

Unbounded Causal-Past Four-Volume and Asymptotic Volume Ticks in de Sitter and Λ CDM Spacetimes

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Abstract

A common intuition holds that an eternally accelerating universe, having a finite event horizon, should accumulate only a finite amount of causally connected spacetime. We show by direct integration that this is incorrect: in de Sitter space the causal-past four-volume of a comoving observer grows without bound, $V_4(t) \sim (4\pi c^4/3H_\Lambda^3) (t - \frac{11}{6}H_\Lambda^{-1})$ as $t \rightarrow \infty$, even though the past-light-cone radius saturates at c/H_Λ . Consequently, crossings of any fixed four-volume increment $\Delta V_4 > 0$ form an infinite sequence whose proper-time spacing converges to the finite constant $\Delta t_\infty = 3H_\Lambda^3 \Delta V_4 / (4\pi c^4)$. For a flat Λ CDM continuation with Planck-2018 baseline parameters and the normalization $\Delta V_4 = f V_4(T_0)$ with $f = 1/6$, the first future increment completes 0.580 Gyr from the present (cosmic age 14.375 Gyr) and the spacing decreases monotonically toward $\Delta t_\infty = 40.1996$ Myr. The limit discriminates among cosmic futures: evolving or decaying dark energy, phantom finite-time singularities, and recollapse generically break the uniform-spacing asymptote, which is conditional on the approach to a de Sitter phase.

1 Introduction

The event horizon of de Sitter space, and its thermodynamics, are classical results [1]. Because the horizon assigns a finite radius c/H_Λ to the region an observer can still influence or be influenced by, it is natural—but wrong—to conclude that the total four-dimensional volume of the observer’s causal past is likewise bounded. The purpose of this note is to make the correct statement sharp, elementary, and quantitative for the spacetimes relevant to our own future.

The object of study is the invariant four-volume of the past light cone of a comoving observer at cosmic time t . For flat FLRW,

$$V_4(t) = \frac{4\pi c}{3} \int_0^t a^3(t') \left[\int_{t'}^t \frac{c \, du}{a(u)} \right]^3 dt', \quad (1)$$

the quantity derived for FLRW geometries by Park and Woodard [2] and central to volume-based approaches to spacetime such as causal sets [3, 4] and unimodular gravity, where accumulated four-volume plays the role of time [5]. A companion paper [7] develops the normalized functional $\mathcal{C}(t) = V_4(t)/V_4(T_0)$ as a diagnostic of the observed expansion history; here we establish its

late-time behavior. Throughout, T_0 denotes the present cosmic age, and the asymptotic Hubble rate of a Λ -dominated future is

$$H_\Lambda = H_0 \sqrt{\Omega_\Lambda}. \quad (2)$$

The note contains three results. First, the exact closed form of the de Sitter causal-past four-volume, Eq. (7), and its linear asymptote with the de Sitter lag $\frac{11}{6} H_\Lambda^{-1}$, Eq. (8), which together correct the common finite-horizon/finite-history intuition. Second, a general proposition (Section 3): for any expansion history approaching a de Sitter phase, in the sense $a(t)e^{-H_\Lambda t} \rightarrow A \in (0, \infty)$ — a condition flat Λ CDM satisfies exactly — fixed four-volume increments define an infinite sequence of crossings whose proper-time spacing converges to the finite constant

$$\Delta t_\infty = \frac{3H_\Lambda^3 \Delta V_4}{4\pi c^4}. \quad (3)$$

Third, a quantitative conditional prediction: for the Planck-2018 baseline and the normalization $\Delta V_4 = V_4(T_0)/6$, the spacing converges to $\Delta t_\infty = 40.20 \pm 1.6$ Myr, a derived constant of the standard cosmological model's future whose value discriminates sharply among alternative fates of the expansion (Section 5). Appendices A–D give the term-by-term integration, the asymptotic analysis, the proof of the proposition, and the numerical-continuation and posterior-propagation procedures.

