A Spin(1,3) Rapidity Theory for Galactic Rotation Curves: Kinematic and Maurer–Cartan Derivations of the Constant-Lagrangian Condition

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Abstract

The empirical Constant-Lagrangian (CL) relation has previously been shown to reproduce a substantial fraction of observed galactic rotation curves, but until now lacked a theoretical derivation. In this work we derive the CL condition from a gravitational rapidity field that generates a local Spin$(\mathbf{1,3})$ rotor describing the galactic vacuum flow. Two independent dynamical branches are developed from the same gravitational rotor, both converging on the identical invariant transport law. The first is the kinematic branch $\chi_g \to G \to U_g \to A_g$, which derives the CL condition from the convective four-acceleration. The second is the Maurer–Cartan branch $\chi_g \to G \to \Omega_\tau \to U_g\Omega_\tau$, which derives the same condition from the transport of the rotor itself. Although these branches describe different geometric objects, both recover the invariant isorapidity condition $D\Gamma/D\tau = 0$, with $\Gamma = \cosh\chi_r \cosh\chi_\phi$, whose weak-field limit reduces to the Constant-Lagrangian law governing the exterior galactic rotation curve. The construction is developed in its native BQ (biquaternion) algebra and subsequently lifted to the Weyl representation of $\mathbf{Cl(1,3)}$, after which equivalent formulations are given in Spacetime Algebra and in the conventional Dirac-matrix formalism. The agreement between these independent representations shows that the resulting isorapidity condition is a representation-independent consequence of the gravitational rapidity rotor rather than a property of a particular algebraic formalism, thereby providing a theoretical foundation for the empirical Constant-Lagrangian postulate.

Keywords: Galactic rotation curves; Spin(1,3); Lorentz rotors; Rapidity; Gravitational rapidity field; Maurer–Cartan connection; Spacetime Algebra; Dirac formalism; Clifford algebra; Galaxy dynamics

Contents

1 Introduction

Rotation curves of disk galaxies remain nearly flat well beyond the radius where the observed baryonic mass predicts a Keplerian decline Rubin et al. (1978, 1980), a result confirmed by extensive surveys such as SPARC Lelli et al. (2016). The two principal approaches attribute this discrepancy either to an unseen dark-matter halo Zwicky (1933); Bertone et al. (2005) or to a modification of gravitational dynamics at low accelerations Milgrom (1983b,a). The approach developed here follows neither direction. Instead, it proposes a gravitational rapidity field as the primary geometric variable describing the vacuum flow. Within this framework, galactic rotation curves are generated from a $\mathrm{Spin}(1,3)$ rotor field, while Newtonian gravity itself remains unchanged and neither dark-matter particles nor phenomenological interpolation functions are introduced. The construction is developed in its native BQ (biquaternion) algebra, and subsequently embedded into the Clifford algebra $\mathrm{Cl}(1,3)$ through the Weyl representation, before being translated into the standard languages of Space-Time Algebra and the Dirac-matrix formalism. The empirical basis of this programme extends from the first direct rotation-curve fits de Haas (2018) to the recent Hubble-inclusive Constant-Lagrangian formulation de Haas (2025).

The gravitational rapidity programme.

The framework developed in this paper is organised around a single $\mathrm{Spin}(1,3)$ rotor field from which the metric, the kinematics and the connection arise as complementary geometric projections. Unlike metric-first or gauge-first formulations of gravity, the present construction takes the scalar gravitational rapidity field $\chi_g(x)$ as its primary object, from which the rotor, the spacetime geometry and the transport equations are generated successively.

The construction proceeds through three stages.

1. *The gravitational rapidity field.* Rather than taking the spacetime metric as the fundamental gravitational variable, the theory begins from the scalar rapidity field
$$\chi_r(r,t) \equiv \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)r\right), \tag{1}$$
which describes the local state of the gravitational vacuum flow. The Newtonian escape flow and the Hubble expansion therefore combine linearly in rapidity space. Since Lorentz rapidity is the natural additive coordinate of hyperbolic velocity space Sommerfeld (1909); Varičak (1912); Rindler (1991); Barrett (2019), it provides the natural starting point for a rotor description of the gravitational field.
2. *The gravitational rotor.* The framework developed here is organised around a local $\mathrm{Spin}(1,3)$ rotor field, the natural geometric object describing Lorentz transformations Hestenes (1966); Doran and Lasenby (2013); Hehl et al. (1976); Blagojević and Hehl (2013). The rapidity field generates this local $\mathrm{Spin}(1,3)$ rotor
$$G = \exp\left(\frac{\chi_g}{2}\hat{\sigma}_g\right),$$
or, for the axisymmetric galactic flow,
$$G = G_\phi(\chi_\phi)\, G_r(\chi_r).$$

The rotor is therefore not introduced independently but is generated directly by the gravitational rapidity field.

3. *Three complementary rotor branches.* The same rotor generates three complementary descriptions of the gravitational field,

$$\chi_g \longrightarrow G \longrightarrow \begin{cases} G^{-1} dR G^{-1} = dR_g \rightarrow \theta^a \rightarrow g_{\mu\nu}, \\ G U_0 G = U_g \rightarrow A_g \rightarrow \dfrac{D\Gamma}{D\tau} = 0, \\ \left(\dfrac{D}{D\tau} G\right) G^{-1} = \Omega_\tau \rightarrow U_g \Omega_\tau \rightarrow \dfrac{D\Gamma}{D\tau} = 0. \end{cases} \tag{2}$$

The first branch constructs the spacetime geometry through the Lorentz-transformed line element and its slow-flow Painlevé–Gullstrand limit. The resulting slow-flow metric reduces to the Painlevé–Gullstrand river form Hamilton and Lisle (2008). That metric route is developed elsewhere and serves here as geometric background. The present paper focuses on the remaining two branches. The kinematic branch derives the Constant-Lagrangian condition from the convective four-acceleration, whereas the Maurer–Cartan branch derives the same condition from the transport of the rotor itself, producing the standard right Maurer–Cartan form associated with the rotor field Sharpe (1997). In both cases the relativistic isorapidity condition

$$\frac{D\Gamma}{D\tau} = 0$$

reduces in the slow-flow limit to

$$v_r^2 + v_\phi^2 = v_L^2,$$

from which the galactic rotation law follows. The three branches therefore represent complementary geometric projections of the same underlying gravitational rotor.

The principal new result of the present work is an independent derivation of the Constant-Lagrangian condition through the Maurer–Cartan connection. Together with the kinematic derivation, this provides two complementary theoretical routes to the same empirical rotation law previously established from SPARC galaxy fits de Haas (2018); de Haas (2025).

Lifting the construction to the Clifford algebra.

The developments described above are carried out in the eight-dimensional BQ–Pauli algebra, whose Lorentz invariant is obtained through the bilinear product $A^T B$. Although sufficient for the native formulation, this algebra does not yet constitute a genuine representation of $\mathrm{Cl}(1,3)$, since ordinary matrix multiplication does not satisfy the Clifford quadratic relation.

Sections 4–5 lift the complete construction to the off-diagonal Weyl representation, thereby embedding the theory into the full Clifford algebra. At that level the quadratic identity

$$\not{A}^2 = A^2 \mathbb{1}$$

holds directly under ordinary matrix multiplication. The gravitational rapidity rotor, the convective branch and the Maurer–Cartan branch are then reconstructed independently within the Clifford framework. Both again recover the same relativistic Constant-Lagrangian condition, demonstrating that the resulting dynamics are representation independent rather than artefacts of the original BQ formulation.

Translation to established Spin$(1,3)$ formalisms.

Having established the Clifford formulation, the final sections translate the theory into the two $\mathrm{Spin}(1,3)$ languages most commonly used today: Space-Time Algebra Hestenes (1966, 2003); Doran and Lasenby (2013); Perez and DeKieviet (2024) and the conventional Dirac-matrix formalism Dirac (1928); Ryder (1996). Unlike conventional presentations Hestenes (1966); Doran and Lasenby (2013); Perez and DeKieviet (2024), which typically begin from the metric, tetrad or spin connection, both translations again take the gravitational rapidity field as the primary object and reconstruct the corresponding rotor, kinematic branch and Maurer–Cartan branch from it. The Dirac formulation is employed purely as a classical representation of the Lorentz group; no spinor field, wave equation or quantum-mechanical interpretation is introduced.

Outline.

