$F = ma$ as a Rindler Modular-Work Identity in Quantum Field Theory

Exact Translation Theorem, Relative-Entropy Completion, and Linearized Entropic Corollary

Russell S. Clark II
Independent Researcher

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Abstract

For the Minkowski vacuum and the right Rindler wedge, the Bisognano–Wichmann theorem fixes the vacuum modular flow to be Lorentz-boost flow. For a translated finite-energy excitation, this implies an exact, acceleration-independent identity for the vacuum-relative one-sided modular energy,

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle K_W\rangle = \frac{2\pi}{\hbar c}E_W(p),$$

where p is displacement normal to the horizon and $E_W(p)$ is the vacuum-subtracted energy contained in the wedge. If the excitation is concentrated well inside the wedge, $E_W(p) \simeq E$. Selecting a Rindler observer of proper acceleration a converts the dimensionless modular generator into the proper-time Hamiltonian $H_a = k_B T_U K_W$, with $T_U = \hbar a/(2\pi c k_B)$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle H_a\rangle = \frac{a}{c^2}E_W(p).$$

Under a specified quasistatic control protocol, this is the external holding-force identity. At the reference orbit, a wedge-contained narrow state of invariant mass M obeys $|F_{\mathrm{hold}}| = Ma$.

The entanglement interpretation is more limited. For a finite state, the exact regulated relation is

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle H_a\rangle = T_U\frac{\mathrm{d}\Delta S_W}{\mathrm{d}p} + k_B T_U\frac{\mathrm{d}}{\mathrm{d}p}S_{\mathrm{rel}}(\rho_{W,p}\|\Omega_W),$$

so $T_U \mathrm{d}S_W$ alone is not generally equal to work. The relative-entropy term is second order only for infinitesimal state variations. In that linear-response sector, the entanglement first law yields $\mathrm{d}(\delta S_W)/\mathrm{d}p = 2\pi k_B \delta E/(\hbar c)$, and for $\delta E = Mc^2$ this reproduces the coefficient in Verlinde's local entropy–displacement ansatz. The result is therefore an exact modular-work identity and a linearized entropic representation of Rindler holding force, not a derivation of acceleration, general equations of motion, or gravity.

Keywords: modular Hamiltonian; Rindler wedge; Bisognano–Wichmann theorem; entanglement first law; relative entropy; Unruh effect; inertia; entropic force

1 Introduction

The thermodynamic structure of horizons has repeatedly suggested that geometry, gravity, and inertia may admit descriptions in terms of information and statistical mechanics. Bekenstein and Hawking identified entropy and temperature for black holes [1, 2]; Jacobson related the Einstein equation to local horizon thermodynamics [3]; and Verlinde proposed that Newtonian inertia and gravity could arise from entropy gradients on holographic screens [4]. These proposals differ sharply in rigor and scope. In particular, the formula

$$\Delta S = 2\pi k_B \frac{Mc}{\hbar} \Delta x \tag{1}$$

was introduced by Verlinde as a screen postulate rather than derived from a specified quantum field theory.

The Rindler wedge offers a setting in which several ingredients are not postulated. The Bisognano–Wichmann theorem identifies the vacuum modular flow of the wedge with Lorentz boosts [5, 6]. The same modular structure gives the KMS form of the Unruh effect [7, 8]. Translation covariance then fixes how the wedge modular energy changes when a localized excitation is displaced relative to the horizon.

The central result of this paper is narrow but exact: the derivative of vacuum-relative wedge modular energy with respect to normal displacement is proportional to the energy presently contained in the wedge. The acceleration a is absent from this theorem. It enters only when the canonical dimensionless modular flow is normalized to the proper time of a selected Rindler observer. This distinction blocks the claim that the factor aE/c^2 is obtained by inserting the same acceleration twice. The acceleration is an input labeling the accelerated frame; the nontrivial QFT statement is the universal modular gradient and its conversion into a generalized Rindler force.

The entropic statement requires greater care. The equality

$$\delta S_W = k_B \delta \langle K_W \rangle \tag{2}$$

is a first-order identity around a fixed reference state. It is not an exact equality for arbitrary finite excitations. At finite displacement in state space, relative entropy supplies the missing term. This revision therefore separates three logically different results:

1. an exact modular-translation identity in a Minkowski vacuum wedge;

2. a conditional quasistatic holding-work interpretation after proper-time normalization;

3. a linearized entanglement-entropy corollary, including a precise derivation of the coefficient in equation (1).

