Pseudo-Trigonometric Functions

A closed notation for piecewise-affine periodic curves

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Abstract. We study two elementary periodic functions — the scaled fractional part $\operatorname{sp}_p x$ (a pure ramp of period p) and its complement $\operatorname{cp}_p x = p - \operatorname{sp}_p x$ — together with the single algebraic identity $\operatorname{sp}_p x + \operatorname{cp}_p x = p$. Because their derivatives are the *constants* $+1$ and -1, they turn a range of nonlinear differential equations into closed one-line solutions, they solve first-order transport and inviscid Burgers equations exactly, and they let one write intricate periodic curves — half-circle chains, staircases, asymmetric arcs, modulated ramps — each from a single algebraic expression. A complex generator $\mathbf{Z}(x) = \operatorname{sp} x + i \operatorname{cp} x$ opens a second, geometric part: its powers $\mathbf{Z}^n$ are explicit algebraic curves of degree n, and the normalized family $\{\mathbf{Z}^\alpha / p^\alpha\}$ covers an eye-shaped region with an exactly empty core bounded by the unit circle. The aim is not to compete with Fourier analysis but to offer a light, exact notation for the angular, piecewise-affine patterns where Fourier series are heavy and only approximate. All identities have been verified symbolically and numerically (sympy, machine precision).

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1 At a glance: one line, one curve

Each curve below is the graph of the one-line expression printed with it. No series, no truncation — the equalities are exact.

2 Definition

Fix a period $p > 0$ and the intervals $[0, p), [p, 2p), \ldots$ For $x \in \mathbb{R}$,

$$\operatorname{sp}_p x = x - p\left\lfloor \frac{x}{p} \right\rfloor, \qquad \operatorname{cp}_p x = p - \operatorname{sp}_p x,$$

where $\operatorname{sp}_p x$ is the residue of x modulo p — the sawtooth wave of signal processing — here promoted to a first-class operator. The period is a free scale, since $\operatorname{sp}_p x = p \operatorname{sp}_1(x/p)$; we fix $p = 5$ in all figures and write $\operatorname{sp} x$ for $\operatorname{sp}_p x$ when the period is understood. The two functions are tied by the linear identity

$$\boxed{\operatorname{sp} x + \operatorname{cp} x = p}$$

in contrast with the circular law $\sin^2 + \cos^2 = 1$. On each open interval $(pk, p(k+1))$ the derivatives are the constants

$$(\operatorname{sp} x)' = +1, \qquad (\operatorname{cp} x)' = -1,$$

with jump discontinuities at the points pk. The pure periodic ramp is $\operatorname{sp} x$ itself, while $x - \operatorname{sp} x = p\lfloor x/p \rfloor$ is the associated staircase.

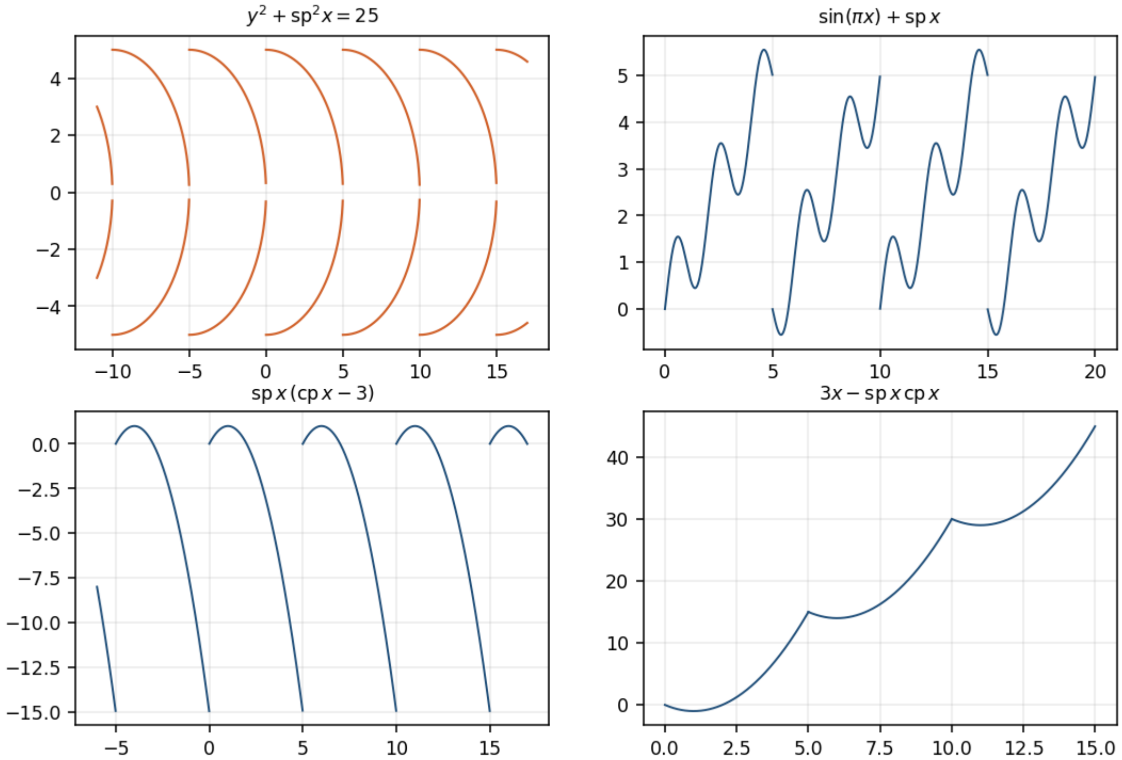

Figure 1: A chain of half-circles; a smooth wave carried by a ramp; asymmetric arcs with angular bounce points; a line modulated by a strict periodic term.

