Q]$$

rather than a bare numerical zero. More generally, a concrete value of such a quantity is represented relative to a specified unit, whereas $\mathrm{Dim}\, Q$ identifies only the dimensional subset to which that unit belongs. The dimensional subset therefore does not determine which particular unit is to be used as its representative.

The choice of a representative unit is consequently a separate metrological question. Kitano has noted that the number and choice of base units are, in principle, partly arbitrary and that the resulting multiplicity of possible unit systems creates a strong need for standardization[19]. The present framework likewise separates the dimensional structure itself from the particular units chosen to represent it: dimensional subsets are established independently of whether, for example, one member of a given subset rather than another is adopted as the internationally agreed reference unit.

Once the dimensional subsets have been established, it therefore becomes necessary to distinguish two subsequent questions: which physical quantities should be selected to form an appropriate base-quantity set, and which representative units should be internationally adopted for those quantities. The former question is considered first in Section 5, where algebraic admissibility and physical preference are treated separately. Representative units of the selected base quantities may then be designated as base units.

5. Algebraic Admissibility and Physical Preference in Base-Quantity Selection

The dimensional subsets derived in the preceding section do not by themselves uniquely determine which physical quantities should be chosen as base quantities. It is therefore useful to distinguish between the **algebraic admissibility** of a candidate set and the physical preference among algebraically admissible alternatives.

Criterion 1: Algebraic admissibility. A candidate set of quantities for use as base quantities is regarded as algebraically admissible, within the system of quantities under consideration, when two conditions are satisfied. First, the dimensions associated with its members must be mutually independent under multiplicative operations; that is, the dimension associated with any member cannot be expressed as a product of powers of the dimensions associated with the other members. Second, the set must be dimensionally complete: the dimension of every quantity considered within the system must be expressible as a product of powers of the dimensions associated with members of the candidate set.

A quantity of dimension one cannot provide an independent dimensional component and is therefore excluded automatically by the first condition. The two conditions above establish only whether a candidate set is algebraically admissible; they do not generally determine a unique set of base quantities. Different admissible sets may represent the same dimensional structure.

Criterion 2: Physical structural preference. Because algebraic admissibility concerns only the unit-based dimensional information represented by Dim, it does not distinguish alternative candidates according to their quantity kind or physical role. Among algebraically admissible alternatives, a quantity is regarded as having physical structural support for base-quantity status when its associated dimension exhibits an established higher-order occurrence, with a net exponent n satisfying $|n| > 1$, in a derived quantity, physical relation, or geometrical construction, and when that occurrence has an established physical role or meaning distinct from the corresponding first-order occurrence.

Purely formal powers introduced by algebraic manipulation do not by themselves satisfy this criterion. Likewise, the mere construction of a statistical moment or variance is not regarded as sufficient evidence of a distinct physical role. Criterion 2 is non-exclusive: if two or more algebraically admissible candidates independently satisfy the criterion, it does not by itself rank them relative to one another.

The following application retains the present SI-oriented dimensional coordinates as the starting admissible choice and examines whether the corresponding physical quantities satisfy

Criterion 2. This procedure is not intended to establish that these quantities constitute the only algebraically admissible basis.

Mass: Universal gravitation between two bodies contains the product of two masses,

$$F = G\frac{m_1 m_2}{r^2}$$

so that the mass dimension occurs to the second power in a physical interaction that is distinct from the gravitational field produced by a single source mass. Mass therefore satisfies Criterion 2 and is retained as a preferred base quantity within the present SI-oriented admissible set.

Length: Mutually orthogonal lengths induce the concepts of area and volume:

$$A \propto L^2$$
$$V \propto L^3$$

These quantities are physically and geometrically distinct from length itself. Length therefore satisfies Criterion 2 through the established geometrical roles of its second- and third-order occurrences in area and volume, and is retained in the preferred set.

Time: It serves as the quantity that differentiates, for example, velocity from acceleration and momentum from force. In particular,

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

so that the inverse time dimension occurs at first and second order, respectively, with physically distinct meanings. The inverse second-order occurrence in acceleration, as distinct from the first-order occurrence in velocity, satisfies Criterion 2.

Electric current: Electric current is retained here as the electromagnetic coordinate of the SI-oriented admissible set. This does not imply that it is algebraically unique; alternative electromagnetic quantities, such as electric charge, may be used in alternative admissible bases. Moreover, established physical relations contain higher powers of electric current; for example, Joule heating is expressed as:

$$P = I^2R$$

Electric charge may likewise satisfy Criterion 2 in an alternative admissible basis, for example through its established second-order occurrence in electrostatic interactions. Criterion 2 therefore does not uniquely rank electric current above electric charge; the present discussion establishes only that electric current has physical structural support within the SI-oriented admissible set considered here. Within the present SI-oriented choice, the established second-order occurrence of current in Joule heating satisfies Criterion 2.

Thermodynamic temperature: According to the Stefan-Boltzmann law, radiant exitance is proportional to the fourth power of thermodynamic temperature:

$$M = \sigma T^4$$

Thus, the fourth power of temperature has an established physical role distinct from temperature itself. Thermodynamic temperature therefore satisfies Criterion 2 and is retained in the preferred SI-oriented set.

Amount of substance: A second-order occurrence arises, for example, in the thermodynamic Hessian:

$$\frac{\partial \mu_i}{\partial n_j} = \frac{\partial^2 G}{\partial n_j \partial n_i}$$

whose dimensional structure contains the inverse square of the amount-of-substance dimension. Because this thermodynamic Hessian characterizes thermodynamic response and stability, the second-order occurrence has an established physical role beyond a merely formal square or statistical moment. Amount of substance therefore satisfies Criterion 2 and is retained in the preferred SI-oriented set.

The corresponding SI base units are the kilogram, metre, second, ampere, kelvin, and mole, and the dimensional symbols used here are M, L, T, I, Θ, and N, respectively.

Plane angle and luminous intensity provide two contrasting applications of these criteria and are considered separately in Sections 5.1 and 5.2.

5.1. Plane Angle as a Preferred Base Quantity

As noted in Section 1, the current SI distinguishes physical plane angle from ordinary length ratios while treating plane angle as a quantity with the unit one for reasons of history and convention[3]. The present framework examines whether this distinction should instead be reflected in a nontrivial angular dimension. This dimensional status is not assumed here; it is determined from the unit relations according to the equivalence-class construction of Section 4 before Criterion 1 is applied.

For a circular sector, the relationship among the arc length l, the radius r, and the central angle θ is derived from the proportion:

$$l : 2\pi r = \theta : 2\pi \text{ rad}$$

Therefore:

$$l = r\theta/\text{rad} \Leftrightarrow \frac{\theta}{\text{rad}} = \frac{l}{r}$$

These relations are unit-explicit equations in the sense defined in Section 3 because the expression involving the physical angle explicitly retains division by the specified angular unit. In this particular case, θ/rad is also the numerical value of the physical plane angle when the radian is selected as its measurement unit[10,11,13].

The numerical value of a physical plane angle depends on the angular unit chosen as its reference. The same physical angle may, for example, be expressed in radians, degrees, gon, or turns. These angular units are interconnected through unit relations[9,24]:

$$1^\circ = \frac{\pi}{180}\ \mathrm{rad}$$
$$1^\mathrm{g} = \frac{\pi}{200}\ \mathrm{rad}$$

and, for one complete turn:

$$1\ \mathrm{tr} = 2\pi\ \mathrm{rad}$$

Consequently, the numerical values of the same physical angle satisfy

$$\frac{\theta}{\mathrm{rad}} = \frac{\pi}{180}\frac{\theta}{^\circ} = \frac{\pi}{200}\frac{\theta}{^\mathrm{g}} = 2\pi\frac{\theta}{\mathrm{tr}}$$

Thus, the ratio l/r in the preceding equation corresponds specifically to the numerical value of θ when the radian is selected as the reference angular unit, rather than to the physical plane angle itself[11,13].