The modular and entropic treatment of force has precedents. Carroll and Remmen analyzed the role of modular Hamiltonians in entropic-gravity arguments [15], and An and Cheng used the entanglement first law in a Rindler-force construction [16]. The exact modular-translation identity (Theorem 1, Eq. (21)) and the finite-state relative-entropy completion (Eq. (42)) are not part of that construction as presented. The contribution here is the explicit translation theorem with a complete sign and unit audit, the distinction between reference and local Rindler force, the finite-state relative-entropy completion, and a strict classification of which part of the Verlinde construction follows as a theorem.

No claim is made that the calculation derives an acceleration, selects a trajectory, explains microscopically why matter possesses inertia, or produces Newtonian or Einstein gravity. The exact result is a generalized-force identity for prescribed accelerated motion on fixed Minkowski spacetime. The phrase *entropic inertia* will be used only for its linearized information-theoretic representation.

2 Rindler modular structure and observer normalization

2.1 The right wedge and the Bisognano–Wichmann generator

Work in $(d+1)$-dimensional Minkowski spacetime with inertial coordinates $(ct, x^1, \mathbf{x}_\perp)$. The right Rindler wedge is

$$W_R = \{x^1 > |ct|\}, \tag{3}$$

and its $t=0$ Cauchy half-space is $\Sigma_R = \{x^1 > 0\}$. Let $\mathcal{A}(W_R)$ be the associated local observable algebra and Ω the Poincaré-invariant vacuum. Under the standard assumptions of algebraic or Wightman QFT, Ω is cyclic and separating for $\mathcal{A}(W_R)$, so Tomita–Takesaki modular flow is defined. The Bisognano–Wichmann property identifies this modular flow with Lorentz boosts preserving W_R [5, 6, 9, 10].

In a regulator or split representation, the one-sided vacuum modular Hamiltonian has the familiar local form

$$K_W = \frac{2\pi}{\hbar c}\int_{x^1>0} x^1 T_{00}(0,\mathbf{x})\,\mathrm{d}^d x + C, \tag{4}$$

where C is state independent. Throughout, the object of the theorem is the full boost generator; the local integral is its bulk contribution in a regulated or split representation. Where boundary or improvement terms are present (Section 8.3), they are to be added to K_W and carried through the translation identity unchanged. In strict continuum AQFT the local algebra is generally type III, so one should not interpret this expression as the logarithm of a trace-class reduced density matrix. The algebraic modular operator and Araki relative entropy are the rigorous objects [11, 10]. For the present calculation it is sufficient to use the vacuum-relative quadratic form

$$\Delta\langle K_W\rangle_\rho := \frac{2\pi}{\hbar c}\int_{x^1>0} x^1\big(\langle T_{00}\rangle_\rho - \langle T_{00}\rangle_\Omega\big)\,\mathrm{d}^d x, \tag{5}$$

whenever the integral exists. The additive constant and the state-independent ultraviolet terms cancel from this difference.

Equation (4) contains no selected proper acceleration. It is canonically normalized by the 2π relation between modular parameter and Lorentz rapidity. This fact is essential for interpreting the later appearance of a.

2.2 From modular parameter to proper Rindler time

Introduce Rindler coordinates $(\eta, \rho, \mathbf{x}_\perp)$ by

$$ct = \rho \sinh \eta, \qquad x^1 = \rho \cosh \eta, \qquad \rho > 0, \tag{6}$$

so that

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \mathrm{d}\mathbf{x}_\perp^2. \tag{7}$$

An orbit of constant ρ has proper acceleration

$$a_{\mathrm{loc}}(\rho) = \frac{c^2}{\rho} \tag{8}$$

and proper time satisfying $\eta = c\tau/\rho = a_{\mathrm{loc}}\tau/c$.

Select a reference orbit with proper acceleration a,

$$\rho_a = \frac{c^2}{a}. \tag{9}$$

The corresponding proper-time Hamiltonian is obtained by converting modular parameter to the reference proper time:

$$H_a = \frac{\hbar a}{2\pi c} K_W = k_B T_U K_W, \qquad T_U = \frac{\hbar a}{2\pi c k_B}. \tag{10}$$

Using the local form,

$$H_a = \frac{a}{c^2} \int_{x^1>0} x^1 T_{00}\, \mathrm{d}^d x \tag{11}$$

up to a state-independent constant.

Writing $x^1 = \rho_a + p$ on the $t = 0$ slice gives the standard shifted Rindler lapse

$$N(p) = \frac{a(\rho_a + p)}{c^2} = 1 + \frac{ap}{c^2}. \tag{12}$$

The local acceleration and Tolman temperature at displacement p are

$$a_{\mathrm{loc}}(p) = \frac{a}{N(p)}, \qquad T_{\mathrm{loc}}(p) = \frac{T_U}{N(p)}. \tag{13}$$

At $p = 0$, the reference and local quantities coincide.