3 Closed-form solutions of nonlinear ODEs

The constant derivatives collapse several nonlinear equations to algebra. With $y = \dfrac{1}{\mathrm{cp}\, x}$ one has $y' = \mathrm{cp}^{-2} x = y^2$, and differentiating repeatedly gives the clean family

$$y = \frac{1}{\mathrm{cp}\, x} \implies y' = y^2, \quad y'' = 2y^3, \quad y''' = 6y^4, \quad y^{(n)} = n!\, y^{n+1}.$$

(The mirror choice $y = 1/\mathrm{sp}\, x$ solves $y' = -y^2$, $y'' = 2y^3$, $y''' = -6y^4$.) Exponential forcing behaves just as cleanly: with $y = e^{ax}\, \mathrm{sp}\, x$,

$$y' - ay = e^{ax}, \qquad y'' - a^2 y = 2a\, e^{ax}, \qquad y''' - a^3 y = 3a^2 e^{ax}.$$

Each right-hand side is an ordinary exponential — no pseudo-term survives.

Arbitrary slope, and composition. The slope $+1$ is not essential: $m\, \mathrm{sp}_p\, x$ is a sawtooth of constant slope m. More usefully, because the inner derivative of sp_p is 1, the chain rule collapses: for any differentiable f,

$$\big(f(\mathrm{sp}_p\, x)\big)' = f'(\mathrm{sp}_p\, x) \qquad \text{on each open interval } (pk, p(k{+}1)).$$

This single identity is what makes the whole gallery mechanical — every curve $f(\mathrm{sp}_p\, x)$ is differentiated simply by differentiating f, with no correction term.

Status of these solutions. All the differential identities above are meant on each open interval of continuity; at the points pk the functions jump, and neither the classical nor the distributional derivative is defined there. The functions are therefore *piecewise-classical* solutions, periodically reconnected — an honourable status, shared with self-similar and piecewise solutions elsewhere in analysis. The notation is exact on each cell and makes no claim across the jumps.

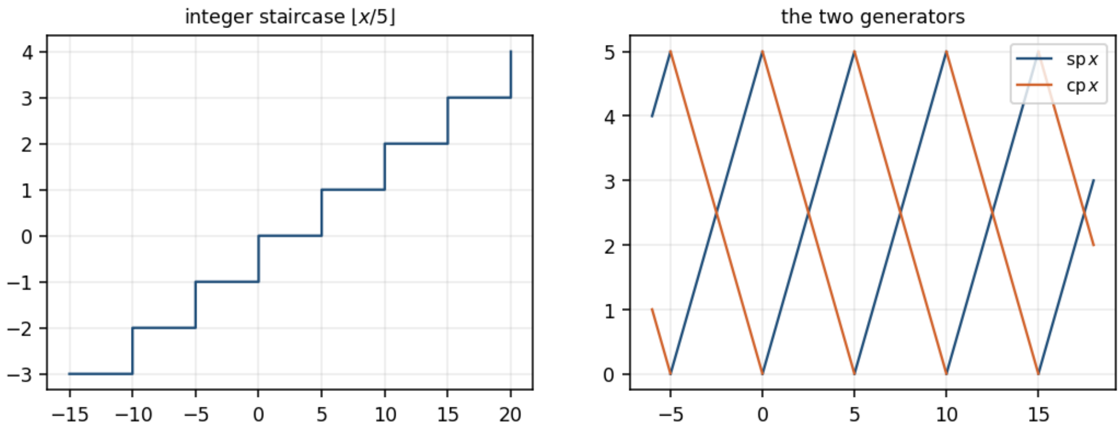

Figure 2: The integer staircase $\lfloor x/5 \rfloor = (x - \mathrm{sp}\, x)/5$, and the two generators $\mathrm{sp}\, x$ (ramp) and $\mathrm{cp}\, x$.

A travelling wave (transport equation). One last closed solution is of a slightly different nature: not an ODE, but a first-order transport partial differential equation, solved by letting the ramp travel.

Proposition 1. *Let $v \in \mathbb{R}$, or more generally $v(t)$ with $X(t) = \int_0^t v(\tau)\, d\tau$. Then $f(x,t) = \mathrm{sp}\big(x - X(t)\big)$ satisfies, on each cell of continuity, $\partial_t f + v(t)\, \partial_x f = 0$, and so does $g(x,t) = \mathrm{cp}(x - X(t))$. The discontinuity front $x = X(t) + pk$ moves at the exact speed $v(t)$.*

Proof. On a cell, $\mathrm{sp}(x - X(t)) = (x - X(t)) - pk$ for a locally constant integer k. Hence $\partial_t f = -X'(t) = -v(t)$ and $\partial_x f = 1$, so $\partial_t f + v(t)\partial_x f = 0$; $\mathrm{cp} = p - \mathrm{sp}$ gives the opposite sign. Verified symbolically and numerically: residual max $\approx 1.8 \times 10^{-11}$ away from the jumps. □

Remark. This is a special case of $f(x,t) = F(x - X(t))$ solving the linear transport equation (method of characteristics): it holds for any piecewise-differentiable F. What sp contributes is (i) an exact sawtooth wave whose jump travels without deformation or Gibbs overshoot, and (ii) that the front sticks exactly to the speed $v(t)$ — a trivial Rankine–Hugoniot condition verified with no extra hypothesis. More generally, if $\xi(x,t)$ is a characteristic coordinate of a field $c(x,t)$, then $\mathrm{sp}(\xi(x,t))$ solves $\partial_t f + c\, \partial_x f = 0$.

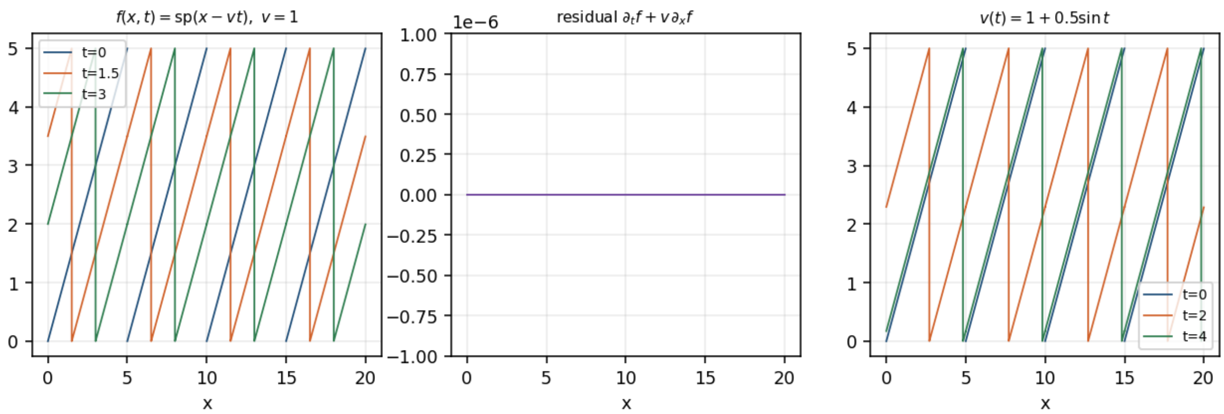

Figure 3: Left: snapshots of $\mathrm{sp}(x - vt)$ at $v = 1$, $t = 0, 1.5, 3$ — the profile glides without deformation. Centre: numerical residual $\partial_t f + v\, \partial_x f$ away from the jumps, of order 10^{-11}. Right: same with variable speed $v(t) = 1 + 0.5 \sin t$.

