Non-Associativity Implies Intrinsic Open Quantum Dynamics:
The Ultramark Sector, Octonionic Decoherence,
and the Gallium/MiniBooNE Anomalies

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Abstract

Main mathematical result. The exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$ (the Albert algebra, 27-dimensional) admits the associative collapse map $\pi_0 : \mathcal{J}_3(\mathbb{O}) \to \mathcal{J}_3(\mathbb{R})$ with a 21-dimensional kernel. The kernel carries precisely the imaginary octonionic degrees of freedom that make the associator non-zero. We show, within a strongly motivated EFT framework, that when these degrees of freedom are activated (non-zero associator), the natural consistent open-system dynamics on the residual associative sector $\mathcal{J}_3(\mathbb{R})$ takes the Lindblad GKSL form whose jump operators are the Fano-plane associators $L_k = \sqrt{\lambda}[A_k, B_k, C_k]$. The chain:

$$\mathcal{J}_3(\mathbb{O}) \xrightarrow{\pi_0} \mathcal{J}_3(\mathbb{R}) \xrightarrow{[A,B,C]\neq 0} \text{Lindblad decoherence}$$

is grounded in the representation theory of $G_2 = \mathrm{Aut}(\mathbb{O})$ and the Nakajima–Zwanzig open-system framework [1]. Complete positivity is verified from the G_2-symmetry of the Fano plane (the jump operators satisfy $\sum_k L_k^\dagger L_k = 28\lambda\,\mathbb{I}$, the completeness relation for a CP map).

Physical application. We develop **non-associative topological neutrino dynamics** (NATND), in which the framework above is applied to neutrino propagation through the octonionic *ultramark* condensate (where by "ultramark" we mean the level-6 constituent of the hierarchical *level model* of the present author [2]). The decoherence rate $\Gamma(E) = g_U^2 E^2/(M_\mathrm{U}^2 + E^2)/M_\mathrm{U}$ is a strongly motivated EFT consequence of the derivative-coupling structure of the associator vertex. It provides an alternative to sterile-neutrino oscillations for the *gallium anomaly* (GALLEX, SAGE, BEST) and the MiniBooNE excess, with the key signature that the deficit is baseline-independent (no L/E pattern) and energy-dependent as $E^2/(M_\mathrm{U}^2 + E^2)$. The ultramark mass coincides numerically with the Totani Fermi-halo excess.

Contents

1 Introduction

Several anomalies in neutrino physics challenge the standard three-flavour oscillation framework. The *gallium anomaly*—a deficit in the electron-neutrino detection rate observed in the calibration experiments GALLEX [4], SAGE [5], and most recently BEST [6, 7]—has persisted at 4–5σ and remains unexplained by conventional mechanisms. The measured ratio

$$R = \frac{N_{\text{obs}}}{N_{\text{pred}}} \approx 0.87 \pm 0.05$$

implies a systematic $\approx 13\%$ deficit of ν_e interactions in gallium. Crucially, BEST measured the deficit simultaneously at two radii (inner vessel $\approx 0.5\,\text{m}$, outer vessel $\approx 1.5\,\text{m}$) and found consistent suppression at both distances, disfavouring a simple oscillation interpretation with $\Delta m^2 \sim 1\text{–}2\,\text{eV}^2$.

In parallel, an independent astrophysical anomaly has recently emerged: Totani [3] reported a statistically significant gamma-ray excess in 15 years of Fermi-LAT data from the Milky Way halo, with a **spectral peak near 20 GeV**. The morphology is consistent with dark matter (DM) annihilation following an NFW profile, though the required cross section exceeds canonical thermal relic values.

In this paper we show that both anomalies receive a natural interpretation within the hierarchical level model introduced in [2]. The key ingredient is the *ultramark*: a level-6 constituent whose mass, derived from the Feigenbaum cascade of the ORC+torsion flow, is $M_{\text{U}} \approx 20.55\,\text{GeV}$. Furthermore, the octonionic structure of level-6 time implies a Lindblad-type decoherence term in the neutrino evolution equation, which modifies the effective ν_e interaction cross section consistently with the gallium observations.

The paper is organised as follows.
Section 2 reviews the level model and introduces level 6.
Section 3 derives the ultramark mass and discusses its connection to the Fermi halo.
Section 4 presents the octonionic decoherence framework and its application to the gallium anomaly.
Section 5 provides the rigorous mathematical foundations of non-associative quantum mechanics, grounding the decoherence mechanism in the published literature. Two independent derivations of $\Gamma(E)$ are presented: a QFT route via the optical theorem (section 5.5) and an open-system Born–Markov route via the ultramark bath correlator (section 5.6); the algebraic decoherence switch (32)–(33) is formalised; and the four physical validity conditions (complete positivity, trace preservation, CPT consistency, entropy production) are verified from the G_2-symmetry of $\mathbb{O}$ (section 5.7).
Section 6 extends the framework to MiniBooNE and MicroBooNE.
Section 7 formulates falsifiable predictions.
Section 8 proposes neutrinos as collective topological supermark states and addresses precision constraints.
Section 9 details neutrino oscillation via ultramark exchange, exploring the topological

interpretation of flavor, the underlying Lagrangian for neutrino-ultramark coupling, and its numerical constraints under dual propagation regimes.

Section 10 extends the level model framework to broader BSM applications, including PeV cosmic neutrinos, the muon anomalous magnetic moment $(g-2)$, the ANITA anomaly, the Cabibbo-Majorana connection, and the formal structure of emergent flavor.

Section 11 provides a systematic mathematical verification and maps the model within the current octonionic literature, evaluating the Cayley-Dickson cascade, multi-scale consistency of Lindblad decoherence, and the status of remaining open issues.

Appendix A develops the cosmological interpretation of the level cascade as successive emergent algebraic phases. It is presented as a *speculative interpretive framework*, not a derivation, and includes the sigmoid $\alpha_{\text{cosmo}}(\tau)$ as order parameter of the Big Wow transition.

Appendix B analyzes a global statistical fit.

Appendix C and Appendix D discuss open problems, appendix E concludes.

2 The Level Model: Extension to Level 6

Remark 2.1 (Speculative status of the level model)**.** The level model is a **speculative, non-mainstream theoretical framework**. Its constituent counts, hidden-quark hypotheses, and hadronic mass predictions are working conjectures, not derivations from established physics. Readers should approach this section with appropriate scepticism. The core physics of this paper — non-associative Lindblad decoherence, the gallium and MiniBooNE predictions, and the Jordan-algebra chain $\mathcal{J}_3(\mathbb{O}) \to \mathcal{J}_3(\mathbb{R}) \to$ GKSL — is *mathematically independent* of the level-model numerology and stands on its own merits. The level model provides the physical context for introducing the ultramark sector; the contact with mainstream physics occurs in sections 4 and 6 to 8 and onwards.

2.1 Review of levels 0–5

The level model [2] is a hierarchical spectrum of fundamental constituents derived from a discrete geometric model of spacetime emergence. Before listing the levels, we state two conceptual principles that govern the entire framework:

Masses as moduli.

The mass formula $M_n = m_e/(4\alpha)^{n-3}$ gives the *modulus* (absolute value) of the mass in the normed division algebra at level n: a real number at level 3 ($\mathbb{R}$), the modulus of a complex number at level 4 ($\mathbb{C}$), the norm of a quaternion at level 5 ($\mathbb{H}$), and the norm of an octonion at level 6 ($\mathbb{O}$). What a 4D observer perceives as the "mass" of a particle is the *real projection* of this algebraic mass vector. The modulus is fixed by the Feigenbaum cascade; the orientation (and hence the observable mass) is determined by the dynamics of the bound state.

Equivalences are dynamic, not mass-equalities.

When we write "an electron is equivalent to two marks," this means the *reaction* $e^- \leftrightarrow \text{mark} + \overline{\text{mark}}$ is kinematically allowed in the algebraic sense (quantum numbers balance). It does *not* mean masses are equal: $m_e = 0.511\,\text{MeV}$ while $M_{\text{mark}} = 17.5\,\text{MeV}$. The mass discrepancy is accounted for by the orientation of the complex mass vector: in the leptonic (destructive) configuration, the two marks' complex phases cancel to leave only m_e; in the hadronic (constructive) configuration, the same two marks contribute $2 \times 17.5\,\text{MeV}$ to a visible hadronic state.

The key structural results relevant here are:

(i) **Mass scaling.** $M_n = m_e/(4\alpha)^{n-3}$ for $n \geq 3$, where $\alpha \approx 1/137.036$ is the fine-structure constant and $4\alpha = 2/(\pi\delta_F^2) \approx 0.02919$, with $\delta_F \approx 4.6692$ the Feigenbaum period-doubling constant.

(ii) **Spin halving.** $\text{Spin}_n = 2^{1-n}$.

(iii) **Dimension doubling.** $\text{Dim}_n = 2^n$.

(iv) **Cayley–Dickson algebraic doubling of mass and time.** Level 3: mass and time $\in \mathbb{R}$; level 4: $\in \mathbb{C}$; level 5: $\in \mathbb{H}$ (quaternions); level 6: $\in \mathbb{O}$ (octonions).

(v) **Charge structure.** $q_{\text{em}} = 2^{2-n}$ for $n \geq 3$. Strong charge $q_s = 1$ first appears at level 4 and halves at each level. Weak charge $q_w = 1$ first appears at level 5.

Remark 2.2 (Methodological clarifications on dimensionality)**.** We advise the reader to distinguish between two distinct types of dimensions throughout the level hierarchy:

(1) **Algebraic vs. spacetime dimensionality.** The progression of Cayley–Dickson algebras ($\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$) refers to the algebraic structure governing the mass-time vector space at each level and should not be conflated with the dimensionality of the physical spacetime manifold. For instance, while level 4 introduces complex numbers ($\mathbb{C}$) as the algebraic base for the mass-time vector, this does not imply a 2-dimensional spacetime; conversely, the spacetime dimension $\text{Dim}_n = 2^n$ doubles independently at each level according to its own rule. The two doublings—algebraic ($\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$) and geometric ($\text{Dim}_n = 2^n$)—are parallel structural features of the Cayley–Dickson cascade, not the same quantity expressed twice.

(2) **Mathematical space vs. physical reality.** The "spacetime dimensionality discussed in this framework refers to the auxiliary mathematical space required for the consistency of the field equations at a given level. Such constructions are essential for the formal definition of the dynamical system but are not necessarily intended as a literal description of the physical manifold observable at macroscopic scales. In particular, the 32-dimensional spacetime invoked at level 6 is a mathematical scaffold for the octonionic algebra of the ultramark sector; it does not imply that 32 macroscopic spatial directions are observable to a 4D observer.

2.2 Level 6: the ultramark sector

Following the pattern, level 6—the *ultramark*—is characterised by:

Spacetime dimension:

$\mathrm{Dim} = 2^5 = 32.$

Spin: $\mathrm{Spin}_6 = 2^{1-5} = 1/16$. The observable real component seen by 4D observers is 1/16; the remaining 7/8 of the octonionic wavefunction carries imaginary time components inaccessible to 4D measurement.

Charges:

$q_{\mathrm{em}} = 1/8$, $q_s = 1/4$, $q_w = 1/2$, $q_u = 1$, where q_u denotes an *ultra-weak charge* first appearing at level 6.

Mass and time:

Octonionic ($\in \mathbb{O}$). The 8-dimensional octonionic algebra is non-associative: the associator $[a, b, c] = (ab)c - a(bc) \neq 0$ in general. This non-associativity has direct physical consequences for quantum evolution (see section 4).

Table 1 summarises the complete level spectrum.

Table 1: The hierarchical level spectrum including the ultramark (level 6, bold row). Each level doubles the spacetime dimension, halves the spin, and scales the mass by $4\alpha \approx 0.0292$. q_u is the ultra-weak charge first appearing at level 6.

Lv.	Particle	Dim	Spin	q_{em}	q_s	q_w	q_u	Mass / structure
0	Substrate	<1	—	0	0	0	0	0 / combinatorial
1	Scalar graviton	1	0	0	0	0	0	0 (massless)
2	Photon	2	1	0	0	0	0	0 (massless)
3	Electron/positron	4	$\frac{1}{2}$	1	0	0	0	$m_e = 0.511\,\mathrm{MeV}$ ($\mathbb{R}$)
4	Mark	8	$\frac{1}{4}$	$\frac{1}{2}$	1	0	0	$m_e/(4\alpha) \approx 17.51\,\mathrm{MeV}$ ($\mathbb{C}$)
5	Supermark	16	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	0	$m_e/(4\alpha)^2 \approx 600\,\mathrm{MeV}$ ($\mathbb{H}$)
6	**Ultramark**	**32**	$\boldsymbol{\frac{1}{16}}$	$\boldsymbol{\frac{1}{8}}$	$\boldsymbol{\frac{1}{4}}$	$\boldsymbol{\frac{1}{2}}$	**1**	$\boldsymbol{m_e/(4\alpha)^3 \approx 20.55\,\mathrm{GeV}}$ ($\mathbb{O}$)

The holographic spin invariant $D \cdot S = 2$. Inspection of the level table reveals a remarkable structural identity: at every level $n \geq 3$, the product of the spacetime dimension D_n and the fundamental spin S_n is constant:

$$D_n \times S_n \;=\; 4 \times \tfrac{1}{2} \;=\; 8 \times \tfrac{1}{4} \;=\; 16 \times \tfrac{1}{8} \;=\; 32 \times \tfrac{1}{16} \;=\; 2. \tag{1}$$

We call (1) the *holographic spin invariant*. It is not an input to the model but an emergent consequence of the Cayley–Dickson doubling rule: each level doubles the spacetime dimension ($D_n = 2^n$) and, to preserve the total *informational spin density* of the network, the fundamental spin halves ($S_n = 2^{2-n}$).

The informational interpretation is as follows. In the framework of Sung-Sik Lee [70], spacetime itself emerges as an extra renormalisation-group coordinate of a many-body quantum system; the geometry carries an information capacity proportional to D. In the framework of Terazawa [69], the spacetime dimension is a dynamical condensate variable that increases at each phase transition. Combining the two: when D doubles at a Cayley–Dickson transition, the "informational bandwidth" of the geometry doubles, while the "quantum granularity" of each matter excitation (its spin S) must halve to keep the product—and thus the total information per node—constant.

The invariant (1) rationalises three seemingly independent features of the level table:

- Why spin is 1/16 at level 6.
- Why the fractional charges halve at each level: $q_w = 1/2^{n-3}$ for levels $n = 3$–6.
- Why the cascade terminates at level 6: a putative level 7 would require spin 1/32 in 64 dimensions (level 7 would have $2^6 = 64$ dimensions), but 64-dimensional spacetime with zero-divisor sedenions has no consistent probabilistic interpretation (Hurwitz theorem).

A formal derivation of (1) from the ORC+torsion action requires the Clifford-algebra analysis of Terazawa [69] applied to the discrete graph setting; this is left to future work. **Remark 2.3** ($\mathcal{N} = 4$ SYM as the zero-associator limit)**.** The level table makes the algebraic status of $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) transparent. SYM is the unique maximally supersymmetric, massless, conformally invariant gauge theory in four dimensions. In the level model, these properties are the exact signature of the *zero-associator limit*: the sector in which the octonionic associator is artificially suppressed $[a, b, c]_{\mathbb{O}} = 0$. In this limit: all particles are massless (because mass generation requires non-vanishing associator, as the norm-deficit $\Delta N(X, Y) = |XY|^2 - |X|^2|Y|^2$ vanishes when $[a, b, c] = 0$); the beta function vanishes (no running coupling because the Cayley–Dickson cascade that breaks conformal symmetry has not been initiated); and the full supersymmetry is preserved. SYM therefore lives entirely within the associative levels 3–5 of the hierarchy. The Amplituhedron of Arkani-Hamed and Trnka [62] — the positive Grassmannian encoding SYM scattering amplitudes — is the geometric realisation of this associative sector. Myo Oo et al. [61] demonstrate that the positive Grassmannian $G_+(k, n)$ is the associative collapse $\pi_0(\mathcal{J}_3^+(\mathbb{O}))$ of the positive cone of the Albert algebra; in the level model, this corresponds to the projection of level-6 physics onto the levels 3–5 observable sector. The full ultramark sector (level 6, octonionic, massive, non-associative) is precisely the structure that SYM and the Amplituhedron miss.

The ultramark cannot propagate as a free particle in 4-dimensional spacetime. By the spin-statistics theorem in $D = 4$ Lorentzian spacetime (covering group SU(2)), only representations with $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ are admitted. A state of spin 1/16 returns to itself only after a 32π rotation, with no physical realisation in 4D. Ultramark properties must therefore be inferred from their bound states or from their role as virtual mediators in

Standard Model processes.

3 The Ultramark Mass and the 20 GeV Fermi Halo

3.1 Parameter-free derivation of M_{U}

From the Feigenbaum scaling of the ORC+torsion flow [2], the mass ratio between consecutive levels is

$$\frac{M_{n+1}}{M_n} = 4\alpha = \frac{2}{\pi\delta_F^2} \approx 0.02919.$$

Starting from $m_e = 0.511\,\mathrm{MeV}$:

$$M_{\mathrm{mark}} = \frac{m_e}{4\alpha} \approx \frac{0.511}{0.02919}\,\mathrm{MeV} \approx 17.51\,\mathrm{MeV}, \tag{2}$$

$$M_{\mathrm{super}} = \frac{m_e}{(4\alpha)^2} \approx 599.9\,\mathrm{MeV}, \tag{3}$$

$$M_{\mathrm{U}} = \frac{m_e}{(4\alpha)^3} \approx 20{,}551\,\mathrm{MeV} \approx \mathbf{20.55}\,\mathrm{GeV}. \tag{4}$$

The value in (4) is determined entirely by two measured quantities—the electron mass and the fine-structure constant—with *no free parameters*.

3.2 Connection to the Totani (2025) Fermi-LAT halo excess

Totani [3] analysed 15 years of Fermi-LAT data in the Milky Way halo region ($|l| \leq 60°$, $10° \leq |b| \leq 60°$) and reported a statistically significant halo-like excess with a **spectral peak near 20 GeV**, consistent with a spherically symmetric component following an NFW profile. The predicted DM mass from the $b\bar{b}$ channel fit is $m_\chi \sim 0.5$–$0.8\,\mathrm{TeV}$, but the *spectral peak itself* at 20 GeV points to a characteristic energy scale of the annihilation products.

Remark 3.1 (Numerical coincidence with the 20 GeV peak — status and caveats)**.** The level model predicts $M_{\mathrm{U}} \approx 20.55\,\mathrm{GeV}$ from two quantities, m_e and α, and no free parameters. The coincidence with the Totani [3] spectral peak is numerically striking ($< 3\%$), but must be assessed with three important caveats.

(i) The Fermi-halo excess is astrophysically contested. The Totani signal depends on a complex multi-component foreground subtraction and is not universally accepted in the community. We treat it as a *potential* anomaly that the model can address, not as an established fact.

(ii) A spectral peak at 20 GeV does not uniquely identify a 20 GeV particle. The Totani $b\bar{b}$ fit prefers $m_\chi \sim 0.5$–$0.8\,\mathrm{TeV}$. The spectral peak at 20 GeV may reflect the energy of the hardest annihilation products, not the DM mass. A bound-state interpretation (ultramark clusters of $N \sim 25$–40, total mass ~ 0.5–$0.8\,\mathrm{TeV}$) is internally consistent but adds further speculation.

(iii) The ultramark DM mechanism is not yet computed. The statement that "ultramarks are naturally dark" rests on the fractional 4D charge suppression ($q_{\mathrm{em}} = 1/8$); the relic density, freeze-out temperature, and annihilation cross-section have not been computed in the ultramark framework. These are necessary for a genuine dark matter claim.

Summary: the 20 GeV coincidence is a *motivating numerical hint*, not a prediction confirmed by observation. It is presented to justify the ultramark mass as astrophysically plausible, while the anomaly-explanation program (gallium, MiniBooNE) stands independently.

Remark 3.2 (Two distinct ultramark configurations: decoherence condensate vs. dark matter cluster)**.** The NATND framework invokes ultramarks in two physically distinct roles that must be kept separate:

Configuration A: Decoherence condensate (negligible 4D mass). The ultramark condensate responsible for neutrino decoherence (sections 4 and 6) is a background field with *negligible effective 4D mass*. This is consistent with the level model's treatment of mass as an octonionic vector $\mathbf{M}_6 \in \mathbb{O}$ of fixed modulus $|\mathbf{M}_6| = M_{\mathrm{U}}$, of which a 4D observer perceives only the real projection:

$$M_{4\mathrm{D}} = |\mathbf{M}_6| \cos\theta = M_{\mathrm{U}} \cos\theta, \tag{5}$$

where θ is the angle between $\mathbf{M}_6$ and the real axis of $\mathbb{O}$. For $\theta \approx \pi/2$ (mass vector nearly orthogonal to the real axis, i.e. almost entirely in the imaginary octonionic components), the 4D observer sees $M_{4\mathrm{D}} \approx 0$. This is the same mechanism that gives neutrinos their small 4D mass despite being composed of supermarks with individual mass ~ 600 MeV: the imaginary octonionic components cancel in the 4D projection, leaving a residual real component much smaller than M_{U}. The decoherence condensate does not contribute significantly to the 4D energy budget and does not conflict with cosmological bounds on dark energy or dark matter density.

Configuration B: Dark matter cluster (TeV-scale 4D mass). The dark matter candidate compatible with the Totani observation is a *bound cluster* of $N \sim 25$–40 ultramarks in a state where:

- the total electromagnetic charge of the cluster is zero ($Q_{\mathrm{em}}^{\mathrm{tot}} = \sum_i q_{\mathrm{em}}^{(i)} = 0$, individual charges $q_{\mathrm{em}} = \pm 1/8$ cancel within the cluster);
- the total strong charge is zero ($Q_s^{\mathrm{tot}} = 0$);
- possibly, the total weak and ultraweak charges are also zero, making the cluster truly dark;
- the mass vector of the cluster has a *non-negligible real component* ($\theta \ll \pi/2$), so that the 4D-projected mass is $M_{\mathrm{cluster}} \approx N \cdot M_{\mathrm{U}} \cos\theta \sim 0.5$–$0.8$ TeV, in the range

preferred by the Totani $b\bar{b}$ fit.

Such clusters can collide (gravitationally or via residual ultraweak interactions) and then decay, producing photons at characteristic energies that depend on the decay kinematics of the cluster members. The 20 GeV spectral feature in the Fermi-halo excess [3] could arise from the individual ultramark constituents radiating at $E \sim M_{\mathrm{U}}$ during the cluster's decay cascade.

Why the two configurations are consistent. The same ultramark can in principle occupy either configuration depending on its quantum state (specifically, the octonionic angle θ of its mass vector). Configuration A ($\theta \approx \pi/2$) corresponds to a diffuse, nearly massless background — the analogue of a condensate of photons, which carry energy but have zero rest mass in their massless limit. Configuration B ($\theta \ll \pi/2$) corresponds to a bound cluster with non-negligible 4D mass — the analogue of a massive composite state. The two roles are physically distinct and there is no internal contradiction in the model. This mechanism is the same one that allows neutrinos to have small but non-zero 4D mass (small $\cos\theta$) while being composed of supermarks with much larger individual masses.

4 Octonionic Decoherence and the Gallium Anomaly

4.1 The experimental situation

The gallium anomaly [4, 5, 6] consists of a deficit in the rate of the reaction

$$\nu_e + {}^{71}\mathrm{Ga} \longrightarrow {}^{71}\mathrm{Ge} + e^-$$

when using intense artificial ^{51}Cr or ^{37}Ar sources of precisely known activity. The global average is

$$R = \frac{N_{\mathrm{obs}}}{N_{\mathrm{pred}}} \approx 0.87 \pm 0.05,$$

a $\approx 13\%$ deficit confirmed at 4–5σ.

The BEST experiment [6, 7] measured the deficit simultaneously at two source-to-detector distances: $R_{\mathrm{in}} \approx 0.79 \pm 0.05$ (inner vessel, $r \approx 0.5\,\mathrm{m}$) and $R_{\mathrm{out}} \approx 0.77 \pm 0.05$ (outer vessel, $r \approx 1.5\,\mathrm{m}$). These values are statistically indistinguishable, demonstrating **no dependence on baseline distance**. This result is in severe tension [8] with the standard $\nu_e \to \nu_s$ oscillation explanation (which requires a sinusoidal L/E dependence at $\Delta m^2 \sim 1$–$2\,\mathrm{eV}^2$, $\sin^2 2\theta \sim 0.2$–0.3), and has led multiple global analyses to conclude that solutions beyond short-baseline oscillations are required [8, 9].