2.3 Entropy, relative entropy, and the first-law limit

In a common ultraviolet regulator, or in a split inclusion that replaces the sharp wedge algebra by an intermediate type-I factor [12], let

$$s_W = \frac{S_W}{k_B} \tag{14}$$

denote the dimensionless von Neumann entropy. For a wedge state ρ_W relative to the vacuum state Ω_W, the exact identity is

$$S_{\text{rel}}(\rho_W \| \Omega_W) = \Delta\langle K_W\rangle - \Delta s_W, \tag{15}$$

where S_{rel} is dimensionless. The combination on the right has a regulator-independent algebraic meaning even when its two terms are not separately defined in the continuum [11, 13, 10].

For a differentiable family

$$\rho_W(\lambda) = \Omega_W + \lambda\delta\rho_W + O(\lambda^2), \tag{16}$$

relative entropy is stationary at $\lambda = 0$ and begins at second order. Therefore

$$\delta s_W = \delta\langle K_W\rangle, \qquad \delta S_W = k_B\delta\langle K_W\rangle. \tag{17}$$

This is the entanglement first law [14]. It is exact as a first derivative, not as a finite-state equality.

3 Exact modular-translation identity

3.1 Statement and assumptions

Let ρ_p be a one-parameter family obtained by translating an admissible excitation in the $+x^1$ direction while the wedge is held fixed. Define its vacuum-subtracted energy-density expectation

$$\epsilon_p(x^1, \mathbf{x}_\perp) := \langle T_{00}(0, x^1, \mathbf{x}_\perp)\rangle_{\rho_p} - \langle T_{00}\rangle_\Omega. \tag{18}$$

Translation covariance means

$$\epsilon_p(x^1, \mathbf{x}_\perp) = \epsilon_0(x^1 - p, \mathbf{x}_\perp). \tag{19}$$

The wedge energy is

$$E_W(p) = \int_{x^1>0} \epsilon_p(\mathbf{x})\, \mathrm{d}^d x. \tag{20}$$

No positivity assumption on the pointwise energy density is needed. The theorem requires only that the relevant integrals and derivatives exist and that $x^1\epsilon_p \to 0$ at spatial infinity. Singular surface energy at the horizon, changing gauge-boundary flux, or other edge contributions must be included separately (see Section 8.3).

Theorem 1 (Wedge modular-translation identity)**.** *For a translation-covariant family satisfying the conditions above,*

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle K_W\rangle_{\rho_p} = \frac{2\pi}{\hbar c}E_W(p). \tag{21}$$

The identity is exact for the translated family. It is kinematic and does not require a quasistatic time evolution.

Proof. Using the vacuum-relative form and translation covariance,

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle K_W\rangle_{\rho_p} = \frac{2\pi}{\hbar c}\int_{x^1>0} x^1 \frac{\partial \epsilon_p}{\partial p}\,\mathrm{d}^d x = -\frac{2\pi}{\hbar c}\int_{x^1>0} x^1 \frac{\partial \epsilon_p}{\partial x^1}\,\mathrm{d}^d x. \tag{22}$$

Integrating by parts in x^1 gives

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle K_W\rangle_{\rho_p} = \frac{2\pi}{\hbar c}\int_{x^1>0} \epsilon_p\,\mathrm{d}^d x - \frac{2\pi}{\hbar c}\int \mathrm{d}^{d-1}x_\perp \big[x^1\epsilon_p\big]_0^\infty. \tag{23}$$

The term at $x^1 = 0$ vanishes because of the explicit factor x^1; the term at infinity vanishes by the assumed falloff. The remaining integral is the wedge energy. □

The same result follows directly from the translation Ward identity. With the active-translation convention

$$U(p) = \exp\Big(-\frac{i}{\hbar}pP_1\Big), \qquad U(p)^\dagger T_{00}(x^1)U(p) = T_{00}(x^1 - p), \tag{24}$$

we have $[P_1, T_{00}] = i\hbar\partial_1 T_{00}$ and

$$\frac{\mathrm{d}}{\mathrm{d}p}\langle K_W\rangle_{\rho_p} = \frac{i}{\hbar}\langle [P_1, K_W]\rangle_{\rho_p} = \frac{2\pi}{\hbar c}E_W(p). \tag{25}$$

This convention fixes the sign ambiguity that arises if active and passive translations are mixed.