A Burgers sawtooth (nonlinear, with a shock). The centred ramp $\operatorname{sp}(x) - p/2$ also gives, in closed form, an exact solution of the inviscid Burgers equation.

Proposition 2. *For $t > -1$, $u(x,t) = \dfrac{\operatorname{sp}(x) - p/2}{1+t}$ satisfies, on each cell of continuity, $\partial_t u + u\,\partial_x u = 0$. The shock at $x = pk$ stays fixed for all t (zero Rankine–Hugoniot speed, by antisymmetry), and its amplitude decays as $p/\big(2(1+t)\big)$.*

Proof. On a cell, $u = (x - pk - \frac{p}{2})/(1+t)$ is affine in x, so $\partial_x u = 1/(1+t)$, $\partial_t u = -u/(1+t)$, and $\partial_t u + u\partial_x u = 0$. At $x = p(k{+}1)$ the left and right values $\pm p/(2(1+t))$ are opposite, so the shock speed $\frac{1}{2}(u_g + u_d) = 0$. At $t = 0$, $u = \operatorname{sp}(x) - p/2$. Verified symbolically (residual exactly zero) and numerically (residual $\approx 3 \times 10^{-9}$ away from the jumps). □

Remark. That a periodic sawtooth is a self-similar $1/t$ solution of Burgers is classical (Burgers 1948; Kida 1979). What this notation adds is the closed one-line formula $\big(\operatorname{sp}(x) - p/2\big)/(1+t)$, exact and verifiable at every t, where the literature usually describes the regime asymptotically.

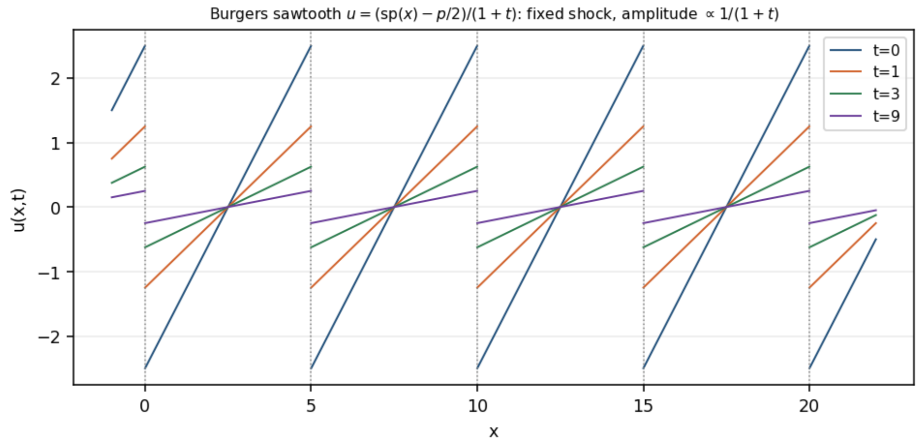

Figure 4: Burgers sawtooth at $t = 0, 1, 3, 9$: the shock stays fixed at $x = pk$ (dotted) while the amplitude decays exactly as $p/(2(1+t))$.

4 Antiderivatives

Within any interval of continuity, integration is equally direct:

$$\int \operatorname{sp} x\, dx = \frac{\operatorname{sp}^2 x}{2} + c, \qquad \int \operatorname{cp} x\, dx = -\frac{\operatorname{cp}^2 x}{2} + c, \qquad \int e^x \operatorname{cp} x\, dx = e^x \operatorname{cp} x + e^x + c.$$

5 Hyperbolic pseudo-functions

Define

$$\operatorname{psh} x = \frac{e^{\operatorname{sp} x} + e^{\operatorname{cp} x}}{2\, e^{p/2}}, \qquad \operatorname{pch} x = \frac{e^{\operatorname{sp} x} - e^{\operatorname{cp} x}}{2\, e^{p/2}}.$$

Since $\operatorname{sp} x + \operatorname{cp} x = p$, writing $u = \operatorname{sp} x - \frac{p}{2}$ gives $\operatorname{cp} x - \frac{p}{2} = -u$, so these are exactly the ordinary hyperbolic functions evaluated on the centred ramp:

$$\boxed{\operatorname{psh} x = \cosh\!\big(\operatorname{sp} x - \tfrac{p}{2}\big), \qquad \operatorname{pch} x = \sinh\!\big(\operatorname{sp} x - \tfrac{p}{2}\big)}$$

whence the Minkowski-type identity and the cross-symmetry of the derivatives follow at once:

$$\operatorname{psh}^2 x - \operatorname{pch}^2 x = 1, \qquad \operatorname{psh}' x = \operatorname{pch} x, \qquad \operatorname{pch}' x = \operatorname{psh} x.$$

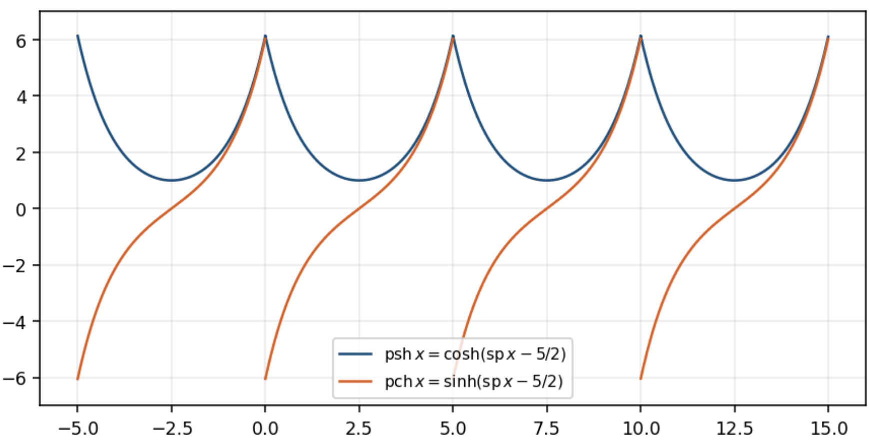

Figure 5: The pseudo-hyperbolic functions, driven by the ramp: they reset at every multiple of 5.