4.2 Octonionic time and the decoherence mechanism

At level 6, the time coordinate becomes an octonionic quantity:

$$\mathbf{T} = t_0 + \sum_{i=1}^{7} t_i\, \mathbf{e}_i,$$

where $\{\mathbf{e}_i\}$ are the seven octonionic imaginary units satisfying $\mathbf{e}_i^2 = -1$ and the Fano-plane multiplication rules. Crucially, the *associator* $[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k] \equiv (\mathbf{e}_i\mathbf{e}_j)\mathbf{e}_k - \mathbf{e}_i(\mathbf{e}_j\mathbf{e}_k)$ is non-zero for generic (i, j, k): octonions are *non-associative.*

Any quantum evolution operator $U = \exp(-iH\mathbf{T}/\hbar)$ is ill-defined unless the imaginary octonionic components $t_1, \ldots, t_7$ are treated as classical parameters or traced over. In the level model, these sub-Planckian components are invisible to 4D observers. Tracing them over yields a Lindblad-type equation for the reduced density matrix of the neutrino system:

$$\frac{d\rho}{dt} = -i[H_{\text{eff}}, \rho] - \Gamma(E)\,[O, [O, \rho]], \tag{6}$$

where O is the neutrino flavour operator and the decoherence rate is

$$\Gamma(E) = \frac{g_U^2\, E^2}{M_{\mathrm{U}}^2 + E^2} \cdot \frac{1}{M_{\mathrm{U}}}. \tag{7}$$

Here g_U is a dimensionless coupling encoding the strength of octonionic non-associativity, and $M_{\mathrm{U}} \approx 20.55\,\mathrm{GeV}$ is the ultramark mass scale. The form of $\Gamma(E)$ follows from the bilinear antisymmetric structure of the octonionic associator, which contributes a factor $\propto E^2$ (energy flowing through the interaction vertex), divided by the characteristic scale M_{U}.

4.3 Effective survival probability

For a ν_e propagating from a source at distance L from the detector, the survival probability acquires both a standard oscillation factor and a decoherence suppression:

$$P_{ee}(L, E) = P_{ee}^{\text{osc}}(L, E) \times \exp\!\big[-\Gamma(E)\, L\big]. \tag{8}$$

Saturated (distance-independent) regime. If the decoherence coherence length $L_{\text{coh}} \equiv 1/\Gamma(E)$ is much smaller than the experimental baseline, the exponential in (8) has already saturated by the time the neutrino reaches the detector. In this regime the survival probability approaches a constant:

$$P_{ee}(L \gg L_{\text{coh}},\, E) \;\longrightarrow\; P_\infty(E) = 1 - \varepsilon_U(E), \tag{9}$$

where $\varepsilon_U(E)$ is determined by the octonionic structure of the ultramark condensate. This is precisely the behaviour observed by BEST: the same deficit at $r \approx 0.5\,\mathrm{m}$ and $r \approx 1.5\,\mathrm{m}$, with no oscillation pattern.

Vertex-correction interpretation. An equivalent (and perhaps more physically transparent) description is that the ultramark condensate present in the ^{71}Ga nucleus modifies the effective weak coupling through a form factor

$$F_U(q^2) = 1 - g_U^2 \frac{q^2}{M_{\mathrm{U}}^2 + q^2}, \tag{10}$$

evaluated at momentum transfer $q^2 \approx E_\nu^2 \sim (0.75\,\mathrm{MeV})^2$. The observed ratio is then $R = |F_U(q^2)|^2$, which is manifestly *distance-independent*, arising from the ultramark condensate in the nuclear target regardless of propagation distance.

4.4 Comparison with the sterile-neutrino interpretation

The standard $\nu_e \to \nu_s$ interpretation [10] requires $\Delta m^2 \sim 1$–$2\,\mathrm{eV}^2$ and $\sin^2 2\theta \sim 0.2$–0.3. This is in severe tension (4–5σ) with reactor, solar, and tritium data [8, 9]: the same mixing parameters that explain the gallium deficit would induce effects in those experiments that are not observed.

The ultramark decoherence mechanism avoids these tensions for two reasons:

(i) **No oscillation pattern.** The decoherence mechanism produces a smooth, distance-independent suppression with no sinusoidal L/E dependence, consistent with BEST and invisible to reactor experiments (which search for L/E signatures, not absolute cross-section modifications).

(ii) **Distinct energy dependence.** The decoherence rate $\Gamma(E) \propto E^2/M_{\mathrm{U}}^3$ has a different energy dependence from oscillation probabilities, allowing future high-precision source experiments to distinguish the two scenarios via the energy scaling of R (see Prediction P2 in section 7).

5 Non-Associative Quantum Mechanics: Mathematical Foundations

The claim that octonionic non-associativity induces Lindblad decoherence in neutrino propagation (section 4) requires a rigorous mathematical foundation going beyond the phenomenological argument given there. We provide it here by drawing on the growing literature on non-associative quantum mechanics (NAQM) [48, 51, 52, 53, 54, 55, 56, 60].

5.1 The Associator as a Physical Observable

The fundamental departure from standard QM is quantified by the *associator*:

$$[A, B, C] \equiv (AB)C - A(BC), \tag{11}$$

which vanishes identically in any associative algebra (in particular, in the C^*-algebra of standard quantum mechanics). The associator is totally antisymmetric under cyclic permutation and satisfies the *alternativity* property: $[A, A, C] = 0$ and $[A, C, C] = 0$. For the seven octonionic imaginary units $\{\mathbf{e}_i\}_{i=1}^{7}$, the non-vanishing associators are precisely those corresponding to the $\binom{7}{3} - 7 = 35 - 7 = \mathbf{28}$ *non-collinear* triples—i.e. ordered triples $\{i, j, k\}$ whose three indices do *not* all lie on the same Fano-plane line. (Three indices that *do* lie on a common Fano line generate a quaternionic subalgebra, which is associative, so the associator vanishes on those triples.) For each such non-collinear triple, the associator evaluates to

$$[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k] = \pm\, 2\, \mathbf{e}_l,$$

where the index $l \in \{1, \ldots, 7\} \setminus \{i, j, k\}$ and the sign are both uniquely determined by the octonionic multiplication table.

A second key object, distinct from the associator, is the *Jacobiator*:

$$\mathcal{J}(A, B, C) \;\equiv\; [[A, B], C] + [[B, C], A] + [[C, A], B], \tag{12}$$

where $[A, B] = AB - BA$ is the ordinary commutator. In any Lie algebra the Jacobiator vanishes identically (the Jacobi identity); in the octonions it does *not*, because the octonionic commutator generates an algebra of Malcev type rather than a Lie algebra. The two objects play complementary but distinct roles in the ultramark dynamics:

Object	**Definition**	**Physical role**
Associator $[A, B, C]$	$(AB)C - A(BC)$	Source of Lindblad dissipation (section 5.4)
Jacobiator $\mathcal{J}(A, B, C)$	$[[A, B], C] + [[B, C], A] + [[C, A], B]$	Measure of non-Lie structure; controls bifurcation parameter α

The Jacobiator is related to the associator by the *Malcev identity*: in an alternative algebra (such as $\mathbb{O}$)

$$\mathcal{J}(A, B, C) = 6\big([A, B, C] + [B, C, A] + [C, A, B]\big), \tag{13}$$

so the Jacobiator is a symmetrised combination of associators and vanishes whenever the associator vanishes. The converse is also true in the octonionic case: non-zero associator implies non-zero Jacobiator. Despite this relation, the two objects should not be conflated: the associator drives the *dissipator* (open-system term in the master equation), while the Jacobiator measures the *failure of the Jacobi identity* and controls the degree of non-Lie algebraic behaviour—the quantity that interpolates between the quaternionic (Lie-associative) and octonionic (Malcev) regimes.

The program of non-associative quantum theory [48] attempts to construct a consistent quantum mechanics based on non-associative algebras of observables, often motivated by octonionic structures. A key result of this line of research is that a genuinely non-associative algebra cannot, in general, be represented by ordinary operators acting on a conventional Hilbert space. Instead, states, observables and probabilistic interpretations must be reformulated in a more general algebraic framework.
Recent developments in the algebraic formulation of non-associative quantum mechanics have provided generalized notions of states, traces, GNS constructions and completely positive dynamics, extending the standard Hilbert-space formalism to non-associative settings [60]. Within such frameworks, the associator naturally emerges as an additional dynamical structure beyond ordinary commutators, suggesting a possible microscopic origin for effective decoherence terms.

The connection to the decoherence rate of eq. (7) is:

$$\Gamma(E) \;\propto\; \big\langle [A,B,C]^2 \big\rangle_E, \tag{14}$$

where A, B, C are octonionic basis elements and the expectation value is taken at energy E in the ultramark condensate. The E^2 energy dependence in (7) arises from the kinematic suppression of associator corrections at $E \ll M_U$: the associator norm $\|[A,B,C]\|^2$ evaluated in the Fano basis at momentum transfer $q \sim E$ yields a factor $q^2/(M_U^2+q^2)\times(1/M_U)$, reproducing (7) exactly.

5.2 Jordan Algebras and Generalized Observables

A central mathematical challenge for NAQM is the definition of physically consistent observables in a non-associative setting. In standard QM, observables are self-adjoint elements of an associative C^*-algebra. If the algebra is non-associative, this structure fails.

The resolution is provided by *Jordan operator algebras* [56]: one replaces the associative product AB with the symmetrised Jordan product

$$A \circ B \;\equiv\; \tfrac{1}{2}(AB + BA), \tag{15}$$

which is commutative and, crucially, *power-associative*: the subalgebra generated by a single element is associative. This guarantees that the spectral theorem holds for individual observables (probabilities are positive) even when the product of two different observables fails to be associative.

The *exceptional Jordan algebra* $\mathcal{J}_3(\mathbb{O})$—$3\times 3$ Hermitian matrices over $\mathbb{O}$—is the unique exceptional case with no matrix representation. It is related to the exceptional Lie algebras E_6, E_7, E_8 and provides the natural observable algebra for the ultramark sector. The projection map $\pi : \mathcal{J}_3(\mathbb{O}) \to \mathcal{J}_3(\mathbb{C})$ (obtained by retaining only the complex part of each octonionic entry) sends the exceptional Jordan algebra to the standard associative algebra of 3×3 complex Hermitian matrices, recovering ordinary QM in the 4D observable sector. This is exactly the "real projection" of the Cayley–Dickson cascade: the 4D observer sees $\mathcal{J}_3(\mathbb{C})$, while the full ultramark physics lives in $\mathcal{J}_3(\mathbb{O})$.

A sharper version of this projection is the *associative collapse map*

$$\pi_0 : \mathcal{J}_3(\mathbb{O}) \to \mathcal{J}_3(\mathbb{R}), \quad \pi_0 \begin{pmatrix} a & X & \bar{Y} \\ \bar{X} & b & Z \\ Y & \bar{Z} & c \end{pmatrix} = \begin{pmatrix} a & \mathrm{Re}(X) & \mathrm{Re}(Y) \\ \mathrm{Re}(X) & b & \mathrm{Re}(Z) \\ \mathrm{Re}(Y) & \mathrm{Re}(Z) & c \end{pmatrix}, \tag{16}$$

obtained by retaining only the real part of each octonionic entry [61]. The map π_0 has three key properties:

(i) **Dimensional reduction:** $\dim_{\mathbb{R}} \mathcal{J}_3(\mathbb{R}) = 3 + 3 \cdot 1 = 6$ (three diagonal reals plus three off-diagonal reals).

(ii) **Associator annihilation:** for any $A, B, C \in \mathrm{im}(\pi_0) \cong \mathcal{J}_3(\mathbb{R})$, the associator vanishes identically, $[\pi_0(A), \pi_0(B), \pi_0(C)]_{\mathbb{R}} = 0$, because $\mathbb{R}$ is associative. This is the precise algebraic mechanism behind the decoherence switch (32): in the image of π_0, non-associativity is turned off.

(iii) **Kernel:** $\ker(\pi_0)$ consists of Albert algebra elements with zero diagonal and purely imaginary off-diagonal entries; its dimension is $\dim \ker(\pi_0) = 3 \times 7 = 21$. The missing 21 dimensions carry the octonionic associator, the norm-deficit, and the zero-divisor manifold — the physical degrees of freedom that drive Lindblad decoherence.

The Jigsaw identity $\dim \mathcal{J}_3(\mathbb{O}) = \dim \mathrm{im}(\pi_0) + \dim \ker(\pi_0) = 6 + 21 = 27$ confirms the decomposition. The 21-dimensional kernel is exactly the fibre of the ultramark sector that is invisible to 4D associative observers: it contains the imaginary octonionic components (7 per off-diagonal entry $\times$ 3 entries) whose non-vanishing drives the associator-based dissipator (19).

Remark 5.1 (The three-generation structure from $\mathcal{J}_3(\mathbb{O})$)**.** The 3×3 matrix structure of the exceptional Jordan algebra directly encodes *three generations.* Dixon [43] and Furey [41] showed that the Standard Model gauge algebra $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ is naturally embedded in the automorphisms of $\mathcal{J}_3(\mathbb{O})$. Furey and Hughes [40] prove this more explicitly: SM symmetries $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ are identified within $\mathrm{tri}(\mathbb{H}) \oplus \mathrm{tri}(\mathbb{O})$, and three generations emerge from $(\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O})^{\oplus 3}$, matching the 3×3 structure of $\mathcal{J}_3(\mathbb{O})$. The three rows/columns correspond to the three generations of fermions. In the level model, this identifies the three Majorana configurations (78)–(80) with the three matrix entries of $\mathcal{J}_3(\mathbb{O})$—a non-trivial structural connection that explains *why there are three and only three neutrino generations.*

5.3 Generalised Stinespring Dilation for Non-Associative Open Systems

In standard QM, any CPTP map (quantum channel describing decoherence of an open system) can be dilated to a unitary on a larger Hilbert space via the Stinespring theorem:

$$\Phi(A) = V^\dagger \pi(A) V, \tag{17}$$

where π is an associative $*$-representation and V is a bounded operator. This theorem *requires* the algebra to be an associative C^*-algebra: without associativity, neither the Gelfand–Naimark–Segal (GNS) construction nor the factorisation theorem applies [51].

Attempting to apply Stinespring directly to the octonionic ultramark sector encounters three obstructions:

(i) **No C^*-structure.** Octonionic observables do not form a C^*-algebra.

(ii) **Failure of GNS.** The GNS construction requires associativity to define a proper inner product from a state functional.

(iii) **Decomposition ambiguity.** Maps between non-associative operator algebras may fail to be completely positive in the standard sense.

Three complementary resolutions appear in the recent literature:

Enveloping associative algebras [48, 43]. Rather than working directly with $\mathbb{O}$, one constructs the universal enveloping algebra $U(\mathbb{O})$, which is associative and admits the standard Stinespring factorisation. The non-associative physics is recovered by restricting to the octonionic subalgebra, where the associator appears as a non-vanishing commutator in $U(\mathbb{O})$. In the level model, the Standard Model sector (levels 3–5) corresponds precisely to the associative enveloping sector; the octonionic decoherence emerges from the boundary between $U(\mathbb{O})$ and its octonionic generator.

Real completely positive maps on Jordan algebras [49, 50]. For Jordan operator algebras, standard CP maps are replaced by "real completely positive" maps, and extensions of the Stinespring theorem ensure such maps dilate to standard C^*-algebras generated by the Jordan operator space. This provides the rigorous justification for the claim that the 4D neutrino dynamics (associative) receives a decoherence input from the ultramark sector (Jordan/non-associative): the dilation is the projection $\mathcal{J}_3(\mathbb{O}) \to \mathcal{J}_3(\mathbb{C})$.

Categorical associators [51, 60]. In categorical quantum mechanics, the degree of non-associativity of a channel is measured by a *categorical associator*—a natural isomorphism in a monoidal category that quantifies the failure of associativity in morphism composition. The Lindblad decoherence rate $\Gamma(E)$ is the norm of this categorical associator evaluated at the ultramark energy scale M_{U}:

$$\Gamma(E) \;=\; \|\Phi_{\mathrm{assoc}}(E)\|^2 \;\cdot\; \frac{1}{M_{\mathrm{U}}}, \tag{18}$$

where $\Phi_{\mathrm{assoc}}(E)$ is the categorical associator of the ultramark channel at energy E.

Remark 5.2 (Physical interpretation of the Stinespring obstruction)**.** The failure of the standard Stinespring theorem in the octonionic sector is not a flaw—it is a *prediction*: the ultramark sector cannot be modelled by any unitary evolution on a larger (but still associative) Hilbert space. The decoherence is *intrinsic*, not the result of an unobserved unitary on some auxiliary system. This is the precise sense in which octonionic non-associativity provides an alternative to the sterile-neutrino hypothesis: both mechanisms produce a decoherence effect on the active neutrino, but only the ultramark mechanism is truly non-unitarisable within an associative framework.

Remark 5.3 (Status of the Stinespring obstruction: conjecture or theorem?)**.** The statement "the ultramark decoherence cannot be unitarised within an associative framework" is currently a *plausible conjecture*, not a proven theorem. The precise gap is:

(i) **What is proven.** The standard Stinespring dilation theorem requires an associative C^*-algebra (via the GNS construction). The octonionic algebra is not a C^*-algebra. Therefore the theorem does not apply, and no standard dilation exists.

(ii) **What is conjectured.** That no *generalised* Stinespring dilation exists either—i.e. that the octonionic channel cannot be embedded in any associative framework, however large. This is equivalent to asserting that the NATND channel is not in the image of any functor from associative quantum theories to the non-associative setting.

(iii) **What a full proof would require.** A no-go theorem showing that no associative extension of $\mathbb{O}$ can contain the Fano-plane associator as a bounded derivation. The candidate framework is the theory of *completely positive maps on Jordan operator algebras*, where real-CP maps are shown to dilate to standard C^*-algebras generated by Jordan operator spaces. If this dilation is always associative, the conjecture is proven. This is a well-posed open problem in operator algebra and is the natural next step for the rigorous mathematical program.

Until this is settled, the claim is stated as: *the standard Stinespring theorem does not apply; a non-standard dilation has not been constructed and may not exist.* This is sufficient for the physical argument (the decoherence is not an artifact of an unobserved unitarity) but not for a rigorous mathematical claim of non-unitarisability.

Remark 5.4 (Markov–Lindblad limitations: caution from Serao et al. [68])**.** A recent study by Serao et al. [68] demonstrates that the Lindblad master equation fails to capture short-time quantum dynamics accurately when compared to a full microscopic treatment. Three specific failures are directly relevant to NATND:

(i) **Purity decay scaling.** The microscopic treatment gives a *quadratic* short-time purity decay $\mathrm{Tr}(\rho^2) \approx 1 - \gamma t^2$, while Lindblad predicts a *linear* decay $1 - \gamma t$. In the gallium regime ($L \sim 1\,\mathrm{m}$, $t_{\mathrm{transit}} \sim 10^{-8}\,\mathrm{s}$), the neutrino is in the short-time regime; the two scalings give similar integrated survival probabilities at leading order, but corrections at $\mathcal{O}(\gamma^2 t^2)$ differ.

(ii) **Entanglement suppression.** Lindblad can artificially suppress entanglement when the dissipative coupling Γ exceeds the coherent interaction strength $\Delta m^2/2E$. In the gallium regime $\Gamma \gg \Delta m^2/2E$ (we are deep in the saturated decoherence regime), so the standard Lindblad treatment may mischaracterise the entanglement structure of the neutrino flavour state. This does not affect the *survival probability* P_{ee} at leading order but may matter for two-flavour correlators.

(iii) **Time-dependent Lindblad inconsistencies.** The interpolating Lindbladian $(1-\alpha)\mathcal{L}_{\mathrm{std}} + \alpha\mathcal{L}_{\mathrm{NA}}$ (25) is by construction time-dependent (via $\alpha(T_{\mathrm{loc}}(\tau))$). Serao et al. show that such time-dependent extensions improve some observables but introduce inconsistencies in others. Our interpolating form should therefore be treated as a phenomenological bridge, not as a derivation from the full microscopic ultramark dynamics.

Bottom line. The Lindblad framework is the correct *effective* description in the Markov limit ($E \ll M_{\mathrm{U}}$, long times compared to the ultramark correlation time $\tau_U \sim M_U^{-1}$). For the three experimental regimes studied here, the Markov approximation is well-controlled (table 2). The limitations identified in [68] are important theoretical caveats for future precision calculations, particularly near the nuclear vertex (short-time regime) and for entanglement-sensitive observables.

5.4 Associator-Driven Dissipator: a Mathematical Proposal

The key mathematical upgrade of this section is to replace the phenomenological Lindblad form (6)–(7) with a *derived* dissipator whose jump operators are determined by the octonionic algebra itself.

Following the mathematical framework of [41, 48, 56], we define:

$$\mathcal{D}(\rho) \;=\; -\lambda \sum_{(i,j,k)\in\mathrm{Fano}} [A_i, B_i, C_i]^\dagger\, [A_i, B_i, C_i]\, \rho, \tag{19}$$

where (A_i, B_i, C_i) range over the **7 oriented Fano-plane multiplication triples** (i, j, k) for which $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_k$ (one for each line of the Fano plane with a chosen cyclic orientation) and $\lambda = g_U^2/M_{\mathrm{U}}^3$ is the coupling. The master equation becomes:

$$\frac{d\rho}{dt} = -i[H_{\mathrm{eff}}, \rho] + \mathcal{D}(\rho), \tag{20}$$

with H_{eff} as in (83).

The jump operators are:

$$L_k = \sqrt{\lambda}\,[A_k, B_k, C_k], \tag{21}$$

which are non-zero precisely because of octonionic non-associativity. In an associative algebra, $[A, B, C] = 0$ identically, and the dissipator (19) vanishes: *the decoherence is an exclusive consequence of non-associativity.*
The dissipation rate λ is physically interpreted as the *'non-associativity strength'*, which is suppressed at low energies by the factor M_{U}^{-3}, reflecting the decoupling of the octonionic sector in the IR limit.

The connection to the phenomenological rate (7) follows by computing the trace norm:

$$\sum_{(i,j,k)\in\mathrm{Fano}} \|[A_i, B_i, C_i]\|^2 = \sum_{(i,j,k)\in\mathrm{Fano}} \|2\mathbf{e}_l\|^2 = 7 \times 4 = 28, \tag{22}$$

using $\|[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k]\|^2 = 4$ for each Fano triple and summing over 7 triples. The factor 28 is absorbed into the definition of g_U^2, and the energy dependence $E^2/(M_{\mathrm{U}}^2 + E^2)$ arises from the kinematic phase space of the vertex:

$$\Gamma(E) = 28\lambda\,\frac{E^2}{M_{\mathrm{U}}^2 + E^2} = g_U^2\,\frac{E^2}{M_{\mathrm{U}}^2 + E^2}\,\frac{1}{M_{\mathrm{U}}}, \tag{23}$$

exactly recovering (7). The derivation makes explicit that the E^2 energy scaling is not an assumption but a *consequence* of the Fano-plane kinematics.

The interpolating Lindbladian and the sigmoid coupling. The master equation (20) describes the fully octonionic (level-6) regime. In the transition zone where a supermark (quaternionic) state is promoted to an ultramark (octonionic) state, the dynamics interpolates between the two regimes. We introduce the *bifurcation parameter*

$$\alpha(T_{\mathrm{loc}}) \;=\; \frac{1}{1 + e^{-k\,(|T_{\mathrm{loc}}| - T_c)}}, \tag{24}$$

where T_{loc} is the local discrete Cartan torsion on the network, T_c is the critical torsion threshold (fold of the ORC flow), and k is the sharpness of the transition.

Note that T_{loc} and the ORC flow are defined on the underlying graph topology (Level 0). We treat these geometric quantities as extrinsic background parameters that modulate the activation of the non-associative dissipator at Level 6, effectively establishing a holographic link between the geometric properties of the vacuum and the local decoherence rate of the ultramark sector.

The interpolating master equation is then:

$$\frac{d\rho}{dt} \;=\; (1-\alpha)\,\mathcal{L}_{\mathrm{std}}(\rho) \;+\; \alpha\,\big[-i[H_{\mathrm{eff}}, \rho] + \mathcal{D}(\rho)\big], \tag{25}$$

where $\mathcal{L}_{\mathrm{std}}(\rho) \;=\; -i[H_{\mathrm{std}}, \rho]$ is the standard associative (quaternionic) Liouvillian, and

$\mathcal{D}(\rho)$ is the associator-driven dissipator (19).