To simplify, I introduce the underbar notation:

$$\theta/\mathrm{rad} = \frac{\theta}{\mathrm{rad}} \to \frac{\theta}{-} = \underline{\theta}$$

Therefore,

$$\underline{\theta} = \frac{l}{r}$$

and

$$\mathrm{Dim}\,\underline{\theta} = 1$$

The cancellation of the length units in l/r therefore establishes that the radian-based numerical angular value $\underline{\theta}$ is a pure number and therefore has dimension one. It does not by itself establish that either the physical plane angle θ or the radian has dimension one[11,13]. Under the equivalence-class construction of Section 4, the radian would belong to the dimension-one subset only if a unit relation

$$\mathrm{rad} = k \cdot 1 \quad (k \in \mathbb{R} \setminus \{0\})$$

were established. The circular-sector relation above provides no such unit relation.

Previous treatments in which plane angle is assigned an independent dimension obtain solid angle as a derived angular quantity with a coherent unit satisfying $\mathrm{sr} = \mathrm{rad}^2$[11,13]. This second-order angular structure also has a practical analogue in the long-standing use of the square degree as a unit of solid angle in astronomy[14,25]. These precedents provide context for a nontrivial angular dimensional structure, but they are not used as a premise for assigning the radian to its dimensional subset in the present derivation. Applying the equivalence-class construction of Section 4 to the unit relations established above places the radian in a subset distinct from dimension one. The corresponding plane-angle dimension is denoted by:

$$\mathrm{Dim(rad)} = \mathsf{A}, \quad \mathsf{A} \neq 1$$

The corresponding dimension of the steradian will be derived below from its unit relation, following the unit-to-dimension hierarchy established in Section 4.

Before deriving this relation independently from the geometry of solid angle, it is necessary to clarify the dimensional interpretation of trigonometric and inverse trigonometric functions. The sine and cosine functions are expressed by Taylor series:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

For the ordinary mathematical definitions of these functions, x is a pure numerical variable. Correspondingly, dimensional homogeneity of the power-series representation requires:

$$\mathrm{Dim}\, x = 1$$

However, this condition alone does not determine which angular unit is involved. As shown above,

$$\frac{\theta}{\mathrm{rad}} \quad \frac{\theta}{{}^\circ} \quad \frac{\theta}{{}^\mathrm{g}} \quad \frac{\theta}{\mathrm{tr}}$$

are all pure numbers. The ordinary mathematical sine and cosine functions conventionally used in scientific equations act on pure numerical arguments corresponding to radian-based numerical angular values[9,12]. Therefore, when x represents the numerical value of a physical plane angle θ,

$$x = \frac{\theta}{\mathrm{rad}} = \underline{\theta}$$

Accordingly, when θ explicitly denotes the dimensional physical plane angle, the corresponding trigonometric expressions are written as:

$$\sin \underline{\theta} = \sin\left(\frac{\theta}{\mathrm{rad}}\right)$$
$$\cos \underline{\theta} = \cos\left(\frac{\theta}{\mathrm{rad}}\right)$$

For example, if the same physical angle is instead represented by its numerical value in degrees,

$$\{\theta_\circ\} = \frac{\theta}{{}^\circ}$$

then the ordinary sine function is evaluated as:

$$\sin\left(\frac{\pi}{180}\{\theta_{\circ}\}\right)$$

rather than as $\sin\{\theta_{\circ}\}$. Thus, the familiar Taylor coefficients apply directly to the radian-based numerical value $\underline{\theta}$, rather than to an arbitrary numerical value of angle expressed in another angular unit[12].

This conclusion is also consistent with the geometrical definition of trigonometric functions. For a circle of radius r centered at the origin and a point (a, b) on the circumference

$$a^2 + b^2 = r^2$$

and

$$\sin x = \frac{b}{r}$$
$$\cos x = \frac{a}{r}$$

Since a, b, and r possess the same dimension,

$$\mathrm{Dim}\left(\frac{b}{r}\right) = \mathrm{Dim}\left(\frac{a}{r}\right) = 1$$

The inverse trigonometric functions likewise return the radian-based numerical angular value rather than dimensional physical angle itself[12,13]. For example,

$$\arcsin\frac{b}{r} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!\cdot(2n+1)}\left(\frac{b}{r}\right)^{2n+1} = \frac{b}{r} + \frac{1}{6}\left(\frac{b}{r}\right)^3 + \frac{3}{40}\left(\frac{b}{r}\right)^5 + \cdots$$

and therefore

$$\arcsin\left(\frac{b}{r}\right) = x = \underline{\theta} = \frac{\theta}{\mathrm{rad}}$$

The corresponding physical angle is

$$\theta = \left(\arcsin\left(\frac{b}{r}\right)\right)\ \mathrm{rad}$$

Thus, ordinary trigonometric functions operate on radian-based numerical angular values, and ordinary inverse trigonometric functions return such numerical values[12]. This does not imply that the corresponding physical plane angle itself has dimension one. If the argument of a mathematical trigonometric function is not interpreted as a physical angular quantity, no explicit angular unit need be introduced.

With this distinction established, the unit structure of solid angle can now be derived directly. Let Ω denote a solid-angle quantity expressed using the steradian as its reference unit:

$$[\Omega] = \mathrm{sr}$$

For a fixed steradian reference unit, writing $\Omega = \{\Omega\}\,\mathrm{sr}$ gives:

$$\mathrm{d}\Omega = \mathrm{d}\{\Omega\}\,\mathrm{sr}$$

so that taking the differential does not alter the reference unit in this representation.

$$[\mathrm{d}\Omega] = \mathrm{sr}$$

On the other hand, let θ and ϕ denote the physical polar and azimuthal plane angles, respectively. Geometrically, the infinitesimal solid angle is

$$\mathrm{d}\Omega = \sin\underline{\theta}\ \mathrm{d}\theta\,\mathrm{d}\phi$$

This expression is a unit-explicit equation in the sense defined in Section 3 because it explicitly retains the unit-divided angular expression θ/rad.

Since $\left[\sin\underline{\theta}\right] = 1$, and $[\mathrm{d}\theta] = [\mathrm{d}\phi] = [\theta] = [\phi] = \mathrm{rad}$, the unit component of the right-hand side is rad^2. Therefore, comparison with the independently specified solid-angle unit $[\mathrm{d}\Omega] = \mathrm{sr}$ gives the unit relation:

$$\mathrm{sr} = \mathrm{rad}^2$$

Only after this unit relation has been established do we apply the dimension-extraction function introduced in Section 4. Thus,

$$\mathrm{Dim}\,\Omega = \mathrm{Dim}(\mathrm{sr}) = \mathrm{Dim}(\mathrm{rad}^2) = (\mathrm{Dim}(\mathrm{rad}))^2 = \mathsf{A}^2$$

Accordingly, the second-order angular dimension of solid angle follows from its unit relation, rather than being assumed beforehand. This unit-to-dimension derivation is consistent with the hierarchy developed in Sections 3 and 4.