Proposition 3 (periodic tracing of the hyperbola)**.** *The complex function* $\psi(x) = \operatorname{psh} x + i \operatorname{pch} x$ *traces, on each cell, the arc of the unit hyperbola* $X^2 - Y^2 = 1$ *corresponding to* $u = \operatorname{sp} x - \frac{p}{2} \in [-\frac{p}{2}, \frac{p}{2})$*, and re-traces exactly the same arc every period. It is the honest hyperbolic counterpart of* $x \mapsto \cos x + i \sin x$ *on the unit circle.*

Proof. $\psi = \cosh u + i \sinh u$ with $u = \operatorname{sp} x - \frac{p}{2}$; by $\cosh^2 u - \sinh^2 u = 1$ the point lies on $X^2 - Y^2 = 1$. Since sp is p-periodic, u sweeps $[-\frac{p}{2}, \frac{p}{2})$ on each cell, so ψ re-traces the same arc. Verified numerically: $\max_x |\operatorname{psh}^2 x - \operatorname{pch}^2 x - 1| = 1.4 \times 10^{-14}$. $\square$

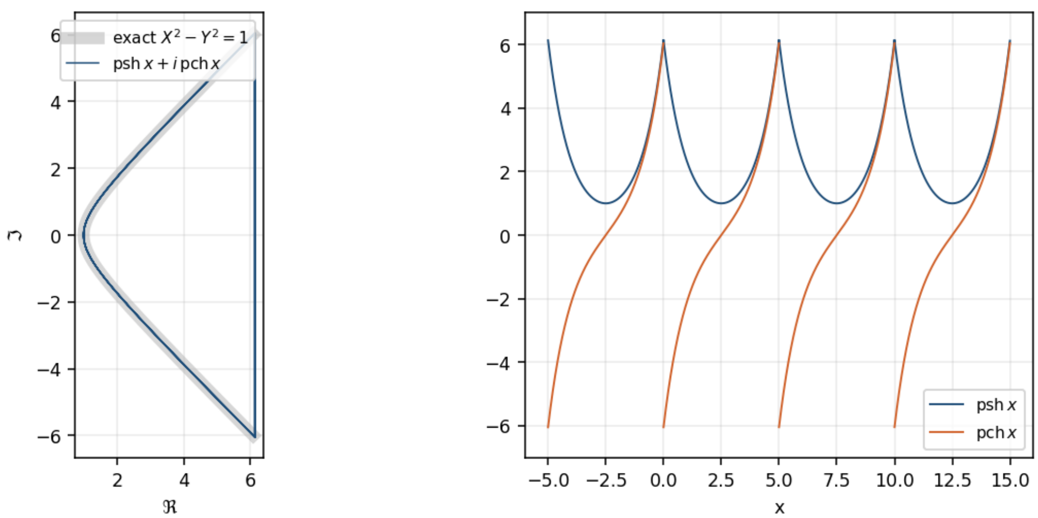

Figure 6: Left: $\operatorname{psh} x + i \operatorname{pch} x$ (4 periods) lands exactly on the hyperbola $X^2 - Y^2 = 1$. Right: the two components against x.

6 The complex generator and its algebraic arcs

Set $\mathbf{Z}(x) = \operatorname{sp} x + i \operatorname{cp} x$. Since $\operatorname{cp} x = p - \operatorname{sp} x$, writing $s = \operatorname{sp} x \in [0, p)$ gives $\mathbf{Z} = s + i(p - s)$, the segment of the line $u + v = p$: $\mathbf{Z}$ is *affine* in s. It is this affine character that makes its powers algebraic.

Proposition 4 (algebraic power arcs)**.** *For every integer* $n \geq 1$*, eliminating* s *between* $u = \Re \mathbf{Z}^n$ *and* $v = \Im \mathbf{Z}^n$ *yields a real polynomial* $P_n(u, v)$ *of degree* n *whose zero locus carries the arc* $\{\mathbf{Z}^n :$

$s \in [0, p)\}$. *For* $p = 5$*:*

$$\begin{aligned}
&P_2: \quad u^2 + 50v - 625 = 0 && \textit{(parabola)},\\
&P_3: \quad (u - v)^3 - 2625(u^2{+}v^2) - 1500\,uv + 187500(u{-}v) + 15\,625\,000 = 0 && \textit{(cubic)},\\
&P_4: \quad v^4 + 40000\,u^3 - 43\,750\,000\,u^2 + 35000\,uv^2\\
&\qquad\quad - 53\,125\,000\,v^2 + 7\,812\,500\,000\,u + 2\,441\,406\,250\,000 = 0 && \textit{(quartic)}.
\end{aligned}$$

Moreover $\mathbf{Z}(p - s) = i\,\overline{\mathbf{Z}(s)}$*, so each curve* $P_n = 0$ *is symmetric about the line through the origin at angle* $n\pi/4$*.*

Proof. $\Re\,\mathbf{Z}^n$ and $\Im\,\mathbf{Z}^n$ are polynomials of degree n in s, so $P_n := \mathrm{Res}_s(u{-}\Re\mathbf{Z}^n,\ v{-}\Im\mathbf{Z}^n)$ has degree n and vanishes on the arc. The forms P_2, P_3, P_4 are the exact resultants, and $P_n(\Re\mathbf{Z}^n, \Im\mathbf{Z}^n) \equiv 0$ identically in s was verified for $n = 2, 3, 4$. The symmetry follows from $\mathbf{Z}(p - s) = i\,\overline{\mathbf{Z}(s)}$, the map $w \mapsto e^{in\pi/2}\bar{w}$ being the reflection of axis-angle $n\pi/4$, realized on the arc by $s \mapsto p - s$. □

Thus $\mathbf{Z}^2$ is a parabola arc, $\mathbf{Z}^3$ a cubic, $\mathbf{Z}^4$ a quartic: the fabric is a pencil of named algebraic curves, one per integer power.