Section 2 develops the gravitational rapidity field and its kinematic route to the Constant-Lagrangian condition at the BQ–Pauli level. Section 3 develops the complementary Maurer–Cartan route. Sections 4 and 5 lift both branches to the Clifford algebra, establishing their representation independence. Finally, Sections 6 and 7 translate the complete construction into the standard languages of Space-Time Algebra and the Dirac-matrix formalism. The appendices collect the algebraic infrastructure, multiplication tables and notation dictionaries used throughout the paper.

2 From the Gravitational Rapidity Field to the Constant-Lagrangian Condition

2.1 The gravitational rapidity field

In the BQ rotor framework (de Haas 2026), gravity is described by Lorentz rapidity fields rather than by gravitational potentials or forces. The rapidity field is taken as the primary gravitational object; the physical velocity field is obtained afterwards through its hyperbolic projection.

For a spherically symmetric mass M embedded in an FRW background with Hubble parameter $H(t)$, the radial gravitational rapidity is composed of a Newtonian contribution and a cosmological contribution,

$$\chi_N(r) \equiv \frac{\sqrt{2GM/r}}{c}, \qquad \chi_H(r,t) \equiv \frac{H(t)r}{c}, \tag{3}$$

which add linearly because they are collinear in rapidity space. Consequently,

$$\chi_r(r,t) \equiv \chi_N(r) - \chi_H(r,t), \tag{4}$$

or

$$\boxed{\chi_r(r,t) \equiv \frac{1}{c}\left(\sqrt{\frac{2GM}{r}} - H(t)r\right).} \tag{5}$$

Because the primary gravitational field is not the velocity but the rapidity, the effective Newtonian riverbed appears as its weak-field projection,

$$v_r = c\tanh\chi_r \approx \sqrt{\frac{2GM}{r}} - H(t)r, \tag{6}$$

while the exact hyperbolic functions furnish the relativistic theory without any additional assumptions.

The complete gravitational rapidity field is written

$$\chi_g = (\chi_r, \chi_\phi),$$

where the source determines the radial component χ_r, while the azimuthal component χ_ϕ remains a dynamical degree of freedom. Its evolution is not prescribed a priori but is determined by the hydrodynamic organization of the gravitational riverbed developed in the following sections.

2.2 The compounded gravitational flow U_g

The gravitational boost rotor.

The gravitational rapidity field

$$\chi_g = (\chi_r, \chi_\phi)$$

generates a local $\mathrm{Spin}(1,3)$ boost rotor. Throughout this paper we adopt the sequential galactic construction

$$G = G_\phi G_r, \qquad G_r = \exp\!\left(\frac{\chi_r}{2}\hat{\sigma}_r\right), \qquad G_\phi = \exp\!\left(\frac{\chi_\phi}{2}\hat{\sigma}_\phi\right), \tag{7}$$

in which the radial and azimuthal rapidities represent two successive Lorentz boosts of the same gravitational flow. Their ordering is therefore an intrinsic part of the rotor construction.

Application of the gravitational boost rotor.

The transport bilinear of the previous section is constructed from the gravitational four-velocity generated by this sequential rotor $G = G_\phi G_r$. Using the non-inverse sandwich appropriate to the BQ construction,

$$U_g = G(u_0\hat{\mathrm{T}})G = G_\phi\big[G_r(c\hat{\mathrm{T}})G_r\big]G_\phi, \qquad u_0 \equiv c. \tag{8}$$

Because the sandwich is non-inverse, the sequential boosts must be evaluated explicitly rather than combined into a single rotor.

Step 1: radial boost.

Direct multiplication gives

$$G_r(c\hat{\mathrm{T}})G_r = c\cosh\chi_r\,\hat{\mathrm{T}} + c\sinh\chi_r\,\hat{\mathrm{I}}_r = c\gamma_r\,\hat{\mathrm{T}} + \gamma_r v_r\,\hat{\mathrm{I}}_r, \tag{9}$$

where $\gamma_r = \cosh\chi_r$ and $\gamma_r v_r = c\sinh\chi_r$.

Step 2: azimuthal boost.

The second boost acts separately on the two terms,

$$G_\phi\,\hat{\mathrm{T}}\,G_\phi = \cosh\chi_\phi\,\hat{\mathrm{T}} + \sinh\chi_\phi\,\hat{\mathrm{J}}_\phi, \qquad G_\phi\,\hat{\mathrm{I}}_r\,G_\phi = \hat{\mathrm{I}}_r. \tag{10}$$

The second identity is exact. Since $\hat{\mathrm{I}}_r = \hat{\mathrm{T}}\hat{\sigma}_r$ and $\hat{\mathrm{T}}$ commutes with the algebra, the non-inverse sandwich leaves $\hat{\mathrm{I}}_r$ invariant. This differs fundamentally from the inverse sandwich used in the Maurer–Cartan connection, where $\hat{\sigma}_r$ rotates into the Thomas–Wigner channel $\hat{\mathrm{K}}_\theta$ (Eq. (47)).

Compound four-velocity.

Combining both boosts,

$$U_g = c\gamma_r\big[\cosh\chi_\phi\,\hat{\mathrm{T}} + \sinh\chi_\phi\,\hat{\mathrm{J}}_\phi\big] + \gamma_r v_r\,\hat{\mathrm{I}}_r = c\gamma_r\gamma_\phi\,\hat{\mathrm{T}} + \gamma_r v_r\,\hat{\mathrm{I}}_r + c\gamma_r\sinh\chi_\phi\,\hat{\mathrm{J}}_\phi, \tag{11}$$

with no $\hat{\mathrm{K}}_\theta$ component generated. Using $\gamma_\phi v_\phi = c\sinh\chi_\phi$ gives

$$\boxed{U_g = u_0\hat{\mathrm{T}} + \mathbf{u}\cdot\hat{\mathbf{K}}, \qquad u_0 = c\gamma_r\gamma_\phi, \quad u_r = \gamma_r v_r, \quad u_\phi = \gamma_r\gamma_\phi v_\phi.} \tag{12}$$

The common factor $\gamma_r\gamma_\phi$ reflects the sequential, non-Abelian composition of the rotor and appears consistently in both U_g and the Maurer–Cartan connection Ω_τ.

2.3 The convective four-acceleration $a = DU_g/D\tau$

Distinct from the flow field $H_g = \partial^T U_g$, which is generated by the full four-gradient, the convective (proper) acceleration is defined through the material derivative acting on the components of U_g:

$$\boxed{A_g \equiv \frac{DU_g}{D\tau} = a_0\hat{\mathrm{T}} + a_r\hat{\mathrm{I}}_r + a_\phi\hat{\mathrm{J}}_\phi, \qquad a_\mu \equiv \frac{Du_\mu}{D\tau} = \frac{u_0}{c}\dot{u}_\mu + u_r u'_\mu,} \tag{13}$$

with $a_\theta = 0$, since $u_\theta \equiv 0$.

The $\hat{\mathrm{T}}$ channel.

Using $u_0 = c\Gamma$ together with $\Gamma = \cosh\chi_r \cosh\chi_\phi$, the time component reduces exactly to

$$\boxed{a_0 = c\,\frac{D\Gamma}{D\tau},} \tag{14}$$

verified directly by symbolic computation. Consequently,

$$a_0 = 0 \qquad \Longleftrightarrow \qquad \frac{D\Gamma}{D\tau} = 0,$$

providing a direct kinematic route to the constant-Lagrangian condition.

The spatial channels.

$$\boxed{a_r = c\cosh\chi_r\Big[c\sinh\chi_r\,\chi_r' + \cosh\chi_\phi\cosh\chi_r\,\dot{\chi}_r\Big],} \tag{15}$$

$$\boxed{\begin{aligned} a_\phi = c^2\sinh\chi_r\Big[\,&\sinh\chi_\phi\sinh\chi_r\,\chi_r' + \cosh\chi_\phi\cosh\chi_r\,\chi_\phi'\Big] \\ &+ c\cosh\chi_\phi\cosh\chi_r\Big[\sinh\chi_\phi\sinh\chi_r\,\dot{\chi}_r + \cosh\chi_\phi\cosh\chi_r\,\dot{\chi}_\phi\Big]. \end{aligned}} \tag{16}$$

2.4 Orthogonality and the dynamical branches

As in the Weyl-level construction, the convective acceleration is identically orthogonal to the four-velocity,

$$\boxed{[U_g^T A_g]_{\hat{1}} \equiv u_0 a_0 - u_r a_r - u_\phi a_\phi = 0,} \tag{17}$$

verified directly by symbolic computation. This orthogonality follows from the invariant norm

$$U_g^T U_g = -c^2\hat{1}, \tag{18}$$

since differentiation along the flow immediately yields

$$0 = \frac{D}{D\tau}\left[U_g^T U_g\right] = 2\,[U_g^T A_g]_{\hat{1}}. \tag{19}$$

The direct calculation and the invariant-norm argument therefore provide independent derivations of Eq. (17). This orthogonality is an intrinsic property of the convective acceleration.

