The analytic frame: drawing heavy algebraic curves by hand

A method for teaching — foundations, proofs and comparison

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Abstract

Many remarkable curves — roses, the lemniscate, the cardioid, the limaçon — have a *heavy* Cartesian equation: implicit polynomials of degree 4, 6, up to 10, hard to plot point by point and often multivalued. This paper presents a method that makes them *drawable by hand*, with no software and no polar paper: in an *analytic frame* (an abscissa axis together with an origin of ordinates O' set off the axis by a distance λ), each of these curves is written as a *single* explicit ordinate $y = f(x)$, which one tabulates and plots with a ruler. We prove the transition formulas and the pivot identity $X^2 + Y^2 = y^2$ — the *engine* guaranteeing that the one-line expression does describe the heavy curve — then the equation of the line, of the circle, the pseudo-derivative, and a proportional relation in the triangle. One section *compares the two methods* on six classical curves. The scope is honest and bounded: this is a method of *construction* and of *teaching*, in the service of simpler access to objects reputed difficult.

1 The frame and its transformation

The analytic frame is defined in the plane by a horizontal abscissa axis with origin O, together with a distinct origin of ordinates, written O', lying off this axis at a distance $\lambda > 0$ ($OO' = \lambda$). To a pair (x, y) of the analytic frame one associates the point M whose coordinates (X, Y) in the Cartesian frame centred at O' are

$$X = \frac{-\,xy}{\sqrt{\lambda^2 + x^2}}\,, \qquad Y = \frac{\lambda\, y}{\sqrt{\lambda^2 + x^2}}\,. \tag{1}$$

2 The engine: why one line is enough

Theorem 1. *For every* (x, y)*, the image point* (1) *satisfies*

$$X^2 + Y^2 = y^2,$$

and its direction from O' *is given by the angle* $\varphi \in (0, \pi)$ *such that*

$$\cos\varphi = \frac{-x}{\sqrt{\lambda^2 + x^2}}, \qquad \sin\varphi = \frac{\lambda}{\sqrt{\lambda^2 + x^2}}, \qquad \textit{i.e.} \qquad \boxed{x = -\lambda \cot\varphi\,.}$$

Thus y is the signed measure $\overline{O'M}$ *(positive if M is after O', negative before, so $|O'M| = |y|$) and x parametrises the angle: as x runs over $\mathbb{R}$, φ describes the interval $(0, \pi)$ once.*

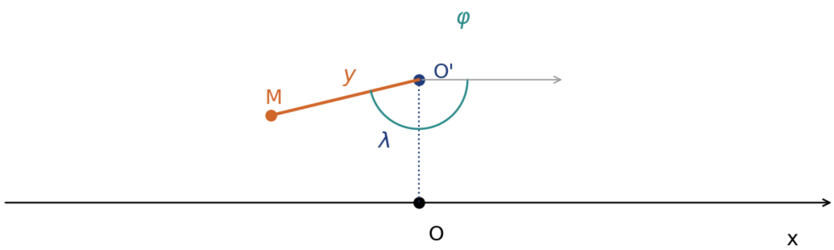

Figure 1: The analytic frame. The origin of ordinates O' lies at distance λ off the axis. The image point M is located from O' by the ordinate $y = \overline{O'M}$ (a signed measure: positive if M is *after* O', negative *before*) and by the angle φ that $O'M$ makes with the horizontal; $|O'M| = |y|$.

Proof. Summing the squares of (1), $X^2 + Y^2 = \dfrac{y^2(x^2+\lambda^2)}{\lambda^2+x^2} = y^2$. For $y \neq 0$, the vector $\big(X, Y\big)/y = \big(-x/\sqrt{\lambda^2+x^2}, \lambda/\sqrt{\lambda^2+x^2}\big)$ is a unit vector with positive second component; it equals $(\cos\varphi, \sin\varphi)$ for a unique $\varphi \in (0, \pi)$, whence $\cot\varphi = -x/\lambda$. The map $x \mapsto -\lambda\cot\varphi$ is an increasing bijection of $\mathbb{R}$ onto $(0, \pi)$. $\square$

Corollary 1. *An ordinate $y = f(x)$ describes exactly the curve of signed radius $r = y$ and angle φ around O', where $r = \overline{O'M}$ (positive after O', negative before). In other words:* tabulating $y = f(x)$ and laying off each y with a ruler from O' is drawing the curve. *The sign of y selects the ray φ (if $y > 0$) or the opposite ray $\varphi + \pi$ (if $y < 0$).*

This is the whole spring of the method: a curve whose (X, Y) equation is a heavy polynomial becomes, in this frame, an explicit one-line function. Theorem 1 guarantees that the two writings describe *the same* locus.

Remark. The frame allows one to associate to an abscissa *zero or several* ordinates. This possibility — absent from an ordinary single-valued Cartesian graph $Y = g(X)$ — is precisely what allows one to draw *closed* curves (the double value $\pm$), as we shall see for the lemniscate.

3 The vertigo: the grid becomes a target

If the frame is polar, then the *graph paper itself* changes nature. The horizontal lines $y = c$ become *concentric circles* (since $y = c$ gives $X^2 + Y^2 = c^2$), and the vertical lines $x = k$ become *rays* issuing from O' (a fixed abscissa fixes the direction). The familiar square grid was only a polar frame in disguise.

Reading the frame on examples

Before the curves, let us learn to place points ($\lambda = 1$). For a fixed abscissa x, all pairs (x, y) read off *one and the same line* through O'; the ordinate y there is the signed measure from O' (positive *after* O', negative *before*). In particular, any zero ordinate $(k, 0)$ gives $M = O'$. Thus $(0, 2)$ lands above the axis and $(0, -3)$ *below*; $(2, 3)$ and $(2, -1)$ lie on the line $x = 2$, on either side of O'; $(-3, 4)$ and $(-3, -2)$ on the line $x = -3$.

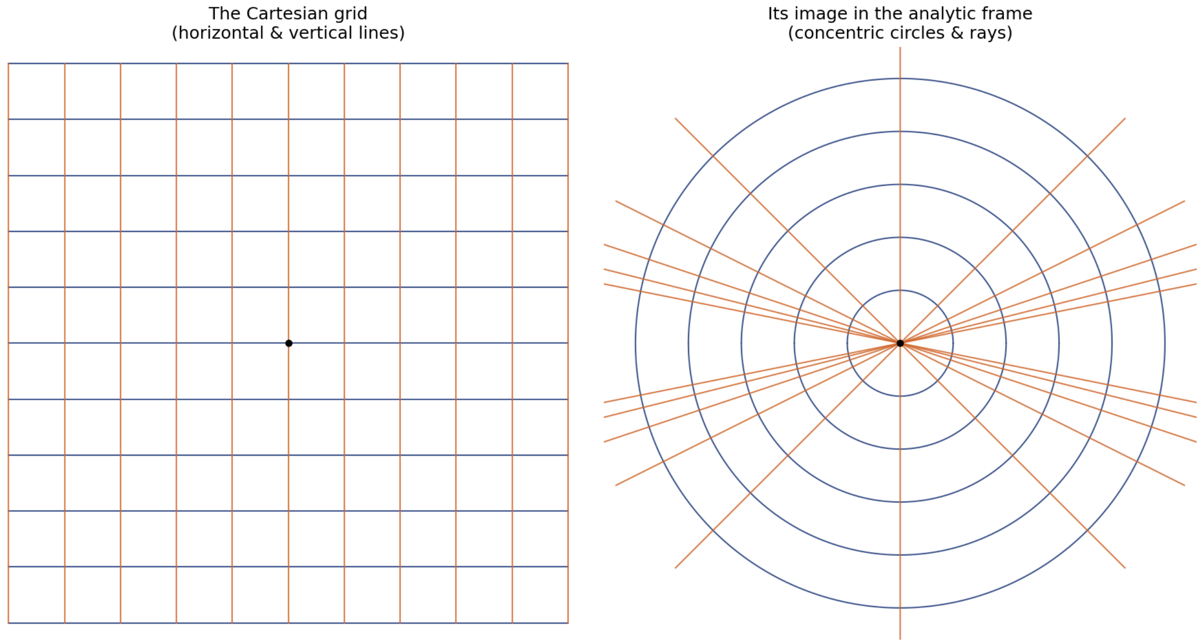

Figure 2: The Cartesian grid (left) and its image in the analytic frame (right): horizontal lines become concentric circles, vertical lines become rays from O'. It is, literally, the passage from graph paper to a polar target.