Corollary 1 (Excitation concentrated inside the wedge)**.** *If the vacuum-subtracted energy density is concentrated well inside W_R and the leakage across $x^1 = 0$ is negligible over the displacement interval, then $E_W(p) = E + \varepsilon_{\text{leak}}(p)$, where E is the total excitation energy and $|\varepsilon_{\text{leak}}| \ll E$. Hence*

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle K_W\rangle = \frac{2\pi E}{\hbar c} + \frac{2\pi}{\hbar c}\varepsilon_{\text{leak}}(p). \tag{26}$$

In the ideal wedge-confined limit, the slope is constant.

Remark 1. *The theorem is not restricted to free or weakly interacting fields. Its structural inputs are translation covariance, the Bisognano–Wichmann form of the wedge modular generator, and control of boundary terms. Interactions are allowed whenever these properties and a suitable renormalized stress tensor hold.*

Remark 2. *Quasistatic motion is unnecessary for the modular-translation identity. It becomes necessary only when a family of translated states is interpreted as an actual controlled process and the change in $\langle H_a\rangle$ is identified with external work.*

4 Gaussian stress-energy profile

A simple profile makes the horizon-leakage correction explicit. In $1 + 1$ dimensions let the vacuum-subtracted energy density on the $t = 0$ slice be

$$\epsilon_{\bar{x}}(x) = \frac{E}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{(x-\rho)^2}{2\sigma^2}\Big), \tag{27}$$

where E is the total energy on the full line, $\bar{x} > 0$ is the packet center measured from the horizon, and σ is its width. This is the narrow-packet stress-energy profile obtained in the usual wave-packet approximation; the theorem itself does not depend on the Gaussian form.

Define the standard normal density and cumulative distribution

$$\phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}, \qquad \Phi(z) = \int_{-\infty}^{z} \phi(u)\,\mathrm{d}u = \frac{1}{2}\big(1 + \mathrm{erf}(z/\sqrt{2})\big). \tag{28}$$

The energy contained in the right wedge is

$$E_W(\rho) = \int_0^\infty \epsilon_{\bar{x}}(x)\,\mathrm{d}x = E\,\Phi(\rho/\sigma). \tag{29}$$

The modular energy is

$$\Delta\langle K_W\rangle_\rho = \frac{2\pi}{\hbar c}\int_0^\infty x\epsilon_{\bar{x}}(x)\,\mathrm{d}x = \frac{2\pi E}{\hbar c}\Big[\bar{x}\Phi(\bar{x}/\sigma) + \sigma\phi(\bar{x}/\sigma)\Big]. \tag{30}$$

Differentiation gives exactly

$$\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\Delta\langle K_W\rangle_{\bar{x}} = \frac{2\pi E}{\hbar c}\Phi(\bar{x}/\sigma) = \frac{2\pi}{\hbar c}E_W(\bar{x}), \tag{31}$$

confirming the theorem. For $\bar{x}/\sigma \gg 1$,

$$\frac{E_W(\bar{x})}{E} = 1 - \frac{\sigma}{\sqrt{2\pi}\bar{x}}e^{-\bar{x}^2/(2\sigma^2)}\Big(1 + O\big(\sigma^2/\bar{x}^2\big)\Big). \tag{32}$$

Thus the correction is Gaussian in the squared distance-to-width ratio, not merely exponential in $\bar{x}/\sigma$.

5 Rindler holding work and the $F = ma$ form

5.1 Proper-time modular energy

Multiplying the modular-translation identity by $k_B T_U = \hbar a/(2\pi c)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle H_a\rangle_{\rho_p} = \frac{a}{c^2}E_W(p). \tag{33}$$

The factors of c are essential. The expression aE is not a force in SI units; the required conversion from energy to inertial mass supplies the factor c^{-2}, so the correct force scale is aE/c^2.

Equation (33) is still a statement about a parameterized family of states. To interpret it as work, impose the following operational protocol:

1. The excitation is instantaneously at rest relative to the Rindler congruence and is centered near the reference orbit $\rho_a = c^2/a$.

2. An external controller translates the center by $\mathrm{d}p$ while preserving the excitation's internal state and invariant mass to the required accuracy.

3. The step is quasistatic relative to the Rindler timescale, $\tau_{\text{move}} \gg c/a$, and small compared with the acceleration length, $|\mathrm{d}p| \ll c^2/a$.
4. Independent energy transfer through radiation, particle production, horizon crossing, or coupling to another reservoir is negligible.

