The BQ–Pauli to BQ–Weyl Lift: Electromagnetism and Relativistic Fluid Dynamics in Cl(1,3)

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Abstract

The BQ–Pauli algebra provides an eight-dimensional 1+3+3+1 framework in which relativistic field equations can be organized through a hierarchy of algebraic products. Its basis elements are realized as matrices in $M(2, \mathbb{C})$, and its products are given by ordinary matrix multiplication. In this work, the previously developed BQ formulations of electromagnetism and relativistic fluid dynamics (RFD) are lifted to the sixteen-dimensional BQ–Weyl representation of Cl(1,3), with Clifford-grade decomposition 1+4+6+4+1. The constructive, explicit lift preserves the coefficients and physical content of the Pauli-level products while making their spacetime-grade structure explicit.

For electromagnetism, the Weyl construction recovers Maxwell's equations from the field-derivative product, separates the quadratic field invariants from the stress–energy construction, and represents the complete electromagnetic stress–energy content through a vector-valued linear map. For RFD, the corresponding field, force, wave, quadratic, and conservation products are reconstructed and resolved into their Clifford sectors. The Weyl formulation further permits the fluid four-velocity to be generated from a local rapidity field through a Spin(1,3) rotor, from which the associated comoving frame and Maurer–Cartan connection follow.

Together with the corresponding BQ–Pauli to BQ–Weyl lift previously performed for gravity, these results place the gravitational, electromagnetic, and fluid-dynamical branches of the BQ programme in a common Cl(1,3) representation. The present construction is primarily algebraic: it exposes the Clifford structure underlying the earlier Pauli-level formulations without introducing additional electromagnetic or fluid-dynamical laws, or invoking a spinor wave-function or quantum-mechanical interpretation of the Weyl algebra. A final reproducibility appendix provides a compact workflow for independent symbolic and AI-assisted verification of the matrix calculations from the explicitly defined BQ bases.

Keywords: BQ algebra; Clifford algebra; Weyl representation; electromagnetism; relativistic fluid dynamics; rapidity; Lorentz rotors; Maurer–Cartan connection

Contents

1 Introduction

The Pauli and Dirac matrix algebras provide closely related representations of relativistic geometry. Pauli's two-component formalism Pauli (1927) and Dirac's four-component relativistic algebra Dirac (1928) are naturally connected through the Clifford algebra of Minkowski spacetime. In particular, the Dirac matrices generate a representation of $\mathrm{Cl}(1,3)$[1], while their even subalgebra is isomorphic to $\mathrm{Cl}^+(1,3) \simeq \mathrm{Cl}(3,0)$, the algebra underlying the Pauli matrices. This relation is standard in Clifford-algebra and spacetime-algebra formulations of relativistic physics Hestenes (1966, 2003); Doran and Lasenby (2003).

The BQ formalism developed in earlier work, de Haas (2020), uses the Pauli algebra in the real eight-dimensional decomposition

$$1 + 3 + 3 + 1, \tag{1}$$

with scalar, vector, bivector-like, and pseudoscalar-like channels. Within this algebra, electromagnetic and relativistic-fluid equations were formulated through a small hierarchy of bilinear, triple, and higher products de Haas (2026a). The compactness of the Pauli representation is useful for these calculations, but it does not make the full spacetime-grade structure explicit.

The purpose of the present paper is therefore to lift this BQ–Pauli construction to the Weyl representation of the Dirac algebra. The lift replaces the Pauli decomposition by the sixteen-dimensional Clifford decomposition

$$1 + 4 + 6 + 4 + 1, \tag{2}$$

and thereby makes the scalar, vector, bivector, trivector, and pseudoscalar sectors explicit. The construction is algebraic: the BQ basis elements and their multiplication rules are mapped into a Weyl basis generated by β_μ, and the Pauli-level product hierarchy is then reconstructed at the $\mathrm{Cl}(1,3)$ level.

Electromagnetism is used first as the reference case, where the resulting structures can be compared directly with the standard Clifford formulation of the electromagnetic field and its stress–energy map Hestenes (1966); Doran and Lasenby (2003). The same procedure is subsequently applied to relativistic fluid dynamics. In the latter case the Weyl lift also permits the velocity field to be generated by a local $\mathrm{Spin}(1,3)$ rapidity rotor and its associated Maurer–Cartan connection. Such rotor and connection constructions are standard ingredients of spin and gauge formulations of relativistic geometry Utiyama (1956); Kibble (1961); Hehl et al. (1976); the point here is not to introduce these structures themselves, but to show how they arise within, and extend, the BQ product hierarchy.

[1] Throughout this work, $\mathrm{Cl}(1,3)$ denotes the real Clifford algebra with signature $(-,+,+,+)$: $\hat\beta_0^2 = -\mathbb{1}$ and $\hat\beta_i^2 = +\mathbb{1}$ for $i = I, J, K$.

Accordingly, the contribution of this work is primarily algebraic. It establishes an explicit passage from the BQ–Pauli formulation to its Weyl/Dirac Clifford representation and uses that passage to expose the grade structure of the electromagnetic and relativistic-fluid objects and products that were compressed at Pauli level. No new spinor dynamics is introduced: the Dirac algebra is used here as a classical spacetime and $\mathrm{Spin}(1,3)$ representation, prior to the introduction of a spinor wave function or quantum-mechanical interpretation. For reproducibility, a final appendix provides a compact workflow for independently reconstructing and verifying the principal BQ–Pauli and BQ–Weyl matrix calculations using symbolic or AI-assisted computational tools.

Relation to the gravitational BQ–Weyl lift.

The BQ–Pauli to BQ–Weyl lift performed in the present work for electromagnetism and relativistic fluid dynamics is the direct parallel of the corresponding lift already carried out for the gravitational sector de Haas (2026c,b). In that construction, the gravitational rapidity field, its associated $\mathrm{Spin}(1,3)$ rotor, the kinematic hierarchy, and the Maurer–Cartan connection were first developed at the native BQ–Pauli level and subsequently reconstructed in the off-diagonal Weyl representation of $\mathrm{Cl}(1,3)$. The Weyl formulation preserved the Pauli-level channel structure and invariant transport law while replacing the BQ bilinear closure by ordinary Clifford multiplication. The present paper applies the same algebraic passage to the previously developed BQ formulations of electromagnetism and relativistic fluid dynamics. Thus the three sectors are treated in parallel:

$$\begin{array}{ccc} \text{BQ–Pauli gravity} & \longrightarrow & \text{BQ–Weyl gravity,} \\ \text{BQ–Pauli electromagnetism} & \longrightarrow & \text{BQ–Weyl electromagnetism,} \\ \text{BQ–Pauli RFD} & \longrightarrow & \text{BQ–Weyl RFD.} \end{array}$$

The purpose of the present lift is therefore not to alter the physical content of the Pauli-level EM and RFD constructions, but to establish their embedding in the same sixteen-dimensional Clifford–Weyl algebra already obtained for gravity. Taken together, the three lifts provide a common $\mathrm{Cl}(1,3)$ representation of the gravitational, electromagnetic, and fluid-dynamical branches of the BQ programme.

2 The BQ–Pauli algebra

2.1 Matrix representation and basic generators

Quaternions admit several 2×2 complex matrix realizations; the one adopted here is chosen for direct compatibility with physics. The space-time four-set is

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3}$$

Comparing to the Pauli matrices $\boldsymbol{\sigma_P} = (\sigma_x, \sigma_y, \sigma_z)$ shows $\hat{I} = i\sigma_z$, $\hat{J} = i\sigma_y$, $\hat{K} = i\sigma_x$ – an axis ordering $(I, J, K) \leftrightarrow (z, y, x)$ chosen, rather than the conventional (x, y, z), because it is the ordering under which $\hat{K}_i = i\hat{\sigma}_i$ holds uniformly (footnote: this reordering is purely a labelling convention and carries no physical content). Equivalently, defining the Hermitian set $\hat{\sigma}_I = \sigma_z, \hat{\sigma}_J = \sigma_y, \hat{\sigma}_K = \sigma_x$, one has $\hat{\sigma}_i = -\hat{\mathrm{T}}\hat{K}_i$ with the central timelike element

$$\hat{\mathrm{T}} = i\hat{1}. \tag{4}$$

The pauliquat set $\hat{\sigma}_\mu = (\hat{1}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K)$ is thus not a starting point but the basis complementary to the physically motivated minquat set $\hat{K}_\mu = (\hat{\mathrm{T}}, \hat{I}, \hat{J}, \hat{K})$ (see Synge (1972) for the categorisation of Minquats (after Minkowski) and Pauliquats). (This representation is additionally the canonical complex-matrix form for Möbius transformations, simultaneously realizing Hamilton's quaternion algebra, $SU(2)$, and spinor transformations – the underlying reason this particular matrix ordering, rather than an arbitrary alternative, was selected.)

Consequences of $\hat{T} = i\hat{1}$.

Since $\hat{\mathrm{T}}$ is the unique non-scalar central element and $\hat{\mathrm{T}}^2 = -\hat{1}$, while $\hat{K}_i = \hat{\mathrm{T}}\hat{\sigma}_i$, the generator squares and Hermiticity split follow immediately:

$$\hat{\sigma}_i^2 = +\hat{1}, \qquad \hat{K}_i^2 = -\hat{1}, \qquad \hat{T}^\dagger = -\hat{T}, \quad \hat{\sigma}_i^\dagger = \hat{\sigma}_i, \quad \hat{K}_i^\dagger = -\hat{K}_i. \tag{5}$$

Consequently the two sectors generate different one-parameter exponentials,

$$e^{a\hat{\sigma}_i} = \hat{1}\cosh a + \hat{\sigma}_i \sinh a \qquad \text{(non-unitary, hyperbolic boost)}, \tag{6}$$

$$e^{a\hat{K}_i} = \hat{1}\cos a + \hat{K}_i \sin a \qquad \text{(unitary, circular rotation)}, \tag{7}$$

so that the distinction between Lorentz boosts and spatial rotations is a direct algebraic consequence of the generator properties, not an additional postulate.

Explicit matrix products.

All products used below are ordinary matrix products in $M_2(\mathbb{C})$. For example, the quaternionic square of $\hat{I}$ is obtained directly as

$$\hat{I}\hat{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\hat{1}. \tag{8}$$

Likewise, the cyclic quaternion product is

$$\hat{I}\hat{J} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \hat{K}, \tag{9}$$

whereas reversal of the order gives

$$\hat{J}\hat{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\hat{K}. \tag{10}$$

Thus the usual quaternion relations $\hat{I}^2 = \hat{J}^2 = \hat{K}^2 = -\hat{1}$ and $\hat{I}\hat{J} = -\hat{J}\hat{I} = \hat{K}$ follow directly from the matrix representation.

The relation between the minquat and pauliquat sectors is equally explicit. Since

$$\hat{\mathrm{T}} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad \hat{\sigma}_I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{11}$$

one finds

$$\hat{\mathrm{T}}\hat{\sigma}_I = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \hat{I}. \tag{12}$$

Hence $\hat{K}_i = \hat{\mathrm{T}}\hat{\sigma}_i$ is not an additional multiplication rule but follows from the same ordinary matrix product. Conversely, because $\hat{\mathrm{T}}^{-1} = -\hat{\mathrm{T}}$,

$$\hat{\sigma}_i = -\hat{\mathrm{T}}\hat{K}_i. \tag{13}$$

These elementary examples illustrate explicitly that the BQ multiplication rules used throughout this work are consequences of the chosen matrix realization in $M_2(\mathbb{C})$.

2.2 The multiplication table of the BQ algebra

The BQ algebra is generated over $\mathbb{R}$ by the eight-element basis

$$\{\hat{1}, \hat{T}, \hat{I}, \hat{J}, \hat{K}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K\}, \tag{14}$$

split into the minquat 4-set $\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ and the pauliquat 4-set $\hat{\sigma}_\mu = (\hat{1}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K)$. The defining relations are

$$\hat{T}^2 = -\hat{1}, \qquad \hat{I}^2 = \hat{J}^2 = \hat{K}^2 = -\hat{1}, \qquad \hat{\sigma}_I^2 = \hat{\sigma}_J^2 = \hat{\sigma}_K^2 = +\hat{1}, \tag{15}$$

$$\hat{T}X = X\hat{T} \text{ for all } X \ (\hat{T} \text{ is central}), \qquad \hat{I} = \hat{T}\hat{\sigma}_I, \quad \hat{J} = \hat{T}\hat{\sigma}_J, \quad \hat{K} = \hat{T}\hat{\sigma}_K, \tag{16}$$

together with the cyclic rules

$$\boxed{\hat{I}\hat{J} = \hat{K}, \qquad \hat{J}\hat{K} = \hat{I}, \qquad \hat{K}\hat{I} = \hat{J},} \tag{17}$$

$$\boxed{\hat{\sigma}_I\hat{\sigma}_J = -\hat{K}, \qquad \hat{\sigma}_J\hat{\sigma}_K = -\hat{I}, \qquad \hat{\sigma}_K\hat{\sigma}_I = -\hat{J},} \tag{18}$$

the opposite cyclic sign between the two sets inherited from $\hat{T}^2 = -\hat{1}$. Every other product in the algebra is generated from these by associativity, which the algebra

Table 1 Full multiplication table of the BQ algebra. Entry (row, col) gives row · col; the algebra is non-commutative, so the table is read with row as the left factor. $\dim_{\mathbb{R}} = 8$; $\hat{T}$ is the unique central, non-identity generator, square -1. Every entry verified directly by 2×2 matrix multiplication against Eqs. (79)–(81).

$\cdot$	$\hat{1}$	$\hat{T}$	$\hat{I}$	$\hat{J}$	$\hat{K}$	$\hat{\sigma}_I$	$\hat{\sigma}_J$	$\hat{\sigma}_K$
$\hat{1}$	$\hat{1}$	$\hat{T}$	$\hat{I}$	$\hat{J}$	$\hat{K}$	$\hat{\sigma}_I$	$\hat{\sigma}_J$	$\hat{\sigma}_K$
$\hat{T}$	$\hat{T}$	$-\hat{1}$	$-\hat{\sigma}_I$	$-\hat{\sigma}_J$	$-\hat{\sigma}_K$	$\hat{I}$	$\hat{J}$	$\hat{K}$
$\hat{I}$	$\hat{I}$	$-\hat{\sigma}_I$	$-\hat{1}$	$\hat{K}$	$-\hat{J}$	$\hat{T}$	$\hat{\sigma}_K$	$-\hat{\sigma}_J$
$\hat{J}$	$\hat{J}$	$-\hat{\sigma}_J$	$-\hat{K}$	$-\hat{1}$	$\hat{I}$	$-\hat{\sigma}_K$	$\hat{T}$	$\hat{\sigma}_I$
$\hat{K}$	$\hat{K}$	$-\hat{\sigma}_K$	$\hat{J}$	$-\hat{I}$	$-\hat{1}$	$\hat{\sigma}_J$	$-\hat{\sigma}_I$	$\hat{T}$
$\hat{\sigma}_I$	$\hat{\sigma}_I$	$\hat{I}$	$\hat{T}$	$\hat{\sigma}_K$	$-\hat{\sigma}_J$	$\hat{1}$	$-\hat{K}$	$\hat{J}$
$\hat{\sigma}_J$	$\hat{\sigma}_J$	$\hat{J}$	$-\hat{\sigma}_K$	$\hat{T}$	$\hat{\sigma}_I$	$\hat{K}$	$\hat{1}$	$-\hat{I}$
$\hat{\sigma}_K$	$\hat{\sigma}_K$	$\hat{K}$	$\hat{\sigma}_J$	$-\hat{\sigma}_I$	$\hat{T}$	$-\hat{J}$	$\hat{I}$	$\hat{1}$

inherits automatically from its matrix realization in $M_2(\mathbb{C})$ but which is not itself part of the defining data below: the multiplication table (Table 1) is the complete structure-constant data of the algebra, stated without reference to any ambient matrix ring.

Three structural remarks follow directly from Table 1 rather than from any external classification. First, the eight basis elements split into two square classes: $\{\hat{T}, \hat{I}, \hat{J}, \hat{K}\}$ each square to -1, while $\{\hat{1}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K\}$ each square to $+1$ – the elliptic/hyperbolic split that later produces circular versus hyperbolic one-parameter exponentials. Second, $\hat{T}$ is the unique non-scalar central element: it commutes with every other basis vector, which is what makes $\hat{T} = i\hat{1}$ a legitimate identification of $\hat{T}$ with the ordinary complex unit acting on the whole algebra, rather than a notational convenience. The channel formulas – the bilinear $A^T B$ and the triple product $E = DC$ – are direct corollaries of this table, obtainable by substituting $A = a_0\hat{T} + \mathbf{a} \cdot \hat{K}$, $B = b_0\hat{T} + \mathbf{b} \cdot \hat{K}$ and expanding term by term against the table; no property of the channel decomposition is assumed beyond what the table already fixes.

Third, the table is asymmetric under transposition. For example,

$$\hat{I}\hat{\sigma}_J = \hat{\sigma}_K, \qquad \hat{\sigma}_J\hat{I} = -\hat{\sigma}_K, \tag{19}$$

and hence

$$\hat{I}\hat{\sigma}_J = -\hat{\sigma}_J\hat{I}. \tag{20}$$

More generally, the three quaternion generators anticommute among themselves when their indices differ, as do the three Pauli generators. Mixed generators with different indices also anticommute. The corresponding same-axis pairs instead commute:

$$\hat{I}\hat{\sigma}_I = \hat{J}\hat{\sigma}_J = \hat{K}\hat{\sigma}_K = \hat{T}, \tag{21}$$

with the same products obtained in the reverse order.

2.3 The transpose bilinear as time-reversal

The superscript T in $C = A^T B$ is not the ordinary matrix transpose (written A^t) but a discrete operation on the real coordinates of the four-vector $A = a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}$: time-reversal and parity are defined by

$$A^T \equiv -a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}, \qquad A^P \equiv a_0\hat{T} - \mathbf{a}\cdot\hat{\mathbf{K}}, \tag{22}$$

satisfying $A^P = -A^T$, $(A^T)^P = -A$, so that $\{\mathrm{id}, \mathsf{T}, \mathsf{P}, \mathsf{TP}\}$ forms the Klein four-group of the disconnected components of $O(1,3)$ relative to $SO^+(1,3)$. Expanding $A^T B$ against Table 1 for $A = a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}$, $B = b_0\hat{T} + \mathbf{b}\cdot\hat{\mathbf{K}}$ gives

$$A^T B = (a_0 b_0 - \mathbf{a}\cdot\mathbf{b})\hat{1} + (\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{K}} + (a_0\mathbf{b} - b_0\mathbf{a})\cdot\hat{\boldsymbol{\sigma}}, \tag{23}$$

so that the scalar channel $[A^T B]_{\hat{1}} = a_0 b_0 - \mathbf{a}\cdot\mathbf{b} = \eta_{\mu\nu}A^\mu B^\nu$ is exactly the Lorentzian metric contraction: time-reversal supplies the single relative sign that converts the Euclidean-signature product AB into the Lorentzian invariant, with $A^P B = -A^T B$ giving the opposite-signature pairing. The superscript A^T is never conjugation or multiplication by the basis element $\hat{T}$.

2.4 Why the minquat and pauliquat sets, not the closed quaternion subalgebra

Exactly one natural four-dimensional real span in the basis closes under multiplication: $\{\hat{1}, \hat{I}, \hat{J}, \hat{K}\} \simeq \mathbb{H}$, Hamilton's quaternion group. Its invariant norm $q^*q = r_0^2 + r_1^2 + r_2^2 + r_3^2$ is positive-definite and cannot become Lorentzian under any real change of coordinates. The minquat and pauliquat four-sets used throughout this paper, $\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ and $\hat{\sigma}_\mu = (\hat{1}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K)$, are useful precisely because they do *not* close: the product of two minquats necessarily populates both sets, $\mathcal{K}\times\mathcal{K} \subset \mathrm{span}_{\mathbb{R}}\{\hat{1}, \hat{I}, \hat{J}, \hat{K}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K\}$. This non-closure is the mechanism, not a defect: it is how a product of two spacetime four-vectors is forced to generate both the Lorentzian scalar invariant and the complementary rotation/boost channels.

2.5 Real coordinates as an explicit convention

Synge's minquat formulation keeps the quaternion basis real and makes the temporal coordinate complex, $R_0 = ib_0$. The present convention reverses this: all coordinates R_μ, P_μ are kept real, and the complex unit is instead carried by the central basis element $\hat{T} = i\hat{1}$ (equivalently $\hat{K}_i = i\hat{\sigma}_i$), so that $Z(\mathrm{BQ}) = \mathrm{span}_{\mathbb{R}}\{\hat{1}, \hat{T}\} \simeq \mathbb{C}$. This is a choice of representation, not a result forced uniquely by the abstract algebra; it is adopted because the coefficients of physical spacetime, momentum, and fluid four-vectors are then real measured quantities throughout, while the Lorentzian and complex structure is carried entirely by the basis itself.

Summary

The BQ algebra is obtained by internalizing the complex unit as the central generator $\hat{\mathrm{T}} = i\hat{1}$, thereby passing from Hamilton's quaternion algebra to its biquaternion extension while keeping all physical coefficients real. The chosen Möbius-compatible matrix realization determines the multiplication table, and the multiplication table in turn generates all bilinear, triple and differential BQ products used throughout this paper. The appendix therefore provides the complete algorithmic foundation required for the subsequent physical developments.

3 The sixteen-element BQ–Weyl Clifford basis

The BQ–Weyl algebra used here is not introduced independently of the BQ–Pauli algebra summarized in Appendix 2. It is obtained by placing the BQ minquat four-set

$$\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K}) \tag{24}$$

into a parity-duplex block construction. The internal entries remain elements of the same eight-dimensional BQ–Pauli algebra, while the additional block structure generates the sixteen-dimensional Clifford grade hierarchy.

The Weyl tetrad is

$$\beta_0 = \begin{pmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{pmatrix}, \qquad \beta_i = \begin{pmatrix} 0 & \hat{K}_i \\ -\hat{K}_i & 0 \end{pmatrix}, \qquad i = I, J, K. \tag{25}$$

Thus the Clifford vector sector is the direct block lift of the BQ minquat set. Using

$$\hat{\mathrm{T}} = i\hat{1}, \qquad \hat{K}_i = \hat{T}\hat{\sigma}_i, \qquad -\hat{T}\hat{K}_i = \hat{\sigma}_i, \tag{26}$$

the products of the tetrad recover the two complementary BQ generator sectors:

$$\alpha_i := \beta_0\beta_i = \begin{pmatrix} \hat{\sigma}_i & 0 \\ 0 & -\hat{\sigma}_i \end{pmatrix}, \tag{27}$$

and, for cyclic (i, j, k),

$$\Sigma_i := \beta_j \beta_k = \begin{pmatrix} -\hat{K}_i & 0 \\ 0 & -\hat{K}_i \end{pmatrix}. \tag{28}$$

The boost bivectors α_i are therefore the Weyl-level realization of the BQ Pauli generators $\hat{\sigma}_i$, while the spin bivectors Σ_i are the corresponding realization of the quaternion rotation generators $\hat{K}_i$.

The passage from BQ–Pauli to BQ–Weyl can consequently be summarized as

$$\underbrace{1+3+3+1}_{\text{BQ–Pauli channels}} \quad \longrightarrow \quad \underbrace{1+4+6+4+1}_{\text{Weyl–Clifford grades}}. \tag{29}$$

The block doubling does not replace the BQ algebra; it uses BQ as its internal 2×2 coefficient algebra and adds the vector, pseudovector and pseudoscalar placements required for the full Clifford hierarchy. The complete sixteen-element basis is generated from the tetrad β_μ alone.

3.1 The sixteen-element basis by Clifford grade

The products generated from the tetrad β_μ close on sixteen real basis elements. These divide into the five Clifford grades

$$1+4+6+4+1,$$

with the six bivectors separating naturally into the three boost generators α_i and the three spin generators Σ_i already identified above. The complete basis is summarized in Table 2.

Table 2 The sixteen-element BQ–Weyl basis organized by Clifford grade.

Grade	Object	Count	Definition
0	scalar	1	$\mathbb{1}$
1	vector	4	$\beta_0,\ \beta_I,\ \beta_J,\ \beta_K$
2	boost bivector	3	$\alpha_i = \beta_0\beta_i$
2	spin bivector	3	$\Sigma_i = \beta_j\beta_k, \quad (i, j, k)$ cyclic
3	pseudovector	4	$\tilde{\beta}_\mu = \chi\beta_\mu, \quad \mu = 0, I, J, K$
4	pseudoscalar	1	$\chi = -\beta_0\beta_I\beta_J\beta_K$
$1+4+3+3+4+1 = 16$.			

Even and odd sectors.

The basis splits equally into an eight-dimensional even sector and an eight-dimensional odd sector,

$$C_{\text{even}} = \operatorname{span}_{\mathbb{R}}\{\mathbb{1}, \alpha_I, \alpha_J, \alpha_K, \Sigma_I, \Sigma_J, \Sigma_K, \chi\}, \tag{30}$$

$$C_{\text{odd}} = \operatorname{span}_{\mathbb{R}}\{\beta_0, \beta_I, \beta_J, \beta_K, \tilde{\beta}_0, \tilde{\beta}_I, \tilde{\beta}_J, \tilde{\beta}_K\}. \tag{31}$$

In the Weyl representation this grading coincides with the block structure: the even elements are block diagonal and the odd elements are block off diagonal.

The even sector is precisely the sector in which the BQ–Pauli 1 + 3 + 3 + 1 structure reappears:

$$\hat{1} \longmapsto \mathbb{1}, \qquad \hat{\sigma}_i \longmapsto \alpha_i, \qquad \hat{K}_i \longmapsto -\Sigma_i, \qquad \hat{T} \longmapsto \chi. \tag{32}$$

The odd sector contains the four lifted minquat vectors β_μ and their pseudoscalar partners $\tilde{\beta}_\mu$.

Squares.

The vector and bivector generators satisfy

$$\beta_0^2 = -\mathbb{1}, \qquad \beta_I^2 = \beta_J^2 = \beta_K^2 = +\mathbb{1}, \tag{33}$$

$$\alpha_I^2 = \alpha_J^2 = \alpha_K^2 = +\mathbb{1}, \qquad \Sigma_I^2 = \Sigma_J^2 = \Sigma_K^2 = -\mathbb{1}. \tag{34}$$

For the pseudovector and pseudoscalar sectors,

$$\tilde{\beta}_0^2 = -\mathbb{1}, \qquad \tilde{\beta}_I^2 = \tilde{\beta}_J^2 = \tilde{\beta}_K^2 = +\mathbb{1}, \qquad \chi^2 = -\mathbb{1}. \tag{35}$$

The bivector square classes preserve the BQ–Pauli distinction: α_i are hyperbolic generators and Σ_i are circular generators.

Explicit BQ block forms.

For reference, the sixteen basis elements are

$$\mathbb{1} = \begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix}, \qquad \beta_0 = \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix}, \tag{36}$$

$$\beta_I = \begin{bmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{bmatrix}, \qquad \beta_J = \begin{bmatrix} 0 & \hat{J} \\ -\hat{J} & 0 \end{bmatrix}, \qquad \beta_K = \begin{bmatrix} 0 & \hat{K} \\ -\hat{K} & 0 \end{bmatrix}, \tag{37}$$

$$\alpha_i = \begin{bmatrix} \hat{\sigma}_i & 0 \\ 0 & -\hat{\sigma}_i \end{bmatrix}, \qquad \Sigma_i = \begin{bmatrix} -\hat{K}_i & 0 \\ 0 & -\hat{K}_i \end{bmatrix}, \tag{38}$$

$$\tilde{\beta}_0 = \begin{bmatrix} 0 & -\hat{1} \\ \hat{1} & 0 \end{bmatrix}, \qquad \tilde{\beta}_i = \begin{bmatrix} 0 & -\hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{bmatrix}, \tag{39}$$

$$\chi = \begin{bmatrix} \hat{T} & 0 \\ 0 & -\hat{T} \end{bmatrix}. \tag{40}$$

Explicit Weyl matrix products.