Longitudinal and transverse acceleration.

Writing Eq. (17) explicitly gives

$$\boxed{u_r a_r + u_\phi a_\phi = u_0 a_0,} \tag{20}$$

so that the longitudinal component of the spatial acceleration is entirely determined by the temporal component a_0. Since

$$a_0 = c\,\frac{D\Gamma}{D\tau},$$

two distinct dynamical branches follow immediately.

Constant-Lagrangian branch. If

$$a_0 = 0 \qquad \left(\frac{D\Gamma}{D\tau} = 0\right),$$

then

$$u_r a_r + u_\phi a_\phi = 0, \tag{21}$$

so the spatial acceleration is purely transverse to the flow, with no longitudinal component.

Non-constant-Lagrangian branch. If

$$a_0 \neq 0,$$

then

$$u_r a_r + u_\phi a_\phi = u_0 a_0 \neq 0, \tag{22}$$

so a longitudinal acceleration necessarily accompanies the change in Γ. The temporal and longitudinal components therefore appear together: equilibrium flows satisfy $D\Gamma/D\tau = 0$, whereas departures from the constant-Lagrangian condition generate a nonzero acceleration along the flow direction.

2.5 The power interpretation of a_0 and the Constant-Lagrangian consequence

a_0 and the power per unit mass.

Since

$$u_0 = c\Gamma,$$

linearity of the material derivative gives

$$a_0 = \frac{Du_0}{D\tau} = c\,\frac{D\Gamma}{D\tau}. \tag{23}$$

The relativistic energy per unit mass carried by the flow is

$$\frac{E}{m} = c\,u_0 = c^2\Gamma,$$

and therefore

$$\frac{1}{m}\frac{DE}{D\tau} = c\,a_0 = c^2\frac{D\Gamma}{D\tau}. \tag{24}$$

Thus $c\,a_0$ is the power per unit mass along the flow, while the spatial components a_r and a_ϕ have the dimensions of force per unit mass. Consequently,

$$a_0 = 0 \qquad \Longleftrightarrow \qquad \frac{D\Gamma}{D\tau} = 0 \tag{25}$$

expresses conservation of the relativistic energy per unit mass along the flow.

The effective riverbed.

The galactic riverbed combines Newtonian infall with the Hubble background through the effective radial profile

$$v_{r,\mathrm{eff}}(r) = \sqrt{\frac{2GM}{r}} - H_z r, \tag{26}$$

where H_z is taken as constant for the stationary galactic flow considered here. The radial gravitational rapidity is defined by

$$\boxed{\chi_r(r) \equiv \frac{v_{r,\mathrm{eff}}(r)}{c}.} \tag{27}$$

Its exact hyperbolic projection gives the physical radial velocity,

$$v_r^{\mathrm{FR}} = c\tanh\chi_r = c\tanh\!\left(\frac{v_{r,\mathrm{eff}}}{c}\right), \tag{28}$$

which remains bounded by c.

Slow-flow limit.

For $|\chi_r|, |\chi_\phi| \ll 1$, the compound Lorentz factor

$$\Gamma = \cosh\chi_r \cosh\chi_\phi$$

reduces to

$$\Gamma \approx 1 + \frac{v_{r,\text{eff}}^2 + v_\phi^2}{2c^2}. \tag{29}$$

Hence

$$c^2(\Gamma - 1) \longrightarrow \frac{1}{2}v_g^2, \qquad \boxed{v_g^2 \equiv v_{r,\text{eff}}^2 + v_\phi^2.} \tag{30}$$

Accordingly, the relativistic invariant carried by Γ reduces to the Newtonian kinetic energy per unit mass of the compound flow. Within this slow-flow approximation,

$$\frac{D\Gamma}{D\tau} = 0 \qquad \Longrightarrow \qquad \frac{Dv_g^2}{D\tau} = 0, \tag{31}$$

so that $v_g^2 = v_L^2$ is conserved along a streamline. This is the Constant-Lagrangian relation used in the empirical galactic rotation-curve model.

Effective potential.

Introduce the effective potential per unit mass

$$\Phi_{\text{eff}}(r) \equiv -\frac{1}{2}v_{r,\text{eff}}(r)^2. \tag{32}$$

For the Newton–Hubble riverbed this gives

$$\begin{aligned}\Phi_{\text{eff}}(r) &= -\frac{1}{2}\left(\sqrt{\frac{2GM}{r}} - H_z r\right)^2 \\ &= \boxed{-\frac{GM}{r} + H_z r\sqrt{\frac{2GM}{r}} - \frac{1}{2}H_z^2 r^2}.\end{aligned} \tag{33}$$

The three terms are respectively the Newtonian contribution, the Newtonian–Hubble cross term, and the pure Hubble contribution. In this sense the effective potential is already encoded in the radial rapidity field through its defining riverbed profile.

The effective Constant-Lagrangian invariant.

Because

$$\frac{1}{2}v_{r,\text{eff}}^2 = -\Phi_{\text{eff}},$$

the slow-flow limit of the relativistic invariant becomes

$$\boxed{c^2(\Gamma - 1) \longrightarrow \frac{1}{2}v_\phi^2 - \Phi_{\rm eff}(r).} \tag{34}$$

Defining the effective specific Lagrangian

$$\mathcal{L}_{\rm eff} \equiv \frac{1}{2}v_\phi^2 - \Phi_{\rm eff}, \tag{35}$$

one obtains

$$a_0 = 0 \qquad \Longleftrightarrow \qquad \frac{D\Gamma}{D\tau} = 0 \qquad \Longrightarrow \qquad \frac{D\mathcal{L}_{\rm eff}}{D\tau} = 0 \tag{36}$$

in the slow-flow limit. Thus the empirical Constant-Lagrangian relation is recovered as the weak-flow projection of the relativistic isorapidity condition, while the exact conservation law remains expressed entirely through the gravitational rapidity field $\chi_g = (\chi_r, \chi_\phi)$.

Bernoulli signature

The conservation of $\frac{1}{2}(v_{r,\rm eff}^2 + v_\phi^2)$ has a Bernoulli-like fluid-dynamical signature: radial and azimuthal kinetic contributions are exchanged along the flow while their sum remains invariant. With $\Phi_{\rm eff} = -\frac{1}{2}v_{r,\rm eff}^2$, however, the same conserved quantity takes the explicit effective-Lagrangian form

$$\mathcal{L}_{\rm CL} = \frac{1}{2}v_\phi^2 - \Phi_{\rm eff} = \text{const.}$$

Thus the isorapidity law $D\Gamma/D\tau = 0$ supplies a relativistic theoretical foundation for the Constant-Lagrangian postulate.

2.6 The exterior rotation law from the relativistic CL condition

Throughout the exterior disk, outside the bulge containing the enclosed mass $M = M(R)$, the slow-flow limit of the relativistic rapidity invariant $\Gamma = \cosh\chi_r \cosh\chi_\phi$ gives, as shown in Sec. 2.5,

$$\boxed{v_\phi^2(r) + v_{r,\rm eff}^2(r) = v_L^2, \qquad R \le r \le r_c.} \tag{37}$$

Here v_L^2 is conserved along the stationary exterior flow. We assume that the relevant exterior streamlines inherit a common boundary value at $r = R$, allowing the streamline invariant to be represented by a single constant over the exterior radial domain. For $r \geq R$, the enclosed bulge mass is treated as constant, $M(r) = M(R) \equiv M$, and the effective radial riverbed profile is

$$v_{r,\mathrm{eff}}(r) = \underbrace{\sqrt{\frac{2GM}{r}}}_{v_{\mathrm{esc}}(r)} - \underbrace{H_z r}_{v_H(r)}, \tag{38}$$

where H_z is taken as effectively constant in the stationary galactic model.

Domain of validity.

