A τ-Vanishing Identity in the Collatz Problem: A Duality Between Base-6 Geometry and Base-2 Algebra

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Abstract

We establish an exact algebraic identity connecting two quantities that have previously been studied separately in the Collatz $(3n+1)$ problem: the cumulative near-conjugacy error E_σ of a base-6 circle-rotation model of the Collatz map (Honarvar Shakibaei Asli [3]), and the terminal correction ε in the classical stopping-time identity. For a convergent Syracuse orbit, the identity is

$$E_\sigma \;=\; 1 - \log_6(5n_0+1) + \bigl(\log_2 n_0 + \varepsilon\bigr)\log_6 2,$$

in which the Syracuse stopping time τ cancels exactly, via the change-of-base formula $\log_2 6 \cdot \log_6 2 = 1$. Consequently, for every convergent Syracuse orbit,

$$E_\sigma = O(1) \iff \varepsilon = O(1) \iff \sum_{i=1}^{\tau} \frac{1}{n_{i-1}} = O(1),$$

so the boundedness of the reciprocal sum becomes the central open question. The identity is verified to machine precision ($\leq 2 \times 10^{-15}$) for every tested $n_0 \leq 77031$.

1 Introduction

The Collatz conjecture asserts that every positive integer n_0 reaches 1 under iteration of the map $C(n) = n/2$ (if n is even) or $C(n) = 3n+1$ (if n is odd). For odd $n_0 \geq 3$ it suffices to study the *Syracuse map*

$$\mathrm{S}(n) = \frac{3n+1}{2^{\nu_2(3n+1)}},$$

where $\nu_2(m) = \max\{k : 2^k \mid m\}$ is the 2-adic valuation. The Syracuse orbit $n_0, n_1, \ldots, n_\tau = 1$ (with $n_i = \mathrm{S}^i(n_0)$) has been studied extensively [1, 2].

Two algebraic frameworks have been developed independently.

1. **Stopping-time identity (Lagarias [1]).** The total Collatz stopping time σ (counting both even and odd steps) satisfies

$$\sigma = \tau \log_2 6 + \log_2 n_0 + \varepsilon, \tag{1}$$

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where

$$\varepsilon = \sum_{i=1}^{\tau} \log_2\left(1 + \frac{1}{3n_{i-1}}\right)$$

is the *terminal correction*.

2. **Near-conjugacy (Honarvar Shakibaei Asli [3]).** The map $\Theta(n) = \{\log_6(n + 1/5)\}$ (fractional part) satisfies

$$\Theta(C(n)) = \Theta(n) + \alpha_n + \varepsilon_{\mathrm{nc}}(n) \pmod 1,$$

where C is the Collatz map, $\alpha = \log_6 3$, $\alpha_n = \alpha$ (if n is odd) or $\alpha - 1$ (if n is even), and $\varepsilon_{\mathrm{nc}}(n) = \log_6(1 + 1/(5n+1))$. The *cumulative near-conjugacy error* is $E_k = \sum_{i=0}^{k-1} \varepsilon_{\mathrm{nc}}(m_i)$, where $m_0, m_1, \ldots$ is the Collatz orbit.

These frameworks operate in different bases (base-6 geometry vs. base-2 algebra) and have not previously been connected. In this note we derive an exact identity linking E_σ and ε, in which τ vanishes entirely.

2 Preliminaries

Definition 2.1. *For odd $n_0 \geq 3$, the Syracuse orbit is $n_0, n_1, \ldots, n_\tau$ with $n_i = \mathrm{S}(n_{i-1})$ and $n_\tau = 1$. Define:*

- *τ = Syracuse stopping time (number of Syracuse steps);*
- *σ = total Collatz stopping time, $\sigma = \tau + a$ where $a = \sum_{i=1}^{\tau} \nu_i$ and $\nu_i = \nu_2(3n_{i-1} + 1)$;*
- *$\varepsilon = \sum_{i=1}^{\tau} \log_2(1 + 1/(3n_{i-1}))$ (terminal correction);*
- *$E_k = \sum_{i=0}^{k-1} \log_6(1 + 1/(5m_i + 1))$ (cumulative near-conjugacy error), where $m_0, m_1, \ldots$ is the Collatz orbit ($m_{i+1} = C(m_i)$).*

Lemma 2.2 (Stopping-time identity)**.** *For odd $n_0 \geq 3$ with $n_\tau = 1$,*

$$\sigma = \tau \log_2 6 + \log_2 n_0 + \varepsilon.$$

Proof. From $n_i = (3n_{i-1} + 1)/2^{\nu_i}$,

$$\log_2 n_i = \log_2(3n_{i-1} + 1) - \nu_i = \log_2 3 + \log_2\bigl(n_{i-1} + \tfrac{1}{3}\bigr) - \nu_i.$$