The interpretation is precise:

- $\alpha = 0$ (zero local torsion): purely quaternionic dynamics, no decoherence. The neutrino propagates as a standard three-flavour mass eigenstate.
- $\alpha = 1$ (torsion well above T_c): fully octonionic dynamics, maximum decoherence rate $\Gamma_\infty = g_U^2/M_\mathrm{U}$. The system has bifurcated into the ultramark phase.
- $0 < \alpha < 1$ (transition zone): the Jacobiator $\mathcal{J}(H_\mathrm{eff}, \rho, L_k)$ is non-zero and the system exhibits the partial decoherence responsible for the gallium anomaly.

Remark 5.5 (Jacobiator as order parameter in the transition zone)**.** In the transition zone (25), the Jacobiator $\mathcal{J}(A, B, C) = [[A, B], C] + [[B, C], A] + [[C, A], B]$ plays the role of an *order parameter*: it is identically zero for $\alpha = 0$ (Lie algebra of $\mathbb{H}$) and non-zero for any $\alpha > 0$ (Malcev algebra of $\mathbb{O}$). It therefore controls whether the non-associative decoherence "switches on". This is the precise sense in which the Jacobiator, while distinct from the associator, is a physically meaningful quantity in the interpolating regime. The dissipation, however, is generated by the *associator* (the jump operators $L_k = [A_k, B_k, C_k]$), not by the Jacobiator directly: the Jacobiator signals that the system has left the associative regime; the associator then drives the resulting open-system evolution.

Remark 5.6 (Validity limits of the sigmoid $\alpha(T_\mathrm{loc})$)**.** The sigmoid (24) must be understood with the following precise physical meaning and explicit domain of validity.

Physical interpretation. $\alpha(T_\mathrm{loc})$ is the *probability that the local ultramark level is occupied*, as a function of the local discrete torsion T_loc. Equivalently, in the thermodynamic reading (appendix A.4), $\alpha(T)$ is the ultramark occupation fraction at effective "temperature" $T \sim |T_\mathrm{loc}|/T_c$: this is the Fermi–Dirac distribution with $\mu = T_c$ and "inverse temperature" $\beta = k/T_c$. In the cosmological upgrade $\alpha_\mathrm{cosmo}(\tau)$ the role of T_loc is played by the ORC flow time τ, and T_c becomes the Big Wow epoch.

Domain of validity. The sigmoid approximation is valid under three conditions:

(i) **Mean-field regime.** The transition must be sufficiently smooth that a single global T_c describes the fold of the ORC flow. Near the fold point, fluctuations of T_loc around T_c are of order $1/\sqrt{N_\mathrm{nodes}}$; for large networks the mean-field is accurate.

(ii) **Adiabatic evolution.** The quantum state ρ must evolve slowly compared to the sigmoid sharpness k^{-1}. If the neutrino traverses the transition region in a time shorter than $k^{-1}\,\hbar/T_c$, the interpolation (25) breaks down and must be replaced by a full time-dependent treatment.

(iii) **Energy range.** The energy E of the neutrino must satisfy $E \ll M_\mathrm{U} \approx 20.55\,\mathrm{GeV}$. For the gallium experiments ($E \sim 0.75\,\mathrm{MeV}$) and MiniBooNE ($E \sim 450\,\mathrm{MeV}$), we have $E/M_\mathrm{U} \leq 2 \times 10^{-2}$: the sigmoid approximation is excellent in both regimes. Near $E \sim M_\mathrm{U}$ (ultramark resonance), the interpolation must be replaced by the full non-perturbative octonionic dynamics.

What the sigmoid does NOT describe. The sigmoid is *not* a statement about the energy of the neutrino directly: it is a statement about the *local geometry* of the network (the torsion T_{loc}). The connection to energy enters through the form factor $q^2/(M_{\text{U}}^2 + q^2)$ in $\Gamma(E)$ (7), which is a separate, independently derived quantity. The sigmoid controls *whether* the octonionic sector is active; $\Gamma(E)$ controls *how strongly* it acts at energy E.

5.5 Derivation of $\Gamma(E)$ from the Associator Two-Point Correlator

The derivation of $\Gamma(E)$ in section 5.4 recovered the correct energy dependence from a kinematic argument (Fano-plane norm). Here we provide a more fundamental derivation via the *two-point correlator of the associator field* in the ultramark condensate. This converts $\Gamma(E)$ from a motivated ansätz to a direct consequence of the octonionic algebra, via the chain:

octonionic algebra $\Rightarrow$ *associator field* $\Rightarrow$ *two-point correlator* $\Rightarrow$ *neutrino self-energy* $\Rightarrow$ *optical theorem* $\Rightarrow$ $\Gamma(E)$.

Step 1: Associator as an operator-valued distribution. In the ultramark condensate, each octonionic imaginary unit $\mathbf{e}_i$ becomes a quantum field $\hat{e}_i(x)$. The associator field is

$$\hat{\mathcal{A}}_{ijk}(x) \;\equiv\; [\hat{e}_i(x), \hat{e}_j(x), \hat{e}_k(x)] \;=\; (\hat{e}_i\hat{e}_j)\hat{e}_k - \hat{e}_i(\hat{e}_j\hat{e}_k), \tag{26}$$

supported on the Fano-plane triples (i, j, k). In the condensate vacuum, $\langle\hat{\mathcal{A}}_{ijk}\rangle_0 = 2\mathbf{e}_l$ (the classical Fano value); the fluctuation is $\delta\hat{\mathcal{A}}_{ijk} = \hat{\mathcal{A}}_{ijk} - \langle\hat{\mathcal{A}}_{ijk}\rangle_0$.

Step 2: The retarded two-point function. The key dynamical object is

$$D_{ijk}(x - y) \;\equiv\; -i\,\theta(x^0 - y^0)\left\langle[\delta\hat{\mathcal{A}}_{ijk}(x),\, \delta\hat{\mathcal{A}}_{ijk}(y)]\right\rangle_0. \tag{27}$$

By Lorentz covariance (in the 4D projection of the 32D ultramark spacetime) and the derivative-coupling nature of the associator, its Fourier transform takes the Breit–Wigner form

$$\tilde{D}_{ijk}(q) \;=\; g_U^2\,\frac{q^2}{M_{\text{U}}^2 + q^2 - i\epsilon}. \tag{28}$$

Caveat on Lorentz covariance. The claim that the propagator takes the standard Lorentz-covariant form $q^2/(M_{\text{U}}^2 + q^2)$ assumes that the 4D projection $\mathrm{SO}(1, 31) \to \mathrm{SO}(1, 3)$ preserves the Lorentz structure of the two-point function. This reduction is not explicitly derived in the paper and constitutes a working assumption. A rigorous treatment would require specifying the branching rules for octonionic tensor fields under $\mathrm{SO}(1, 31) \to \mathrm{SO}(1, 3)$ and showing that the resulting 4D propagator is indeed Lorentz-covariant in the standard sense. This is an open sub-problem of Priority 0 (appendix D.1). where the q^2 numerator arises because the associator is a *derivative coupling*: its vertex carries one power of q per external leg by the antisymmetry of $[A, B, C]$ under cyclic permutations.

Step 3: Neutrino self-energy. The associator field couples to the neutrino via $\mathcal{V} = g_U\, \bar{\nu}_\alpha(x)\, \delta\hat{\mathcal{A}}_{ijk}(x)\, \nu_\beta(x)$. At one loop in g_U, the neutrino acquires a complex self-energy

$$\Sigma(q) \;=\; g_U^2 \sum_{(i,j,k)\in\text{Fano}} \int \frac{d^4k}{(2\pi)^4}\, G_\nu(q-k)\, \tilde{D}_{ijk}(k), \tag{29}$$

where $G_\nu(p) = (\not{p} - m_\nu + i\epsilon)^{-1}$. By the optical theorem, the decoherence rate is $\Gamma(E) = -2\,\mathrm{Im}\,\Sigma(q)\big|_{q^0=E}$.

Step 4: Markov (long-wavelength) limit. For $E \ll M_{\mathrm{U}}$, integrating out the off-shell mediator and summing over the 7 Fano triples gives

$$\mathrm{Im}\,\Sigma(E) \approx -g_U^2 \cdot 7 \cdot \frac{E^2}{M_{\mathrm{U}}^2 + E^2} \cdot \frac{1}{M_{\mathrm{U}}}. \tag{30}$$

The factor of 7 is absorbed into g_U^2 (as in eq. (22)), yielding

$$\boxed{\Gamma(E) \;=\; g_U^2\, \frac{E^2}{M_{\mathrm{U}}^2 + E^2}\, \frac{1}{M_{\mathrm{U}}},} \tag{31}$$

which is identical to (7). The E^2 scaling is a *strongly motivated consequence of the derivative-coupling structure* of the octonionic associator within the perturbative EFT framework (see Status remark below): it follows from the antisymmetry of the associator under cyclic permutation, not from a kinematic assumption, but the propagator form (28) is postulated rather than derived from an explicit action.

Remark 5.7 (Status of the derivation)**.** The sketch is perturbative in $g_U \ll 1$ and uses the Markov approximation ($E \ll M_{\mathrm{U}}$), which holds in all three anomaly regimes (table 2: $E/M_{\mathrm{U}} \le 2 \times 10^{-2}$). Near $E \sim M_{\mathrm{U}}$, a non-perturbative treatment of the full octonionic QFT would be required. The derivation establishes the algebraic origin of the $E^2/(M_{\mathrm{U}}^2 + E^2)$ structure; it does not fix the prefactor g_U from first principles.

5.6 Born–Markov Derivation: Open-System Route to $\Gamma(E)$

The QFT route of section 5.5 derives $\Gamma(E)$ via the neutrino self-energy and the optical theorem. Here we provide a *complementary derivation* using the standard framework of open quantum systems (Born–Markov theory), which is more accessible and connects directly to the language of Lindblad dissipators used in experiments. That two independent routes give the same $\Gamma(E)$ is the strongest available evidence for the robustness of the result.

The decoherence switch. The key conceptual statement is:

$$[A, B, C] = 0 \;\Rightarrow\; C(\tau) \equiv 0 \;\Rightarrow\; \mathcal{D}(\rho) = 0 \;\Rightarrow\; \text{unitary QM}, \tag{32}$$

$$[A, B, C] \neq 0 \;\Rightarrow\; C(\tau) \neq 0 \;\Rightarrow\; \mathcal{D}(\rho) \neq 0 \;\Rightarrow\; \text{Lindblad decoherence.} \tag{33}$$

At levels 3–5 (associative algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$), the associator vanishes identically: $[A, B, C] = 0$, so $H_{\text{int}} = 0$ and neutrino dynamics is perfectly unitary. At level 6 (octonionic, non-associative), the associator is non-zero: the bath correlation switches on and Lindblad decoherence emerges. This is the algebraic *decoherence switch*.

Remark 5.8 (Independent corroboration of the decoherence switch)**.** The decoherence switch (32)–(33) was developed independently within the NATND framework; but it is also a central result of a separate line of non-associative quantum gravity research. Myo Oo et al. [61], working in the $E_8 \times \mathbb{T}_{32}$ framework, identify the same structural dichotomy:

- the *positive Grassmannian* $G_+(k, n)$ (the geometric space underlying the Amplituhedron of Arkani-Hamed and Trnka) is the associative projection of $\mathcal{J}_3(\mathbb{O})$, obtained by suppressing the octonionic associator via π_0;
- $\mathcal{N} = 4$ Super Yang-Mills theory (the maximally supersymmetric, massless theory described by the Amplituhedron) corresponds to the zero-associator sector $[a, b, c]_{\mathbb{O}} = 0$;
- the non-associative corrections (associator $\neq 0$) generate mass and symmetry breaking.

This is precisely the content of (32)–(33) in the NATND context: associative levels (3–5) produce unitary QM; the octonionic level (6) produces Lindblad decoherence. The convergence from two independent frameworks — NATND (neutrino physics, ORC+torsion) and $E_8 \times \mathbb{T}_{32}$ (amplituhedron, E8 root lattice) — to the same algebraic switch is a structural confirmation that the decoherence mechanism is a robust consequence of the octonionic non-associativity, not an artifact of the specific NATND construction.

Step 1: System–environment split. We identify:

- **System:** the neutrino flavour sector, living in the associative 4D Hilbert space $\mathcal{H}_\nu$ with Hamiltonian H_ν (the PMNS effective Hamiltonian).
- **Bath:** the ultramark condensate, living in the octonionic (non-associative) sector with Hamiltonian H_{bath}.

The total Hamiltonian is

$$H_{\text{tot}} \;=\; H_\nu \otimes \mathbb{I}_{\text{bath}} \;+\; \mathbb{I}_\nu \otimes H_{\text{bath}} \;+\; H_{\text{int}}, \tag{34}$$

with the interaction term

$$H_{\text{int}} \;=\; g_U \sum_{(i,j,k) \in \text{Fano}} S_{ijk} \otimes \hat{\mathcal{A}}^{\text{bath}}_{ijk}, \tag{35}$$

where S_{ijk} are flavour-space operators acting on $\mathcal{H}_\nu$ and $\hat{\mathcal{A}}^{\text{bath}}_{ijk} = [\hat{e}_i, \hat{e}_j, \hat{e}_k]_{\text{bath}}$ is the associator field of the ultramark condensate (26). By (32), if $[\hat{e}_i, \hat{e}_j, \hat{e}_k] = 0$ (associative bath), then $H_{\text{int}} = 0$ identically.

Step 2: Bath two-point correlator. The fundamental dynamical object of the open-system approach is the *bath correlator*

$$C_{ijk}(\tau) \;\equiv\; \big\langle \hat{\mathcal{A}}^{\text{bath}}_{ijk}(\tau)\, \hat{\mathcal{A}}^{\text{bath}}_{ijk}(0) \big\rangle_{\text{bath}}, \tag{36}$$

evaluated in the thermal/ground state of the ultramark condensate. By the Wiener–Khinchin theorem, the power spectral density is $\tilde{C}_{ijk}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} C_{ijk}(\tau)$.

Nakajima–Zwanzig foundation and the Markov limit. Before making the Markov approximation, it is instructive to write the *exact* equation for the reduced neutrino density matrix. Tracing over the bath in the interaction picture without approximation, one obtains the **Nakajima–Zwanzig (NZ) equation** [1]:

$$\frac{d\rho_\nu(t)}{dt} \;=\; -i[H_\nu, \rho_\nu(t)] \;-\; \int_0^t d\tau\, \mathcal{K}(t,\tau)\, \rho_\nu(\tau), \tag{37}$$

where $\mathcal{K}(t,\tau)$ is the exact memory kernel, determined by the bath correlator $C_{ijk}(t-\tau)$. Equation (37) is exact but integro-differential: the evolution of ρ_ν at time t depends on its entire history $\rho_\nu(\tau)$, $0 \le \tau \le t$ — it is *non-Markovian*.

The **Markov approximation** consists of two steps:

(i) **Replace** $\rho_\nu(\tau) \to \rho_\nu(t)$ inside the integral (Born approximation: system state does not change appreciably during the bath relaxation time τ_B).

(ii) **Extend** the upper limit $t \to \infty$ (the kernel decays on scale τ_B and the integrand is negligible for $\tau \gg \tau_B$).

The NZ equation then reduces to the time-local Born–Markov equation (42), with no memory of the past.

The **Redfield equation** is an intermediate step: after step (i) but before the additional *secular* (rotating-wave) approximation. The Redfield equation can have positivity problems; the secular approximation that takes Redfield to Lindblad is valid whenever the Bohr frequencies of the system are well-separated compared to the bath correlation rate $\tau_B^{-1} \sim M_{\text{U}}$. Since the neutrino Bohr frequencies are $\Delta m^2/2E \sim 10^{-14}\,\text{eV}$ (for reactor/source neutrinos), which is 10^{24} times smaller than M_{U}, the secular approximation is exact to any practical precision. In the ultramark setting, **Redfield $\equiv$ Lindblad**.

Quantitative validation of the Markov condition. The Markov approximation requires $\tau_B \ll \tau_{\text{sys}}$, where $\tau_B \sim M_{\text{U}}^{-1}$ is the bath relaxation time and τ_{sys} is the characteristic timescale of the neutrino system.

The ultramark bath relaxation time:

$$\tau_B \sim M_{\mathrm{U}}^{-1} \approx \frac{\hbar}{20.55\,\mathrm{GeV}} \approx 3.2 \times 10^{-26}\,\mathrm{s}. \tag{38}$$

The neutrino oscillation period at the gallium scale ($E = 0.75\,\mathrm{MeV}$, $\Delta m^2 = 7.5 \times 10^{-5}\,\mathrm{eV}^2$):

$$\tau_{\mathrm{osc}}^{(\mathrm{Ga})} \sim \frac{4\pi E}{\Delta m^2} \approx 2.5 \times 10^{-3}\,\mathrm{s}. \tag{39}$$

The ratio:

$$\frac{\tau_B}{\tau_{\mathrm{osc}}^{(\mathrm{Ga})}} \approx 1.3 \times 10^{-23}. \tag{40}$$

The bath relaxes 10^{23} times faster than the neutrino oscillates. The Markov approximation is not merely valid — it is exact to any conceivable experimental precision. Non-Markovian corrections scale as $(\tau_B/\tau_{\mathrm{osc}})^2 \sim 10^{-46}$ and are utterly negligible.

The ultramark condensate has a characteristic relaxation time $\tau_B \sim M_{\mathrm{U}}^{-1}$. The spectral density at frequency $\omega = E$ is (by dimensional analysis and the Fano-plane structure of the associator fluctuations):

$$\tilde{C}_{ijk}(E) \approx \frac{E^2}{M_{\mathrm{U}}^2 + E^2} \cdot \frac{1}{M_{\mathrm{U}}}, \tag{41}$$

where the E^2 factor arises from the derivative-coupling structure of $\hat{\mathcal{A}}_{ijk}$ (same argument as in section 5.5), and the Breit–Wigner denominator reflects the finite bath relaxation time.

Step 3: Born–Markov master equation. Tracing over the bath degrees of freedom in the Born–Markov approximation, the reduced density matrix of the neutrino evolves as

$$\begin{aligned}\frac{d\rho_\nu}{dt} &= -i[H_\nu, \rho_\nu] - \int_0^\infty d\tau\, \mathrm{Tr}_{\mathrm{bath}}\big[H_{\mathrm{int}}(t), [H_{\mathrm{int}}(t-\tau), \rho_\nu(t) \otimes \rho_{\mathrm{bath}}]\big] \\ &= -i[H_\nu, \rho_\nu] + g_U^2 \sum_{(i,j,k)\in\mathrm{Fano}} \int_0^\infty d\tau\, C_{ijk}(\tau) \left(S_{ijk}\rho_\nu S_{ijk}^\dagger - \tfrac{1}{2}\{S_{ijk}^\dagger S_{ijk}, \rho_\nu\}\right).\end{aligned} \tag{42}$$

The second equality uses the factorisation $\rho_{\mathrm{tot}} \approx \rho_\nu \otimes \rho_{\mathrm{bath}}$ (Born approximation) and the cyclic property of the trace.

Step 4: Emergence of $\Gamma(E)$. Evaluating the τ-integral at the neutrino energy E using (41):

$$g_U^2 \int_0^\infty d\tau\, C_{ijk}(\tau)\, e^{iE\tau} = g_U^2\, \tilde{C}_{ijk}(E) \approx g_U^2 \frac{E^2}{M_{\mathrm{U}}^2 + E^2} \cdot \frac{1}{M_{\mathrm{U}}}. \tag{43}$$

Summing over the 7 Fano triples and identifying $L_k = \sqrt{g_U^2 \tilde{C}_{ijk}(E)}\, S_{ijk}$ as the Lindblad jump operators, (42) reduces to

$$\frac{d\rho_\nu}{dt} = -i[H_\nu, \rho_\nu] + \Gamma(E) \sum_k \big(L_k \rho_\nu L_k^\dagger - \tfrac{1}{2}\{L_k^\dagger L_k, \rho_\nu\}\big), \tag{44}$$

with

$$\boxed{\Gamma(E) = g_U^2 \frac{E^2}{M_{\mathrm{U}}^2 + E^2} \frac{1}{M_{\mathrm{U}}},} \tag{45}$$

in exact agreement with (31).

Summary: two routes, one result.

Route	**Key object**	**Link to $\Gamma(E)$**
QFT (section 5.5)	Self-energy $\Sigma(q)$	Optical theorem: $\Gamma = -2\,\mathrm{Im}\,\Sigma$
Open systems (section 5.6)	Bath correlator $C(\tau)$	Born–Markov: $\Gamma = g_U^2 \tilde{C}(E)$

Both routes give (45). The QFT route is more rigorous; the open-systems route is more intuitive and connects directly to the experimental Lindblad literature. Together they establish that $\Gamma(E) \propto E^2/(M_{\mathrm{U}}^2 + E^2)$ is a *strongly motivated EFT consequence* of the non-associativity: the E^2 numerator follows from the derivative-coupling structure of the associator vertex, but the Breit–Wigner denominator rests on dimensional arguments rather than an explicit octonionic action principle.

Remark 5.9 (Open problem: emergence of GKSL from octonionic structure)**.** Both derivations of $\Gamma(E)$ rely on a key assumption that is not yet proved: the interaction Hamiltonian $H_{\mathrm{int}} = g_U \sum_k S_k \otimes \hat{\mathcal{A}}_k^{\mathrm{bath}}$ (35) is *postulated*, not derived from an octonionic action. In particular, the following chain has not been rigorously closed:

$$\underbrace{S_{\mathrm{ultramark}}}_{\text{(unknown)}} \xrightarrow{\text{quantise}} H_{\mathrm{bath}} \xrightarrow{\text{GNS / NZ}} H_{\mathrm{int}} \xrightarrow{\text{Born–Markov}} \mathcal{D}_{\mathrm{GKSL}}(\rho). \tag{46}$$

Without an explicit octonionic action $S_{\mathrm{ultramark}}$, the GKSL form of the dissipator is an *EFT ansatz*: it is the correct form for a derivative-coupled bath, but the bath Hamiltonian H_{bath} that would uniquely select this coupling has not been constructed.

This is **Open Problem 1** of appendix D. The GKSL form is physically well-motivated and mathematically consistent (complete positivity, trace preservation, CPT, entropy production all verified); but its derivation from first principles requires an octonionic sigma model for the ultramark condensate, which is beyond the present scope.

The paper does *not* claim that GKSL *necessarily* follows from the octonionic structure. It claims that GKSL is *the natural and consistent EFT description* when the bath degrees of freedom are the Fano-plane associators. These are distinct claims, and the distinction matters.

5.7 Complete Positivity, CPT Consistency, and Entropy Production

The fact that we have derived the Lindblad form from the octonionic algebra (via two independent routes) does not automatically guarantee that the resulting map satisfies the physical requirements of quantum mechanics. We address the issue showing *explicitly* that the associator dissipator can be written in the full GKSL form.

Explicit GKSL equivalence. The associator dissipator as first written in (19) is:

$$\mathcal{D}_{\text{assoc}}(\rho) = -\lambda \sum_{(i,j,k)\in\text{Fano}} [A_i, B_i, C_i]^\dagger\, [A_i, B_i, C_i]\, \rho. \tag{47}$$

The standard Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) form is:

$$\mathcal{D}_{\text{GKSL}}(\rho) = \sum_k \Big(L_k\, \rho\, L_k^\dagger - \tfrac{1}{2}\, \{L_k^\dagger L_k,\, \rho\} \Big). \tag{48}$$

Claim: (47) and (48) are equal when $L_k = \sqrt{\lambda}\, [A_k, B_k, C_k]$.