4 The equation of the line

Proposition 1. *The Cartesian line $Y = aX + b$ has, in the analytic frame, the equation*

$$y = \frac{b\sqrt{\lambda^2 + x^2}}{\lambda + ax}. \tag{2}$$

Proof. Substituting (1) into $Y = aX + b$ and multiplying by $\sqrt{\lambda^2 + x^2}$: $\lambda y + axy = b\sqrt{\lambda^2 + x^2}$, i.e. $y(\lambda + ax) = b\sqrt{\lambda^2 + x^2}$. □

Remark. The value $\lambda + ax = 0$ is not a spurious pole: it gives the asymptotic direction of the line.

5 Circles and semicircles

Proposition 2. *The constant $y = c$ describes the arc $X^2 + Y^2 = c^2$ (upper semicircle of radius c for $c > 0$). The circle with centre (a, b) and radius $\sqrt{a^2 + b^2}$ (passing through O') has equation*

$$y = \frac{2(\lambda b - ax)}{\sqrt{\lambda^2 + x^2}}.$$

Proof. The first follows from $X^2 + Y^2 = y^2 = c^2$ (Thm 1), with $Y > 0$. For the second, the circle reads $X^2 + Y^2 - 2aX - 2bY = 0$; inserting $X^2 + Y^2 = y^2$ and (1), then dividing by y, one obtains the stated ordinate. □

6 The pseudo-derivative

Set $\lambda = 1$ and $y = f(x)$, so $Y = f/\sqrt{1 + x^2}$, $X = -xf/\sqrt{1 + x^2}$. The *pseudo-derivative* $f_p'(x)$ is the slope of the image curve, that is $Y'(x)/X'(x)$.

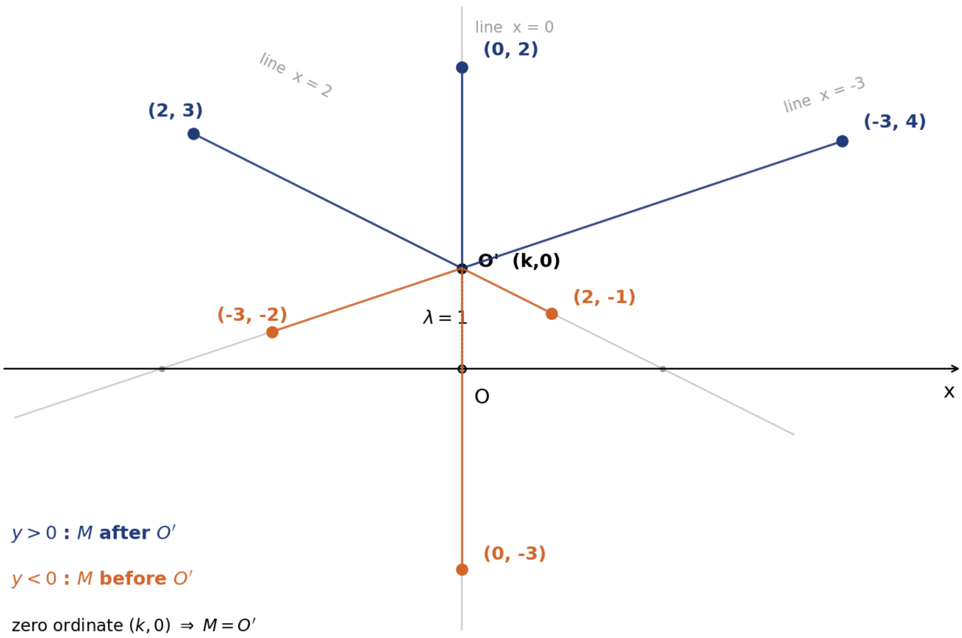

Figure 3: Six points placed in the analytic frame ($\lambda = 1$). Blue: $y > 0$ (M after O'); orange: $y < 0$ (M before O'). Points with the same abscissa share the line through O' at angle φ; the zero ordinate $(k, 0)$ is sent to O'.

Proposition 3. $f_p'(x) = -\dfrac{(1+x^2)f'(x) - xf(x)}{x(1+x^2)f'(x) + f(x)}$.

Proof. One has $Y' = (1+x^2)^{-3/2}\big[(1+x^2)f' - xf\big]$ and $X' = -(1+x^2)^{-3/2}\big[x(1+x^2)f' + f\big]$; the factor $(1+x^2)^{-3/2}$ cancels in the ratio. □

Remark (an "à la L'Hôpital" reading). Forming the slope $\mathrm{d}Y/\mathrm{d}X = Y'/X'$ as a ratio of derivatives is exactly the gesture of L'Hôpital's rule. For instance, as $x \to e$:

$$\lim_{x\to e} \frac{x^2\sqrt{\ln x} - e^2}{x\sqrt{\ln x} - e} = \frac{5e}{3}, \qquad \lim_{x\to e} \frac{x - e}{x\sqrt{\ln x} - e} = \frac{2}{3}$$

(here $M'(e) = \frac{5e}{2}$, $N'(e) = 1$, $D'(e) = \frac{3}{2}$).

7 A proportional relation in the triangle

Write $I = O'$ ($OI = \lambda$) and let D be the point of the axis with abscissa OD.

Proposition 4. *With $IA = b$ ($x = 0$), $IB = b/a$ ($x \to \infty$), $IH = b\,(ID)/(\lambda + a\,OD)$ ($x = OD$) and $ID = \sqrt{\lambda^2 + (OD)^2}$, one has*

$$\frac{OI}{IA} + \frac{OD}{IB} = \frac{ID}{IH}.$$

Proof. By (2), $y(0) = b$, $y(x) \to b/a$ as $x \to \infty$, and $y(OD) = b\sqrt{\lambda^2 + (OD)^2}/(\lambda + a\,OD) = b\,(ID)/(\lambda + a\,OD)$. Hence $\frac{OI}{IA} + \frac{OD}{IB} = \frac{\lambda + a\,OD}{b} = \frac{ID}{IH}$. □

8 Comparing the two methods: heavy equations drawn in one line

Here is the practical heart. To draw a rose or a lemniscate, the classical route forces one either to solve *point by point* a high-degree implicit Cartesian equation (multivalued, delicate by hand),

