A Spectral Coherence Threshold for Prime Egyptian Fractions, with an Exact Search for the Four-Term Equation

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Abstract

We study the Diophantine equation

$$\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}, \qquad p_1, p_2, p_3, p_4 \text{ distinct primes},$$

a special case of the Sierpiński–Schinzel family of unit-fraction equations. We introduce a spectral coherence threshold R, defined from a normalized Dirichlet-type exponential sum over the primes involved, and prove that $R = 1/2$ is its exact transition value, obtained via the Kronecker–Weyl equidistribution theorem applied to the $\mathbb{Q}$-linear independence of logarithms of distinct primes. For $N = 3$ terms we derive an exact closed-form identity linking this threshold to the gaps between consecutive primes, yielding a short and independent proof that no three consecutive primes satisfy $1/p_1 = 1/p_2 + 1/p_3$. For $N = 4$ terms, we combine the spectral classification with an exact, exhaustive divisor-based algorithm for the two-term equation $1/x + 1/y = a/b$, and report a systematic computational search covering more than 6×10^8 configurations with $p_1 \leq 5431$. No exact solution is found; we document the closest near-solution, in which the three larger denominators form an arithmetic progression, and we prove that an exact solution in arithmetic progression is structurally impossible. Finally, we record an elementary divisibility argument showing that no solution exists in distinct primes for any number of terms $k \geq 2$; we state this generalization explicitly to clarify its scope, since it applies uniformly to denominators restricted to single primes and does not address the harder case of composite denominators that is the substance of the Sierpiński–Schinzel conjectures.

Contents

1 Introduction

Egyptian fractions – representations of a rational number as a sum of distinct unit fractions – form a classical topic in number theory with long-standing open questions when the denominators are restricted to primes. The Sierpiński and Schinzel conjectures [1, 2] concern the solvability of

$$\frac{n}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$$

for a fixed integer $n > 1$ and prime denominators. The closely related case $n = 1$,

$$\frac{1}{p_1} = \frac{1}{p_2} + \cdots + \frac{1}{p_k},$$

is classical for $k = 2$ (no solution in distinct primes, by an elementary divisibility argument, Section 7) and has the well-known composite-denominator identity $1/2 = 1/3 + 1/6$ for $k = 3$. The case $k = 4$ with all four denominators prime and distinct is the main object of this paper.

Our approach combines three tools, kept explicitly separate throughout.

1. An *analytic classification criterion* (Section 2): a normalized exponential sum over the primes involved has an exact infimum formula, vanishing exactly at a sharp threshold $R = 1/2$, which coincides precisely with the unit-fraction equality condition.

2. An *exact computational method* (Sections 3–5): a divisor-based algorithm solving the two-term equation $1/x + 1/y = a/b$ completely, used to conduct an exhaustive search for near-solutions of the four-term equation, together with a structural obstruction ruling out solutions in arithmetic progression.

3. An *elementary divisibility observation* (Section 7): a short argument, valid for any $k \geq 2$, showing that $1/p_1$ can never equal a sum of $k - 1$ unit fractions with distinct prime denominators. We state this generalization explicitly, since restricting it to $k = 4$ alone would understate – and restricting it to prime denominators throughout would overstate – what it actually shows.

We keep these three results clearly separated, since conflating a proved general theorem, a classical search technique, and an elementary but general divisibility fact would overstate what each establishes on its own.

1.1 Summary of results

- **Theorem 2.1.** An exact closed-form formula for $\inf_{t\in\mathbb{R}} \rho_N(\sigma, t)$, identifying $R = 1/2$ as the sharp coherence threshold.

- **Theorem 3.1.** For $N = 3$, an exact algebraic identity linking the spectral threshold to the gaps between three primes in arithmetic position, yielding a new proof that no three consecutive primes satisfy $1/p_1 = 1/p_2 + 1/p_3$.

- **Proposition 4.1.** An exact, exhaustive divisor-based solution method for $1/x + 1/y = a/b$.
- **Theorem 6.1.** No solution of the four-term equation exists with p_2, p_3, p_4 in arithmetic progression.
- **Section 5.** A systematic computational search covering more than 6×10^8 configurations, reporting the closest known near-solution.
- **Theorem 7.1.** For every $k \geq 2$, the equation $1/p_1 = 1/p_2 + \cdots + 1/p_k$ has no solution in distinct primes.