Writing $\log_2(n_{i-1} + 1/3) = \log_2 n_{i-1} + \log_2(1 + 1/(3n_{i-1}))$ and telescoping over $i = 1, \ldots, \tau$,

$$0 = \log_2 n_0 + \tau \log_2 3 - a + \varepsilon.$$

With $a = \sigma - \tau$ this gives $0 = \log_2 n_0 + \tau \log_2 6 - \sigma + \varepsilon$. □

3 The Telescoping Identity

Proposition 3.1 (Closed form for $\varepsilon_{\mathrm{nc}}$)**.** *For all $n \geq 1$,*

$$\varepsilon_{\mathrm{nc}}(n) = \log_6(5n+2) - \log_6(5n+1).$$

Proof. Set $g(n) := \log_6(5n+1)$. We verify the telescoping relation $\varepsilon_{\mathrm{nc}}(n) = g(C(n)) - g(n) - \alpha_n$, where C is the Collatz map and $\alpha_n = \alpha$ (if n is odd) or $\alpha - 1$ (if n is even).

Odd n: $C(n) = 3n+1$, so

$$g(C(n)) - g(n) = \log_6 \frac{5(3n+1)+1}{5n+1} = \log_6 \frac{3(5n+2)}{5n+1} = \log_6 3 + \log_6 \frac{5n+2}{5n+1} = \alpha + \varepsilon_{\mathrm{nc}}(n).$$

Even n: $C(n) = n/2$, so

$$g(C(n)) - g(n) = \log_6 \frac{5n/2+1}{5n+1} = \log_6 \frac{5n+2}{2(5n+1)} = -\log_6 2 + \varepsilon_{\mathrm{nc}}(n) = (\alpha - 1) + \varepsilon_{\mathrm{nc}}(n),$$

using $\alpha = \log_6 3 = 1 - \log_6 2$. □

Proposition 3.2 (Telescoping)**.** *For the Collatz orbit $m_0 = n_0, m_1, \ldots, m_k$,*

$$E_k = \log_6(5m_k+1) - \log_6(5m_0+1) + a_k - k\alpha,$$

where a_k is the number of even steps among the first k Collatz steps.

Proof. Set $g(n) = \log_6(5n+1)$. By Proposition 3.1, $\varepsilon_{\mathrm{nc}}(m_i) = g(m_{i+1}) - g(m_i) - \alpha_i$, where $\alpha_i = \alpha - 1$ (even step) or α (odd step). Summing over $i = 0, \ldots, k-1$,

$$E_k = g(m_k) - g(m_0) - \sum_{i=0}^{k-1} \alpha_i.$$

With a_k even steps, $\sum_{i=0}^{k-1} \alpha_i = a_k(\alpha - 1) + (k - a_k)\alpha = k\alpha - a_k$. Hence $E_k = g(m_k) - g(m_0) + a_k - k\alpha$. □

4 The τ-Vanishing Identity

Theorem 4.1 (τ-Vanishing Identity)**.** *For odd $n_0 \geq 3$ reaching 1 in σ total Collatz steps,*

$$\boxed{E_\sigma = 1 - \log_6(5n_0+1) + (\log_2 n_0 + \varepsilon) \cdot \log_6 2.}$$

In particular, E_σ depends on n_0 and ε but is independent of τ.

Proof. Apply Proposition 3.2 with $k = \sigma$ (so $m_\sigma = 1$ and $a_\sigma = a$):

$$\begin{aligned} E_\sigma &= \log_6 6 - \log_6(5n_0+1) + a - \sigma\alpha \\ &= 1 - \log_6(5n_0+1) + a - \sigma\alpha. \end{aligned}$$

Since $a = \sigma - \tau$ and $\alpha = 1 - \log_6 2$,

$$a - \sigma\alpha = (\sigma - \tau) - \sigma(1 - \log_6 2) = \sigma \log_6 2 - \tau.$$

By Lemma 2.2, $\sigma = \tau \log_2 6 + \log_2 n_0 + \varepsilon$, so

$$\sigma \log_6 2 = (\tau \log_2 6 + \log_2 n_0 + \varepsilon) \log_6 2 = \tau \cdot \underbrace{(\log_2 6 \cdot \log_6 2)}_{=1} + (\log_2 n_0 + \varepsilon) \log_6 2.$$

Substituting,

$$E_\sigma = 1 - \log_6(5n_0 + 1) + \tau + (\log_2 n_0 + \varepsilon) \log_6 2 - \tau,$$

and the τ terms cancel. ☐

Corollary 4.2 (Asymptotic form). *As $n_0 \to \infty$,*

$$E_\sigma = (1 - \log_6 5) + \varepsilon \cdot \log_6 2 + O(1/n_0).$$

Proof. Since $\log_2 n_0 \cdot \log_6 2 = \log_6 n_0$ and $\log_6(5n_0+1) = \log_6 n_0 + \log_6 5 + O(1/n_0)$, the n_0-dependent terms reduce to the constant $1 - \log_6 5$ up to $O(1/n_0)$. ☐

Corollary 4.3 (Conditional equivalence). *For a convergent Syracuse orbit (i.e., $n_\tau = 1$),*

$$E_\sigma = O(1) \iff \varepsilon = O(1) \iff \sum_{i=1}^{\tau} \frac{1}{n_{i-1}} = O(1).$$

Proof. By Theorem 4.1, $E_\sigma = c_1(n_0) + \varepsilon \log_6 2$, where

$$c_1(n_0) = 1 - \log_6(5n_0 + 1) + \log_2 n_0 \cdot \log_6 2 = 1 - \log_6 5 + O(1/n_0) = O(1).$$

Since $\log_6 2 > 0$ is a constant, $E_\sigma = O(1) \iff \varepsilon = O(1)$.