The two contributions to Eq. (38) scale oppositely with radius: $v_{\mathrm{esc}}(r)$ falls as $r^{-1/2}$ while $v_H(r)$ grows linearly, so the two must cross at a critical radius r_c defined by $v_{\mathrm{esc}}(r_c) = v_H(r_c)$,

$$\sqrt{\frac{2GM}{r_c}} = H_z r_c \qquad \Longrightarrow \qquad r_c = \left(\frac{2GM}{H_z^2}\right)^{1/3}. \tag{39}$$

At r_c the riverbed itself momentarily stalls, $v_{r,\mathrm{eff}}(r_c) = 0$, and beyond it the Hubble term dominates and $v_{r,\mathrm{eff}}$ changes character from inward-dominated to outward-dominated flow.

We consequently restrict the physical interpretation of the exterior rotation law to $R \leq r \leq r_c$. Although the squared expression can be continued algebraically beyond r_c, such a continuation would describe a Hubble-dominated outward branch and is not assumed to belong to the same stationary CL flow. Outside the r_c range the riverbed picture itself needs to be re-examined rather than merely extrapolated.

Boundary condition at $r = R$.

The integration constant v_L^2 in Eq. (37) is fixed at the bulge–disk boundary by the same virial closure used in the earlier CL derivation: the orbit at $r = R$ is assumed to be virialized with respect to the effective potential Φ_{eff} defined through $v_{r,\mathrm{eff}}^2 = -2\Phi_{\mathrm{eff}}$, so that

$$K_\phi(R) = -\tfrac{1}{2} m \Phi_{\mathrm{eff}}(R).$$

With $K_\phi(R) = \frac{1}{2} m v_\phi^2(R)$ and $\Phi_{\mathrm{eff}}(R) = -\frac{1}{2} v_{r,\mathrm{eff}}^2(R)$, this gives

$$\tfrac{1}{2} v_\phi^2(R) = \tfrac{1}{4} v_{r,\mathrm{eff}}^2(R) \qquad \Longrightarrow \qquad \boxed{v_\phi^2(R) = \tfrac{1}{2} v_{r,\mathrm{eff}}^2(R).}$$

The conserved slow-flow invariant evaluated at the boundary is thus

$$v_L^2 = v_\phi^2(R) + v_{r,\mathrm{eff}}^2(R) = \tfrac{1}{2} v_{r,\mathrm{eff}}^2(R) + v_{r,\mathrm{eff}}^2(R) = \tfrac{3}{2} v_{r,\mathrm{eff}}^2(R), \tag{40}$$

$$\boxed{v_L^2 = \frac{3}{2}\left(\sqrt{\frac{2GM}{R}} - H_z R\right)^2.} \tag{41}$$

Remark on the virial closure.

The relation $v_\phi^2(R) = \frac{1}{2}v_{r,\mathrm{eff}}^2(R)$ is exact for a Keplerian circular orbit when $H_z = 0$. Here it is retained as a boundary closure for the composite Newton–Hubble profile at a radius where the Keplerian contribution is assumed to dominate. Because $H_z R \ll v_{\mathrm{esc}}(R)$ at the bulge–disk boundary of any realistic galaxy, the Keplerian term dominates $\Phi_{\mathrm{eff}}(R)$ overwhelmingly and the Keplerian virial ratio is an excellent approximation there; the closure should accordingly be read as a physically motivated boundary condition tied specifically to the radius R at which the flow is still Keplerian-dominated, rather than as a virial identity assumed to hold in the same form at every radius of the exterior disk.

Exterior azimuthal velocity.

At an arbitrary exterior radius $R \le r \le r_c$, conservation of the slow-flow rapidity invariant gives

$$v_\phi^2(r) + v_{r,\mathrm{eff}}^2(r) = \tfrac{3}{2}v_{r,\mathrm{eff}}^2(R),$$

so that

$$\boxed{v_\phi^2(r) = \frac{3}{2}v_{r,\mathrm{eff}}^2(R) - v_{r,\mathrm{eff}}^2(r).} \tag{42}$$

Substituting the explicit riverbed profile, Eq. (38), gives the exterior rotation law

$$\boxed{v_\phi^2(r) = \frac{3}{2}\left(\sqrt{\frac{2GM}{R}} - H_z R\right)^2 - \left(\sqrt{\frac{2GM}{r}} - H_z r\right)^2, \qquad R \le r \le r_c.} \tag{43}$$

Equation (43) is the fitting formula used throughout the remainder of this work: given a bulge mass M, a bulge–disk boundary radius R, and the local Hubble parameter H_z, once M and R are specified and H_z is fixed, it determines the exterior rotation curve without introducing a further halo or interpolation parameter over the full radial range $R \le r \le r_c$ within which the underlying riverbed construction remains valid. The earlier Constant-Lagrangian rotation law is thereby recovered as the slow-flow limit of the relativistic conservation of the composed rapidity invariant Γ.

Empirical results

The inclusion of the Hubble term $H_z r$ in Eq. (38) is a comparatively recent refinement. The original formulation of the Constant-Lagrangian postulate, tested in de Haas (2018) against the complete SPARC database of 175 late-type galaxies using the $v_H = 0$ (equivalently $H_z = 0$) reduction of $v_{r,\mathrm{eff}}(r)$, already produced a striking result using only elementary curve fitting: of the 175 galaxies, 77 – about 44% – admitted a single continuous fit to the measured rotation curve within the reported measurement uncertainties, quantified there by the Root Mean Weighted

Residual Sum of Squares (RMWRSS); a further 18 galaxies were fit almost as well, and 13 additional galaxies required a pair of crossing fit curves rather than one, a feature that itself correlated with identifiable features of galactic structure and dynamics. This direct-match rate indicated that the agreement was unlikely to be attributable to isolated accidental fits.

The subsequent introduction of the Hubble term, and its identification as the physical boundary condition closing the riverbed at large radius (Sec. 2.6), extended this result rather than superseding it: the Newtonian/CL core of the model established in de Haas (2018) remains the inner engine of Eq. (38), while H_z supplies the outer closure that the $v_H = 0$ formulation lacked. The quantitative gain from this extension is documented in de Haas (2025), where the same two-parameter (R, M) prescription – now with H_z fixed at its cosmological value rather than dropped – achieves reduced chi-squared values in the range $\chi^2_\nu \simeq 0.05$–0.4 across a subset of the SPARC sample, with the extended per-galaxy variants averaging $\langle\chi^2_\nu\rangle \approx 0.67$ against $\langle\chi^2_\nu\rangle \approx 1.1$ for the basic two-parameter fit – a systematic improvement over the $H_z = 0$ result associated with retaining the Hubble contribution in the radial riverbed profile, while the virial condition at R fixes the conserved value v_L^2.

3 The Maurer–Cartan connection at the BQ–Pauli level

3.1 The exact connection of the ordered galactic rotor

Throughout this section, $\hat{\sigma}_r$, $\hat{\sigma}_\phi$ and $\hat{\mathrm{K}}_\theta$ denote fixed generators of the chosen local BQ frame. The material derivative D_τ therefore acts on the rapidity fields but not on the generators. A separate connection associated with the transport of a coordinate-dependent polar basis is not included in the definition below.

BCH content.

For the ordered rotor

$$G = G_\phi G_r, \qquad G_\phi = \exp\left(\frac{\chi_\phi}{2}\hat{\sigma}_\phi\right), \qquad G_r = \exp\left(\frac{\chi_r}{2}\hat{\sigma}_r\right),$$

the relation to a local single-exponential representation is determined by the Baker–Campbell–Hausdorff expansion

$$e^A e^B = e^C, \qquad C = A + B + \frac{1}{2}[A, B] + \cdots,$$

with

$$A = \frac{\chi_\phi}{2}\hat{\sigma}_\phi, \qquad B = \frac{\chi_r}{2}\hat{\sigma}_r.$$

Direct BQ multiplication gives

$$\hat{\sigma}_\phi\hat{\sigma}_r = -\hat{\mathrm{K}}_\theta, \qquad [\hat{\sigma}_\phi, \hat{\sigma}_r] = -2\hat{\mathrm{K}}_\theta,$$

and hence

$$\frac{1}{2}[A,B] = -\frac{\chi_\phi \chi_r}{4}\hat{\mathrm{K}}_\theta. \tag{44}$$

To first non-Abelian order, the logarithm of the ordered rotor is therefore

$$C = \frac{\chi_\phi}{2}\hat{\sigma}_\phi + \frac{\chi_r}{2}\hat{\sigma}_r - \frac{\chi_\phi \chi_r}{4}\hat{\mathrm{K}}_\theta + \cdots . \tag{45}$$

Thus a Thomas–Wigner rotational channel is already present in the ordered composition of the two non-collinear boosts. It is neither introduced by differentiation nor postulated as an independent rotational field. The Maurer–Cartan connection differentiates this non-Abelian rotor structure along the flow.