2 Exact causal-past four-volume in de Sitter space

Let $a(t) = \exp(H_\Lambda t)$. The physical radius of the past-light-cone slice at time $t' \leq t$ is

$$R(t', t) = e^{H_\Lambda t'} \int_{t'}^t c e^{-H_\Lambda u} du = \frac{c}{H_\Lambda} [1 - e^{-H_\Lambda(t-t')}], \quad (4)$$

so $R \rightarrow c/H_\Lambda$ as $t - t' \rightarrow \infty$: the horizon radius saturates. The four-volume, however, is the accumulation of these slices through unbounded proper time,

$$V_4(t) = c \int_0^t \frac{4\pi}{3} R^3(t', t) dt'. \quad (5)$$

Substituting Eq. (4) into Eq. (5),

$$V_4(t) = \frac{4\pi c^4}{3H_\Lambda^3} \int_0^t [1 - e^{-H_\Lambda s}]^3 ds, \quad (6)$$

which integrates in closed form:

$$V_4(t) = \frac{4\pi c^4}{3H_\Lambda^3} \left[t - \frac{3}{H_\Lambda} (1 - e^{-H_\Lambda t}) + \frac{3}{2H_\Lambda} (1 - e^{-2H_\Lambda t}) - \frac{1}{3H_\Lambda} (1 - e^{-3H_\Lambda t}) \right]. \quad (7)$$

The late-time behavior is linear with a fixed lag,

$$V_4(t) \sim \frac{4\pi c^4}{3H_\Lambda^3} \left(t - \frac{11}{6H_\Lambda} \right), \quad t \rightarrow \infty, \quad (8)$$

so the causal four-volume is unbounded even though the event-horizon radius is finite. The resolution of the apparent paradox is dimensional: the spatial cross-section saturates, but it continues to be swept through unlimited proper time.

3 Fixed four-volume increments have a finite asymptotic spacing

Differentiating Eq. (6),

$$\frac{dV_4}{dt} = \frac{4\pi c^4}{3H_\Lambda^3} [1 - e^{-H_\Lambda t}]^3 \longrightarrow \frac{4\pi c^4}{3H_\Lambda^3}. \quad (9)$$

For a general expansion history the same accumulation rate follows by differentiating Eq. (1) with respect to its endpoint:

$$V_4'(T) = \frac{4\pi c^2}{a(T)} \int_0^T a^3(t) \chi^2(t, T) dt, \quad \chi(t, T) = \int_t^T \frac{c du}{a(u)}. \quad (10)$$

Proposition 1. *Let $a(t)$ be an expansion history approaching a de Sitter phase in the sense*

$$a(t) e^{-H_\Lambda t} \longrightarrow A \in (0, \infty) \quad \text{for some } H_\Lambda > 0, \quad (11)$$

let $\Delta V_4 > 0$ be fixed, and let t_n be the successive solutions of $V_4(t_n) = V_4(t_0) + n \Delta V_4$. Then the sequence $\{t_n\}$ is infinite and

$$\boxed{\lim_{n \rightarrow \infty} (t_{n+1} - t_n) = \Delta t_\infty = \frac{3H_\Lambda^3 \Delta V_4}{4\pi c^4}.} \quad (12)$$

The proof — dominated convergence of the rate (10) to the de Sitter limit $4\pi c^4/3H_\Lambda^3$, followed by the mean-value theorem — is given in Appendix C.

Remark. Convergence of $H(t)$ to H_Λ alone does not imply hypothesis (11): the history $H = H_\Lambda + 1/(1+t)$ gives $a \propto (1+t) e^{H_\Lambda t}$, for which the prefactor diverges. A sufficient reformulation is $H - H_\Lambda \in L^1$. The proposition may well extend to such histories, but the proof uses the stronger hypothesis (11) — which flat Λ CDM satisfies exactly, since

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}H_\Lambda t\right) \implies A = \left(\frac{\Omega_m}{4\Omega_\Lambda}\right)^{1/3}. \quad (13)$$

For the increment normalized by the present causal four-volume, $\Delta V_4 = f V_4(T_0)$ with $0 < f \leq 1$, the limiting spacing takes the compact form

$$\Delta t_\infty(f) = \frac{3f H_\Lambda^3 V_4(T_0)}{4\pi c^4}, \quad (14)$$

which is linear in f and fully determined by the measured expansion history through the two numbers H_Λ and $V_4(T_0)$. Uncertainty propagation is immediate: Eq. (14) may be evaluated for every sample of a cosmological posterior chain, exactly as the fraction ages of Ref. [7] are propagated.

4 Numerical continuation of the baseline Λ CDM cosmology

We adopt the Planck-2018 base- Λ CDM parameters $H_0 = 67.36 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m = 0.3153$, $\Omega_r = 9.209 \times 10^{-5}$, $\Omega_\Lambda = 1 - \Omega_m - \Omega_r$ [8]. Direct quadrature of Eq. (1) (numerical methods as in Ref. [7]) gives

$$T_0 = 13.795181 \text{ Gyr}, \quad H_\Lambda = 0.0570002 \text{ Gyr}^{-1}, \quad V_4(T_0) = 5455.92 c^4 \text{ Gyr}^4. \quad (15)$$