Under these conditions the change in the expectation of the proper-time Hamiltonian is the mean external work normalized to the reference observer's clock:

$$\mathrm{d}W_{\text{ext}}^{(a)} = \mathrm{d}\Delta\langle H_a\rangle. \tag{34}$$

The generalized force supplied by the controller is therefore

$$F_{\text{ext}}^{(a)}(p) := \frac{\mathrm{d}W_{\text{ext}}^{(a)}}{\mathrm{d}p} = \frac{a}{c^2}E_W(p). \tag{35}$$

The force exerted by the accelerated frame on the excitation is opposite in sign,

$$F_{\text{frame}}^{(a)}(p) = -\frac{\partial}{\partial p}\Delta\langle H_a\rangle = -\frac{a}{c^2}E_W(p). \tag{36}$$

Thus outward displacement requires positive controller work, while the effective Rindler force is directed toward the horizon.

5.2 Reference force and local force

Away from the reference orbit, equation (35) is a redshifted force conjugate to displacement with energy measured by the reference clock. The local proper force is

$$F_{\text{ext}}^{\text{loc}}(p) = \frac{F_{\text{ext}}^{(a)}(p)}{N(p)} = \frac{a_{\text{loc}}(p)}{c^2}E_W(p), \tag{37}$$

using the local acceleration. At $p = 0$, $N = 1$ and $a_{\text{loc}} = a$.

For a narrow, instantaneously comoving state with well-defined invariant mass M, the rest-frame total energy is

$$E_{\text{rest}} = Mc^2. \tag{38}$$

Here M denotes the invariant mass of a narrow, instantaneously comoving excitation whose stress-energy is concentrated inside the wedge. For a composite body the wedge energy E_W includes internal kinetic and stress contributions; the identification $E_W \simeq Mc^2$ requires that the body be unstressed and at rest in the instantaneous comoving frame, so that the vacuum-subtracted wedge energy reduces to the rest energy. Where that fails, the exact statement remains the force identity with E_W, and the coefficient is E_W/c^2 rather than M. If the packet is concentrated within the wedge so that $E_W \simeq E_{\text{rest}}$, then at the reference orbit

$$F_{\text{ext}}^{\text{loc}} = Ma, \qquad F_{\text{frame}}^{\text{loc}} = -Ma. \tag{39}$$

More generally, the local force is $F_{\text{ext}}^{\text{loc}} = Ma_{\text{loc}}$.

Equation (39) is a holding-force or virtual-work identity. It does not derive the dynamical equation $M\ddot{x} = F_{\text{net}}$, determine the acceleration, or explain why a freely moving body follows an inertial trajectory. The acceleration has already been selected by the Rindler congruence. What the modular calculation fixes is the universal coefficient E/c^2 multiplying that prescribed acceleration.

5.3 Why the result is not an algebraic tautology

The result is limited, but it is not obtained by placing a into the modular Hamiltonian and reading it back out. The logical order is

$$\text{vacuum wedge modular structure} \longrightarrow \frac{\mathrm{d}\Delta\langle K_W\rangle}{\mathrm{d}p} = \frac{2\pi E}{\hbar c} \longrightarrow H_a = \frac{\hbar a}{2\pi c} K_W. \tag{40}$$

The first arrow is independent of a. The second converts the canonical boost normalization into the proper-time normalization of a chosen observer. The construction does not predict a, but neither does it assume the coefficient E/c^2 or the 2π cancellation that produces the holding-force form.

6 Relative-entropy completion of the entropic relation

6.1 Exact finite-state identity

For a finite state in a common regulator or split representation, differentiate the relative-entropy identity with respect to displacement at fixed wedge:

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle K_W\rangle = \frac{1}{k_B}\frac{\mathrm{d}\Delta S_W}{\mathrm{d}p} + \frac{\mathrm{d}}{\mathrm{d}p} S_{\text{rel}}(\rho_{W,p}\|\Omega_W). \tag{41}$$

Multiplying by $k_B T_U$ gives the exact regulated decomposition

$$F^{(a)}_{\text{ext}}(p) = T_U \frac{\mathrm{d}\Delta S_W}{\mathrm{d}p} + k_B T_U \frac{\mathrm{d}}{\mathrm{d}p} S_{\text{rel}}(\rho_{W,p}\|\Omega_W). \tag{42}$$

At a general Rindler position the local form is

$$F^{\text{loc}}_{\text{ext}}(p) = T_{\text{loc}}(p) \frac{\mathrm{d}\Delta S_W}{\mathrm{d}p} + k_B T_{\text{loc}}(p) \frac{\mathrm{d}}{\mathrm{d}p} S_{\text{rel}}(\rho_{W,p}\|\Omega_W). \tag{43}$$

Equation (42) is the correct finite-state replacement for $\mathrm{d}W = T_U \mathrm{d}S_W$. Relative entropy is nonnegative, but its derivative with respect to position need not have a fixed sign. Therefore positivity alone does not permit the second term to be discarded.