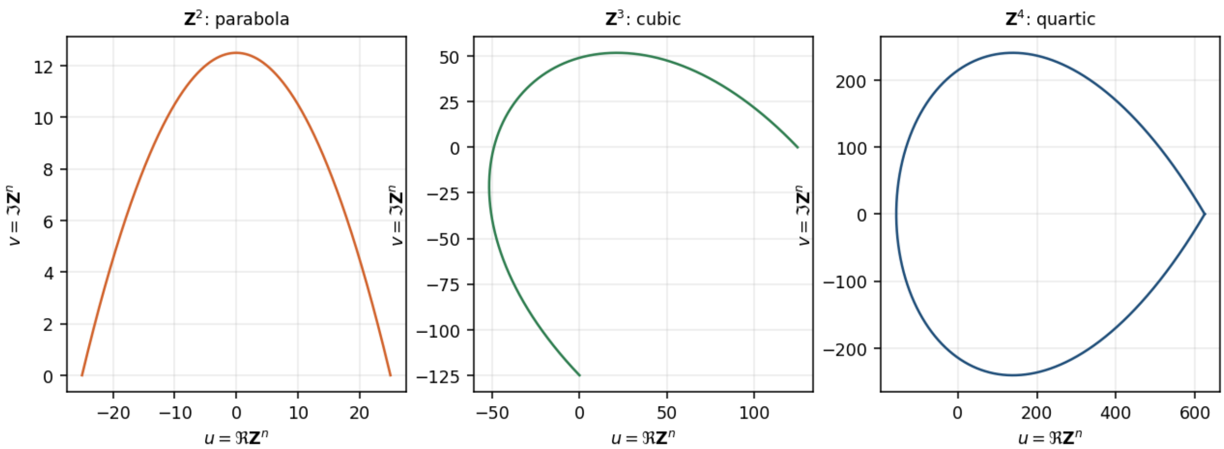

Figure 7: Power arcs $\mathbf{Z}^n$ for $n = 2, 3, 4$: parabola, cubic, quartic — exact algebraic curves of degree n.

Two elementary algebraic complements, both verified:

Proposition 5 (Möbius image)**.** $1/\mathbf{Z}$ *traces an arc of the circle of centre* $(\frac{1}{10}, -\frac{1}{10})$ *and radius* $\frac{\sqrt{2}}{10}$*. (Inversion sends the line* $u + v = 5$*, not through the origin, to a circle through the origin; radial residual* 1.4×10^{-16}*.)*

Proposition 6 (phase filter)**.** *For any period* p*,* $e^{i\frac{2\pi}{p}\,\mathrm{sp}_p x} = e^{i\frac{2\pi}{p}x}$*.*

Proof. $\mathrm{sp}_p\,x = x - p\lfloor x/p \rfloor$, so $\frac{2\pi}{p}\,\mathrm{sp}_p\,x = \frac{2\pi}{p}x - 2\pi\lfloor x/p \rfloor$ and $e^{-2\pi i\lfloor x/p \rfloor} = 1$. Verified: max discrepancy 5.6×10^{-15}. □

Remark (which combinations carry content)*.* The phase filter tells us the sawtooth vanishes the moment it is placed inside a full-turn phase tuned to its period. It therefore adds new geometric content only as an *algebraic coordinate* (a power, a product, a real or imaginary part), never as the *angle* of a tuned exponential. Practical corollary: among combinations $A(x){+}iB(x)$, those built on affine or algebraic ingredients (cp, psh / pch, inversion) remain exact algebraic curves; replacing one by sin or cos gives handsome but transcendental curves, out of reach of any polynomial equation $P_n(u, v) = 0$.

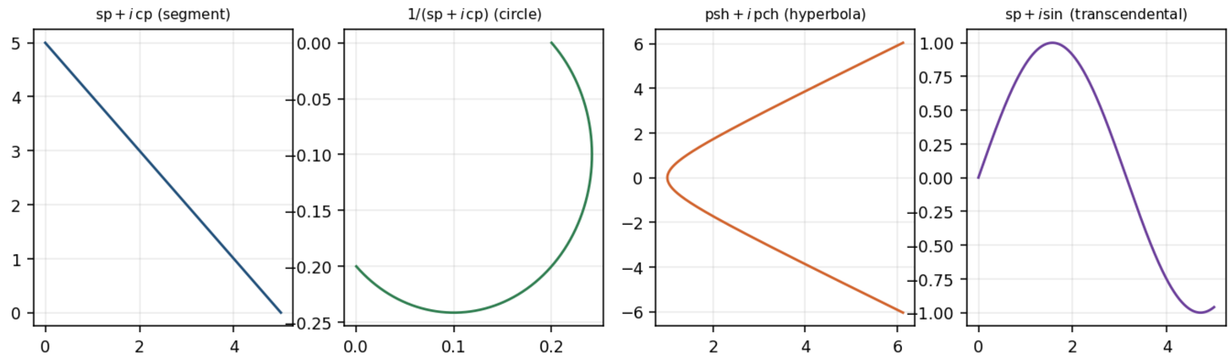

Figure 8: Map of complex combinations at $p = 5$: segment $\mathrm{sp} + i\,\mathrm{cp}$, circle arc $1/(\mathrm{sp} + i\,\mathrm{cp})$, hyperbola arc $\mathrm{psh} + i\,\mathrm{pch}$ — all algebraic; $\mathrm{sp} + i \sin$ is transcendental.