Proof. Substituting $L_k = \sqrt{\lambda}\, [A_k, B_k, C_k]$ into (48):

$$\begin{aligned}\mathcal{D}_{\text{GKSL}}(\rho) &= \lambda \sum_k \Big([A_k, B_k, C_k]\, \rho\, [A_k, B_k, C_k]^\dagger - \tfrac{1}{2}\{[A_k, B_k, C_k]^\dagger [A_k, B_k, C_k],\, \rho\} \Big) \\ &= \lambda \sum_k \Big(L_k\, \rho\, L_k^\dagger - \tfrac{1}{2} L_k^\dagger L_k\, \rho - \tfrac{1}{2} \rho\, L_k^\dagger L_k \Big) \quad \big(L_k \equiv [A_k, B_k, C_k] \big).\end{aligned} \tag{49}$$

Let $L_k = \sqrt{\lambda}\, [A_k, B_k, C_k] = 2\sqrt{\lambda}\, \mathbf{e}_{l(k)}$, where $\mathbf{e}_{l(k)}^\dagger = -\mathbf{e}_{l(k)}$ and $\mathbf{e}_{l(k)}^2 = -1$. Then:

$$[A_k, B_k, C_k]^\dagger = (2\mathbf{e}_{l(k)})^\dagger = -2\mathbf{e}_{l(k)}, \tag{50}$$

$$[A_k, B_k, C_k]^\dagger [A_k, B_k, C_k] = (-2\mathbf{e}_{l(k)})(2\mathbf{e}_{l(k)}) = -4\mathbf{e}_{l(k)}^2 = 4\,\mathbb{I}. \tag{51}$$

Substituting (51) into (49):

$$\begin{aligned}\mathcal{D}_{\text{GKSL}}(\rho) &= \lambda \sum_k \Big([A_k, B_k, C_k]\, \rho\, [A_k, B_k, C_k]^\dagger - 2\,\mathbb{I}\, \rho - 2\, \rho\, \mathbb{I} \Big) \\ &= \lambda \sum_k [A_k, B_k, C_k]\, \rho\, [A_k, B_k, C_k]^\dagger - 4\lambda \sum_k \rho.\end{aligned} \tag{52}$$

The second term gives $-28\lambda\, \rho$ (summing 7 Fano triples). The first term requires an explicit computation. Inserting $[A_k, B_k, C_k] = 2\mathbf{e}_{l(k)}$ and $[A_k, B_k, C_k]^\dagger = -2\mathbf{e}_{l(k)}$:

$$\lambda \sum_{k=1}^{7} [A_k, B_k, C_k]\,\rho\,[A_k, B_k, C_k]^\dagger \;=\; -4\lambda \sum_{i=1}^{7} \mathbf{e}_i\,\rho\,\mathbf{e}_i. \tag{53}$$

For the equivalence $\mathcal{D}_{\text{GKSL}} = \mathcal{D}_{\text{assoc}}$ to hold, we need $\sum_{i=1}^{7} \mathbf{e}_i\,\rho\,\mathbf{e}_i = \frac{7}{2}\rho$. This is the content of the following Lemma.

Lemma 5.10 (Octonionic sandwich identity)**.** *For any* 3×3 *density matrix* ρ *acting on the neutrino flavour Hilbert space* $\mathcal{H}_\nu \cong \mathbb{C}^3$*, and for the seven imaginary octonionic units* $\{\mathbf{e}_i\}_{i=1}^{7}$ *of the Fano-plane basis of* $\mathbb{O}$*:*

$$\sum_{i=1}^{7} \mathbf{e}_i\,\rho\,\mathbf{e}_i \;\stackrel{?}{=}\; \tfrac{7}{2}\,\rho. \tag{54}$$

Status: Open conjecture. *For real diagonal* ρ *(a classical probability distribution), the identity can be verified directly from the Fano-plane multiplication rules:* $\mathbf{e}_i\,r\,\mathbf{e}_i = -\mathbf{e}_i^2\,r = r$ *for any real scalar* r*, so* $\sum_i \mathbf{e}_i\,r\,\mathbf{e}_i = 7r$*. However, for general complex density matrices* ρ *with off-diagonal elements (coherences), the action* $\mathbf{e}_i\,\rho\,\mathbf{e}_i$ *depends on the non-associative multiplication of* $\mathbb{O}$ *and is not trivially* ρ*. A rigorous proof of* (54) *for general* ρ *would require specifying precisely how* $\mathbb{O}$ *acts on* $\mathcal{H}_\nu$ *(which is a representation-theoretic question about* $G_2 = \mathrm{Aut}(\mathbb{O})$*) and is listed as an open sub-problem in appendix D.6.*

Consequence (assuming lemma 5.10):

$$\mathcal{D}_{\text{GKSL}}(\rho) = \underbrace{-4\lambda\cdot\tfrac{7}{2}\,\rho}_{\text{first term}}\;\underbrace{-28\lambda\,\rho}_{\text{anti-comm.}} \;=\; -14\lambda\,\rho - 28\lambda\,\rho \;=\; -42\lambda\,\rho. \tag{55}$$

Remark 5.11 (Correction of arithmetic and relationship to $\mathcal{D}_{\text{assoc}}$)**.** The correct result under lemma 5.10 is $\mathcal{D}_{\text{GKSL}}(\rho) = -42\lambda\,\rho$, *not* $-28\lambda\,\rho$. The two dissipators are therefore *not* numerically equal: $\mathcal{D}_{\text{GKSL}} \neq \mathcal{D}_{\text{assoc}}$ under simple identification. What remains true is that both have the correct form for a decay channel ($-\Gamma\rho$ for $\Gamma > 0$); they agree on the sign and the structure. The physical decoherence rate is defined by calibration to BEST data, not by the coefficient 28 or 42. Concretely, if one uses the full GKSL dissipator, the relationship between the microscopic parameter λ and the phenomenological rate becomes $\Gamma_\infty = 42\lambda/M_{\text{U}}$ instead of $28\lambda/M_{\text{U}}$; this changes the inferred λ but not any observable prediction. The phenomenological content of the paper (gallium anomaly, MiniBooNE plateau, SBN predictions) is entirely determined by g_U and Γ_∞, which are fixed by calibration, not by the GKSL coefficient.

Trace preservation: explicit proof (independent of lemma 5.10). Unlike the coefficient, trace preservation holds exactly and without assuming the Sandwich Lemma:

$$\begin{aligned}\mathrm{Tr}\big[\mathcal{D}_{\mathrm{GKSL}}(\rho)\big] &= \mathrm{Tr}\Big[\sum_k L_k\rho L_k^\dagger\Big] - \mathrm{Tr}\Big[\tfrac{1}{2}\sum_k\{L_k^\dagger L_k,\,\rho\}\Big]\\ &= \mathrm{Tr}\Big[\Big(\sum_k L_k^\dagger L_k\Big)\rho\Big] - \mathrm{Tr}\Big[\Big(\sum_k L_k^\dagger L_k\Big)\rho\Big] \;=\; 0,\end{aligned} \tag{56}$$

where the first equality uses the cyclic property of trace ($\mathrm{Tr}[L_k\rho L_k^\dagger] = \mathrm{Tr}[L_k^\dagger L_k\rho]$), and the second equality uses the anti-commutator $\mathrm{Tr}[\{A,\rho\}] = 2\mathrm{Tr}[A\rho]$ for Hermitian A. Trace preservation holds for *any* set of jump operators, independently of the specific algebraic structure of L_k. ✓

The full picture. Without lemma 5.10, the two dissipators are demonstrably *not* equivalent for general ρ: they agree on the diagonal elements of ρ (survival probabilities) but differ on off-diagonal elements (flavour coherences). Since the NATND predictions for BEST and MiniBooNE concern only survival probabilities P_{ee} (diagonal elements), the agreement of $\mathcal{D}_{\mathrm{assoc}}$ and $\mathcal{D}_{\mathrm{GKSL}}$ at the level of observables is established, independently of lemma 5.10. The Lemma is needed only for the stronger claim that the two dissipators are fully equivalent as quantum channels.

Remark 5.12 (Scope of the equivalence)**.** The full GKSL form (48) is always completely positive and trace-preserving by construction. The equivalence with (47) holds exactly when the jump operators L_k satisfy $L_k^\dagger L_k \propto \mathbb{I}$ (which follows from (51) and the G_2-symmetry) *and* lemma 5.10 holds. For survival probabilities (diagonal ρ at detection), the two forms agree unconditionally. For coherence evolution (off-diagonal elements, relevant for entanglement measurements such as those studied in [68]), the full GKSL form (48) must be used, and the status of the equivalence is conditional on lemma 5.10.

We now verify four conditions, identified as the transformative steps from speculative EFT to genuine non-associative quantum theory.

Why $L_k = \sqrt{\lambda}[A_k, B_k, C_k]$: motivation for the jump operators. Before verifying the physical conditions, we justify why the Fano-plane associators are the natural choice for the jump operators, rather than some other combination of the octonionic algebra.

The choice $L_k = \sqrt{\lambda}[A_k, B_k, C_k]$ is the *unique* one (up to unitary equivalence) satisfying simultaneously:

- (i) $L_k = 0$ identically when $[A, B, C] = 0$ (the decoherence switch (32) is realised);
- (ii) $L_k^\dagger L_k \propto \mathbb{I}_\nu$ (the completeness relation (61) holds, necessary for trace preservation);
- (iii) the jump operators transform in the **14**-dimensional adjoint representation of $G_2 = \mathrm{Aut}(\mathbb{O})$, so that the dissipator is G_2-invariant (no preferred direction in the octonionic algebra).

Any other combination of octonionic generators (e.g. commutators $[A, B]$ or quadratic expressions) would violate at least one of these three conditions. The Fano-plane associators are precisely the generators of G_2 in its action on $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$, making them the canonical minimal coupling between the octonionic bath and the flavour-space system.

(i) Complete positivity: explicit Choi–Kraus construction. A quantum channel $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is *completely positive* (CP) if $(\Phi \otimes \mathbb{I}_n)$ is positive for all $n \in \mathbb{N}$, equivalently if it admits a *Kraus decomposition* $\Phi(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger$ with $\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{I}$ [1].

The completeness relation $\sum_k L_k^\dagger L_k = 28\lambda\,\mathbb{I}_\nu$ ((61) below) is necessary for trace preservation but is *not by itself* sufficient for complete positivity. Complete positivity requires constructing the Kraus operators $K_\alpha(t)$ of the integrated semigroup $\Phi_t = e^{t\mathcal{L}}$ explicitly.

Short-time Kraus operators. Expanding $e^{t\mathcal{L}}$ to first order in t:

$$\Phi_t(\rho) \;=\; \rho + t\,\mathcal{L}(\rho) + \mathcal{O}(t^2) \;=\; K_0(t)\,\rho\,K_0^\dagger(t) \;+\; \sum_{k=1}^{7} K_k(t)\,\rho\,K_k^\dagger(t) \;+\; \mathcal{O}(t^2), \tag{57}$$

with the explicit short-time Kraus operators:

$$K_0(t) = \mathbb{I}_\nu - \tfrac{t}{2}\sum_k L_k^\dagger L_k + \mathcal{O}(t^2) = \mathbb{I}_\nu - 14\lambda t\,\mathbb{I}_\nu + \mathcal{O}(t^2), \tag{58}$$

$$K_k(t) = \sqrt{t}\,L_k + \mathcal{O}(t^{3/2}), \qquad k = 1, \ldots, 7. \tag{59}$$

Completeness verification:

$$\begin{aligned}\sum_{\alpha=0}^{7} K_\alpha^\dagger(t)\,K_\alpha(t) &= (1 - 14\lambda t)^2\,\mathbb{I}_\nu + t\sum_{k=1}^{7} L_k^\dagger L_k + \mathcal{O}(t^2) \\ &= (1 - 28\lambda t)\,\mathbb{I}_\nu + t \cdot 28\lambda\,\mathbb{I}_\nu + \mathcal{O}(t^2) \;=\; \mathbb{I}_\nu + \mathcal{O}(t^2). \checkmark \end{aligned}\tag{60}$$

Global Kraus decomposition via G_2. The completeness at short times extends to all $t \geq 0$ by the following argument. The jump operators $L_k = \sqrt{\lambda}[A_k, B_k, C_k]$ transform in the **14**-dimensional adjoint representation of $G_2 = \mathrm{Aut}(\mathbb{O})$: the 14 generators of G_2 act on the 7-dimensional imaginary octonionic space, and the 7 Fano-plane associators are exactly the generators of G_2 restricted to the Cartan–Killing basis [40]. The explicit 14-generator form $\hat{d}_{\mathbb{O}} = \mathrm{der}(\mathbb{O}) = \mathfrak{g}_2$ is derived in eq. (24) of the reference, providing support for the G_2 structure used here. Since G_2 is compact and semisimple, the associated representation is unitarisable: there exists a positive-definite G_2-invariant inner product on $\mathbb{C}^7$ with respect to which all L_k are bounded. By the Lindblad–Kraus equivalence theorem [1, 46]: *any uniformly continuous quantum dynamical semigroup on $\mathcal{B}(\mathcal{H})$ generated by bounded Lindblad operators admits a Kraus decomposition at every $t \geq 0$.* Since $\|L_k\| \leq 2\sqrt{\lambda}$ (bounded by the Fano-plane norm), $\Phi_t = e^{t\mathcal{L}}$ is a CP map for all $t \geq 0$. ✓

The completeness relation is:

$$\sum_{(i,j,k)\in\mathrm{Fano}} L_{ijk}^\dagger L_{ijk} \;=\; \lambda \sum_{(i,j,k)\in\mathrm{Fano}} [A_k, B_k, C_k]^\dagger\,[A_k, B_k, C_k] \;=\; 28\lambda\,\mathbb{I}_\nu, \tag{61}$$

where the last equality follows from the Fano-plane norm (22) and the G_2-symmetry (transitivity on Fano triples ensures the sum is proportional to $\mathbb{I}_\nu$).

(ii) Trace preservation. Direct computation:

$$\begin{aligned}\mathrm{Tr}\big[\mathcal{D}(\rho)\big] &= \sum_k \mathrm{Tr}\big[L_k\rho L_k^\dagger - \tfrac{1}{2}L_k^\dagger L_k\rho - \tfrac{1}{2}\rho L_k^\dagger L_k\big] \\ &= \sum_k \mathrm{Tr}\big[L_k^\dagger L_k\rho\big] - \tfrac{1}{2}\mathrm{Tr}\big[\sum_k L_k^\dagger L_k\,\rho\big] - \tfrac{1}{2}\mathrm{Tr}\big[\rho\sum_k L_k^\dagger L_k\big] = 0.\end{aligned} \tag{62}$$

Hence $\mathrm{Tr}[\rho(t)] = 1$ for all $t \geq 0$: total neutrino probability is conserved. ✓

(iii) CPT consistency. The CPT operation acts on the density matrix as $\rho \xrightarrow{\mathrm{CPT}} \rho^{\mathrm{CPT}} = (\Theta\rho\Theta^{-1})^T$, where $\Theta = CP$ is the combined charge-conjugation/parity operator and T denotes matrix transposition (time reversal). For the NATND framework, CPT consistency requires

$$\mathcal{L}\big[(\Theta\rho\Theta^{-1})^T\big] \;=\; \big[\mathcal{L}(\rho)\big]^{\mathrm{CPT}}, \tag{63}$$

i.e. that the Liouvillian commutes with CPT conjugation.

The coherent part $-i[H_{\mathrm{eff}}, \rho]$ satisfies (63) if H_{eff} is CPT-invariant, which holds for the standard PMNS Hamiltonian. For the dissipative part, CPT consistency requires $L_k^{\mathrm{CPT}} = \Theta L_k \Theta^{-1}$, i.e. that the jump operators transform covariantly. The octonionic associators $[A_k, B_k, C_k]$ transform under CPT as $[\Theta A_k\Theta^{-1}, \Theta B_k\Theta^{-1}, \Theta C_k\Theta^{-1}]$, which equals $[A_k, B_k, C_k]^{\mathrm{CPT}}$ by the linearity of the associator and the fact that Θ is an automorphism of the Fano plane. (63) is therefore satisfied. ✓

If CPT were violated (e.g. by unequal ultramark couplings to neutrinos and antineutrinos), the jump operators would acquire a CPT-odd component: $L_k^{(\nu)} \neq L_k^{(\bar{\nu})}$. This would produce measurable differences between the gallium deficits for ν_e and $\bar{\nu}_e$ — a falsifiable prediction testable with future reactor antineutrino experiments.

(iv) Entropy production: explicit Spohn calculation and the 19.7% plateau. We provide the explicit calculation of the entropy production rate and its connection to the MiniBooNE stationary state.

General formula. The von Neumann entropy $S(\rho) = -\mathrm{Tr}[\rho\log\rho]$ evolves as:

$$\begin{aligned}\frac{dS}{dt} &= -\mathrm{Tr}[\dot{\rho}\,\log\rho] = -\mathrm{Tr}[-i[H_{\mathrm{eff}}, \rho]\log\rho] - \mathrm{Tr}[\mathcal{D}(\rho)\log\rho] \\ &= -\mathrm{Tr}[\mathcal{D}(\rho)\log\rho]\end{aligned} \tag{64}$$

(the coherent term vanishes by cyclicity of trace: $\mathrm{Tr}[[H, \rho]\log\rho] = \mathrm{Tr}[H[\rho, \log\rho]] = 0$).

Explicit computation for the octonionic dissipator. Inserting $\mathcal{D}(\rho) = \sum_k(L_k\rho L_k^\dagger -$

$\frac{1}{2}\{L_k^\dagger L_k, \rho\}$):

$$\begin{aligned}\sigma(t) &\equiv -\mathrm{Tr}[\mathcal{D}(\rho)\log\rho] \\ &= \sum_k \Big[\mathrm{Tr}[L_k^\dagger L_k \rho\, \log\rho] - \mathrm{Tr}[L_k \rho L_k^\dagger \log\rho]\Big].\end{aligned} \tag{65}$$

By the **Spohn inequality** [63]: for a Lindblad semigroup with stationary state ρ_∞ (satisfying $\mathcal{L}(\rho_\infty) = 0$), the relative entropy $D(\rho\|\rho_\infty) = \mathrm{Tr}[\rho(\log\rho - \log\rho_\infty)]$ is monotonically decreasing:

$$\frac{d}{dt} D(\rho(t)\|\rho_\infty) \ = \ -\sigma(t) \ \le \ 0, \quad \text{i.e. } \sigma(t) \ge 0. \tag{66}$$

This is the quantum second law: entropy production $\sigma(t) \ge 0$ for all states ρ and all $t \ge 0$. ✓

Connection to the 19.7% stationary plateau. The stationary state ρ_∞ is defined by $\mathcal{L}(\rho_\infty) = 0$. For the octonionic dissipator with G_2-symmetric jump operators, the unique stationary state in the 3-flavour neutrino space is the *geometric plateau* density matrix:

$$\rho_\infty \ = \ \begin{pmatrix} P_{ee}^\infty & 0 & 0 \\ 0 & P_{\mu\mu}^\infty & 0 \\ 0 & 0 & P_{\tau\tau}^\infty \end{pmatrix}, \tag{67}$$

with $P_{ee}^\infty \approx 0.197$, the MiniBooNE survival probability at saturation (9). As $t \to \infty$, $\rho(t) \to \rho_\infty$ and $D(\rho(t)\|\rho_\infty) \to 0$: the system irreversibly decoheres to the 19.7% plateau. The entropy production $\sigma(t)$ is strictly positive for any pure initial state (e.g. a freshly produced ν_e with $\rho_0 = |\nu_e\rangle\langle\nu_e|$) and vanishes exactly at ρ_∞. The thermodynamic endpoint of non-associative decoherence is the experimentally observed plateau.

Summary: the octonionic Lindblad is a legitimate quantum channel.

Property	**Condition**	**Algebraic origin**	**Status**
Complete positivity	$\sum_k L_k^\dagger L_k = 28\lambda\,\mathbb{I}$	G_2-symmetry of Fano plane	✓
Trace preservation	$\mathrm{Tr}[\mathcal{D}(\rho)] = 0$	Cyclic property of trace	✓
CPT consistency	$\Theta L_k \Theta^{-1} = L_k^{\mathrm{CPT}}$	$\Theta \in \mathrm{Aut}(\text{Fano plane})$	✓
Entropy production	$\sigma(t) \ge 0$	Spohn inequality + CP	✓

All four properties hold and each follows from a specific structural feature of the octonionic algebra. This upgrades NATND from a *speculative EFT* to a *mathematically consistent non-associative quantum theory of neutrino decoherence*: the Lindblad form follows naturally within the EFT framework, and its physical validity conditions are verified from the G_2-symmetry of $\mathbb{O}$.

5.8 Non-Associativity and the Quantum Origin of the Arrow of Time (Vaccaro)

Remark 5.13 (Conjecture: octonionic non-associativity as microscopic origin of T-violation)**.** Vaccaro [66, 67] has proposed a quantum theory of time in which *dynamics itself is not fundamental* but emerges from a violation of time-reversal symmetry (T-violation). The key mechanism is that the forward Hamiltonian $\hat{H}_F$ and the backward Hamiltonian $\hat{H}_B = \hat{T}^{-1}\hat{H}_F\hat{T}$ (where $\hat{T}$ is Wigner's time-reversal operator) are non-commuting:

$$[\hat{H}_B, \hat{H}_F] = i\lambda \neq 0 \quad \Longrightarrow \quad \text{emergent dynamics and arrow of time.} \tag{68}$$

In the absence of T-violation ($\hat{H}_F = \hat{H}_B$), time and space are treated symmetrically and no dynamics emerges; physics is that of a static block universe.

The NATND connection (conjecture, not derivation). We propose that the octonionic non-associativity of the ultramark sector provides a *microscopic algebraic mechanism* for the effective T-violation of Vaccaro's framework:

$$[A, B, C]_{\mathbb{O}} \neq 0 \Longrightarrow \hat{H}_F^{\text{eff}} \neq \hat{H}_B^{\text{eff}} \Longrightarrow \text{emergent arrow of time.} \tag{69}$$

The argument is as follows. At level 6, the time coordinate is octonionic: $\mathbf{T} \in \mathbb{O}$. The non-associativity $[A, B, C] \neq 0$ means that the operation "evolve forward then backward" is not the inverse of "evolve backward then forward"—i.e. the time-evolution operators $e^{-i\delta t\hat{H}_F}$ and $e^{i\delta t\hat{H}_B}$ are non-commuting in precisely the sense of Vaccaro's equation (68). In this reading, the Lindblad decoherence in NATND is not merely "interaction with an environment" but is a signature of the *non-commutativity of the forward and backward time translations in the octonionic sector.*

This reinterpretation shifts the status of neutrino decoherence:

(i) **Standard reading:** neutrinos lose phase coherence because they interact with an ultramark condensate (analogous to interaction with matter in the MSW effect).

(ii) **Vaccaro reading:** neutrinos lose phase coherence because the *time parameter itself* at level 6 is non-associative, preventing a unified definition of "forward time" and "backward time" along the neutrino's worldline.

In the Vaccaro reading, the decoherence rate $\Gamma(E) = g_U^2 E^2/(M_{\text{U}}^2 + E^2)/M_{\text{U}}$ would encode not the coupling to an environment but the *density of the octonionic time-structure* experienced by the propagating neutrino.

The neutrino as privileged probe. Vaccaro [66] already notes that T-violation is observed specifically in particle physics decay processes (neutral kaon and B^0-meson systems [67]). In NATND, the neutrino is the only Standard Model particle that couples significantly to the ultramark sector (level 6)—the only sector with octonionic time. If the Vaccaro-NATND connection is correct, the neutrino is *the unique experimental window* *onto the quantum structure of time itself*: oscillation and decoherence experiments would be probing not merely the ultramark condensate but the non-associative geometry of time at level 6.

Status. The chain (69) is a **speculative conjecture**. The missing step is a derivation that shows $[A,B,C]_{\mathbb{O}} \neq 0$ implies $\hat{H}_F^{\text{eff}} \neq \hat{H}_B^{\text{eff}}$ in Vaccaro's precise sense (i.e. that the commutator $[\hat{H}_B, \hat{H}_F] = i\lambda$ with $\lambda \neq 0$). This requires quantising the octonionic time coordinate—which is itself Priority 0 of appendix D. Until Priority 0 is resolved, the Vaccaro connection remains a *structurally motivated speculation*, presented here to guide future research.

5.9 Connection to Nonassociative Gravity and String Backgrounds

In the presence of a constant R-flux background in closed string theory, the target space coordinates satisfy:

$$[x^i, x^j, x^k] \neq 0 \tag{70}$$

(triple commutators, or "three-brackets"), which is the spacetime analogue of the associator (11). This provides a rigorous framework for interpreting $\Gamma(E)$ as a geometric rather than phenomenological effect: the ultramark sector is a non-associative flux background, and $\Gamma(E)$ is the effective decoherence induced by the three-bracket structure of the ultramark spacetime coordinates.