The corresponding numerical value equation is:

$$\frac{\Omega}{\mathrm{sr}} = \iint \sin\underline{\theta}\ \mathrm{d}\underline{\theta}\,\mathrm{d}\underline{\phi}$$

In analogy with $\underline{\theta} = \theta/\mathrm{rad}$, I introduce the second-order underbar notation

$$\frac{\Omega}{\mathrm{sr}} = \frac{\Omega}{\mathrm{rad}^2} \to \frac{\Omega}{{}_2} \to \underline{\Omega}_2$$

Therefore, the area S of a spherical surface subtending the solid angle Ω at radius r may consequently be written as:

$$S = \underline{\Omega}_2 r^2$$

Thus, solid angle is a geometrically distinct quantity whose unit structure is generated by a second-order product of the plane-angle unit:

$$\mathrm{rad} \;\rightarrow\; \mathrm{rad}^2 = \mathrm{sr}$$

The corresponding dimensional structure then follows as:

$$\mathsf{A} \;\rightarrow\; \mathsf{A}^2$$

The relation $\mathrm{sr} = \mathrm{rad}^2$ determines the unit structure of solid angle relative to that of plane angle. It is compatible with an additional identification of the radian with the unit one, but it does not imply such an identification; that would require an additional unit relation of the type defined in Section 4. The preceding construction establishes a nontrivial angular dimensional subset and the corresponding second-order dimensional structure of solid angle. Once this angular dimension is established, the SI-oriented candidate set containing only mass, length, time, electric current, thermodynamic temperature, and amount of substance is no longer dimensionally complete for the system considered here, because none of those dimensions can generate the angular dimension. Inclusion of plane angle therefore satisfies Criterion 1. At the same time, the geometrically distinct second-order structure of solid angle provides the physical structural preference specified in Criterion 2. Together, these results support the selection of plane angle as a preferred base quantity and solid angle as a derived quantity within the present framework.

Independent dimensional treatments of plane angle, the distinction between a physical angle and its radian-based numerical value, and second-order treatments of solid angle have been discussed previously[10–14], with more recent work addressing both the current SI treatment and the practical realization of angular units[15,26]. In particular, the 19th meeting of the Consultative Committee for Length noted that participating laboratories generally do not realize plane-angle units through the geometrical length-ratio representation in the SI Brochure, but instead mainly use circular-closure methods[27]. Lewis and Coveney subsequently developed this practical dimensional-metrology viewpoint in greater detail[26]. These practical-realization arguments are logically independent of, but compatible with, the unit-based dimensional construction developed here. Taken together, the independent agreement between practical realization and the present algebraic construction strengthens the case for reconsidering the present SI treatment of angle.

An important practical feature of the proposed reform is that its adoption would not require a discontinuity in the numerical realization of plane angle. Existing relations among angular units, such as a complete revolution corresponding to 2π rad, remain unchanged, as do numerical calibration results obtained using established angle standards. The principal change concerns the dimensional and unit representation of angular quantities and of equations in which angular units are conventionally suppressed. Accordingly, existing calibration procedures, uncertainty evaluations, and key-comparison results need not be repeated merely because the radian is assigned a nontrivial dimension; if the proposed framework were adopted, changes would instead arise primarily in dimensional formulae, coherent derived-unit expressions, and associated metrological notation or metadata.

The present treatment differs from the current SI status of the radian, but is proposed as a candidate framework for resolving the dimensional ambiguities that have motivated continuing discussion of angle within the SI. The CCU has considered positions ranging from introducing the radian as a base unit to retaining the status quo, without reaching consensus; the subsequent task group improved the description of angle while leaving the radian's status as a derived unit unchanged[28]. The unit-based construction developed here is intended to contribute to this continuing discussion by providing a systematic basis on which a future revision of the SI dimensional and base–derived unit structure could be considered.

The consequences of this treatment for rotational kinematics and dynamics, periodic quantities, quantum-mechanical relations, and radiometric equations are considered collectively in Section 7.

5.2. Luminous Intensity as a Derived Quantity

Luminous intensity provides a contrasting application of the two criteria. Its relation to radiant intensity is governed by the established photometric–radiometric correspondence through spectral weighting of the visual response[3,6,7]. The dimensional consequences of this correspondence are examined below.

As discussed in Section 1, the current SI fixes the numerical value of the defining constant K_{cd} to exactly 683 when expressed in $\mathrm{lm/W}$ for monochromatic radiation of frequency $\nu_{\mathrm{cd}} = 540 \times 10^{12}$ Hz[3]. The effect of this definition is that a luminous intensity of $1\ \mathrm{cd}$ corresponds exactly to a radiant intensity of $(1/683)\ \mathrm{W/sr}$ in the same direction for this reference radiation[3]. In the generalized unit-relation sense defined in Section 3, this exact correspondence can therefore be written as:

$$1\ \mathrm{cd} = \frac{1}{683}\ \mathrm{W/sr}$$

Since $1/683$ is a non-zero pure number and introduces no additional unit component, the candela and watt per steradian belong to the same dimensional subset within the present framework. Therefore,

$$\mathrm{Dim(cd)} = \mathrm{Dim(W/sr)}$$

Since the candela and watt per steradian are the respective units of luminous intensity and radiant intensity, this relation establishes their common unit-based dimensional structure at the reference radiation.

The same result can be expressed in terms of the defining constant K_{cd}. In the current SI, $K_{\mathrm{cd}} = 683\ \mathrm{lm/W} = 683\ \mathrm{cd \cdot sr/W}$. Applying the dimensional relation obtained above gives:

$$\mathrm{Dim}(K_{\mathrm{cd}}) = 1$$

within the present framework.

As discussed in Section 1, the maximum luminous efficacy for photopic vision is:

$$K_{\mathrm{m}} = K_{\mathrm{cd}} \frac{V(\lambda_{\mathrm{m}})}{V(\lambda_{\mathrm{cd}})}$$

Since the ratio $V(\lambda_{\mathrm{m}})/V(\lambda_{\mathrm{cd}})$ is a pure number,

$$\mathrm{Dim}(K_{\mathrm{m}}) = \mathrm{Dim}(K_{\mathrm{cd}}) = 1$$

The BIPM/CIE relation for photopic luminous intensity introduced in Section 1,

$$I_{\mathrm{v}} = K_{\mathrm{m}} \int_{0\ \mathrm{m}}^{\infty\ \mathrm{m}} I_{\mathrm{e},\lambda}(\lambda) V(\lambda)\, \mathrm{d}\lambda$$

is consistent with the same dimensional classification. Since $V(\lambda)$ is a dimension-one weighting function whose values are pure numbers and therefore introduce no additional unit component, and

$$\mathrm{Dim}\big(I_{\mathrm{e},\lambda}(\lambda)\, \mathrm{d}\lambda\big) = \mathrm{Dim}(I_{\mathrm{e}})$$

the dimensional rules for integration established in Section 4 give:

$$\mathrm{Dim}(I_{\mathrm{v}}) = \mathrm{Dim}(I_{\mathrm{e}})$$

This equality refers only to the unit-based dimensional structure of the two quantities. Luminous intensity and radiant intensity remain quantities of different kinds, because the former incorporates the specified physiological weighting of the visual response.

As established in Section 5.1, $\mathrm{sr} = \mathrm{rad}^2$, $\mathrm{Dim(rad)} = \mathsf{A}$. Therefore,

$$\mathrm{Dim}(I_{\mathrm{v}}) = \mathsf{ML^2T^{-3}A^{-2}}$$

The candela consequently becomes a scaled unit of a derived quantity rather than a base unit:

$$\mathrm{Dim(cd)} = \mathrm{Dim(W/sr)} = \mathsf{ML^2T^{-3}A^{-2}}$$

This classification differs from the current SI, in which the candela remains a base unit and K_{cd} is one of the seven defining constants. The numerical value of K_{cd} is fixed exactly at 683 when expressed in $\mathrm{lm/W} = \mathrm{cd \cdot sr/W}$[3]. The present proposal does not alter this numerical photometric–radiometric correspondence; it instead assigns a different dimensional interpretation to the unit structures connected by that correspondence. The status of the candela within the hierarchy of SI units has also been reconsidered recently from different perspectives. Lehman et al. proposed a three-category organization of SI units in which the candela is classified among physiologically relevant derived units[29]. Finkelstein and Whitehead distinguish the candela, kelvin, and mole conceptually from the kilogram, metre, second, and ampere, and propose assigning the former group to a separate category of “scale-spanning units” rather than treating all seven base units as playing the same structural role[30]. Their proposal differs from the present framework, which does not introduce an additional category of units but instead classifies luminous intensity as dimensionally derived through the photometric–radiometric unit relation.