As at the BQ–Pauli level, the multiplication rules of the Weyl basis are not imposed independently: they follow from ordinary matrix multiplication, now in $M_4(\mathbb{C})$. A few representative products make the construction explicit. First, the timelike generator satisfies

$$\begin{aligned}\beta_0^2 &= \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix} = \begin{bmatrix} \hat{T}^2 & 0 \\ 0 & \hat{T}^2 \end{bmatrix} \\ &= \begin{bmatrix} -\hat{1} & 0 \\ 0 & -\hat{1} \end{bmatrix} = -\mathbb{1}. \end{aligned} \tag{41}$$

In full 4×4 form this is

$$\begin{aligned} &\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\ &\qquad = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\mathbb{1}. \end{aligned} \tag{42}$$

The spatial generators have the opposite square. For example,

$$\beta_I^2 = \begin{bmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{bmatrix} = \begin{bmatrix} -\hat{I}^2 & 0 \\ 0 & -\hat{I}^2 \end{bmatrix} = \mathbb{1}, \tag{43}$$

since $\hat{I}^2 = -\hat{1}$. Thus the matrix representation directly gives

$$\boxed{\beta_0^2 = -\mathbb{1}, \qquad \beta_i^2 = +\mathbb{1},} \tag{44}$$

and hence the $(-, +, +, +)$ Clifford signature used throughout this work.

Products of different Weyl vectors generate the six bivectors. For a timelike–spacelike pair,

$$\begin{aligned}\beta_0\beta_I &= \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{bmatrix} = \begin{bmatrix} -\hat{T}\hat{I} & 0 \\ 0 & \hat{T}\hat{I} \end{bmatrix} \\ &= \begin{bmatrix} \hat{\sigma}_I & 0 \\ 0 & -\hat{\sigma}_I \end{bmatrix} = \alpha_I, \end{aligned} \tag{45}$$

where $\hat{T}\hat{I} = -\hat{\sigma}_I$. In full matrix form,

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \alpha_I.
\end{aligned} \tag{46}$$

Reversing the order gives

$$\beta_I \beta_0 = -\alpha_I, \tag{47}$$

so that

$$\beta_0 \beta_I = -\beta_I \beta_0. \tag{48}$$

A product of two different spatial vectors generates the complementary rotation bivector. Using $\hat{J}\hat{K} = \hat{I}$,

$$\begin{aligned}
\beta_J \beta_K &= \begin{bmatrix} 0 & \hat{J} \\ -\hat{J} & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{K} \\ -\hat{K} & 0 \end{bmatrix} = \begin{bmatrix} -\hat{J}\hat{K} & 0 \\ 0 & -\hat{J}\hat{K} \end{bmatrix} \\
&= \begin{bmatrix} -\hat{I} & 0 \\ 0 & -\hat{I} \end{bmatrix} = \Sigma_I.
\end{aligned} \tag{49}$$

In full 4×4 form,

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \Sigma_I.
\end{aligned} \tag{50}$$

Again the reversed product changes sign,

$$\beta_K \beta_J = -\Sigma_I. \tag{51}$$

These examples display directly the basic Clifford construction underlying the complete multiplication table:

$$\boxed{\beta_0^2 = -\mathbb{1}, \qquad \beta_i^2 = +\mathbb{1}, \qquad \beta_0 \beta_i = \alpha_i, \qquad \beta_j \beta_k = \Sigma_i \quad (i, j, k \text{ cyclic}).} \tag{52}$$

Thus the four Weyl vectors and six Weyl bivectors are not independent matrix structures introduced by definition: the bivector sector is generated directly by pairwise multiplication of the vector generators.

The remaining trivector and pseudoscalar sectors follow in the same way from higher products, as summarized by the complete multiplication table below. The grade-4 element is generated in the same way by the product of all four Weyl vectors. Using the block definitions above,

$$\beta_0\beta_I\beta_J\beta_K = \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{J} \\ -\hat{J} & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{K} \\ -\hat{K} & 0 \end{bmatrix}. \tag{53}$$

The first two factors give

$$\beta_0\beta_I = \begin{bmatrix} -\hat{T}\hat{I} & 0 \\ 0 & \hat{T}\hat{I} \end{bmatrix} = \begin{bmatrix} \hat{\sigma}_I & 0 \\ 0 & -\hat{\sigma}_I \end{bmatrix} = \alpha_I. \tag{54}$$

Multiplication by β_J then gives

$$\begin{aligned} \beta_0\beta_I\beta_J &= \begin{bmatrix} \hat{\sigma}_I & 0 \\ 0 & -\hat{\sigma}_I \end{bmatrix} \begin{bmatrix} 0 & \hat{J} \\ -\hat{J} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \hat{\sigma}_I\hat{J} \\ \hat{\sigma}_I\hat{J} & 0 \end{bmatrix}. \end{aligned} \tag{55}$$

Since

$$\hat{\sigma}_I = -\hat{T}\hat{I}, \qquad \hat{I}\hat{J} = \hat{K}, \qquad \hat{T}\hat{K} = -\hat{\sigma}_K, \tag{56}$$

one has

$$\hat{\sigma}_I\hat{J} = -\hat{T}\hat{I}\hat{J} = -\hat{T}\hat{K} = \hat{\sigma}_K. \tag{57}$$

Therefore

$$\beta_0\beta_I\beta_J = \begin{bmatrix} 0 & \hat{\sigma}_K \\ \hat{\sigma}_K & 0 \end{bmatrix} = -\tilde{\beta}_K. \tag{58}$$

So the product of three β's produces a pseudo-β, $\tilde{\beta}$. Finally,

$$\begin{aligned} \beta_0\beta_I\beta_J\beta_K &= \begin{bmatrix} 0 & \hat{\sigma}_K \\ \hat{\sigma}_K & 0 \end{bmatrix} \begin{bmatrix} 0 & \hat{K} \\ -\hat{K} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\hat{\sigma}_K\hat{K} & 0 \\ 0 & \hat{\sigma}_K\hat{K} \end{bmatrix}. \end{aligned} \tag{59}$$

Using $\hat{K} = \hat{T}\hat{\sigma}_K$ and $\hat{\sigma}_K^2 = \hat{1}$,

$$\hat{\sigma}_K\hat{K} = \hat{\sigma}_K\hat{T}\hat{\sigma}_K = \hat{T}, \tag{60}$$

because $\hat{T}$ is central. Hence

$$\boxed{\beta_0\beta_I\beta_J\beta_K = \begin{bmatrix} -\hat{T} & 0 \\ 0 & \hat{T} \end{bmatrix} = -\chi.} \tag{61}$$

Equivalently, the orientation convention adopted throughout this work is

$$\boxed{\chi = -\beta_0\beta_I\beta_J\beta_K,} \tag{62}$$

or, in its explicit 4x4 complex matrix multiplication:

$$\begin{aligned} &\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = -\chi. \end{aligned} \tag{63}$$

The explicit products also clarify the relation of the present construction to two common formulations of relativistic algebra. In contrast to an abstract STA/GA presentation, where the Clifford geometric product is introduced intrinsically and a matrix representation may be chosen subsequently, the BQ construction used here is representation-first: its basis elements are explicit complex matrices and all products are ordinary matrix products. At the same time, this places the construction directly alongside the standard Pauli–Dirac matrix formalism, since the resulting $M_2(\mathbb{C})$ and Weyl $M_4(\mathbb{C})$ generators are explicit matrix realizations of the corresponding Pauli and Clifford structures. The distinction is therefore one of algebraic realization and calculational organization, not a claim of an inequivalent underlying Clifford algebra.

3.2 The complete multiplication table

Table 3 gives the ordered product XY for every pair of basis elements, with the row element acting from the left and the column element from the right. Every entry was verified by direct 4×4 matrix multiplication.

With real coefficients, the sixteen matrices span the sixteen-dimensional real Clifford algebra represented inside $M_4(\mathbb{C})$. They do not span all of $M_4(\mathbb{C})$ as a real vector space, whose real dimension is 32. The multiplication table is nevertheless exhaustive for the real BQ–Weyl algebra, since every product of two basis elements closes within the same sixteen-element set.

Structural remarks.
The tetrad satisfies

$$\beta_\mu\beta_\nu + \beta_\nu\beta_\mu = 2\eta_{\mu\nu}\mathbb{1}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1), \tag{64}$$

and hence

$$\beta_\mu\beta_\nu = -\beta_\nu\beta_\mu, \qquad \mu \neq \nu. \tag{65}$$

The pseudoscalar anticommutes with the odd sector,

$$\chi\beta_\mu = -\beta_\mu\chi, \qquad \chi\tilde{\beta}_\mu = -\tilde{\beta}_\mu\chi, \tag{66}$$

and commutes with the even bivector sector,

$$\chi\alpha_i = \alpha_i\chi, \qquad \chi\Sigma_i = \Sigma_i\chi. \tag{67}$$

The multiplication rules therefore respect the parity grading,

$$\mathcal{C}_{\mathrm{even}}\mathcal{C}_{\mathrm{even}} \subset \mathcal{C}_{\mathrm{even}}, \qquad \mathcal{C}_{\mathrm{even}}\mathcal{C}_{\mathrm{odd}} \subset \mathcal{C}_{\mathrm{odd}}, \tag{68}$$

$$\mathcal{C}_{\mathrm{odd}}\mathcal{C}_{\mathrm{even}} \subset \mathcal{C}_{\mathrm{odd}}, \qquad \mathcal{C}_{\mathrm{odd}}\mathcal{C}_{\mathrm{odd}} \subset \mathcal{C}_{\mathrm{even}}. \tag{69}$$

Thus the full table displays the $1 + 4 + 6 + 4 + 1$ Clifford hierarchy, while its even block-diagonal sector preserves the original $1 + 3 + 3 + 1$ BQ–Pauli structure.

3.3 Why the BQ construction is carried out in the Weyl representation

The BQ construction presented in this appendix is developed in the Weyl representation because it preserves the algebraic structure inherited from the BQ–Pauli algebra. The Weyl tetrad

$$\beta_\mu = (\beta_0, \beta_I, \beta_J, \beta_K) \tag{70}$$

is obtained directly by parity-duplex lifting of the BQ minquat basis

$$\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K}), \tag{71}$$

so that every Clifford generator follows algorithmically from products of the tetrad.

The resulting hierarchy

$$1 + 3 + 3 + 1 \quad \longrightarrow \quad 1 + 4 + 6 + 4 + 1 \tag{72}$$

is therefore completely transparent: the BQ–Pauli algebra forms the internal even structure, while the block doubling generates the odd vector and pseudovector sectors required for the full Clifford algebra. The even and odd sectors remain explicitly separated throughout the construction.

This feature is illustrated by the Weyl momentum operator,

$$\not{P}_W = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix}, \tag{73}$$

which consists simply of the parity-duplex pairing of the original BQ four-vector P. The complete Clifford basis is then generated by successive products of the four matrices β_μ.

By contrast, the Dirac representation mixes the parity and time-reversal duplexes into the block structure,

$$\not{P}_D = \begin{bmatrix} p_0\hat{T} & \mathbf{p}\cdot\hat{\mathbf{K}} \\ -\mathbf{p}\cdot\hat{\mathbf{K}} & -p_0\hat{T} \end{bmatrix}, \tag{74}$$

so that the direct correspondence between the block entries and the original BQ generators is no longer manifest. The Dirac representation is therefore less transparent from the standpoint of the algebraic construction, even though it is often the more convenient representation for the description of massive fermions.

The two representations are mathematically equivalent, being related by a similarity transformation, and therefore generate the same Clifford algebra. Their difference lies only in the organization of the basis. The Weyl representation preserves the BQ hierarchy and exposes the algorithmic generation of the Clifford grades, whereas the Dirac representation rearranges the same algebra into a form optimized for the standard Dirac equation. Thus, the Weyl representation is construction-oriented, whereas the Dirac representation is application-oriented.

For this reason the algebraic development of the present work is carried out almost entirely in the Weyl representation. Once the Clifford basis and its multiplication properties have been established, one may pass to the Dirac representation whenever the physical application makes that representation more convenient.

Table 3 Complete multiplication table for the sixteen-element Weyl-representation basis. Read as row $\times$ column.

$\times$	$\mathbb{1}$	β_0	β_I	β_J	β_K	α_I	α_J	α_K	Σ_I	Σ_J	Σ_K	$\tilde{\beta}_0$	$\tilde{\beta}_I$	$\tilde{\beta}_J$	$\tilde{\beta}_K$	χ
$\mathbb{1}$	$\mathbb{1}$	β_0	β_I	β_J	β_K	α_I	α_J	α_K	Σ_I	Σ_J	Σ_K	$\tilde{\beta}_0$	$\tilde{\beta}_I$	$\tilde{\beta}_J$	$\tilde{\beta}_K$	χ
β_0	β_0	$-\mathbb{1}$	α_I	α_J	α_K	$-\beta_I$	$-\beta_J$	$-\beta_K$	$-\tilde{\beta}_I$	$-\tilde{\beta}_J$	$-\tilde{\beta}_K$	χ	Σ_I	Σ_J	Σ_K	$-\tilde{\beta}_0$
β_I	β_I	$-\alpha_I$	$\mathbb{1}$	Σ_K	$-\Sigma_J$	$-\beta_0$	$\tilde{\beta}_K$	$-\tilde{\beta}_J$	$-\tilde{\beta}_0$	$-\beta_K$	β_J	$-\Sigma_I$	$-\chi$	$-\alpha_K$	α_J	$-\tilde{\beta}_I$
β_J	β_J	$-\alpha_J$	$-\Sigma_K$	$\mathbb{1}$	Σ_I	$-\tilde{\beta}_K$	$-\beta_0$	$\tilde{\beta}_I$	β_K	$-\tilde{\beta}_0$	$-\beta_I$	$-\Sigma_J$	α_K	$-\chi$	$-\alpha_I$	$-\tilde{\beta}_J$
β_K	β_K	$-\alpha_K$	Σ_J	$-\Sigma_I$	$\mathbb{1}$	$\tilde{\beta}_J$	$-\tilde{\beta}_I$	$-\beta_0$	$-\beta_J$	β_I	$-\tilde{\beta}_0$	$-\Sigma_K$	$-\alpha_J$	α_I	$-\chi$	$-\tilde{\beta}_K$
α_I	α_I	β_I	β_0	$-\tilde{\beta}_K$	$\tilde{\beta}_J$	$\mathbb{1}$	Σ_K	$-\Sigma_J$	$-\chi$	$-\alpha_K$	α_J	$\tilde{\beta}_I$	$\tilde{\beta}_0$	β_K	$-\beta_J$	$-\Sigma_I$
α_J	α_J	β_J	$\tilde{\beta}_K$	β_0	$-\tilde{\beta}_I$	$-\Sigma_K$	$\mathbb{1}$	Σ_I	α_K	$-\chi$	$-\alpha_I$	$\tilde{\beta}_J$	$-\beta_K$	$\tilde{\beta}_0$	β_I	$-\Sigma_J$
α_K	α_K	β_K	$-\tilde{\beta}_J$	$\tilde{\beta}_I$	β_0	Σ_J	$-\Sigma_I$	$\mathbb{1}$	$-\alpha_J$	α_I	$-\chi$	$\tilde{\beta}_K$	β_J	$-\beta_I$	$\tilde{\beta}_0$	$-\Sigma_K$
Σ_I	Σ_I	$-\tilde{\beta}_I$	$-\tilde{\beta}_0$	$-\beta_K$	β_J	$-\chi$	$-\alpha_K$	α_J	$-\mathbb{1}$	$-\Sigma_K$	Σ_J	β_I	β_0	$-\tilde{\beta}_K$	$\tilde{\beta}_J$	α_I
Σ_J	Σ_J	$-\tilde{\beta}_J$	β_K	$-\tilde{\beta}_0$	$-\beta_I$	α_K	$-\chi$	$-\alpha_I$	Σ_K	$-\mathbb{1}$	$-\Sigma_I$	β_J	$\tilde{\beta}_K$	β_0	$-\tilde{\beta}_I$	α_J
Σ_K	Σ_K	$-\tilde{\beta}_K$	$-\beta_J$	β_I	$-\tilde{\beta}_0$	$-\alpha_J$	α_I	$-\chi$	$-\Sigma_J$	Σ_I	$-\mathbb{1}$	β_K	$-\tilde{\beta}_J$	$\tilde{\beta}_I$	β_0	α_K
$\tilde{\beta}_0$	$\tilde{\beta}_0$	$-\chi$	$-\Sigma_I$	$-\Sigma_J$	$-\Sigma_K$	$-\tilde{\beta}_I$	$-\tilde{\beta}_J$	$-\tilde{\beta}_K$	β_I	β_J	β_K	$-\mathbb{1}$	α_I	α_J	α_K	β_0
$\tilde{\beta}_I$	$\tilde{\beta}_I$	Σ_I	χ	α_K	$-\alpha_J$	$-\tilde{\beta}_0$	$-\beta_K$	β_J	β_0	$-\tilde{\beta}_K$	$\tilde{\beta}_J$	$-\alpha_I$	$\mathbb{1}$	Σ_K	$-\Sigma_J$	β_I
$\tilde{\beta}_J$	$\tilde{\beta}_J$	Σ_J	$-\alpha_K$	χ	α_I	β_K	$-\tilde{\beta}_0$	$-\beta_I$	$\tilde{\beta}_K$	β_0	$-\tilde{\beta}_I$	$-\alpha_J$	$-\Sigma_K$	$\mathbb{1}$	Σ_I	β_J
$\tilde{\beta}_K$	$\tilde{\beta}_K$	Σ_K	α_J	$-\alpha_I$	χ	$-\beta_J$	β_I	$-\tilde{\beta}_0$	$-\tilde{\beta}_J$	$\tilde{\beta}_I$	β_0	$-\alpha_K$	Σ_J	$-\Sigma_I$	$\mathbb{1}$	β_K
χ	χ	$\tilde{\beta}_0$	$\tilde{\beta}_I$	$\tilde{\beta}_J$	$\tilde{\beta}_K$	$-\Sigma_I$	$-\Sigma_J$	$-\Sigma_K$	α_I	α_J	α_K	$-\beta_0$	$-\beta_I$	$-\beta_J$	$-\beta_K$	$-\mathbb{1}$

4 BQ-Weyl EM

4.1 The electromagnetic product $\not{B} = \not{\partial}\not{A}$

As a direct application of the Weyl multiplication table and the bilinear product rule of Appendix A.2, consider the electromagnetic four-potential

$$\not{A} = A_0\hat{\beta}_0 + \mathbf{A}\cdot\boldsymbol{\beta}, \qquad A_0 = \frac{\phi}{c}, \tag{75}$$

and the Weyl vector derivative

$$\not{\partial} = \partial_0\hat{\beta}_0 + \boldsymbol{\nabla}\cdot\boldsymbol{\beta}, \qquad \partial_0 = -\frac{1}{c}\partial_t. \tag{76}$$

Their Clifford product is

$$\boxed{\not{B} \equiv \not{\partial}\not{A} = \left(\partial_0\hat{\beta}_0 + \boldsymbol{\nabla}\cdot\boldsymbol{\beta}\right)\left(A_0\hat{\beta}_0 + \mathbf{A}\cdot\boldsymbol{\beta}\right).} \tag{77}$$

Applying the multiplication table of section 3 gives directly

$$\boxed{\begin{aligned}\not{B} = {} & (-\partial_0 A_0 + \boldsymbol{\nabla}\cdot\mathbf{A})\,\not{1} \\ & + (\partial_0\mathbf{A} - \boldsymbol{\nabla}A_0)\cdot\boldsymbol{\alpha} + (\boldsymbol{\nabla}\times\mathbf{A})\cdot\boldsymbol{\Sigma}.\end{aligned}} \tag{78}$$

Thus the product of the two Weyl vectors populates only the scalar and bivector sectors,

$$\Lambda^1\Lambda^1 \subset \Lambda^0 \oplus \Lambda^2, \tag{79}$$

with the bivector sector separating naturally into the boost-like $\hat{\alpha}_i$ and rotation-like $\hat{\Sigma}_i$ channels.

Using

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial_t\mathbf{A}, \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{80}$$

together with $A_0 = \phi/c$ and $\partial_0 = -c^{-1}\partial_t$, the three channels become

$$[\not{B}]_{\not{1}} = \frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\mathbf{A}, \tag{81a}$$

$$[\not{B}]_{\alpha} = \frac{\mathbf{E}}{c}, \tag{81b}$$

$$[\not{B}]_{\Sigma} = \mathbf{B} \tag{81c}$$

Hence

$$\boxed{\not{B} = \left(\frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\mathbf{A}\right)\not{1} + \frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma}.} \tag{82}$$

The scalar channel is the Lorenz-gauge combination. Therefore, under

$$\frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\mathbf{A} = 0, \tag{83}$$

the electromagnetic product reduces to the pure bivector field

$$\boxed{\slashed{B} = \slashed{\partial}\slashed{A} = \frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma}.} \tag{84}$$

The BQ–Weyl multiplication table therefore resolves the single product $\slashed{\partial}\slashed{A}$ directly into three geometrically distinct channels: the scalar gauge channel, the electric boost-bivector channel $\boldsymbol{\alpha}$, and the magnetic rotation-bivector channel $\boldsymbol{\Sigma}$. Under the Lorenz condition the scalar channel vanishes, leaving the electric and magnetic fields as the two complementary sectors of the Weyl bivector field.

4.2 The electromagnetic triple product $\slashed{\partial}\slashed{B} = \mu_0\slashed{J}$

The electromagnetic field obtained in the preceding subsection is

$$\slashed{B} = b_1\slashed{1} + \mathbf{b}_\alpha\cdot\boldsymbol{\alpha} + \mathbf{b}_\Sigma\cdot\boldsymbol{\Sigma}, \tag{85}$$

with

$$b_1 = \frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\mathbf{A}, \qquad \mathbf{b}_\alpha = \frac{\mathbf{E}}{c}, \qquad \mathbf{b}_\Sigma = \mathbf{B}. \tag{86}$$

The next product in the Weyl hierarchy is therefore

$$\boxed{\slashed{\partial}\slashed{B} = \left(\partial_0\hat{\beta}_0 + \boldsymbol{\nabla}\cdot\boldsymbol{\beta}\right)\left(b_1\slashed{1} + \mathbf{b}_\alpha\cdot\boldsymbol{\alpha} + \mathbf{b}_\Sigma\cdot\boldsymbol{\Sigma}\right),} \tag{87}$$

where

$$\partial_0 = -\frac{1}{c}\partial_t.$$

Applying the general Weyl triple-product rule of Appendix A.2 gives

$$\slashed{\partial}\slashed{B} = K_0\hat{\beta}_0 + \mathbf{K}\cdot\boldsymbol{\beta} + R_0\tilde{\hat{\beta}}_0 + \mathbf{R}\cdot\tilde{\boldsymbol{\beta}}, \tag{88}$$

with

$$K_0 = \partial_0 b_1 - \boldsymbol{\nabla}\cdot\mathbf{b}_\alpha, \tag{89a}$$
$$\mathbf{K} = \boldsymbol{\nabla}b_1 - \partial_0\mathbf{b}_\alpha - \boldsymbol{\nabla}\times\mathbf{b}_\Sigma, \tag{89b}$$
$$R_0 = -\boldsymbol{\nabla}\cdot\mathbf{b}_\Sigma, \tag{89c}$$
$$\mathbf{R} = -\partial_0\mathbf{b}_\Sigma + \boldsymbol{\nabla}\times\mathbf{b}_\alpha. \tag{89d}$$

Substituting $\mathbf{b}_\alpha = \mathbf{E}/c$, $\mathbf{b}_\Sigma = \mathbf{B}$, and $\partial_0 = -c^{-1}\partial_t$ yields

$$K_0 = -\frac{1}{c}\partial_t b_1 - \frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{E}, \tag{90a}$$

$$\mathbf{K} = \boldsymbol{\nabla} b_1 + \frac{1}{c^2}\partial_t\mathbf{E} - \boldsymbol{\nabla}\times\mathbf{B}, \tag{90b}$$

$$R_0 = -\boldsymbol{\nabla}\cdot\mathbf{B}, \tag{90c}$$

$$\mathbf{R} = \frac{1}{c}\left(\partial_t\mathbf{B} + \boldsymbol{\nabla}\times\mathbf{E}\right). \tag{90d}$$

Under the Lorenz condition

$$b_1 = \frac{1}{c^2}\partial_t\phi + \boldsymbol{\nabla}\cdot\mathbf{A} = 0, \tag{91}$$

the vector channels reduce to

$$K_0 = -\frac{1}{c}\boldsymbol{\nabla}\cdot\mathbf{E}, \tag{92a}$$

$$\mathbf{K} = -\left(\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E}\right), \tag{92b}$$

whereas the pseudovector channels are

$$R_0 = -\boldsymbol{\nabla}\cdot\mathbf{B}, \tag{93a}$$

$$\mathbf{R} = \frac{1}{c}\left(\partial_t\mathbf{B} + \boldsymbol{\nabla}\times\mathbf{E}\right). \tag{93b}$$

Using the Maxwell equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} = \mu_0 c^2\rho, \qquad \boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E} = \mu_0\mathbf{J}, \tag{94}$$

together with

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{E} + \partial_t\mathbf{B} = 0, \tag{95}$$

the four Weyl channels become

$$K_0 = -\mu_0 c\rho, \qquad \mathbf{K} = -\mu_0\mathbf{J}, \qquad R_0 = 0, \qquad \mathbf{R} = \mathbf{0}. \tag{96}$$

Defining the physical four-current with the conventional positive components,

$$\boxed{\not{J} \equiv c\rho\,\hat{\beta}_0 + \mathbf{J}\cdot\boldsymbol{\beta},} \tag{97}$$

the complete electromagnetic field equation therefore takes the form

$$\boxed{\not{\partial}\not{B} = -\mu_0\not{J}.} \tag{98}$$

The Weyl triple product thus separates Maxwell's equations by Clifford grade. The vector sector

$$\{\hat{\beta}_0, \hat{\beta}_i\}$$

contains the sourced equations, Gauss' electric law and the Ampère–Maxwell law, while the pseudovector sector

$$\{\tilde{\hat{\beta}}_0, \tilde{\hat{\beta}}_i\}$$

contains the homogeneous equations, Gauss' magnetic law and Faraday's law. In the absence of magnetic charge and magnetic current the complete pseudovector sector vanishes, leaving the electromagnetic source entirely in the vector sector.

Thus the BQ–Weyl multiplication hierarchy

$$\not{A} \xrightarrow{\not{\partial}} \not{B} \xrightarrow{\not{\partial}} -\mu_0 \not{J}$$

recovers all four Maxwell equations from successive Clifford products, with the vector and pseudovector sectors separating the sourced and homogeneous equations automatically.

