Modeling of a Hybrid Dynamical System with Fission–Fusion Cycle Derived from a Modulated Arithmetico-Geometric Sequence

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Abstract

This work presents the construction of a continuous model derived from an original discrete sequence alternating between geometric and arithmetic progression, modulated by a periodic cosine function. The sequence is interpreted as an energy reservoir whose charge (arithmetic) and discharge (geometric) are controlled by a hysteresis logic. The model is enriched by coupling with a mechanical oscillator, heat production via damping, conversion into nuclear kinetic energy (fusion), and reinjection into the reservoir, realizing a self-sustained fission–fusion cycle. Each equation is derived, physically justified, and linked to the original sequence. The article details the modeling, theoretical justifications, energy couplings, and global dynamics of the system.

1 Introduction

Mathematical sequences constitute fundamental tools for modeling discrete phenomena. Among classical sequences, arithmetic and geometric progressions occupy a central position due to their simplicity and dynamic richness. Hybrid sequences, combining multiple progression mechanisms, are however much less studied. In previous work, we introduced an original sequence defined piecewise according to the parity of the index, alternating between geometric progression (even terms) and arithmetic progression (odd terms), all modulated by a periodic cosine function of period 5. This work was published on the viXra platform, where the fundamental properties of convergence, pseudo-periodicity, and parametric sensitivity were established.

The present article extends this discrete approach by proposing a continuous modeling of the same phenomenon. The objective is to construct a hybrid dynamical system whose behavior reproduces, in continuous time, the alternations and modulations present in the discrete sequence. To this end, we interpret the amplitude of the sequence as energy stored in a reservoir. The geometric progression corresponds to a rapid discharge phase (amplification), while the arithmetic progression corresponds to a slow charge phase (accumulation). The switching between these two regimes is no longer temporal (based on the parity of n) but energetic: it is triggered by thresholds on the stored energy.

We then enrich this system by endowing the reservoir with an internal energy source: nuclear fission. In accumulation mode, the reservoir fills via a fission process, which we model as a source proportional to the available energy. In discharge mode, the reservoir feeds a mechanical oscillator, whose damping produces heat. This heat is converted into kinetic energy of nuclei, modeling a fusion process. A fraction of this energy is reinjected into the reservoir, thus ensuring a self-sustained cycle. This closed cycle constitutes the core of the model.

The article is structured as follows. Section 2 recalls the definition and properties of the original discrete sequence. Section 3 presents the transition to continuous variables and the switching logic. Section 4 presents the energy reservoir equation, detailing each term and its justification. Section 5 introduces the mechanical oscillator and its coupling with the reservoir. Section 6 addresses heat production and its link with nuclear kinetic energy. Section 7 presents the feedback mechanism and the fission–fusion loop. Section 8 discusses the global dynamics and operating regimes. Section 9 concludes and opens perspectives. A summary table of all equations is provided in the appendix.

2 The Original Discrete Sequence: Definition and Properties

2.1 Definition of the Sequence

Let $a, q, r \in \mathbb{R}$ be real parameters, where a represents the initial amplitude, q the geometric ratio, and r the arithmetic increment. The sequence $\{U_n\}_{n=0}^{\infty}$ is defined piecewise according to the parity of the index n:

$$U_n = \begin{cases} a \cdot q^{n/2} \cdot \cos\left(\frac{2\pi n}{5}\right) & \text{if } n \equiv 0 \pmod 2 \\ (a \cdot q^{(n-1)/2} + r) \cdot \cos\left(\frac{2\pi n}{5}\right) & \text{if } n \equiv 1 \pmod 2 \end{cases} \tag{1}$$

This definition shows that even terms follow a geometric progression with ratio $q^{1/2}$, while odd terms follow an arithmetico-geometric progression. In both cases, the modulation by the cosine of period 5 introduces an oscillatory component.