The same dimensional classification extends to the corresponding photometric quantities. For example,

$$\mathrm{Dim(lm)} = \mathrm{Dim(cd \cdot sr)} = \mathrm{Dim\,W}$$
$$\mathrm{Dim(lm \cdot s)} = \mathrm{Dim\,J}$$
$$\mathrm{Dim(lx)} = \mathrm{Dim(W/m^2)}$$

and

$$\mathrm{Dim}(\mathrm{cd/m^2}) = \mathrm{Dim}\big(\mathrm{W/(m^2 sr)}\big)$$

These equalities under Dim express only equality of unit-based dimensional structure; they do not imply identity of quantity kind or numerical equality between photometric and radiometric quantities. Their numerical correspondence remains determined by the relevant spectral weighting function[6,7].

Consequently, luminous intensity does not provide an additional independent dimensional component under Criterion 1, because its unit structure belongs to the same dimensional subset as that of radiant intensity within the present framework. Furthermore, it does not exhibit an established higher-order physical structure of the type specified in Criterion 2 that would provide an independent reason for preferential base-quantity status. Luminous intensity is therefore classified as a derived quantity within the present framework, while remaining a quantity kind distinct from radiant intensity.

6. Reformulation of Logarithmic Levels as Numerical Indices

Regarding the neper and decibel, the current metrological framework establishes a specific quantity termed the level L, expressed respectively as:

$$\frac{L}{\mathrm{Np}} = \ln\frac{F}{F_0}$$

$$\frac{L}{\mathrm{dB}} = 10\lg\frac{P}{P_0} = 20\lg\frac{F}{F_0}$$

where

$$\lg x = \log_{10} x = \frac{\ln x}{\ln 10}.$$

In the conventional definitions, F/F_0 and P/P_0 are ratios of quantities of the same kind and therefore reduce algebraically to pure numbers. The logarithmic functions consequently act on pure numerical arguments, and their numerical outputs are pure numbers and therefore have dimension one[31]. Hence,

$$\mathrm{Dim}\left(\frac{L}{\mathrm{Np}}\right) = \mathrm{Dim}\left(\frac{L}{\mathrm{dB}}\right) = 1$$

which establishes:

$$\mathrm{Dim}\, L = \mathrm{Dim}(\mathrm{Np}) = \mathrm{Dim}(\mathrm{dB})$$

but does not, from these defining equations alone, determine the common dimensional subset before a convention for the logarithmic units is supplied. Within the unit-based construction of Section 4, this leaves the dimensional assignment to be specified separately from the logarithmic relation itself.

Historically, the neper was adopted as a special name for the number one for logarithmic quantities defined using natural logarithms[8]. In the current SI Brochure, the neper, bel, and decibel are listed as non-SI units used to express specified logarithmic ratio quantities[3]. The current SI also requires that the quantity concerned and any reference value used be specified, so the conventional notation is fully capable of retaining the physical context of the logarithmic ratio. The question considered here is therefore not whether such information can be represented conventionally, but whether a separate logarithmic unit is required within the present unit-based dimensional construction when the defining logarithmic relation itself yields a pure numerical result.

A simpler treatment within the present framework is to represent logarithmic levels directly by numerical index mappings. The symbols Np and dB are retained in these function names to preserve an explicit correspondence with the established neperian and decadic conventions. Their use as function labels is specific to the proposed notation; whenever the conventional units are discussed, Np and dB retain their established meanings as unit symbols. To distinguish the proposed numerical index mappings from the conventional unit symbols, the former are written explicitly as functions, $\mathrm{Np}(\cdot)$ and $\mathrm{dB}(\cdot)$, throughout this subsection. To retain explicitly the type of physical ratio and the corresponding reference quantity, let F and P be field quantity and power quantity, respectively, and define

$$\mathrm{Np}_{F_0}(F) \coloneqq \ln\frac{F}{F_0}$$
$$\mathrm{Np}_{P_0}(P) \coloneqq \frac{1}{2}\ln\frac{P}{P_0}$$

Likewise, the corresponding decadic indices are defined as:

$$\mathrm{dB}_{F_0}(F) \coloneqq 20\,\mathrm{lg}\frac{F}{F_0}$$
$$\mathrm{dB}_{P_0}(P) \coloneqq 10\,\mathrm{lg}\frac{P}{P_0}$$

The defining numerical functions are the same as those conventionally used for field and power levels[31]; what changes in the present framework is the metrological role assigned to Np and dB.

Here, they function as labels for numerical index mappings rather than as measurement units. This reformulation is not intended to recover information that is unavailable in the conventional notation, since conventional level quantities together with their specified reference quantities already retain that information[3,31]. Rather, it consolidates the physical quantity, its reference quantity, and the logarithmic convention directly in the mapping notation while treating the resulting value as a pure numerical index. Thus, after the index mapping has been defined, a value may also be reported compactly in a prefix form, for example

$$\mathrm{Np}(F)\ 3$$
$$\mathrm{dB}(P)\ 30$$

analogously to expressions such as $\mathrm{pH}\ 7.00$ discussed in Section 2. In these expressions, $\mathrm{Np}(F)$ and $\mathrm{dB}(P)$ identify the type of numerical index and are not factors multiplying the following number or symbols for measurement units. For mathematical derivations, the equivalent equation forms

$$\mathrm{Np}(F) = 3$$
$$\mathrm{dB}(P) = 30$$

are used to avoid ambiguity.

Because the arguments of the logarithms are ratios of quantities of the same kind, they reduce algebraically to pure numbers, and the resulting index values are likewise pure numbers and therefore have dimension one:

$$\mathrm{Dim}\left(\mathrm{Np}_{F_0}(F)\right) = \mathrm{Dim}\left(\mathrm{Np}_{P_0}(P)\right) = \mathrm{Dim}\left(\mathrm{dB}_{F_0}(F)\right) = \mathrm{Dim}\left(\mathrm{dB}_{P_0}(P)\right) = 1$$

This assignment applies to the outputs of the index mappings; the labels Np and dB themselves are not physical quantities and are therefore not assigned independent dimensions.

An important consequence of this reinterpretation is that the algebraic relation between the numerical index values must be distinguished from the conventional relation between the corresponding unit symbols. Under the conventional treatment as measurement units, the established relation is[3,31]:

$$1\ \mathrm{Np} = (20 \lg e)\ \mathrm{dB} \approx 8.686\ \mathrm{dB}$$
$$1\ \mathrm{dB} = \frac{\ln 10}{20}\ \mathrm{Np}$$

By contrast, when Np and dB denote the numerical index mappings defined above, their values for the same underlying logarithmic ratio satisfy:

$$\mathrm{dB}(X) = (20 \lg e)\mathrm{Np}(X) \approx 8.686\ \mathrm{Np}(X)$$

where X denotes either the corresponding field or power quantity with its specified reference quantity.

The coefficient in the index-value relation is therefore the reciprocal of that appearing when the unit symbols themselves are related. This is not a contradiction, but a direct consequence of the distinction between a unit relation and the corresponding relation between numerical values established in Section 3. The reversal is therefore not presented as evidence in favour of the proposed formulation, but as a necessary consistency condition and a practical caution when translating conventional unit-based expressions into the proposed index notation. Conventional unit-conversion equations cannot be transferred unchanged to the proposed index notation.

7. Consequences of the Proposed Angular Dimension for Selected Physical Equations

As established in Section 5.1, the plane angle is assigned the independent dimension A, with $\mathrm{sr} = \mathrm{rad}^2$. Consequently, familiar equations in which the conventional SI treatment permits the radian or steradian to be reduced to the unit one take different unit-explicit forms when the nontrivial angular dimension proposed here is adopted. The equations considered below are therefore not presented as corrections to mathematically invalid conventional expressions, but as consequences of applying the dimensional framework developed in the preceding sections. In accordance with the terminology introduced in Section 3, equations that retain explicit division by these angular units are treated herein as unit-explicit equations. In the vector-, tensor-, and operator-valued applications below, the dimensional construction is applied to the associated unit components, while their respective mathematical algebra remains unchanged[1,12].