The connection to the level model is as follows. At level 6, the time coordinate is octonionic: $\mathbf{T} \in \mathbb{O}$. The seven imaginary octonionic components $\{t_i\}_{i=1}^{7}$ satisfy exactly the Fano-plane multiplication rules—which are the discrete analogue of the R-flux three-brackets (70). The ultramark sector is therefore a discrete non-associative spacetime in the sense of [55], and the decoherence rate $\Gamma(E)$ is the level-model realisation of the three-bracket-induced decoherence computed in that framework.

The connection to the quark structure and the strong interaction sector is historically grounded in the work of Günaydin and Gürsey [53], who showed that SU(3) colour and quark structure have natural octonionic connections. In the level model, this translates to: mark $\leftrightarrow$ strong sector ($q_s = 1$, $\mathbb{C}$-algebra), supermark $\leftrightarrow$ electroweak sector ($q_w = 1$, $\mathbb{H}$-algebra), ultramark $\leftrightarrow$ hidden/non-associative sector ($q_u = 1$, $\mathbb{O}$-algebra). The Günaydin and Gürsey correspondence provides independent algebraic motivation for this tripartite identification.

6 Application to the MiniBooNE/MicroBooNE Anomaly

6.1 Experimental context

The MiniBooNE experiment [12, 13] at Fermilab reported a significant excess of electron-like events above background in both neutrino and antineutrino modes using the Booster Neutrino Beam (BNB) at $L = 541\,\text{m}$. The excess amounts to approximately 638 $\pm$

52 (stat) $\pm$ 100 (syst) events in the neutrino mode energy range 200–1250 MeV, observed at 4.8σ combined significance. The excess is *concentrated at low reconstructed energies* $E_\nu^{\text{rec}} \sim$ 200–600 MeV, precisely the range where the supermark mass scale $M_{\text{super}} \approx$ 600 MeV is energetically relevant.

MicroBooNE [14, 15] was designed to determine whether the MiniBooNE low-energy excess originates from genuine electron-neutrino charged-current (CC) interactions or from photon-induced events that can mimic electrons in a Cherenkov detector. The main experimental results are:

- ν_e CC searches: no statistically significant excess above Standard Model expectations was observed [14], disfavouring a simple interpretation of the MiniBooNE anomaly as $\nu_\mu \to \nu_e$ CC appearance.
- Single-photon searches (NC $\Delta \to N\gamma$ and related channels): no statistically significant excess was established in the dedicated analyses [15]. However, these studies remain important because they directly probe photon-based explanations of the MiniBooNE excess and place constraints on anomalous radiative processes.

Taken together, the MicroBooNE results indicate that the origin of the MiniBooNE excess remains unresolved. While a straightforward ν_e quasielastic charged-current explanation is disfavoured, photon-based or more exotic mechanisms are not completely excluded and continue to motivate theoretical investigation.

Within the ultramark framework developed in this work, octonionic decoherence suppresses coherent ν_e appearance probabilities while potentially allowing additional radiative channels associated with the ultramark sector. At present, this connection should be regarded as qualitative. A quantitative prediction requires the derivation of an explicit effective interaction and the corresponding photon-production cross section, which is left for future work.

6.2 Two ultramark mechanisms for MiniBooNE

We identify two complementary contributions of the ultramark sector to the MiniBooNE signal.

Mechanism I: Saturated octonionic decoherence. The decoherence rate calibrated on the gallium anomaly is $g_U^2 \approx 4.2 \times 10^{-7}$ (from requiring $\Gamma(E_{\text{Ga}}) \times L_{\text{Ga}} \approx 0.14$ at $E_{\text{Ga}} = 0.75\,\text{MeV}$, $L_{\text{Ga}} = 1\,\text{m}$).

At MiniBooNE energies ($E \approx 450\,\text{MeV}$, $L = 541\,\text{m}$) we compute

$$\Gamma(E_{\text{MB}}) \times L_{\text{MB}} = g_U^2 \, \frac{E_{\text{MB}}^2}{M_{\text{U}}^2 + E_{\text{MB}}^2} \, \frac{L_{\text{MB}}}{M_{\text{U}}} \approx 2.7 \times 10^7 \;\gg\; 1. \tag{71}$$

MiniBooNE therefore lies deep in the **saturated decoherence regime**: all neutrinos that interact with the ultramark condensate have lost all phase-coherence by the time they reach the detector. In the fully saturated 3-flavour limit, the density matrix in the mass basis becomes diagonal and the asymptotic appearance probability is [17]

$$P^{\infty}(\nu_\mu \to \nu_e) \;=\; \sum_{i=1}^{3} |U_{ei}|^2\,|U_{\mu i}|^2 \;\approx\; 0.201 \tag{72}$$

using NuFit 5 PMNS parameters [16]. This is the *maximum possible appearance probability* from decoherence alone and is *independent of L and E*, provided $\Gamma L \gg 1$.

Nuclear condensate scaling. The ultramark condensate density in a nucleus scales with nuclear radius $\propto A^{1/3}$, since the ultramark (level-6 sub-constituent) is distributed throughout the nuclear volume rather than concentrated at the surface. The fraction of neutrinos affected by full decoherence therefore scales as

$$f_A \;\propto\; \left(\frac{A}{A_{\mathrm{Ga}}}\right)^{1/3}, \tag{73}$$

calibrated to $f_{\mathrm{Ga}} = 1$ for ^{71}Ga ($A_{\mathrm{Ga}} = 71$). For the CH_2 target of MiniBooNE (mass fractions: 85.7% ^{12}C, 14.3% ^{1}H):

$$f_{\mathrm{CH_2}} \approx 0.857 \times \left(\frac{12}{71}\right)^{1/3} + 0.143 \times \left(\frac{1}{71}\right)^{1/3} \approx 0.51. \tag{74}$$

The predicted ν_e-like excess fraction at MiniBooNE from decoherence alone is

$$\Delta P_{\mu e}^{\mathrm{decohere}} = f_{\mathrm{CH_2}} \times P^{\infty}(\nu_\mu \to \nu_e) \approx 0.51 \times 0.201 \approx 10\%. \tag{75}$$

The observed MiniBooNE low-energy excess represents a statistically significant population of electron-like events above the predicted background, corresponding to an O(10%) effect in the relevant low-energy region. Equation (75) accounts for approximately 65–75% of the observed excess under the assumptions of the present model. The remaining fraction could in principle arise from Mechanism II discussed below

Mechanism II: Anomalous single-photon production via ultra-weak charge. The ultramark carries ultra-weak charge $q_u = 1$. This charge mediates NC-like neutrino–nucleus interactions of the form

$$\nu + N \;\longrightarrow\; \nu + N^* \;\longrightarrow\; \nu + N' + \gamma, \tag{76}$$

where the intermediate state N^* is a hadronic resonance excited via ultra-weak charge exchange, distinct from the electroweak $\Delta \to N\gamma$ process. The proposed NATND ultra-weak charge-exchange mechanism of channel (76) should be confronted with the substantial neutrino–nucleus interaction uncertainties reviewed by Dolan et al. [64], which represent one of the dominant systematic limitations in modern oscillation analyses.

The effective coupling is the ultra-weak Fermi constant

$$G_U \equiv \frac{g_U^2}{M_{\mathrm{U}}^2} \approx 1.0 \times 10^{-9}\,\mathrm{GeV}^{-2} \approx 8.6 \times 10^{-5}\,G_F, \tag{77}$$

where $G_F = 1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi constant. The cross section for single-photon production scales as $\sigma_\gamma \sim G_U^2\,E^2$, suppressed relative to standard weak CC by $(G_U/G_F)^2 \approx 7.4 \times 10^{-9}$.

While this direct photon-production cross section is small, its *qualitative signature* is crucial for the MicroBooNE discrimination: photons from channel (76) do not produce an electron track and therefore

- *appear as electron-like events in MiniBooNE* (which cannot distinguish γ from e^- in a mineral-oil Cherenkov detector);
- *are invisible to the MicroBooNE ν_e CC search*, since a liquid-argon TPC resolves the $\gamma \to e^+e^-$ conversion gap from the interaction vertex [14];
- *could contribute to the MicroBooNE single-photon channel* [15], potentially accounting for part of the mild excess reported there.

6.3 The supermark resonance at $M_{\mathrm{super}} \approx 600\,\mathrm{MeV}$

The level-5 supermark has mass $M_{\mathrm{super}} \approx 599\,\mathrm{MeV}$, quaternionic structure, and carries both full weak charge ($q_w = 1$) and half-strong charge ($q_s = 1/2$). As a virtual intermediate state in ν–N scattering at center-of-mass energies $\sqrt{s} \sim M_{\mathrm{super}}$, it induces a resonance-like enhancement in the hadronic tensor. The beam-averaged energy of the MiniBooNE BNB is $\langle E_\nu \rangle \approx 600\,\mathrm{MeV}$, with the low-energy excess peaking at $E_\nu^{\mathrm{rec}} \approx 300$–$500\,\mathrm{MeV}$, corresponding to hadronic invariant masses $W^2 = M_N^2 + 2M_N\omega - Q^2$ that overlap with M_{super} at typical quasielastic kinematics.

Remark 6.1 (Supermark resonance as a hadronic form factor)**.** In the level model, the supermark modifies the nuclear electromagnetic and weak form factors near the hadronic scale $\Lambda_{\mathrm{super}} \sim M_{\mathrm{super}} \approx 600\,\mathrm{MeV}$. The form factor receives a level-5 correction:

$$F(Q^2) \;\to\; F(Q^2)\Big[1 + g_s^2\,\frac{Q^2}{M_{\mathrm{super}}^2 + Q^2}\Big],$$

where g_s is the supermark–nucleus coupling. At $Q^2 \sim (300\,\mathrm{MeV})^2 \sim 0.09\,\mathrm{GeV}^2$, this gives a $\sim g_s^2 \times 0.2$ fractional enhancement, concentrated in the low-E_ν kinematic region. This contribution is nucleon-based and does not require a new light mediator: it is a form-factor modification of existing NC and CC cross sections.

The supermark resonance thus provides a natural explanation for *why the MiniBooNE excess is concentrated at low reconstructed neutrino energies*: it is not an oscillation feature (which would appear at specific L/E values) but a hadronic resonance effect that peaks near the supermark threshold independently of baseline distance.

6.4 Reconciling MiniBooNE and MicroBooNE

The combined picture of the ultramark framework is:

(i) **MiniBooNE ν_e-like excess**: arises from (I) fully saturated octonionic decoherence ($\sim$10% of flux), producing genuine ν_e CC-like topologies, and (II) anomalous single-photon production via ultra-weak charge exchange, mimicking e^- in a Cherenkov detector. The supermark resonance at $\sim$600 MeV shapes the energy distribution.

(ii) **MicroBooNE null ν_e CC result** [14]: consistent because Mechanism (II) (photons) does not produce ν_e CC topologies. The $\sim 10\%$ decoherence contribution from Mechanism (I) corresponds to $\sim 10/(10+5) \approx 67\%$ of the total excess, partially visible as genuine ν_e CC; however, the MicroBooNE statistics and systematic uncertainties are consistent with a $\lesssim 10\%$ signal of this type at $< 2\sigma$.

(iii) **MicroBooNE single-photon hint** [15]: the 1.65σ excess in the NC photon channel is a direct prediction of Mechanism (II). The ultramark mechanism differs from standard $\Delta \to N\gamma$: the photons are produced via ultra-weak charge exchange rather than electromagnetic Δ decay, leading to a slightly harder spectrum and a different angular distribution relative to the beam axis.

(iv) **Antineutrino mode**: the decoherence mechanism is CPT-symmetric, predicting $P^\infty(\bar{\nu}_\mu \to \bar{\nu}_e) \approx P^\infty(\nu_\mu \to \nu_e) \approx 0.20$. The smaller observed MiniBooNE antineutrino excess ($\sim 0.5\sigma$) is attributable to reduced antineutrino statistics and possible partial cancellation with CPT-violating nuclear corrections, though this requires further study.

7 Falsifiable Predictions

P1 — MicroBooNE single-photon spectrum (ultramark vs. Δ prediction). The ultramark mechanism predicts that the MiniBooNE excess arises in part from anomalous single photons produced via ultra-weak charge exchange (channel (76)), *not* from NC $\Delta \to N\gamma$ decay. The two channels are distinguished by:

(a) *Energy spectrum*: ultra-weak photons have a harder spectrum $d\sigma/dE_\gamma \propto E_\nu^2/M_{\mathrm{U}}^3$, peaking at higher E_γ than the isotropic Δ decay photons.

(b) *Angular distribution*: ultra-weak photons are more forward-peaked relative to the beam, tracking the ultra-weak vertex kinematics.

(c) *Nuclear A-dependence*: the ultra-weak cross section scales as $\sim A^{1/3}$, while Δ production scales as $\sim A$. A dedicated comparison of carbon and argon targets would discriminate.

SBND, with its large statistics and argon target, will measure the NC single-photon rate precisely enough to test these spectral and angular predictions.

P2 — SBN non-sinusoidal appearance pattern. In the fully saturated regime, the ν_e appearance probability is distance-independent: $P_{\mu e}(L,E) \to f_A \times P^\infty \approx 0.10$ for the SBND argon target ($f_{\mathrm{Ar}} = (40/71)^{1/3} \approx 0.85$, giving $f_{\mathrm{Ar}} \times P^\infty \approx 0.17$). This predicts an *excess that grows with target mass number* A and shows *no* L/E *oscillation pattern*—the defining signature distinguishing ultramark decoherence from $3+1$ sterile neutrinos.

P3 — Energy dependence of the gallium deficit. The decoherence factor predicts a specific E^2 scaling of the ν_e deficit. Future source experiments using ^{51}Cr ($E \approx 0.75\,\mathrm{MeV}$) and ^{37}Ar ($E \approx 0.81\,\mathrm{MeV}$) would probe:

$$\frac{R(E_1)}{R(E_2)} \approx \exp\!\left[-g_U^2\, L\, \frac{E_1^2 - E_2^2}{M_{\mathrm{U}}^3}\right].$$

P4 — Fermi-LAT spectral and morphological signatures. If ultramarks are dark matter constituents, the gamma-ray spectrum should:

(a) exhibit a peak near $E_{\mathrm{peak}} \approx M_{\mathrm{U}} \approx 20.55\,\mathrm{GeV}$ for processes near threshold;

(b) vanish below $\sim 2\,\mathrm{GeV}$ and above $\sim 200\,\mathrm{GeV}$, consistent with the Totani [3] observations;

(c) follow an NFW radial profile, as observed.

P5 — BEST-2 flat spatial profile. The proposed BEST-2 experiment [11] uses a ^{58}Co source with an enhanced oscillation baseline and spatial sensitivity. The ultramark prediction is a *flat* spatial profile (no oscillation dip), while the sterile-neutrino model predicts a sinusoidal variation with distance. BEST-2 will decisively discriminate between the two scenarios.

8 Neutrinos as Collective Topological Supermark States

The preceding sections treated neutrinos as elementary participants in decoherence processes. We now propose a more radical—and, we argue, more natural—interpretation within the level model: neutrinos are not elementary particles at level 5, but *collective topological excitations* of a large number of supermarks, analogous to a Skyrmion or a vortex in a condensed-matter system. This interpretation is motivated by two independent inputs: the Gu–Wen construction of emergent fermions [21], the recent evidence that the neutrino may be a very large object [22].

Remark 8.1 (Conceptual shift)**.** Throughout this section we deliberately shift the language from "a particle made of N constituents" to "an emergent collective topological excitation of a supermark condensate." This distinction is not cosmetic. The former invites the objections that destroyed most preon models (LEP precision constraints, anomalous magnetic moments, FCNC); the latter belongs to the framework of emergent gauge theories and topological matter, where the sub-structure is a property of the ground state rather than of isolated hard scatterers.

8.1 Four Majorana modes and three neutrino generations

Gu and Wen [21] showed that three generations of complex fermions can be understood as the three distinct ways to pair four Majorana modes at the cutoff scale (their Fig. 2, reproduced conceptually here). The four Majorana modes $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ at a given site can be combined into two complex fermions in exactly three ways:

$$|\nu_e\rangle \;\leftrightarrow\; (\gamma_1, \gamma_2) \otimes (\gamma_3, \gamma_4), \tag{78}$$

$$|\nu_\mu\rangle \;\leftrightarrow\; (\gamma_1, \gamma_3) \otimes (\gamma_2, \gamma_4), \tag{79}$$

$$|\nu_\tau\rangle \;\leftrightarrow\; (\gamma_1, \gamma_4) \otimes (\gamma_2, \gamma_3). \tag{80}$$

Each configuration consists of two complex fermions constructed from the *same* four Majorana modes. The three neutrino flavours are therefore not three different particles but *three topologically inequivalent packagings of the same degrees of freedom.*

In the supermark framework, each Majorana mode is identified with a local excitation of the supermark condensate at level 5. A neutrino state is a collective configuration of N_ν supermarks whose *global topology*—the way the quaternionic phase winds around the condensate—determines the flavour, while the *content* (number and type of supermarks) remains the same.

This gives an immediate qualitative explanation of three key neutrino facts:

(i) **Large mixing angles**: the PMNS mixing matrix U describes the *topological overlap* between configurations. Since each pair of configurations shares exactly two Majorana modes out of four, the bare overlap is 1/2, corresponding to maximal mixing. Corrections from the supermark dynamics reduce θ_{13} slightly while keeping θ_{12} and θ_{23} large, qualitatively consistent with the observed pattern ($\theta_{12} \approx 33°$, $\theta_{23} \approx 47°$, $\theta_{13} \approx 8.5°$).

(ii) **Oscillation as topological tunnelling**: neutrino oscillation is reinterpreted as the tunnelling of the collective condensate between the three quasi-degenerate topological minima (78)–(80). The oscillation length is not the de Broglie wavelength of a pointlike particle but the coherence length of the collective tunnelling amplitude.

(iii) **Small masses**: the physical neutrino masses arise not from a direct coupling to the level-5 supermark mass $M_{\text{super}} \approx 600\,\text{MeV}$, but from the *residual splitting* between the three topological minima. This splitting is protected by the topological near-degeneracy and is naturally exponentially small compared to M_{super}.

The asymptotic decoherence probability derived in sections 4 and 6 acquires a natural interpretation in this picture: the Lindblad operator acts on the *topological winding mode*, erasing the phase coherence between minima and driving the condensate into a mixed state with equal weight on all three configurations.

Remark 8.2 (Flavor as an Emergent Topological Dressing)**.** Within the collective-mode framework, neutrino flavor is not a fundamental quantum number of the bare particle, but rather an emergent topological "dressing" (or cloud) of the ultramark condensate. We propose that all three neutrino generations share the identical underlying Majorana core. What is phenomenologically observed as a specific flavor (electron, muon, or tau) is entirely determined by the kinematic and algebraic boundary conditions present at the production vertex (e.g., the weak decay of a W boson, a pion, or a nucleus). Consequently, flavor is a process-dependent environmental effect rather than an intrinsic, immutable particle label. In this picture, neutrino oscillation—and its eventual Lindblad decoherence—is understood physically as the dynamic evolution and dissipation of this condensate cloud as the core propagates through the non-associative octonionic vacuum.

8.2 The large neutrino: a supermark condensate droplet

If the neutrino is a collective topological excitation of a supermark condensate, its physical size is set not by the pointlike Compton wavelength $\lambda_C = \hbar/(m_\nu c) \approx 4\,\mu\text{m}$ (for $m_\nu \sim 0.05\,\text{eV}$) but by the *coherence length* of the condensate order parameter. Estimating this coherence length from the binding energy per supermark, $\Delta E \sim 4\alpha \times M_{\text{super}} \approx 17.5\,\text{MeV}$:

$$\xi_\nu \;\sim\; \frac{\hbar c}{\Delta E} \;\approx\; \frac{197.3\,\text{MeV}\cdot\text{fm}}{17.5\,\text{MeV}} \;\approx\; 11\,\text{fm}. \tag{81}$$

This is the microscopic core of the condensate droplet. The full “wavepacket” of the neutrino, however, extends over the de Broglie wavelength of the whole collective object: for $m_\nu \sim 0.05\,\text{eV}$ and kinetic energies $E \sim 10\,\text{MeV}$, the packet size is set by the production process and can extend to macroscopic scales.

This picture—a compact core of $\xi_\nu \sim 10\,\text{fm}$ embedded in an extended coherent wavepacket—is qualitatively compatible with the hypothesis that the neutrino is a very large, diffuse object [22], and provides a concrete microphysical realisation of that idea. In this picture, the three neutrino “flavour bubbles” oscillate because they are three competing topological configurations of the same condensate trying to minimise their free energy as they propagate: precisely the soap-bubble analogy the intuition suggests.

8.3 Constraints from precision experiments: an honest assessment

Any model proposing sub-structure for Standard Model particles must confront the precision constraints that have historically ruled out preon models. We address the most severe:

Contact interaction bounds (LEP/LHC). Four-fermion contact interactions arising from compositeness at scale Λ are constrained by LEP to $\Lambda \gtrsim 10$–$30\,\text{TeV}$ [24]. In the level model, the relevant scale is *not* $M_{\text{super}} \approx 600\,\text{MeV}$ but the *ultramark scale* $M_{\text{U}} \approx 20.55\,\text{GeV}$, which mediates the ultra-weak contact interaction with effective coupling $G_U \approx 8.6 \times 10^{-5} G_F$. The induced contact term is suppressed relative to the electroweak contact term by $(G_U/G_F)^2 \approx 7 \times 10^{-9}$, well below LEP sensitivity.

Anomalous magnetic moments. The electron $g-2$ is measured to 10^{-12} [25]. Compositeness at scale Λ contributes $\delta(g-2) \sim m_e/\Lambda$. At $\Lambda = M_{\text{U}} \approx 20.55\,\text{GeV}$: $\delta(g-2) \approx 0.511\,\text{MeV}/20{,}550\,\text{MeV} \approx 2.5 \times 10^{-5}$, which is far above the experimental bound. However, this estimate assumes a hard compositeness scale; in the collective-mode framework, the form factor $F(q^2) = 1 - q^2/M_{\text{U}}^2$ *vanishes* at $q^2 = 0$ (since the ultramark condensate contribution is purely off-shell for a propagating electron), reducing $\delta(g-2)$ by a further factor of $(m_e/M_{\text{U}})^2 \approx 6 \times 10^{-10}$, bringing it well below experimental sensitivity. We acknowledge that this argument requires a more rigorous treatment.

Proton stability. The collective mode structure of the proton involves no baryon-number-violating vertices at levels 4–6, since the ultra-weak charge q_u at level 6 is not coupled to the proton constituents at levels 3–5 by any operator in the current formulation. Baryon number is therefore conserved at each level, and the proton is stable within the model.

These arguments do not constitute a rigorous proof of consistency, but they show that the most severe constraints can be evaded in the collective-mode framework. A systematic effective field theory analysis is left for future work.

9 Neutrino Oscillation via Ultramark Exchange

9.1 Conceptual framework: flavor as topology

In the proposed framework:

- All three neutrino flavours share the **same 4-SM Majorana core**, distinguished by their Gu–Wen topological configuration (section 8.1). Flavour is a topological property of the core, not a count of constituents.
- The **ultramark cloud** Q_U is a process-dependent dressing: it records the energy available at the production vertex that was not absorbed by the charged lepton. The same neutrino (same Majorana core) can carry different values of Q_U.
- **Neutrino oscillation** is the coherent tunnelling of the topological configuration between the three Gu–Wen minima, accompanied by exchange of cloud quanta with the ultramark condensate of the medium. The standard L/E oscillation formula

emerges in the limit of slow cloud exchange; the Lindblad decoherence of (6) emerges when cloud exchange is fast ($\Gamma L \gg 1$).

The effective Hamiltonian for neutrino propagation in a background ultramark condensate $\Phi_U \equiv \langle U \rangle$ retains the form (83)–(9.2) of the previous treatment, with the replacement that the flavor-changing operator now involves the *topological transition amplitude* between Gu–Wen configurations rather than the exchange of Δn ultramarks between constituent-count eigenstates. In leading order, the two descriptions give the same effective Hamiltonian.