2.2 Unified Form with Indicator Functions

The sequence can be rewritten compactly by introducing parity indicator functions:

$$\delta_e(n) = \frac{1 + (-1)^n}{2}, \quad \delta_o(n) = \frac{1 - (-1)^n}{2} \tag{2}$$

These functions equal 1 if n is even and 0 otherwise, and vice versa for odd n. The sequence then becomes:

$$U_n = \left[a \cdot q^{\lfloor n/2 \rfloor} \cdot \delta_e(n) + \left(a \cdot q^{\lfloor (n-1)/2 \rfloor} + r\right) \cdot \delta_o(n)\right] \cdot \cos\left(\frac{2\pi n}{5}\right) \tag{3}$$

This unified form will be particularly useful for the transition to the continuous model, as it reveals the alternation structure as a linear combination of two regimes.

2.3 Fundamental Properties of the Sequence

The sequence possesses remarkable properties, which we briefly recall.

2.3.1 Convergence and Divergence

The asymptotic behavior of the sequence is governed by the parameter q:

Theorem 1. The sequence $\{U_n\}$ exhibits the following convergence properties:

1. If $|q| < 1$, then $\lim_{n\to\infty} U_n = 0$ (absolute convergence).
2. If $|q| > 1$, then $\lim_{n\to\infty} |U_n| = \infty$ (exponential divergence).
3. If $|q| = 1$, the sequence is bounded if $r = 0$, and grows linearly if $r \neq 0$.

The proof relies on bounding the even and odd subsequences. For the even subsequence:

$$|U_{2k}| = |a| \cdot |q|^k \cdot \left|\cos\left(\frac{4\pi k}{5}\right)\right| \leq |a| \cdot |q|^k \tag{4}$$

For the odd subsequence:

$$|U_{2k+1}| = |a \cdot q^k + r| \cdot \left|\cos\left(\frac{2\pi(2k+1)}{5}\right)\right| \leq |a| \cdot |q|^k + |r| \tag{5}$$

The dominant behavior is thus indeed that of $|q|^k$.

2.3.2 Pseudo-periodicity

The sequence satisfies a pseudo-periodicity relation of order 5, due to the cosine modulation:

$$U_{n+5} = \begin{cases} q \cdot U_{n+3} & \text{for } n \text{ even} \\ q \cdot U_{n+3} + r \cdot \cos\left(\frac{2\pi(n+5)}{5}\right) & \text{for } n \text{ odd} \end{cases} \tag{6}$$

This relation expresses the fact that the sequence is not strictly periodic, but its evolution over a window of 5 steps is dictated by the parameters q and r.

2.4 Energy Interpretation

The key step in our modeling consists of interpreting the amplitude of the sequence as an energy. In the sequence U_n, the even terms grow geometrically, evoking a runaway or amplification phenomenon. The odd terms grow arithmetically, evoking a regular and controlled accumulation. We therefore propose to set:

$$E_n \equiv \text{amplitude of the sequence at time } n \tag{7}$$

and to consider that the two regimes correspond to two operating modes of the same energy reservoir. This reservoir fills slowly (arithmetic mode) and empties rapidly (geometric mode). This interpretation is the foundation of the transition to the continuous model.

3 Transition to a Continuous Model: Variables and Switching

3.1 Introduction of Continuous Variables

To move from discrete to continuous, we introduce the following variables:

- $E(t)$: the energy stored in the reservoir (Joule), continuous equivalent of the amplitude.
- $\theta(t)$: an internal phase modulating energy exchanges. We set $\theta(t) = \omega t$, with $\omega = 2\pi/5$, to recover the period 5 of the sequence.
- $\gamma(t) \in \{0, 1\}$: a logical variable indicating the system's operating mode. $\gamma = 0$ corresponds to accumulation mode (arithmetic), $\gamma = 1$ to discharge mode (geometric).

The phase θ thus evolves according to:

$$\dot{\theta} = \omega \tag{8}$$

This equation defines an internal clock that paces the system. The function $\cos^2(\theta)$ (rather than $\cos(\theta)$) will be used in couplings to guarantee positivity of power terms, consistent with energy balances.

3.2 Hysteresis Switching Logic

In the discrete sequence, switching between the two regimes is imposed by the parity of n, i.e., by time. In the continuous model, we want switching to be dictated by the system state, and more precisely by the energy level $E(t)$. We therefore introduce two thresholds:

- A: low threshold, below which the system switches to accumulation mode.
- C: high threshold, above which the system switches to discharge mode.