The underbar notation introduced in Section 5.1 is used below as shorthand for explicit division by a specified unit. When the denominator is a measurement unit of the same kind as the quantity in the numerator, the resulting expression may represent its conventional numerical quantity value. When the underbar is applied to another quantity expression, however, it denotes only the explicitly unit-divided expression and should not be interpreted as a numerical quantity value.

7.1. Infinitesimal Rotations in Three-Dimensional Space

Let us consider a vector $\mathbf{r}$ that undergoes a rotation by an angle θ around an axis defined by a unit vector $\hat{\mathbf{n}}$, resulting in a new vector $\mathbf{r}'$ (Figure 2). According to Rodrigues' rotation formula, $\mathbf{r}'$ is expressed as:

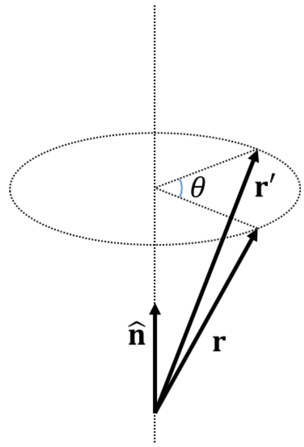

Figure 2. Rotation of $\mathbf{r}$ to $\mathbf{r}'$ around the axis.

$$\begin{aligned}\mathbf{r}' &= (\cos\underline{\theta})\mathbf{r} + (1-\cos\underline{\theta})(\mathbf{r}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\sin\underline{\theta})\hat{\mathbf{n}}\times\mathbf{r} \\ &= \mathbf{r} + (1-\cos\underline{\theta})\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{r}) + (\sin\underline{\theta})\hat{\mathbf{n}}\times\mathbf{r}\end{aligned}$$

As $\hat{\mathbf{n}}$ and $\mathbf{r}$ are constant vectors, $\mathrm{d}\mathbf{l}/\mathrm{d}\underline{\theta}$ can be expressed by the derivative of $\mathbf{r}'$ with respect to $\underline{\theta}$ at the initial state $\underline{\theta} = 0$:

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\underline{\theta}} = \left.\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}\underline{\theta}}\right|_{\underline{\theta}=0} = \hat{\mathbf{n}}\times\mathbf{r}$$

Therefore, the infinitesimal arc vector $\mathrm{d}\mathbf{l}$ is obtained as:

$$\mathrm{d}\mathbf{l} = \mathrm{d}\underline{\theta}\,\hat{\mathbf{n}}\times\mathbf{r} = \mathrm{d}\theta\,\hat{\mathbf{n}}\times\mathbf{r}/\mathrm{rad}$$

This result can also be derived by applying the infinitesimal approximation of $\mathrm{d}\underline{\theta}$ to the displacement $\mathbf{r}' - \mathbf{r}$. More precisely, to first order in $\mathrm{d}\underline{\theta}$:

$$\begin{gathered}1-\cos\mathrm{d}\underline{\theta} = O(\mathrm{d}\underline{\theta}^2) \\ \sin\mathrm{d}\underline{\theta} = \mathrm{d}\underline{\theta} + O(\mathrm{d}\underline{\theta}^3)\end{gathered}$$

Thus, the term containing $1 - \cos \mathrm{d}\underline{\theta}$ is negligible to first order, whereas the term containing $\sin \mathrm{d}\underline{\theta}$ yields the preceding infinitesimal displacement.

Here, since $\hat{\mathbf{n}}$ is a unit vector—a quantity of dimension one—in the direction of the rotation axis, it can be treated as defining the direction of the infinitesimal rotation vector. Denoting this vector as $\mathrm{d}\boldsymbol{\theta} = \mathrm{d}\theta\, \hat{\mathbf{n}}$, we obtain:

$$\mathrm{d}\mathbf{l} = \mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r} = \mathrm{d}\boldsymbol{\theta} \times \mathbf{r}/\mathrm{rad}$$

Consequently, the tangential velocity $\mathbf{v}$ is:

$$\begin{aligned}\mathbf{v} = \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t} = \frac{\mathrm{d}\underline{\boldsymbol{\theta}}}{\mathrm{d}t} \times \mathbf{r} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} \times \mathbf{r}/\mathrm{rad} \\ = \boldsymbol{\omega} \times \mathbf{r}/\mathrm{rad}\end{aligned}$$

Letting $\boldsymbol{\omega}/\mathrm{rad} = \underline{\boldsymbol{\omega}}$, we obtain:

$$\mathbf{v} = \underline{\boldsymbol{\omega}} \times \mathbf{r}$$

Alternatively:

$$\mathbf{v} = \boldsymbol{\omega} \times \underline{\mathbf{r}}$$

Similarly, the acceleration $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ becomes:

$$\begin{aligned}\mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega} \times \mathbf{r})/\mathrm{rad} \\ &= \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)/\mathrm{rad} \\ &= (\boldsymbol{\alpha} \times \mathbf{r})/\mathrm{rad} + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \mathbf{r})\end{aligned}$$

Defining $\underline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/\mathrm{rad}$, we obtain:

$$\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \mathbf{r})$$

For motion in which $\boldsymbol{\omega} \cdot \mathbf{r} = \mathbf{0}\ \mathrm{m} \cdot \mathrm{rad/s}$, the vector triple-product identity gives:

$$\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} - \|\underline{\boldsymbol{\omega}}\|^2 \mathbf{r}$$

where the first term represents tangential acceleration $\mathbf{a}_\mathbf{t}$ and the second term represents centripetal acceleration $\mathbf{a}_\mathbf{c}$. Because the underbar notation in these expressions represents explicit division by the angular unit, these equations are unit-explicit equations in the sense defined in Section 3. Similar explicit-radian formulations of rotational kinematics have been discussed previously[12,32].

7.2. Torque

Consider a situation in which an external force $\mathbf{F}$ is applied to a rigid body, causing it to rotate by an infinitesimal displacement $\mathbf{dl}$. The resulting infinitesimal work $\mathrm{d}W$ is expressed as:

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{l}$$

Since $\mathbf{dl} = \mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}$, this becomes:

$$\mathrm{d}W = \mathbf{F} \cdot \left(\mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}\right)$$

Using the scalar triple-product identity,

$$\begin{aligned} \mathrm{d}W &= \mathrm{d}\underline{\boldsymbol{\theta}} \cdot (\mathbf{r} \times \mathbf{F}) = (\mathbf{r} \times \mathbf{F}) \cdot \mathrm{d}\underline{\boldsymbol{\theta}} \\ &= (\mathbf{r} \times \mathbf{F}) \cdot \mathrm{d}\boldsymbol{\theta}/\mathrm{rad} \end{aligned}$$

Within the present framework, torque is the quantity conjugate to the physical angular displacement $\boldsymbol{\theta}$, such that

$$\mathrm{d}W = \boldsymbol{\tau} \cdot \mathrm{d}\boldsymbol{\theta}$$

Therefore:

$$\boldsymbol{\tau} = \frac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}} = \mathbf{r} \times \mathbf{F}/\mathrm{rad} = \underline{\mathbf{r}} \times \mathbf{F}$$

The familiar expression

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

corresponds to the form obtained when the explicit radian factor is suppressed. Within the present framework, the former expression is a unit-explicit equation in the sense defined in Section 3.

Dimensional analysis yields:

$$\begin{aligned} \mathrm{Dim}(\boldsymbol{\tau}) &= \mathrm{Dim}(\mathbf{r} \times \mathbf{F}/\mathrm{rad}) \\ &= \frac{\mathsf{L} \cdot \mathsf{MLT}^{-2}}{\mathsf{A}} \\ &= \mathsf{ML}^{2}\mathsf{T}^{-2}\mathsf{A}^{-1} \end{aligned}$$

Accordingly, the coherent unit of torque becomes, within the present framework:

$$\mathrm{J}/\mathrm{rad} = \mathrm{N} \cdot \mathrm{m}/\mathrm{rad}$$

This angle-dependent assignment for torque was previously proposed by Quincey[32]. Leonard subsequently contested this conclusion[33], to which Quincey replied[34].