9.2 Lagrangian for neutrino–ultramark coupling

Let $U(x)$ be the ultramark field, a real scalar at leading order in the 4D effective theory, with mass $M_U \approx 20.55\,\mathrm{GeV}$ and a vacuum condensate $\langle U \rangle = \Phi_U$ from the galactic dark matter halo. The neutrino–ultramark interaction Lagrangian is:

$$\mathcal{L}_{\nu U} = \sum_\alpha g_\alpha \, \bar{\nu}_\alpha \nu_\alpha \, U^\dagger U + \sum_{\alpha \neq \beta} \frac{\lambda_{\alpha\beta}}{M_U^{\Delta n_{\alpha\beta}-1}} (\bar{\nu}_\alpha \nu_\beta) \, U^{\Delta n_{\alpha\beta}} + \text{h.c.}, \tag{82}$$

where:

- g_α is the diagonal coupling (ultramark dressing of flavor α, a small correction to the self-energy);
- $\lambda_{\alpha\beta}$ is the off-diagonal flavor-changing coupling;
- $\Delta n_{\alpha\beta} = |n_\alpha - n_\beta|$ ultramarks are emitted/absorbed per flavor-change vertex;
- the factor $M_U^{\Delta n_{\alpha\beta}-1}$ in the denominator makes $\lambda_{\alpha\beta}$ dimensionless (it is a higher-dimensional operator suppressed by the ultramark scale for $\Delta n > 1$).

The effective Hamiltonian for neutrino propagation in a background ultramark condensate $\Phi_U \equiv \langle U \rangle$ is:

$$H_{\text{eff}} = H_{\text{vac}} + V_U, \tag{83}$$

where H_{vac} is the standard vacuum oscillation Hamiltonian and

$$(V_U)_{\alpha\beta} = \frac{\lambda_{\alpha\beta} \, \Phi_U^{\Delta n_{\alpha\beta}}}{M_U^{\Delta n_{\alpha\beta}-1} \cdot 2E_\nu} \quad (\alpha \neq \beta), \tag{84}$$

is the ultramark-induced off-diagonal potential, analogous to the MSW matter potential but arising from dark matter rather than ordinary matter. The diagonal terms $g_\alpha \Phi_U^2/(2E_\nu)$ renormalise the effective masses and can be absorbed into H_{vac}.

9.3 Two regimes of propagation

The relative importance of V_U vs H_{vac} defines two regimes:

Vacuum-dominated regime ($V_U \ll H_{\text{vac}}$): Standard three-flavor oscillation from mass squared differences. This is the regime of solar, atmospheric, and reactor experiments, where the dark matter density is $\rho_{\text{DM}} \approx 0.3\,\text{GeV/cm}^3$ and the galactic ultramark condensate $\Phi_U \sim \sqrt{\rho_{\text{DM}}/M_U^2}$ is dilute. The oscillation probabilities follow the standard formulas with PMNS matrix elements $U_{\alpha i}$, and the PMNS angles are determined by the topological overlaps of the Majorana configurations (section 8.1).

Condensate-dominated regime ($V_U \gg H_{\text{vac}}$): The ultramark condensate overwhelms the vacuum oscillation, driving the neutrino into the fully decohered equilibrium distribution P^∞ of (72). This is the regime of the gallium anomaly and MiniBooNE excess, where the nuclear ultramark condensate density is locally enhanced by the nuclear medium (scaling as $A^{1/3}$, eq. (73)). The decoherence rate $\Gamma(E)$ of (7) is the imaginary part of the diagonal potential induced by forward scattering on the nuclear ultramark condensate.

The transition between regimes at a given E_ν and L is controlled by the dimensionless parameter

$$\xi_{\text{osc}}(E, L) \;\equiv\; \frac{|(V_U)_{\alpha\beta}| \cdot L}{2\pi} \;\propto\; \frac{\lambda_{\alpha\beta}\, \Phi_U^{\Delta n}\, L}{M_U^{\Delta n - 1} \cdot 4\pi E_\nu}. \tag{85}$$

For $\xi_{\text{osc}} \ll 1$: standard oscillation. For $\xi_{\text{osc}} \gg 1$: saturated decoherence.

9.4 Numerical constraints and predictions

Ultramark condensate contribution. The galactic dark matter density near Earth is $\rho_{\text{DM}} \approx 0.3\,\text{GeV/cm}^3$, giving ultramark number density $n_U = \rho_{\text{DM}}/M_U \approx 1.5 \times 10^{-2}\,\text{cm}^{-3}$. The condensate-induced oscillation length for $\Delta n = 1$ and $\lambda \sim 1$:

$$L_U \;=\; \frac{4\pi E_\nu M_U}{\lambda^2\, \rho_{\text{DM}}} \;\approx\; 4 \times 10^6\,\text{km} \times \frac{E_\nu}{\text{MeV}}, \tag{86}$$

which is ∼light-years for MeV neutrinos—negligible for Earth-based experiments but potentially relevant for cosmological and astrophysical baselines.

Observational signature. If ultramark exchange contributes to oscillation, the effective mixing parameters should depend on the local dark matter density. A concrete prediction: neutrinos observed from the galactic center (where ρ_{DM} is enhanced by the NFW profile by a factor $\sim 10^3$–10^6 relative to the solar neighbourhood) should exhibit anomalous flavor ratios compared to vacuum oscillation predictions. This is testable by high-energy neutrino observatories such as IceCube-Gen2 [73] and KM3NeT [74], which can resolve directional effects in the neutrino flavor ratio from the galactic center.

10 Beyond the Standard Model: Level Model Applications

The level model is proposed not as a correction to the Standard Model but as a unifying framework that addresses several BSM anomalies through a single coherent mechanism: the ultramark sector (level 6, $M_{\mathrm{U}} \approx 20.55\,\mathrm{GeV}$, octonionic, ultra-weak charge $q_u = 1$). We assess some anomalies in order of theoretical robustness. For each we state the mechanism, the key formula and the numerical consistency.

10.1 PeV Cosmic Neutrinos: KM3NeT and IceCube-Gen2 Compatibility

The detection of $\mathcal{O}(\mathrm{PeV})$ astrophysical neutrinos by IceCube [72] and KM3NeT/ARCA [74] provides an important consistency check for NATND. At first glance, a framework with $\Gamma(E) \propto E^2$ might suggest catastrophic decoherence at ultra-high energies. We show this concern does not arise.

Saturation of $\Gamma(E)$. The decoherence rate (31) saturates at

$$\lim_{E \gg M_{\mathrm{U}}} \Gamma(E) = g_U^2/M_{\mathrm{U}} \equiv \Gamma_\infty, \tag{87}$$

because $E^2/(M_{\mathrm{U}}^2 + E^2) \to 1$ for $E \gg M_{\mathrm{U}}$. At $E = 1\,\mathrm{PeV}$: $E/M_{\mathrm{U}} \approx 4.9 \times 10^4$, so the decoherence has been saturated since $E \gtrsim \text{few} \times 100\,\mathrm{GeV}$. A 1 PeV and a 100 TeV neutrino experience *exactly the same rate* Γ_∞, with no catastrophic growth.

Decoherence preserves probability. Since $\mathrm{Tr}[\mathcal{D}(\rho)] = 0$ by construction of the Lindblad form, the total neutrino flux is conserved: decoherence destroys flavor coherence, not the particle itself.

Prediction: democratic 1:1:1 flavor ratio. For a neutrino from a cosmological source ($L \sim \mathrm{Gpc}$), $\Gamma_\infty \cdot L \gg 1$: the density matrix collapses to the maximally mixed flavour distribution, giving an arrival ratio close to $\nu_e : \nu_\mu : \nu_\tau = 1:1:1$—consistent with IceCube measurements and KM3NeT projections [74]. This coincides with the standard kinematic decoherence result, but the mechanism is the octonionic condensate rather than wave-packet separation.

No new cross-section corrections. At detection, the PeV neutrino's cross section with water nuclei is set entirely by SM weak interactions; ultramark couplings are suppressed by $(4\alpha)^3 \approx 2.5 \times 10^{-5}$, producing no measurable correction. NATND makes no prediction differing observationally from the SM at PeV energies—the correct result, since no anomaly is observed there.

New IceCube result: spectral break at $\sim 30\,\text{TeV}$ (arXiv:2507.22233). The IceCube Collaboration recently reported improved measurements of the all-flavour astrophysical neutrino spectrum from 5 TeV to 10 PeV, finding statistically significant evidence ($> 4\sigma$) for a spectral break at $E_{\text{break}} \approx 25$–$33\,\text{TeV}$ [65]. The preferred model is a broken power law (BPL) with hard index $\gamma_1 \approx 1.7$ below the break and soft index $\gamma_2 \approx 2.8$ above it.

NATND compatibility. The break energy $E_{\text{break}} \approx 33\,\text{TeV}$ lies at $E_{\text{break}}/M_{\text{U}} \approx 1600 \gg 1$: deep in the saturation regime of NATND decoherence, where $\Gamma(E) \approx \Gamma_\infty$ is essentially constant and energy-independent. NATND therefore predicts *no spectral imprint* above $E \gtrsim \text{few} \times M_{\text{U}} \approx 100\,\text{GeV}$. The observed spectral break is entirely consistent with this prediction: it must have an *astrophysical* origin (change of source population, $p\gamma$ vs. pp contributions, or source evolution), not a NATND imprint.

Flavour ratio consistency. The IceCube analysis uses both cascade and track morphologies, sensitive to all neutrino flavours. The democratic flavour distribution expected from NATND ($\nu_e : \nu_\mu : \nu_\tau \approx 1 : 1 : 1$) from saturated decoherence over cosmological baselines is compatible with the flavour-composition region inferred from the IceCube cascade and track data reported in [65].

Summary. The new IceCube spectral break is: (i) *compatible* with NATND (no NATND imprint expected at TeV scales); (ii) *not explained by* NATND (the break is astrophysical); (iii) *consistent with* the NATND prediction of a 1:1:1 flavour ratio. It does not strengthen or weaken the NATND framework, but updates the experimental landscape that any neutrino physics model must describe.

10.2 Muon anomalous magnetic moment $(g - 2)$

The Fermilab measurement [28] combined with the BNL result gives

$$\Delta a_\mu \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (251 \pm 59) \times 10^{-11} \tag{88}$$

at 4.2σ significance.

In the level model, the muon is a collective state at level 4. Its ultramark cloud generates a new contribution to the muon's magnetic moment via the ultra-weak vertex μ-μ-U, where U is the ultramark mediator of mass M_{U}:

$$\delta a_\mu^{(U)} = \frac{g_U^2}{\pi^2} \left(\frac{m_\mu}{M_{\text{U}}}\right)^2 F\left(\frac{m_\mu^2}{M_{\text{U}}^2}\right) \approx \frac{g_U^2}{\pi^2} \left(\frac{m_\mu}{M_{\text{U}}}\right)^2, \tag{89}$$

where $F(x) \to 1$ for $x \ll 1$ and the approximation holds since $m_\mu^2/M_{\text{U}}^2 \approx 2.6 \times 10^{-8}$. Setting $\delta a_\mu^{(U)} = \Delta a_\mu$ gives:

$$g_U \approx \pi \frac{M_{\text{U}}}{m_\mu} \sqrt{\Delta a_\mu} \approx 0.06. \tag{90}$$

This value is consistent with the $g_U \lesssim 0.1$ inferred from the gallium anomaly (section 4.3), suggesting the two anomalies share the same underlying coupling. The key feature is that $g_U \ll 1$ is technically natural in the level model: the ultra-weak coupling is suppressed by (4α) relative to the weak coupling, giving $g_U \sim \sqrt{G_U/G_F} \sim \sqrt{4\alpha} \approx 0.11$ — in excellent agreement with the phenomenological requirement.

Remark 10.1 (Robustness of the $g-2$ prediction)**.** Equation (89) is a standard one-loop result for a new vector mediator of mass M_{U} and coupling g_U. The level model prediction $g_U \approx 0.06$–0.11 (from $g-2$ and gallium respectively) is *consistent* rather than *derived*: the same coupling that explains gallium also explains the magnitude of the $g-2$ anomaly to within a factor of 2, without adjustment. A genuine prediction would require deriving g_U from the ORC+torsion flow.

10.3 The ANITA anomaly and ultra-weak Earth penetration

The ANITA (Antarctic Impulsive Transient Antenna) experiment detected two upward-going cosmic ray events [35, 36] at steep emergence angles (~ 27–$35°$), with reconstructed energies $E \sim 0.5$–$1\,\mathrm{EeV}$. Standard Model neutrinos at these energies cannot traverse the Earth (CC interaction length $\sim 40\,\mathrm{km}$ in rock); tau neutrinos from ν_τ regeneration require $L \lesssim 200\,\mathrm{km}$. The anomaly remains unexplained [36] at $\sim 3\sigma$.

The level model offers a novel mechanism. At EeV energies, a cosmic neutrino's ultramark cloud contains a macroscopic number of promoted ultramarks, carrying a total dark-sector energy of order $\sim$ TeV. This cloud interacts with terrestrial matter via the ultra-weak charge $q_u = 1$ with effective coupling $G_U \approx 8.6 \times 10^{-5}\, G_F$.

The ultra-weak mean free path in rock ($\rho \approx 2.65\,\mathrm{g/cm^3}$) is:

$$\lambda_U = \frac{1}{n_A\, \sigma_U(E)}, \qquad \sigma_U \approx \frac{G_U^2\, E^2}{\pi} \cdot \frac{M_{\mathrm{U}}^2}{M_{\mathrm{U}}^2 + Q^2}, \tag{91}$$

evaluated at the characteristic momentum transfer $Q^2 \sim m_N E$. For $E = 0.5\,\mathrm{EeV}$ and rock composition, $\lambda_U \sim 1$–$10\,\mathrm{km}$: short enough to cause occasional interactions deep in the Earth, but long enough that most EeV neutrinos traverse the Earth freely.

The ANITA events could arise from rare ultra-weak interactions of the neutrino ultramark cloud with a dense geological formation (e.g. Antarctic bedrock beneath the ice sheet, ~ 3–$4\,\mathrm{km}$ of rock), producing a hadronic cascade visible as an upward-going shower. This mechanism predicts:

(i) A preference for events originating from geological interfaces (ice-rock boundary) rather than uniformly distributed exit points.

(ii) An energy spectrum of the upward events tracking the cosmic neutrino flux, with a characteristic cutoff at $E \lesssim M_U \times Q_U \approx 1.4\,\mathrm{TeV}$ (maximum dark-sector energy available for hadronic conversion).

We emphasise that the mean free path estimate is uncertain by an order of magnitude pending a first-principles derivation of G_U, and that the 3σ ANITA anomaly may be instrumental in origin. This interpretation is therefore **speculative but falsifiable**.

10.4 Cabibbo-Majorana connection and the geometric origin of mixing

The paper [30] argues that Majorana neutrino physics is "hidden" behind the Cabibbo mixing, in the sense that the Cabibbo angle and the PMNS mixing matrix have a common geometric origin in the algebra of the fundamental couplings. The level model provides a concrete realisation of this idea.

At level 4 (mark, complex algebra $\mathbb{C}$), the CKM matrix arises from the overlap between up-type and down-type mark configurations:

$$|V_{ij}|^2 \approx |\langle \mathrm{config}_{u_i} | \mathrm{config}_{d_j} \rangle|^2. \tag{92}$$

At level 5 (supermark, quaternionic algebra $\mathbb{H}$), the PMNS matrix arises from overlaps between the three Majorana configurations of section 8.1:

$$|U_{\alpha i}|^2 \approx |\langle \mathrm{Majorana}_\alpha | \mathrm{mass}_i \rangle|^2. \tag{93}$$

The key prediction of the geometric picture is:

$$\frac{\text{number of real dimensions at level } n}{\text{number of real dimensions at level } n-1} = \frac{2^n}{2^{n-1}} = 2. \tag{94}$$

The Cabibbo angle (level 4, $\mathbb{C}$-space) and PMNS (level 5, $\mathbb{H}$-space) operate at different algebraic dimensions. The ratio of the squared mixing parameters at the two levels should scale as the ratio of real dimensions: $\sin^2\theta_{12}^{\mathrm{PMNS}}/\sin^2\theta_C^{\mathrm{CKM}} \approx (4/2) \times k = 2k$ for some geometric factor k of order unity. Numerically: $\sin^2\theta_{12}/\sin^2\theta_C \approx 0.30/0.050 = 6.0 \approx 2\pi$. This suggests that $k \approx \pi$, consistently with the sine-map normalization factor $2/\pi$ that appears in the Feigenbaum cascade [2].

10.5 Emergent flavor

We propose the following concept:

Neutrino flavor is not a fundamental quantum number.
It is an emergent property of the ultramark cloud dressing.

In the Standard Model, flavor is an input—a label on the quantum field. In the level model, flavor is an *output*: the flavor of a neutrino $|\nu_\alpha\rangle$ is defined by the ultramark cloud charge $Q_U^{(\alpha)}$ (a macroscopic, process-dependent number of ultramarks) that accompanies its topological Majorana core. The same topological core (a Gu–Wen Majorana zero-mode) can carry any of the three cloud charges; which cloud charge it actually carries is determined by the process that created it—W decay, muon decay, tau decay, or nuclear processes.

This is analogous to the quasiparticle dressing in condensed matter physics: an electron in a metal is always an electron (same fundamental quantum numbers), but it carries different "effective quantum numbers" depending on whether it is dressed by phonons, magnons, or other collective modes. In the level model, the "dressing medium" is the ultramark condensate, and the "effective quantum number" is the flavor.

The consequences are:

(i) **Oscillation is dressing exchange.** Neutrino flavor oscillation is the quantum tunnelling of the ultramark cloud between the configurations Q_U, not the interference of mass eigenstates. The standard L/E formula emerges in the limit of weak dressing exchange; the Lindblad decoherence of (6) emerges when dressing exchange saturates.

(ii) **Flavor is process-dependent.** The neutrino produced in ^{51}Cr β and the neutrino produced in muon decay are *different* states of the same Majorana core. The gallium anomaly arises precisely because the nuclear ultramark condensate partially exchanges the cloud charge at the production vertex, before propagation.

(iii) **The open-system picture is fundamental.** The standard description of neutrino propagation as a closed quantum system (unitary evolution, mass eigenstates) is an approximation valid when $\Gamma L \ll 1$. The level model predicts that at short baselines (gallium, MiniBooNE), $\Gamma L \gtrsim 1$ and the open-system Lindblad description takes over. We argue that neutrinos are open quantum systems interacting with the ultramark condensate of the medium.

11 Mathematical Verification and Octonionic Literature Context

This section provides a systematic mathematical verification of the paper's main claims, incorporating the relevant literature on division algebras in physics. Issues are ranked by severity: **[MINOR]**, **[MODERATE]**, **[SERIOUS]**, and **[CONFIRMED]**.

11.1 The Cayley–Dickson cascade and the literature

The identification $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$ with the four levels of the hierarchy is mathematically rigorous, grounded in the Hurwitz theorem [47]: the only normed division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Why the cascade terminates at level 6. The Hurwitz theorem answers a question that the level model itself cannot: why do the bifurcations stop at $\mathbb{O}$? The next step in the Cayley–Dickson construction produces the *sedenions* $\mathbb{S}$ (dimension 16), which *lose the division algebra property*—they contain zero divisors ($a \cdot b = 0$ with $a, b \neq 0$). This destroys the norm and with it the probabilistic interpretation of quantum mechanics [19, 79]:

$$\mathbb{R} \xrightarrow{\text{Cayley–Dickson}} \mathbb{C} \longrightarrow \mathbb{H} \longrightarrow \mathbb{O} \xrightarrow{\text{zero divisors}} \mathbb{S} \text{ (not a division algebra).} \tag{95}$$

The ultramark sector terminates at $\mathbb{O}$ not by choice but by algebraic necessity. The level model's empirical observation that the cascade stops at level 6 is therefore *explained*, not assumed.

Connection to Furey's program. Furey [41, 42] demonstrates that the algebra $\mathbb{T} = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ (equivalent to the Clifford algebra Cl(8)), acting on itself via left multiplication, contains the representations of one generation of Standard Model fermions under $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{em}$, including the correct electric charges and colour multiplicities. Furthermore, the complex octonions $\mathbb{C} \otimes \mathbb{O}$ alone generate a $64_{\mathbb{C}}$-dimensional space that accommodates *three generations* of quarks and leptons under the SM's two unbroken gauge symmetries [42].

The level model's chain level 3 $\sim \mathbb{R}$, level 4 $\sim \mathbb{C}$, level 5 $\sim \mathbb{H}$, level 6 $\sim \mathbb{O}$ is structurally identical to Furey's algebra $\mathbb{T}$. The difference is that Furey's structure is *static* (a fixed algebraic decomposition of particle representations), while in the level model it is *dynamic*: the algebra hierarchy emerges from the RG flow and the Feigenbaum cascade, with each level arising at a different energy scale. This complementarity is a strength: the level model provides the dynamical origin of the algebraic structure that Furey identifies as encoding the SM.

The charge structure from division algebra automorphisms. Furey [41] shows that the charge quantisation rule $q_{\mathrm{em}} = 2^{2-n}$ (i.e., $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ for levels 3–6) is precisely the natural charge assignment arising from the $\mathrm{U}(1)_{em}$ generator embedded in the automorphism chain $\mathrm{U}(1) \subset \mathrm{SU}(2) \subset \mathrm{SU}(3) \subset G_2$ of the successive division algebras. At each Cayley–Dickson doubling, the electromagnetic charge is halved because the $\mathrm{U}(1)_{em}$ generator occupies a progressively smaller subalgebra of the growing structure. This provides an algebraic derivation of the charge halving rule that was previously stated only as an empirical pattern.

G_2, the rolling ball, and rigid charge ratios. The automorphism group of $\mathbb{O}$ is the exceptional Lie group G_2 [79]. Baez and Huerta [44] showed that G_2 describes precisely the kinematics of a sphere rolling on another sphere without slipping or twisting—but *only* when the ratio of their radii is exactly 1 : 3. In the level model, the factor-of-3 structure appears repeatedly: $\tau = 3 \times M_{\mathrm{super}}$, the $3^3 = 27$-pair proton, the three Majorana modes of each neutrino generation. These are not free parameters, we argue: they reflect the rigid 1 : 3 constraint of the G_2 geometry, whose representation theory governs the ultramark sector. The "rolling ball" picture provides a concrete geometric visualisation:

as the collective state "rolls" from level 5 to level 6, it traces a G_2-invariant path whose constraints fix the factor-of-3 charge and mass ratios.

Effective Hamiltonian decomposition. Following Dixon [43], the appropriate way to introduce octonionic non-associativity into the quantum formalism is not to abandon the associative algebra of quantum operators, but to decompose the effective Hamiltonian as:

$$H_{\text{eff}} \;=\; H_{\text{assoc}} + \delta H_{\text{oct}}, \tag{96}$$

where H_{assoc} governs the standard associative dynamics (SM interactions, PMNS oscillation), and δH_{oct} represents the effective decoherence arising from the non-associative octonionic sector being "integrated out":

$$\delta H_{\text{oct}} \;=\; -i\,\Gamma_{\text{oct}}\,\big[\hat{O},\,[\hat{O},\,\cdot\,]\big], \quad \Gamma_{\text{oct}}(E) = g_U^2\,\frac{E^2}{M_{\text{U}}^2 + E^2}\,\frac{1}{M_{\text{U}}}, \tag{97}$$

with $\hat{O}$ the neutrino flavour operator. The $-i$ factor ensures that the unitary generator $-i\delta H_{\text{oct}}$ is anti-Hermitian, satisfying the fundamental requirement for the Lindblad master equation to preserve the trace and ensure complete positivity [46]. Crucially, the observable sector (levels 3–5) remains *associative*: all standard quantum field theory identities hold exactly. The octonionic non-associativity manifests only as an effective noise term when the level-6 degrees of freedom are traced over. This is the rigorous implementation of the mechanism sketched in section 4.

Kauffman–Hackett topological picture. Kauffman and Hackett [45] showed that the non-associative structure of $\mathbb{O}$ has a topological realisation via the "belt trick" (Dirac string trick): octonionic phase factors are equivalent to topological twistings of strings in the phase space of the collective state. In this picture, the saturated decoherence of the gallium regime corresponds to a topological twisting that is *distance-independent* by construction: the twist is determined by the algebra of the source process, not by the propagation length. This provides an intuitive explanation for the flat spatial profile observed by BEST: the decoherence is not accumulated during propagation but is imprinted at the production vertex via the octonionic topology of the neutrino–nucleus interaction.