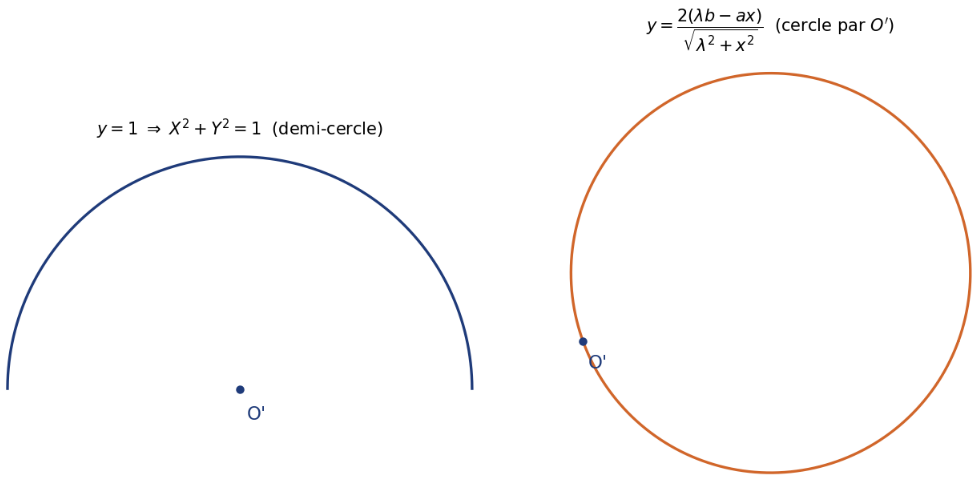

Figure 4: Left: $y = 1$ gives the unit semicircle. Right: a one-line ordinate gives a whole circle passing through O'.

or to set up a polar grid. The analytic frame offers a third way: for each of these curves, a *single* explicit ordinate $y = f(x)$, which one tabulates and lays off with a ruler on the offset axis — with no software. Theorem 1 certifies that this line describes exactly the heavy curve. The table below puts the two writings side by side (writing $\varphi = \operatorname{arccot}(-x/\lambda)$, so that each y is indeed a function of x).

Curve	**Cartesian equation (heavy)**	**Degree**	**Analytic ordinate**
3-petal rose	$(X^2+Y^2)^2 = X(X^2-3Y^2)$	4	$y = \cos 3\varphi$
4-petal rose	$(X^2+Y^2)^3 = (X^2-Y^2)^2$	6	$y = \cos 2\varphi$
8-petal rose	implicit polynomial (see text)	10	$y = \cos 4\varphi$
Lemniscate	$(X^2+Y^2)^2 = X^2-Y^2$	4	$y = \pm\sqrt{\cos 2\varphi}$
Cardioid	$(X^2+Y^2-X)^2 = X^2+Y^2$	4	$y = 1+\cos\varphi$
Looped limaçon	$(X^2+Y^2-X)^2 = \frac{1}{4}(X^2+Y^2)$	4	$y = \frac{1}{2}+\cos\varphi$

Table 1: Left, the equation one would have to plot; right, the one-line ordinate that draws it. Each equivalence was verified symbolically (the polynomial on the left vanishes identically on the parametrisation on the right). For the 8-petal rose, elimination yields a polynomial of degree 10.

What the comparison shows. The simple writing $y = \cos 3\varphi$ (one line, differentiable, tabulable) and the heavy writing $(X^2+Y^2)^2 = X(X^2-3Y^2)$ (an implicit quartic) denote the *same* locus. The method does not "simplify" the algebra by magic: it *shifts* the description into a frame where drawing becomes an elementary gesture — tabulate a function and lay it off with a ruler. This is exactly what makes it useful in class: a student who could not tackle a degree-10 equation nonetheless draws the eight-petal rose.

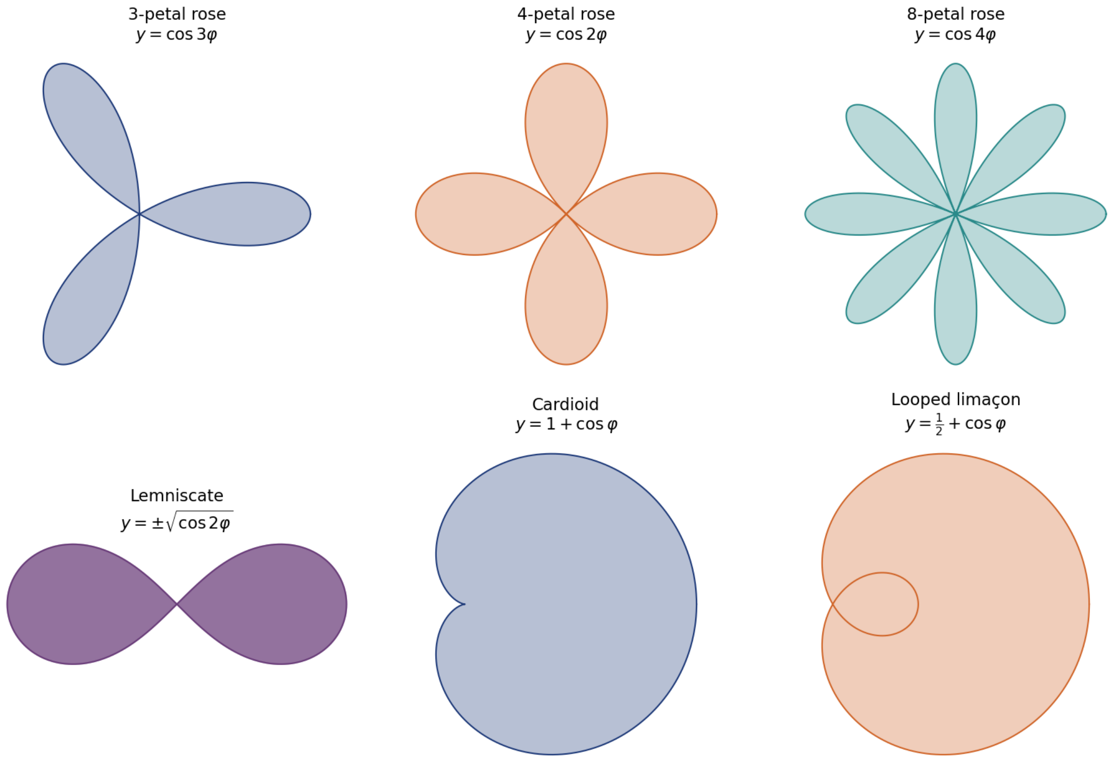

Figure 5: The six curves of the table, drawn (and *filled*) by their one-line ordinate. The lemniscate illustrates the double value ($\pm$): it is the frame's ability to associate several ordinates to one abscissa that closes the curve — which a single-valued Cartesian graph cannot do.

9 High-school functions, brand-new curves

Here is perhaps the most striking argument in class. Take the functions that *every high-school student* already knows how to plot, and look at what they become in the analytic frame. Two surprises answer each other.

A known curve, made trivial. The circle is not a function in the Cartesian frame: one must cut it into two pieces, $y = +\sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$. In the analytic frame it is simply $y = \pm 1$ — two constants against two square roots.

Curves found nowhere. The same elementary functions — x^2, $x^2 - 1$, $\dfrac{1}{1+x^2}$, x^3 — draw there figures absent from any textbook: a wing, a looped drop, an egg, a leaf. None bears a name; all are written in one line and drawn with a ruler.
The message for the teacher fits in one sentence: *the same function, the same gesture, a brand-new curve* — with no polar coordinates, no algebraic equation, within a student's reach.

10 Gallery: one line, one curve

The trigonometric functions of the original paper fit here too: cos and sin give the same rose, one being the *rotation* of the other.

11 The catalogue of the original paper, redrawn

The eight curves below are those of the initial paper, redrawn in the analytic frame ($\lambda = 1$). Each is a *single* ordinate $y = f(x)$: the line $y = x$ gives a drop, the inverses $1/x$, $1/|x|$, $1/x^2$ give parabolas and esses, the translations $1/(x-1)$, $1/(x+1)$ the same esses *tilted*, while $x/(x^2-1)$

crosses its branches and $x/(x^2+1)$ closes into a double loop. One reads at a glance the frame's effect: a pole of f (e.g. at $x=1$) sends the point to infinity along the corresponding abscissa line.