7.3. Angular Momentum

Since $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$, the torque becomes:

$$\boldsymbol{\tau} = \underline{\mathbf{r}} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

Because

$$\frac{\mathrm{d}\underline{\mathbf{r}}}{\mathrm{d}t} \times \mathbf{p} = \frac{\mathbf{v}}{\mathrm{rad}} \times m\mathbf{v} = \mathbf{0}\ \mathrm{kg} \cdot \mathrm{m}^2/(\mathrm{s}^2\mathrm{rad})$$

, we obtain:

$$\boldsymbol{\tau} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{\mathbf{r}} \times \mathbf{p}\right)$$

Accordingly, the angular momentum $\mathbf{L}$ is identified as:

$$\mathbf{L} = \int \boldsymbol{\tau}\, \mathrm{d}t = \underline{\mathbf{r}} \times \mathbf{p} = \mathbf{r} \times \mathbf{p}/\mathrm{rad}$$

Therefore, the familiar expression $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ corresponds to the radian-suppressed form. Dimensional analysis gives:

$$\begin{aligned} \mathrm{Dim}(\mathbf{L}) &= \frac{\mathrm{Dim}(\mathbf{r}) \cdot \mathrm{Dim}(\mathbf{p})}{\mathrm{Dim}(\mathrm{rad})} \\ &= \frac{\mathsf{L} \cdot \mathsf{MLT}^{-1}}{\mathsf{A}} \\ &= \mathsf{ML}^2\mathsf{T}^{-1}\mathsf{A}^{-1} \end{aligned}$$

This corresponding coherent unit is

$$\mathrm{J} \cdot \mathrm{s}/\mathrm{rad}$$

Furthermore, substituting $\mathbf{p} = m\mathbf{v} = m\underline{\boldsymbol{\omega}} \times \mathbf{r}$ and assuming $\underline{\boldsymbol{\omega}} \cdot \mathbf{r} = 0\ \mathrm{m/s}$, we obtain:

$$\mathbf{L} = \underline{\mathbf{r}} \times \left(m\underline{\boldsymbol{\omega}} \times \mathbf{r}\right) = m\left\|\underline{\mathbf{r}}\right\|^2 \boldsymbol{\omega} = m\|\mathbf{r}\|^2\boldsymbol{\omega}/\mathrm{rad}^2$$

Dimensional analysis gives:

$$\mathrm{Dim}\left(m\left\|\underline{\mathbf{r}}\right\|^2 \boldsymbol{\omega}\right) = \mathsf{ML}^2\mathsf{T}^{-1}\mathsf{A}^{-1}$$

consistent with the dimension obtained above.

The explicit-radian units of angular momentum proposed here likewise correspond to one side of the published Quincey-Leonard debate[32–34].

7.4. Moment of Inertia

As discussed in Section 7.1, the acceleration of a point undergoing rotational motion is:

$$\mathbf{a} = \underline{\boldsymbol{\alpha}} \times \mathbf{r} - \left\|\underline{\boldsymbol{\omega}}\right\|^2 \mathbf{r}$$

Substituting this into the expression for torque yields:

$$\begin{aligned}\boldsymbol{\tau} &= \underline{\mathbf{r}} \times m\mathbf{a} \\ &= m\underline{\mathbf{r}} \times (\underline{\boldsymbol{\alpha}} \times \mathbf{r}) - m\underline{\mathbf{r}} \times \left(\|\underline{\boldsymbol{\omega}}\|^2 \mathbf{r}\right)\end{aligned}$$

The second term vanishes because $\underline{\mathbf{r}} \times \mathbf{r} = \mathbf{r} \times \mathbf{r}/\mathrm{rad} = \mathbf{0}\ \mathrm{m}^2/\mathrm{rad}$. When the rotation axis is perpendicular to $\mathbf{r}$, so that $\underline{\mathbf{r}} \cdot \underline{\boldsymbol{\alpha}} = 0\ \mathrm{m}/(\mathrm{s}^2\mathrm{rad}^2)$, the first term reduces to:

$$\boldsymbol{\tau} = m\|\mathbf{r}\|^2 \boldsymbol{\alpha}/\mathrm{rad}^2$$

This leads to:

$$\boldsymbol{\tau} = I\boldsymbol{\alpha}$$

where I is the moment of inertia defined by:

$$I = m\|\underline{\mathbf{r}}\|^2 = m\|\mathbf{r}\|^2/\mathrm{rad}^2$$

More generally, if $\mathbf{I}$ denotes the inertia tensor, its components I_{ij} are written as:

$$\begin{aligned}I_{ij} &= \sum_{\alpha} m_{\alpha} \left[\delta_{ij}\left(\sum_{k} \underline{x_{\alpha k}}^2\right) - \underline{x_{\alpha i}}\,\underline{x_{\alpha j}}\right] \\ &= \sum_{\alpha} m_{\alpha} \left[\delta_{ij}\left(\sum_{k} x_{\alpha k}^2\right) - x_{\alpha i} x_{\alpha j}\right]/\mathrm{rad}^2\end{aligned}$$

where δ_{ij} is the Kronecker delta. Therefore,

$$\mathrm{Dim}\, I = \mathsf{ML^2A^{-2}}$$

and its coherent unit within the present framework is $\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{rad}^2$.

7.5. Periodic Quantities and Dimensional Consistency of Angular Units

Consider a system undergoing periodic motion with angular velocity ω. Since one full rotation corresponds to an angle of 2π rad, the period T is:

$$T = \frac{2\pi\ \mathrm{rad}}{\omega}$$

Accordingly, the frequency ν, defined as the reciprocal of T, is:

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi\ \mathrm{rad}} = \frac{\underline{\omega}}{2\pi}$$

Thus, ordinary frequency and angular frequency remain distinct quantities:

$$\mathrm{Dim}\,\nu = \mathsf{T}^{-1}$$

whereas

$$\mathrm{Dim}\,\omega = \mathsf{T}^{-1}\mathsf{A}$$

The familiar expression $\nu = \omega/(2\pi)$ is recovered when the radian is suppressed. This distinction allows the conventional definition $1\ \mathrm{Hz} = 1\ \mathrm{s}^{-1}$ to remain unchanged[35].

Likewise, the wavenumber $\tilde{\nu}$ is:

$$\tilde{\nu} = 1/\lambda$$

and the angular wavenumber k is:

$$k = 2\pi\tilde{\nu}\ \mathrm{rad} = \frac{2\pi\ \mathrm{rad}}{\lambda}$$

The angular wavelength $\lambdabar$, defined as the reciprocal of k, is therefore:

$$\lambdabar = \frac{1}{k} = \frac{\lambda}{2\pi\ \mathrm{rad}}$$

In a similar manner, the energy E of a quantum can be expressed as:

$$E = h\nu = \frac{h}{2\pi\ \mathrm{rad}}\omega$$

Here, the Planck constant h is retained as the action-valued constant, with

$$\mathrm{Dim}\,h = \mathsf{ML}^2\mathsf{T}^{-1}$$

Within the present framework, the reduced Planck constant, or Dirac constant, is instead assigned specifically to angular momentum and defined as:

$$\hbar \equiv \frac{h}{2\pi\ \mathrm{rad}}$$

Therefore,

$$E = \hbar\omega$$

with

$$\mathrm{Dim}\,\hbar = \mathsf{ML}^2\mathsf{T}^{-1}\mathsf{A}^{-1}$$

which is the same as that of angular momentum $\mathbf{J}$. Thus:

$$\|\mathbf{J}\| = \sqrt{l(l+1)}\hbar$$

where l is a numerical quantum number and therefore introduces no additional unit component. This assignment differs from the conventional SI treatment, in which the reduced Planck constant is defined as $h/(2\pi)$ and therefore has the same action dimension as h. In the present framework, the explicit angular unit is retained in the definition of $\hbar$, so that h remains associated with action whereas $\hbar$ is associated with angular momentum. The action-valued quantity $h/(2\pi)$ remains well defined, but no separate symbol is introduced for it here. The distinction between the action and angular-momentum roles conventionally represented by $\hbar$ has previously been discussed by Quincey[32]. For example, the position–momentum commutation relation has the dimension of action and is therefore written as:

$$[\hat{x}, \hat{p}] = i\frac{h}{2\pi}$$

Using the angular-momentum constant defined above,

$$\frac{h}{2\pi} = \hbar\,\mathrm{rad}$$

and hence the same relation may be written as:

$$[\hat{x}, \hat{p}] = i\hbar\,\mathrm{rad}$$

Alternatively, introducing the unit-divided position operator $\underline{\hat{x}} = \hat{x}/\mathrm{rad}$ gives:

$$[\underline{\hat{x}}, \hat{p}] = i\hbar$$

Here, $\underline{\hat{x}}$ is not the numerical value of the position operator; it is merely shorthand for the explicitly unit-divided expression $\hat{x}/\mathrm{rad}$.

7.6. Planck's Law

Planck's law describes the spectral energy density of blackbody radiation at a frequency ν and thermodynamic temperature T as:

$$u_\nu(\nu, T)\,\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,\mathrm{d}\nu$$

Performing dimensional analysis on the components:

$$\begin{cases} \mathrm{Dim}(h) = \mathsf{ML^2T^{-1}} \\ \mathrm{Dim}\left(\frac{\nu}{c}\right) = \mathsf{L^{-1}} \\ \mathrm{Dim}(h\nu) = \mathrm{Dim}(k_\mathrm{B}T) = \mathsf{ML^2T^{-2}} \\ \mathrm{Dim}(\mathrm{d}\nu) = \mathsf{T^{-1}} \end{cases}$$

we obtain:

$$\mathrm{Dim}(u_\nu\ \mathrm{d}\nu) = \mathsf{ML^{-1}T^{-2}} = \frac{\mathsf{ML^{2}T^{-2}}}{\mathsf{L^{3}}}$$

which corresponds to the dimension of energy density.

To obtain spectral radiance, it is more direct to use the standard relation between radiation energy density and radiance rather than deriving it from an infinitesimal volume $c\ \mathrm{d}A\ \mathrm{d}t$. For a radiation field,

$$u_\nu = \frac{1}{c}\int B_\nu\ \mathrm{d}\Omega$$

For an isotropic blackbody field, B_ν is independent of direction, and therefore:

$$u_\nu = \frac{4\pi\ \mathrm{sr}}{c} B_\nu$$

Hence,

$$B_\nu(\nu, T)\ \mathrm{d}\nu = \frac{2h\nu^3}{c^2}\cdot\frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}}-1}\ \mathrm{sr}^{-1}\ \mathrm{d}\nu$$

Within the present framework, the explicit factor sr^{-1} is retained because the steradian is no longer reducible to the number one. Accordingly,

$$\mathrm{Dim}(B_\nu(\nu, T)\ \mathrm{d}\nu) = \mathsf{MT^{-3}A^{-2}} = \frac{\mathsf{ML^{2}T^{-2}}}{\mathsf{T}}\cdot\frac{1}{\mathsf{L^{2}}}\cdot\frac{1}{\mathsf{A^{2}}}$$

Integrating over all frequencies yields the total blackbody radiance $L(T)$:

$$L(T) = \int_{0\ \mathrm{Hz}}^{\infty\ \mathrm{Hz}} B_\nu(\nu, T)\ d\nu$$

and therefore:

$$\begin{aligned} L &= \frac{2\pi^4 {k_\mathrm{B}}^4}{15c^2h^3}T^4\ \mathrm{sr}^{-1} \\ &= \frac{\sigma}{\pi}T^4/\mathrm{sr} \end{aligned}$$

Thus, within the present framework, the conventional Stefan–Boltzmann expression for blackbody radiance

$$L(T) = \frac{\sigma}{\pi}T^4$$

is rewritten with the explicit solid-angle unit as:

$$L(T) = \frac{\sigma}{\pi}\frac{T^4}{\mathrm{sr}}$$

This is a unit-explicit equation in the sense defined in Section 3.

Table 1 Comparison of conventional forms with the corresponding forms in the present framework.

Equations	Conventional equation	Form in the present framework
	(Dimensional analysis)	
Arc length l	$l = r\theta$ (L ≠ L · A)	$l = r\theta/\mathrm{rad} = r\underline{\theta}$ (L = L · A · A^{-1} = L · 1)
Area on spherical surface A	$A = \Omega r^2$ (L^2 ≠ A^2 · L^2)	$A = \Omega r^2/\mathrm{sr} = \underline{\Omega}_2 r^2$ (L^2 = A^2 · L^2 · A^{-2} = 1 · L^2)
Trigonometric and inverse trigonometric functions	$a = \sin\theta \Leftrightarrow \theta = \arcsin a$ (On the equation of θ, A ≠ 1)	$a = \sin\underline{\theta} \Leftrightarrow \underline{\theta} = \arcsin a$ (On the equation of θ, 1 = 1)
Period T	$T = \dfrac{2\pi}{\omega}$ (T ≠ TA^{-1})	$T = \dfrac{2\pi\ \mathrm{rad}}{\omega} = \dfrac{2\pi}{\underline{\omega}}$ (T = A · TA^{-1} = T)
Angular frequency ω	$\omega = 2\pi\nu$ (T^{-1}A ≠ T^{-1})	$\omega/\mathrm{rad} = \underline{\omega} = 2\pi\nu$ (T^{-1}A · A^{-1} = T^{-1} = T^{-1})
Angular wavenumber k	$k = 2\pi\tilde{\nu} = \dfrac{2\pi}{\lambda}$ (L^{-1}A ≠ L^{-1})	$k = 2\pi\tilde{\nu}\ \mathrm{rad} = \dfrac{2\pi\ \mathrm{rad}}{\lambda}$ (L^{-1}A = L^{-1} · A = A · L^{-1})
Angular wavelength $\bar{\lambda}$	$\bar{\lambda} = \dfrac{\lambda}{2\pi}$ (LA^{-1} ≠ L)	$\bar{\lambda} = \dfrac{\lambda}{2\pi\ \mathrm{rad}} = \dfrac{\underline{\lambda}}{2\pi}$ (LA^{-1} = L · A^{-1})
Wave equation $u(\mathbf{x}, t)$ of amplitude A	$u(\mathbf{x}, t) = \dfrac{A}{r} e^{i(\omega t \pm \mathbf{k}\cdot\mathbf{x})}$ (The exponent part is A ≠ 1)	$u(\mathbf{x}, t) = \dfrac{A}{r} e^{i\{\underline{\omega} t \pm \underline{\mathbf{k}}\cdot\mathbf{x}\}}$ (The exponent part is 1)
Infinitesimal displacement $\mathrm{d}\mathbf{l}$ by infinitesimal rotation	$\mathrm{d}\mathbf{l} = \mathrm{d}\boldsymbol{\theta} \times \mathbf{r}$ (L ≠ A · L)	$\mathrm{d}\mathbf{l} = \mathrm{d}\boldsymbol{\theta} \times \mathbf{r}/\mathrm{rad} = \mathrm{d}\underline{\boldsymbol{\theta}} \times \mathbf{r}$ (L = A · L · A^{-1} = 1 · L)
Tangential velocity $\mathbf{v}$	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ (LT^{-1} ≠ T^{-1}A · L)	$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}/\mathrm{rad} = \underline{\boldsymbol{\omega}} \times \mathbf{r}$ (LT^{-1} = T^{-1}A · L · A^{-1} = T^{-1} · L)
Tangential acceleration $\mathbf{a_t}$	$\mathbf{a_t} = \boldsymbol{\alpha} \times \mathbf{r}$ (LT^{-2} ≠ T^{-2}A · L)	$\mathbf{a_t} = \boldsymbol{\alpha} \times \mathbf{r}/\mathrm{rad} = \underline{\boldsymbol{\alpha}} \times \mathbf{r}$ (LT^{-2} = T^{-2}A · L · A^{-1} = T^{-2} · L)
Centripetal acceleration $\mathbf{a_c}$	$\mathbf{a_c} = \lVert\boldsymbol{\omega}\rVert^2\mathbf{r}$ (LT^{-2} ≠ T^{-2}A^2 · L)	$\mathbf{a_c} = \lVert\boldsymbol{\omega}\rVert^2\mathbf{r}/\mathrm{rad}^2 = \lVert\underline{\boldsymbol{\omega}}\rVert^2\mathbf{r}$ (LT^{-2} = T^{-2}A^2 · L · A^{-2} = T^{-2} · L)
Torque $\boldsymbol{\tau}$	$\boldsymbol{\tau} = \dfrac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}} = \mathbf{r} \times \mathbf{F}$ (ML^{2}T^{-2}A^{-1} ≠ L · MLT^{-2})	$\boldsymbol{\tau} = \dfrac{\mathrm{d}W}{\mathrm{d}\boldsymbol{\theta}} = \mathbf{r} \times \mathbf{F}/\mathrm{rad} = \underline{\mathbf{r}} \times \mathbf{F}$ (ML^{2}T^{-2}A^{-1} = L · MLT^{-2} · A^{-1})
Angular momentum $\mathbf{L}$	$\mathbf{L} = \displaystyle\int \boldsymbol{\tau}\,\mathrm{d}t = \mathbf{r} \times \mathbf{p}$ (ML^{2}T^{-1}A^{-1} ≠ L · MLT^{-1})	$\mathbf{L} = \displaystyle\int \boldsymbol{\tau}\,\mathrm{d}t = \mathbf{r} \times \mathbf{p}/\mathrm{rad} = \underline{\mathbf{r}} \times \mathbf{p}$ (ML^{2}T^{-1}A^{-1} = L · MLT^{-1} · A^{-1})
	$\mathbf{L} = \mathbf{r} \times (m\boldsymbol{\omega} \times \mathbf{r}) = m\lVert\mathbf{r}\rVert^2\boldsymbol{\omega}$ (ML^{2}T^{-1}A^{-1} ≠ M · L^2 · AT^{-1})	$\mathbf{L} = \underline{\mathbf{r}} \times (m\underline{\boldsymbol{\omega}} \times \mathbf{r}) = m\lVert\underline{\mathbf{r}}\rVert^2\boldsymbol{\omega}$ (ML^{2}T^{-1}A^{-1} = M · L^{2}A^{-2} · AT^{-1})