4.3 The Lorentz-force product $\not{J}\not{B}$

The electromagnetic four-current and field bivector are

$$\not{J} = c\rho\,\hat{\beta}_0 + \mathbf{J}\cdot\boldsymbol{\beta}, \tag{99}$$

and, under the Lorenz condition,

$$\not{B} = \frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma}. \tag{100}$$

Their Clifford product is therefore

$$\boxed{\not{J}\not{B} = \left(c\rho\,\hat{\beta}_0 + \mathbf{J}\cdot\boldsymbol{\beta}\right)\left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma}\right).} \tag{101}$$

Since a Clifford vector multiplied by a bivector produces vector and trivector grades, the BQ–Weyl multiplication table gives

$$\not{J}\not{B} = F_0\hat{\beta}_0 + \mathbf{F}\cdot\boldsymbol{\beta} + R_0\tilde{\hat{\beta}}_0 + \mathbf{R}\cdot\tilde{\boldsymbol{\beta}}, \tag{102}$$

where

$$F_0 = -\frac{1}{c}\,\mathbf{J}\cdot\mathbf{E}, \tag{103a}$$

$$\mathbf{F} = -\left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right), \tag{103b}$$

$$R_0 = -\mathbf{J} \cdot \mathbf{B}, \tag{103c}$$

$$\mathbf{R} = -c\rho\, \mathbf{B} + \frac{1}{c}\mathbf{J} \times \mathbf{E}. \tag{103d}$$

Thus

$$\boxed{\begin{aligned} \not{J}\not{\beta} = &- \frac{\mathbf{J} \cdot \mathbf{E}}{c}\,\hat{\beta}_0 - (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \cdot \boldsymbol{\beta} \\ &- (\mathbf{J} \cdot \mathbf{B})\,\tilde{\hat{\beta}}_0 + \left(-c\rho\,\mathbf{B} + \frac{1}{c}\mathbf{J} \times \mathbf{E}\right) \cdot \tilde{\boldsymbol{\beta}}. \end{aligned}} \tag{104}$$

The vector sector contains the standard electromagnetic power and Lorentz-force densities. Defining

$$\mathcal{P} \equiv \mathbf{J} \cdot \mathbf{E}, \qquad \mathbf{f}_{\mathrm{L}} \equiv \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{105}$$

the vector projection of the full product is

$$\boxed{\left[\not{J}\not{\beta}\right]_{\text{vector}} = -\frac{\mathcal{P}}{c}\,\hat{\beta}_0 - \mathbf{f}_{\mathrm{L}} \cdot \boldsymbol{\beta}.} \tag{106}$$

Hence the physical Lorentz four-force density may be written, with the present BQ–Weyl sign convention, as

$$\boxed{\not{f}_{\mathrm{L}} \equiv -\left[\not{J}\not{\beta}\right]_{\text{vector}} = \frac{\mathbf{J} \cdot \mathbf{E}}{c}\,\hat{\beta}_0 + (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \cdot \boldsymbol{\beta}.} \tag{107}$$

The full Clifford product contains in addition the pseudovector (trivector) sector

$$\boxed{\left[\not{J}\not{\beta}\right]_{\text{pseudo}} = -(\mathbf{J} \cdot \mathbf{B})\,\tilde{\hat{\beta}}_0 + \left(-c\rho\,\mathbf{B} + \frac{1}{c}\mathbf{J} \times \mathbf{E}\right) \cdot \tilde{\boldsymbol{\beta}}.} \tag{108}$$

The distinction is important. The conventional Lorentz force is the vector projection of the Clifford product, whereas $\not{J}\not{\beta}$ itself contains the additional trivector contribution required by the complete BQ–Weyl multiplication algebra. Thus

$$\boxed{\Lambda^1\Lambda^2 \longrightarrow \Lambda^1 \oplus \Lambda^3,} \tag{109}$$

with the grade-1 channel carrying electromagnetic power and force, and the grade-3 channel carrying the complementary pseudovector content of the same current–field product.

Reduction to the Lorentz four-force.

The full BQ–Weyl product $\not{J}\not{B}$ contains both vector and pseudovector sectors,

$$\not{J}\not{B} = \left[\not{J}\not{B}\right]_{\text{vector}} + \left[\not{J}\not{B}\right]_{\text{pseudo}}. \tag{110}$$

The conventional Lorentz four-force density is carried by the vector projection alone. With the present BQ–Weyl sign convention,

$$\boxed{\not{F}_{\mathrm{L}} = -\left[\not{J}\not{B}\right]_{\text{vector}} = \frac{\mathbf{J}\cdot\mathbf{E}}{c}\,\hat{\beta}_0 + (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\cdot\boldsymbol{\beta}.} \tag{111}$$

This identification does not require the pseudovector part of the Clifford product to vanish. In general,

$$\boxed{\not{J}\not{B} = -\not{F}_{\mathrm{L}} + \left[\not{J}\not{B}\right]_{\text{pseudo}},} \tag{112}$$

where

$$\left[\not{J}\not{B}\right]_{\text{pseudo}} = -(\mathbf{J}\cdot\mathbf{B})\,\tilde{\hat{\beta}}_0 + \left(-c\rho\,\mathbf{B} + \frac{1}{c}\mathbf{J}\times\mathbf{E}\right)\cdot\tilde{\boldsymbol{\beta}}. \tag{113}$$

The pseudovector sector is therefore complementary Clifford content of the complete current–field product and is not part of the conventional Lorentz four-force.

Only for the special class of electromagnetic configurations satisfying

$$\left[\not{J}\not{B}\right]_{\text{pseudo}} = 0, \tag{114}$$

or explicitly

$$\mathbf{J}\cdot\mathbf{B} = 0, \qquad \mathbf{J}\times\mathbf{E} = c^2\rho\,\mathbf{B}, \tag{115}$$

does the complete Clifford product itself reduce to

$$\boxed{\not{J}\not{B} = -\not{F}_{\mathrm{L}}.} \tag{116}$$

These are additional restrictions on the electromagnetic configuration, not general consequences of Maxwell's equations. Thus, in the general case, the Lorentz force is obtained by grade projection, $\not{F}_{\mathrm{L}} = -\langle\not{J}\not{B}\rangle_1$, while the BQ–Weyl multiplication table retains the independent grade-3 content of the full product.

4.4 The electromagnetic field product $\not{B}\not{B}$

The electromagnetic field is represented at the Weyl level by the pure bivector

$$\not{B} = \frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma}. \tag{117}$$

Its square is therefore

$$\boxed{\not{B}\not{B} = \left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma}\right)\left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma}\right).} \tag{118}$$

Applying the BQ–Weyl multiplication table of Appendix 3, the pure electric and magnetic products contribute to the scalar channel,

$$\left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha}\right)^2 = \frac{E^2}{c^2}\,\mathbb{1}, \qquad (\mathbf{B}\cdot\boldsymbol{\Sigma})^2 = -B^2\,\mathbb{1}. \tag{119}$$

The mixed products contribute only to the pseudoscalar channel. Thus

$$\not{B}^2 = \left(\frac{E^2}{c^2} - B^2\right)\mathbb{1} + \left[\left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha}\right)(\mathbf{B}\cdot\boldsymbol{\Sigma}) + (\mathbf{B}\cdot\boldsymbol{\Sigma})\left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha}\right)\right]. \tag{120}$$

With the pseudoscalar defined by

$$\hat{\chi} = -\hat{\beta}_0\hat{\beta}_I\hat{\beta}_J\hat{\beta}_K, \tag{121}$$

the multiplication table gives

$$\hat{\alpha}_i\hat{\Sigma}_i = -\hat{\chi}, \qquad \hat{\Sigma}_i\hat{\alpha}_i = -\hat{\chi}, \tag{122}$$

for each cyclic spatial index i. Consequently, the mixed contribution is

$$\boxed{\left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha}\right)(\mathbf{B}\cdot\boldsymbol{\Sigma}) + (\mathbf{B}\cdot\boldsymbol{\Sigma})\left(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha}\right) = -\frac{2}{c}(\mathbf{E}\cdot\mathbf{B})\,\hat{\chi}.} \tag{123}$$

Hence

$$\boxed{\not{B}\not{B} = \left(\frac{E^2}{c^2} - B^2\right)\mathbb{1} - \frac{2}{c}(\mathbf{E}\cdot\mathbf{B})\,\hat{\chi}.} \tag{124}$$

The square of the electromagnetic bivector therefore occupies only the scalar and pseudoscalar sectors of the sixteen-element Weyl algebra. All vector and pseudovector channels vanish identically, while the bivector contributions generated by unequal component products cancel between the two identical field factors. The complete square therefore closes on

$$\boxed{\{\mathbb{1}, \hat{\chi}\}.} \tag{125}$$

The two surviving coefficients are the two Lorentz invariants of the electromagnetic field. Defining

$$\mathcal{I}_1 = B^2 - \frac{E^2}{c^2}, \qquad \mathcal{I}_2 = \frac{\mathbf{E}\cdot\mathbf{B}}{c}, \tag{126}$$

Eq. (124) becomes

$$\boxed{\not{\beta}^2 = -\mathcal{I}_1\,\not{1} - 2\mathcal{I}_2\,\hat{\chi}.} \tag{127}$$

Thus the BQ–Weyl multiplication table packages the two independent electromagnetic Lorentz invariants into two distinct algebraic channels of a single product: the scalar channel contains the electric–magnetic norm difference, while the pseudoscalar channel contains the electric–magnetic inner product, with its sign fixed unambiguously by the definition $\hat{\chi} = -\hat{\beta}_0\hat{\beta}_I\hat{\beta}_J\hat{\beta}_K$.

4.5 The electromagnetic energy–momentum product $\not{\beta}\hat{\beta}_0\not{\beta}$

At the BQ–Pauli level the quadratic field product

$$B^T B = -\left(B^2 + \frac{E^2}{c^2}\right)\hat{1} - \frac{2}{c}(\mathbf{E}\times\mathbf{B})\cdot\hat{\boldsymbol{\sigma}} \tag{128}$$

contains the electromagnetic energy density and Poynting vector,

$$-\frac{1}{2\mu_0}B^T B = u_{\mathrm{EM}}\hat{1} + \frac{\mathbf{S}}{c}\cdot\hat{\boldsymbol{\sigma}}, \tag{129}$$

where

$$u_{\mathrm{EM}} = \frac{1}{2\mu_0}\left(B^2 + \frac{E^2}{c^2}\right), \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}. \tag{130}$$

The Pauli object $B^T B$ occupies the $\{\hat{1}, \hat{\boldsymbol{\sigma}}\}$ sector. Multiplication by the timelike basis element $\hat{T}$ maps this into the $\{\hat{T}, \hat{\mathbf{K}}\}$ spacetime sector. Using

$$\hat{T}\hat{\boldsymbol{\sigma}} = \hat{\mathbf{K}}, \tag{131}$$

one obtains

$$\boxed{\hat{T}B^T B = -\left(B^2 + \frac{E^2}{c^2}\right)\hat{T} - \frac{2}{c}(\mathbf{E}\times\mathbf{B})\cdot\hat{\mathbf{K}}.} \tag{132}$$

Consequently,

$$\boxed{-\frac{1}{2\mu_0}\hat{T}B^T B = u_{\mathrm{EM}}\hat{T} + \frac{\mathbf{S}}{c}\cdot\hat{\mathbf{K}}.} \tag{133}$$

Thus the same Pauli energy–Poynting content can be written directly in the $(\hat{T}, \hat{\mathbf{K}})$ four-vector sector. Since $\hat{T}$ commutes with all elements of the BQ–Pauli basis,

$$\hat{T}B^T B = B^T\hat{T}B. \tag{134}$$

Under the Pauli–Weyl doubling, the corresponding spacetime basis is

$$\hat{T} \longrightarrow \hat{\beta}_0, \qquad \hat{\mathbf{K}} \longrightarrow \boldsymbol{\beta}, \tag{135}$$

while the electromagnetic field itself becomes the Weyl bivector

$$\not{B} = \frac{\mathbf{E}}{c} \cdot \boldsymbol{\alpha} + \mathbf{B} \cdot \boldsymbol{\Sigma}. \tag{136}$$

The Weyl counterpart of Eq. (132) is generated directly by inserting the timelike Weyl vector $\hat{\beta}_0$ between the two field bivectors,

$$\boxed{\not{B}\hat{\beta}_0\not{B} = \left(\frac{\mathbf{E}}{c} \cdot \boldsymbol{\alpha} + \mathbf{B} \cdot \boldsymbol{\Sigma}\right)\hat{\beta}_0\left(\frac{\mathbf{E}}{c} \cdot \boldsymbol{\alpha} + \mathbf{B} \cdot \boldsymbol{\Sigma}\right).} \tag{137}$$

Direct application of the BQ–Weyl multiplication table to the complete sandwich product shows that the pure electric and magnetic terms combine in the timelike vector channel as

$$\left[\not{B}\hat{\beta}_0\not{B}\right]_{\beta_0} = -\left(B^2 + \frac{E^2}{c^2}\right)\hat{\beta}_0, \tag{138}$$

whereas the electric–magnetic cross terms combine in the spatial vector channel as

$$\left[\not{B}\hat{\beta}_0\not{B}\right]_{\boldsymbol{\beta}} = -\frac{2}{c}(\mathbf{E} \times \mathbf{B}) \cdot \boldsymbol{\beta}. \tag{139}$$

No other Weyl channels survive in the complete product. Hence

$$\boxed{\not{B}\hat{\beta}_0\not{B} = -\left(B^2 + \frac{E^2}{c^2}\right)\hat{\beta}_0 - \frac{2}{c}(\mathbf{E} \times \mathbf{B}) \cdot \boldsymbol{\beta}.} \tag{140}$$

Multiplication by $-1/(2\mu_0)$ therefore gives the Weyl energy–momentum four-vector associated with the timelike direction $\hat{\beta}_0$,

$$\boxed{-\frac{1}{2\mu_0}\not{B}\hat{\beta}_0\not{B} = u_{\mathrm{EM}}\hat{\beta}_0 + \frac{\mathbf{S}}{c} \cdot \boldsymbol{\beta}.} \tag{141}$$

The Pauli and Weyl constructions therefore correspond directly,

$$\boxed{\mathcal{L}\left(B^T \hat{T} B\right) = \not{B}\hat{\beta}_0\not{B},} \tag{142}$$

where $\mathcal{L}$ denotes the Pauli–Weyl lift. In channel form,

$$\{\hat{T}, \hat{\mathbf{K}}\} \longrightarrow \{\hat{\beta}_0, \boldsymbol{\beta}\}. \tag{143}$$

Thus the Pauli operation

$$\hat{T} : \{\hat{1}, \hat{\boldsymbol{\sigma}}\} \longrightarrow \{\hat{T}, \hat{\mathbf{K}}\} \tag{144}$$

is represented after Weyl doubling by the explicit timelike insertion $\hat{\beta}_0$. The resulting object lies entirely in the grade-1 spacetime-vector sector and gives the electromagnetic energy–momentum four-vector relative to the timelike direction $\hat{\beta}_0$.

Together with the ordinary field square, the Weyl algebra therefore distinguishes two quadratic electromagnetic constructions:

$$\boxed{\begin{array}{rl} BB \longrightarrow \not{B}\not{B}, & \text{electromagnetic field invariants,} \\ B^T\hat{T}B \longrightarrow \not{B}\hat{\beta}_0\not{B}, & \text{energy density and Poynting four-vector.} \end{array}} \tag{145}$$

The Pauli–Weyl doubling thus preserves both quadratic field constructions while placing the second explicitly in the grade-1 spacetime-vector sector of $\mathrm{Cl}(1,3)$.

4.6 The complete electromagnetic stress–energy map $\not{T}(\hat{\beta}_\mu)$

The preceding result for $\not{B}\hat{\beta}_0\not{B}$ is the timelike member of a more general BQ–Weyl construction. For any Weyl spacetime vector

$$a = a^\mu \hat{\beta}_\mu, \tag{146}$$

define the electromagnetic stress–energy map

$$\boxed{\not{T}(a) = -\frac{1}{2\mu_0}\not{B}\, a\, \not{B}.} \tag{147}$$

Since the map is linear in its vector argument,

$$\not{T}(a) = a^\mu \not{T}(\hat{\beta}_\mu), \tag{148}$$

so that its complete content is determined by the four basis insertions

$$\not{T}(\hat{\beta}_\mu) = -\frac{1}{2\mu_0}\not{B}\hat{\beta}_\mu\not{B}, \qquad \mu = 0, I, J, K. \tag{149}$$

The timelike insertion has already been evaluated in Sec. 4.5. It gives

$$\boxed{\not{T}(\hat{\beta}_0) = u_{\mathrm{EM}}\hat{\beta}_0 + \frac{\mathbf{S}}{c}\cdot\boldsymbol{\beta},} \tag{150}$$

where

$$u_{\mathrm{EM}} = \frac{1}{2\mu_0}\left(\frac{E^2}{c^2} + B^2\right), \qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}. \tag{151}$$

Thus the $\hat{\beta}_0$ insertion generates the energy density together with the three components of the Poynting vector.

The spatial insertions

$$\not{T}(\hat{\beta}_i) = -\frac{1}{2\mu_0}\not{\beta}\hat{\beta}_i\not{\beta}, \qquad i = I, J, K, \tag{152}$$

generate the remaining three columns of the electromagnetic stress–energy structure. Direct application of the BQ–Weyl multiplication table gives

$$\boxed{\not{T}(\hat{\beta}_i) = -\frac{S_i}{c}\,\hat{\beta}_0 + \sum_j \sigma_{ji}\hat{\beta}_j,} \tag{153}$$

where

$$\boxed{\sigma_{ij} = \frac{1}{\mu_0}\left[\frac{E_iE_j}{c^2} + B_iB_j - \frac{1}{2}\delta_{ij}\left(\frac{E^2}{c^2} + B^2\right)\right]} \tag{154}$$

is the Maxwell stress tensor in the present conventions.

Explicitly, the three spatial basis insertions are therefore

$$\not{T}(\hat{\beta}_I) = -\frac{S_I}{c}\hat{\beta}_0 + \sigma_{II}\hat{\beta}_I + \sigma_{JI}\hat{\beta}_J + \sigma_{KI}\hat{\beta}_K, \tag{155a}$$

$$\not{T}(\hat{\beta}_J) = -\frac{S_J}{c}\hat{\beta}_0 + \sigma_{IJ}\hat{\beta}_I + \sigma_{JJ}\hat{\beta}_J + \sigma_{KJ}\hat{\beta}_K, \tag{155b}$$

$$\not{T}(\hat{\beta}_K) = -\frac{S_K}{c}\hat{\beta}_0 + \sigma_{IK}\hat{\beta}_I + \sigma_{JK}\hat{\beta}_J + \sigma_{KK}\hat{\beta}_K. \tag{155c}$$

The four insertions can consequently be collected into the single array

$$\boxed{\begin{array}{c|cccc} & \hat{\beta}_0 & \hat{\beta}_I & \hat{\beta}_J & \hat{\beta}_K \\ \hline \not{T}(\hat{\beta}_\mu) & u_{\mathrm{EM}}\hat{\beta}_0 + \dfrac{\mathbf{S}}{c}\cdot\boldsymbol{\beta} & -\dfrac{S_I}{c}\hat{\beta}_0 + \sigma_{jI}\hat{\beta}_j & -\dfrac{S_J}{c}\hat{\beta}_0 + \sigma_{jJ}\hat{\beta}_j & -\dfrac{S_K}{c}\hat{\beta}_0 + \sigma_{jK}\hat{\beta}_j \end{array}} \tag{156}$$

with summation over the spatial index j in the last three columns.

Equivalently, the complete coefficient structure generated by the four BQ–Weyl sandwiches may be displayed as

$$\boxed{T_{\mathrm{EM}} = \begin{pmatrix} u_{\mathrm{EM}} & -\dfrac{S_I}{c} & -\dfrac{S_J}{c} & -\dfrac{S_K}{c} \\ \dfrac{S_I}{c} & \sigma_{II} & \sigma_{IJ} & \sigma_{IK} \\ \dfrac{S_J}{c} & \sigma_{JI} & \sigma_{JJ} & \sigma_{JK} \\ \dfrac{S_K}{c} & \sigma_{KI} & \sigma_{KJ} & \sigma_{KK} \end{pmatrix},} \tag{157}$$

where the relative signs of the temporal–spatial entries reflect the placement of the Lorentzian index in the present $\hat{\beta}_\mu$ convention.

The significance of the construction is that no separate tensor object has to be introduced at the level of the BQ–Weyl multiplication. The same electromagnetic bivector $\not{B}$ is used on both sides of each basis vector, and the four choices

$$a = \hat{\beta}_\mu \tag{158}$$

generate the four energy–momentum columns:

$$\boxed{\begin{array}{lll} -\dfrac{1}{2\mu_0}\not{B}\hat{\beta}_0\not{B} & \longrightarrow & \text{energy density and Poynting flux,} \\ -\dfrac{1}{2\mu_0}\not{B}\hat{\beta}_i\not{B} & \longrightarrow & \text{momentum density and Maxwell stress.} \end{array}} \tag{159}$$

Thus the single BQ–Weyl sandwich rule

$$\boxed{\not{T}(a) = -\frac{1}{2\mu_0}\not{B}\, a\, \not{B}} \tag{160}$$

extends the previously obtained energy–Poynting four-vector to the complete electromagnetic stress–energy map. The timelike insertion $a = \hat{\beta}_0$ selects its energy–transport content, while the three spatial insertions $a = \hat{\beta}_i$ expose the corresponding momentum–stress content.

4.7 Lorentz-transformation properties of the electromagnetic stress–energy map

The Lorentz-transformation properties of the BQ–Weyl electromagnetic products follow directly from the same Pauli–Weyl rotor construction used above for the transformation of a spacetime four-vector. For a boost of rapidity ψ in the I direction, introduce the BQ–Pauli boost rotor

$$U = \exp\left(\frac{\psi}{2}\hat{I}\right), \qquad U^{-1} = \exp\left(-\frac{\psi}{2}\hat{I}\right), \tag{161}$$

with

$$\cosh\psi = \gamma, \qquad \sinh\psi = \gamma\frac{v}{c}. \tag{162}$$

Under the Pauli–Weyl doubling this produces the Weyl transformation matrix

$$\boxed{\Lambda = \begin{pmatrix} U^{-1} & 0 \\ 0 & U \end{pmatrix} = \operatorname{diag}\left(e^{-\psi/2}, e^{+\psi/2}, e^{+\psi/2}, e^{-\psi/2}\right),} \tag{163}$$

with inverse

$$\Lambda^{-1} = \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}. \tag{164}$$

The explicit diagonal form in Eq. (163) fixes the boost convention used below.

Every BQ–Weyl object transforms by the corresponding similarity operation,

$$\boxed{\not{X}^{L} = \Lambda \not{X} \Lambda^{-1}.} \tag{165}$$

Applied to the Weyl vector basis, this gives

$$\hat{\beta}_0^L = \Lambda\hat{\beta}_0\Lambda^{-1} = \cosh\psi\,\hat{\beta}_0 - \sinh\psi\,\hat{\beta}_I, \tag{166}$$

$$\hat{\beta}_I^L = \Lambda\hat{\beta}_I\Lambda^{-1} = -\sinh\psi\,\hat{\beta}_0 + \cosh\psi\,\hat{\beta}_I, \tag{167}$$

$$\hat{\beta}_J^L = \hat{\beta}_J, \qquad \hat{\beta}_K^L = \hat{\beta}_K. \tag{168}$$

Thus

$$(\hat{\beta}_\mu)^L = \Lambda_\mu{}^\nu \hat{\beta}_\nu, \tag{169}$$

with the usual Lorentz boost matrix represented internally by the Weyl similarity transformation.

Lorentz transformation of the electromagnetic bivector.

The electromagnetic field

$$\not{B} = \frac{\mathbf{E}}{c}\cdot\boldsymbol{\alpha} + \mathbf{B}\cdot\boldsymbol{\Sigma} \tag{170}$$

transforms by the same similarity rule,

$$\boxed{\not{B}^{L} = \Lambda \not{B} \Lambda^{-1}.} \tag{171}$$

For the I-directed boost of Eq. (163), expansion of $\not{B}^L$ again on the $\{\boldsymbol{\alpha}, \boldsymbol{\Sigma}\}$ bivector basis gives

$$E_I^L = E_I, \tag{172a}$$

$$E_J^L = \gamma(E_J - vB_K), \tag{172b}$$

$$E_K^L = \gamma(E_K + vB_J), \tag{172c}$$

and

$$B_I^L = B_I, \tag{173a}$$

$$B_J^L = \gamma\left(B_J + \frac{v}{c^2}E_K\right), \tag{173b}$$

$$B_K^L = \gamma\left(B_K - \frac{v}{c^2}E_J\right). \tag{173c}$$

Thus the BQ–Weyl similarity transformation reproduces the standard Lorentz mixing of the electric and magnetic components.

Lorentz invariance of the field square.

The field square transforms as

$$\begin{aligned}(\not{B}^L)^2 &= (\Lambda\not{B}\Lambda^{-1})(\Lambda\not{B}\Lambda^{-1}) \\ &= \Lambda\not{B}\underbrace{\Lambda^{-1}\Lambda}_{\not{1}}\not{B}\Lambda^{-1} \\ &= \Lambda\not{B}^2\Lambda^{-1}.\end{aligned} \tag{174}$$

From Sec. 4.4,

$$\not{B}^2 = -\mathcal{I}_1\not{1} - 2\mathcal{I}_2\hat{\chi}, \tag{175}$$

where

$$\mathcal{I}_1 = B^2 - \frac{E^2}{c^2}, \qquad \mathcal{I}_2 = \frac{\mathbf{E}\cdot\mathbf{B}}{c}. \tag{176}$$

The scalar $\not{1}$ commutes trivially with Λ. The pseudoscalar $\hat{\chi}$ also commutes with the even Lorentz rotor Λ,

$$[\Lambda, \hat{\chi}] = 0. \tag{177}$$

Consequently,

$$\boxed{(\not{B}^L)^2 = \not{B}^2,} \tag{178}$$

and therefore

$$\boxed{\mathcal{I}_1^L = \mathcal{I}_1, \qquad \mathcal{I}_2^L = \mathcal{I}_2.} \tag{179}$$

The scalar and pseudoscalar channels of $\not{B}^2$ therefore carry the two Lorentz invariants of the electromagnetic field.

Lorentz covariance of the stress–energy map.

Consider now the BQ–Weyl stress–energy map introduced in Sec. 4.6,

$$\boxed{\not{T}(a) = -\frac{1}{2\mu_0}\not{B}\, a\, \not{B},} \tag{180}$$

where a is an arbitrary Weyl four-vector. Under the same Lorentz transformation,

$$a^L = \Lambda a\Lambda^{-1}, \qquad \not{B}^L = \Lambda\not{B}\Lambda^{-1}. \tag{181}$$

The transformed stress–energy map is therefore

$$\begin{aligned}\not{T}^L(a^L) &= -\frac{1}{2\mu_0}\not{B}^L a^L \not{B}^L \\ &= -\frac{1}{2\mu_0}(\Lambda\not{B}\Lambda^{-1})(\Lambda a\Lambda^{-1})(\Lambda\not{B}\Lambda^{-1})\end{aligned}$$

$$= -\frac{1}{2\mu_0}\Lambda \not{B}\, \underbrace{\Lambda^{-1}\Lambda}_{\mathbb{1}}\, a\, \underbrace{\Lambda^{-1}\Lambda}_{\mathbb{1}}\, \not{B}\Lambda^{-1}$$

$$= \Lambda\left(-\frac{1}{2\mu_0}\not{B}a\not{B}\right)\Lambda^{-1}. \tag{182}$$

Hence

$$\boxed{\not{T}^{L}(a^{L}) = \Lambda\not{T}(a)\Lambda^{-1}.} \tag{183}$$

Equation (183) is the master Lorentz-covariance relation for the electromagnetic stress–energy construction. It shows that $\not{T}$ is a vector-valued linear map: when its vector argument a is Lorentz transformed, the output transforms as a Weyl four-vector under the same similarity operation.

The four basis insertions and the rank-two structure.

Taking

$$a = \hat{\beta}_\mu \tag{184}$$

gives the four basis-resolved stress–energy vectors

$$\not{T}(\hat{\beta}_\mu) = -\frac{1}{2\mu_0}\not{B}\hat{\beta}_\mu\not{B}. \tag{185}$$

Expand each output on the Weyl vector basis,

$$\boxed{\not{T}(\hat{\beta}_\mu) = T^{\nu}{}_{\mu}\hat{\beta}_\nu.} \tag{186}$$

The index μ labels the vector inserted between the two field bivectors, whereas ν labels the vector channel of the resulting BQ–Weyl product.