The switching law is then given by:

$$\gamma(t) = \begin{cases} 0 & \text{if } E(t) \leq A \\ 1 & \text{if } E(t) \geq C \\ \gamma(t^-) & \text{if } A < E(t) < C \end{cases} \tag{9}$$

This is a hysteresis law: the system retains memory of its previous state as long as it lies in the intermediate zone $A < E < C$. This avoids spurious switchings (chattering) that would occur if a single threshold were used.

One can also express this law more compactly using the sign function and memory:

$$\gamma(t) = \frac{1 + \operatorname{sgn}(E(t) - C)}{2} + \frac{1 - \operatorname{sgn}(E(t) - A)}{2} \cdot \gamma(t^-) \tag{10}$$

This formulation highlights the role of memory: the first term forces $\gamma = 1$ if $E \geq C$, the second forces $\gamma = 0$ if $E \leq A$, and in the intermediate zone, only the term with $\gamma(t^-)$ remains.

4 Modeling the Energy Reservoir

4.1 Power Balance

The energy reservoir $E(t)$ follows a power balance of the form:

$$\dot{E} = \text{power in} - \text{power out} + \text{feedback} \tag{11}$$

We will detail each of these terms.

4.2 Input Power: Accumulation Mode ($\gamma = 0$)

When the system is in accumulation mode ($\gamma = 0$), the reservoir fills. We propose a charging model with saturation:

$$\dot{E}_{\text{in}} = \alpha_0 (C - E)\cos^2(\theta) + \beta \tag{12}$$

The justifications for this term are as follows:

- α_0 is a charge rate coefficient. The larger it is, the faster the reservoir fills.
- The term $(C - E)$ translates a saturation effect: the charge rate decreases as the reservoir approaches its maximum capacity C. This is a classical phenomenon in physics (capacitor charging, reservoir filling, etc.).
- $\cos^2(\theta)$ modulates the charging efficiency as a function of phase. Thus, charging is not uniform in time; it is more efficient at certain moments of the cycle.
- β is a permanent minimal input. Even when $\cos^2(\theta) = 0$, the reservoir receives a constant flux β. This ensures the system does not remain stuck at zero energy.

4.3 Output Power: Discharge Mode ($\gamma = 1$)

When the system is in discharge mode ($\gamma = 1$), the reservoir empties to feed the oscillator. The extracted power is:

$$P_{\text{ext}} = \alpha_1 (E - A)\cos^2(\theta) \tag{13}$$

Thus, the output power is:

$$\dot{E}_{\text{out}} = -P_{\text{ext}} = -\alpha_1 (E - A)\cos^2(\theta) \tag{14}$$

The justifications are:

- α_1 is an extraction coefficient. It determines the rate at which energy is withdrawn from the reservoir.
- The term $(E - A)$ ensures that discharge stops when the reservoir reaches the low threshold A. The residual energy A is not extractable.
- $\cos^2(\theta)$ modulates the extraction, which is therefore periodic.

4.4 Complete Reservoir Equation

Combining the two modes using the variable γ, we obtain:

$$\dot{E} = [\alpha_0(C - E)\cos^2(\theta) + \beta](1 - \gamma) - \alpha_1(E - A)\cos^2(\theta)\gamma \tag{15}$$

This equation is the core of the system. It describes a reservoir that charges in mode $\gamma = 0$ and discharges in mode $\gamma = 1$. The switching between the two modes is dictated by the hysteresis law (9).

4.5 Link with the Discrete Sequence

Equation (15) is the continuous analog of the sequence U_n. Indeed, the geometric progression (discharge) corresponds to the exponential decay of E when $\gamma = 1$, while the arithmetic progression (charge) corresponds to the saturated growth of E when $\gamma = 0$. The squared cosine replaces the simple cosine of the sequence to ensure positivity of exchanges.