12 Bridge to the pseudo-trigonometric functions: surface functions

The frame marries another object: the sawtooth *pseudo-trigonometric* functions $\operatorname{sp} x = x - 5\lfloor x/5 \rfloor$ and $\operatorname{cp} x = 5 - \operatorname{sp} x$ (with constant derivatives $+1$ and -1). Dropped into the analytic frame, these ramps generate *surface functions* — filled, angular lobes, re-glued at each period. Two elementary and independent constructions meet here in a single image.

This meeting is fruitful. Because the sawtooth is *asymmetric* (it rises, then drops at once), a polar equation built on it generates *pinwheels* and *turbines* that seem to turn — which no symmetric sin / cos rose produces. This is the geometric signature proper to these functions.
Other combinations of sp/cp, dropped into the frame, give yet more surface functions: sp^2, the symmetric arch $|\mathrm{sp} - \frac{5}{2}|$ (a heart), the twin lobe $\mathrm{sp} - \mathrm{cp}$, the flame psh, and the mixes $\mathrm{sp} \cdot \sin$, $\mathrm{sp} \cdot \cos$.

13 The complex plane and the inner product in the frame

The frame does not merely resemble the complex plane: it *is* it. A point (x, y) encodes the complex number z of modulus $|y|$ and argument $\varphi = \operatorname{arccot}(-x/\lambda)$ (measured around O'). All the identities below were verified numerically.

The modulus is read off the ordinate. $|z| = |y|$ (Theorem 1): the ordinate *is* the modulus.

Multiplying means multiplying the ordinates and composing the abscissas. For $z_1 z_2$,

$$|y_{\text{prod}}| = |y_1\, y_2|, \qquad x_{\text{prod}} = \frac{x_1 x_2 - \lambda^2}{x_1 + x_2}.$$

The law $x_1 * x_2 = (x_1 x_2 - \lambda^2)/(x_1 + x_2)$ is a *group law*: endowed with it, the abscissas form the rotation group (it is the addition of cotangents, x being a "cotangent" coordinate on the circle).

Inner product and orthogonality.

$$\langle M_1, M_2 \rangle = y_1 y_2 \frac{x_1 x_2 + \lambda^2}{\sqrt{\lambda^2 + x_1^2}\,\sqrt{\lambda^2 + x_2^2}}, \qquad \text{whence} \qquad \boxed{M_1 \perp M_2 \iff x_1 x_2 = -\lambda^2.}$$

This is the exact echo of "perpendicular slopes $\Leftrightarrow m_1 m_2 = -1$", carried onto the abscissa.
Where is i ? The number i (modulus 1, argument $\frac{\pi}{2}$) has abscissa $x = -\lambda \cot \frac{\pi}{2} = 0$: so $i \leftrightarrow (0, 1)$, straight above O', and $-i \leftrightarrow (0, -1)$. The vertical axis $x = 0$ *is* the imaginary axis; the reals, in turn, are pushed to $x = \pm\infty$. Better still: multiplying by i amounts to the transformation

$$x \longmapsto -\frac{\lambda^2}{x}$$

on the abscissa (a quarter turn), preserving the modulus. Orthogonality then reappears as a special case: $z \perp iz$ always, since $x \cdot (-\lambda^2/x) = -\lambda^2$.
These formulas, so clean, say one simple and honest thing — the analytic frame is a *re-coordinatisation of the complex plane* — and that is precisely what makes its elegance.

14 Fractals of circles and spheres, and their equations

Here is an extension that crowns the chapter. Start from a circle of centre c and radius r; on its circumference place k circles ($k = 1, 2, 3, \dots$), equally spaced, each with centre *on* the parent circle and radius ρr ($0 < \rho < 1$); then *iterate*. In the limit, a fractal.

The equation of a circle in the frame. Each circle is written in one line. A circle whose centre lies at distance d from O' (angle θ), of radius s, has in the frame the (two-valued) ordinate

$$y = d\cos(\varphi-\theta) \;\pm\; \sqrt{s^2 - d^2\sin^2(\varphi-\theta)}\,, \qquad \varphi = \operatorname{arccot}(-x/\lambda). \tag{3}$$

The $\pm$ is the frame's *double image*: the two intersections of the ray from O' with the circle. Two limiting cases are worth reading: a circle *centred at* O' ($d = 0$) gives $y = \pm s$, a constant — this is "the circle $= y = \pm 1$" at free scale; and for a centre at distance $d = 2$ of radius $s = r$ one recovers exactly

$$y_k = 2\cos(\varphi-\theta_k) \;\pm\; \sqrt{r^2 - 4\sin^2(\varphi-\theta_k)}\,,$$

the equation of the first-generation small circles (identity verified numerically, error $\sim 10^{-15}$).

The equation of the fractal. The fractal is not a polynomial of infinite degree: it is a *union* of ordinates all of the form (3), enumerated by a simple recursion. In complex notation, the centres and radii obey the iterated function system

$$c_{\text{child}} = c_{\text{parent}} + s_{\text{parent}}\, e^{\,i(\varphi_0+2\pi j/k)}, \qquad s_{\text{child}} = \rho\, s_{\text{parent}} \qquad (j = 0,\dots,k-1),$$

and the fractal is

$$\boxed{F = \bigcup_{(c,s)} \Big\{\, y = |c|\cos(\varphi - \arg c) \pm \sqrt{s^2 - |c|^2\sin^2(\varphi-\arg c)}\,\Big\}.}$$

Thus the frame *absorbs* the complexity of a repetitive pattern: the same simple ordinate, copied and rescaled, instead of an equation collapsing under an exploding degree. The *dimension* is $\dim F = \log k/\log(1/\rho)$, and for $k = 3$, $\rho = \frac{1}{2}$ one obtains exactly the Sierpiński triangle.

And spheres. In the frame extended to space, a *sphere* is also a constant ($f = \pm 1$), like the circle. The same construction — k spheres whose centres lie on the surface of the parent sphere, iterated — gives fractals in three dimensions. With 4 spheres (vertices of a tetrahedron) and $\rho = \frac{1}{2}$, one obtains the *Sierpiński tetrahedron*; the octahedron (6) and the icosahedron (12) give its cousins. The dimension remains $\log k/\log(1/\rho)$.

Remark. These objects belong to the classical family of self-similar iterated-function-system attractors (of which Sierpiński is the canonical example): this is what gives them an exact equation and a closed-form dimension. The frame's contribution is to write each piece as a *one-line ordinate* (3), so that the whole fractal reads as a union of these ordinates.

15 Status and scope

Let us put it plainly. The value of the analytic frame is a *drawing method*: it makes accessible by hand, on an axis and with a ruler, algebraic curves whose Cartesian equation is heavy. Theorem 1 is its engine and its guarantee of exactness; one notes, in all honesty, that the simple ordinate coincides with the polar writing of the curve — the frame's own contribution is to *host that simplicity within a Cartesian-graph gesture*, tabulable and manual, and, through its several-images convention ($\pm$), to carry closed curves that an ordinary graph cannot draw. This is not a new analysis: no orthogonality, no spectral theory; it is a tool of writing and of teaching, in the service of democratised access to objects reputed difficult.