Because the input basis transforms as a four-vector and the output is itself a four-vector, Eq. (183) induces the mixed rank-two transformation law

$$\boxed{T'^{\rho}{}_{\mu} = \Lambda^{\rho}{}_{\nu}T^{\nu}{}_{\sigma}(\Lambda^{-1})^{\sigma}{}_{\mu}.} \tag{187}$$

Thus the rank-two character of electromagnetic stress–energy need not be imposed separately on the BQ–Weyl product. It emerges from the combination of two facts: $\not{T}$ accepts a Lorentz four-vector as its argument and returns a Lorentz four-vector as its output.

The timelike insertion and the observer-resolved energy–Poynting vector.

The special choice

$$a = \hat{\beta}_0 \tag{188}$$

selects the timelike member of the stress–energy map,

$$\slashed{T}(\hat{\beta}_0) = -\frac{1}{2\mu_0}\slashed{B}\hat{\beta}_0\slashed{B}. \tag{189}$$

As derived above,

$$\boxed{\slashed{T}(\hat{\beta}_0) = u_{\mathrm{EM}}\hat{\beta}_0 + \frac{\mathbf{S}}{c}\cdot\boldsymbol{\beta}.} \tag{190}$$

The insertion $\hat{\beta}_0$ therefore selects the electromagnetic energy density and energy flux relative to the corresponding timelike direction.

This object is covariant, but it is not a Lorentz scalar. If both the field and the timelike vector are transformed,

$$\hat{\beta}_0^L = \Lambda\hat{\beta}_0\Lambda^{-1}, \tag{191}$$

then

$$\begin{aligned}\slashed{T}^L(\hat{\beta}_0^L) &= -\frac{1}{2\mu_0}\slashed{B}^L\hat{\beta}_0^L\slashed{B}^L \\ &= \Lambda\slashed{T}(\hat{\beta}_0)\Lambda^{-1}.\end{aligned} \tag{192}$$

Hence

$$\boxed{\slashed{T}^L(\hat{\beta}_0^L) = \Lambda\left(u_{\mathrm{EM}}\hat{\beta}_0 + \frac{\mathbf{S}}{c}\cdot\boldsymbol{\beta}\right)\Lambda^{-1}.} \tag{193}$$

The energy density and Poynting vector therefore change between Lorentz-related observers, while the complete energy–momentum four-vector transforms covariantly.

By contrast, transforming $\slashed{B}$ while holding $\hat{\beta}_0$ fixed does not represent the simultaneous Lorentz transformation of the complete map argument. In general,

$$\slashed{B}^L\hat{\beta}_0\slashed{B}^L \neq \Lambda\left(\slashed{B}\hat{\beta}_0\slashed{B}\right)\Lambda^{-1}, \tag{194}$$

because the fixed $\hat{\beta}_0$ cannot be absorbed through the intermediate factors of $\Lambda^{-1}\Lambda$. This distinction makes the observer dependence of the energy–Poynting decomposition explicit within the BQ–Weyl product itself.

Invariant, covariant, and observer-resolved structures.

The electromagnetic products considered above therefore exhibit three distinct Lorentz-transformation properties:

$$\boxed{\begin{aligned} \not{B}^2 \quad &\longrightarrow \quad \text{Lorentz invariant,} \\ \not{T}(a) = -\frac{1}{2\mu_0}\not{B}a\not{B} \quad &\longrightarrow \quad \text{Lorentz-covariant vector-valued map,} \\ \not{T}(\hat{\beta}_0) = -\frac{1}{2\mu_0}\not{B}\hat{\beta}_0\not{B} \quad &\longrightarrow \quad \text{observer-resolved energy–Poynting four-vector.} \end{aligned}} \tag{195}$$

The first contains the two observer-independent electromagnetic field invariants. The second is the complete electromagnetic stress–energy map, whose coefficients form a mixed rank-two Lorentz tensor. The third is the timelike projection of that map and therefore resolves the energy density and Poynting flux associated with a specified timelike direction.

The BQ–Weyl construction consequently separates invariance from covariance without introducing an additional transformation rule for the stress–energy tensor. Both follow from the same similarity transformation

$$\not{X}^L = \Lambda \not{X} \Lambda^{-1} \tag{196}$$

already governing the Weyl four-vector and bivector sectors: the field square closes on Lorentz-invariant scalar and pseudoscalar channels, whereas insertion of a transforming vector between the two field bivectors generates a Lorentz-covariant vector-valued map and, for $a = \hat{\beta}_\mu$, the complete electromagnetic stress–energy structure.

This also clarifies the relation to the corresponding BQ–Pauli construction. At the Pauli level the energy density and Poynting vector are generated by the mixed product $B^T B$. Although this product contains the correct physical energy–Poynting content, its transformation is not itself a single similarity transformation, because the time adjoint reverses the rotor action on the first factor. After Pauli–Weyl doubling, the same physical content is instead contained in

$$\boxed{\not{T}(a) = -\frac{1}{2\mu_0}\not{B}\, a\, \not{B},} \tag{197}$$

where the observer or spacetime direction is represented explicitly by the vector argument a. Its transformation,

$$a^L = \Lambda a \Lambda^{-1}, \qquad \not{T}^L(a^L) = \Lambda \not{T}(a) \Lambda^{-1}, \tag{198}$$

is consequently an ordinary Weyl similarity transformation. The observer dependence of electromagnetic energy and momentum has therefore not disappeared in the Weyl lift; rather, it has been relocated from the transformation properties of the Pauli mixed product to an explicit Lorentz-covariant vector argument. In particular, $a = \hat{\beta}_0$ selects the energy–Poynting four-vector associated with that timelike direction, whereas the complete set $a = \hat{\beta}_\mu$ generates the rank-two stress–energy map. The Pauli–Weyl doubling thus converts the Pauli energy–Poynting construction into the covariant map

$$\boxed{\not{T} : V \longrightarrow V, \qquad a \longmapsto -\frac{1}{2\mu_0} \not{B}\, a\, \not{B},} \tag{199}$$

making the distinction between Lorentz-invariant field content, Lorentz-covariant stress–energy, and observer-resolved energy–momentum explicit within a single multiplication framework.

4.8 From the basis sandwich to the physical four-velocity stress–energy map

The products

$$\not{B}\hat{\beta}_\mu\not{B} \tag{200}$$

derived above provide a convenient basis resolution of the electromagnetic stress–energy structure. The individual $\hat{\beta}_\mu$, however, are basis elements of the BQ–Weyl four-vector sector rather than additional physical ingredients of the electromagnetic system. The construction can therefore be formulated more intrinsically by replacing the basis insertion by a physical BQ–Weyl four-vector.

Let the observer or material four-velocity be

$$\boxed{\not{U} = U^\mu \hat{\beta}_\mu, \qquad \not{U}^2 = -c^2 \not{1}.} \tag{201}$$

The corresponding dimensionless timelike direction is simply $\not{U}/c$. The electromagnetic energy–momentum vector associated with this timelike direction is then defined by

$$\boxed{\not{P}_{\mathrm{EM}}(\not{U}) = -\frac{1}{2\mu_0} \not{B} \left(\frac{\not{U}}{c}\right) \not{B} = -\frac{1}{2\mu_0 c} \not{B}\, \not{U}\, \not{B}.} \tag{202}$$

The product is entirely internal to the BQ–Weyl algebra,

$$\Lambda^2\, \Lambda^1\, \Lambda^2 \longrightarrow \Lambda^1, \tag{203}$$

and maps the electromagnetic bivector and a physical timelike four-vector onto an electromagnetic energy–momentum four-vector.

For the fiducial observer,

$$\not{U}_0 = c\hat{\beta}_0, \tag{204}$$

Eq. (202) immediately reduces to

$$
\begin{aligned}
\not{P}_{\mathrm{EM}}(\not{U}_0) &= -\frac{1}{2\mu_0 c}\not{B}(c\hat{\beta}_0)\not{B} \\
&= -\frac{1}{2\mu_0}\not{B}\hat{\beta}_0\not{B} \\
&= u_{\mathrm{EM}}\hat{\beta}_0 + \frac{\mathbf{S}}{c}\cdot\boldsymbol{\beta}.
\end{aligned} \tag{205}
$$

Thus the previously derived $\not{B}\hat{\beta}_0\not{B}$ product is recovered as the special case in which the physical observer four-velocity is aligned with the fiducial timelike basis direction.

More generally, inserting

$$
\not{U} = U^\mu\hat{\beta}_\mu \tag{206}
$$

into Eq. (202) and using linearity gives

$$
\begin{aligned}
\not{P}_{\mathrm{EM}}(\not{U}) &= -\frac{1}{2\mu_0 c}\not{B}\left(U^\mu\hat{\beta}_\mu\right)\not{B} \\
&= \frac{U^\mu}{c}\left[-\frac{1}{2\mu_0}\not{B}\hat{\beta}_\mu\not{B}\right].
\end{aligned} \tag{207}
$$

Introducing the general stress–energy map

$$
\boxed{\not{T}(a) = -\frac{1}{2\mu_0}\not{B}\,a\,\not{B}, \qquad a\in\Lambda^1,} \tag{208}
$$

the observer-resolved energy–momentum vector may therefore be written compactly as

$$
\boxed{\not{P}_{\mathrm{EM}}(\not{U}) = \not{T}\left(\frac{\not{U}}{c}\right) = \frac{U^\mu}{c}\not{T}(\hat{\beta}_\mu).} \tag{209}
$$

The basis sandwiches consequently retain an important but secondary role. They resolve the general map on the Weyl spacetime basis,

$$
\not{T}(\hat{\beta}_\mu) = -\frac{1}{2\mu_0}\not{B}\hat{\beta}_\mu\not{B} = T^\nu{}_\mu\hat{\beta}_\nu, \tag{210}
$$

and hence expose the components $T^\nu{}_\mu$ of the complete electromagnetic stress–energy map. They are therefore not separate physical constructions, but the four basis evaluations of the single vector-valued map

$$
\boxed{\not{T}:\Lambda^1\longrightarrow\Lambda^1, \qquad a\longmapsto -\frac{1}{2\mu_0}\not{B}\,a\,\not{B}.} \tag{211}
$$

The physical four-velocity then selects from this map the energy–momentum vector associated with a particular observer,

$$\boxed{\not{U} \quad \longrightarrow \quad \not{P}_{\mathrm{EM}}(\not{U}) = \not{T}\left(\frac{\not{U}}{c}\right).} \tag{212}$$

This formulation also makes Lorentz covariance immediate. Under the BQ–Weyl similarity transformation,

$$\not{B}^L = \Lambda \not{B} \Lambda^{-1}, \qquad \not{U}^L = \Lambda \not{U} \Lambda^{-1}, \tag{213}$$

one obtains

$$\begin{aligned} \not{P}^L_{\mathrm{EM}}(\not{U}^L) &= -\frac{1}{2\mu_0 c} \not{B}^L \not{U}^L \not{B}^L \\ &= -\frac{1}{2\mu_0 c} (\Lambda \not{B} \Lambda^{-1})(\Lambda \not{U} \Lambda^{-1})(\Lambda \not{B} \Lambda^{-1}) \\ &= \Lambda \left[-\frac{1}{2\mu_0 c} \not{B} \not{U} \not{B} \right] \Lambda^{-1}. \end{aligned} \tag{214}$$

Hence

$$\boxed{\not{P}^L_{\mathrm{EM}}(\not{U}^L) = \Lambda \not{P}_{\mathrm{EM}}(\not{U}) \Lambda^{-1}.} \tag{215}$$

The observer-resolved electromagnetic energy–momentum therefore transforms as a BQ–Weyl four-vector when both the field and the observer four-velocity are transformed by the same Lorentz rotor.

The passage from $\not{B}\hat{\beta}_\mu\not{B}$ to $\not{B}\not{U}\not{B}$ consequently changes the interpretation rather than the underlying algebra. The former is the basis resolution required to display the components of the rank-two stress–energy map, whereas the latter is its direct evaluation on a physical four-velocity. In this ordering the $\hat{\beta}_\mu$ enter only as the basis used to expand the physical BQ–Weyl vector,

$$\not{U} = U^\mu \hat{\beta}_\mu, \tag{216}$$

and not as independent physical ingredients of the electromagnetic product. The complete relation is therefore

$$\boxed{\begin{array}{rll} \not{T}(a) = -\dfrac{1}{2\mu_0} \not{B}\, a\, \not{B}, & & a \in \Lambda^1, \\ \not{P}_{\mathrm{EM}}(\not{U}) = \not{T}\left(\dfrac{\not{U}}{c}\right) = -\dfrac{1}{2\mu_0 c} \not{B}\, \not{U}\, \not{B}, & & \text{physical observer,} \\ \not{T}(\hat{\beta}_\mu) = T^\nu{}_\mu \hat{\beta}_\nu, & & \text{basis resolution.} \end{array}} \tag{217}$$

Thus the BQ–Weyl stress–energy construction may be regarded fundamentally as a field–vector–field map. The physical product $\not{B}\not{U}\not{B}$ supplies its observer-resolved energy–momentum content, while the four products $\not{B}\hat{\beta}_\mu\not{B}$ merely resolve that same map into its spacetime components.

4.9 The current-resolved stress–energy product $\not{B}\not{J}\not{B}$

The general stress–energy map introduced above,

$$\not{T}(a) = -\frac{1}{2\mu_0}\not{B}\, a\, \not{B}, \qquad a \in \Lambda^1, \tag{218}$$

may be evaluated not only on a basis vector $\hat{\beta}_\mu$ or on a physical four-velocity $\not{U}$, but also directly on the electromagnetic four-current $\not{J}$. This does not introduce a new independent stress–energy construction; rather, it is a particular physical evaluation of the already established linear map $\not{T}(a)$. It gives the current-resolved product

$$\boxed{\not{T}(\not{J}) = -\frac{1}{2\mu_0}\not{B}\,\not{J}\,\not{B}.} \tag{219}$$

For a purely convective charge current,

$$\boxed{\not{J} = \rho_0\not{U},} \tag{220}$$

where ρ_0 is the proper charge density and therefore a Lorentz scalar. Since ρ_0 commutes with all elements of the BQ–Weyl algebra,

$$\begin{aligned}\not{B}\not{J}\not{B} &= \not{B}(\rho_0\not{U})\not{B}\\ &= \rho_0\not{B}\not{U}\not{B}.\end{aligned} \tag{221}$$

Consequently,

$$\boxed{\not{T}(\not{J}) = \rho_0\not{T}(\not{U}).} \tag{222}$$

This relation is therefore an immediate consequence of the linearity of the general map $\not{T}(a)$.

The reduction in Eq. (222) applies specifically to a convective current $\not{J} = \rho_0\not{U}$, with a single proper charge density carried by the flow. It is not a general decomposition of currents containing independent conduction, polarization, or magnetization contributions.

If the four-velocity is expanded on the Weyl vector basis,

$$\not{U} = U^\mu\hat{\beta}_\mu, \tag{223}$$

linearity further gives

$$\boxed{\not{T}(\not{J}) = \rho_0 U^\mu\not{T}(\hat{\beta}_\mu) = \rho_0 U^\mu T^\nu{}_\mu\hat{\beta}_\nu.} \tag{224}$$

Thus, for a convective current, $\not{B}\not{J}\not{B}$ is the electromagnetic stress–energy map evaluated on the charge-current vector. In tensor language its coefficient structure corresponds to the contraction

$$T^{\nu}{}_{\mu}J^{\mu} = \rho_0 T^{\nu}{}_{\mu}U^{\mu}. \tag{225}$$

The physical interpretation of this product must be distinguished from that of the Lorentz-force product. The product

$$\not{J}\not{B} \tag{226}$$

is linear in the electromagnetic field and contains the Lorentz-force interaction of the field with the charge current, together with the additional BQ–Weyl channel identified above. By contrast,

$$\not{B}\not{J}\not{B} \tag{227}$$

is quadratic in the field and does not represent the Lorentz four-force. It evaluates the electromagnetic stress–energy map along the vector direction selected by the current. The two products therefore encode distinct aspects of the charge–field relation,

$$\boxed{\begin{array}{lll} \not{J}\not{B} & \longrightarrow & \text{field action on the charge current,} \\ -\dfrac{1}{2\mu_0}\not{B}\not{J}\not{B} & \longrightarrow & \not{T}(\not{J}), \quad \text{field stress–energy resolved on the current.} \end{array}} \tag{228}$$

For a convective current, this result can be related directly to the flow-resolved electromagnetic energy–momentum vector introduced above,

$$\not{P}_{\mathrm{EM}}(\not{U}) = -\frac{1}{2\mu_0 c}\not{B}\not{U}\not{B} = \not{T}\left(\frac{\not{U}}{c}\right). \tag{229}$$

Using $\not{J} = \rho_0\not{U}$ then gives

$$\boxed{-\frac{1}{2\mu_0}\not{B}\not{J}\not{B} = c\rho_0\,\not{P}_{\mathrm{EM}}(\not{U}), \qquad \not{J} = \rho_0\not{U}.} \tag{230}$$

Thus, under the convective-current assumption, the current-resolved product contains no independent electromagnetic observable beyond the flow-resolved energy–momentum vector: it is that vector weighted by the proper charge density, with the factor c arising from the normalization $\not{P}_{\mathrm{EM}}(\not{U}) = \not{T}(\not{U}/c)$.

The relation becomes particularly transparent in the local rest frame of the charge flow,

$$\not{U}_0 = c\hat{\beta}_0, \qquad \not{J}_0 = \rho_0 c\hat{\beta}_0. \tag{231}$$

Then

$$\not{P}_{\text{EM}}(\not{U}_0) = u_{\text{EM}}\hat{\beta}_0 + \frac{\mathbf{S}}{c} \cdot \boldsymbol{\beta}, \tag{232}$$

and therefore

$$\boxed{-\frac{1}{2\mu_0 c}\not{B}\not{J}_0\not{B} = \rho_0 u_{\text{EM}}\hat{\beta}_0 + \rho_0\frac{\mathbf{S}}{c} \cdot \boldsymbol{\beta}, \qquad \not{J}_0 = \rho_0 c\hat{\beta}_0.} \tag{233}$$

The temporal and spatial channels are consequently the electromagnetic energy density and Poynting flux resolved on the local charge flow and weighted by its proper charge density. This is a current-weighted resolution of the already established field energy–momentum content, rather than a separate interaction energy or force.

The current-resolved product also inherits directly the Lorentz covariance of the general stress–energy map. With

$$\not{B}^L = \Lambda\not{B}\Lambda^{-1}, \qquad \not{J}^L = \Lambda\not{J}\Lambda^{-1}, \tag{234}$$

one obtains

$$\begin{aligned} \not{T}^L(\not{J}^L) &= -\frac{1}{2\mu_0}\not{B}^L\not{J}^L\not{B}^L \\ &= -\frac{1}{2\mu_0}(\Lambda\not{B}\Lambda^{-1})(\Lambda\not{J}\Lambda^{-1})(\Lambda\not{B}\Lambda^{-1}) \\ &= \Lambda\not{T}(\not{J})\Lambda^{-1}. \end{aligned} \tag{235}$$

Hence

$$\boxed{\not{T}^L(\not{J}^L) = \Lambda\not{T}(\not{J})\Lambda^{-1}.} \tag{236}$$

This covariance is likewise not an additional transformation law: it is the specialization to $a = \not{J}$ of the general covariance already established for $\not{T}(a)$.

The product $\not{B}\not{J}\not{B}$ therefore serves primarily to clarify the internal organization of the BQ–Weyl stress–energy map. The sequence

$$\boxed{\begin{array}{rcl} a = \hat{\beta}_\mu & \longrightarrow & \text{basis resolution of } T^\nu{}_\mu, \\ a = \not{U}/c & \longrightarrow & \text{flow-resolved electromagnetic energy–momentum}, \\ a = \not{J} & \longrightarrow & \text{current-resolved electromagnetic stress–energy} \end{array}} \tag{237}$$

places the basis, flow, and current evaluations within one and the same field–vector–field multiplication rule. For a convective current the last two are related simply by $\not{J} = \rho_0\not{U}$, so that the current-resolved product is a charge-density-weighted form of the flow-resolved result.

In this hierarchy $\not{J}\not{B}$ belongs to the force branch, whereas $\not{B}\not{J}\not{B}$ belongs to the stress–energy branch. In standard electrodynamics these structures are related through local energy–momentum balance. That differential relation has not been assumed in the present construction; deriving it directly from the BQ–Weyl multiplication and differentiation scheme constitutes a separate consistency test of the relation between the two branches.

4.10 The Hodge dual and electromagnetic duality rotation

The BQ–Weyl formulation does not require the Hodge dual in order to combine the homogeneous and sourced Maxwell equations. Both are already contained in the channel structure of the single BQ–Weyl product

$$\boxed{\not{\partial}\,\not{B} = -\mu_0 \not{J}.} \tag{238}$$

As shown above, direct application of the Weyl multiplication table to this product generates the complete Maxwell system in its nonvanishing Weyl channels. This single-product structure is already present at the BQ–Pauli level; the Pauli–Weyl doubling resolves it within the 1 + 4 + 6 + 4 + 1 Weyl basis.

The Hodge dual therefore enters here for a different reason. In the standard covariant tensor formulation of electrodynamics, the sourced Maxwell equations may be written as $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, while the homogeneous equations $\partial_{[\lambda} F_{\mu\nu]} = 0$ may equivalently be expressed, after introducing the Hodge-dual field tensor, as $\partial_\mu {}^\star F^{\mu\nu} = 0$. The Hodge dual is thus not required for the completeness of Maxwell's equations themselves, but provides the familiar parallel divergence form for their homogeneous sector.

In the BQ–Weyl formulation no such separate dual-field equation is required: the sourced and homogeneous Maxwell equations are already contained in the channel structure of the single product $\not{\partial}\,\not{B} = -\mu_0 \not{J}$. The Hodge dual therefore enters here instead as an additional algebraic operation already contained in the BQ–Weyl multiplication table. The grade-4 element

$$\hat{\chi} = -\hat{\beta}_0 \hat{\beta}_I \hat{\beta}_J \hat{\beta}_K, \qquad \hat{\chi}^2 = -\mathbb{1}, \tag{239}$$

acts directly on the six bivector basis elements according to

$$\boxed{\hat{\chi}\hat{\alpha}_i = -\hat{\Sigma}_i, \qquad \hat{\chi}\hat{\Sigma}_i = +\hat{\alpha}_i.} \tag{240}$$

These signs follow directly from the definition of $\hat{\chi}$ and the BQ–Weyl multiplication table.

For the electromagnetic bivector

$$\not{B} = \frac{\mathbf{E}}{c} \cdot \boldsymbol{\alpha} + \mathbf{B} \cdot \boldsymbol{\Sigma}, \tag{241}$$

this gives

$$\hat{\chi}\not{B} = \frac{\mathbf{E}}{c} \cdot (\hat{\chi}\boldsymbol{\alpha}) + \mathbf{B} \cdot (\hat{\chi}\boldsymbol{\Sigma}) \\ = -\frac{\mathbf{E}}{c} \cdot \boldsymbol{\Sigma} + \mathbf{B} \cdot \boldsymbol{\alpha}. \tag{242}$$

Hence

$$\boxed{\hat{\chi}\not{B} = \mathbf{B} \cdot \boldsymbol{\alpha} - \frac{\mathbf{E}}{c} \cdot \boldsymbol{\Sigma}.} \tag{243}$$

This is the Hodge-dual electromagnetic bivector corresponding to the orientation fixed by the present BQ–Weyl basis. In terms of its electric and magnetic parts,

$$\boxed{\frac{\mathbf{E}}{c} \longrightarrow \mathbf{B}, \qquad \mathbf{B} \longrightarrow -\frac{\mathbf{E}}{c}.} \tag{244}$$

Since $\hat{\chi}^2 = -\not{1}$, the same pseudoscalar generates the continuous electric–magnetic duality rotation

$$\boxed{e^{\theta\hat{\chi}} = \cos\theta\,\not{1} + \sin\theta\,\hat{\chi}.} \tag{245}$$

Acting on the field bivector,

$$\not{B}_\theta = e^{\theta\hat{\chi}}\not{B} = \cos\theta\,\not{B} + \sin\theta\,\hat{\chi}\not{B}, \tag{246}$$

and therefore

$$\boxed{\begin{aligned} \not{B}_\theta = &\left(\frac{\mathbf{E}}{c}\cos\theta + \mathbf{B}\sin\theta\right) \cdot \boldsymbol{\alpha} \\ &+ \left(\mathbf{B}\cos\theta - \frac{\mathbf{E}}{c}\sin\theta\right) \cdot \boldsymbol{\Sigma}. \end{aligned}} \tag{247}$$

Thus

$$\boxed{\begin{aligned} \frac{\mathbf{E}_\theta}{c} &= \frac{\mathbf{E}}{c}\cos\theta + \mathbf{B}\sin\theta, \\ \mathbf{B}_\theta &= \mathbf{B}\cos\theta - \frac{\mathbf{E}}{c}\sin\theta. \end{aligned}} \tag{248}$$

At $\theta = \pi/2$ the continuous rotation reduces directly to Eq. (244),

$$\frac{\mathbf{E}}{c} \longrightarrow \mathbf{B}, \qquad \mathbf{B} \longrightarrow -\frac{\mathbf{E}}{c}. \tag{249}$$

The role of $\hat{\chi}$ is therefore distinct from that of $\not{\partial}$: the BQ–Weyl product $\not{\partial}\not{B}$ already contains the complete Maxwell system, whereas $\hat{\chi}$ exposes the internal duality structure of the electromagnetic bivector itself. Together with the quadratic field product, the Weyl multiplication table thus makes three complementary structures explicit,

$$\boxed{\begin{array}{lcl} \not\partial \not{B} & \longrightarrow & \text{Maxwell field equations,} \\ \not{B}\,\not{B} & \longrightarrow & \text{electromagnetic field invariants,} \\ \hat{\chi}\,\not{B} & \longrightarrow & \text{Hodge dual and electric–magnetic duality.} \end{array}} \tag{250}$$

The last structure is the BQ–Weyl counterpart of the standard pseudoscalar formulation of Hodge duality in Spacetime Algebra and of its γ^5 representation in the Dirac algebra. Schematically,

$$\boxed{\begin{array}{ccc} \text{BQ–Weyl} & \text{STA} & \text{Dirac} \\ \hat{\chi} & I & \text{pseudoscalar} \sim i\gamma^5 \\ \hat{\chi}\,\not{B} & IF & \gamma^5 \not{F} \\ e^{\theta\hat{\chi}}\not{B} & e^{\theta I}F & \text{Dirac representation of duality rotation} \end{array}} \tag{251}$$

where the relative signs and factors of i in the STA and Dirac columns depend on their respective pseudoscalar and γ^5 conventions. The BQ–Weyl signs used above, by contrast, are fixed throughout by $\hat{\chi} = -\hat{\beta}_0\hat{\beta}_I\hat{\beta}_J\hat{\beta}_K$ and the multiplication table of Appendix 3.

5 Lift of the relativistic fluid field from BQ–Pauli to BQ–Weyl algebra

The relativistic-fluid construction of de Haas (2026a) is formulated at the BQ–Pauli level in terms of the four-momentum density

$$G = \frac{\varepsilon}{c}\hat{T} + \mathbf{g}\cdot\hat{\mathbf{K}}, \tag{252}$$

and its momentum-vorticity field

$$H = \partial^T G. \tag{253}$$

The purpose of the present section is to place the same construction in the sixteen-element BQ–Weyl algebra. No new fluid-dynamical assumption is introduced; only the algebraic resolution of the same coefficients is changed.

The Pauli–Weyl passage used here is the parity-duplex construction described in Appendix 3. It should not be identified with a single multiplicative embedding of the complete BQ–Pauli algebra. At the Pauli level the same three-element $\hat{\mathbf{K}}$

set occurs both as the spatial carrier of a four-vector and as one of the channels produced by a bilinear product. The Weyl doubling separates these two roles.