The exact QFT statement is modular work. Its decomposition into an entropy difference and relative entropy depends on the reference state and, for the separate entropy term, on a common regulator. This is why a finite excitation cannot generally be said to pay its entire Rindler work cost through a change in entanglement entropy. Pure von Neumann entropy is therefore insufficient outside the linear-response regime.

6.2 Linearized entropic corollary

Let

$$\rho_{W,p}(\lambda) = \Omega_W + \lambda \delta\rho_{W,p} + O(\lambda^2). \tag{44}$$

At fixed p, $S_{\text{rel}} = O(\lambda^2)$. Taking the coefficient of λ in equation (42) yields

$$T_U \frac{\mathrm{d}}{\mathrm{d}p} \delta S_W(p) = \frac{a}{c^2} \delta E_W(p). \tag{45}$$

For a perturbation concentrated inside the wedge,

$$\frac{\mathrm{d}}{\mathrm{d}p} \delta S_W = \frac{2\pi k_B}{\hbar c} \delta E. \tag{46}$$

This is the valid entanglement first-law statement. It is linear in the state perturbation, although the displacement derivative itself can be evaluated at any p for which the translated linearized state remains admissible.

A further qualification is necessary. If the excitation energy begins only at second order in the state-amplitude parameter, then equation (46) is trivial at first order. At the first nonzero order, relative entropy can contribute at the same order as modular energy, and the pure entropy formula does not follow.

6.3 Coherent states as a finite-state diagnostic: worked free-field example

Free-field coherent states provide a direct, fully explicit example. Consider a free massive scalar field in $1+1$ dimensions. A coherent state is generated by a Weyl unitary $W(f) = \mathrm{e}^{i\phi(f)}$ acting on the vacuum, where f is a real test function (a classical solution of the Klein–Gordon equation). The vacuum-subtracted energy density is quadratic in the coherent amplitude.

When the test function f is supported strictly inside the right wedge, the Weyl unitary factors as an operator affiliated with $\mathcal{A}(W_R)$ times an operator affiliated with the commutant. Consequently the reduced density matrix on the wedge differs from the vacuum reduced density matrix by a unitary conjugation, and the von Neumann entanglement entropy is unchanged:

$$\Delta S_W = 0. \tag{47}$$

(The same conclusion holds, up to a boundary term of the same order as the leakage $\varepsilon_{\text{leak}}$ of Corollary 2, for a Gaussian profile whose tails cross the horizon; the diagnostic concerns the structural impossibility of attributing all finite modular work to ΔS_W, so one may equally take f compactly supported inside the wedge.)

The modular energy is nonzero. Writing the packet center as $\bar{x} = \rho_a + p$ (so that $\mathrm{d}/\mathrm{d}p = \mathrm{d}/\mathrm{d}\bar{x}$), the modular energy is given by the same integral that appears in equation (30). Relative entropy therefore saturates the modular energy:

$$S_{\text{rel}}(\rho_{W,p} \| \Omega_W) = \Delta\langle K_W \rangle_p. \tag{48}$$

Differentiating with respect to the translation parameter yields

$$\frac{\mathrm{d}}{\mathrm{d}p}S_{\mathrm{rel}} = \frac{2\pi}{\hbar c}E_W(p), \tag{49}$$

and the entire finite modular-work gradient is carried by the relative-entropy term:

$$F_{\mathrm{ext}}^{(a)} = k_B T_U \frac{\mathrm{d}}{\mathrm{d}p}S_{\mathrm{rel}}. \tag{50}$$

This example rules out an unrestricted identification of finite Rindler work with $T_U \Delta S_W$. It does not contradict the entanglement first law, because both the coherent-state stress energy and the relative entropy begin at second order in the coherent amplitude [17]. The same conclusion holds for any free-field coherent state generated by a test function supported inside the wedge.

7 The precise status of Verlinde's local entropy law

Verlinde's local screen relation (1) can now be compared with the linearized result. For a linearized, wedge-localized perturbation whose first-order rest-energy coefficient is

$$\delta E = Mc^2, \tag{51}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}p}\delta S_W = 2\pi k_B \frac{Mc}{\hbar}. \tag{52}$$

The sign depends on the orientation assigned to p; the invariant statement is the magnitude of the coefficient. In the present convention, p increases away from the Rindler horizon and the modular weight increases with p.