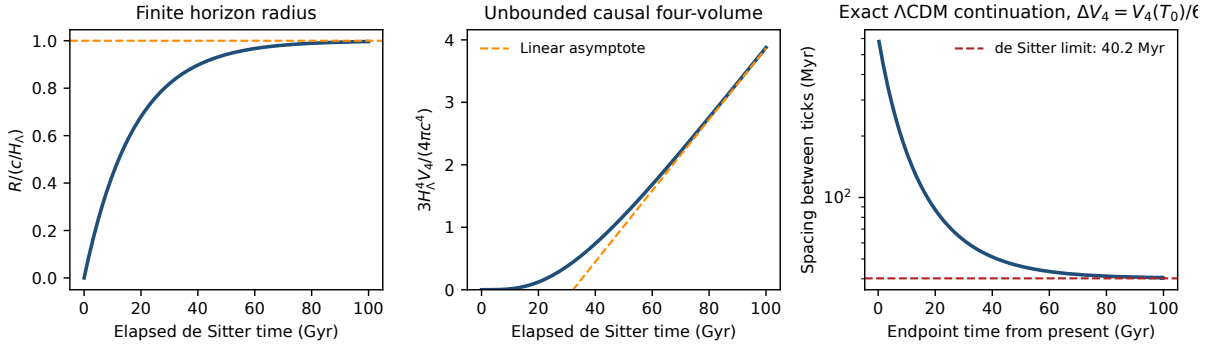


Figure 1 | Finite horizon, unbounded four-volume, and the asymptotic increment spacing. Left: the de Sitter past-light-cone radius approaches c/H_Λ , Eq. (4). Center: the corresponding causal four-volume, Eq. (7), becomes linear and remains unbounded, with the asymptote of Eq. (8) shown dashed. Right: exact successive $V_4(T_0)/6$ crossings in the continued Planck-baseline flat Λ CDM history converge toward the analytic 40.2 Myr limit; the curve uses finite differences between actual crossings, not a local-derivative approximation.

Equation (14) then yields, for $f = 1/6$,

$$\Delta t_\infty(1/6) = 40.1996 \text{ Myr} \simeq 40.2 \text{ Myr}. \quad (16)$$

Exact integration of the continued flat Λ CDM history places the first future crossing of $V_4 = \frac{7}{6}V_4(T_0)$ at cosmic age

$$t_1 = 14.375154 \text{ Gyr}, \quad (17)$$

i.e. 0.579974 Gyr from the present, with the subsequent spacings 524.7, 480.9, 445.3, 415.6, 390.5 Myr, decreasing monotonically toward the limit of Eq. (16). Figure 1 summarizes the three statements of this note: the horizon radius saturates; the four-volume becomes linear and unbounded; and the exact finite-difference spacings between successive $V_4(T_0)/6$ crossings converge to the analytic de Sitter limit, reaching 40.45 Myr by 100 Gyr from the present. The convergence is slow because the approach of Eq. (9) to its limit is itself controlled by $e^{-H_\Lambda t}$ with $H_\Lambda^{-1} \simeq 17.5 \text{ Gyr}$.

5 Discussion

Four features of the result deserve emphasis.

(i) *A finite horizon does not imply a finite causal history.* The saturating spatial cross-section is integrated through unlimited proper time, so $V_4(t) \rightarrow \infty$; the normalized functional $\mathcal{C}(t) = V_4(t)/V_4(T_0)$ of Ref. [7] exceeds unity in finite time and grows without bound, and an eternally de Sitter future contains infinitely many fixed-volume crossings.

(ii) *Volume-uniformity replaces time-uniformity.* Early increments depend in detail on radiation, matter, and the onset of acceleration—the first six future spacings above span 580 to 390 Myr—while late increments converge to the single constant of Eq. (14). Asymptotically, four-volume time and proper time become affinely related, which is the FLRW realization of the linear late-time relation between unimodular time and proper time [5].

(iii) *The limit classifies cosmic futures through $\dot{V}_4$.* Successive crossings obey $t_{n+1} - t_n = \Delta V_4 / V'_4(\xi_n)$ exactly, so the organizing quantity is the asymptotic behavior of the four-volume accumulation rate V'_4 of Eq. (10). Four regimes exhaust the possibilities. (a) $V'_4 \rightarrow 4\pi c^4 / 3H_\Lambda^3 > 0$ (approach to de Sitter): finite uniform spacing, Eq. (12); decaying dark energy with $H \rightarrow H'_\Lambda \neq H_\Lambda$ shifts the limit by the cube of the ratio. (b) $V'_4 \rightarrow \infty$: spacing tends to *zero*. This includes quintessence-like decelerating futures with $H \rightarrow 0$, since any power-law expansion gives $V_4 \propto t^4$ (the self-similar law of Ref. [7]) and hence $V'_4 \propto t^3$. (c) $V'_4 \rightarrow 0$, if realized: spacing grows without bound. (d) Finite total future four-volume — phantom histories reaching a finite-time singularity, or recollapse to a crunch: only finitely many crossings occur and the sequence terminates. Because causal pasts are nested along a future-directed worldline, $I^-(p_{t_1}) \subset I^-(p_{t_2})$ for $t_2 > t_1$, V_4 is monotone nondecreasing in every case: no history reverses the sequence. Quantitatively, for the de Sitter class, $\Delta t_\infty \propto H_\Lambda^3 V_4(T_0)$ implies the sensitivity