The minquat four-vector carrier

$$(\hat{T}, \hat{\mathbf{K}}) \tag{254}$$

is lifted to the Weyl vector tetrad

$$(\hat{\beta}_0, \boldsymbol{\beta}), \tag{255}$$

whereas the two three-dimensional channels produced by a Pauli four-vector bilinear are represented by the two Weyl bivector sectors,

$$\hat{\boldsymbol{\sigma}} \longrightarrow \boldsymbol{\alpha}, \qquad \hat{\mathbf{K}}\big|_{\text{bilinear}} \longrightarrow \boldsymbol{\Sigma}. \tag{256}$$

Explicitly,

$$\hat{\alpha}_i = \hat{\beta}_0 \hat{\beta}_i, \qquad \hat{\Sigma}_i = \hat{\beta}_j \hat{\beta}_k, \qquad (i, j, k) \text{ cyclic.} \tag{257}$$

Thus the Pauli $1 + 3 + 3 + 1$ carrier is not copied unchanged into a larger matrix space; the Weyl doubling resolves its vector and bivector roles within the full

$$1 + 4 + 6 + 4 + 1 \tag{258}$$

Clifford hierarchy.

5.1 The momentum four-vector and the Weyl fluid field $\not{H} = \not{\partial} \not{G}$

At the BQ–Pauli level the relativistic four-momentum density is

$$G = \frac{\varepsilon}{c}\hat{T} + \mathbf{g} \cdot \hat{\mathbf{K}}, \tag{259}$$

where ε is the energy density and $\mathbf{g}$ the momentum density. Its BQ–Weyl four-vector counterpart is

$$\boxed{\not{G} = \frac{\varepsilon}{c}\hat{\beta}_0 + \mathbf{g} \cdot \boldsymbol{\beta}.} \tag{260}$$

Using the same Weyl derivative convention as in the electromagnetic construction,

$$\boxed{\not{\partial} = \partial_0 \hat{\beta}_0 + \boldsymbol{\nabla} \cdot \boldsymbol{\beta}, \qquad \partial_0 = -\frac{1}{c}\partial_t,} \tag{261}$$

the Weyl momentum-vorticity field is defined directly by the BQ–Weyl product

$$\boxed{\not{H} \equiv \not{\partial}\, \not{G}.} \tag{262}$$

Writing the product out explicitly,

$$\not{H} = \left(-\frac{1}{c}\partial_t\hat{\beta}_0 + \nabla\cdot\boldsymbol{\beta}\right)\left(\frac{\varepsilon}{c}\hat{\beta}_0 + \mathbf{g}\cdot\boldsymbol{\beta}\right). \tag{263}$$

Application of the BQ–Weyl multiplication table gives three nonvanishing channel sets,

$$\boxed{\not{H} = \sigma_E \not{1} + \boldsymbol{\Omega}\cdot\boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{\Pi}\cdot\boldsymbol{\alpha},} \tag{264}$$

where

$$\sigma_E = \frac{1}{c^2}\partial_t\varepsilon + \nabla\cdot\mathbf{g}, \tag{265}$$

$$\boldsymbol{\Omega} = \nabla\times\mathbf{g}, \tag{266}$$

$$\boldsymbol{\Pi} = \partial_t\mathbf{g} + \nabla\varepsilon. \tag{267}$$

The result is the direct fluid counterpart of the electromagnetic Weyl field decomposition. The scalar channel carries the energy–momentum continuity coefficient σ_E, the $\boldsymbol{\Sigma}$ sector carries the momentum vorticity $\boldsymbol{\Omega}$, and the $\boldsymbol{\alpha}$ sector carries the momentum–energy gradient $\boldsymbol{\Pi}/c$.

At the BQ–Pauli level the same coefficients occur in

$$H = -\sigma_E\hat{1} + \boldsymbol{\Omega}\cdot\hat{\mathbf{K}} - \frac{1}{c}\boldsymbol{\Pi}\cdot\hat{\boldsymbol{\sigma}}. \tag{268}$$

The Pauli–Weyl channel correspondence is therefore

$$\boxed{\begin{array}{ccc} -\sigma_E\hat{1} & \longleftrightarrow & +\sigma_E\not{1}, \\ \boldsymbol{\Omega}\cdot\hat{\mathbf{K}} & \longleftrightarrow & \boldsymbol{\Omega}\cdot\boldsymbol{\Sigma}, \\ -\frac{1}{c}\boldsymbol{\Pi}\cdot\hat{\boldsymbol{\sigma}} & \longleftrightarrow & -\frac{1}{c}\boldsymbol{\Pi}\cdot\boldsymbol{\alpha}. \end{array}} \tag{269}$$

The sign reversal in the scalar placement is a consequence of the parity-duplex Weyl realization together with $\partial_0 = -c^{-1}\partial_t$; it does not alter the physical coefficient σ_E. The two three-dimensional Pauli channels, by contrast, become explicitly the spatial-rotation bivectors $\boldsymbol{\Sigma}$ and boost bivectors $\boldsymbol{\alpha}$.

The field therefore occupies only the even scalar-plus-bivector sector,

$$\boxed{\not{H} \in \operatorname{span}\{\not{1}, \boldsymbol{\alpha}, \boldsymbol{\Sigma}\}.} \tag{270}$$

No Weyl vector, pseudovector or pseudoscalar channels are generated by this bilinear product.

The conserved field condition.

Because the scalar and six bivector basis elements are linearly independent,

$$\not{H} = 0 \tag{271}$$

is equivalent to the simultaneous vanishing of the same three physical coefficients obtained at the BQ–Pauli level,

$$\frac{1}{c^2}\partial_t \varepsilon + \boldsymbol{\nabla}\cdot\mathbf{g} = 0, \tag{272a}$$

$$\boldsymbol{\nabla}\times\mathbf{g} = 0, \tag{272b}$$

$$\partial_t \mathbf{g} + \boldsymbol{\nabla}\varepsilon = 0. \tag{272c}$$

Thus

$$\boxed{H = 0 \qquad \Longleftrightarrow \qquad \not{H} = 0} \tag{273}$$

at the level of the physical channel equations, despite the different algebraic placement of those channels.

For the dust-type restriction used in the BQ–Pauli RFD construction,

$$\mathbf{g} = \rho_0 \mathbf{u}, \qquad \varepsilon = \rho_0 c u_0, \tag{274}$$

and constant ρ_0, Eqs. (272) reduce to

$$\boldsymbol{\nabla}\cdot\mathbf{u} = -\frac{1}{c}\partial_t u_0, \tag{275}$$

$$\boldsymbol{\nabla}\times\mathbf{u} = 0, \tag{276}$$

$$\partial_t \mathbf{u} = -c\,\boldsymbol{\nabla} u_0. \tag{277}$$

In the slow-flow limit,

$$\mathbf{u} \simeq \mathbf{v}, \qquad u_0 \simeq c + \frac{v^2}{2c}, \tag{278}$$

these become

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \tag{279}$$

$$\boldsymbol{\nabla}\times\mathbf{v} = 0, \tag{280}$$

$$\partial_t \mathbf{v} = -\boldsymbol{\nabla}\left(\frac{v^2}{2}\right). \tag{281}$$

For a potential flow

$$\mathbf{v} = \boldsymbol{\nabla}\Phi, \tag{282}$$

the last equation gives

$$\boldsymbol{\nabla}\left(\partial_t \Phi + \frac{v^2}{2}\right) = 0, \tag{283}$$

and hence

$$\boxed{\partial_t \Phi + \frac{v^2}{2} = \text{constant.}} \tag{284}$$

The BQ–Weyl lift therefore preserves the complete physical content of the Pauli field condition while resolving its algebraic structure more sharply:

$$\boxed{\begin{array}{ccl} \text{BQ–Pauli} & & \text{BQ–Weyl} \\ G \in \{\hat{T}, \hat{\mathbf{K}}\} & \longrightarrow & \not{G} \in \{\hat{\beta}_\mu\} \\ H = \partial^T G & \longrightarrow & \not{H} = \not{\partial}\not{G} \\ \hat{1} & \longrightarrow & \not{1} \\ \hat{\mathbf{K}} \text{ channel} & \longrightarrow & \boldsymbol{\Sigma} \\ \hat{\boldsymbol{\sigma}} \text{ channel} & \longrightarrow & \boldsymbol{\alpha}. \end{array}} \tag{285}$$

The essential point is therefore not a coefficient-by-coefficient matrix embedding of the Pauli algebra. The parity-duplex doubling separates the Pauli four-vector carrier from the two bilinear generator channels: the former becomes the Weyl grade-1 tetrad, while the latter become the six explicit grade-2 bivectors of the BQ–Weyl algebra.

5.2 Quadratic fluid-field products at the BQ–Weyl level

The BQ–Pauli relativistic-fluid construction generates two natural quadratic field products, HH and $H^T H$. Their Weyl counterparts follow the same distinction already encountered for the electromagnetic products BB and $B^T B$.

At the Weyl level the fluid field derived above is

$$\boxed{\not{H} = \sigma_E \not{1} + \boldsymbol{\Omega} \cdot \boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{\Pi} \cdot \alpha.} \tag{286}$$

The rotational field $\boldsymbol{\Omega}$ therefore occupies the $\boldsymbol{\Sigma}$ bivector sector, while the momentum–energy field $-\boldsymbol{\Pi}/c$ occupies the $\boldsymbol{\alpha}$ bivector sector.

This gives the direct correspondence with the electromagnetic Weyl field

$$\not{B} = \frac{\mathbf{E}}{c} \cdot \alpha + \mathbf{B} \cdot \boldsymbol{\Sigma} \tag{287}$$

through

$$\boxed{\mathbf{B} \longleftrightarrow \boldsymbol{\Omega}, \qquad \frac{\mathbf{E}}{c} \longleftrightarrow -\frac{\boldsymbol{\Pi}}{c}.} \tag{288}$$

5.2.1 Locally conserved fluid: $\sigma_E = 0$

For a locally conserved fluid,

$$\sigma_E = 0, \tag{289}$$

the Weyl fluid field is the pure bivector

$$\boxed{\not{H}_0 = \boldsymbol{\Omega} \cdot \boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{\Pi} \cdot \boldsymbol{\alpha}.} \tag{290}$$

The invariant quadratic field product $\not{H}_0\not{H}_0$.

The first quadratic product is

$$\boxed{\not{H}_0\not{H}_0 = \left(\boldsymbol{\Omega} \cdot \boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{\Pi} \cdot \boldsymbol{\alpha}\right)\left(\boldsymbol{\Omega} \cdot \boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{\Pi} \cdot \boldsymbol{\alpha}\right).} \tag{291}$$

Applying the BQ–Weyl multiplication table, the pure $\boldsymbol{\Sigma}$ and $\boldsymbol{\alpha}$ products give

$$(\boldsymbol{\Omega} \cdot \boldsymbol{\Sigma})^2 = -\Omega^2 \not{1}, \tag{292}$$

and

$$\left(-\frac{1}{c}\boldsymbol{\Pi} \cdot \boldsymbol{\alpha}\right)^2 = \frac{\Pi^2}{c^2}\not{1}. \tag{293}$$

The mixed products reduce to the pseudoscalar channel. With

$$\hat{\chi} = -\hat{\beta}_0\hat{\beta}_I\hat{\beta}_J\hat{\beta}_K, \tag{294}$$

and the multiplication-table convention

$$\hat{\alpha}_i\hat{\Sigma}_i = \hat{\Sigma}_i\hat{\alpha}_i = -\hat{\chi}, \tag{295}$$

one obtains

$$\boxed{\not{H}_0^2 = \left(\frac{\Pi^2}{c^2} - \Omega^2\right)\not{1} + \frac{2}{c}\left(\boldsymbol{\Omega} \cdot \boldsymbol{\Pi}\right)\hat{\chi}.} \tag{296}$$

The Weyl square therefore closes entirely on the scalar and pseudoscalar channels,

$$\boxed{\not{H}_0^2 \in \{\not{1}, \hat{\chi}\}.} \tag{297}$$

The two coefficients are the direct Weyl counterparts of the two quadratic structures obtained at the BQ–Pauli level,

$$HH = \left(\frac{\Pi^2}{c^2} - \Omega^2\right)\hat{1} - \frac{2}{c}\left(\boldsymbol{\Omega} \cdot \boldsymbol{\Pi}\right)\hat{T}. \tag{298}$$

Thus, in this quadratic channel,

$$\boxed{\hat{1} \longrightarrow \not{1}, \qquad \hat{T} \longrightarrow -\hat{\chi}.} \tag{299}$$

The correspondence is exactly parallel to the electromagnetic field square,

$$\boxed{\begin{aligned} \not{B}^2 &= \left(\frac{E^2}{c^2} - B^2\right)\not{1} - \frac{2}{c}(\mathbf{E}\cdot\mathbf{B})\hat{\chi}, \\ \not{H}_0^2 &= \left(\frac{\Pi^2}{c^2} - \Omega^2\right)\not{1} + \frac{2}{c}(\mathbf{\Omega}\cdot\mathbf{\Pi})\hat{\chi}, \end{aligned}} \tag{300}$$

where the relative sign of the pseudoscalar term follows directly from the field correspondence $\mathbf{E}/c \leftrightarrow -\mathbf{\Pi}/c$.

Under a Weyl Lorentz transformation

$$\not{H}_0^L = \Lambda \not{H}_0 \Lambda^{-1}, \tag{301}$$

the field square transforms as

$$(\not{H}_0^L)^2 = \Lambda \not{H}_0^2 \Lambda^{-1}. \tag{302}$$

Since both $\not{1}$ and $\hat{\chi}$ commute with the even Lorentz rotor,

$$\boxed{(\not{H}_0^L)^2 = \not{H}_0^2.} \tag{303}$$

The two coefficients of Eq. (296) are therefore the invariant quadratic fluid-field structures at the Weyl level.

The fluid energy–transport product.

The second BQ–Pauli quadratic product is

$$H^T H = -\left(\Omega^2 + \frac{\Pi^2}{c^2}\right)\hat{1} + \frac{2}{c}\left(\mathbf{\Pi}\times\mathbf{\Omega}\right)\cdot\hat{\boldsymbol{\sigma}}. \tag{304}$$

Defining

$$u_H = -\left(\Omega^2 + \frac{\Pi^2}{c^2}\right), \qquad \mathbf{S}_H = \mathbf{\Pi}\times\mathbf{\Omega}, \tag{305}$$

this is

$$H^T H = u_H\hat{1} + \frac{2}{c}\mathbf{S}_H\cdot\hat{\boldsymbol{\sigma}}. \tag{306}$$

As in the electromagnetic case, the mixed Pauli product $H^T H$ contains the correct energy–transport coefficients but is not itself a Lorentz-covariant four-vector. Multiplication by the central Pauli time element places these coefficients in the spacetime carrier,

$$\hat{T} H^T H = u_H \hat{T} + \frac{2}{c}\mathbf{S}_H \cdot \hat{\mathbf{K}}, \tag{307}$$

where $\hat{T}\hat{\boldsymbol{\sigma}} = \hat{\mathbf{K}}$.

The Weyl lift follows the same mechanism already obtained for the electromagnetic relation $\hat{T} B^T B \;\to\; \not{B}\hat{\beta}_0\not{B}$. For the fluid field, the corresponding Weyl sandwich is

$$\boxed{\not{H}_0 \hat{\beta}_0 \not{H}_0.} \tag{308}$$

Direct application of the BQ–Weyl multiplication table gives

$$\boxed{\not{H}_0 \hat{\beta}_0 \not{H}_0 = -\left(\Omega^2 + \frac{\Pi^2}{c^2}\right)\hat{\beta}_0 + \frac{2}{c}\left(\boldsymbol{\Pi} \times \boldsymbol{\Omega}\right) \cdot \boldsymbol{\beta}.} \tag{309}$$

Equivalently,

$$\boxed{\not{H}_0 \hat{\beta}_0 \not{H}_0 = u_H \hat{\beta}_0 + \frac{2}{c}\mathbf{S}_H \cdot \boldsymbol{\beta}.} \tag{310}$$

The direct Pauli–Weyl correspondence is therefore

$$\boxed{\mathcal{L}\left(\hat{T} H^T H\right) = \not{H}_0 \hat{\beta}_0 \not{H}_0,} \tag{311}$$

under the spacetime-channel lift

$$\hat{T} \longrightarrow \hat{\beta}_0, \qquad \hat{\mathbf{K}} \longrightarrow \boldsymbol{\beta}. \tag{312}$$

The parallel with the electromagnetic construction is therefore complete:

Electromagnetic field	Relativistic fluid field
$BB \longrightarrow \not{B}\not{B}$	$HH \longrightarrow \not{H}_0\not{H}_0$
$\hat{T} B^T B \longrightarrow \not{B}\hat{\beta}_0\not{B}$	$\hat{T} H^T H \longrightarrow \not{H}_0\hat{\beta}_0\not{H}_0$
$\mathbf{E}/c$	$-\boldsymbol{\Pi}/c$
$\mathbf{B}$	$\boldsymbol{\Omega}$
$\mathbf{E} \times \mathbf{B}$	$-\boldsymbol{\Pi} \times \boldsymbol{\Omega}$

(313)

The sign in the last line follows from the field correspondence $\mathbf{E} \leftrightarrow -\boldsymbol{\Pi}$; the fluid transport vector has been defined as $\mathbf{S}_H = \boldsymbol{\Pi} \times \boldsymbol{\Omega}$.

The Weyl lift therefore separates the same two quadratic roles as in electromagnetism:

$$\boxed{\begin{aligned} \not{H}_0^2 &\quad\longrightarrow\quad \text{invariant quadratic fluid-field structure,} \\ \not{H}_0\hat{\beta}_0\not{H}_0 &\quad\longrightarrow\quad \text{frame-resolved fluid-field energy and transport.} \end{aligned}} \tag{314}$$

5.2.2 General non-conservative fluid: $\sigma_E \neq 0$

For the general field

$$\not{H} = \sigma_E \mathbb{1} + \mathbf{\Omega} \cdot \mathbf{\Sigma} - \frac{1}{c}\mathbf{\Pi} \cdot \boldsymbol{\alpha}, \tag{315}$$

the scalar component $\sigma_E \mathbb{1}$ generates additional couplings in the quadratic products.

The direct field square becomes

$$\boxed{\begin{aligned} \not{H}^2 = &\left(\sigma_E^2 - \Omega^2 + \frac{\Pi^2}{c^2}\right)\mathbb{1} \\ &+ 2\sigma_E \mathbf{\Omega} \cdot \mathbf{\Sigma} - \frac{2\sigma_E}{c}\mathbf{\Pi} \cdot \boldsymbol{\alpha} + \frac{2}{c}\left(\mathbf{\Omega} \cdot \mathbf{\Pi}\right)\hat{\chi}. \end{aligned}} \tag{316}$$

The locally conserved result Eq. (296) is recovered immediately when $\sigma_E = 0$.

Compared with the BQ–Pauli result,

$$\begin{aligned} HH = &\left(\sigma_E^2 - \Omega^2 + \frac{\Pi^2}{c^2}\right)\hat{1} - \frac{2}{c}(\mathbf{\Omega} \cdot \mathbf{\Pi})\hat{\mathcal{T}} \\ &- 2\sigma_E \mathbf{\Omega} \cdot \hat{\mathbf{K}} + \frac{2\sigma_E}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}, \end{aligned} \tag{317}$$

the Weyl doubling resolves the same four coefficient structures into the scalar, pseudoscalar, rotation-bivector and boost-bivector channels of the sixteen-element basis.

The corresponding Weyl transport sandwich is

$$\boxed{\begin{aligned} \not{H}\hat{\beta}_0\not{H} = &\left(\sigma_E^2 - \Omega^2 - \frac{\Pi^2}{c^2}\right)\hat{\beta}_0 \\ &+ \frac{2}{c}\left(\mathbf{\Pi} \times \mathbf{\Omega}\right) \cdot \boldsymbol{\beta} - 2\sigma_E \mathbf{\Omega} \cdot \widetilde{\boldsymbol{\beta}}. \end{aligned}} \tag{318}$$

where

$$\widetilde{\beta}_\mu = \hat{\chi}\hat{\beta}_\mu \tag{319}$$

denotes the Weyl pseudovector basis.

For $\sigma_E = 0$ the pseudovector channel disappears and Eq. (318) reduces to the genuine grade-1 transport vector of Eq. (309). When local energy non-conservation is retained, however, the scalar–vorticity coupling $\sigma_E \mathbf{\Omega}$ is resolved by the Weyl doubling into an additional pseudovector channel. Thus

$$\boxed{\sigma_E = 0 : \qquad \not{H}\hat{\beta}_0\not{H} \in \Lambda^1,} \tag{320}$$

whereas

$$\boxed{\sigma_E \neq 0 : \qquad \not{H}\hat{\beta}_0\not{H} \in \Lambda^1 \oplus \Lambda^3.} \tag{321}$$

This is an algebraic distinction that is compressed at the BQ–Pauli level. There the corresponding $H^T H$ product places the $\sigma_E \mathbf{\Omega}$ coupling in the ordinary $\hat{\mathbf{K}}$ channel. After Weyl doubling, the full $1 + 4 + 6 + 4 + 1$ structure separates the locally conserved energy–transport vector from the additional pseudovector content generated when the scalar channel σ_E is nonzero. The electromagnetic parallel is therefore exact only for the pure-bivector sector $\sigma_E = 0$; the non-conservative fluid field contains the additional scalar degree of freedom and consequently explores a larger part of the Weyl algebra.

5.3 The fluid force product $\not{U}\not{H} = \not{F}$

The BQ–Pauli force equation is

$$UH = F, \tag{322}$$

with

$$U = u_0\hat{T} + \mathbf{u} \cdot \hat{\mathbf{K}}, \tag{323}$$

and

$$H = -\sigma_E \hat{1} + \mathbf{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}. \tag{324}$$

Its four Pauli channels are

$$F_{\hat{1}} = -\mathbf{u} \cdot \mathbf{\Omega}, \tag{325}$$

$$F_{\hat{T}} = -\sigma_E u_0 - \frac{1}{c}\mathbf{u} \cdot \mathbf{\Pi}, \tag{326}$$

$$\mathbf{F}_{\hat{K}} = \mathbf{u} \times \mathbf{\Omega} - \frac{u_0}{c}\mathbf{\Pi} - \sigma_E \mathbf{u}, \tag{327}$$

$$\mathbf{F}_{\hat{\sigma}} = -\frac{1}{c}\mathbf{u} \times \mathbf{\Pi} - u_0 \mathbf{\Omega}. \tag{328}$$

At the BQ–Weyl level the four-velocity is the Weyl vector

$$\boxed{\not{U} = u_0\hat{\beta}_0 + \mathbf{u} \cdot \boldsymbol{\beta},} \tag{329}$$

while the fluid field obtained above is

$$\boxed{\not{H} = \sigma_E \not{1} + \mathbf{\Omega} \cdot \mathbf{\Sigma} - \frac{1}{c}\mathbf{\Pi} \cdot \alpha.} \tag{330}$$

The Weyl force product is therefore

$$\boxed{\not{F} \equiv \not{\Psi}\not{H} = \left(u_0 \hat{\beta}_0 + \mathbf{u} \cdot \boldsymbol{\beta}\right) \left(\sigma_E \not{1} + \mathbf{\Omega} \cdot \mathbf{\Sigma} - \frac{1}{c}\mathbf{\Pi} \cdot \alpha\right).} \tag{331}$$

Applying the BQ–Weyl multiplication table and collecting equal basis elements gives

$$\boxed{\begin{aligned} \not{\Psi}\not{H} = & \left(\sigma_E u_0 + \frac{1}{c}\mathbf{u} \cdot \mathbf{\Pi}\right) \hat{\beta}_0 \\ & + \left(\sigma_E \mathbf{u} + \frac{u_0}{c}\mathbf{\Pi} - \mathbf{u} \times \mathbf{\Omega}\right) \cdot \boldsymbol{\beta} \\ & - (\mathbf{u} \cdot \mathbf{\Omega})\, \widetilde{\beta}_0 \\ & - \left(u_0 \mathbf{\Omega} + \frac{1}{c}\mathbf{u} \times \mathbf{\Pi}\right) \cdot \widetilde{\boldsymbol{\beta}}. \end{aligned}} \tag{332}$$

Thus the complete product occupies only the vector and pseudovector sectors,

$$\boxed{\not{F} = \not{F}_\beta + \widetilde{\not{F}}_\beta,} \tag{333}$$

with

$$\begin{aligned} \not{F}_\beta = & \left(\sigma_E u_0 + \frac{1}{c}\mathbf{u} \cdot \mathbf{\Pi}\right) \hat{\beta}_0 \\ & + \left(\sigma_E \mathbf{u} + \frac{u_0}{c}\mathbf{\Pi} - \mathbf{u} \times \mathbf{\Omega}\right) \cdot \boldsymbol{\beta}, \end{aligned} \tag{334}$$

$$\begin{aligned} \widetilde{\not{F}}_\beta = & -(\mathbf{u} \cdot \mathbf{\Omega})\, \widetilde{\beta}_0 \\ & - \left(u_0 \mathbf{\Omega} + \frac{1}{c}\mathbf{u} \times \mathbf{\Pi}\right) \cdot \widetilde{\boldsymbol{\beta}}. \end{aligned} \tag{335}$$

Comparison with Eqs. (325)–(328) gives the direct channel correspondence

$$\boxed{\begin{aligned} [\not{\Psi}\not{H}]_{\beta_0} &= -F_{\hat{T}}, \\ [\not{\Psi}\not{H}]_{\beta} &= -\mathbf{F}_{\hat{K}}, \\ [\not{\Psi}\not{H}]_{\widetilde{\beta}_0} &= F_{\hat{1}}, \\ [\not{\Psi}\not{H}]_{\widetilde{\beta}} &= \mathbf{F}_{\hat{\sigma}}. \end{aligned}} \tag{336}$$

Equivalently,

$$\boxed{\begin{array}{lll} \text{BQ–Pauli channel} & & \text{BQ–Weyl channel} \\ \hat{T} & \longrightarrow & -\hat{\beta}_0 \\ \hat{\mathbf{K}} & \longrightarrow & -\boldsymbol{\beta} \\ \hat{1} & \longrightarrow & \widetilde{\beta}_0 \\ \hat{\boldsymbol{\sigma}} & \longrightarrow & \widetilde{\boldsymbol{\beta}}. \end{array}} \tag{337}$$

For a locally conserved fluid,

$$\sigma_E = 0, \tag{338}$$

the result reduces to

$$\boxed{\begin{aligned} \not{\psi}\not{H}_0 = {} & \frac{1}{c}(\mathbf{u}\cdot\boldsymbol{\Pi})\,\hat{\beta}_0 + \left(\frac{u_0}{c}\boldsymbol{\Pi} - \mathbf{u}\times\boldsymbol{\Omega}\right)\cdot\boldsymbol{\beta} \\ & - (\mathbf{u}\cdot\boldsymbol{\Omega})\,\widetilde{\beta}_0 - \left(u_0\boldsymbol{\Omega} + \frac{1}{c}\mathbf{u}\times\boldsymbol{\Pi}\right)\cdot\widetilde{\boldsymbol{\beta}}. \end{aligned}} \tag{339}$$

The lift therefore preserves all four coefficient structures of the BQ–Pauli force product while resolving them into the odd part of the sixteen-element Weyl algebra,

$$\boxed{\not{\psi} \in \Lambda^1, \qquad \not{H} \in \Lambda^0 \oplus \Lambda^2, \qquad \not{\psi}\not{H} \in \Lambda^1 \oplus \Lambda^3.} \tag{340}$$

The Pauli $\hat{T}$- and $\hat{\mathbf{K}}$-coefficients become Weyl vector channels, whereas the Pauli $\hat{1}$- and $\hat{\boldsymbol{\sigma}}$-coefficients become the corresponding pseudovector channels. No scalar, bivector, or pseudoscalar channels occur in the Weyl force product.