Corollary 2 (Linearized Rindler entropy–displacement law). *Within a Minkowski vacuum Rindler wedge, for infinitesimal state variations with nonzero first-order energy and negligible horizon leakage, the coefficient in Verlinde's local entropy–displacement relation follows from the Bisognano–Wichmann modular Hamiltonian and the entanglement first law.*

This is a genuine promotion of a restricted formula from postulate to corollary, but it does not promote the full holographic-screen framework to a theorem. The distinctions are summarized in Table 1.

The Rindler horizon is therefore a canonical algebraic thermodynamic interface: its modular flow, KMS normalization, and linearized entropy response are fixed by QFT. Calling it a finite holographic information-storage screen requires additional assumptions that are not contained in the Bisognano–Wichmann theorem.

8 Domain of validity and failure modes

8.1 Exact theorem: required structure

The exact modular-translation theorem requires:

Table 1: Status of the ingredients commonly grouped under the entropic-force construction.

Statement	**Status in the present framework**
Vacuum wedge modular flow is Lorentz-boost flow	Bisognano–Wichmann theorem under standard QFT assumptions.
The reference Unruh temperature is $T_U = \hbar a/(2\pi c k_B)$	KMS/Unruh result after choosing the observer's proper-time normalization.
$\mathrm{d}\Delta\langle K_W\rangle/\mathrm{d}p = 2\pi E/(\hbar c)$ for a wedge-contained translated excitation	Exact modular-translation identity, subject to finite-energy and boundary-domain conditions.
$\lvert F_{\mathrm{hold}}\rvert = aE/c^2 = Ma$ at the reference orbit	Quasistatic generalized-force corollary for prescribed accelerated motion.
$\mathrm{d}(\delta S_W)/\mathrm{d}p = 2\pi k_B Mc/\hbar$	Linearized entanglement-first-law corollary; not a universal finite-state entropy equality.
$F = T\nabla S$ with no relative-entropy term for arbitrary excitations	Not established; false in general for finite states.
Every causal or spatial surface is a holographic screen	Not established.
The number of screen degrees of freedom is proportional to area with a specified Newton constant	Not established here.
Equipartition of screen energy	Not established.
Newtonian gravity or the Einstein equation follows	Not derived by this calculation.
A microscopic vacuum mechanism explains why matter has inertia	Not supplied by the modular identity.

- a Poincaré-invariant vacuum and a QFT satisfying the Bisognano–Wichmann property for the wedge;
- translation covariance in the normal direction;
- a renormalized stress tensor, or equivalent Poincaré generators, for which the modular-energy expectation difference is defined;
- finite vacuum-subtracted wedge energy and sufficient falloff at spatial infinity;
- no unaccounted singular horizon term or boundary charge that changes during the translation.

Strict compact localization is neither required nor generally available in relativistic QFT. The exact theorem automatically uses the energy $E_W(p)$ actually contained in the wedge. Replacing it by the global energy E is an approximation controlled by the leakage term.

8.2 Interactions

The theorem is not a free-field artifact. The Bisognano–Wichmann result was formulated within axiomatic interacting QFT, and the translation proof uses symmetry rather than Gaussian dynamics. Interactions can alter the state, stress-energy distribution, and admissible operator domains, but they do not change the coefficient in the modular-translation identity when the stated structural assumptions remain valid.

8.3 Gauge theories and edge structure

Gauge theories require a choice of regional observable algebra and can possess electric-flux sectors or edge contributions. It is not generally valid to declare such terms state independent. For a neutral compact excitation translated well away from the horizon without changing boundary flux, the bulk modular-translation identity applies. If the protocol changes the flux sector or induces horizon-supported terms, those contributions must be added to the modular generator and to the force balance. The theorem then applies to the complete generator, not merely the bulk stress-tensor integral.

8.4 Finite-density and nonvacuum reference states

The local form of the modular Hamiltonian is special to the vacuum wedge. At finite density, finite temperature with respect to inertial time, or in a generic nonequilibrium state, the regional modular Hamiltonian is usually nonlocal. A universal displacement slope proportional only to the total energy does not follow. Extension to such states requires the actual modular generator and cannot be obtained by replacing the vacuum occupation numbers in the present formula.