Exact sequential connection.

Using the right Maurer–Cartan convention

$$\Omega_\tau = (D_\tau G)G^{-1},$$

the product rule for $G = G_\phi G_r$ gives

$$\boxed{\Omega_\tau = \Omega_\phi + G_\phi \Omega_r G_\phi^{-1}}, \qquad \Omega_\phi = \frac{1}{2}\frac{D\chi_\phi}{D\tau}\hat{\sigma}_\phi, \qquad \Omega_r = \frac{1}{2}\frac{D\chi_r}{D\tau}\hat{\sigma}_r. \tag{46}$$

The present construction adopts the right Maurer–Cartan form

$$\Omega_\tau = (D_\tau G)G^{-1},$$

rather than the left form $G^{-1}(D_\tau G)$. The two are related by conjugation and therefore contain the same geometric information, but they describe it in different frames. The right Maurer–Cartan connection is expressed in the external spacetime frame and naturally accompanies the evolution of the physical vacuum-flow field generated by the rotor. For the ordered decomposition $G = G_\phi G_r$ it yields

$$\Omega_\tau = \Omega_\phi + G_\phi \Omega_r G_\phi^{-1},$$

so that the radial boost is transported through the azimuthal rotor, producing the Thomas–Wigner channel directly. The left Maurer–Cartan connection would instead transport the azimuthal boost through the radial one,

$$G^{-1}(D_\tau G) = G_r^{-1}\Omega_\phi G_r + \Omega_r,$$

an equivalent description in a different frame, but one that is less natural for the riverbed interpretation adopted here, because the radial flow is the causation and the azimuthal flow the self-organised reaction and not the other way around.

The noncommutative content resides in the adjoint action of G_ϕ on the radial boost generator. Direct BQ multiplication gives

$$\boxed{G_\phi \hat{\sigma}_r G_\phi^{-1} = \cosh \chi_\phi \, \hat{\sigma}_r - \sinh \chi_\phi \, \hat{\mathrm{K}}_\theta}. \tag{47}$$

The exact proper-time Maurer–Cartan connection of the ordered rotor is therefore

$$\boxed{\Omega_\tau = \frac{1}{2}\frac{D\chi_\phi}{D\tau}\hat{\sigma}_\phi + \frac{1}{2}\cosh \chi_\phi \frac{D\chi_r}{D\tau}\hat{\sigma}_r - \frac{1}{2}\sinh \chi_\phi \frac{D\chi_r}{D\tau}\hat{\mathrm{K}}_\theta}. \tag{48}$$

The first term describes the proper-time evolution of the azimuthal boost. The second is the radial boost rate expressed in the frame already transformed by G_ϕ, while the third is the induced Thomas–Wigner rotational channel generated by the ordered composition of the radial and azimuthal boosts. Its subsequent coupling to U_g supplies the flow-weighted transport studied in the following subsection.

3.2 The flow-weighted connection and recovery of the Constant-Lagrangian condition

Write the proper-time Maurer–Cartan connection obtained in Eq. (48) as

$$\Omega_\tau = \boldsymbol{\omega}_\sigma + \boldsymbol{\omega}_K = \omega_r \hat{\sigma}_r + \omega_\phi \hat{\sigma}_\phi + \omega_\theta \hat{\mathrm{K}}_\theta, \tag{49}$$

where

$$\omega_r = \frac{1}{2}\gamma_\phi \frac{D\chi_r}{D\tau}, \qquad \omega_\phi = \frac{1}{2}\frac{D\chi_\phi}{D\tau}, \qquad \omega_\theta = -\frac{1}{2}\gamma_\phi \frac{v_\phi}{c}\frac{D\chi_r}{D\tau}. \tag{50}$$

The boost and rotation sectors are therefore

$$\boldsymbol{\omega}_\sigma = \omega_r \hat{\sigma}_r + \omega_\phi \hat{\sigma}_\phi, \qquad \boldsymbol{\omega}_K = \omega_\theta \hat{\mathrm{K}}_\theta. \tag{51}$$

Applying the BQ triple-product rule to the compound four-velocity U_g and the connection Ω_τ gives

$$U_g \Omega_\tau = e_T \hat{\mathrm{T}} + \mathbf{e}_K \cdot \hat{\mathbf{K}} + \mathbf{e}_\sigma \cdot \hat{\boldsymbol{\sigma}}, \qquad e_1 = 0, \tag{52}$$

with

$$e_T = \mathbf{u} \cdot \boldsymbol{\omega}_\sigma = u_r \omega_r + u_\phi \omega_\phi, \tag{53a}$$
$$\mathbf{e}_K = u_0 \boldsymbol{\omega}_\sigma + \mathbf{u} \times \boldsymbol{\omega}_K, \tag{53b}$$
$$\mathbf{e}_\sigma = \mathbf{u} \times \boldsymbol{\omega}_\sigma - u_0 \boldsymbol{\omega}_K. \tag{53c}$$

These expressions are evaluated using the compound components u_0, u_r, u_ϕ of Eq. (12).

Explicit channel structure.

The time channel is

$$\boxed{[U_g\Omega_\tau]_{\hat{\mathrm{T}}} = \frac{1}{2}\gamma_r\gamma_\phi\left(v_r\frac{D\chi_r}{D\tau} + v_\phi\frac{D\chi_\phi}{D\tau}\right).} \tag{54}$$

The radial and azimuthal parts of the $\hat{\mathbf{K}}$ channel reduce to

$$\boxed{[U_g\Omega_\tau]_{\hat{\mathrm{I}}_r} = \frac{1}{2}c\gamma_r\frac{D\chi_r}{D\tau}}, \tag{55a}$$

$$\boxed{[U_g\Omega_\tau]_{\hat{\mathrm{J}}_\phi} = \frac{1}{2}c\gamma_r\gamma_\phi\frac{D\chi_\phi}{D\tau} + \frac{1}{2}\gamma_r v_r \sinh\chi_\phi\frac{D\chi_r}{D\tau}}. \tag{55b}$$

In the radial channel, the identity

$$\gamma_\phi^2 - \gamma_\phi^2\frac{v_\phi^2}{c^2} = \gamma_\phi^2 - \sinh^2\chi_\phi = 1$$

removes the explicit orbital dependence.

The only nonvanishing $\hat{\sigma}$ contribution is

$$\boxed{[U_g\Omega_\tau]_{\hat{\sigma}_\theta} = \frac{1}{2}\gamma_r v_r\frac{D\chi_\phi}{D\tau}.} \tag{56}$$

Here the two terms proportional to $D\chi_r/D\tau$ cancel exactly between the boost and rotational parts of the connection.

The colatitude θ_0 enters only through $v_\phi = r\sin\theta_0\,\dot{\phi}$. Once χ_r, χ_ϕ and their material derivatives are specified, the algebraic channel expressions contain no further explicit dependence on θ_0.

Recovery of the relativistic CL condition.

Using

$$v_r = c\tanh\chi_r, \qquad v_\phi = c\tanh\chi_\phi,$$

the time channel becomes

$$[U_g\Omega_\tau]_{\hat{\mathrm{T}}} = \frac{c\gamma_r\gamma_\phi}{2}\left(\tanh\chi_r\frac{D\chi_r}{D\tau} + \tanh\chi_\phi\frac{D\chi_\phi}{D\tau}\right). \tag{57}$$

For

$$\Gamma = \gamma_r\gamma_\phi = \cosh\chi_r\cosh\chi_\phi,$$

the chain rule gives

$$\frac{1}{\Gamma}\frac{D\Gamma}{D\tau} = \tanh\chi_r\frac{D\chi_r}{D\tau} + \tanh\chi_\phi\frac{D\chi_\phi}{D\tau}. \tag{58}$$

Consequently,

$$\boxed{[U_g\Omega_\tau]_{\hat{\mathrm{T}}} = \frac{c}{2}\frac{D\Gamma}{D\tau}.} \tag{59}$$

The vanishing of the time channel is therefore exactly equivalent to the relativistic Constant-Lagrangian condition,

$$\boxed{[U_g\Omega_\tau]_{\hat{\mathrm{T}}} = 0 \qquad \Longleftrightarrow \qquad \frac{D\Gamma}{D\tau} = 0.} \tag{60}$$

Unconditional orthogonality of the vector channel.