The four coefficient structures obtained here are identical, under the Pauli–Weyl channel correspondence above, to those derived and analysed at the BQ–Pauli level in de Haas (2026a). Their physical interpretation is therefore unchanged and will not be repeated here. In particular, the Pauli channels associated with helicity, power balance, translational momentum transport, and rotational transport are reproduced coefficient by coefficient by the Weyl product $\not{\psi}\not{H}$. The new content of the present subsection is algebraic: the Pauli–Weyl lift resolves these four coefficient structures into the grade-1 and grade-3 sectors of $\mathrm{Cl}(1,3)$,

$$\not{\psi}\not{H} \in \Lambda^1 \oplus \Lambda^3, \tag{341}$$

without altering the fluid-dynamical content established in de Haas (2026a). We therefore refer to that work for the detailed interpretation of the individual channels.

5.4 The fluid field equation $\partial\!\!\!/ H\!\!\!/$

The third level of the BQ–Pauli fluid hierarchy is

$$\partial H = F', \tag{342}$$

with

$$\partial = -\frac{1}{c}\partial_t \hat{T} + \nabla \cdot \hat{\mathbf{K}}, \tag{343}$$

and

$$H = -\sigma_E \hat{1} + \mathbf{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}. \tag{344}$$

Its Pauli channel coefficients are

$$F'_1 = -\nabla \cdot \mathbf{\Omega}, \tag{345}$$

$$F'_T = \frac{1}{c}\left(\partial_t \sigma_E - \nabla \cdot \mathbf{\Pi}\right), \tag{346}$$

$$\mathbf{F}'_K = \nabla \times \mathbf{\Omega} - \nabla \sigma_E + \frac{1}{c^2}\partial_t \mathbf{\Pi}, \tag{347}$$

$$\mathbf{F}'_\sigma = \frac{1}{c}\left(\partial_t \mathbf{\Omega} - \nabla \times \mathbf{\Pi}\right). \tag{348}$$

At the BQ–Weyl level, using

$$\boxed{\partial\!\!\!/ = -\frac{1}{c}\partial_t \hat{\beta}_0 + \nabla \cdot \boldsymbol{\beta},} \tag{349}$$

and

$$\boxed{H\!\!\!/ = \sigma_E 1\!\!\!/ + \mathbf{\Omega} \cdot \mathbf{\Sigma} - \frac{1}{c}\mathbf{\Pi} \cdot \boldsymbol{\alpha},} \tag{350}$$

the field product is

$$\boxed{\partial\!\!\!/ H\!\!\!/ = \left(-\frac{1}{c}\partial_t \hat{\beta}_0 + \nabla \cdot \boldsymbol{\beta}\right)\left(\sigma_E 1\!\!\!/ + \mathbf{\Omega} \cdot \mathbf{\Sigma} - \frac{1}{c}\mathbf{\Pi} \cdot \boldsymbol{\alpha}\right).} \tag{351}$$

Direct application of the BQ–Weyl multiplication table gives

$$\boxed{\begin{aligned} \partial\!\!\!/ H\!\!\!/ = &-\frac{1}{c}\left(\partial_t \sigma_E - \nabla \cdot \mathbf{\Pi}\right)\hat{\beta}_0 \\ &-\left(\nabla \times \mathbf{\Omega} - \nabla \sigma_E + \frac{1}{c^2}\partial_t \mathbf{\Pi}\right) \cdot \boldsymbol{\beta} \\ &-\left(\nabla \cdot \mathbf{\Omega}\right)\widetilde{\beta}_0 \\ &+\frac{1}{c}\left(\partial_t \mathbf{\Omega} - \nabla \times \mathbf{\Pi}\right) \cdot \widetilde{\boldsymbol{\beta}}. \end{aligned}} \tag{352}$$

Thus the Pauli–Weyl channel correspondence is again

$$\boxed{\begin{aligned} [\partial\!\!\!/ H\!\!\!/]_{\beta_0} &= -F'_T, \\ [\partial\!\!\!/ H\!\!\!/]_{\boldsymbol{\beta}} &= -\mathbf{F}'_K, \\ [\partial\!\!\!/ H\!\!\!/]_{\widetilde{\beta}_0} &= F'_1, \\ [\partial\!\!\!/ H\!\!\!/]_{\widetilde{\boldsymbol{\beta}}} &= \mathbf{F}'_\sigma. \end{aligned}} \tag{353}$$

Equivalently,

$$\boxed{\begin{array}{ccl} \text{BQ–Pauli channel} & & \text{BQ–Weyl channel} \\ \hat{T} & \longrightarrow & -\hat{\beta}_0 \\ \hat{\mathbf{K}} & \longrightarrow & -\boldsymbol{\beta} \\ \hat{1} & \longrightarrow & \widetilde{\beta}_0 \\ \hat{\boldsymbol{\sigma}} & \longrightarrow & \widetilde{\boldsymbol{\beta}}. \end{array}} \tag{354}$$

Substitution of

$$\sigma_E = \frac{1}{c^2}\partial_t \varepsilon + \boldsymbol{\nabla}\cdot\mathbf{g}, \qquad \boldsymbol{\Omega} = \boldsymbol{\nabla}\times\mathbf{g}, \qquad \boldsymbol{\Pi} = \partial_t \mathbf{g} + \boldsymbol{\nabla}\varepsilon \tag{355}$$

makes the two pseudovector channels vanish identically:

$$[\partial\!\!\!/ H\!\!\!/]_{\widetilde{\beta}_0} = -\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{g}) = 0, \tag{356}$$

$$[\partial\!\!\!/ H\!\!\!/]_{\widetilde{\boldsymbol{\beta}}} = \frac{1}{c}\left[\partial_t(\boldsymbol{\nabla}\times\mathbf{g}) - \boldsymbol{\nabla}\times(\partial_t\mathbf{g} + \boldsymbol{\nabla}\varepsilon)\right] = 0. \tag{357}$$

The timelike vector channel reduces to

$$\begin{aligned} [\partial\!\!\!/ H\!\!\!/]_{\beta_0} &= -\frac{1}{c}\left[\partial_t\left(\frac{1}{c^2}\partial_t\varepsilon + \boldsymbol{\nabla}\cdot\mathbf{g}\right) - \boldsymbol{\nabla}\cdot(\partial_t\mathbf{g} + \boldsymbol{\nabla}\varepsilon)\right]\hat{\beta}_0 \\ &= -\frac{1}{c}\left(\frac{1}{c^2}\partial_t^2\varepsilon - \nabla^2\varepsilon\right)\hat{\beta}_0, \end{aligned} \tag{358}$$

while the spatial vector channel becomes

$$\begin{aligned} [\partial\!\!\!/ H\!\!\!/]_{\boldsymbol{\beta}} = &-\left[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{g}) - \boldsymbol{\nabla}\left(\frac{1}{c^2}\partial_t\varepsilon + \boldsymbol{\nabla}\cdot\mathbf{g}\right)\right. \\ &\left. + \frac{1}{c^2}\partial_t\left(\partial_t\mathbf{g} + \boldsymbol{\nabla}\varepsilon\right)\right]\cdot\boldsymbol{\beta} \\ = &-\left(\frac{1}{c^2}\partial_t^2\mathbf{g} - \nabla^2\mathbf{g}\right)\cdot\boldsymbol{\beta}. \end{aligned} \tag{359}$$

Defining

$$\Box \equiv \frac{1}{c^2}\partial_t^2 - \nabla^2, \tag{360}$$

the complete Weyl product therefore reduces to

$$\boxed{\not{\partial}\not{H} = -\frac{1}{c}\Box\varepsilon\,\hat{\beta}_0 - (\Box\mathbf{g})\cdot\boldsymbol{\beta}.} \tag{361}$$

Since

$$\not{G} = \frac{\varepsilon}{c}\hat{\beta}_0 + \mathbf{g}\cdot\boldsymbol{\beta}, \tag{362}$$

this may be written compactly as

$$\boxed{\not{\partial}\not{H} = -\Box\not{G}.} \tag{363}$$

If the BQ–Weyl source four-vector is defined by

$$\not{F}' = \frac{1}{c}F_T'\hat{\beta}_0 + \mathbf{F}_K'\cdot\boldsymbol{\beta}, \tag{364}$$

with the coefficient normalization inherited from the Pauli result, then the Weyl field equation takes the compact form

$$\boxed{\not{\partial}\not{H} = -\not{F}',} \tag{365}$$

or equivalently

$$\boxed{\Box\not{G} = \not{F}'.} \tag{366}$$

In the source-free case,

$$\not{F}' = 0, \tag{367}$$

one obtains

$$\boxed{\Box\not{G} = 0,} \tag{368}$$

with component equations

$$\boxed{\Box\varepsilon = 0, \qquad \Box\mathbf{g} = 0.} \tag{369}$$

The Weyl lift therefore preserves the four Pauli coefficient structures exactly while resolving them into

$$\not{\partial}\not{H} \in \Lambda^1 \oplus \Lambda^3. \tag{370}$$

For a field generated by $\not{H} = \not{\partial}\not{G}$, the grade-3 sector vanishes identically, leaving the pure grade-1 wave equation

$$\boxed{\not{\partial}\not{H} = -\Box\not{G}.} \tag{371}$$

The detailed interpretation of the corresponding Pauli channels and their fluid-dynamical reductions is unchanged and is given in de Haas (2026a); the present result establishes their BQ–Weyl grade resolution.

5.4.1 Acoustic medium response in the grade-1 sector

The BQ–Pauli fluid construction further distinguishes vacuum propagation from propagation in an acoustic medium by introducing the acoustic refractive index

$$\boxed{n_s = \frac{c}{c_s},} \tag{372}$$

where c_s is the acoustic propagation speed. This medium modification can be carried over directly to the BQ–Weyl representation.

From the preceding subsection,

$$\not{\partial}\not{H} = -\Box\not{G}, \qquad \Box = \frac{1}{c^2}\partial_t^2 - \nabla^2, \tag{373}$$

after the grade-3 identities have vanished identically. The remaining field equation therefore lies entirely in the Weyl grade-1 sector,

$$\boxed{\not{\partial}\not{H} \in \Lambda^1.} \tag{374}$$

For vacuum propagation,

$$\not{F}' = 0, \tag{375}$$

and hence

$$\boxed{\Box\not{G} = 0,} \tag{376}$$

corresponding to propagation with the invariant speed c.

For an acoustic medium, the BQ–Pauli source term is

$$\boxed{\not{F}' = -\frac{n_s^2 - 1}{c^2}\partial_t^2\not{G}.} \tag{377}$$

Since $\not{G}$ is a Weyl vector, the source remains entirely within the same grade,

$$\not{F}' \in \Lambda^1. \tag{378}$$

Using

$$\Box\not{G} = \not{F}', \tag{379}$$

one obtains

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\not{G} = -\frac{n_s^2 - 1}{c^2}\partial_t^2\not{G}, \tag{380}$$

and therefore

$$\boxed{\left(\frac{n_s^2}{c^2}\partial_t^2 - \nabla^2\right)\not{G} = 0.} \tag{381}$$

With

$$n_s = \frac{c}{c_s}, \tag{382}$$

this becomes

$$\boxed{\left(\frac{1}{c_s^2}\partial_t^2 - \nabla^2\right) \not{G} = 0.} \tag{383}$$

The acoustic refractive index therefore does not introduce an additional Clifford grade. Rather, it modifies the propagation operator within the already isolated grade-1 dynamical sector:

$$\boxed{\begin{aligned} \Lambda^3 &: [\not{\partial}\not{H}]_{\Lambda^3} = 0, \\ \Lambda^1 \text{ vacuum} &: \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right) \not{G} = 0, \\ \Lambda^1 \text{ medium} &: \left(\frac{n_s^2}{c^2}\partial_t^2 - \nabla^2\right) \not{G} = 0, \qquad n_s = \frac{c}{c_s}. \end{aligned}} \tag{384}$$

Thus the Pauli–Weyl lift separates two independent structures: the Clifford grading determines the algebraic sector in which the field equation resides, while n_s determines the propagation speed within that sector. The detailed acoustic interpretation of n_s and of the corresponding medium source is given in de Haas (2026a); here only its placement within the BQ–Weyl grade structure is required.

5.5 The stress–energy equation $\not{\partial}(\not{U}\not{G}) = \not{F}$

We next lift the BQ–Pauli stress–energy equation

$$\partial(U^T G) = F \tag{385}$$

to the BQ–Weyl algebra. At the Pauli level,

$$U^T G = \mathcal{L}\,\hat{1} + \mathbf{T}_K \cdot \hat{\mathbf{K}} + \mathbf{T}_\sigma \cdot \hat{\boldsymbol{\sigma}}, \tag{386}$$

with

$$\mathcal{L} = \frac{u_0}{c}\varepsilon - \mathbf{u}\cdot\mathbf{g}, \tag{387}$$

$$\mathbf{T}_K = \mathbf{u} \times \mathbf{g}, \tag{388}$$

$$\mathbf{T}_\sigma = u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}. \tag{389}$$

The detailed interpretation of these coefficient structures is given in de Haas (2026a).

At the Weyl level,

$$\not{U} = u_0\hat{\beta}_0 + \mathbf{u}\cdot\boldsymbol{\beta}, \qquad \not{G} = \frac{\varepsilon}{c}\hat{\beta}_0 + \mathbf{g}\cdot\boldsymbol{\beta}. \tag{390}$$

Using the BQ–Weyl vector product

$$\boxed{\begin{aligned}\not{A}\not{B} = &-(a_0 b_0 - \mathbf{a}\cdot\mathbf{b})\,\not{1} + (\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\Sigma}\\ &-(a_0\mathbf{b} - b_0\mathbf{a})\cdot\boldsymbol{\alpha},\end{aligned}} \tag{391}$$

one obtains directly

$$\boxed{\not{T}_{\mathrm{fd}} \equiv \not{U}\not{G} = -\mathcal{L}\,\not{1} + \mathbf{T}_K\cdot\boldsymbol{\Sigma} + \mathbf{T}_\sigma\cdot\boldsymbol{\alpha}.} \tag{392}$$

Thus

$$\not{T}_{\mathrm{fd}} \in \Lambda^0 \oplus \Lambda^2. \tag{393}$$

The minus sign of the scalar channel is intrinsic to the Weyl lift. Since

$$\hat{\beta}_0^2 = -\not{1}, \qquad \hat{\beta}_i^2 = +\not{1}, \tag{394}$$

the Pauli Lorentz scalar $\mathcal{L}\hat{1}$ appears as $-\mathcal{L}\not{1}$ in the direct Weyl product. In particular,

$$U^T U = c^2\hat{1} \qquad \longrightarrow \qquad \not{U}\not{U} = -c^2\not{1}, \tag{395}$$

and the same scalar-sign reversal applies to the mixed bilinear $U^T G \to \not{U}\not{G}$.

Using

$$\not{\partial} = -\frac{1}{c}\partial_t\hat{\beta}_0 + \boldsymbol{\nabla}\cdot\boldsymbol{\beta}, \tag{396}$$

the Weyl stress–energy product is

$$\boxed{\not{F} \equiv \not{\partial}\,\not{T}_{\mathrm{fd}} = \not{\partial}\left(-\mathcal{L}\,\not{1} + \mathbf{T}_K\cdot\boldsymbol{\Sigma} + \mathbf{T}_\sigma\cdot\boldsymbol{\alpha}\right).} \tag{397}$$

Since

$$\Lambda^1\left(\Lambda^0 \oplus \Lambda^2\right) \subset \Lambda^1 \oplus \Lambda^3, \tag{398}$$

one has

$$\boxed{\not{F} \in \Lambda^1 \oplus \Lambda^3.} \tag{399}$$

Direct application of the BQ–Weyl multiplication table gives

$$\begin{aligned}\not{F} = &\left(\frac{1}{c}\partial_t\mathcal{L} - \boldsymbol{\nabla}\cdot\mathbf{T}_\sigma\right)\hat{\beta}_0\\ &+\left(-\boldsymbol{\nabla}\times\mathbf{T}_K - \boldsymbol{\nabla}\mathcal{L} + \frac{1}{c}\partial_t\mathbf{T}_\sigma\right)\cdot\boldsymbol{\beta}\end{aligned}$$

$$
\begin{aligned}
&- (\boldsymbol{\nabla}\cdot\mathbf{T}_K)\,\widetilde{\beta}_0 \\
&+ \left(\boldsymbol{\nabla}\times\mathbf{T}_\sigma + \frac{1}{c}\partial_t\mathbf{T}_K\right)\cdot\widetilde{\boldsymbol{\beta}}.
\end{aligned} \tag{400}
$$

The relation to the Pauli channel coefficients is therefore

$$
\boxed{\begin{aligned}
[\not{F}]_{\beta_0} &= -F_T, \\
[\not{F}]_{\boldsymbol{\beta}} &= -\mathbf{F}_K, \\
[\not{F}]_{\widetilde{\beta}_0} &= F_1, \\
[\not{F}]_{\widetilde{\boldsymbol{\beta}}} &= \mathbf{F}_\sigma.
\end{aligned}} \tag{401}
$$

Thus the four Pauli coefficient structures are preserved, while their placement in the Weyl algebra follows the same vector–pseudovector resolution already found for $\not{U}\not{H}$ and $\not{\partial}\not{H}$.

Substitution of Eqs. (387)–(389) gives

$$
[\not{F}]_{\beta_0} = \left[\frac{1}{c}\partial_t\left(\frac{u_0}{c}\varepsilon - \mathbf{u}\cdot\mathbf{g}\right) - \boldsymbol{\nabla}\cdot\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right)\right]\hat{\beta}_0, \tag{402}
$$

$$
\begin{aligned}
[\not{F}]_{\boldsymbol{\beta}} = \Big[&-\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{g}) - \boldsymbol{\nabla}\left(\frac{u_0}{c}\varepsilon - \mathbf{u}\cdot\mathbf{g}\right) \\
&+\frac{1}{c}\partial_t\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right)\Big]\cdot\boldsymbol{\beta},
\end{aligned} \tag{403}
$$

$$
[\not{F}]_{\widetilde{\beta}_0} = -\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{g})\,\widetilde{\beta}_0, \tag{404}
$$

$$
[\not{F}]_{\widetilde{\boldsymbol{\beta}}} = \left[\boldsymbol{\nabla}\times\left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right) + \frac{1}{c}\partial_t(\mathbf{u}\times\mathbf{g})\right]\cdot\widetilde{\boldsymbol{\beta}}. \tag{405}
$$

Single-component reduction.

For

$$
\mathbf{g} = \rho_0\mathbf{u}, \qquad \varepsilon = \rho_0 c\, u_0, \tag{406}
$$

one has

$$
\mathbf{T}_K = 0, \qquad \mathbf{T}_\sigma = 0, \tag{407}
$$

while

$$
\begin{aligned}
\mathcal{L} &= \frac{u_0}{c}\varepsilon - \mathbf{u}\cdot\mathbf{g} \\
&= \rho_0\left(u_0^2 - \mathbf{u}^2\right) = \rho_0 c^2 \equiv \varepsilon_0.
\end{aligned} \tag{408}
$$

The Pauli and Weyl scalar bilinears are therefore

$$
\boxed{U^T G = +\varepsilon_0\hat{1}, \qquad \not{U}\not{G} = -\varepsilon_0\not{1}.} \tag{409}
$$

Accordingly,

$$\boxed{\not{T}_{\mathrm{fd}} = -\mathcal{L}\,\not{1} = -\varepsilon_0\not{1}.} \tag{410}$$

The grade-3 sector then vanishes identically and

$$\begin{aligned} \not{F} &= \not{\partial}(-\mathcal{L}\not{1}) \\ &= -\not{\partial}\mathcal{L} \\ &= \frac{1}{c}\partial_t\mathcal{L}\,\hat{\beta}_0 - \nabla\mathcal{L}\cdot\boldsymbol{\beta}. \end{aligned} \tag{411}$$

Hence

$$\boxed{\not{F} = -\not{\partial}\mathcal{L}.} \tag{412}$$

The coefficient relation with the Pauli equation $F = \partial\mathcal{L}$ is again

$$F_T \longrightarrow -F_T\,\hat{\beta}_0, \qquad \mathbf{F}_K \longrightarrow -\mathbf{F}_K\cdot\boldsymbol{\beta}. \tag{413}$$

In the source-free case,

$$\not{F} = 0, \tag{414}$$

and therefore

$$\boxed{\not{\partial}\mathcal{L} = 0 \qquad \Longrightarrow \qquad \mathcal{L} = \text{constant.}} \tag{415}$$

The overall Weyl sign therefore leaves the source-free constant-Lagrangian condition unchanged.

Barotropic extension.

Using

$$c_s^2 = c^2\left(\frac{\partial p}{\partial\varepsilon_0}\right)_s, \qquad n_s = \frac{c}{c_s}, \tag{416}$$

one has

$$\not{\partial}\varepsilon_0 = n_s^2\not{\partial}p. \tag{417}$$

For a pressure perturbation δp around a constant background, the single-component Weyl equation becomes

$$\boxed{\not{F} = -n_s^2\not{\partial}(\delta p).} \tag{418}$$

Explicitly,

$$\boxed{\not{F} = n_s^2\left[\frac{1}{c}\partial_t(\delta p)\,\hat{\beta}_0 - \nabla(\delta p)\cdot\boldsymbol{\beta}\right].} \tag{419}$$

The acoustic factor modifies only the coefficients within the grade-1 sector,

$$\not{F} \in \Lambda^1. \tag{420}$$

The complete basic lift is therefore

$$\boxed{\begin{aligned} U^T G &= \mathcal{L}\,\hat{1} + \mathbf{T}_K \cdot \hat{\mathbf{K}} + \mathbf{T}_\sigma \cdot \hat{\boldsymbol{\sigma}}, \\ \not{\psi}\not{G} &= -\mathcal{L}\,\not{1} + \mathbf{T}_K \cdot \boldsymbol{\Sigma} + \mathbf{T}_\sigma \cdot \boldsymbol{\alpha}, \\ \partial(U^T G) &\longrightarrow \not{\partial}(\not{\psi}\not{G}) \in \Lambda^1 \oplus \Lambda^3. \end{aligned}} \tag{421}$$

The physical coefficient structures are the same as those analysed in Sec. 4.5 of de Haas (2026a); their detailed interpretation is not repeated here. The Weyl lift adds the explicit Clifford-grade resolution and, in particular, makes the scalar sign reversal $+\mathcal{L}\hat{1} \to -\mathcal{L}\not{1}$ manifest.

Conservation-law interpretation of the Weyl channels.

The physical interpretation of the four coefficient equations is unchanged by the Pauli–Weyl lift. The detailed derivation and fluid-dynamical interpretation are given in de Haas (2026a); here it is sufficient to record how those conservation equations are organized by the Weyl grade structure.

To avoid ambiguity with the various action-density conventions used for relativistic perfect fluids, we denote the scalar coefficient of the BQ bilinear by

$$\boxed{\mathcal{L}_{\mathrm{BQ}} \equiv \frac{u_0}{c}\varepsilon - \mathbf{u} \cdot \mathbf{g}.} \tag{422}$$

At the Pauli level this occurs as $+\mathcal{L}_{\mathrm{BQ}}\hat{1}$, whereas the direct Weyl product contains $-\mathcal{L}_{\mathrm{BQ}}\not{1}$:

$$U^T G \supset +\mathcal{L}_{\mathrm{BQ}}\hat{1}, \qquad \not{\psi}\not{G} \supset -\mathcal{L}_{\mathrm{BQ}}\not{1}. \tag{423}$$

The sign change is therefore an algebraic consequence of the Weyl Clifford metric and does not change the coefficient $\mathcal{L}_{\mathrm{BQ}}$ itself.

With

$$\mathbf{T}_K = \mathbf{u} \times \mathbf{g}, \qquad \mathbf{T}_\sigma = u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}, \tag{424}$$

the source-free Pauli equations and their Weyl placement may be summarized as follows:

The apparent sign reversal of the Λ^1 coefficients in the direct Weyl product does not alter these conservation laws. It follows from

$$+\mathcal{L}_{\mathrm{BQ}}\hat{1} \longrightarrow -\mathcal{L}_{\mathrm{BQ}}\not{1} \tag{425}$$

together with the corresponding Weyl channel signs derived above. Thus the Pauli equations in the second column are retained as the physical coefficient equations, while their Weyl representatives carry the signs fixed by the BQ–Weyl multiplication table.

Table 4 Conservation-law interpretation of the coefficient equations of $\partial(U^T G) = F$ and their organization after the BQ–Weyl lift. The detailed physical interpretation is given in de Haas (2026a).

Pauli channel	Source-free content	Interpretation	BQ–Weyl $Cl(1,3)$ sector
$\hat{1}$	$\nabla \cdot \mathbf{T}_K = 0$	Angular-momentum conservation	$\Lambda^3 : \widetilde{\beta}_0$
$\hat{T}$	$-\frac{1}{c}\partial_t \mathcal{L}_{\mathrm{BQ}} = \nabla \cdot \mathbf{T}_\sigma$	Energy conservation	$\Lambda^1 : \hat{\beta}_0$
$\hat{\mathbf{K}}$	$\frac{1}{c}\partial_t \mathbf{T}_\sigma = -\nabla \mathcal{L}_{\mathrm{BQ}} + \nabla \times \mathbf{T}_K$	Momentum conservation / Euler	$\Lambda^1 : \boldsymbol{\beta}$
$\hat{\boldsymbol{\sigma}}$	$\nabla \times \mathbf{T}_\sigma = 0$	Energy-flux irrotationality	$\Lambda^3 : \widetilde{\boldsymbol{\beta}}$

For the barotropic extension used in de Haas (2026a), the momentum channel contains

$$\boxed{\mathbf{F}_K = n_s^2 \nabla(\delta p), \qquad n_s = \frac{c}{c_s},} \tag{426}$$

subject to the thermodynamic assumptions under which $\partial\varepsilon_0 = n_s^2 \partial p$ is introduced. The factor n_s modifies the coefficient equation but introduces no new Clifford grade.

The Weyl lift consequently exposes a compact grade-level organization of the four Pauli conservation channels:

$$\boxed{\begin{aligned} &\Lambda^1 \ : \text{energy and momentum conservation,} \\ &\Lambda^3 \ : \text{angular-momentum and energy-flux constraints.} \end{aligned}} \tag{427}$$

The Weyl grading therefore reorganizes, rather than replaces, the four conservation equations obtained at the BQ–Pauli level.

5.6 Rapidity-field and rotor representation of the Weyl four-velocity

The Weyl four-velocity field may be parametrized directly by a local rapidity field. We first consider a scalar rapidity field for motion along a fixed spatial direction, then introduce the directed rapidity field required for a general three-dimensional flow, and finally make explicit the local rotor field mediating the map from rapidity to four-velocity.

Scalar rapidity field.

Let

$$\psi = \psi(\mathbf{r}, t) \tag{428}$$

be a scalar rapidity field associated with a fixed boost generator $\hat{\alpha}_i$, satisfying

$$\hat{\alpha}_i^2 = \mathbb{1}. \tag{429}$$

The corresponding Weyl four-velocity is

$$\boxed{\Psi(\mathbf{r},t) = c\,\hat{\beta}_0 \exp[-\psi(\mathbf{r},t)\hat{\alpha}_i]\;.} \tag{430}$$

Since

$$e^{-\psi\hat{\alpha}_i} = \cosh\psi\,\mathbb{1} - \sinh\psi\,\hat{\alpha}_i, \tag{431}$$

and

$$\hat{\alpha}_i = \hat{\beta}_0\hat{\beta}_i, \qquad \hat{\beta}_0^2 = -\mathbb{1}, \tag{432}$$

one has

$$\hat{\beta}_0\hat{\alpha}_i = -\hat{\beta}_i. \tag{433}$$

Consequently,

$$\begin{aligned}\Psi &= c\hat{\beta}_0\left(\cosh\psi\,\mathbb{1} - \sinh\psi\,\hat{\alpha}_i\right)\\ &= c\cosh\psi\,\hat{\beta}_0 + c\sinh\psi\,\hat{\beta}_i.\end{aligned} \tag{434}$$

Thus

$$\boxed{\Psi = u_0\hat{\beta}_0 + u_i\hat{\beta}_i, \qquad u_0 = c\cosh\psi, \qquad u_i = c\sinh\psi.} \tag{435}$$

With the standard rapidity relations

$$\gamma = \cosh\psi, \qquad \gamma\frac{v_i}{c} = \sinh\psi, \tag{436}$$

this gives

$$\boxed{u_0 = \gamma c, \qquad u_i = \gamma v_i, \qquad \frac{v_i}{c} = \tanh\psi.} \tag{437}$$

The inverse relation is

$$\boxed{\psi = \operatorname{artanh}\left(\frac{v_i}{c}\right) = \operatorname{arsinh}\left(\frac{u_i}{c}\right).} \tag{438}$$

The Weyl norm follows directly:

$$\begin{aligned}\Psi^2 &= c^2\left(\cosh\psi\,\hat{\beta}_0 + \sinh\psi\,\hat{\beta}_i\right)^2\\ &= c^2\left(-\cosh^2\psi + \sinh^2\psi\right)\mathbb{1}\\ &= -c^2\mathbb{1}.\end{aligned} \tag{439}$$

Hence

$$\boxed{\Psi^2 = -c^2\mathbb{1},} \tag{440}$$

so that the four-velocity normalization is built into the rapidity parametrization.