8.5 Curved spacetime and gravity

The derivation is performed on fixed Minkowski spacetime and contains no gravitational backreaction. Near a smooth spacetime point, a sufficiently small causal neighborhood can approximate a Rindler wedge, but the exact global boost symmetry and exact Bisognano–Wichmann generator are then lost. Curvature, state dependence, finite-size effects, and the choice of local vacuum introduce corrections. Applying the result near a stationary bifurcate Killing horizon requires a separate treatment of the Killing normalization, redshift, and gravitational constraints.

Nothing in the present calculation derives the Einstein equation, Newton's inverse-square law, spatial curvature, gravitational radiation, or a holographic area law. Those require additional dynamical and geometric input.

8.6 Rapid motion, radiation, and equation-of-motion dynamics

The modular translation theorem compares states on a fixed slice and remains kinematic. The work interpretation assumes a controlled slow process. Rapid displacement can generate radiation, detector excitation, particle production, dissipative forces, and nonlocal memory. In that regime,

$$\Delta\langle H_a\rangle \neq W_{\mathrm{controller}} \tag{53}$$

without a full energy-balance calculation. A dynamical theory of inertia would need a real-time influence functional or response kernel that predicts the trajectory and accounts for dissipation and fluctuations. The present paper does not supply that theory.

9 Interpretation

The calculation supports a precise but limited statement: the Minkowski vacuum wedge carries a canonical modular energy whose spatial gradient assigns the universal inertial weight E/c^2 to a localized excitation. Once a Rindler proper-time normalization is selected, the gradient is the external force required to maintain prescribed accelerated motion. In linear response around the vacuum, the same gradient can be represented by a change in wedge entanglement entropy.

This result has three levels of interpretation:

1. **Exact structural result.** Translation covariance and the Bisognano–Wichmann generator imply the modular-translation identity. This is independent of any entropic-force postulate.
2. **Operational work result.** Under the control assumptions of the quasistatic protocol, the proper-time modular-energy gradient is the quasistatic holding force. This produces the form $F = Ma$ for a body at the reference Rindler orbit.
3. **Linearized information-theoretic representation.** The entanglement first law converts the first-order modular-energy gradient into the entropy gradient that recovers Verlinde's coefficient. For finite states, relative entropy generally carries part or all of the modular work.

The calculation does not establish that inertia is physically caused by the production or rearrangement of entanglement. Such a causal claim would require a microscopic matter–vacuum interaction model and a real-time derivation of the response. The safe conclusion is that inertial holding work admits a canonical modular and, at linear order, entanglement-thermodynamic representation. The representation is nontrivial because its normalization is fixed by QFT, but it is not yet a constitutive mechanism.

The term *entropic inertia* is therefore defensible only with the following meaning:

> In the Minkowski vacuum Rindler setting, the quasistatic force required to maintain a localized excitation on a prescribed accelerated trajectory is the proper-time gradient of vacuum modular energy; for infinitesimal state variations, this gradient equals the Unruh temperature times the wedge entanglement-entropy gradient. For finite states the exact relation requires the relative-entropy completion.

Any stronger claim must be separately derived.

10 Conclusion

For a translated excitation in the right Rindler wedge, the vacuum-relative modular energy obeys the exact identity

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle K_W\rangle = \frac{2\pi}{\hbar c}E_W(p). \tag{54}$$

The theorem follows from the Bisognano–Wichmann modular generator and translation covariance. It contains no chosen acceleration and is valid beyond free-field theory when the structural assumptions hold.

Choosing a reference observer of proper acceleration a gives

$$\frac{\mathrm{d}}{\mathrm{d}p}\Delta\langle H_a\rangle = \frac{a}{c^2}E_W(p). \tag{55}$$

Under a quasistatic holding protocol and for a wedge-contained body of invariant mass M, this yields the local reference-orbit force magnitude Ma. It is a virtual-work identity for prescribed acceleration, not an equation-of-motion derivation.

For finite states, the exact information-theoretic relation includes relative entropy,

$$F_{\text{hold}} = T_U\frac{\mathrm{d}\Delta S_W}{\mathrm{d}p} + k_B T_U\frac{\mathrm{d}S_{\text{rel}}}{\mathrm{d}p}. \tag{56}$$

Only in the first-order state-variation limit does the relative-entropy term drop out. In that domain,

$$\frac{\mathrm{d}}{\mathrm{d}p}\delta S_W = 2\pi k_B\frac{Mc}{\hbar}, \tag{57}$$

which derives the coefficient in Verlinde's local entropy–displacement ansatz from QFT modular structure. The pure-entropy formula is false in general for finite states; the relative-entropy completion is required. The full holographic-screen postulate, equipartition, gravity, and a microscopic explanation of inertia remain outside the result.

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