Let

$$(U_g\Omega_\tau)_{\mathrm{vec}} \equiv e_T\hat{\mathrm{T}} + \mathbf{e}_K \cdot \hat{\mathbf{K}} \tag{61}$$

denote the $\{\hat{\mathrm{T}}, \hat{\mathbf{K}}\}$-type part of the flow-weighted connection. From Eq. (53),

$$\begin{aligned} \mathbf{u} \cdot \mathbf{e}_K &= \mathbf{u} \cdot (u_0\omega_\sigma + \mathbf{u} \times \omega_K) \\ &= u_0\, \mathbf{u} \cdot \omega_\sigma = u_0 e_T, \end{aligned} \tag{62}$$

because $\mathbf{u} \cdot (\mathbf{u} \times \omega_K) = 0$. Hence

$$\boxed{u_0 e_T - \mathbf{u} \cdot \mathbf{e}_K = 0.} \tag{63}$$

This is an unconditional structural identity: the vector part of $U_g\Omega_\tau$ is Lorentz-orthogonal to U_g, independently of the CL condition and independently of the particular values of $\omega_r, \omega_\phi, \omega_\theta$.

On the Constant-Lagrangian branch, Eq. (60) gives $e_T = 0$, and therefore

$$\frac{D\Gamma}{D\tau} = 0 \qquad \Longrightarrow \qquad \mathbf{u} \cdot \mathbf{e}_K = 0. \tag{64}$$

Thus the CL condition removes the component of the $\hat{\mathbf{K}}$ channel parallel to the spatial flow. Any surviving $\hat{\mathbf{K}}$ contribution is spatially transverse to $\mathbf{u}$ and may be interpreted as the rotational or deflective part of the transported connection, while the separate $\hat{\sigma}$ channel retains the additional bivector content displayed in Eq. (56).

3.3 Isorapidity and longitudinally force-free transport

The compound Lorentz factor

$$\Gamma = \gamma_r\gamma_\phi = \cosh\chi_r \cosh\chi_\phi$$

combines the radial and azimuthal rapidities of the galactic flow into a single relativistic quantity. The condition

$$\frac{D\Gamma}{D\tau} = 0$$

states that this compound rapidity is conserved along each flow line, although χ_r and χ_ϕ may individually vary. In the slow-flow limit this reduces to the Constant-Lagrangian invariant

$$v_{r,\mathrm{eff}}^2 + v_\phi^2 = v_L^2,$$

showing that the total riverbed speed is conserved while radial and azimuthal motion continuously exchange.

The transport bilinear satisfies

$$[U_g \Omega_\tau]_{\hat{\mathrm{T}}} = e_T = \frac{c}{2}\frac{D\Gamma}{D\tau},$$

together with the unconditional identity

$$\mathbf{u} \cdot \mathbf{e}_K = u_0 e_T.$$

Hence

$$\boxed{\frac{D\Gamma}{D\tau} = 0 \quad \Longrightarrow \quad e_T = 0 \quad \Longrightarrow \quad \mathbf{u} \cdot \mathbf{e}_K = 0.} \tag{65}$$

The Constant-Lagrangian condition therefore removes the component of the transported connection parallel to the flow. The surviving $\hat{\mathbf{K}}$-channel is everywhere transverse to the local riverbed velocity.

This transverse transport should not be interpreted as the absence of acceleration. A steady spiral flow necessarily changes direction while maintaining its speed. Indeed, from

$$v_g^2 = v_{r,\mathrm{eff}}^2 + v_\phi^2 = \mathrm{const},$$

it follows immediately that

$$\mathbf{v}_g \cdot \frac{D\mathbf{v}_g}{D\tau} = \frac{1}{2}\frac{D}{D\tau}\left(v_g^2\right) = 0,$$

so the material acceleration is everywhere perpendicular to the flow velocity. The situation is directly analogous to uniform circular motion, where the centripetal acceleration continuously redirects the velocity vector without changing its magnitude. For the galactic riverbed, the surviving transverse channel generated by the noncommutativity of the radial and azimuthal boosts performs precisely this role: it bends the riverbed into its spiral trajectory while preserving the Constant-Lagrangian speed.

Outside the CL branch,

$$\boxed{\mathbf{u} \cdot \mathbf{e}_K = u_0 e_T = \frac{c^2 \Gamma}{2}\frac{D\Gamma}{D\tau},} \tag{66}$$

so departures from isorapidity produce a genuine longitudinal transport component in addition to the unavoidable transverse one. The quantity $D\Gamma/D\tau$ therefore measures deviations from the stationary, Constant-Lagrangian spiral flow.

The above results provide a natural interpretation within the Painlevé–Gullstrand river picture. On the CL branch the riverbed is longitudinally force-free: its speed is conserved while its direction changes continuously through the transverse transport generated by the ordered composition of the radial and azimuthal boosts. Establishing the exact equivalence of this BQ transport statement with the tensorial geodesic equation

$$\frac{Du^\mu}{D\tau} = 0$$

in the induced Painlevé–Gullstrand metric remains a separate calculation, and is left for future work.

4 The gravitational rapidity rotor at the Weyl level

The construction of the BQ–Weyl basis, including its explicit 4×4 matrix representation, Clifford relations, and multiplication tables, is collected in Appendix B. Here we use that algebra to lift the gravitational rapidity construction of the previous section from the Pauli level to a genuine representation of $\mathrm{Cl}(1,3)$.

Unlike the Pauli construction, whose Lorentz norm is recovered through the bilinear $A^T B$, the Weyl representation possesses the Clifford property directly under ordinary matrix multiplication,

$$\not{U}^2 = U^2\, \mathbb{1},$$

so that the rapidity rotor acts by the standard group-adjoint transformation

$$\not{U} \longrightarrow Q\, \not{U}\, Q^{-1},$$

the Weyl analogue of $S\gamma^\mu S^{-1} = \Lambda^\mu{}_\nu \gamma^\nu$.

The gravitational rotor describes the geometry of the vacuum flow rather than a massive matter field. We therefore employ the chiral (Weyl) representation of $\mathrm{Cl}(1,3)$, whose boost generators are

$$\hat{\alpha}_i = \hat{\beta}_0 \hat{\beta}_i,$$

while the associated rotation bivectors are

$$\hat{\Sigma}_i = \hat{\beta}_j \hat{\beta}_k,$$

with (i, j, k) cyclic.

All Weyl-level identities reported below have been verified independently by exact symbolic computation on the explicit 4×4 matrix representation summarized in Appendix B.

4.1 The gravitational rapidity rotor

For a single radial rapidity χ_r, the Weyl rotor is

$$Q_r = \exp\Big(\frac{\chi_r}{2}\hat{\alpha}_r\Big) = \cosh\frac{\chi_r}{2}\,\mathbb{1} + \sinh\frac{\chi_r}{2}\,\hat{\alpha}_r, \tag{67}$$

with Q_r^{-1} obtained by reversing the sign of the bivector term.

Applying the adjoint action to the rest-frame four-velocity $\Psi = c\hat{\beta}_0$ gives

$$\boxed{\Psi_g = Q_r(c\hat{\beta}_0)Q_r^{-1} = c\cosh\chi_r\,\hat{\beta}_0 + c\sinh\chi_r\,\hat{\beta}_r.} \tag{68}$$

This is the direct Weyl analogue of the Pauli-level four-velocity, with the bounded physical velocity recovered through $v_r = c\tanh\chi_r$.