For the second equivalence, write $x_i = 1/(3n_{i-1})$. Since $n_{i-1} \geq 3$, we have $0 < x_i \leq 1/9$, and $\log_2(1 + x) = x/\ln 2 + O(x^2)$ uniformly on $[0, 1/9]$. Hence

$$\varepsilon = \sum_{i=1}^{\tau} \log_2(1 + x_i) = \frac{1}{3 \ln 2} \sum_{i=1}^{\tau} \frac{1}{n_{i-1}} + O\left(\sum_{i=1}^{\tau} \frac{1}{n_{i-1}^2} \right).$$

The values $n_0, n_1, \ldots, n_{\tau-1}$ are distinct positive odd integers, so $\sum_{i=1}^{\tau} 1/n_{i-1}^2 \leq \sum_{k=1}^{\infty} (2k-1)^{-2} = O(1)$. Thus $\varepsilon = \frac{1}{3 \ln 2} \sum_{i=1}^{\tau} 1/n_{i-1} + O(1)$, and $\varepsilon = O(1) \iff \sum_{i=1}^{\tau} 1/n_{i-1} = O(1)$. ☐

Remark 4.4 (Relation to the Collatz conjecture). *Corollary 4.3 shows that, for orbits that already converge to 1, the three quantities E_σ, ε, and $\sum_i 1/n_{i-1}$ are simultaneously bounded or unbounded. This reframes the Collatz problem—which asks whether all orbits converge—as the question of whether $\sum_i 1/n_{i-1}$ is uniformly bounded over all convergent Syracuse orbits. We emphasize that this is a reframing, not a full equivalence: the forward implication ("if all orbits converge, then $\sum 1/n_i = O(1)$") and the backward implication both require additional argumentation. For a convergent orbit, the elementary bound*

$$\sum_{i=1}^{\tau} \frac{1}{n_{i-1}} \leq \frac{1}{3} \ln \tau + O(1)$$

holds unconditionally, because $n_1, \ldots, n_{\tau-1}$ are distinct odd integers not divisible by 3 (see the companion paper [4]). In particular, no bound of the form $O(\log \log n_0)$ is currently established for all convergent orbits.

n_0	σ	E_σ (direct sum)	E_σ (formula)	Difference
7	16	0.1896449158	0.1896449158	1.1×10^{-16}
27	111	0.1988567067	0.1988567067	1.1×10^{-15}
97	118	0.1960424826	0.1960424826	5.0×10^{-16}
871	178	0.1832180215	0.1832180215	5.0×10^{-16}
6171	261	0.1993448650	0.1993448650	1.8×10^{-15}
77031	350	0.2203964402	0.2203964402	6.4×10^{-16}

Table 1: Verification of Theorem 4.1. Differences are at the level of floating-point roundoff.

5 Numerical Verification

The identity of Theorem 4.1 was verified by computing E_σ both directly (summing $\varepsilon_{\mathrm{nc}}(m_i)$ over the Collatz orbit) and via the closed form, in IEEE 754 double precision.

6 Discussion

The vanishing of τ in Theorem 4.1 follows from the duality between base-6 geometry (the near-conjugacy framework) and base-2 algebra (the stopping-time identity). The change-of-base formula $\log_2 6 \cdot \log_6 2 = 1$, trivial in isolation, is exactly what eliminates τ—the most difficult quantity to control in the Collatz problem. We note that Theorem 4.1 is a purely algebraic identity and does not rely on any boundedness claim in [3].

Corollary 4.3 and Remark 4.4 identify the uniform boundedness of $\sum_i 1/n_{i-1}$ over convergent Syracuse orbits as the key remaining question. Recent work of Tao [2] establishes that almost all orbits (in logarithmic density) attain almost bounded values; whether the identity above helps close the gap between “almost all” and “all” remains open.

References

[1] J. C. Lagarias, *The $3x + 1$ problem and its generalizations*, Amer. Math. Monthly **92** (1985), 3–23.

[2] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, Forum Math. Pi **10** (2022), e12.

[3] B. Honarvar Shakibaei Asli, *An explicit near-conjugacy between the Collatz map and a circle rotation*, arXiv:2601.04289, 2026.

[4] Companion paper: *Structural constraints on Syracuse orbits: alternation, bit consumption, and the critical boundary* (in preparation).