5 Coupling with the Mechanical Oscillator

5.1 Principle

The energy extracted from the reservoir is converted into mechanical motion. We model this motion by a damped forced harmonic oscillator. The general equation of such an oscillator is:

$$\ddot{x} + \mu\dot{x} + \omega_0^2 x = F(t) \tag{16}$$

where $x(t)$ is the position, μ the damping coefficient, ω_0 the natural pulsation, and $F(t)$ the external force.

5.2 Thermal Damping

The damping μ is of thermal origin: mechanical energy is dissipated as heat. We propose to relate μ to the energy extracted from the reservoir:

$$\mu = \alpha_3(E - A)\cos^2(\theta) \tag{17}$$

Thus, the more charged the reservoir, the stronger the damping. The parameter α_3 is a thermal dissipation coefficient.

5.3 Excitation Force

The external force is proportional to the power extracted from the reservoir:

$$F(t) = K_{\text{force}} \cdot P_{\text{ext}} = K_{\text{force}}\, \alpha_1(E - A)\cos^2(\theta)\gamma \tag{18}$$

The coefficient K_{force} ensures unit conversion (power $\to$ force). If set to 1, we identify force and power, which is an acceptable simplification.

5.4 Complete Oscillator Equation

Substituting μ and $F(t)$, we obtain:

$$\ddot{x} + \alpha_3(E - A)\cos^2(\theta)\dot{x} + \omega_0^2 x = K_{\text{force}}\,\alpha_1(E - A)\cos^2(\theta)\gamma \tag{19}$$

This equation is the second pillar of the system. It describes how the energy extracted from the reservoir is converted into oscillatory motion, and how this motion is dissipated as heat.

5.5 First-Order Form

For numerical simulations, it is useful to rewrite the oscillator as a first-order system. Setting $v = \dot{x}$, we have:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\alpha_3(E - A)\cos^2(\theta)v - \omega_0^2 x + K_{\text{force}}\,\alpha_1(E - A)\cos^2(\theta)\gamma \end{cases} \tag{20}$$

6 Energy Balance and Heat Production

6.1 Power Extracted from the Reservoir

The power extracted from the reservoir in discharge mode is:

$$P_{\text{ext}} = \alpha_1(E - A)\cos^2(\theta) \tag{21}$$

This power is divided into two parts: one part maintains the mechanical oscillation, and the other is dissipated as heat.

6.2 Useful Mechanical Power

The useful mechanical power is the product of force and velocity:

$$P_{\text{osc}} = F(t) \cdot v = K_{\text{force}}\,\alpha_1(E - A)\cos^2(\theta)\gamma \cdot v \tag{22}$$

6.3 Power Dissipated as Heat

The power dissipated by damping is:

$$P_{\text{heat}} = \mu v^2 = \alpha_3(E - A)\cos^2(\theta)v^2 \tag{23}$$

This power is positive and represents the mechanical energy converted into thermal energy.

6.4 Energy Conservation

The power balance can be written as:

$$P_{\text{ext}} = P_{\text{osc}} + P_{\text{heat}} \tag{24}$$

Substituting the expressions, we obtain:

$$\alpha_1(E-A)\cos^2(\theta) = K_{\text{force}}\,\alpha_1(E-A)\cos^2(\theta)\gamma v + \alpha_3(E-A)\cos^2(\theta)v^2 \tag{25}$$

This identity is satisfied if appropriate proportionality relations are imposed among the coefficients.

7 Thermonuclear Coupling: Heat $\rightarrow$ Nuclear Kinetic Energy

7.1 Temperature and Equipartition

The heat produced by damping raises the temperature of the medium. We consider this medium to be a plasma, and we use the equipartition theorem to relate temperature to the average kinetic energy of nuclei.

The average kinetic energy of a nucleus in a plasma at temperature T is:

$$K_{\text{nuc}} = \frac{3}{2}k_B T \tag{26}$$

where $k_B = 1.38 \times 10^{-23}$ J/K is the Boltzmann constant.

We define $K(t)$ as the average kinetic energy of nuclei in the plasma. This is a macroscopic variable representing the thermal agitation of the medium.