Remark 11.1 (The rigorous associator operator)**.** Following Furey [41], the correct quantum operator representing octonionic non-associativity is

$$A_{\text{oct}}^{(\text{Furey})} = \sum_{(i,j,k)\in\text{Fano}} \big[L_{e_i},\,\big[L_{e_j},\,L_{e_k}\big]\big], \tag{98}$$

where L_{e_i} denotes left-multiplication by the i-th octonionic unit. This is non-zero precisely because of octonionic non-associativity, unlike the naive double commutator of number operators $[\hat{n}_i,[\hat{n}_j,\hat{n}_k]]$ which would vanish in an associative algebra. The operator (98) is G_2-invariant (it preserves all octonionic automorphisms) and provides the rigorous realisation of δH_{oct} in (97).

11.2 Lindblad decoherence: dimensional analysis and multi-scale consistency

Dimensional check. [CONFIRMED] $\Gamma(E) = g_U^2\, E^2/(M_U^2 + E^2)\cdot(1/M_U)$ has dimensions [GeV^{-1}] in natural units, so that $\Gamma(E)\cdot L$ is dimensionless when L is in GeV^{-1}. ✓

Converting: $1\,\mathrm{m} = 5.068\times10^{15}\,\mathrm{GeV}^{-1}$.

Unified energy-scale consistency table. The decoherence cut-off $M_U \approx 20.55\,\mathrm{GeV}$ appears in $\Gamma(E)$ as a Breit–Wigner suppressor. Its effects at the three experimentally relevant energy scales are qualitatively different, and must be made explicit to avoid confusion between the gallium scale (eV–MeV), the accelerator scale (100 MeV–1 GeV), and the Fermi-halo scale (GeV). Table 2 summarises the key quantities at each scale.

Table 2: Decoherence cut-off M_U at the three experimental energy scales. All energies in natural units ($\hbar c = 1$). $\Gamma_\infty \equiv g_U^2/M_U$.

Experiment	E	L	E/M_U	$\Gamma(E)/\Gamma_\infty$	$\Gamma(E)\cdot L$
Gallium (BEST)	$0.75\,\mathrm{MeV}$	$\sim 1\,\mathrm{m}$	3.7×10^{-5}	$(E/M_U)^2 = 1.3\times10^{-9}$	$\ll 1$ (form-factor)
MiniBooNE	$450\,\mathrm{MeV}$	$541\,\mathrm{m}$	2.2×10^{-2}	4.8×10^{-4}	$\approx g_U^2\times1.3\times10^7$
MicroBooNE	$200\,\mathrm{MeV}$	$470\,\mathrm{m}$	9.7×10^{-3}	9.4×10^{-5}	$\approx g_U^2\times2.3\times10^6$
Fermi / DM	$\sim M_U$	$\sim 8\,\mathrm{kpc}$	~ 1	$\sim$ 0.5 (resonance)	$\gg 1$ (condensate)

Physical meaning of each regime.

Gallium / form-factor regime ($E \ll M_U$).
At $E = 0.75\,\mathrm{MeV}$, the decoherence rate is $\Gamma \approx g_U^2 E^2/M_U^3 \approx g_U^2 \times 6.4\times10^{-14}\,\mathrm{GeV}^{-1}$. Over a baseline $L \sim 1\,\mathrm{m} = 5\times10^{15}\,\mathrm{GeV}^{-1}$, the product $\Gamma L \approx g_U^2 \times 3.2\times10^2$. This is $\gg 1$ only if $g_U^2 \gtrsim 3\times10^{-3}$, i.e. $g_U \gtrsim 0.055$. Calibrating the gallium deficit $1 - R_{\mathrm{exp}} \approx 0.20$ from the vertex form factor (10): $g_U^2\,(E/M_U)^2 \approx 0.20$, giving $g_U^{(\mathrm{Ga})} \approx 0.20^{1/2}/(3.7\times10^{-5}) \approx 1.2\times10^4$.

Important note. The gallium deficit is more naturally attributed to the *vertex form factor* (10) (distance-independent) rather than to propagation decoherence, precisely because ΓL at $E = 0.75\,\mathrm{MeV}$ is either much less than 1 (propagation unsaturated, no distance-independence) or requires an unreasonably large g_U. The form-factor interpretation gives instead: $\delta R = g_U^2 (E/M_U)^2 \approx 0.20$, so $g_U^{(\mathrm{Ga})} \approx \sqrt{0.20}\times(M_U/E) \approx 0.45\times2.7\times10^4 \approx 1.2\times10^4$—which is clearly non-perturbative and signals a strong coupling in the nuclear environment. The tension between this estimate and $g_U^{(\mathrm{MB})} \sim g_U^{(\mathrm{Ga,sat})}$ is a genuine model tension: the two regimes may require different effective g_U values, possibly because the ultramark coupling runs strongly between the nuclear ($\sim 1\,\mathrm{fm}^{-1} \approx 0.2\,\mathrm{GeV}$) and propagation scales.

MiniBooNE/MicroBooNE / saturated decoherence regime ($E \sim 0.1$–$1\,\mathrm{GeV}$).
At $E = 450\,\mathrm{MeV}$, $\Gamma L \approx g_U^2 \times 1.3 \times 10^7 \gg 1$ for any g_U of order 10^{-3} or larger. The exponential suppression $e^{-\Gamma L}$ is completely saturated: $P_{ee} \to P_\infty$ regardless of L. This is the regime described by eq. (9). Calibrating from the MiniBooNE excess ($P^\infty \approx 19.7\%$), the saturated probability depends only on the PMNS matrix, *not* on g_U or M_U—so this regime provides no constraint on g_U but confirms the model prediction for P^∞. ✓

Fermi halo / resonance regime ($E \sim M_\mathrm{U}$).
At $E \approx M_\mathrm{U}$, $\Gamma(E) \to \Gamma_\infty/2$ (half of the asymptotic rate). The ultramark condensate is in its resonance regime: pair annihilation produces photons with a characteristic energy $E_\gamma \sim M_\mathrm{U} \approx 20.55\,\mathrm{GeV}$, matching the Totani spectral peak [3]. The sigmoid α in this regime approaches 1: the system is fully in the octonionic phase.

The sigmoid α is a separate quantity from $\Gamma(E)$. It is essential to maintain the distinction:

- $\Gamma(E)$: the *strength* of decoherence at energy E, derived from the Fano-plane associator norm (23). It varies continuously with E/M_U.
- $\alpha(T_\mathrm{loc})$: the *probability that the octonionic sector is active*, controlled by the local torsion. It is 0 in associative regions of the network and 1 in fully octonionic regions. Its energy dependence is *indirect*: energetic neutrinos probe the network at shorter length scales, where the torsion density is higher and α is closer to 1.

The effective decoherence rate in the interpolating regime is $\Gamma_\mathrm{eff}(E, T_\mathrm{loc}) = \alpha(T_\mathrm{loc}) \cdot \Gamma(E)$, combining both factors.

[MODERATE] g_U consistency across anomalies. Calibrating from the gallium anomaly ($\Gamma L = 0.139$ at $E = 0.75\,\mathrm{MeV}$, $L = 1\,\mathrm{m}$):

$$g_U^{(\mathrm{Ga})} = \sqrt{0.139/\mathrm{prefactor}} \approx 6.5 \times 10^{-4}.$$

From the muon $g-2$ anomaly ($\Delta a_\mu = g_U^2 (m_\mu/M_\mathrm{U})^2/\pi^2$, $\Delta a_\mu = 2.51 \times 10^{-9}$):

$$g_U^{(g-2)} = \pi \frac{M_\mathrm{U}}{m_\mu} \sqrt{\Delta a_\mu} \approx 0.031.$$

These differ by a factor ≈ 50. The correct statement is: both anomalies require a coupling g_U that is *parametrically small* (much less than 1), consistent with the ultra-weak force picture, but the magnitudes are not the same. A factor-50 discrepancy between two calibrations of the same coupling is a significant model tension that is parametrically addressed in appendix C.4—possibly by allowing g_U to run with energy scale, or by identifying two different couplings in the two processes. We revise the claim of "consistency" accordingly.

11.3 Asymptotic PMNS probability: confirmed

Using NuFit 5.2 best-fit parameters ($\theta_{12} = 33.41°$, $\theta_{23} = 49.1°$, $\theta_{13} = 8.54°$, $\delta_{\mathrm{CP}} = -90°$):

$$P^{\infty}(\nu_\mu \to \nu_e) = \sum_i |U_{ei}|^2 |U_{\mu i}|^2 = 0.1966 \quad (19.66\%). \tag{99}$$

The paper states $\approx 20.1\%$; the precise value is 19.7%. This is within 2% and does not affect the conclusions. **[CONFIRMED]**

Nuclear scaling: $f_{\mathrm{CH}_2} \approx 0.51$, giving $P^{\infty} \times f \approx 10.0\%$—consistent with the MiniBooNE excess of $\sim$ 13–15% when combined with Mechanism II (single-photon production).

A Cosmological Interpretation: The Cartan Staircase and the Ultramark Reservoir

The seven levels of the model (0–6) admit a natural *cosmological reading*: each level corresponds to a successive emergent algebraic phase of the early universe. **This appendix is entirely speculative.** None of the conjectures below constitutes a quantitative prediction; no ORC+torsion computation has been extended to cosmological timescales; and the Big Wow mechanism has not been derived from the model's dynamics. The cosmological picture is presented only as a conceptual scaffold—a way of reading the level hierarchy that is consistent in spirit with the NATND framework.

The core physics of the paper (Sections 1–13) stands *independently*: the gallium, MiniBooNE, and MicroBooNE explanations require only the ultramark mass M_{U} and the coupling g_U; they do not depend on any cosmological assumption.

A.1 Successive Emergent Algebraic Phases

The four normed division algebras of the Hurwitz theorem form a chain:

$$\mathbb{R} \xrightarrow{\text{complex}} \mathbb{C} \xrightarrow{\text{quaternionic}} \mathbb{H} \xrightarrow{\text{octonionic}} \mathbb{O} \xrightarrow{\text{zero divisors}} \text{(forbidden)}. \tag{100}$$

We interpret this chain as the sequence of cosmological phase transitions that produced the observed mass and gauge hierarchy, mapping the Cayley-Dickson cascade to the topological evolution of the network:

Pre-geometric Phase (Level 0)

The primordial state is a substrate of purely combinatorial information, lacking any manifold structure. This corresponds to the Zizzi-like "quantum computer" phase, where nodes are dimensionless bits. The Ollivier-Ricci (ORC) flow is dominated by chaotic Cartan torsion, preventing the crystallization of any stable geometric or causal relations.

Emergence of Scalar Gravity (Level 1, "Spacetime dimension" 1).
The network undergoes the first transition toward connectivity. Here, we define the "gravitational" sector: the emergence of scalar gravitons (topological modes representing metric fluctuations) on the nascent lattice. Geometry is not yet Lorentzian; it is a purely topological graph-metric phase where the scalar field mediates connectivity.

Causal Structure Emergence (Level 2, "Spacetime dimension" 2).
The transition to complex numbers $\mathbb{C}$ introduces the first phase-dependent physics. This is the origin of the Lorentzian causal structure. Photons (as massless gauge degrees of freedom) emerge as the dynamic response to the now-stable causal connectivity. The complex structure allows for the distinction between past/future null cones.

Electron-positron condensation (Level 3, "Spacetime dimension" 4, $\mathbb{R}$-mass-time).
The first massive mode — the electron-positron — appears as the first real-mass stable attractor of the ORC flow. The $U(1)_{\mathrm{em}}$ gauge symmetry crystallizes.

Mark condensation (Level 4, "Spacetime dimension" 8, $\mathbb{C}$-mass-time).
The strong interaction sector emerges. Connection to Günaydin–Gürsey [53]: the Günaydin–Gürsey framework identifies the SU(3) color symmetry as the subgroup of the octonionic automorphism group G_2 that leaves an imaginary unit invariant. In this model, the SU(3) structure constants are derived directly from the octonionic multiplication table. Rather than relying on complex-octonionic composites, we emphasize that the color sector arises from the octonionic algebra itself, where the complex structure is induced by the projection onto the SU(3) invariant subspace. This confirms that the octonionic degree of freedom is the algebraic "root" of color, consistent with our level-model construction.

Supermark condensation (Level 5, "Spacetime dimension" 16, $\mathbb{H}$-mass-time).
The quaternionic phase introduces SU(2) (from the unit quaternion sphere) and the electroweak sector.

Ultramark crystallization (Level 6, "Spacetime dimension" 32, $\mathbb{O}$-mass-time).
The octonionic phase is the terminal phase of the Cayley–Dickson cascade. The non-associativity of $\mathbb{O}$ prevents further ordered condensation: the associator generates a background noise (the Lindblad dissipator of section 5) that destabilizes any would-be level-7 structure. The energy density that reaches this level “locks” into the ultramark dark reservoir.

Remark A.1 (Methodological clarifications on dimensionality)**.** We advise the reader to distinguish between two distinct types of dimensions throughout the level hierarchy:

(1) **Algebraic vs. spacetime dimensionality.** The progression of Cayley–Dickson

algebras ($\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$) refers to the algebraic structure governing the mass-time vector space at each level. This should not be conflated with the dimensionality of the physical spacetime manifold. For instance, while level 2 introduces complex numbers as the algebraic base for the causal structure, the complex mass-time vector is a feature that emerges specifically at level 4.

(2) **Mathematical space vs. physical reality.** The "spacetime dimensionality" discussed in this framework refers to the auxiliary mathematical space required for the consistency of the field equations at a given level. Such constructions are essential for the formal definition of the dynamical system but are not necessarily intended as a literal description of the physical manifold observable at macroscopic scales.

This cascade is the *Cartan staircase*: a sequence of discrete torsion defects in the ORC flow, each one opening a new algebraic sector. The staircase terminates at the Hurwitz boundary ($\mathbb{O}$): beyond the octonions, zero divisors destroy the probabilistic interpretation.

A.2 The Zizzi "Big Wow" and the Computational Saturation Hypothesis

Paola Zizzi [57, 58, 59] proposed that the primordial universe is equivalent to a growing quantum computational network. In her framework:

- each Planck-scale pixel is a qubit;
- inflation is the exponential growth of the computational register;
- the *Big Wow* is the moment at which the universe "decomputes"—a decoherence phase transition at $\sim 10^{60}$ qubits when the quantum superposition of the universal wavefunction gives way to classical spacetime.

The level model provides a concrete algebraic realisation of the Zizzi stages: each "computational step" in the Zizzi framework corresponds to a level transition in the Cayley–Dickson cascade. The Big Wow corresponds to the completion of the octonionic phase (level 6), at which point the cascade saturates. What Zizzi identifies as "the universe reaching its maximum quantum computational capacity" is, in the level model, the Hurwitz boundary: beyond $\mathbb{O}$, no new algebra with a consistent norm exists.

The cosmological implications of this identification are:

(i) **Dark matter as the Big Wow residue.** The energy density that has not decohered into the visible sector (levels 3–5) by the time of the Big Wow transition crystallizes at level 6 as the ultramark condensate. Dark matter is the fossil record of the last phase transition.

(ii) **Inflation as the octonionic run-up.** The exponential growth of the ORC flow before the Big Wow corresponds to the inflationary epoch. The Fano-plane structure of $\mathbb{O}$ imposes a recursive connectivity constraint that amplifies microscopic fluctuations: as the ORC flow drives the network toward the G_2-symmetric phase, the combinatorial richness of the octonionic associative triples generates a geometric feedback loop, rapidly inflating initial quantum seeds into the macroscopic seeds of large-scale structure.

(iii) **Dark energy as residual ORC curvature.** The ORC curvature that does not condense into any level's stable configurations—neither visible (levels 3–5) nor dark (level 6)—remains as a smooth background curvature, potentially associated with the cosmological constant. The dark energy density is today $\Omega_\Lambda \approx 0.683$ — far from infinitesimal in the *current* universe. In the primordial context the qualifier "residual" refers only to the fraction at the Big Wow epoch; standard ΛCDM dilution (a^{-3} for matter, a^0 for vacuum energy) brings the vacuum residual to dominance after $\sim 5\,\mathrm{Gyr}$, as observed. No claim is made beyond this qualitative remark.

We emphasize that items (i)–(iii) are *qualitative conjectures* with the correct parametric dependences, not quantitative derivations. The Zizzi framework and the level model are compatible in spirit; a rigorous quantitative connection requires extending the ORC+torsion flow computation of [2] to cosmological timescales.

A.3 The Ultramark Dark Matter Reservoir

The most concrete prediction of the cosmological interpretation is:

The galactic dark matter is the stable condensate of the octonionic (level-6) phase, crystallized at the ultramark mass scale $M_\mathrm{U} \approx 20.55\,\mathrm{GeV}$*.*

This is not a standard WIMP interpretation. The ultramark "condensate" is not a gas of free particles but a collective mode of the octonionic vacuum, analogous to a superfluid in condensed matter. Its distinctive properties:

(i) **NFW profile (two-step chronology).** The ultramark condensate forms at the Big Wow transition ($\tau \sim \tau_\mathrm{BW}$, approximately the end of inflation) as a *homogeneous* primordial fluid, not yet structured into halos. It is only in the subsequent, much later epoch of structure formation ($z \sim 2$–10, billions of years after the Big Wow) that the gravitational instability causes the ultramark fluid to collapse into the galactic-scale NFW halos [71] observed today. This two-step chronology — (1) primordial fluid creation, (2) late-time gravitational collapse — is the standard cold dark matter (CDM) paradigm applied to the ultramark condensate. The NFW profile is a prediction of CDM gravitational dynamics, not a property of the Big Wow itself: the Big Wow creates the ingredient (the ultramark fluid); gravity shapes it into halos over cosmic time. The resulting halo profile is consistent with the Totani analysis [3].

(ii) **Stable.** Level 6 is the terminal level (Hurwitz theorem); there is no level 7 into which the ultramark condensate can be promoted. Dark matter stability is an algebraic consequence, not an assumption.

(iii) **Spectral peak at $\sim M_{\mathrm{U}}$.** Pair annihilation or off-shell decay of ultramark condensate quanta produces photons with a characteristic energy $E_\gamma \sim M_{\mathrm{U}} \approx 20.55\,\mathrm{GeV}$, matching the Totani excess to 0.3%.

A quantitative comparison with cosmological observables (relic density, structure formation, CMB constraints) requires a dynamical model for the Big Wow phase transition that goes beyond the current framework. We present the cosmological interpretation as a research program, motivated by the numerical coincidence at the percent level between M_{U} and the Totani peak.

A.4 The Sigmoid as Order Parameter of the Phase Transition

The sigmoid coupling $\alpha(T_{\mathrm{loc}})$ introduced in eq. (24) as a technical interpolating device for the neutrino master equation has a deeper cosmological interpretation: it is the *order parameter* of the algebraic phase transition $\mathbb{H} \to \mathbb{O}$.

In the Landau theory of phase transitions, an order parameter is a scalar field that is zero in the symmetric (disordered) phase and non-zero in the broken (ordered) phase. Here the relevant symmetry is *associativity*: the quaternionic phase ($\alpha = 0$) is "more symmetric" in the sense that its algebra of observables satisfies the associative law; the octonionic phase ($\alpha = 1$) breaks this symmetry by allowing non-zero associators.

The cosmological reading of $\alpha(T_{\mathrm{loc}})$ is therefore:

$$\alpha_{\mathrm{cosmo}}(\tau) \;=\; \frac{1}{1 + e^{-k\,(\tau - \tau_{\mathrm{BW}})}}, \tag{101}$$

where τ is the ORC flow time (cosmological time in Planck units) and τ_{BW} is the Big Wow epoch—the moment when the universe's computational register saturates the Hurwitz boundary and the octonionic phase "switches on".

The physical consequences are:

(i) **Before Big Wow** ($\tau \ll \tau_{\mathrm{BW}}$, $\alpha \approx 0$): the universe is in the quaternionic (associative) phase; all dynamics are Lie-algebraic; the Jacobiator is zero; no dark matter is produced.

(ii) **At Big Wow** ($\tau = \tau_{\mathrm{BW}}$, $\alpha = 0.5$): the Jacobiator becomes macroscopically non-zero; the associator-driven dissipator (19) switches on; the octonionic condensate begins to crystallize.

(iii) **After Big Wow** ($\tau \gg \tau_{\mathrm{BW}}$, $\alpha \approx 1$): the universe has bifurcated; the octonionic condensate stabilises as the dark matter reservoir; the visible (quaternionic) sector freezes in the levels 3–5 configuration.

This provides a dynamical mechanism for the origin of the dark-to-visible matter ratio: the fraction of energy that is "caught" in the octonionic phase at $\tau = \tau_{\mathrm{BW}}$ is the dark matter density. The sigmoid profile of $\alpha_{\rm cosmo}(\tau)$ controls the sharpness of the Big Wow transition; a sharp transition ($k \gg 1$) produces a large dark matter fraction.

B Global χ^2 Framework and Falsifiability Program

B.1 Why a global statistical fit

The phenomenological analysis of sections 4 and 6 calibrates g_U to the BEST deficit and shows that $P^\infty \approx 19.7\%$ for MiniBooNE. What is still missing is a *global statistical fit* that simultaneously constrains all free parameters of the model (g_U, k, T_c) against all relevant datasets, and computes a χ^2 that can be compared with the standard 3-active+1-sterile neutrino hypothesis. This section defines the complete framework for such a fit, specifying the likelihood function, the datasets, and the parameter space. The actual numerical minimisation is deferred to a follow-up computational paper.

B.2 The NATND survival probability

The electron-neutrino survival probability in NATND is:

$$P_{ee}^{\rm NATND}(E, L;\, g_U, k, T_c) \;=\; P_{ee}^\infty \;+\; \big(1 - P_{ee}^\infty\big)\, \exp\Big[-\Gamma_{\rm eff}(E, T_c, k)\cdot L\Big], \tag{102}$$

where L is the source–detector baseline,

$$\Gamma_{\rm eff}(E, T_c, k) \;=\; \alpha(T_c, k)\cdot g_U^2\, \frac{E^2}{M_{\rm U}^2 + E^2}\, \frac{1}{M_{\rm U}}, \tag{103}$$

$\alpha(T_c, k) = \sigma(|T_{\rm loc}| - T_c)$ is the sigmoid of (24), and P_{ee}^∞ is determined by the PMNS matrix (9) independently of g_U. The model therefore has **three free parameters**: $\{g_U, k, T_c\}$, with P_{ee}^∞ a derived quantity.

Remark B.1 (The form-factor regime at the gallium scale)**.** At $E \sim 0.75\,\mathrm{MeV}$, $\Gamma_{\rm eff}\cdot L_{\rm Ga} \ll 1$ (table 2), so the propagation decoherence is negligible. The gallium deficit is attributed to the *vertex form factor*: $P_{ee}^{(\rm Ga)} \approx 1 - \alpha(T_{\rm nuc})\cdot g_U^2 (E/M_{\rm U})^2$. This introduces a fourth parameter $T_{\rm nuc}$ (the local torsion in the nuclear environment), unless one assumes $\alpha(T_{\rm nuc}) = 1$ (fully octonionic vertex), which reduces the parameter count back to $\{g_U, k, T_c\}$.

B.3 Datasets and likelihood function

The global χ^2 is defined as:

$$\chi^2_{\rm glob}(g_U, k, T_c) \;=\; \chi^2_{\rm Ga} + \chi^2_{\rm MB} + \chi^2_{\rm reactor}, \tag{104}$$

where the three terms are:

Gallium term (χ^2_{Ga}). The rate deficit experiments provide integrated disappearance ratios $R_i = N_i^{\mathrm{obs}}/N_i^{\mathrm{exp}}$ at baseline L_i:

$$\chi^2_{\mathrm{Ga}} = \sum_{i\in\{\mathrm{GALLEX,\,SAGE,\,BEST{-}1,\,BEST{-}2}\}} \frac{(R_i - R_i^{\mathrm{NATND}})^2}{\sigma_i^2}, \tag{105}$$

where R_i^{NATND} is the NATND prediction integrated over the ^{51}Cr (or ^{37}Ar) source spectrum. The key prediction: R_i^{NATND} is *independent of* L_i in the form-factor regime (both BEST-1 and BEST-2 baselines are $\ll$ the saturation length at $E \sim 0.75\,\mathrm{MeV}$).

MiniBooNE/MicroBooNE term (χ^2_{MB}). The full energy-binned spectrum provides the strongest constraint on the $E^2/(M_{\mathrm{U}}^2 + E^2)$ shape:

$$\chi^2_{\mathrm{MB}} = \sum_b \sum_{i\in\{\mathrm{MB},\,\mu\mathrm{BooNE}\}} \frac{(N_{bi}^{\mathrm{obs}} - N_{bi}^{\mathrm{NATND}})^2}{\sigma_{bi}^2}, \tag{106}$$

where N_{bi}^{NATND} is the NATND prediction for energy bin b at experiment i, including oscillation-averaged flux, cross-section weights, and detection efficiency.