$$\frac{\partial \Delta t_\infty}{\partial H_0} = -0.59 \frac{\text{Myr}}{\text{km s}^{-1} \text{Mpc}^{-1}}, \quad \frac{\partial \Delta t_\infty}{\partial \Omega_m} = -211 \text{ Myr}, \quad (18)$$

so the Planck-2018 marginal uncertainties $(\sigma_{H_0}, \sigma_{\Omega_m}) = (0.54, 0.0073)$ [8] propagate to

$$\boxed{\Delta t_\infty(1/6) = 40.20 \pm 1.6 \text{ Myr}} \quad (19)$$

(added in quadrature, neglecting the H_0 – Ω_m correlation; a full chain propagation follows Appendix D). Equation (19) is the note’s falsifiable content: it is a derived constant of the concordance model’s future, pinned at the few-percent level by present data, and any measured departure of the asymptotic expansion rate from H_Λ displaces it by the cube of the ratio. The asymptotic spacing is therefore a compact scalar label for the class of the future expansion history.

(iv) *The spacing is a derived scale, not an observed cycle.* No known phenomenon is coupled to causal-volume boundaries, and Eq. (16) is presented strictly as a mathematical consequence of the measured expansion history and the stated normalization, in which the fraction f is a free bookkeeping parameter. Its value is conceptual and discriminatory: it corrects the finite-horizon/finite-volume intuition and makes the late-time behavior of causal four-volume under an eternal cosmological constant explicit and quantitative.

A related construction deserves explicit contrast. Causal *diamonds* — regions bounded both to the past and to the future — approach a finite four-volume in asymptotically de Sitter spacetimes [6]. There is no contradiction with the present result: a diamond anchored to a finite worldline segment saturates because its future cap closes at the horizon scale, whereas the causal past from the initial surface, considered here, keeps accumulating an entire horizon-scale spatial volume per unit proper time. The two regions answer different questions, and only the latter grows without bound.

(v) *The pair of papers tests the expansion history from both temporal directions.* The present four-volume $V_4(T_0)$ that fixes the increment here is the same quantity that normalizes the diagnostic clock of Ref. [7], whose past-facing fingerprints ($\rho_{1/6}$ and the w_0 -blind fraction f_*) discriminate Λ CDM from evolving dark energy in current posterior chains. One frozen formalism therefore yields both a diagnostic of the measured past and a derived constant of the conditional future, Eq. (19); any resolution of the present dynamical-dark-energy debate propagates through the shared normalization into a definite revision of the asymptotic beat, so the two papers can be confirmed or refuted together as DESI DR3, Euclid, and standard-siren posteriors arrive.

A Term-by-term integration of the de Sitter four-volume

Expanding the cube in Eq. (6),

$$[1 - e^{-H_\Lambda s}]^3 = 1 - 3e^{-H_\Lambda s} + 3e^{-2H_\Lambda s} - e^{-3H_\Lambda s}, \quad (20)$$

and integrating each term over $s \in [0, t]$,

$$\int_0^t ds = t, \quad (21)$$

$$\int_0^t e^{-kH_\Lambda s} ds = \frac{1 - e^{-kH_\Lambda t}}{kH_\Lambda}, \quad k = 1, 2, 3, \quad (22)$$

gives directly

$$V_4(t) = \frac{4\pi c^4}{3H_\Lambda^3} \left[t - \frac{3}{H_\Lambda} (1 - e^{-H_\Lambda t}) + \frac{3}{2H_\Lambda} (1 - e^{-2H_\Lambda t}) - \frac{1}{3H_\Lambda} (1 - e^{-3H_\Lambda t}) \right], \quad (23)$$

which is Eq. (7) of the main text.