Moment of inertia I	$I = \frac{\lVert\boldsymbol{\tau}\rVert}{\lVert\boldsymbol{\alpha}\rVert} = \sum_i m_i \lVert\mathbf{r}_i\rVert^2$ $(\mathrm{ML^2A^{-2}} \neq \mathrm{M} \cdot \mathrm{L^2})$	$I = \frac{\lVert\boldsymbol{\tau}\rVert}{\lVert\boldsymbol{\alpha}\rVert} = \sum_i m_i \left\lVert\underline{\mathbf{r}_i}\right\rVert^2$ $(\mathrm{ML^2A^{-2}} = \mathrm{M} \cdot \mathrm{L^2A^{-2}})$
Stefan–Boltzmann law for blackbody radiance	$L(T) = \frac{\sigma}{\pi} T^4$ $(\mathrm{MT^{-3}A^{-2}} \neq \mathrm{MT^{-3}\Theta^{-4}} \cdot \Theta^4)$	$L(T) = \frac{\sigma}{\pi} T^4 / \mathrm{sr}$ $(\mathrm{MT^{-3}A^{-2}} = \mathrm{MT^{-3}\Theta^{-4}} \cdot \Theta^4 \cdot \mathrm{A^{-2}})$
Reduced Planck constant $\hbar$	$\hbar = \frac{h\nu}{\omega} = \frac{h}{2\pi}$ $(\mathrm{ML^2T^{-1}A^{-1}} \neq \mathrm{ML^2T^{-1}})$	$\hbar = \frac{h\nu}{\omega} = \frac{h}{2\pi\ \mathrm{rad}}$ $(\mathrm{ML^2T^{-1}A^{-1}} = \mathrm{ML^2T^{-1}} \cdot \mathrm{A^{-1}})$
Logarithmic levels	$\frac{L}{\mathrm{Np}} = \ln\frac{F}{F_0}$ ($\mathrm{Dim}\,L\,/\,\mathrm{Dim(Np)} = 1$, $\mathrm{Dim}\,L$ is undetermined)	$\mathrm{Np}_{F_0}(F) = \ln\frac{F}{F_0}, \quad \mathrm{Np}_{P_0}(P) = \frac{1}{2}\ln\frac{P}{P_0}$ $(\mathrm{Dim}\left(\mathrm{Np}_{F_0}(F)\right) = \mathrm{Dim}\left(\mathrm{Np}_{P_0}(P)\right) = 1)$
	$\frac{L}{\mathrm{dB}} = 10\lg\frac{P}{P_0} = 20\lg\frac{F}{F_0}$ ($\mathrm{Dim}\,L\,/\,\mathrm{Dim(dB)} = 1$, $\mathrm{Dim}\,L$ is undetermined)	$\mathrm{dB}_{F_0}(F) = 20\lg\frac{F}{F_0}, \quad \mathrm{dB}_{P_0}(P) = 10\lg\frac{P}{P_0}$ $(\mathrm{Dim}\left(\mathrm{dB}_{P_0}(P)\right) = \mathrm{Dim}\left(\mathrm{dB}_{F_0}(F)\right) = 1)$

8. Conclusion

In this study, a systematic framework has been proposed in which quantity dimensions are constructed from unit relations before base-quantity selection is considered. This unit-first logical ordering distinguishes the construction of dimensional structure from the subsequent assessment of algebraic admissibility and physical preference. The conventional definition of a numerical value equation was retained, while the broader concept of a unit-explicit equation was introduced for relations in which specified unit components are explicitly retained when their elimination would remove physically relevant information. Unit relations were then used to partition unit expressions into dimensional subsets, providing the basis for the dimension-extraction function Dim. This classification concerns unit-based dimensional structure and does not, in general, identify quantity kind.

Base-quantity selection was formulated as a two-stage procedure separating algebraic admissibility from physical structural preference. Under an SI-oriented application, mass, length, time, electric current, thermodynamic temperature, and amount of substance are retained. Plane angle is additionally selected as a preferred base quantity because its unit relations establish a distinct angular dimensional subset required for dimensional completeness, while solid angle provides the corresponding second-order angular structure. Luminous intensity, by contrast, is classified as a derived quantity having the same unit-based dimensional structure as radiant intensity while remaining a distinct quantity kind.

Retaining angular units explicitly leads to corresponding reformulations of equations involving rotational motion, periodic quantities, angular momentum, quantum-mechanical relations, and radiometry. In particular, the Planck constant retains the dimension of action, whereas the reduced Planck constant is assigned specifically to angular momentum when the angular unit is retained explicitly. Logarithmic levels are likewise represented as dimension-one numerical index mappings, with their physical interpretation retained through the quantity arguments and reference quantities.

These structural changes do not require a discontinuity in the numerical realization of established quantities. Existing numerical angle values, relations among angular units, and calibration results remain valid; the principal changes concern dimensional interpretation and the algebraic representation of equations in which angular units are conventionally suppressed. Overall, the proposed framework provides a common unit-based basis for resolving dimensional ambiguities and reconsidering the dimensional and base–derived unit structure of the SI.

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