Several extensions are natural, given here as a *programme* (not as established results):

- make λ dynamic (rotation, dilation, vibration): $\lambda(t)$ generates a family of curves, hence an animation, perfectly well defined;

- introduce a second angle to pass to three dimensions — a spherical-type reparametrisation giving surfaces, the “filled” body obtained by letting the radius sweep an interval;
- the general relation on the ellipse (segments AH, AB, AC according to the tilt of the major axis), whose circle case recovers the law of cosines (Al-Kashi): it deserves a complete proof before being stated as a theorem.

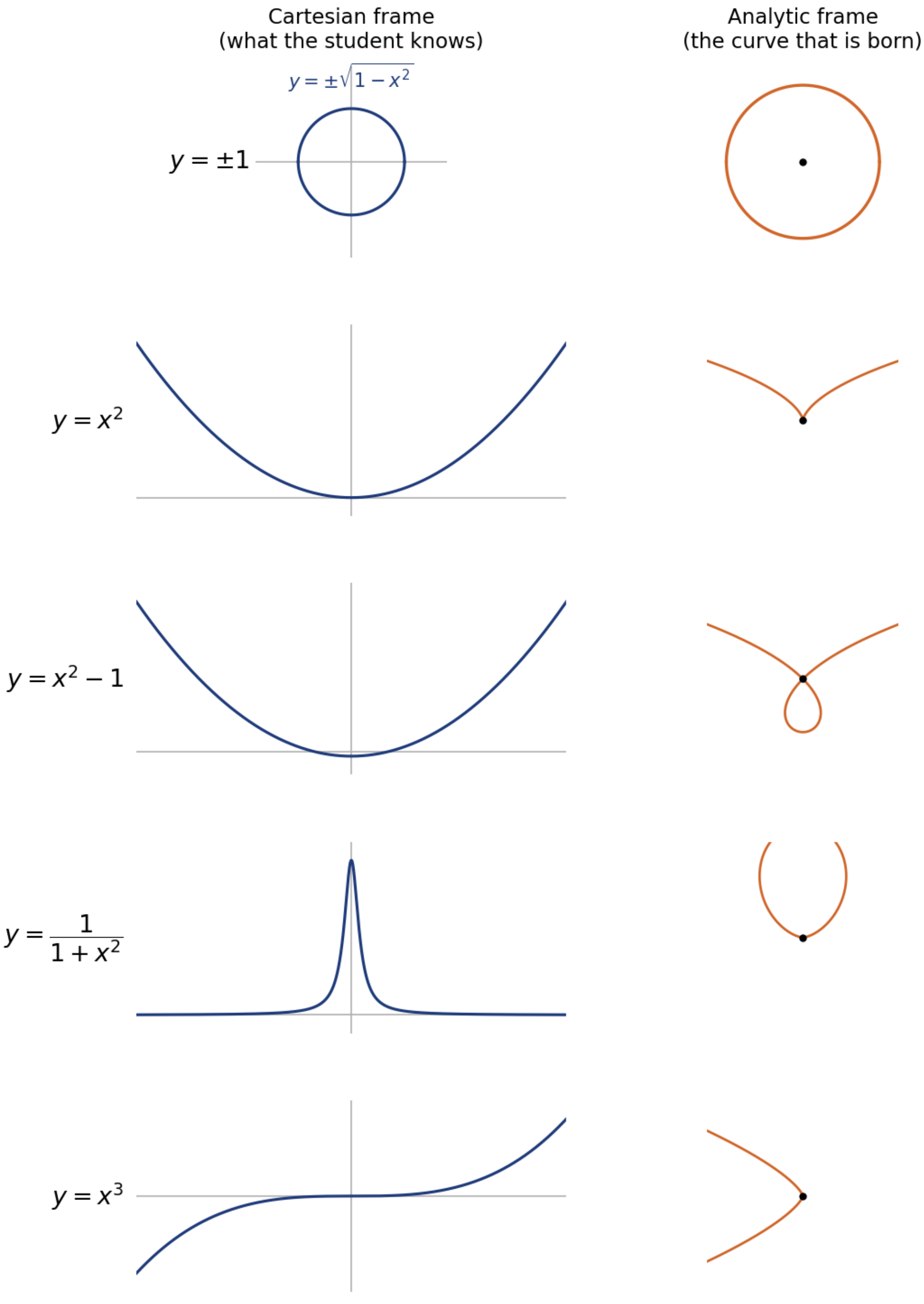

Figure 6: The same function, two frames. Left, the graph the student knows; right, the curve born in the analytic frame. The circle, impossible to write as a single function in Cartesian, becomes $y = \pm 1$; $x^2 - 1$ generates a looped drop, $1/(1 + x^2)$ an egg, x^3 a leaf.

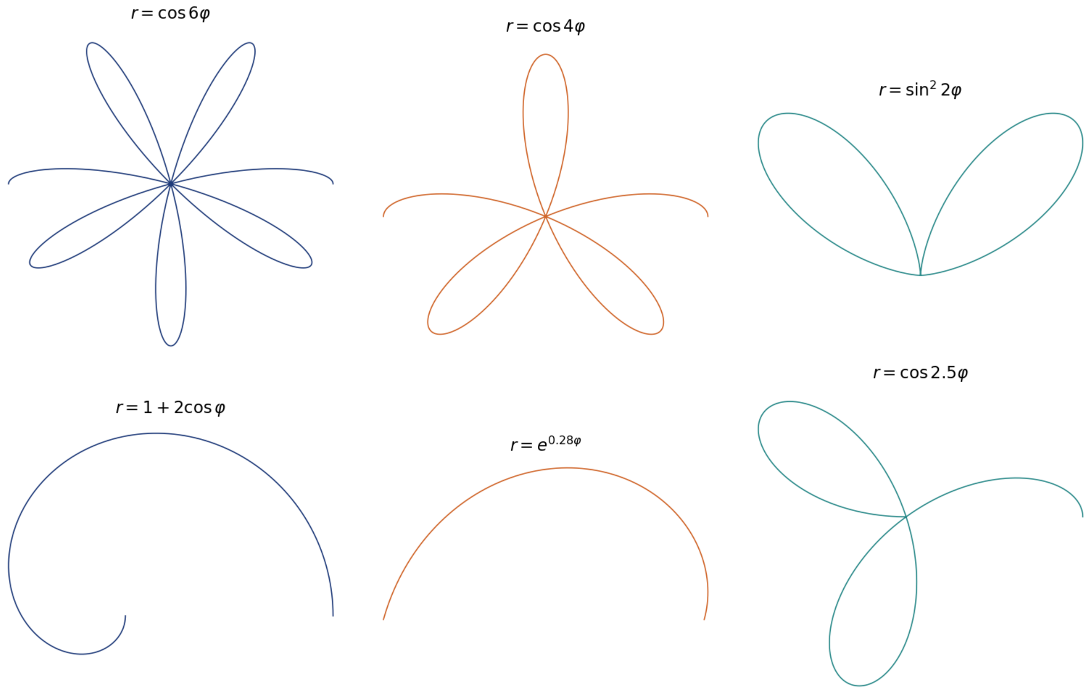

Figure 7: Further curves, each produced by a single ordinate $y = f(x)$ ($\lambda = 1$): roses, twin lobe, limaçon, logarithmic spiral, fractional rose.