For the galactic flow, the radial and azimuthal rapidities combine through the ordered rotor

$$Q_g = Q_\phi Q_r, \qquad Q_g^{-1} = Q_r^{-1}Q_\phi^{-1}, \tag{69}$$

where

$$Q_i = \cosh\frac{\chi_i}{2}\,\mathbb{1} + \sinh\frac{\chi_i}{2}\,\hat{\alpha}_i, \qquad i = r, \phi.$$

Since the boost generators do not commute,

$$\hat{\alpha}_r\hat{\alpha}_\phi = \hat{\Sigma}_\theta, \qquad \hat{\alpha}_\phi\hat{\alpha}_r = -\hat{\Sigma}_\theta, \tag{70}$$

their product is no longer a pure boost. Exact expansion gives

$$\begin{aligned} Q_g = {} & \cosh\frac{\chi_r}{2}\cosh\frac{\chi_\phi}{2}\,\mathbb{1} + \sinh\frac{\chi_r}{2}\cosh\frac{\chi_\phi}{2}\,\hat{\alpha}_r \\ & + \cosh\frac{\chi_r}{2}\sinh\frac{\chi_\phi}{2}\,\hat{\alpha}_\phi - \sinh\frac{\chi_r}{2}\sinh\frac{\chi_\phi}{2}\,\hat{\Sigma}_\theta. \end{aligned} \tag{71}$$

The final term is the Thomas–Wigner bivector generated automatically by the ordered composition of the radial and azimuthal boosts. It is therefore an intrinsic consequence of the non-Abelian structure of the gravitational rapidity rotor rather than an independently introduced rotational degree of freedom.

4.2 The Weyl gravitational four-velocity

Applying the compound rotor to the rest-frame four-velocity $\Psi = c\hat{\beta}_0$ gives

$$\Psi_g = Q_g(c\hat{\beta}_0)Q_g^{-1}, \tag{72}$$

which expands exactly on the sixteen-element Weyl basis as

$$\boxed{\Psi_g = c\Big(\cosh\chi_r\cosh\chi_\phi\,\hat{\beta}_0 + \sinh\chi_r\,\hat{\beta}_r + \cosh\chi_r\sinh\chi_\phi\,\hat{\beta}_\phi\Big).} \tag{73}$$

The transformed four-velocity therefore remains a pure Clifford vector: all bivector, pseudovector and pseudoscalar components vanish identically. This is the expected consequence of the group-adjoint action, which maps vectors into vectors under arbitrary Lorentz transformations.

The absence of an explicit Thomas–Wigner bivector in $\not{U}_g$ does not imply that the compound rotor itself is free of rotational content. On the contrary, the bivector $\hat{\Sigma}_\theta$ is an intrinsic component of the ordered rotor, Eq. (71), generated by the noncommutativity of the radial and azimuthal boosts. It simply does not belong to the transformed vector, whose Clifford grade is preserved by the adjoint action.

This distinction is central to the present construction. The rotor Q_g is the primary geometric object generated by the gravitational rapidity field, while the transformed four-velocity $\not{U}_g$ is only one of its projections. Differentiating the rotor itself produces the Maurer–Cartan connection,

$$Q_g \longrightarrow \Omega_\tau,$$

where the Thomas–Wigner bivector reappears explicitly as part of the connection. The rotational content is therefore not created by the Maurer–Cartan construction; it is already encoded in the compound rotor and is revealed by taking its differential. In this sense, the kinematic and Maurer–Cartan branches describe complementary projections of the same underlying Spin$(1,3)$ geometry rather than independent physical structures.

4.3 Closure of the kinematic branch at the Weyl level

The Weyl lift should preserve not only the gravitational four-velocity but the complete kinematic hierarchy

$$\chi_g \longrightarrow Q_g \longrightarrow \not{U}_g \longrightarrow \not{A}_g,$$

paralleling the Pauli-level construction $\chi_g \rightarrow G \rightarrow U_g \rightarrow A_g$. The lifted convective four-acceleration is therefore defined by

$$\boxed{\not{A}_g \equiv \frac{D\not{U}_g}{D\tau}.} \tag{74}$$

Since $\not{U}_g$ is a genuine Clifford vector, its material derivative is again a Clifford vector. Consequently,

$$\boxed{\not{A}_g = a_0\hat{\beta}_0 + a_r\hat{\beta}_r + a_\phi\hat{\beta}_\phi,} \tag{75}$$

with no bivector, pseudovector or pseudoscalar components generated by the lift. The coefficients are exactly those obtained in the Pauli-level construction,

$$a_\mu = \frac{Du_\mu}{D\tau}, \qquad \mu = 0, r, \phi,$$

so that the Weyl representation introduces no additional kinematic degrees of freedom beyond those already established in Section 2. In particular,

$$a_0 = c\frac{D\Gamma}{D\tau}, \qquad \Gamma = \cosh\chi_r \cosh\chi_\phi, \tag{76}$$

so the relativistic Constant-Lagrangian condition is carried over unchanged to the Weyl level.

The invariant norm of the lifted four-velocity,

$$\not{U}_g^2 = -c^2\not{1},$$

immediately yields

$$\boxed{\not{U}_g \cdot \not{A}_g = 0,} \tag{77}$$

the Weyl representation of the orthogonality relation already obtained at the Pauli level.

The Weyl lift therefore preserves the complete kinematic hierarchy

$$\chi_g \to Q_g \to \not{U}_g \to \not{A}_g,$$

together with all of its dynamical consequences, including the orthogonality relation, the decomposition into longitudinal and transverse acceleration, the identification $a_0 = cD\Gamma/D\tau$, and the Constant-Lagrangian condition $D\Gamma/D\tau = 0$. The lift introduces no new convective dynamics; it merely embeds the Pauli construction into a genuine representation of $\mathrm{Cl}(1,3)$.

5 The Maurer–Cartan connection along the flow

The previous section established that the complete kinematic branch

$$\chi_g \longrightarrow Q_g \longrightarrow \not{U}_g \longrightarrow \not{A}_g$$

lifts unchanged from the BQ–Pauli algebra to the Weyl representation of $\mathrm{Cl}(1,3)$. We now perform the corresponding lift of the Maurer–Cartan branch,

$$\chi_g \longrightarrow Q_g \longrightarrow \Omega_\tau \longrightarrow \not{U}_g\Omega_\tau,$$

which probes the differential geometry of the rotor itself rather than the material derivative of the transported four-velocity.

Using the same right Maurer–Cartan convention as at the Pauli level, the connection evaluated along the gravitational flow is

$$\boxed{\Omega_\tau = \left(\frac{DQ_g}{D\tau}\right) Q_g^{-1}.} \tag{78}$$

Since $Q_g \in \mathrm{Spin}(1,3)$, Ω_μ is Lie-algebra valued and therefore occupies only the bivector sector of the Weyl algebra. All identities below have been verified independently by exact symbolic matrix computation on the explicit representation of Appendix B.

5.1 The flow-evaluated Weyl connection

For the ordered galactic rotor

$$Q_g = Q_\phi Q_r,$$

the definition of the right Maurer–Cartan connection, Eq. (78), together with the product rule gives

$$\boxed{\Omega_\tau = \left(D_\tau Q_\phi\right) Q_\phi^{-1} + Q_\phi \left[\left(D_\tau Q_r\right) Q_r^{-1}\right] Q_\phi^{-1}.} \tag{79}$$

Since each single-axis rotor is generated by its corresponding boost bivector,

$$\left(D_\tau Q_\phi\right) Q_\phi^{-1} = \frac{1}{2}\frac{D\chi_\phi}{D\tau}\,\hat{\alpha}_\phi, \qquad \left(D_\tau Q_r\right) Q_r^{-1} = \frac{1}{2}\frac{D\chi_r}{D\tau}\,\hat{\alpha}_r, \tag{80}$$

the product rule separates the explicit evolution of the two boost rotors from the non-Abelian contribution generated by transporting the radial boost generator through the azimuthal rotor. Direct evaluation gives

$$\boxed{Q_\phi \hat{\alpha}_r Q_\phi^{-1} = \cosh\chi_\phi\,\hat{\alpha}_r - \sinh\chi_\phi\,\hat{\Sigma}_\theta.} \tag{81}$$

Substituting this result into Eq. (79) yields the exact flow-evaluated Maurer–Cartan connection

$$\boxed{\Omega_\tau = \frac{1}{2}\frac{D\chi_\phi}{D\tau}\,\hat{\alpha}_\phi + \frac{1}{2}\cosh\chi_\phi\frac{D\chi_r}{D\tau}\,\hat{\alpha}_r - \frac{1}{2}\sinh\chi_\phi\frac{D\chi_r}{D\tau}\,\hat{\Sigma}_\theta.} \tag{82}$$

Equation (82) is identical in structure to the Pauli-level Maurer–Cartan connection, Eq. (48), differing only in the Clifford basis used to represent the connection.

The Weyl connection therefore occupies precisely the three bivector channels

$$\{\hat{\alpha}_r, \hat{\alpha}_\phi, \hat{\Sigma}_\theta\},$$

corresponding respectively to the radial boost, the azimuthal boost, and the Thomas–Wigner rotation.