7.2 Temperature Evolution Equation

The plasma temperature evolves according to a thermal balance:

$$\dot{T} = \frac{1}{C_v}(P_{\text{heat}} - P_{\text{losses}}) \tag{27}$$

where C_v is the heat capacity of the plasma (J/K). Thermal losses are modeled by a linear term:

$$P_{\text{losses}} = \delta' T \tag{28}$$

where δ' is a loss coefficient (W/K). Thus:

$$\dot{T} = \frac{1}{C_v}\left(\alpha_3(E-A)\cos^2(\theta)v^2 - \delta' T\right) \tag{29}$$

7.3 Evolution Equation for $K(t)$

Differentiating $K = \frac{3}{2}k_B T$, we obtain:

$$\dot{K} = \frac{3}{2}k_B\dot{T} = \frac{3k_B}{2C_v}\left(\alpha_3(E-A)\cos^2(\theta)v^2 - \delta' T\right) \tag{30}$$

Setting:

$$\beta = \frac{3k_B}{2C_v}, \quad \delta = \frac{3k_B\delta'}{2C_v} \tag{31}$$

and replacing T by $\frac{2K}{3k_B}$, we obtain:

$$\dot{K} = \beta\alpha_3(E - A)\cos^2(\theta)v^2 - \delta K \tag{32}$$

This equation is fundamental: it describes how the average kinetic energy of nuclei (K) is fed by the heat produced by the oscillator, and how it decreases through thermal losses.

7.4 Physical Interpretation

The parameter β represents the fraction of dissipated heat that is effectively converted into thermal agitation of nuclei. The parameter δ is the loss rate of this energy. Thus, $K(t)$ is a thermal energy reservoir that can be used subsequently to feed the cycle.

8 The Feedback Mechanism: Closing the Fission–Fusion Cycle

8.1 Principle

The feedback mechanism is the process that reinjects a fraction of the energy K (from fusion) into the fission reservoir. It thus ensures the closure of the cycle and the self-sustenance of the system.

8.2 Feedback Modeling

The feedback is simply written as:

$$\dot{E}_{\text{feedback}} = \eta K \tag{33}$$

where $\eta \in [0, 1]$ is a conversion efficiency. This term represents the energy from fusion that is converted into fission energy and reinjected into the reservoir.

8.3 Reservoir Equation with Feedback

Adding the feedback term to equation (15), we obtain the complete reservoir equation:

$$\dot{E} = [\alpha_0(C - E)\cos^2(\theta) + \beta](1 - \gamma) - \alpha_1(E - A)\cos^2(\theta)\gamma + \eta K \tag{34}$$

This equation is the core of the fission–fusion cycle. It shows that the reservoir is fed both by fission (mode $\gamma = 0$) and by fusion feedback (term ηK).

8.4 Complete Cycle Interpretation

The complete system cycle is as follows:

1. **Fission ($\gamma = 0$):** The reservoir fills via fission, according to a saturated dynamics modulated by $\cos^2(\theta)$.
2. **Switch to discharge:** When E reaches the high threshold C, the system switches to mode $\gamma = 1$.

3. **Discharge ($\gamma = 1$):** The reservoir empties, feeding the mechanical oscillator.
4. **Oscillation:** The extracted energy is converted into mechanical motion.
5. **Heat production:** The damping of the oscillator produces heat, proportional to $\alpha_3(E-A)\cos^2(\theta)v^2$.
6. **Plasma heating:** The heat raises the temperature, increasing the average kinetic energy K of nuclei.
7. **Fusion:** The energy K is partially converted into useful energy via the feedback mechanism.
8. **Reinjection:** The feedback reinjects ηK into the reservoir.
9. **Return to fission:** When E drops back to A, the system returns to mode $\gamma = 0$.

This cycle is self-sustained as long as losses are compensated by feedback.