2 The spectral coherence threshold

Let $p_1 < p_2 < \cdots < p_N$ be distinct primes. For $\sigma > 0$ and $t \in \mathbb{R}$, define the normalized exponential sum

$$\rho_N(\sigma, t) \;=\; \frac{\left|\sum_{k=1}^{N} p_k^{-\sigma} e^{2it \ln p_k}\right|}{\sum_{k=1}^{N} p_k^{-\sigma}}.$$

Lemma 2.1 (Exact infimum formula)**.** *Let $D(\sigma) = \sum_{k=1}^{N} p_k^{-\sigma}$ and $R = p_1^{-\sigma}/D(\sigma)$. Then*

$$\inf_{t \in \mathbb{R}} \rho_N(\sigma, t) \;=\; \max\bigl(0,\, 2R - 1\bigr).$$

Proof. By unique factorization, the numbers $\ln(p_k/p_1)$ for $k = 2, \ldots, N$ are linearly independent over $\mathbb{Q}$. By the Kronecker–Weyl equidistribution theorem, the vector of phases

$$\bigl(2t \ln(p_2/p_1), \ldots, 2t \ln(p_N/p_1)\bigr) \pmod{2\pi}$$

is dense in the torus $[0, 2\pi)^{N-1}$ as t ranges over $\mathbb{R}$.

Write $a_k = (p_1/p_k)^{\sigma}$ for $k \geq 2$ and $S = \sum_{k=2}^{N} a_k$. Factor out $p_1^{-\sigma}$:

$$\sum_{k=1}^{N} p_k^{-\sigma} e^{2it \ln p_k} = p_1^{-\sigma}\Bigl(1 + \sum_{k=2}^{N} a_k e^{i\theta_k}\Bigr), \qquad \theta_k = 2t \ln(p_k/p_1).$$

By density of the phase vector, the bracket attains values arbitrarily close to $1 - S$ (all $\theta_k = \pi$ simultaneously) as its infimum in modulus if $S < 1$, and can be made arbitrarily close to 0 if $S \geq 1$. Dividing by $D(\sigma) = p_1^{-\sigma}(1 + S)$:

$$\inf_{t} \rho_N(\sigma, t) = \begin{cases} \dfrac{1 - S}{1 + S}, & S < 1, \\ 0, & S \geq 1. \end{cases}$$

Since $R = 1/(1 + S)$, we have $(1 - S)/(1 + S) = 2R - 1$ whenever $S < 1$ (equivalently $R > 1/2$), and the two cases combine into $\max(0, 2R - 1)$. □

Proposition 2.2 (Sharpness of the threshold)**.** *The infimum in Lemma 2.1 vanishes identically for $R \leq 1/2$ and is strictly positive for $R > 1/2$. The value $R = 1/2$ is thus the exact transition point of the coherence infimum, read off directly from the formula rather than fixed as an external convention.*

Remark 2.3. For $N = 3$ and the boundary substitution $(p_1, p_2, p_3) = (2, 3, 6)$ – formally allowing the composite denominator 6 to test the transition against the classical identity $1/2 = 1/3 + 1/6$ – one finds $R = 1/2$ exactly, consistent with Theorem 2.1 predicting a vanishing infimum precisely at equality.

2.1 Classification of prime tuples by proximity to $R = 1/2$

For any finite family of prime tuples of fixed length N, each tuple can be ranked by $|R-1/2|$: tuples with R closest to $1/2$ are closest to the coherence transition identified by Theorem 2.1. Table 1 lists the four consecutive prime triplets with $p_3 \leq 10000$ closest to this threshold, among 1227 triplets examined.

Triplet (p_1, p_2, p_3)	R	$\lvert R - 1/2\rvert$
$(3, 5, 7)$	0.49296	0.00704
$(2, 3, 5)$	0.48387	0.01613
$(5, 7, 11)$	0.46108	0.03892
$(7, 11, 13)$	0.45981	0.04019

Table 1: Consecutive prime triplets closest to the coherence transition $R = 1/2$.

3 An exact spectral–gap identity for three primes

We now specialize to $N = 3$ and relate the threshold R directly to the gaps between three primes in arithmetic position.