7 The covered region: the "eye"

Normalize $F(s, \alpha) = \mathbf{Z}(s)/p^\alpha = \rho(s)^\alpha e^{i\alpha\theta(s)}$, with $\rho = |\mathbf{Z}|/p \in [\frac{1}{\sqrt{2}}, 1]$, $\theta \in (0, \frac{\pi}{2}]$, and $r^\star = 2^{-A/2}$. Consider the family $\{F(s, \alpha)\} \cup \{\overline{F(s, \alpha)}\}$, $s \in (0, p)$, $\alpha \in [1, A]$.

Theorem 1 (boundary, core, collar)**.** *Let $A \geq 5$. The image satisfies:* (1) *it lies in the annulus $\{r^\star \leq |z| \leq 1\}$;* (2) *its outer boundary is exactly the unit circle $|z| = 1$, swept by the edge $s = 0$ where $F(0, \alpha) = e^{i\alpha\pi/2}$ winds $\lfloor (A-1)/4 \rfloor$ times;* (3) *there is a threshold radius r_{fill} with $\{r_{\text{fill}} \leq |z| \leq 1\}$ full, empirically $r_{\text{fill}} \approx r^\star 2^{4/A}$ (fit $< 0.3\%$ over $A \in [7, 45]$), so the relative collar width is $\approx (4 \ln 2)/A \to 0$;* (4) *for $r^\star < r < r_{\text{fill}}$ the circle $|z| = r$ is only partially covered (arcs separated by spiraled gaps), and $|z| = r^\star$ is reached only tangentially.*

Idea of proof. $|F| = \rho^\alpha \in [2^{-A/2}, 1]$. On $s = 0$, $F(0, \alpha) = e^{i\alpha\pi/2}$ sweeps an argument of length $(A-1)\pi/2 \geq 2\pi$, covering the unit circle (nothing has modulus > 1). At fixed r, $\rho^\alpha = r$ gives $\alpha(s) = \ln r / \ln \rho(s)$; the total variation of $\alpha(s)\theta(s)$ reaches 2π at $r = r_{\text{fill}}$, above which all angles are attained (full circle) and below which they are not (gaps). The constant 4 is numerical, with no closed proof to date. □

Corollary 1 (a spiraled, non-constant boundary)**.** *The locus of closest-approach points, $r = 2^{-2\varphi/\pi}$, is a logarithmic spiral crossing every concentric circle at the constant angle $\beta = \arctan(2 \ln 2/\pi) \approx 23.8°$, inside the empirical $10°$–$40°$ range of spiral-galaxy arms; the resemblance is of form, not of mechanism — the object remains $z \mapsto z^\alpha$ on a segment.*

Remark (absence of caustic). The Jacobian $J = \alpha\,\rho^{2\alpha-1} g(s)$ with $g(s) = \rho'\theta - \rho\theta' \ln \rho < 0$ on $(0, 5)$ has constant sign: the fabric has no fold and no interior caustic. Its structure is algebraic arcs (α constant) crossed by logarithmic spirals (s constant), the bright edges being the images of the domain boundaries (unit circle at $s = 0$, an "almond" at $\alpha = 1$, a cusp as $s \to 5$). An earlier caustic conjecture is explicitly refuted by this computation.

8 A gallery of heavier patterns

The advantage grows with complexity. Each curve below is again a *single* closed expression. Reproducing any of them with Fourier series would require summing many harmonics (and would still ring at every corner), whereas here the corner structure is built in exactly.

9 Two-dimensional and polar patterns

Fed into parametric or polar equations, the same ramp produces shapes that would each demand a long Fourier development — one closed expression apiece.

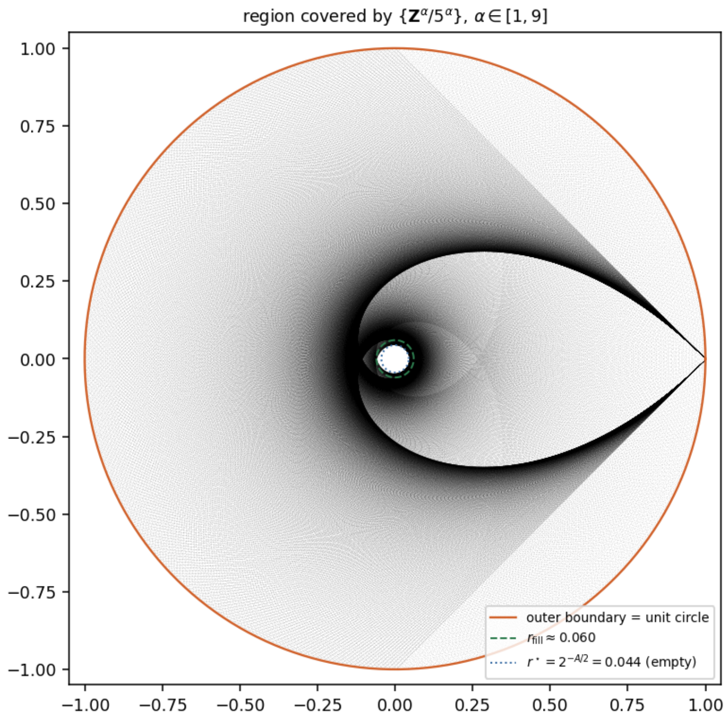

Figure 9: Region covered by $\{\mathbf{Z}^{\alpha}/5^{\alpha}\}$, $\alpha \in [1, 9]$: a full annulus bounded by the unit circle, an exactly empty core $|z| < r^{\star} = 2^{-A/2}$, and a lacunary spiraled collar below $r_{\rm fill}$.

10 Relation to Fourier analysis

The two tools address different geometries and are best seen as complementary. Fourier series excel on smooth-but-spectrally-rich signals; the pseudo-trigonometric notation excels on angular-but-structured patterns, where it is exact, finite, and manipulable by hand. The ramp admits the Fourier series

$$\operatorname{sp} x = \frac{p}{2} - \frac{p}{\pi} \sum_{n \geq 1} \frac{1}{n} \sin \frac{2\pi n x}{p}.$$

This converges in mean square and pointwise away from the jumps, but *not uniformly*: near each discontinuity the partial sums overshoot by the Wilbraham–Gibbs constant $\left(\frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt - 1\right) \approx 8.95\%$ of the jump, for *every* truncation order. The closed symbol $\operatorname{sp} x$ has neither the infinite length nor the corner overshoot of its Fourier series.