9 Global Dynamics of the System

9.1 Complete System

The complete system consists of the following equations:

$$\boxed{\begin{aligned} \dot{\theta} &= \omega \\ \dot{E} &= [\alpha_0(C-E)\cos^2(\theta)+\beta](1-\gamma) - \alpha_1(E-A)\cos^2(\theta)\gamma + \eta K \\ \dot{x} &= v \\ \dot{v} &= -\alpha_3(E-A)\cos^2(\theta)v - \omega_0^2 x + K_{\text{force}}\,\alpha_1(E-A)\cos^2(\theta)\gamma \\ \dot{K} &= \beta\alpha_3(E-A)\cos^2(\theta)v^2 - \delta K \\ \gamma &= \text{hysteresis law as in (9)} \end{aligned}} \tag{35}$$

9.2 Role of Each Equation

- $\dot{\theta} = \omega$: internal clock, modulates all couplings.
- $\dot{E}$: energy balance of the reservoir, with charge, discharge, and feedback.
- $\dot{x} = v$: definition of velocity.
- $\dot{v}$: oscillator dynamics, with thermal damping and excitation force.
- $\dot{K}$: evolution of nuclear kinetic energy, fed by heat and subject to losses.
- γ: hysteresis switching logic.

9.3 Link with the Discrete Sequence

The system (35) is a continuous generalization of the sequence U_n. The variable γ plays the role of the indicator functions δ_e and δ_o, but instead of depending on the parity of n, it depends on the state of the reservoir. The modulation by $\cos^2(\theta)$ replaces the simple cosine of the sequence. Finally, the feedback ηK has no equivalent in the sequence, but it enriches the model by giving it an internal energy source.

9.4 Operating Regimes

The system can exhibit several regimes depending on parameter values:

- **Damped regime:** If losses are too large, the system shuts down, and E remains below A.
- **Cyclic regime:** If gains compensate losses, the system oscillates between A and C.
- **Runaway regime:** If the feedback η is too strong, the system may grow indefinitely.

The choice of parameters $\alpha_0, \alpha_1, \alpha_3, \beta, \delta, \eta$ determines the equilibrium regime.

10 Conclusion

We have constructed a complete continuous model derived from an original discrete sequence. The resulting hybrid dynamical system couples an energy reservoir, a mechanical oscillator, heat production, conversion into nuclear kinetic energy, and feedback to the reservoir. Switching between modes is ensured by an energy hysteresis, which faithfully reproduces the alternation between geometric and arithmetic progression of the sequence.

Each equation possesses a coherent physical interpretation within the proposed framework. The model is self-sustained and can exhibit cyclic or divergent regimes depending on parameters. It opens perspectives in plasma physics, theoretical energetics, and hybrid system modeling.

Future work will include stability analysis of the cycle, bifurcation analysis, systematic numerical simulation, and extension to multidimensional systems.

Summary Table of Equations

References

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[2] Hall, R. R. (2018). *Sequences and Series in Number Theory*. Cambridge University Press.

[3] Hardy, G. H., & Wright, E. M. (2008). *An Introduction to the Theory of Numbers* (6th ed.). Oxford University Press.

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Equation	**Expression**	**Role**
Phase	$\dot{\theta} = \omega$	Internal clock
Reservoir (basic)	$\dot{E} = [\alpha_0(C - E)\cos^2(\theta) + \beta](1 - \gamma) - \alpha_1(E - A)\cos^2(\theta)\gamma$	Reservoir balance
Reservoir (with feedback)	$\dot{E} = [\alpha_0(C - E)\cos^2(\theta) + \beta](1 - \gamma) - \alpha_1(E - A)\cos^2(\theta)\gamma + \eta K$	Fission–fusion cycle
Velocity	$\dot{x} = v$	Definition
Oscillator	$\dot{v} = -\alpha_3(E - A)\cos^2(\theta)v - \omega_0^2 x + K_{\text{force}}\alpha_1(E - A)\cos^2(\theta)\gamma$	Mechanical dynamics
Kinetic energy	$\dot{K} = \beta\alpha_3(E - A)\cos^2(\theta)v^2 - \delta K$	Fusion
Temperature	$T = 2K/3k_B$	K–T link

Table 1: Summary of all equations in the model.

[5] Oppenheim, A. V., & Schafer, R. W. (1999). *Discrete-Time Signal Processing* (2nd ed.). Prentice Hall.

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