Directed rapidity field.

For a general three-dimensional flow, the scalar $\psi(\mathbf{r},t)$ determines the magnitude of the local rapidity but not its spatial direction. Introduce therefore the local unit flow direction

$$\hat{\mathbf{n}}(\mathbf{r},t) = \frac{\mathbf{v}(\mathbf{r},t)}{|\mathbf{v}(\mathbf{r},t)|}, \qquad \hat{\mathbf{n}}^2 = 1, \tag{441}$$

and define the corresponding directed rapidity field by

$$\boxed{\boldsymbol{\psi}(\mathbf{r},t) \equiv \psi(\mathbf{r},t)\,\hat{\mathbf{n}}(\mathbf{r},t).} \tag{442}$$

The associated local boost generator is

$$\boxed{\boldsymbol{\psi}\cdot\boldsymbol{\alpha} = \psi\,\hat{\mathbf{n}}\cdot\boldsymbol{\alpha},} \tag{443}$$

with

$$(\hat{\mathbf{n}}\cdot\boldsymbol{\alpha})^2 = \mathbb{1}. \tag{444}$$

It follows that

$$\exp(-\boldsymbol{\psi}\cdot\boldsymbol{\alpha}) = \cosh\psi\,\mathbb{1} - \sinh\psi\,\hat{\mathbf{n}}\cdot\boldsymbol{\alpha}. \tag{445}$$

The general Weyl four-velocity field can therefore be written

$$\boxed{\not{\Psi}(\mathbf{r},t) = c\,\hat{\beta}_0 \exp[-\boldsymbol{\psi}(\mathbf{r},t)\cdot\boldsymbol{\alpha}]\,.} \tag{446}$$

Using

$$\hat{\beta}_0\,(\hat{\mathbf{n}}\cdot\boldsymbol{\alpha}) = -\hat{\mathbf{n}}\cdot\boldsymbol{\beta}, \tag{447}$$

one obtains

$$\boxed{\not{\Psi} = c\cosh\psi\,\hat{\beta}_0 + c\sinh\psi\,\hat{\mathbf{n}}\cdot\boldsymbol{\beta}.} \tag{448}$$

Hence

$$\boxed{u_0 = c\cosh\psi, \qquad \mathbf{u} = c\sinh\psi\,\hat{\mathbf{n}}, \qquad \mathbf{v} = c\tanh\psi\,\hat{\mathbf{n}}.} \tag{449}$$

Conversely,

$$\boxed{\psi = \operatorname{arsinh}\left(\frac{|\mathbf{u}|}{c}\right) = \operatorname{artanh}\left(\frac{|\mathbf{v}|}{c}\right), \qquad \hat{\mathbf{n}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{v}}{|\mathbf{v}|}.} \tag{450}$$

Away from the zero-flow point, the directed rapidity field and the normalized Weyl four-velocity are therefore in one-to-one correspondence,

$$\boxed{\boldsymbol{\psi}(\mathbf{r},t) \quad \longleftrightarrow \quad \not{\Psi}(\mathbf{r},t).} \tag{451}$$

The local fluid rapidity rotor.

The rapidity-to-four-velocity map may be resolved through a local BQ–Weyl rotor field. Define

$$\boxed{Q_\psi(\mathbf{r},t) \equiv \exp\left[\frac{1}{2}\boldsymbol{\psi}(\mathbf{r},t)\cdot\boldsymbol{\alpha}\right],} \tag{452}$$

with inverse

$$Q_\psi^{-1}(\mathbf{r},t) = \exp\left[-\frac{1}{2}\boldsymbol{\psi}(\mathbf{r},t)\cdot\boldsymbol{\alpha}\right]. \tag{453}$$

The Weyl four-velocity is then generated from the timelike basis vector by

$$\boxed{\Psi(\mathbf{r},t) = c\,Q_\psi(\mathbf{r},t)\hat{\beta}_0 Q_\psi^{-1}(\mathbf{r},t).} \tag{454}$$

Since the boost generator anticommutes with $\hat{\beta}_0$,

$$Q_\psi\hat{\beta}_0 = \hat{\beta}_0 Q_\psi^{-1}, \tag{455}$$

and therefore

$$\begin{aligned} Q_\psi\hat{\beta}_0 Q_\psi^{-1} &= \hat{\beta}_0 Q_\psi^{-2} \\ &= \hat{\beta}_0 \exp[-\boldsymbol{\psi}\cdot\boldsymbol{\alpha}] . \end{aligned} \tag{456}$$

Hence the rotor and exponential forms are identical:

$$\boxed{\begin{aligned} \Psi &= cQ_\psi\hat{\beta}_0 Q_\psi^{-1} \\ &= c\hat{\beta}_0 e^{-\boldsymbol{\psi}\cdot\boldsymbol{\alpha}} \\ &= c\cosh\psi\,\hat{\beta}_0 + c\sinh\psi\,\hat{\mathbf{n}}\cdot\boldsymbol{\beta}. \end{aligned}} \tag{457}$$

The local construction may therefore be written as the two-step map

$$\boxed{\boldsymbol{\psi}(\mathbf{r},t) \xrightarrow{\exp} Q_\psi(\mathbf{r},t) \xrightarrow{\mathrm{Ad}_{Q_\psi}} \Psi(\mathbf{r},t),} \tag{458}$$

where

$$\mathrm{Ad}_{Q_\psi}(\hat{\beta}_0) = Q_\psi\hat{\beta}_0 Q_\psi^{-1}. \tag{459}$$

Thus $\boldsymbol{\psi}$ is the local rapidity field, Q_ψ its associated local rotor field, and Ψ the corresponding Weyl four-velocity field.

For a fixed flow direction this reduces to

$$\boxed{\psi(\mathbf{r},t) \longrightarrow Q_\psi(\mathbf{r},t) = e^{\frac{1}{2}\psi(\mathbf{r},t)\hat{\alpha}_i} \longrightarrow \Psi(\mathbf{r},t).} \tag{460}$$

Insertion into the RFD hierarchy.

For the single-component fluid relation

$$\not{G} = \rho_0 \not{U}, \tag{461}$$

the momentum-density field becomes

$$\boxed{\not{G}(\mathbf{r}, t) = \rho_0 c\, Q_\psi(\mathbf{r}, t)\hat{\beta}_0 Q_\psi^{-1}(\mathbf{r}, t)} \tag{462}$$

or, equivalently,

$$\boxed{\not{G}(\mathbf{r}, t) = \rho_0 c\, \hat{\beta}_0 \exp[-\boldsymbol{\psi}(\mathbf{r}, t) \cdot \boldsymbol{\alpha}]\,.} \tag{463}$$

The Weyl-level RFD hierarchy may therefore be written in the rapidity-generated form

$$\boxed{\begin{aligned} &\boldsymbol{\psi} \longrightarrow Q_\psi \longrightarrow \not{U} \longrightarrow \not{G} \longrightarrow \not{H} = \not{\partial}\not{G} \\ &\quad\longrightarrow \left\{\not{U}\not{H},\ \not{\partial}\not{H},\ \not{\partial}(\not{U}\not{G})\right\}. \end{aligned}} \tag{464}$$

Accordingly, the introduction of Q_ψ adds no new dynamical assumption to the RFD hierarchy. It resolves the exact rapidity–four-velocity parametrization into the local sequence

$$\boxed{\text{directed rapidity field} \;\longrightarrow\; \text{local rotor field} \;\longrightarrow\; \text{normalized Weyl four-velocity.}} \tag{465}$$

For a fixed flow direction the scalar field $\psi(\mathbf{r}, t)$ is sufficient; for a general three-dimensional flow the complete local field is the directed rapidity $\boldsymbol{\psi} = \psi\hat{\mathbf{n}}$.

5.7 From the local rapidity rotor to transport dynamics

The replacement of the fluid four-velocity field by the directed rapidity field,

$$\boldsymbol{\psi}(x) \longrightarrow Q_\psi(x) \longrightarrow \not{U}(x), \tag{466}$$

is, at the kinematic level, an exact reparametrization. With

$$Q_\psi(x) = \exp\left[\frac{1}{2}\boldsymbol{\psi}(x) \cdot \boldsymbol{\alpha}\right], \tag{467}$$

the normalized Weyl four-velocity is

$$\boxed{\not{U}(x) = c\, Q_\psi(x)\hat{\beta}_0 Q_\psi^{-1}(x).} \tag{468}$$

At this level, choosing $\boldsymbol{\psi}$ rather than $\not{U}$ as the primary field introduces no additional dynamics.

The local rotor representation, however, permits differentiation to be performed before projection onto the four-velocity. Define the right Maurer–Cartan connection of the local fluid rotor by

$$\boxed{\mathfrak{Q}_\mu^{(\psi)} \equiv (\partial_\mu Q_\psi) Q_\psi^{-1}.} \tag{469}$$

Since $Q_\psi \in \mathfrak{spin}(1,3)$, the logarithmic derivative belongs to the Lorentz bivector algebra,

$$\mathfrak{Q}_\mu^{(\psi)} \in \mathfrak{spin}(1,3) \simeq \Lambda^2. \tag{470}$$

Differentiating Eq. (468) gives

$$\begin{aligned}\partial_\mu \Psi &= c(\partial_\mu Q_\psi)\hat{\beta}_0 Q_\psi^{-1} + c Q_\psi \hat{\beta}_0 \partial_\mu Q_\psi^{-1} \\ &= \mathfrak{Q}_\mu^{(\psi)} \Psi - \Psi \mathfrak{Q}_\mu^{(\psi)}.\end{aligned} \tag{471}$$

Hence

$$\boxed{\partial_\mu \Psi = \left[\mathfrak{Q}_\mu^{(\psi)}, \Psi\right].} \tag{472}$$

The spacetime variation of the four-velocity can therefore be written as the adjoint action of the local Maurer–Cartan connection on Ψ.

Along a fluid worldline, with

$$\frac{D}{D\tau} = U^\mu \partial_\mu, \tag{473}$$

define

$$\boxed{\mathfrak{Q}_\tau^{(\psi)} \equiv \left(\frac{DQ_\psi}{D\tau}\right) Q_\psi^{-1} = U^\mu \mathfrak{Q}_\mu^{(\psi)}.} \tag{474}$$

Equation (472) then gives

$$\boxed{\frac{D\Psi}{D\tau} = \left[\mathfrak{Q}_\tau^{(\psi)}, \Psi\right].} \tag{475}$$

Thus the four-acceleration may equivalently be represented through the proper-time evolution of the local rapidity rotor.

Fixed boost direction.

For a scalar rapidity field with fixed boost direction,

$$Q_\psi(x) = \exp\left[\frac{1}{2}\psi(x)\hat{\alpha}_i\right], \tag{476}$$

the generator is fixed and commutes with its derivative. The connection therefore reduces exactly to

$$\boxed{\slashed{\boldsymbol{\Omega}}_{\mu}^{(\psi)} = \frac{1}{2}(\partial_{\mu}\psi)\hat{\alpha}_i.} \tag{477}$$

Along the flow,

$$\boxed{\slashed{\boldsymbol{\Omega}}_{\tau}^{(\psi)} = \frac{1}{2}\frac{D\psi}{D\tau}\,\hat{\alpha}_i.} \tag{478}$$

In this case the Maurer–Cartan connection contains the gradient of the scalar rapidity field entirely in the boost-bivector channel.

Directed rapidity field and noncommuting boosts.

For a general directed rapidity field,

$$\boldsymbol{\psi}(x) = \psi(x)\hat{\mathbf{n}}(x), \tag{479}$$

write

$$A(x) = \frac{1}{2}\boldsymbol{\psi}(x)\cdot\boldsymbol{\alpha}, \qquad Q_{\psi} = e^{A}. \tag{480}$$

When the local flow direction varies, A and $\partial_{\mu}A$ need not commute. Consequently,

$$\slashed{\boldsymbol{\Omega}}_{\mu}^{(\psi)} \neq \partial_{\mu}A \tag{481}$$

in general. The exact logarithmic derivative is

$$\boxed{\slashed{\boldsymbol{\Omega}}_{\mu}^{(\psi)} = \int_0^1 e^{sA}(\partial_{\mu}A)e^{-sA}\,ds.} \tag{482}$$

Equivalently,

$$\boxed{\slashed{\boldsymbol{\Omega}}_{\mu}^{(\psi)} = \partial_{\mu}A + \frac{1}{2!}[A,\partial_{\mu}A] + \frac{1}{3!}[A,[A,\partial_{\mu}A]] + \cdots.} \tag{483}$$

Because differently directed boost generators do not commute,

$$[\hat{\alpha}_i,\hat{\alpha}_j] \propto \hat{\Sigma}_k, \qquad i \neq j, \tag{484}$$

the connection generated by a spatially varying directed rapidity field need not remain entirely in the boost-bivector sector. In general it may be decomposed as

$$\boxed{\slashed{\boldsymbol{\Omega}}_{\mu}^{(\psi)} = \slashed{\boldsymbol{\Omega}}_{\mu,\alpha}^{(\psi)} + \slashed{\boldsymbol{\Omega}}_{\mu,\Sigma}^{(\psi)},} \tag{485}$$

where

$$\slashed{\boldsymbol{\Omega}}_{\mu,\alpha}^{(\psi)} \in \mathrm{span}\{\hat{\alpha}_i\}, \qquad \slashed{\boldsymbol{\Omega}}_{\mu,\Sigma}^{(\psi)} \in \mathrm{span}\{\hat{\Sigma}_i\}. \tag{486}$$

The local variation of the rapidity magnitude and direction is therefore encoded in a single Lorentz-bivector connection.

Maurer–Cartan identity.

The connection of Eq. (469) is not an independent field introduced in addition to Q_ψ. Since it is generated directly from the local rotor,

$$\slashed{\Omega}^{(\psi)} = (dQ_\psi)Q_\psi^{-1}, \tag{487}$$

it satisfies the corresponding Maurer–Cartan identity. With the right-invariant convention used here,

$$\boxed{d\slashed{\Omega}^{(\psi)} - \slashed{\Omega}^{(\psi)} \wedge \slashed{\Omega}^{(\psi)} = 0,} \tag{488}$$

or, in components,

$$\boxed{\partial_\mu \slashed{\Omega}_\nu^{(\psi)} - \partial_\nu \slashed{\Omega}_\mu^{(\psi)} - \left[\slashed{\Omega}_\mu^{(\psi)}, \slashed{\Omega}_\nu^{(\psi)}\right] = 0.} \tag{489}$$

Thus the introduction of the local rapidity rotor does not by itself add independent degrees of freedom or a new curvature field. Rather, it exposes the differential structure already implicit in the local rapidity field.

From velocity dynamics to rotor transport.

The distinction between a velocity-first and a rapidity-first formulation may now be stated as

$$\boxed{\begin{array}{ll} \text{velocity-first:} & \slashed{U} \longrightarrow \partial_\mu \slashed{U} \longrightarrow \text{fluid dynamics,} \\ \text{rapidity-first:} & \boldsymbol{\slashed{\psi}} \longrightarrow Q_\psi \longrightarrow \slashed{\Omega}_\mu^{(\psi)} \longrightarrow \text{transport dynamics.} \end{array}} \tag{490}$$

The two routes are not independent, since

$$\partial_\mu \slashed{U} = [\slashed{\Omega}_\mu^{(\psi)}, \slashed{U}]. \tag{491}$$

At the level of the exact parametrization they therefore contain the same kinematic information. The rapidity-first route, however, provides the additional rotor-level objects on which transport conditions and Maurer–Cartan relations can be formulated directly.

The resulting extension of the RFD hierarchy may consequently be displayed as

$$\boxed{\begin{array}{ll} & \nearrow \slashed{U} \longrightarrow \slashed{G} \longrightarrow \slashed{H} \longrightarrow \cdots \\ \boldsymbol{\slashed{\psi}} \longrightarrow Q_\psi & \\ & \searrow \slashed{\Omega}_\mu^{(\psi)} \longrightarrow \slashed{\Omega}_\tau^{(\psi)} \longrightarrow \dfrac{D\slashed{U}}{D\tau}. \end{array}} \tag{492}$$

Accordingly, taking the local rapidity field as primary does not by itself modify the established RFD equations. Its significance is that the same primary field generates both the four-velocity kinematic branch and a local rotor-transport branch. If the RFD dynamics can subsequently be expressed, constrained, or derived directly through $\not{\Omega}_\mu^{(\psi)}$ or $\not{\Omega}_\tau^{(\psi)}$, the rapidity-first formulation becomes more than a change of variables: *this potentially changes the architecture of the theory*.

Introducing the local rapidity rotor exposes, alongside the established kinematic field hierarchy, a parallel Maurer–Cartan transport hierarchy. The former generates the fluid and derived field variables, whereas the latter encodes the local evolution of the rotor-generated four-velocity. At the RFD level this is a reformulation rather than an additional dynamical assumption, but it provides the algebraic architecture through which transport dynamics can be expressed at the rotor level.

5.8 General conclusions for the BQ–Weyl fluid hierarchy

The preceding sections lift the BQ–Pauli relativistic-fluid construction of de Haas (2026a) to the sixteen-element BQ–Weyl representation of $Cl(1,3)$. The principal RFD relations retain the same coefficient content,

$$\boxed{\not{H} = \not{\partial}\,\not{G}, \qquad \not{\Psi}\,\not{H} = \not{F}, \qquad \not{\partial}\,\not{H} = \not{F}', \qquad \not{\partial}(\not{\Psi}\,\not{G}) = \not{F},} \tag{493}$$

but are now resolved in the Clifford decomposition

$$\boxed{Cl(1,3) = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3 \oplus \Lambda^4, \qquad 1+4+6+4+1 = 16.} \tag{494}$$

No new fluid equation is introduced by this lift. The continuity, irrotationality, Bernoulli, force, wave and conservation relations remain those derived and interpreted at BQ–Pauli level in de Haas (2026a). The additional content of the Weyl formulation is the explicit Clifford-grade organization of these channels, their quadratic completion, and their connection to the local rapidity–rotor representation of the fluid four-velocity.

5.8.1 Weyl resolution of the RFD hierarchy

The momentum-density field is a Weyl four-vector,

$$\not{G} = G^\mu \hat{\beta}_\mu \in \Lambda^1, \tag{495}$$

and its first derivative gives

$$\boxed{\not{H} = \not{\partial}\,\not{G} = -\sigma_E \not{1} + \mathbf{\Omega} \cdot \mathbf{\Sigma} - \frac{1}{c}\mathbf{\Pi} \cdot \boldsymbol{\alpha} \in \Lambda^0 \oplus \Lambda^2.} \tag{496}$$

The scalar channel contains the compression/energy-divergence coefficient σ_E, while the six-dimensional bivector sector contains the rotational field $\mathbf{\Omega}$ and the momentum-gradient field $\mathbf{\Pi}/c$. The three Pauli channels of H are therefore retained, but the two vector-like three-component sectors are now recognized as the two complementary parts of a single Weyl bivector.

The force equation couples the spacetime vector $\not{\mathcal{U}}$ to this scalar–bivector field,

$$\Lambda^1 \left(\Lambda^0 \oplus \Lambda^2\right) \longrightarrow \Lambda^1 \oplus \Lambda^3, \tag{497}$$

and consequently

$$\boxed{\not{\mathcal{U}}\,\not{H} = \not{F}^{(1)} + \not{F}^{(3)} \in \Lambda^1 \oplus \Lambda^3.} \tag{498}$$

The four physical Pauli channels derived in de Haas (2026a) thereby reorganize into two relativistic Clifford sectors. The power and Euler–Lamb momentum terms form the temporal and spatial components of the vector sector Λ^1, whereas the helicity and rotational-transport terms occupy the four-dimensional trivector sector Λ^3:

$$\boxed{\begin{array}{rl} \text{power + Euler–Lamb momentum} & \longrightarrow \Lambda^1, \\ \text{helicity + rotational transport} & \longrightarrow \Lambda^3. \end{array}} \tag{499}$$

This gives a direct Weyl-level organization of the physical content identified at Pauli level. The Λ^1 sector collects the energetic and momentum-transfer content of the fluid interaction, while Λ^3 collects its topological and rotational content. In particular, the scalar helicity channel of the Pauli product is not lost under the lift: it becomes part of the relativistic trivector sector together with rotational transport. Likewise, the Pauli time and spatial momentum channels become the two components of one spacetime vector.

Among the four dynamical products inherited directly from the BQ–Pauli RFD hierarchy, $\not{\mathcal{U}}\not{H}$ is therefore distinguished by the simultaneous presence of all four physical sectors: helicity, power, Euler–Lamb momentum and rotational transport. The Weyl lift shows that this four-channel Pauli structure is equivalently a vector–trivector decomposition of a single Clifford product.

The second field derivative initially permits the same odd-grade structure,

$$\not{\partial}\,\not{H} \in \Lambda^1 \oplus \Lambda^3. \tag{500}$$

The differential identities derived above cancel the trivector sector, however, leaving

$$\boxed{\not{\partial}\,\not{H} = \not{F}' = F_0'\hat{\beta}_0 + \mathbf{F}' \cdot \boldsymbol{\beta} \in \Lambda^1,} \tag{501}$$

with coefficients proportional to

$$F_0' \sim \Box\varepsilon, \qquad \mathbf{F}' \sim \Box\mathbf{g}. \tag{502}$$

The wave branch therefore contains only spacetime-vector propagation after the differential cancellations are imposed. In the acoustic extension, replacing the characteristic propagation speed c by c_s through

$$n_s = \frac{c}{c_s} \tag{503}$$

changes the differential coefficients but not their Clifford grade. The acoustic refractive index therefore belongs to the propagation law rather than to the algebraic grade structure.

Finally, under the single-component condition

$$\not{G} = \rho_0 \not{\Psi}, \tag{504}$$

the bilinear product reduces to the Lorentz-scalar Lagrangian channel,

$$\boxed{\not{\Psi}\,\not{G} = \mathcal{L}\,\not{1} \in \Lambda^0,} \tag{505}$$

where the sign of $\mathcal{L}$ follows from the Weyl Clifford-norm convention established above. Its derivative is therefore purely vectorial,

$$\boxed{\not{\partial}(\not{\Psi}\,\not{G}) = \not{\partial}\mathcal{L} = F_0\hat{\beta}_0 + \mathbf{F}\cdot\boldsymbol{\beta} \in \Lambda^1.} \tag{506}$$

Thus the Lagrangian/conservation branch passes from a Lorentz scalar to its spacetime-vector gradient, while the force branch $\not{\Psi}\not{H}$ retains the additional trivector content associated with helicity and rotation.

5.8.2 Quadratic field and stress–energy branch

The Weyl lift also resolves the quadratic RFD field constructions. Writing

$$\not{H} = -\sigma_E\not{1} + \not{H}_2, \qquad \not{H}_2 = \boldsymbol{\Omega}\cdot\boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{\Pi}\cdot\alpha, \tag{507}$$

gives

$$\not{H}^2 = \sigma_E^2\not{1} - 2\sigma_E\not{H}_2 + \not{H}_2^2. \tag{508}$$

Since the square of the bivector part satisfies

$$\not{H}_2^2 \in \Lambda^0 \oplus \Lambda^4, \tag{509}$$

the general quadratic field product occupies

$$\boxed{\not{H}^2 \in \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4.} \tag{510}$$

The grade-2 contribution arises solely from the product of the scalar coefficient σ_E with the bivector part $\slashed{H}_2$.

Under the conserved-field condition

$$\sigma_E = 0, \tag{511}$$

the RFD field becomes a pure bivector and the quadratic structure reduces to

$$\boxed{\slashed{H}^2 \in \Lambda^0 \oplus \Lambda^4.} \tag{512}$$

This is precisely the Clifford-grade structure obtained for the electromagnetic bivector,

$$\slashed{B}^2 \in \Lambda^0 \oplus \Lambda^4. \tag{513}$$

The analogy is algebraic rather than an identification of the two physical fields: in both cases the square of a pure bivector packages its two intrinsic quadratic coefficients into scalar and pseudoscalar channels.

The complementary quadratic construction is obtained by inserting a spacetime vector. For an arbitrary $a \in \Lambda^1$, define

$$\boxed{\slashed{T}_H(a) \equiv -\frac{1}{2}\, \slashed{H}\, a\, \slashed{H},} \tag{514}$$

up to the overall normalization adopted for the RFD quadratic field quantity. In particular, for the four basis vectors $a = \hat{\beta}_\mu$ and for $\sigma_E = 0$,

$$\boxed{\slashed{T}_H(\hat{\beta}_\mu) = -\frac{1}{2}\slashed{H}\hat{\beta}_\mu\slashed{H} \in \Lambda^1.} \tag{515}$$

The four values $\mu = 0, I, J, K$ therefore form the complete vector-resolved quadratic map of the pure-bivector RFD field.

The $\mu = 0$ member contains the energy–transport content already present in the Pauli quadratic construction, while the spatial members complete the corresponding quadratic field-stress resolution. The construction is directly parallel to the electromagnetic Weyl map

$$\slashed{T}_{\mathrm{EM}}(\hat{\beta}_\mu) = -\frac{1}{2\mu_0}\slashed{B}\hat{\beta}_\mu\slashed{B}, \tag{516}$$

so that

$$\boxed{\begin{array}{cc} \text{electromagnetic field} & \text{RFD field} \\ \slashed{B}^2 & \slashed{H}^2 \\ \slashed{B}\hat{\beta}_\mu\slashed{B} & \slashed{H}\hat{\beta}_\mu\slashed{H} \\ \text{field invariants} & \text{quadratic fluid-field channels} \\ \text{and EM stress–energy} & \text{and RFD field-stress map.} \end{array}} \tag{517}$$

The last line should not be read as identifying $\slashed{T}_H(\hat{\beta}_\mu)$ with the standard material stress–energy tensor of a perfect fluid. The field $\slashed{H} = \slashed{\partial}\slashed{G}$ is itself derivative in the momentum-density field, so that $\slashed{H}\hat{\beta}_\mu\slashed{H}$ is quadratic in the RFD field gradients. It is therefore more precisely a quadratic stress–energy-type map of the field $\slashed{H}$. Its temporal member resolves field energy and transport, while its spatial members expose the corresponding field-stress structure.