One structural boundary should be stated plainly: these functions are a tool of construction, not of spectral analysis. They carry no orthogonality, inner product, convolution, or transform theory; unlike the Fourier basis, they do not diagonalise translation. Their role is to write angular periodic shapes exactly, not to decompose arbitrary signals into them.

11 Conclusion

Pseudo-trigonometric functions are an elementary but genuinely useful notation. The underlying object — the scaled fractional part — is classical; the contribution is the gesture of treating it as a first-class primitive with a constant derivative, so that it enters differential equations, transport and Burgers equations, and curve equations directly. The complex part (the generator $\mathbf{Z}$ and the eye region) is elementary complex analysis: the image of a segment under $z \mapsto z^{\alpha}$, together with the observations that integer powers give named algebraic curves and that the normalized family draws a region with an exactly empty core bounded by the unit circle. Presented as such — a useful notation and a few verified facts of elementary algebraic geometry — the whole finds its

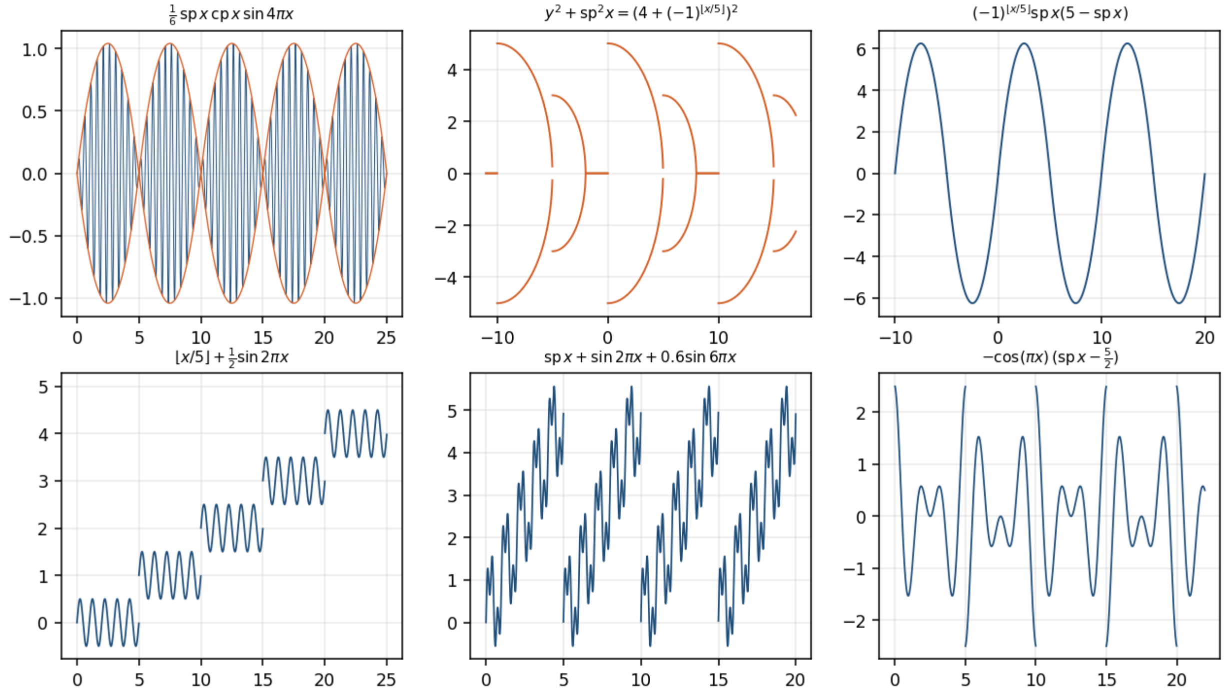

Figure 10: Wave-packet train inside a parabolic envelope; half-circles of alternating radius; alternating parabolic humps; a quantized oscillation on a staircase; a ramp woven with two ripples; an asymmetric shock train.

place in teaching, in fast plotting of angular periodic shapes, and as an expository object with a rigorous core.

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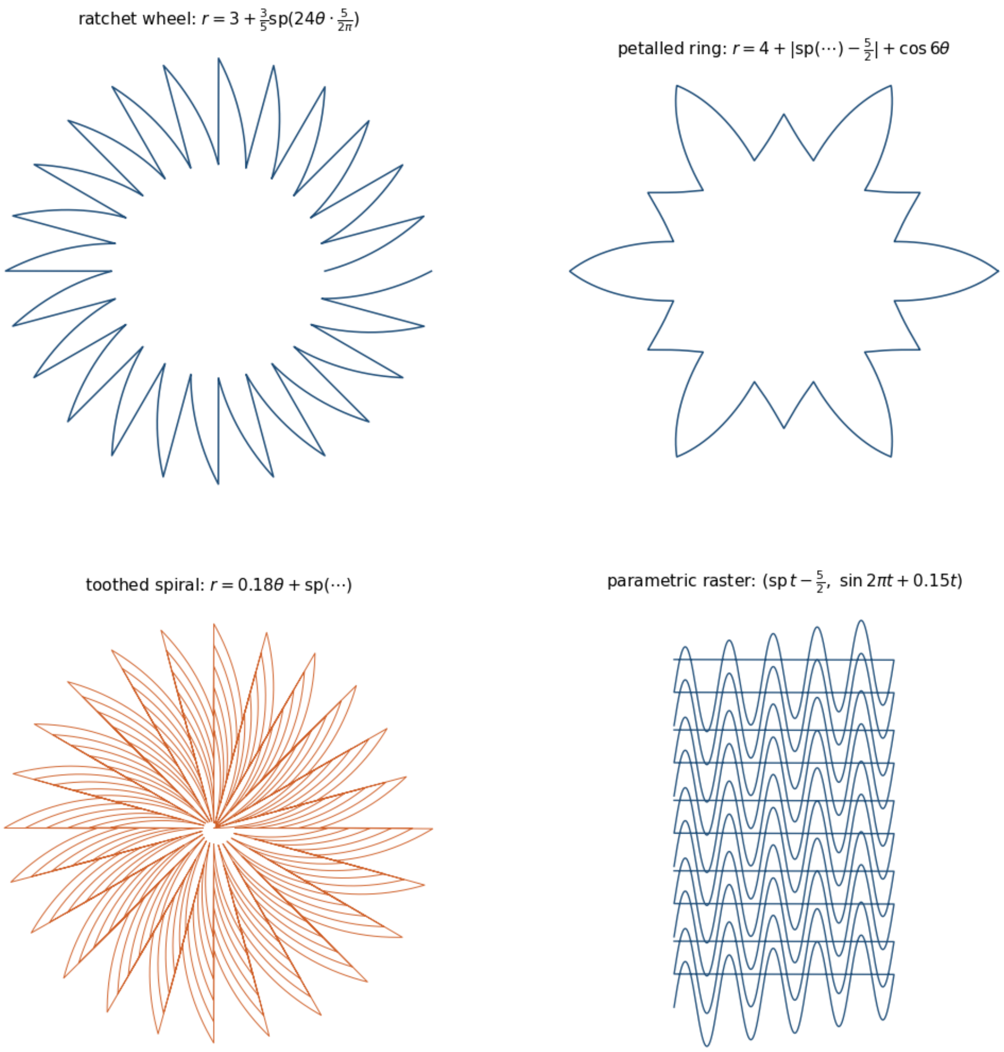