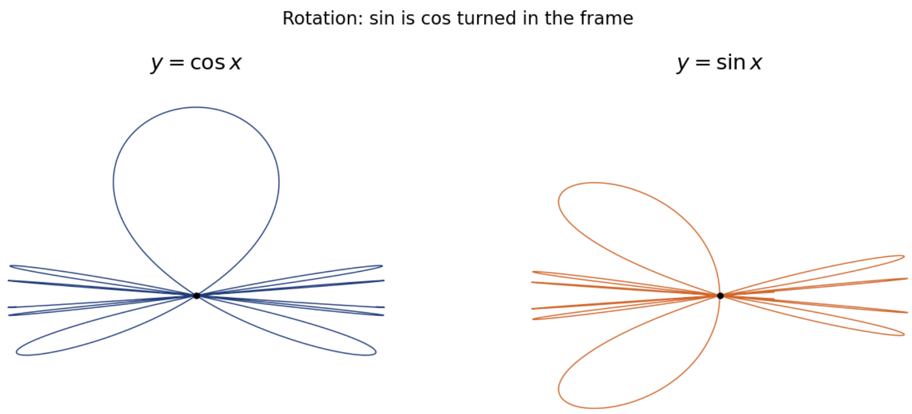

Figure 8: $y = \cos x$ and $y = \sin x$ in the frame: the same figure, turned. The large petal of cos tips to the lower position for sin — a mere rotation, due to the phase shift of the angle.

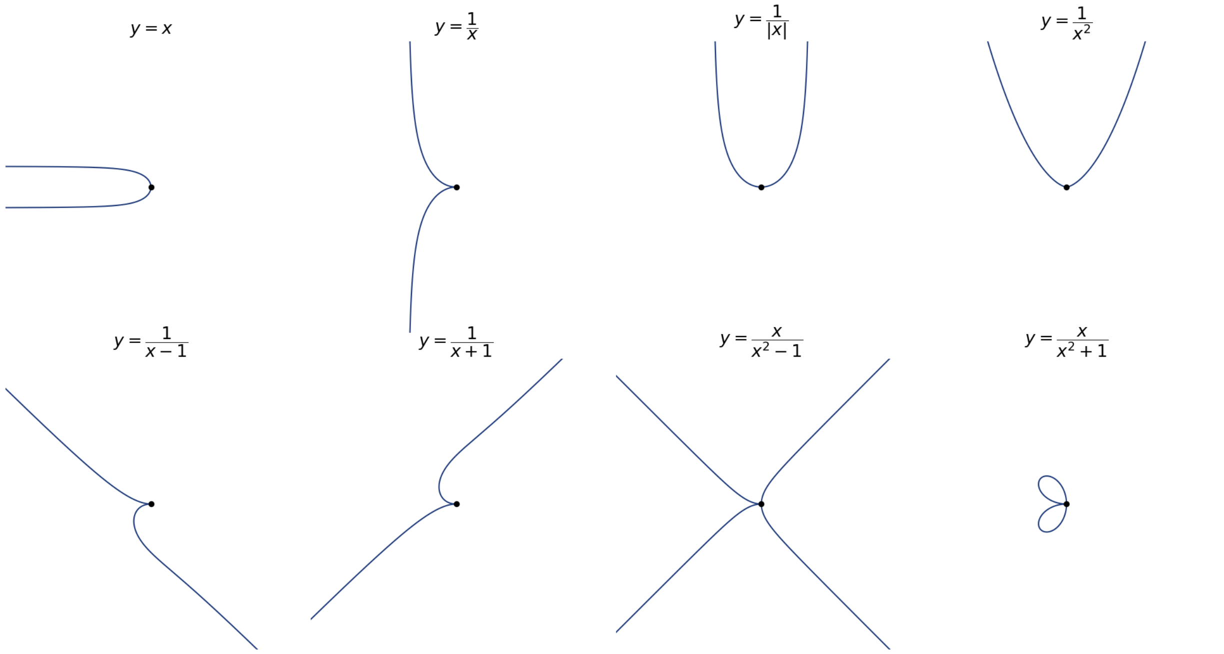

Figure 9: The catalogue of the original paper in the analytic frame. Left to right, top: $y = x$, $y = 1/x$, $y = 1/|x|$, $y = 1/x^2$; bottom: $y = 1/(x-1)$, $y = 1/(x+1)$, $y = x/(x^2-1)$, $y = x/(x^2+1)$. The black dot marks O'.

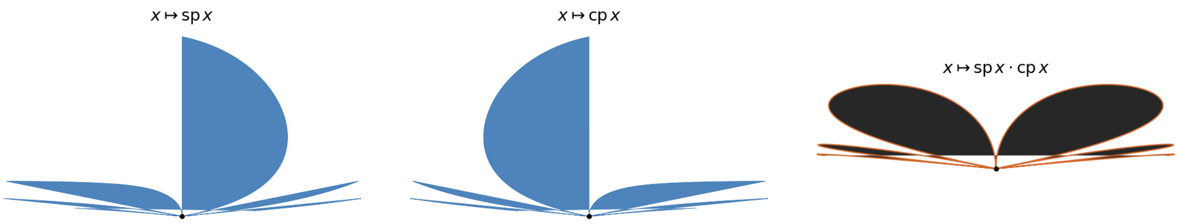

Figure 10: The surface functions of the article: $x \mapsto \operatorname{sp} x$, $x \mapsto \operatorname{cp} x$ and $x \mapsto \operatorname{sp} x \cdot \operatorname{cp} x$ drawn in the analytic frame. The sawtooth (piecewise linear) becomes a filled petal that resets at each period.

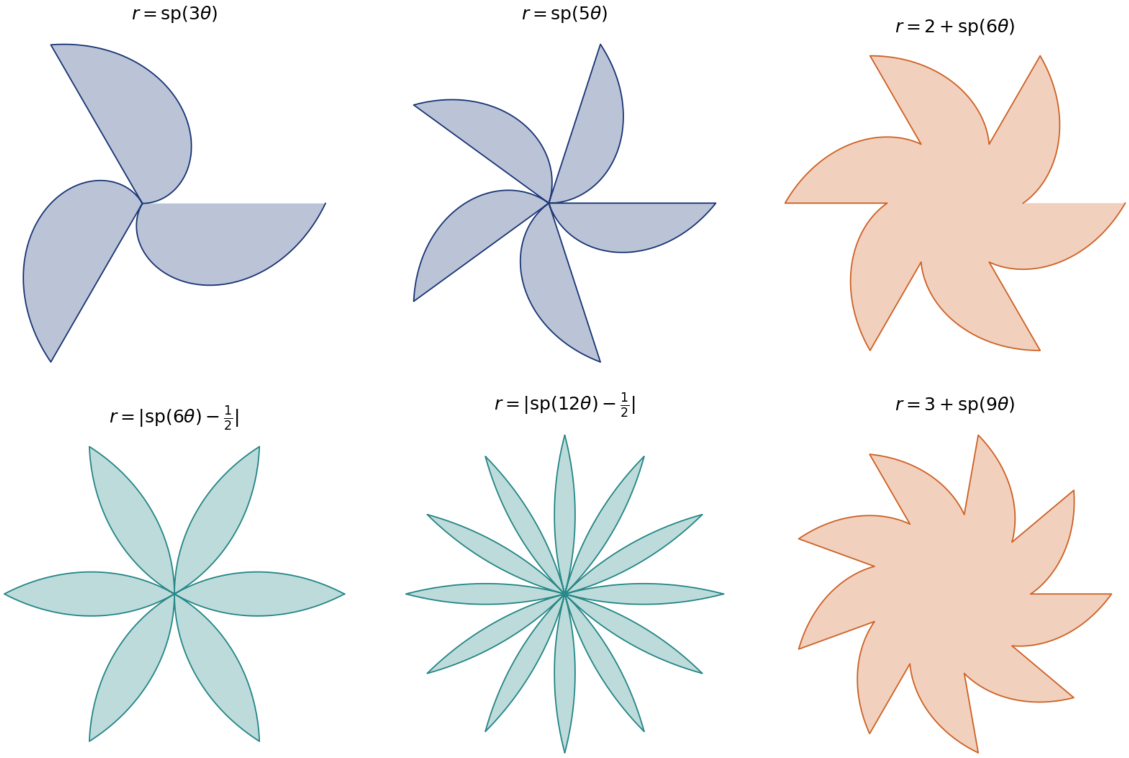

Figure 11: Sawtooth pinwheels, turbines and flowers (polar). Asymmetric blades $r = \mathrm{sp}(3\theta)$, $\mathrm{sp}(5\theta)$; filled turbines $r = 2 + \mathrm{sp}(6\theta)$, $3 + \mathrm{sp}(9\theta)$; flowers with sharp petals $r = |\mathrm{sp}(6\theta) - \frac{1}{2}|$, $|\mathrm{sp}(12\theta) - \frac{1}{2}|$.