The resolving vector need not be a fixed basis element. Choosing the local fluid four-velocity gives

$$\boxed{\slashed{T}_H\left(\frac{\slashed{U}}{c}\right) = -\frac{1}{2c}\slashed{H}\,\slashed{U}\,\slashed{H},} \tag{518}$$

which resolves the quadratic field content relative to the local fluid direction. Thus the two quadratic operations have distinct roles:

$$\boxed{\begin{array}{rcl} \slashed{H}^2 & \longrightarrow & \text{intrinsic quadratic field channels,} \\ \slashed{H}\,a\,\slashed{H} & \longrightarrow & \text{vector-resolved quadratic field content.} \end{array}} \tag{519}$$

5.8.3 Clifford-grade organization and physical content

The complete grade activity of the BQ–Weyl RFD hierarchy can now be collected in a single table:

RFD object/product	Λ^0	Λ^1	Λ^2	Λ^3	Λ^4
$\slashed{G}$	0	$\neq 0$	0	0	0
$\slashed{H} = \slashed{\partial}\slashed{G}$	$\neq 0$	0	$\neq 0$	0	0
$\slashed{U}\slashed{H} = \slashed{F}$	0	$\neq 0$	0	$\neq 0$	0
$\slashed{\partial}\slashed{H} = \slashed{F}'$	0	$\neq 0$	0	0	0
$\slashed{U}\slashed{G} = \mathcal{L}\slashed{1}$	$\neq 0$	0	0	0	0
$\slashed{\partial}(\slashed{U}\slashed{G})$	0	$\neq 0$	0	0	0
$\slashed{H}^2$	$\neq 0$	0	$\neq 0$	0	$\neq 0$
$\slashed{H}^2$ $(\sigma_E = 0)$	$\neq 0$	0	0	0	$\neq 0$
$\slashed{T}_H(\hat{\beta}_\mu)$ $(\sigma_E = 0)$	0	$\neq 0$	0	0	0

(520)

where the reduced rows assume the single-component conditions and differential identities specified in their respective derivations.

The table makes explicit a structural distinction that is only compressed into channels at Pauli level. The primary differential sequence alternates between odd and even Clifford sectors,

$$\boxed{\slashed{G} \in \Lambda^1 \xrightarrow{\slashed{\partial}} \slashed{H} \in \Lambda^0 \oplus \Lambda^2 \xrightarrow{\slashed{U}} \slashed{U}\slashed{H} \in \Lambda^1 \oplus \Lambda^3.} \tag{521}$$

The field $\not{H}$ therefore occupies the even algebra, while its coupling to the fluid four-velocity transfers its content into the odd algebra.

Physically, this odd-grade resolution is not merely a classification. The four Pauli sectors discussed in de Haas (2026a) correspond to four recognizable parts of relativistic fluid dynamics. Helicity encodes the topological organization of the momentum/vorticity field; the power channel encodes energetic transfer; the Euler–Lamb channel contains the momentum equation and its vortex-force structure; and the rotational channel contains vorticity and rotational transport. At Weyl level these four sectors acquire the more compact organization

$$\boxed{\begin{array}{rl} \text{energetics + momentum} & \longrightarrow \Lambda^1, \\ \text{topology + rotation} & \longrightarrow \Lambda^3. \end{array}} \tag{522}$$

This is the principal interpretive gain of the Weyl lift for the force equation. Power and momentum are not independent algebraic channels but the temporal and spatial parts of one relativistic vector. Likewise, helicity and rotational transport are not isolated remnants of the three-dimensional Pauli product but combine within the complementary trivector sector. The Weyl algebra therefore retains the physical interpretation of the Pauli channels while exposing their relativistic geometric relation.

The remaining products occupy more restricted sectors. The wave equation and the gradient of the scalar Lagrangian reduce to Λ^1; the intrinsic field $\not{H}$ occupies $\Lambda^0 \oplus \Lambda^2$; and the pure-bivector quadratic square occupies $\Lambda^0 \oplus \Lambda^4$. The quadratic sandwich $\not{H}\hat{\beta}_\mu\not{H}$ returns the pure-bivector field content to Λ^1, thereby supplying its vector-resolved energy–transport and field-stress completion.

The BQ–Weyl hierarchy can consequently be summarized physically and algebraically as

$$\boxed{\begin{array}{rl} \Lambda^0 & : \text{scalar field and Lagrangian content,} \\ \Lambda^1 & : \text{four-momentum, power, momentum transport,} \\ & : \text{wave and conservation content,} \\ \Lambda^2 & : \text{rotational and momentum-gradient field content,} \\ \Lambda^3 & : \text{helicity and rotational-transport content,} \\ \Lambda^4 & : \text{pseudoscalar quadratic field content.} \end{array}} \tag{523}$$

The grade decomposition therefore does not replace the fluid interpretation developed at Pauli level; it provides its $Cl(1,3)$ organization. Each entry should be read as an assignment of physical content to a grade, not as a coordinate-by-coordinate separation within it. In particular, the trivector sector Λ^3 carries both helicity and rotational-transport content, but its four components do not resolve these into separate coordinates: as the explicit channel form of $\not{U}\not{H}$ shows, each Λ^3 component is already a fixed linear combination of $\mathbf{\Omega}$, $\mathbf{\Pi}$ and U^μ, so helicity and rotational transport remain genuinely coupled within Λ^3 rather than occupying distinct components of it. The same caution applies wherever a single grade in the table above is listed as carrying more than one named physical effect.

5.8.4 Rapidity–rotor and Maurer–Cartan representation

The Weyl formulation further permits the kinematic four-velocity to be generated from a local directed rapidity field rather than taken as the first algebraic object. Let

$$\boldsymbol{\psi}(x) = \psi(x)\,\hat{\mathbf{n}}(x) \tag{524}$$

be the local directed rapidity and define the corresponding boost rotor

$$\boxed{Q_\psi(x) = \exp\left[\frac{1}{2}\boldsymbol{\psi}(x)\cdot\boldsymbol{\alpha}\right].} \tag{525}$$

The fluid four-velocity is then generated by

$$\boxed{\not{U} = cQ_\psi\hat{\beta}_0 Q_\psi^{-1}.} \tag{526}$$

For a fixed local direction this reproduces

$$\not{U} = c\cosh\psi\,\hat{\beta}_0 + c\sinh\psi\,\hat{\mathbf{n}}\cdot\boldsymbol{\beta}, \tag{527}$$

and hence

$$\gamma = \cosh\psi, \qquad \frac{v}{c} = \tanh\psi. \tag{528}$$

The RFD field hierarchy can therefore be rooted one level lower in the local rapidity field,

$$\boxed{\boldsymbol{\psi} \longrightarrow Q_\psi \longrightarrow \not{U} \longrightarrow \not{G} \longrightarrow \not{H} \longrightarrow \left\{\not{U}\not{H},\, \not{\partial}\not{H},\, \not{H}^2,\, \not{T}_H\right\}.} \tag{529}$$

Because $Q_\psi(x)$ is now a local rotor field, its spacetime variation defines the Maurer–Cartan connection

$$\boxed{\not{\Omega}_\mu^{(\psi)} = (\partial_\mu Q_\psi)Q_\psi^{-1} \in \Lambda^2.} \tag{530}$$

Differentiating the rotor representation of $\not{U}$ gives the corresponding transport equation

$$\boxed{\partial_\mu\not{U} = [\not{\Omega}_\mu^{(\psi)}, \not{U}].} \tag{531}$$

Along a fluid worldline this reduces to

$$\boxed{\not{\Omega}_\tau^{(\psi)} = \left(\frac{DQ_\psi}{D\tau}\right) Q_\psi^{-1}, \qquad \frac{D\not{U}}{D\tau} = [\not{\Omega}_\tau^{(\psi)}, \not{U}].} \tag{532}$$

The Weyl formulation consequently admits two parallel constructions generated by the same local rotor:

$$\boxed{\not{\psi} \longrightarrow Q_\psi \begin{cases} \longrightarrow \not{U} \longrightarrow \not{G} \longrightarrow \not{H} \longrightarrow \left\{ \not{U}\not{H},\, \not{\partial}\not{H},\, \not{H}^2,\, \not{T}_H \right\}, \\ \longrightarrow \not{\Omega}_\mu^{(\psi)} \longrightarrow \not{\Omega}_\tau^{(\psi)} \longrightarrow \dfrac{D\not{U}}{D\tau}. \end{cases}} \tag{533}$$

The upper branch is the BQ–Weyl representation of the established RFD field hierarchy; the lower branch expresses the transport of the same locally generated four-velocity through the connection of its rapidity rotor.

At the present RFD level, the rapidity–rotor–connection construction should therefore be understood as a structural reformulation rather than as an additional fluid-dynamical law. It does, however, make explicit a hierarchy that is not visible when $\not{U}$ is simply taken as the starting field:

$$\boxed{\text{local rapidity} \longrightarrow \text{rotor} \longrightarrow \begin{cases} \text{fluid kinematics and field hierarchy,} \\ \text{local Lorentz transport.} \end{cases}} \tag{534}$$

The BQ–Weyl lift thus preserves the fluid physics established in de Haas (2026a), while resolving that physics into the $1 + 4 + 6 + 4 + 1$ Clifford hierarchy and connecting the kinematic four-velocity to the local $\mathrm{Spin}(1,3)$ rotor from which both $\not{U}$ and its Maurer–Cartan transport may be generated.

6 Conclusion

This work has lifted the BQ–Pauli formulations of electromagnetism and relativistic fluid dynamics to the sixteen-dimensional BQ–Weyl representation of $\mathrm{Cl}(1,3)$. The Pauli $1 + 3 + 3 + 1$ channel structure is thereby resolved into the Clifford decomposition

$$1 + 4 + 6 + 4 + 1, \tag{535}$$

while the coefficients and physical content of the original BQ products are preserved. The lift thus makes explicit the Clifford grades that remain compressed at the Pauli level.

For electromagnetism, the Weyl representation recovers Maxwell's equations from the field-derivative product and distinguishes directly between the quadratic field invariants and the vector-valued stress–energy map. For relativistic fluid dynamics, the corresponding lift resolves the field, force, wave, quadratic, and conservation products into their respective Clifford sectors. These results provide the Weyl-level counterpart of the BQ–Pauli RFD hierarchy developed previously de Haas (2026a), rather than introducing an independent fluid dynamics.

The Weyl formulation further allows the fluid four-velocity to be generated from a local rapidity field through a $\mathrm{Spin}(1,3)$ rotor. The same rotor generates the associated comoving frame and Maurer–Cartan connection, placing the kinematic and local-frame transport descriptions within a common algebraic hierarchy. At the present level this extends the algebraic organization of the RFD description, but does not by itself introduce an additional dynamical law.

Together with the corresponding BQ–Pauli to BQ–Weyl lift already performed for gravity de Haas (2026c,b), the present results establish a common $\mathrm{Cl}(1,3)$ representation for the gravitational, electromagnetic, and fluid-dynamical branches of the BQ programme. This common representation permits their previously developed Pauli-level product structures to be compared within a single Clifford–Weyl algebraic framework.

Two natural extensions remain open: whether the rapidity-generated Maurer–Cartan structure can acquire a dynamical role beyond the present RFD reformulation, and whether the Weyl stress–energy construction can be extended explicitly to the corresponding local energy–momentum balance relation.

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A General vector products in the BQ-algebra

A.1 General vector products in the BQ-Pauli representation

Let

$$A = a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}, \qquad B = b_0\hat{T} + \mathbf{b}\cdot\hat{\mathbf{K}}, \tag{536}$$

and similarly

$$D = d_0\hat{T} + \mathbf{d}\cdot\hat{\mathbf{K}}, \qquad F = f_0\hat{T} + \mathbf{f}\cdot\hat{\mathbf{K}}, \tag{537}$$

be arbitrary space-time minquats.

Unlike the sixteen-dimensional Weyl representation, the eight-dimensional BQ-Pauli algebra has no exterior grading to organise the product hierarchy; instead, successive products are organised by the minquat/ pauliquat split itself. A minquat carries only the $\{\hat{T}, \hat{\mathbf{K}}\}$ channels; the bilinear product of two minquats already populates the complementary $\{\hat{1}, \hat{\boldsymbol{\sigma}}\}$ channels together with $\hat{\mathbf{K}}$, and the triple product returns content to every one of the eight channels at once – the BQ-Pauli algebra does not separate into alternating even/odd sectors the way the Weyl algebra does.

A.1.1 Bilinear product

$$\boxed{C = A^T B = c_1 \hat{1} + \mathbf{c}_K \cdot \hat{\mathbf{K}} + \mathbf{c}_\sigma \cdot \hat{\boldsymbol{\sigma}},} \tag{538}$$

where

$$c_1 = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}, \tag{539}$$

$$\mathbf{c}_K = \mathbf{a} \times \mathbf{b}, \tag{540}$$

$$\mathbf{c}_\sigma = a_0 \mathbf{b} - b_0 \mathbf{a}. \tag{541}$$

No $\hat{T}$-channel is generated: the bilinear of two minquats populates exactly the scalar, $\hat{\mathbf{K}}$, and $\hat{\boldsymbol{\sigma}}$ channels, verified by exact symbolic expansion and reconstruction on the eight-element basis.

A.1.2 Triple product

$$\boxed{E = DC = e_1 \hat{1} + e_T \hat{T} + \mathbf{e}_K \cdot \hat{\mathbf{K}} + \mathbf{e}_\sigma \cdot \hat{\boldsymbol{\sigma}},} \tag{542}$$

with

$$e_1 = -\mathbf{d} \cdot \mathbf{c}_K, \tag{543}$$

$$e_T = d_0 c_1 + \mathbf{d} \cdot \mathbf{c}_\sigma, \tag{544}$$

$$\mathbf{e}_K = \mathbf{d}\, c_1 + d_0 \mathbf{c}_\sigma + \mathbf{d} \times \mathbf{c}_K, \tag{545}$$

$$\mathbf{e}_\sigma = \mathbf{d} \times \mathbf{c}_\sigma - d_0 \mathbf{c}_K. \tag{546}$$

The triple product of three minquats populates all eight channels of the algebra – the $\hat{T}$-channel, absent from C, is regenerated here, so that E is a fully general BQ element rather than one confined to a proper subset of channels, verified by exact symbolic expansion and reconstruction.

A.1.3 Quadruple product

$$\boxed{G = F^T E = g_1 \hat{1} + g_T \hat{T} + \mathbf{g}_K \cdot \hat{\mathbf{K}} + \mathbf{g}_\sigma \cdot \hat{\boldsymbol{\sigma}},} \tag{547}$$

with

$$g_1 = f_0 e_T - \mathbf{f} \cdot \mathbf{e}_K, \tag{548}$$

$$g_T = -f_0 e_1 + \mathbf{f} \cdot \mathbf{e}_\sigma, \tag{549}$$

$$\mathbf{g}_K = -f_0 \mathbf{e}_\sigma + \mathbf{f}\, e_1 + \mathbf{f} \times \mathbf{e}_K, \tag{550}$$

$$\mathbf{g}_\sigma = f_0 \mathbf{e}_K - \mathbf{f}\, e_T + \mathbf{f} \times \mathbf{e}_\sigma. \tag{551}$$

As with the triple product, all eight channels remain populated in general: the BQ-Pauli algebra, having no exterior-algebra grading to terminate the hierarchy, does not exhibit an analogue of the Weyl-level alternation between the even and odd subalgebras. All three decompositions were verified by exact symbolic expansion and reconstruction on the eight-element BQ basis, with identically zero residuals.

A.2 General vector products in the Weyl representation

Let

$$\not{A} = a_0\beta_0 + \mathbf{a}\cdot\boldsymbol{\beta}, \qquad \not{B} = b_0\beta_0 + \mathbf{b}\cdot\boldsymbol{\beta}, \tag{552}$$

and similarly

$$\not{D} = d_0\beta_0 + \mathbf{d}\cdot\boldsymbol{\beta}, \qquad \not{F} = f_0\beta_0 + \mathbf{f}\cdot\boldsymbol{\beta}, \tag{553}$$

be arbitrary space-time vectors in the Weyl basis.

The following results are purely algebraic consequences of the Clifford grading. Products of an even (odd) number of vectors belong to the even (odd) subalgebra,

$$\Lambda^0 \to \Lambda^1 \to (\Lambda^0 \oplus \Lambda^2) \to (\Lambda^1 \oplus \Lambda^3) \to (\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4), \tag{554}$$

which, in the BQ basis, becomes

$$\not{1} \to \beta_\mu \to \{\not{1}, \boldsymbol{\alpha}, \boldsymbol{\Sigma}\} \to \{\beta_\mu, \tilde{\beta}_\mu\} \to \{\not{1}, \boldsymbol{\alpha}, \boldsymbol{\Sigma}, \chi\}. \tag{555}$$

A.2.1 Bilinear product

The product of two vectors contains only scalar and bivector components,

$$\boxed{\not{C} = \not{A}\not{B} = c_1\not{1} + \mathbf{c}_\alpha\cdot\boldsymbol{\alpha} + \mathbf{c}_\Sigma\cdot\boldsymbol{\Sigma},} \tag{556}$$

where

$$c_1 = -a_0 b_0 + \mathbf{a}\cdot\mathbf{b}, \tag{557}$$

$$\mathbf{c}_\alpha = a_0\mathbf{b} - \mathbf{a}\, b_0, \tag{558}$$

$$\mathbf{c}_\Sigma = \mathbf{a}\times\mathbf{b}. \tag{559}$$

Thus

$$V\,V = \Lambda^0 \oplus \Lambda^2.$$

A.2.2 Triple product

Multiplication by a third vector gives

$$\boxed{\not{E} = \not{D}(\not{A}\not{B}) = K_0\beta_0 + \mathbf{K}\cdot\boldsymbol{\beta} + R_0\tilde{\beta}_0 + \mathbf{R}\cdot\tilde{\boldsymbol{\beta}},} \tag{560}$$

with

$$K_0 = d_0 c_1 - \mathbf{d}\cdot\mathbf{c}_\alpha, \tag{561}$$
$$\mathbf{K} = \mathbf{d}\,c_1 - d_0\mathbf{c}_\alpha - \mathbf{d}\times\mathbf{c}_\Sigma, \tag{562}$$
$$R_0 = -\mathbf{d}\cdot\mathbf{c}_\Sigma, \tag{563}$$
$$\mathbf{R} = -d_0\mathbf{c}_\Sigma + \mathbf{d}\times\mathbf{c}_\alpha. \tag{564}$$

Thus

$$V(\Lambda^0 \oplus \Lambda^2) = \Lambda^1 \oplus \Lambda^3.$$

A.2.3 Quadruple product

Multiplication by a fourth vector returns the result to the even subalgebra,

$$\boxed{\not{G} = \not{F}\not{D}(\not{A}\not{B}) = g_1\not{1} + \mathbf{g}_\alpha\cdot\boldsymbol{\alpha} + \mathbf{g}_\Sigma\cdot\boldsymbol{\Sigma} + g_\chi\chi,} \tag{565}$$

where

$$g_1 = -f_0K_0 + \mathbf{f}\cdot\mathbf{K}, \tag{566}$$
$$\mathbf{g}_\alpha = -\mathbf{f}K_0 + f_0\mathbf{K} - \mathbf{f}\times\mathbf{R}, \tag{567}$$
$$\mathbf{g}_\Sigma = f_0\mathbf{R} - \mathbf{f}R_0 + \mathbf{f}\times\mathbf{K}, \tag{568}$$
$$g_\chi = f_0R_0 - \mathbf{f}\cdot\mathbf{R}. \tag{569}$$

Hence

$$(\Lambda^1 \oplus \Lambda^3)V = \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4,$$

so that the pseudoscalar χ appears for the first time only in a product containing four independent vectors.

All three decompositions were verified by exact symbolic expansion and reconstruction on the sixteen-element Weyl basis, with identically zero residuals.

B Reproducing the BQ calculations with computational tools

The BQ–Pauli and BQ–Weyl constructions used in this work are given as explicit complex matrix representations. Consequently, the algebraic derivations can be reproduced independently, without dedicated BQ software, using a computer-algebra system, a programming language with symbolic matrix support, or an AI-assisted mathematical environment capable of executing such calculations.

A reference implementation of the basis matrices, multiplication table, and principal algebraic checks is provided as a companion script **?**, archived with a persistent identifier. The script is intended as a reproducibility aid and independent implementation of the definitions in this paper; the mathematical definitions and identities stated in the paper remain authoritative. This appendix gives a compact procedure both for reproducing individual results and for using the same matrix representation to investigate further products.

B.1 Starting point

Any reproduction should begin from the explicit BQ–Pauli and BQ–Weyl matrices defined in the main text. At Pauli level these are the matrices $\hat{1}, \hat{T}, \hat{I}, \hat{J}, \hat{K}$ and the complementary $\hat{\sigma}_i$ set, with

$$\hat{K}_i = \hat{T}\hat{\sigma}_i. \tag{570}$$

At Weyl level the sixteen matrices form the decomposition

$$\boxed{\{\mathbb{1}\} \oplus \{\beta_\mu\} \oplus \{\alpha_i, \Sigma_i\} \oplus \{\tilde{\beta}_\mu\} \oplus \{\chi\},} \tag{571}$$

corresponding to

$$\Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \Lambda^3 \oplus \Lambda^4. \tag{572}$$

The basic relations

$$\beta_0^2 = -\mathbb{1}, \qquad \beta_i^2 = +\mathbb{1}, \tag{573}$$

$$\beta_0\beta_i = \alpha_i, \qquad \beta_j\beta_k = \Sigma_i \quad (i, j, k \text{ cyclic}), \tag{574}$$

and

$$\boxed{\beta_0\beta_I\beta_J\beta_K = -\chi} \tag{575}$$

should first be reproduced directly by matrix multiplication. These provide a minimal consistency test of the implementation before the field products are evaluated.

The matrices should be entered from their definitions in the paper, rather than reconstructed from remembered Pauli, Dirac, STA, or GA conventions. This makes any subsequent discrepancy traceable to a specific definition or computational step.

B.2 General principles

The following principles apply to symbolic, numerical, and AI-assisted reproduction of the calculations in this work:

1. **Use the explicit matrices.** All BQ products in this work are ordinary complex matrix products. No additional multiplication operation is to be introduced.
2. **Execute the calculation.** Where verification is intended, the matrix multiplication should be carried out by an exact symbolic engine whenever possible. Algebra generated only as natural-language or unexecuted symbolic reasoning should not be treated as computational verification.

3. **Preserve the conventions.** Keep the basis ordering (I, J, K), all factors of i, the signature
$$\beta_0^2 = -\mathbb{1}, \qquad \beta_i^2 = +\mathbb{1}, \tag{576}$$
and the orientation
$$\chi = -\beta_0\beta_I\beta_J\beta_K \tag{577}$$
exactly as defined in the paper.
4. **Do not translate before calculating.** Standard Pauli, Dirac, Clifford, STA, or GA conventions may be used for comparison, but they should not replace the conventions of the calculation being checked.
5. **Keep algebraic levels distinct.** BQ–Pauli objects and their BQ–Weyl lifts should not be identified merely because they contain corresponding coefficients. The explicit lift defined in the main text should be followed.
6. **Respect the defined involutions and adjoints.** Operations denoted by T, P, PT, Hermitian conjugation, or other adjoints should be implemented according to their explicit definitions in the paper and should not be replaced by a conventionally similar operation.
7. **Do not repair disagreements silently.** If a calculation disagrees with a stated equation, retain the result and report the discrepancy together with the definitions used. Check the basis ordering, signature, orientation, factors of i, derivative action, and adjoint before proposing a correction.

B.3 Recommended verification procedure

For an algebraic identity in the paper, the recommended procedure is:

1. express each BQ object in the explicitly defined matrix basis, retaining symbolic coefficients for its physical components;
2. perform the required matrix products exactly, without first imposing physical simplifications;
3. where differential operators occur, specify explicitly which factors they act upon and apply the corresponding product rule before simplifying;
4. expand and simplify the resulting matrix element by element;
5. decompose the result back onto the stated BQ basis and, at Weyl level, identify the Clifford grades present;
6. calculate independently the matrix representing the stated result and form the difference between the two matrices;
7. simplify this difference symbolically; an exact verification requires the zero matrix;
8. only after the algebraic identity has been established, compare its coefficients with their physical interpretation or with a standard formulation.

The procedure deliberately separates three levels:

$$\boxed{\text{matrix identity} \;\longrightarrow\; \text{BQ/Clifford decomposition} \;\longrightarrow\; \text{physical interpretation.}} \tag{578}$$

A correspondence at the third level does not by itself establish an identity at the first.

B.4 Representative reproduction tasks

The procedure can be tested on calculations of increasing complexity. Representative tasks include:

1. reconstructing the complete multiplication table of the sixteen BQ–Weyl basis matrices from ordinary matrix multiplication;
2. verifying the Clifford anticommutation relations of the β_μ generators and the duality relations generated by χ;
3. reconstructing the electromagnetic field product from $\not{\partial}$ and $\not{A}$ and verifying its decomposition on the Weyl bivector basis;
4. calculating the quadratic electromagnetic products and separating their scalar and pseudoscalar channels;
5. calculating $\not{B}\, a\, \not{B}$ for a general Weyl vector $a = a^\mu \beta_\mu$, decomposing the result on the vector basis, and checking its Lorentz transformation law;
6. reconstructing $\not{H} = \not{\partial}\not{G}$ and verifying its $\Lambda^0 \oplus \Lambda^2$ decomposition;
7. calculating $\not{U}\not{H}$ and separating its Λ^1 and Λ^3 components;
8. calculating $\not{\partial}\not{H}$ and checking explicitly the cancellations derived in the RFD section;
9. reconstructing the local rapidity rotor and calculating its action on the reference timelike vector and the associated Maurer–Cartan connection.

Once the calculations derived in the paper have been reproduced, the same procedure may be applied to products not considered here. Such calculations should be identified as extensions of the paper rather than as reproductions of its results.

B.5 Suggested instruction for an AI-assisted environment

When the paper itself is supplied to an AI-assisted mathematical environment, the following compact instruction can be used:

> *Use the explicit BQ matrices and conventions defined in this paper as the starting point and do not substitute external Pauli, Dirac, Clifford, STA, or GA conventions. Compute the requested product by executed symbolic matrix multiplication. Preserve the (I, J, K) basis ordering, metric signature, orientation, factors of i, and the defined adjoint operations. Where differential operators occur, state explicitly what they act upon and apply the product rule. Decompose the resulting matrix on the stated BQ basis and identify its Clifford grades. Verify a claimed identity by simplifying the matrix difference between its two sides to the zero matrix. Distinguish exact symbolic verification from numerical testing. If a discrepancy remains, report it explicitly rather than changing the conventions to obtain the expected result. Only after completing the internal BQ calculation should the result be translated into another formalism or physically interpreted.*

A specific task can then be appended, for example:

Verify Eq. [Y] of the paper. Show the basis decomposition of the result and identify each non-vanishing Clifford grade.

B.6 Comparison with standard formulations

Comparison with standard Pauli, Dirac, Clifford, or spacetime-algebra notation should follow, rather than precede, internal verification. A suitable second-stage instruction is:

Having verified the result using the BQ definitions of the paper, translate it into the requested standard notation. State explicitly the basis identification, metric-signature convention, orientation, factors of i, normalization, and relevant adjoint operation. Separate a mere change of representation or notation from any difference in the mathematical content.

This ordering is important because superficially similar matrix and Clifford conventions can differ in signature, orientation, basis ordering, factors of i, and adjoint definitions. Translating only after the internal calculation makes clear which properties follow from the BQ construction and which arise from the chosen target representation.

B.7 Exact, numerical, and AI-assisted checks

Three levels of computational checking should be distinguished. An *exact symbolic verification* reduces the difference between the two sides of an identity to the zero matrix for symbolic coefficients. A *numerical check* evaluates the same difference for specified numerical values and should report the resulting residual. An *AI-assisted derivation* may organize, generate, or interpret either calculation, but its textual agreement with an expected result does not constitute an independent verification.

For the central algebraic results of this paper, exact symbolic matrix evaluation from the stated definitions is therefore preferred. Conversely, a non-zero result should not automatically be interpreted as an error in either the paper or the computational tool. The basis ordering, metric signature, orientation, factors of i, derivative action, and choice of adjoint should first be checked, since these are common sources of apparent discrepancies.

The practical purpose of this appendix is therefore to make the BQ formalism directly reproducible from its matrix definitions. The same workflow can be implemented in different computational environments, while the resulting identities remain independent of the particular software or AI system used.