Let $p_1 = x$, $p_2 = x + a$, $p_3 = x + a + b$ with $a, b > 0$. Define

$$R = \frac{1/x}{1/x + 1/(x + a) + 1/(x + a + b)}.$$

Theorem 3.1 (Exact spectral–gap identity)**.** *With the notation above,*

$$R - \frac{1}{2} \;=\; \frac{a(a + b) - x^2}{2\big(3x^2 + 2(2a + b)x + a(a + b)\big)}. \tag{$*$}$$

In particular, $R = 1/2$ if and only if $x^2 = a(a + b)$.

Proof. Clearing denominators, the numerator of R is $N = (x+a)(x+a+b)$ and the total denominator is

$$D = (x + a)(x + a + b) + x(x + a + b) + x(x + a) = 3x^2 + 2(2a + b)x + a(a + b).$$

Then

$$R - \frac{1}{2} = \frac{2N - D}{2D} = \frac{2a(a + b) - \big(x^2 + a(a + b)\big)}{2D} = \frac{a(a + b) - x^2}{2D},$$

which is $(*)$. □

Remark 3.2. For $(p_1, p_2, p_3) = (3, 5, 7)$ we have $a = 2$, $b = 2$, $x = 3$, so $a(a + b) - x^2 = 8 - 9 = -1$, the smallest possible nonzero value; this explains why this triplet attains the closest proximity to $R = 1/2$ in Table 1.

Theorem 3.3. *Let $p_1 < p_2 < p_3$ be consecutive primes. Then $1/p_1 \neq 1/p_2 + 1/p_3$.*

Proof. By Theorem 3.1, equality would force $x^2 = a(a + b)$ with $x = p_1$. Since x is prime, the positive divisors of x^2 are exactly $1, x, x^2$, and $a \mid x^2$, so $a \in \{1, x, x^2\}$.

Case x odd. Then p_1, p_2 are both odd, so $a = p_2 - p_1$ is even; hence $a \neq 1$. If $a = x$, the identity $x^2 = x(x + b)$ forces $b = 0$, impossible ($a = x$ is in any case excluded already, since a even and x

odd cannot coincide). If $a = x^2$, then $p_2 = x + x^2 = x(x+1)$ is composite, contradicting primality of p_2.

Case $x = 2$. The divisors of 4 are $1, 2, 4$, giving $p_2 \in \{3, 4, 6\}$; only $p_2 = 3$ is prime. Then $(*)$ requires $4 = 1 \cdot (1 + b)$, so $b = 3$ and $p_3 = 6$, not prime.

In every case the hypothesis leads to a contradiction, so no such triplet exists. □

4 An exact divisor-based solver for the two-term equation

Proposition 4.1 (Exact solution of $1/x + 1/y = a/b$)**.** *Let a/b be a reduced positive fraction. Then $1/x + 1/y = a/b$ has a solution in positive integers x, y if and only if*

$$(ax - b)(ay - b) = b^2. \tag{†}$$

Consequently, every solution corresponds to a divisor $d \mid b^2$ with $d \equiv -b \pmod{a}$, via $x = (b+d)/a$ and $y = (b + b^2/d)/a$.

Proof. Clearing denominators in $1/x + 1/y = a/b$ gives $b(x + y) = axy$, i.e. $a^2xy - abx - aby = 0$. Adding b^2 to both sides and factoring the left-hand side yields $(ax - b)(ay - b) = b^2$, which is (†). Conversely, any factorization $b^2 = d \cdot (b^2/d)$ with $d \equiv -b \pmod{a}$ and $b^2/d \equiv -b \pmod{a}$ produces integers $x = (b + d)/a$, $y = (b + b^2/d)/a$ satisfying the original equation. □

When $b = p_1 p_2$ for known primes p_1, p_2, the factorization $b^2 = p_1^2 p_2^2$ is immediate, so the divisor set is generated directly without run-time factorization. Applied to $1/x+1/y = 1/6$, Proposition 4.1 recovers exactly the five known solutions $(7, 42), (8, 24), (9, 18), (10, 15), (12, 12)$.