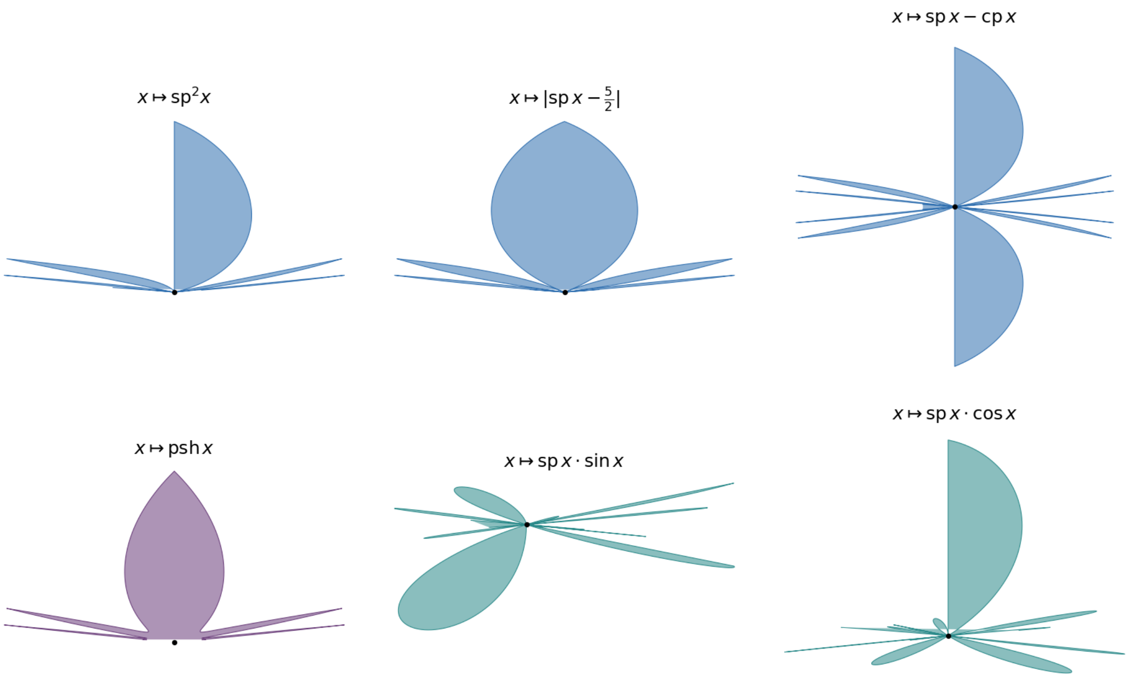

Figure 12: New surface functions in the frame. Top: $\mathrm{sp}^2 x$, $|\mathrm{sp}\, x - \frac{5}{2}|$ (heart), $\mathrm{sp}\, x - \mathrm{cp}\, x$ (twin lobe). Bottom: $\mathrm{psh}\, x$ (flame), $\mathrm{sp}\, x \cdot \sin x$, $\mathrm{sp}\, x \cdot \cos x$.

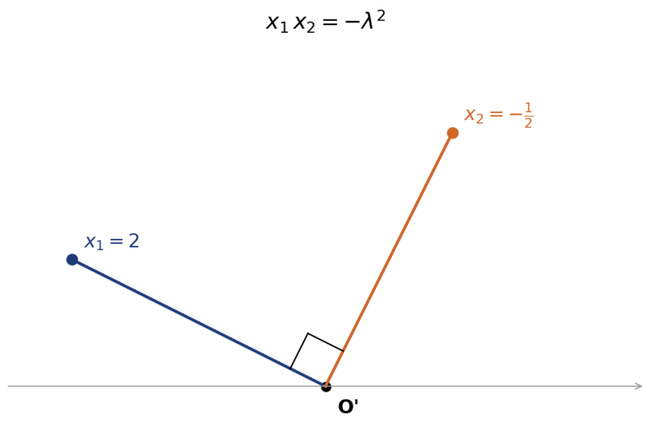

Figure 13: Two orthogonal directions from O': their abscissas satisfy $x_1 x_2 = -\lambda^2$ (here $\lambda = 1$, $x_1 = 2$, $x_2 = -\frac{1}{2}$).

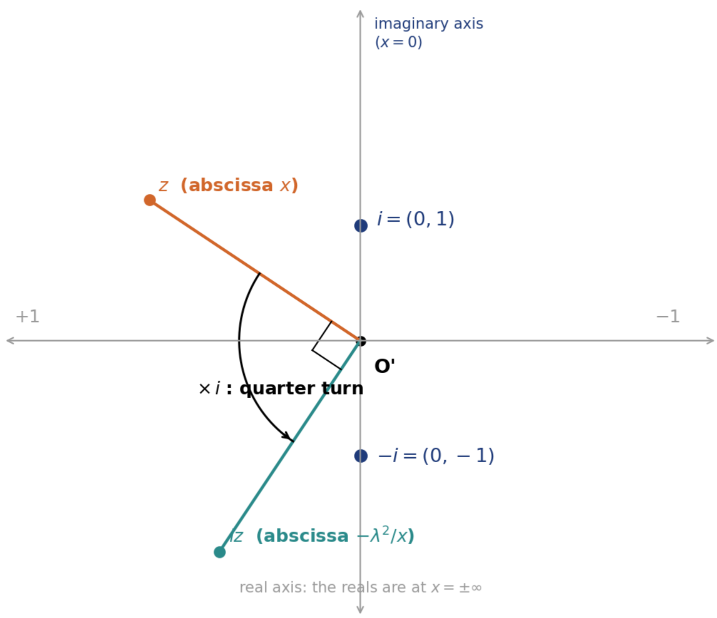

Figure 14: The complex plane in the frame. $i \leftrightarrow (0, 1)$ and $-i \leftrightarrow (0, -1)$ on the imaginary axis $x = 0$; the reals are at $x = \pm\infty$. Multiplying by i rotates by a quarter turn ($x \mapsto -\lambda^2/x$), and $z \perp iz$.