Figure 11: Ratchet wheel (24 teeth), petalled ring, toothed spiral, and a parametric raster weave.

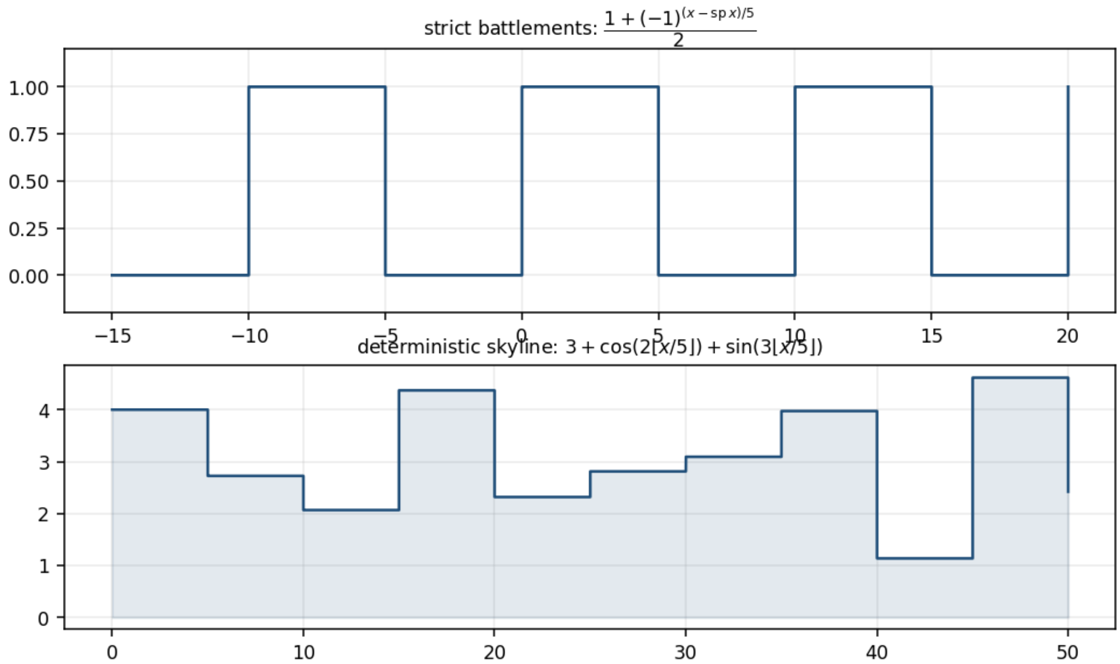

Figure 12: Strict battlements — an exact square wave $\frac{1}{2}\left(1+(-1)^{(x-\mathrm{sp}\,x)/5}\right)$ — and a deterministic skyline, each block a constant height set by $\lfloor x/5 \rfloor$.

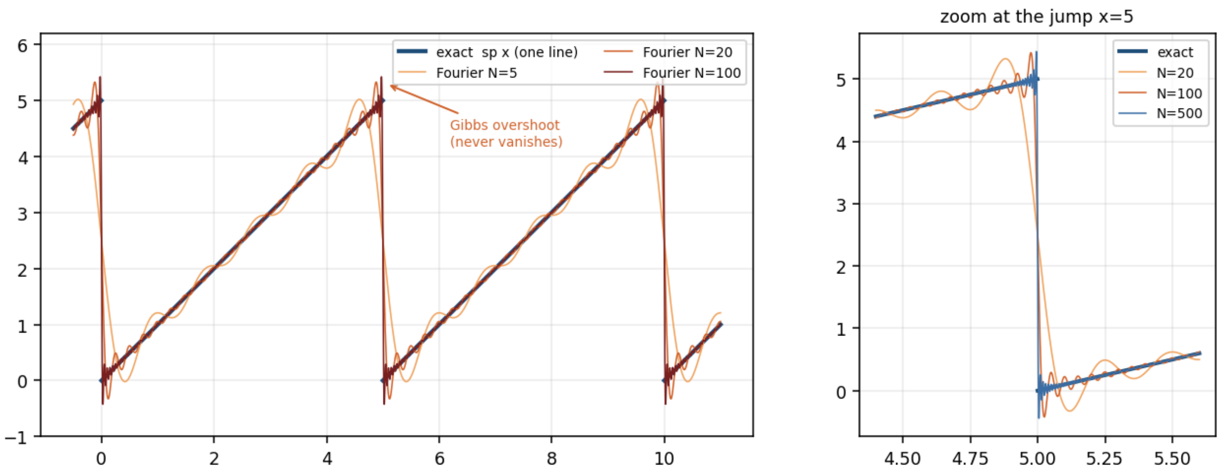

Figure 13: The exact ramp (thick) against Fourier truncations: the $\approx 9\%$ Gibbs overshoot persists as N grows; the zoom at $x=5$ shows it from $N=20$ to $N=500$.