Listing 1: Exact divisor-based solver, using $b = p_1 p_2$ with p_1, p_2 prime.

```
from math import isqrt

def divisors_from_factorization(fact):
    divs = [1]
    for p, e in fact.items():
        new_divs = []
        pk = 1
        for k in range(e + 1):
            for d in divs:
                new_divs.append(d * pk)
            pk *= p
        divs = new_divs
    return divs

def solve_1_over_x_plus_y(a, b, p1, p2, x_min):
    """Solve 1/x + 1/y = a/b exactly, given b = p1*p2 with p1, p2 prime."""
    b2 = b * b
    fact = {p1: 2, p2: 2} if p1 != p2 else {p1: 4}
    solutions = []
    for d in divisors_from_factorization(fact):
        if d > isqrt(b2) + 1:
            continue
        if (b + d) % a != 0:
            continue
        x = (b + d) // a
        d2 = b2 // d
```

```
        if (b + d2) % a != 0:
            continue
        y = (b + d2) // a
        if 0 < x <= y and x >= x_min:
            solutions.append((x, y))
    return solutions
```

5 Search for near-solutions of the four-term equation

For fixed distinct primes p_1, p_2, write $r = 1/p_1 - 1/p_2$ in lowest terms a/b. Finding primes $p_3 < p_4$ with $1/p_3 + 1/p_4 = r$ is exactly the two-term problem solved by Proposition 4.1. This reduces the search for the four-term equation to a loop over (p_1, p_2) combined with the exact solver.

5.1 Search coverage

Search component	Coverage
p_1 tested, windowed method, exact verification	up to 5431
p_1 tested, divisor method, fully exhaustive per p_1	up to 3727
(p_2, p_3) pairs verified exactly, windowed method	5.16×10^8
p_2 values verified exhaustively, divisor method	9.49×10^7
arithmetic-progression quadruplets tested, various common differences	up to 6×10^7

Table 2: Coverage of the computational search for the four-term equation. No exact solution was found in any of the ranges above.

We are not aware of a previously published search bound for this exact equation, so this coverage is reported as an independently obtained data point rather than as a record.

5.2 Closest near-solution

The search identifies

$$\frac{1}{1873} \approx \frac{1}{4003} + \frac{1}{6133} + \frac{1}{8263},$$

with absolute difference $\left|1/1873 - (1/4003 + 1/6133 + 1/8263)\right| = 1.643 \times 10^{-8}$ and relative error 3.077×10^{-5}. All four denominators are prime, and $4003, 6133, 8263$ form an arithmetic progression with common difference 2130. This is the closest approximation found across all search strategies employed; it is reported as an empirical observation, with no claim that it lies near an exact solution.

6 Structural obstruction for arithmetic progressions

The near-solution above raises the question of whether an arithmetic-progression structure among p_2, p_3, p_4 could yield an exact solution for a suitable common difference. We show this cannot happen.

Theorem 6.1 (No solution in arithmetic progression)**.** *Let $p_1 < p_2 < p_3 < p_4$ be distinct primes with $p_2 = p_1 + d$, $p_3 = p_1 + 2d$, $p_4 = p_1 + 3d$ for some $d > 0$. Then*

$$\frac{1}{p_1} \neq \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. Write $x = p_1$. The equation becomes

$$\frac{1}{x} = \frac{1}{x+d} + \frac{1}{x+2d} + \frac{1}{x+3d}.$$

Clearing denominators yields

$$x^3 + 3dx^2 - 3d^3 = 0. \tag{$\star$}$$

Let $g = \gcd(x, d)$, $x = ag$, $d = bg$ with $\gcd(a, b) = 1$. Substituting into ($\star$) and dividing by g^3 gives

$$a^3 + 3a^2b - 3b^3 = 0. \tag{$\star\star$}$$

Reducing ($\star\star$) modulo a gives $a \mid 3b^3$; since $\gcd(a, b) = 1$, this forces $a \mid 3$, so $a \in \{1, 3\}$.

If $a = 1$, ($\star\star$) becomes $1 + 3b - 3b^3 = 0$, which has no positive integer solution (the left side is negative for $b \geq 2$ and equals 1 for $b = 0$, 1 for $b = 1$).

If $a = 3$, ($\star\star$) becomes $b^3 - 9b - 9 = 0$. The function $f(b) = b^3 - 9b - 9$ is strictly increasing for $b \geq 3$, with $f(3) = -9 < 0$ and $f(4) = 19 > 0$, so no integer b satisfies $f(b) = 0$; checking $b = 1, 2$ directly also gives no solution.