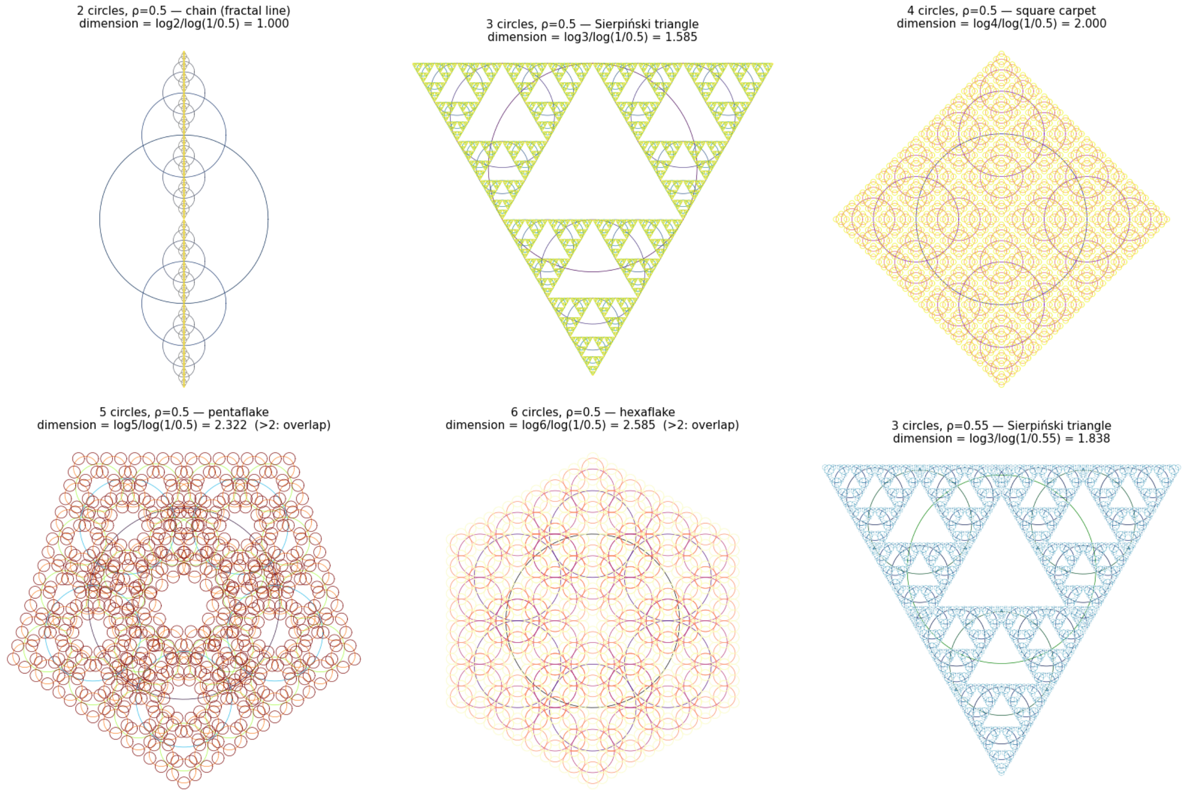

Figure 15: The circle fractal for $k = 2, \ldots, 6$ (colour = iteration level). With 3 circles, the Sierpiński triangle. Each pattern is the attractor $F = \bigcup_j f_j(F)$, of dimension $\log k/\log(1/\rho)$.

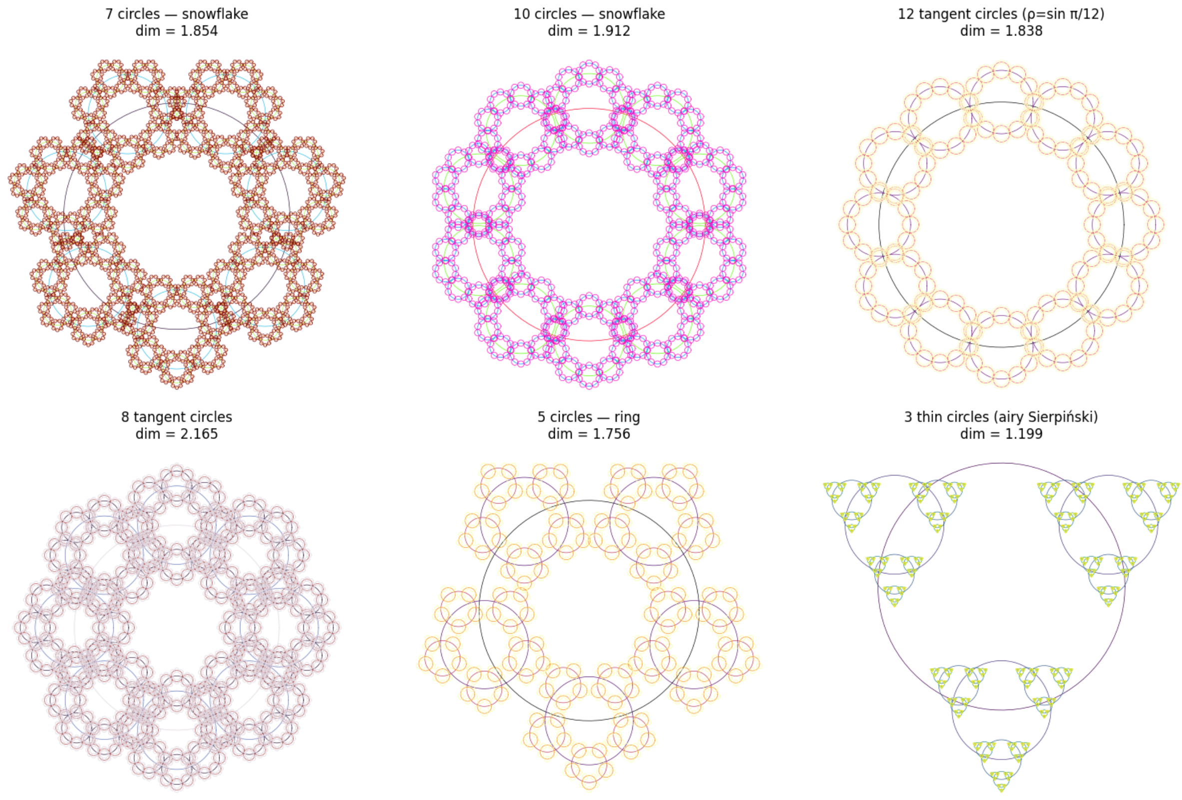

Figure 16: Other circle fractals: snowflakes ($k = 7, 10$), tangent-children variants ($\rho = \sin(\pi/k)$, $k = 8, 12$), a ring ($k = 5$) and an airy Sierpiński ($k = 3$, $\rho = 0.4$).

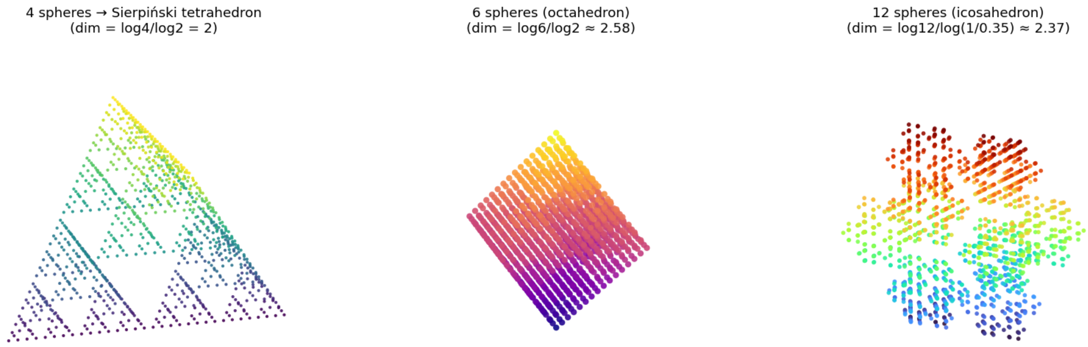

Figure 17: Sphere fractals. Four spheres (tetrahedron) give the Sierpiński tetrahedron; six (octahedron) and twelve (icosahedron) their analogues. Colour = height.

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