B Asymptotic lag and rate of convergence

As $t \rightarrow \infty$ the exponentials in Eq. (23) vanish and the bracket approaches

$$t - \frac{3}{H_\Lambda} + \frac{3}{2H_\Lambda} - \frac{1}{3H_\Lambda} = t - \frac{18 - 9 + 2}{6H_\Lambda} = t - \frac{11}{6H_\Lambda}, \quad (24)$$

establishing the de Sitter lag of Eq. (8). The intercept is specific to the pure de Sitter solution integrated from $t = 0$; asymptotically de Sitter histories share the *slope* of the asymptote — and hence the increment spacing of Eq. (12) — but each carries its own solution-dependent lag. The residual is

$$V_4(t) - \frac{4\pi c^4}{3H_\Lambda^3} \left(t - \frac{11}{6H_\Lambda} \right) = \frac{4\pi c^4}{3H_\Lambda^4} \left[3e^{-H_\Lambda t} - \frac{3}{2}e^{-2H_\Lambda t} + \frac{1}{3}e^{-3H_\Lambda t} \right] = O(e^{-H_\Lambda t}), \quad (25)$$

so the approach to the linear asymptote—and hence the convergence of the increment spacing to Δt_∞ —is exponential with the single rate H_Λ . For the baseline value $H_\Lambda^{-1} = 17.54$ Gyr, this is the origin of the slow convergence visible in Fig. 1: the spacing reaches 40.45 Myr only 100 Gyr from the present.

C Proof of Proposition 1

Proof. Write $a(t) = A e^{H_\Lambda t} [1 + \epsilon(t)]$ with $\epsilon(t) \rightarrow 0$ by hypothesis (11), and substitute $s = T - t$ in the rate formula (10). For every fixed s the integrand converges pointwise, as $T \rightarrow \infty$, to the pure de Sitter integrand

$$\frac{c^2}{H_\Lambda^2} e^{-H_\Lambda s} [1 - e^{-H_\Lambda s}]^2. \quad (26)$$

Choosing t_* with $|\epsilon| \leq \frac{1}{2}$ for $t \geq t_*$, the integrand is dominated for $s \leq T - t_*$ by a constant multiple of $e^{-H_\Lambda s}$, while the early-time segment $t \leq t_*$ contributes at most $\mathcal{O}(e^{-H_\Lambda T})$, because $a^3 \chi^2$ is bounded there and the prefactor is $1/a(T)$. Dominated convergence therefore yields

$$V_4'(T) \longrightarrow \frac{4\pi c^4}{H_\Lambda^3} \int_0^\infty e^{-x} [1 - e^{-x}]^2 dx = \frac{4\pi c^4}{3H_\Lambda^3} > 0, \quad (27)$$

in agreement with Eq. (9), so $V_4(t) \rightarrow \infty$ and every level $V_4(t_0) + n \Delta V_4$ is attained: the sequence $\{t_n\}$ is infinite.

By the mean-value theorem there is $\xi_n \in (t_n, t_{n+1})$ with

$$t_{n+1} - t_n = \frac{\Delta V_4}{V_4'(\xi_n)}; \quad (28)$$

since $\xi_n \rightarrow \infty$ and $V_4'(\xi_n) \rightarrow 4\pi c^4/(3H_\Lambda^3)$ by Eq. (27), Eq. (12) follows. $\square$

D Numerical continuation and posterior propagation

The exact Λ CDM continuation of Section 4 evaluates Eq. (1) on a quadratic scale-factor grid $a_j = a_{\max} [j/(N-1)]^2$ extended to $a_{\max} = 400$ ($N = 600,001$ nodes), using the moment decomposition of Ref. [7]: with $\eta(a)$ the conformal time and

$$M_k(a) = \int_0^a a'^3 \eta^k(a') \frac{da'}{a' H(a')}, \quad k = 0, 1, 2, 3, \quad (29)$$

the four-volume on the whole grid is

$$V_4(a) \propto \eta^3(a) M_0(a) - 3\eta^2(a) M_1(a) + 3\eta(a) M_2(a) - M_3(a). \quad (30)$$

Crossing times t_n of the levels $V_4(T_0)(1 + n/6)$ are obtained by monotone interpolation of V_4 against the cumulative age integral; the spacings quoted in Section 4 are finite differences $t_{n+1} - t_n$ of actual crossings, not local-derivative approximations.

Parameter uncertainty is propagated by evaluating

$$\Delta t_\infty = \frac{3H_\Lambda^3 \Delta V_4}{4\pi c^4}, \quad H_\Lambda = H_0 \sqrt{1 - \Omega_m - \Omega_r}, \quad \Delta V_4 = f V_4(T_0), \quad (31)$$

for every sample of a cosmological posterior chain, exactly as the causal-fraction ages are propagated in Ref. [7]; the quadrature estimate of Eq. (19) is the linearized special case of this procedure.

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