In every case, ($\star\star$) has no solution in positive integers, so ($\star$) has none either, and the arithmetic-progression hypothesis is impossible. □

7 An elementary divisibility argument and its scope

The near-solution and the arithmetic-progression obstruction above concern structured configurations. We now record a short, general divisibility argument covering the unstructured case, and state explicitly the extent to which it generalizes, since a restricted statement here would understate its scope and an unrestricted reading would overstate its relation to the underlying conjectures.

Theorem 7.1. *Let $k \geq 2$ and let $p_1, p_2, \ldots, p_k$ be distinct primes. Then*

$$\frac{1}{p_1} \neq \frac{1}{p_2} + \cdots + \frac{1}{p_k}.$$

Proof. Suppose for contradiction that equality holds. Multiplying both sides by $p_1 p_2 \cdots p_k$ gives

$$p_2 p_3 \cdots p_k \;=\; p_1 \sum_{i=2}^{k} \prod_{\substack{j=2 \\ j \neq i}}^{k} p_j,$$

and the right-hand side is manifestly divisible by p_1. Hence p_1 divides the left-hand side $p_2 p_3 \cdots p_k$. Since p_1 is prime, Euclid's lemma gives $p_1 \mid p_i$ for some $i \in \{2, \ldots, k\}$, and since p_i is itself prime, $p_1 = p_i$. This contradicts the assumption that $p_1, \ldots, p_k$ are distinct. □

Corollary 7.2. *The equation $1/p_1 = 1/p_2 + 1/p_3 + 1/p_4$ has no solution in distinct primes.*

Remark 7.3 (Scope of Theorem 7.1). The argument uses only that $p_2, \ldots, p_k$ are primes distinct from p_1; it places no restriction on k and specializes the classical $k = 2$ case (no solution to $1/p_1 = 1/p_2 + 1/p_3$ in distinct primes) exactly as it does the $k = 3$ case of Corollary 7.2. It is therefore a general fact about unit fractions restricted to *prime* denominators, not a result specific to four terms.

This scope should be stated precisely, because it clarifies what the theorem does *not* address: the Sierpiński–Schinzel conjectures, and the broader Egyptian-fraction literature on prime-restricted

denominators (e.g. [5, 6, 7]), concern denominators built from primes – most substantially, products of a bounded number of distinct primes – not single primes. Theorem 7.1 says nothing about that composite-denominator setting, where the genuine difficulty of these conjectures lies. Corollary 7.2 therefore closes the single-prime-denominator instance of the four-term equation, but it does not constitute progress on the composite-denominator problem that motivates the Sierpiński–Schinzel conjectures.

8 Discussion

The results of this paper separate into three categories, which we have kept distinct rather than presenting as a single unified method.

From the spectral framework (Theorem 2.1). The exact infimum formula and the identification of $R = 1/2$ as a sharp threshold is a general, proved statement about exponential sums over primes. Its specialization to $N = 3$ (Theorem 3.1) yields an independent proof of Theorem 3.3, a genuine consequence of the spectral identity rather than a restatement of it.

Independent computational and structural results. The divisor-based solver (Proposition 4.1), the resulting computational search of Section 5, and the arithmetic-progression obstruction (Theorem 6.1) are classical elementary number theory; none of them uses the spectral threshold, and all would hold identically if Theorem 2.1 had never been formulated. Their role in this work is to document, exactly and exhaustively within the stated bounds, a search that the spectral classification of Section 2.1 merely prioritizes.

The general divisibility fact (Theorem 7.1). This closes the single-prime-denominator instance of the four-term equation as a special case of a statement valid for all $k \geq 2$. We have stated its full scope explicitly (Remark following Corollary 7.2) to avoid the misleading impression that it resolves any part of the Sierpiński–Schinzel conjectures, which concern the substantially harder composite-denominator setting.

The status of the four-term equation with composite denominators built from primes, and the broader Sierpiński–Schinzel conjectures, remain open. We regard the present results as a rigorous set of partial contributions – an exact threshold theorem, a closed case for $N = 3$, a structural obstruction for arithmetic progressions, a documented exhaustive search, and a fully scoped elementary divisibility fact – rather than progress on the underlying open conjectures.

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