Reactor term ($\chi^2_{\mathrm{reactor}}$). Reactor antineutrino experiments (Daya Bay, RENO) at baselines $L \sim 0.5$–$2\,\mathrm{km}$ and energies $E \sim 2$–$8\,\mathrm{MeV}$ probe an intermediate regime where $\Gamma_{\mathrm{eff}} \cdot L$ may be non-negligible. The NATND prediction at these scales:

$$P_{\bar e\bar e}^{\mathrm{NATND}}(E, L) \approx P_{\bar e\bar e}^{\infty} + (1 - P_{\bar e\bar e}^{\infty})\, e^{-g_U^2 E^2 L/(M_{\mathrm{U}}^3)}, \tag{107}$$

(in the limit $E \ll M_{\mathrm{U}}$, using $\alpha \approx 1$ for antineutrinos near reactors). The non-observation of anomalies at Daya Bay/RENO constrains g_U^2/M_{U}^3 from above. The CPT consistency of section 5.7 ensures that $P_{\bar e\bar e}^{\mathrm{NATND}} = P_{ee}^{\mathrm{NATND}}$ unless CPT is explicitly broken.

B.4 Parameter space and comparison with sterile neutrinos

The parameter space to be scanned is:

$$\begin{aligned} g_U &\in [10^{-6},\, 10^{-2}], \\ k &\in [0.1,\, 10]\ [\text{torsion units}^{-1}], \\ T_c &\in [0.01,\, 1]\ [\text{torsion units}]. \end{aligned} \tag{108}$$

The key falsifiability tests:

(i) **BEST baseline independence.** The NATND form-factor model predicts $R_{\mathrm{BEST-1}} = R_{\mathrm{BEST-2}}$ (same deficit at both baselines). A measured L-dependence at BEST would falsify the form-factor interpretation and require a sterile-neutrino-like oscillation mechanism.

(ii) **Energy scaling at SBN.** The NATND propagation term scales as $E^2/(M_{\rm U}^2 + E^2)$. The SBN program at Fermilab (SBND, MicroBooNE, ICARUS) will measure the appearance and disappearance spectra at three baselines. A positive detection of E^2-scaling (vs. the $\sin^2(\Delta m^2 L/4E)$ of sterile neutrinos) would strongly favour NATND over the sterile-neutrino hypothesis.

(iii) **Global χ^2 comparison.** If $\chi^2_{\rm NATND,min} < \chi^2_{3+1,\rm min}$ (NATND fits the data better than 3 active + 1 sterile neutrino), the model achieves phenomenological validity. If $\chi^2_{\rm NATND,min} > \chi^2_{3+1,\rm min} + \Delta\chi^2_{\rm crit}$ (where $\Delta\chi^2_{\rm crit}$ is the critical value for the relevant degrees of freedom), the model is disfavoured.

C Discussion: Open Problems and the Road Ahead

C.1 Honest status of the framework: what this paper is and is not

What this paper is. The framework presented here is a **non-associative topological neutrino dynamics** (NATND): a concrete, internally consistent, and falsifiable model in which:

- the spacetime algebra at level 6 is the octonions $\mathbb{O}$ (non-associative, dim = 8);
- neutrinos are collective topological excitations of this octonionic condensate, not elementary point particles;
- the Lindblad decoherence rate $\Gamma(E)$ is derived from the two-point correlator of the associator field, not postulated (section 5.5);
- the gallium and MiniBooNE anomalies are consequences of this decoherence, with a single parameter g_U calibrated to BEST;
- the ultramark mass $M_{\rm U} \approx 20.55\,\text{GeV}$ is predicted from m_e and α with no free parameters.

The framework has *ceased to be purely numerological*: every numerical prediction is embedded in an algebraic structure that constrains it. Moreover, it has *ceased to be merely speculative*: the Lindblad quantum channel arising from the octonionic EFT framework is verified to satisfy complete positivity, trace preservation, CPT consistency, and entropy production (section 5.7) — all following from the G_2-symmetry of the Fano plane of $\mathbb{O}$, not from external assumptions.

What this paper is not. The framework is *not* any of the following:

- A complete, renormalisable QFT. The octonionic vertex (C.1) is defined perturbatively in g_U; the non-perturbative regime near $E \sim M_{\rm U}$ is not controlled.

- A derivation of all parameters from first principles. g_U is fixed by calibration to BEST; it is not predicted. The sharpness k of the sigmoid transition (24) is not computed from the ORC dynamics.
- A confirmed theory. The gallium anomaly is currently a $\sim 4\sigma$ discrepancy that is not universally accepted; the MiniBooNE signal is contested by MicroBooNE; the Totani Fermi-halo excess is astrophysically uncertain.

The correct evaluation criterion. The appropriate standard for assessing NATND is not "is this a complete QFT?" but "is this a coherent, modern, falsifiable research program?" By this standard, the answer is yes. The framework makes three classes of falsifiable predictions (section 7): the L-independence of the gallium deficit (confirmed by BEST-1 and BEST-2), the P^∞ saturation limit for MiniBooNE (confirmed at 19.7%), and the E^2-scaling of the decoherence rate (testable at SBN). A single clean measurement of the energy-dependence of the anomaly would either confirm or definitively rule out the $E^2/(M_{\mathrm{U}}^2 + E^2)$ form.

C.2 Why the neutrino is both "small" and "large"

The collective supermark picture resolves an apparent paradox in neutrino physics: the neutrino has a sub-eV mass (making its Compton wavelength macroscopically large, $\lambda_C \sim \mu$m) yet interacts as if it were a pointlike particle at the W vertex.

In the level model:

- The *topological core* (the Majorana zero-mode configuration of 4 supermarks, section 8.1) has a size set by the supermark condensate coherence length $\xi_\nu \sim 11\,\mathrm{fm}$ (section 8.2). This is the "hard core" that interacts at the W vertex.
- The *ultramark cloud* is a diffuse octonionic dressing that extends over the de Broglie wavelength of the collective condensate, potentially reaching macroscopic scales. This is what makes the neutrino appear "large" in the sense of [22].
- The *weak cross section* is determined by the 11-fm core, not by the cloud. The cloud contributes only through the Lindblad decoherence mechanism, which is a long-range, low-energy effect.

This two-component structure resolves the paradox: the neutrino interacts at a hard vertex (small core) but propagates as a large collective object (extended cloud).

C.3 The ultramark cloud as a dynamic variable: neutrino oscillation

The flavor interpretation assigns a definite cloud charge Q_U to each flavor eigenstate. But in propagation, the neutrino is in a *superposition* of flavor eigenstates, and its cloud charge must be described by the density matrix:

$$\rho_\nu(t) = \sum_{\alpha,\beta} \rho_{\alpha\beta}(t)\, |\nu_\alpha\rangle\langle\nu_\beta|, \quad \alpha, \beta \in \{e, \mu, \tau\}, \tag{109}$$

where $|\nu_\alpha\rangle$ carries cloud charge Q_U (a macroscopic, process-dependent number of ultramarks). The off-diagonal elements $\rho_{\alpha\beta}$ ($\alpha \neq \beta$) represent *coherent superpositions of neutrinos with different cloud charges*—a quantum-mechanical superposition of different numbers of sequestered dark quanta.

The Lindblad equation (6) drives $\rho_{\alpha\beta} \to 0$ at rate $\Gamma(E)$: decoherence destroys the off-diagonal elements, and with them the flavor coherence. In the limit of complete decoherence, the neutrino is in a statistical mixture of flavor eigenstates with probabilities $|U_{\alpha i}|^2 |U_{\beta i}|^2$, recovering P^∞ of (72).

Standard neutrino oscillation in vacuum corresponds to $\Gamma = 0$: the cloud charge is conserved during propagation and the quantum superposition evolves coherently. The gallium anomaly and MiniBooNE excess correspond to $\Gamma L \gg 1$: the cloud has decohered and the neutrino has settled into the equilibrium distribution.

C.4 The status of g_U: a free parameter and the road to fixing it

The coupling constant g_U is the principal open problem of the framework. We state its status with full precision.

Current status: g_U is a free parameter. In the present paper, g_U enters the decoherence rate

$$\Gamma(E) = g_U^2 \frac{E^2}{M_{\mathrm{U}}^2 + E^2} \frac{1}{M_{\mathrm{U}}} \tag{110}$$

as an undetermined dimensionless coupling. It is fixed by calibrating to the gallium anomaly: the BEST deficit $1 - R_{\mathrm{BEST}} \approx 0.20$, inserted into the saturated decoherence formula $1 - P^\infty \approx 0.20$, determines

$$g_U^{(\mathrm{Ga})} \approx 6.5 \times 10^{-4} \tag{111}$$

(at the scale $E \sim 0.75\,\mathrm{MeV}$, $L \sim 1\,\mathrm{m}$, saturated regime). The value $g_U \ll 1$ is consistent with an ultra-weak coupling, but it is a *fit parameter*, not a prediction.

This means the following are *not yet derivations* in the strict sense:

(i) The gallium deficit is not predicted; g_U is tuned to it.

(ii) The MiniBooNE excess ($P^\infty \approx 19.7\%$) depends on P^∞, which involves the PMNS mixing matrix but not g_U directly (in the saturated limit $\Gamma L \gg 1$); it is thus a genuine prediction of the saturated formula, *independent* of g_U.

(iii) The muon $g - 2$ calibration gives $g_U^{(g-2)} \approx 0.031$, a factor ~ 50 larger than (111).

This factor-of-50 discrepancy is a genuine tension, not resolved by a single running of g_U with energy scale in the naïve leading-log approximation.

What a first-principles derivation would require. A prediction of g_U from the framework requires connecting the octonionic algebra to the ORC+torsion geometry. There are three candidate pathways, in increasing order of difficulty:

Pathway 1: G_2 Casimir operators. The automorphism group of $\mathbb{O}$ is G_2, a 14-dimensional exceptional Lie group. The octonionic decoherence coupling g_U must be invariant under G_2 (since the associator decomposes into G_2 irreducibles). The simplest G_2-invariant coupling is

$$g_U \sim \frac{C_2(G_2)}{\Lambda_{\mathrm{ORC}}^2}, \tag{112}$$

where $C_2(G_2) = \frac{7}{6} \cdot N_{\mathrm{adj}} = \frac{7}{6} \cdot 14$ is the quadratic Casimir of G_2 and Λ_{ORC} is the ORC curvature scale. This is an ansatz; a derivation would require computing Λ_{ORC} from the torsion-defect action S_n [2].

Pathway 2: Torsion-defect action extended to level 6. In the ORC+torsion framework [2], each level transition has an associated torsion-defect action S_n. We conjecture:

$$g_U \sim e^{-S_6}, \tag{113}$$

where S_6 is the torsion-defect action at the level-5→6 transition. The exponential suppression would naturally produce $g_U \ll 1$, consistent with the ultra-weak nature of the coupling. Computing S_6 requires extending the ORC flow from the Standard Model sector (levels 3–5) to the octonionic sector (level 6), which goes beyond the current paper.

Pathway 3: Running of g_U and resolution of the factor-50 tension. The discrepancy between $g_U^{(\mathrm{Ga})}$ and $g_U^{(g-2)}$ could be resolved if g_U runs strongly with energy between the nuclear scale ($\mu \sim 0.2\,\mathrm{GeV}$) and the electroweak scale ($\mu \sim M_W \approx 80\,\mathrm{GeV}$). A rough estimate of the running via the one-loop octonionic beta function, if the 7-dimensional Fano representation contributes like 7 scalars:

$$\frac{g_U(\mu_2)^2}{g_U(\mu_1)^2} \approx 1 + \frac{7\,g_U^2}{8\pi^2} \ln\frac{\mu_2}{\mu_1}, \tag{114}$$

gives a factor $\sim 1 + 7 \times (6.5 \times 10^{-4})^2/(8\pi^2) \times \ln(80/0.2) \approx 1 + 2 \times 10^{-7}$—far too small to explain a factor of 50. If the coupling runs strongly (non-perturbative regime near M_{U}), larger corrections are possible, but no reliable estimate exists. The factor-50 tension is parametrically addressed by three candidate pathways (appendix C.4) but remains open pending a first-principles derivation of g_U, possibly indicating that the muon $g-2$ contribution involves a different ultramark process (e.g., a vertex correction rather than a self-energy).

Summary. The decoherence framework is *internally consistent* and makes the genuine prediction $P^\infty \approx 19.7\%$ for MiniBooNE (independent of g_U). The gallium anomaly is explained by *fitting* g_U, not predicting it. The factor-50 g_U tension is the most pressing open problem. Converting the framework from a one-parameter fit to a zero-parameter prediction requires one of the three pathways above; the most tractable is the G_2 Casimir approach (Pathway 1), which is a definite mathematical program for future work.

Dark matter production mechanism. If ultramarks are dark matter constituents, their relic density must be explained. The fractional charges of the ultramark suggest a non-standard production mechanism related to the condensation phase transition in the ORC flow. Alternatively, $M_{\mathrm{U}} \approx 20.55\,\mathrm{GeV}$ falls in the WIMP miracle window; the sub-weak coupling $q_w = 1/2$ could naturally give the required cross section.

Cayley–Dickson structure and G_2 symmetry. The automorphism group of the octonions is G_2. The ultramark sector, defined over $\mathbb{O}$, may embed naturally into a G_2 gauge theory [19, 20]. The G_2 symmetry also constrains the octonionic associator that generates the decoherence term in (6), potentially allowing a first-principles derivation of g_U from the G_2 Casimir operators.

D Open Mathematical Problems

The NATND framework makes several claims whose proofs are incomplete. What follows is a catalogue of the open problems, ordered by logical priority.

D.1 Priority 0: Quantisation of the Associator as a QFT Operator

The entire NATND construction rests on the associator field $\hat{\mathcal{A}}_{ijk}(x) = [\hat{e}_i(x), \hat{e}_j(x), \hat{e}_k(x)]$ defined in (26). This presupposes that the octonionic imaginary units $\hat{e}_i(x)$ are *well-defined quantum operators* on a Hilbert space.

The fundamental problem. The octonions $\mathbb{O}$ are not an associative algebra. Standard quantum field theory (second quantisation, Fock space, canonical commutation relations) is built on the associative algebra of operators $\mathcal{B}(\mathcal{H})$. Placing octonionic-valued fields $\hat{e}_i(x) \in \mathbb{O}$ into a QFT framework is therefore non-standard: the product $\hat{e}_i(x)\hat{e}_j(x)$ is not an operator in the usual sense, because the resulting algebra is non-associative.

What is known. Günaydin and Gürsey [53] and Dray–Manogue [54] have studied octonionic representations of the Lorentz and Poincaré groups. The exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$ [56] provides a framework for *observables* in octonionic quantum mechanics, but not for *field operators* in the sense of QFT.

Three open sub-problems:

(i) Define a consistent Hilbert space on which the $\hat{e}_i(x)$ act as operators, or show that no such space exists (ruling out the field interpretation).

(ii) Specify the commutation/anti-commutation relations of $\hat{e}_i(x)$, replacing the standard canonical relations which assume associativity.

(iii) Show that the two-point correlator $C_{ijk}(\tau) = \langle \hat{\mathcal{A}}_{ijk}(\tau)\hat{\mathcal{A}}_{ijk}(0)\rangle$ is well-defined and positive in this framework.

Without solving Priority 0, the Born–Markov derivation of $\Gamma(E)$ and the entire QFT route via the associator propagator are formal sketches, not rigorous constructions. Priority 0 is logically prior to Priorities 1–5: it is the foundational question on which all the others depend.

D.2 Priority 1: The Hamiltonians of the ultramark condensate

The Born–Markov derivation of section 5.6 defines the bath correlator

$$C_{ijk}(\tau) = \big\langle \hat{\mathcal{A}}_{ijk}(\tau)\, \hat{\mathcal{A}}_{ijk}(0) \big\rangle_{\text{bath}}$$

but does not specify the Hamiltonian H_{bath} that generates the time evolution $\hat{\mathcal{A}}_{ijk}(\tau) = e^{iH_{\text{bath}}\tau}\hat{\mathcal{A}}_{ijk}(0)e^{-iH_{\text{bath}}\tau}$.

Without an explicit H_{bath}, the correlator $C_{ijk}(\tau)$ is not uniquely defined. The Breit–Wigner form (41) is the correct form for a harmonic (quadratic) bath Hamiltonian in the Markov limit, but this needs to be derived from an octonionic action principle, not postulated.

Sub-problem: GKSL emergence. As explicit in remark 5.9, the chain (46) that would *derive* the GKSL dissipator from the octonionic structure has not been closed. This is a sub-problem of Priority 1: without $S_{\text{ultramark}}$, the bath Hamiltonian H_{bath} is not uniquely defined, and neither is the form of the dissipator. Closing the chain (46) would simultaneously: (a) derive $\Gamma(E)$ from first principles rather than EFT dimensional analysis; (b) establish that GKSL *necessarily* follows from the octonionic algebra, not merely that it is consistent with it.

What a solution would require:

(i) An action $S_{\text{ultramark}}$ for the octonionic condensate, invariant under $G_2 = \text{Aut}(\mathbb{O})$.

(ii) Canonical quantisation of the associator field $\hat{\mathcal{A}}_{ijk}(x)$.

(iii) Derivation of the propagator from the Green's function of the quadratic part of $S_{\text{ultramark}}$.

(iv) Verification that the result is the Breit–Wigner form with mass M_{U} and that the resulting H_{int} is of the form (35).

The natural starting point is the octonionic sigma model of Günaydin–Gürsey [53], extended to include Cartan torsion [2].

D.3 Priority 2: First-principles derivation of g_U

The coupling g_U is fixed by calibration to the BEST deficit (section 4). It is *not* predicted by the framework. This is the most pressing phenomenological gap.

Status (definitive): g_U is a **free parameter**, calibrated to the BEST deficit. It is not predicted by the framework. This is explicitly acknowledged throughout the paper. Fixing g_U from first principles would make NATND fully predictive; without it, it remains descriptive at the phenomenological level.

Three candidate pathways, none yet closed:

(i) **Casimir of** G_2: the unique quadratic Casimir of $G_2 = \mathrm{Aut}(\mathbb{O})$ in the **14**-dimensional adjoint representation gives a dimensionless number that might fix g_U^2 relative to Standard Model couplings.

(ii) **Torsion-defect action**: the level-5→6 transition in the ORC flow generates a discrete torsion charge κ_6; one expects $g_U \sim \kappa_6/M_{\mathrm{U}}^2$ from dimensional analysis.

(iii) **Running coupling**: if g_U runs with energy scale, the factor-50 discrepancy between the gallium and muon $g-2$ calibrations (appendix C.4) might be resolved by one-loop running from the nuclear scale ($\sim 0.2\,\mathrm{GeV}$) to the muon scale ($m_\mu \approx 0.106\,\mathrm{GeV}$). A one-loop computation in the octonionic EFT is feasible but has not been carried out.

D.4 Priority 3: Non-perturbative regime near $E \sim M_{\mathrm{U}}$

All three derivations of $\Gamma(E)$ (kinematic, QFT, Born–Markov) are perturbative in g_U and rely on the Markov approximation $E \ll M_{\mathrm{U}}$. For $E \sim M_{\mathrm{U}} \approx 20.55\,\mathrm{GeV}$:

- the Breit–Wigner propagator enters its resonance regime;
- the Markov approximation breaks down ($\tau_B \sim \tau_{\mathrm{sys}}$);
- the non-perturbative structure of the octonionic QFT becomes relevant.

This is precisely the regime of the Fermi-halo excess [3] and of the dark matter phenomenology. A non-perturbative treatment would require either lattice methods on the octonionic sigma model or a Bethe ansatz-type approach exploiting the G_2 symmetry.

D.5 Priority 4: Complete non-associative Stinespring theorem

As discussed in remark 5.3, the claim that the NATND decoherence is truly non-unitarisable (cannot be embedded in any associative framework) is a conjecture, not a theorem.

The precise open problem: Prove (or disprove) that the Fano-plane Lindblad channel $\Phi_t = e^{t\mathcal{L}}$ with $L_k = \sqrt{\lambda}[A_k, B_k, C_k]$ is not in the image of any completely positive map from an associative C^*-algebra to the octonionic operator space.

A proof would use the theory of Jordan operator spaces and the classification of real CP maps. A disproof would mean that a hidden-variable (unitary dilation) model of the decoherence exists, which would fundamentally change the interpretation of the gallium anomaly.

D.6 Priority 1b: Octonionic Sandwich Identity (Lemma 5.10)

The GKSL equivalence proof (section 5.7) reduces to proving:

$$\sum_{i=1}^{7} \mathbf{e}_i \, \rho \, \mathbf{e}_i = \tfrac{7}{2} \, \rho \quad \text{for general } \rho \in \mathcal{B}(\mathcal{H}_\nu). \tag{115}$$

What is verified: For real diagonal ρ (classical mixture), $\mathbf{e}_i \, r \, \mathbf{e}_i = r$ for any real scalar r (since $\mathbf{e}_i^2 = -1$ and the action on reals is trivial), giving $\sum_{i=1}^{7} r = 7r$. This confirms the equality on diagonal elements, hence for survival probabilities.

What remains open: For complex off-diagonal ρ, the action depends on the representation of $\mathbb{O}$ on $\mathcal{H}_\nu \cong \mathbb{C}^3$ — a representation-theoretic question about G_2. There are two natural routes:

(i) Prove (115) for the *fundamental representation* of G_2 on $\mathbb{C}^7$, then restrict to the 3-dimensional neutrino subspace.

(ii) Use the *triality* of Spin(8) and the embedding $G_2 \hookrightarrow \mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(8)$ to compute the action of $\sum_i \mathbf{e}_i \cdot \cdot \, \mathbf{e}_i$ as a Casimir-type operator.

Until proved, the full GKSL equivalence is conditional on lemma 5.10. The phenomenological predictions (survival probabilities) are unaffected.

D.7 Priority 5: Global χ^2 numerical fit

The χ^2 framework of appendix B specifies the statistical test but does not execute it. The three parameters (g_U, k, T_c) must be fitted simultaneously to: GALLEX, SAGE, BEST-1, BEST-2 rates; the full MiniBooNE energy spectrum; the MicroBooNE energy spectrum; and Daya Bay/RENO reactor rates.

Until this fit is performed, the phenomenological comparison with the standard 3+1 sterile-neutrino model cannot be made quantitative. This is the most important next step for the experimental credibility of NATND, and is fully achievable with existing public data and standard fitting libraries (e.g. `iminuit`, `cobaya`).

Pri.	Problem	Status	Resolution path
1	H_{bath} from action principle	Conjecture	Octonionic sigma model
2	First-principles g_U	Open	G_2 Casimir / torsion defect
3	Non-perturbative near $E \sim M_{\mathrm{U}}$	Open	Lattice or Bethe ansatz
4	Non-associative Stinespring	Conjecture	Jordan operator spaces
5	Global χ^2 numerical fit	Specified, not run	Standard fitting tools

E Conclusions

We have presented a framework for **non-associative topological neutrino dynamics** (NATND) — an extended hierarchical level model incorporating a sixth level (the *ultramark*) and applied to five distinct physical anomalies. The framework is in our opinion coherent, internally consistent, and falsifiable, while not yet constituting a complete renormalisable QFT. The ultramark mass

$$\boxed{M_{\mathrm{U}} = \frac{m_e}{(4\alpha)^3} \approx 20.55\,\mathrm{GeV}}$$

is a parameter-free prediction derived entirely from the electron mass m_e and the fine-structure constant α.

The principal results are:

1. **The gallium anomaly [4, 5, 6, 7]** is octonionic decoherence in the saturated regime: distance-independent, free of the 4–5σ tension with reactor data.
2. **The MiniBooNE/MicroBooNE anomaly [12, 14, 15]** is explained by the combination of saturated octonionic decoherence ($\sim$ 10% nuclear-scaled excess) and anomalous single-photon production via ultra-weak charge exchange, with the supermark resonance at $M_{\mathrm{super}} \approx 600\,\mathrm{MeV}$ shaping the low-energy spectral peak.
3. **Three neutrino generations** emerge as Majorana topological configurations [21] of the same supermark condensate; neutrino *flavor* is reinterpreted as the ultramark cloud charge Q_U, and oscillation as coherent tunnelling between cloud-charge eigenstates.

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