Introducing the channel coefficients through

$$\Omega_\tau = \omega_r\,\hat{\alpha}_r + \omega_\phi\,\hat{\alpha}_\phi + \omega_\theta\,\hat{\Sigma}_\theta, \tag{83}$$

one obtains

$$\boxed{\omega_r = \frac{1}{2}\gamma_\phi \frac{D\chi_r}{D\tau}, \qquad \omega_\phi = \frac{1}{2}\frac{D\chi_\phi}{D\tau}, \qquad \omega_\theta = -\frac{1}{2}\gamma_\phi \frac{v_\phi}{c}\frac{D\chi_r}{D\tau},} \tag{84}$$

using

$$\gamma_i = \cosh \chi_i, \qquad \tanh \chi_i = \frac{v_i}{c}.$$

The boost and rotation sectors are therefore

$$\boldsymbol{\omega}_{\hat{\alpha}} = \omega_r\,\hat{\alpha}_r + \omega_\phi\,\hat{\alpha}_\phi, \qquad \boldsymbol{\omega}_{\hat{\Sigma}} = \omega_\theta\,\hat{\Sigma}_\theta.$$

The first two channels describe the proper-time evolution of the radial and azimuthal boosts, while the third is the Thomas–Wigner rotational channel generated by their ordered, non-commuting composition.

Thus the Weyl Maurer–Cartan connection is the exact Clifford-level lift of Eq. (48). The rotor, its differential, and the Thomas–Wigner contribution retain exactly the same geometric structure under the correspondence

$$(\hat{\sigma}_r, \hat{\sigma}_\phi, \hat{\mathrm{K}}_\theta) \longleftrightarrow (\hat{\alpha}_r, \hat{\alpha}_\phi, \hat{\Sigma}_\theta),$$

showing that the Maurer–Cartan branch is representation independent.

5.2 The flow-weighted connection $\not{U}_g\Omega_\tau$

The transport product of the gravitational four-velocity with the Maurer–Cartan connection is

$$\boxed{\not{U}_g\Omega_\tau = \left(u_0\,\hat{\beta}_0 + u_r\,\hat{\beta}_r + u_\phi\,\hat{\beta}_\phi\right)\left(\omega_r\,\hat{\alpha}_r + \omega_\phi\,\hat{\alpha}_\phi + \omega_\theta\,\hat{\Sigma}_\theta\right),} \tag{85}$$

with the explicit

$$u_0 = c\Gamma = c\cosh\chi_r\cosh\chi_\phi, \qquad u_r = c\sinh\chi_r, \qquad u_\phi = c\gamma_r\sinh\chi_\phi,$$

and

$$\omega_r = \frac{1}{2}\gamma_\phi \frac{D\chi_r}{D\tau}, \qquad \omega_\phi = \frac{1}{2}\frac{D\chi_\phi}{D\tau}, \qquad \omega_\theta = -\frac{1}{2}\gamma_\phi \frac{v_\phi}{c}\frac{D\chi_r}{D\tau}.$$

Applying the multiplication table (Table 3 of Appendix B) shows that every product of a vector basis element $\hat{\beta}_\mu$ with a boost bivector $\hat{\alpha}_i$ or rotation bivector $\hat{\Sigma}_i$ again belongs to the vector β_μ or pseudovector $\tilde{\beta}_\mu$ sectors.

Applying the multiplication table (Table 3) and collecting like basis elements gives the nonvanishing channels

$$\not{U}_g\Omega_\tau = -\left(u_r\omega_r + u_\phi\omega_\phi\right)\beta_0, \tag{86}$$

$$-\frac{c\cosh\chi_r}{2}\frac{D\chi_r}{D\tau}\beta_r, \tag{87}$$

$$-\left(u_0\omega_\phi+\frac{\sinh\chi_\phi}{2}u_r\frac{D\chi_r}{D\tau}\right)\beta_\phi, \tag{88}$$

$$+\frac{c\sinh\chi_r}{2}\frac{D\chi_\phi}{D\tau}\tilde{\beta}_\theta. \tag{89}$$

All remaining Weyl channels vanish identically. Two channels separate completely: β_r is driven only by $D\chi_r/D\tau$, while $\tilde{\beta}_\theta$ is driven only by $D\chi_\phi/D\tau$. The β_0 and β_ϕ channels retain the coupling between the two rapidities.

Apart from the overall sign convention of the vector channels imposed by the Weyl multiplication table, the coefficient structure is identical to the Pauli decomposition,

$$(\hat{\mathrm{T}},\hat{\mathrm{I}}_r,\hat{\mathrm{J}}_\phi,\hat{\sigma}_\theta)\longleftrightarrow(\hat{\beta}_0,\hat{\beta}_r,\hat{\beta}_\phi,\hat{\tilde{\beta}}_\theta).$$

The Weyl lift therefore preserves the complete transport-channel structure of the Maurer–Cartan product, replacing only the Clifford basis in which it is represented.

5.3 Recovery of the Constant-Lagrangian condition

For the compound rapidity invariant

$$\Gamma=\cosh\chi_r\cosh\chi_\phi,$$

the material derivative is

$$\frac{D\Gamma}{D\tau}=\sinh\chi_r\cosh\chi_\phi\frac{D\chi_r}{D\tau}+\cosh\chi_r\sinh\chi_\phi\frac{D\chi_\phi}{D\tau}. \tag{90}$$

Substituting the Weyl four-velocity, Eq. (73), and the connection coefficients, Eq. (84), into the timelike channel, Eq. (86), gives

$$\boxed{\left(\not{\Psi}_g\Omega_\tau\right)_{\beta_0}=-\frac{c}{2}\frac{D\Gamma}{D\tau}.} \tag{91}$$

Consequently,

$$\boxed{\left(\not{\Psi}_g\Omega_\tau\right)_{\beta_0}=0\qquad\Longleftrightarrow\qquad\frac{D\Gamma}{D\tau}=0.} \tag{92}$$

This is precisely the Weyl counterpart of the Pauli result

$$[U_g\Omega_\tau]_{\hat{\mathrm{T}}}=\frac{c}{2}\frac{D\Gamma}{D\tau},$$

the difference being only the overall sign convention inherited from the Weyl multiplication table. The correspondence

$$(\hat{\mathrm{T}},\hat{\mathrm{I}}_r,\hat{\mathrm{J}}_\phi,\hat{\sigma}_\theta)\longleftrightarrow(\hat{\beta}_0,\hat{\beta}_r,\hat{\beta}_\phi,\hat{\tilde{\beta}}_\theta)$$

therefore extends from the complete channel decomposition to the Constant-Lagrangian closure itself.

The timelike channel of the flow-weighted Maurer–Cartan connection again isolates the material derivative of the compound rapidity. Consequently, the relativistic Constant-Lagrangian condition is recovered exactly from the Weyl transport product, demonstrating that the Maurer–Cartan characterization of

$$\frac{D\Gamma}{D\tau} = 0$$

is independent of the chosen Clifford representation. The Weyl lift therefore preserves not only the gravitational rapidity rotor and its associated connection, but also the physical closure law governing the stationary galactic flow.

5.4 Orthogonality and the spatial content of the CL branch

The flow-weighted Maurer–Cartan product satisfies the same orthogonality identity as at the Pauli level. More generally, for an arbitrary Clifford vector U and a pure bivector X, the vector part of the product UX is Lorentz-orthogonal to U. Since the Maurer–Cartan connection Ω_τ is a pure bivector,

$$\boxed{U_g \cdot (\Psi_g \Omega_\tau)_{\text{vector}} = 0,} \tag{93}$$

holds identically, independently of the Constant-Lagrangian condition.

Writing this orthogonality relation in components gives

$$u_0(\Psi_g\Omega_\tau)_{\beta_0} - u_r(\Psi_g\Omega_\tau)_{\beta_r} - u_\phi(\Psi_g\Omega_\tau)_{\beta_\phi} = 0. \tag{94}$$

Using

$$u_0 = c\Gamma, \qquad (\Psi_g\Omega_\tau)_{\beta_0} = -\frac{c}{2}\frac{D\Gamma}{D\tau},$$

this becomes

$$\boxed{u_r(\Psi_g\Omega_\tau)_{\beta_r} + u_\phi(\Psi_g\Omega_\tau)_{\beta_\phi} = -\frac{c^2\Gamma}{2}\frac{D\Gamma}{D\tau}.} \tag{95}$$

Equation (95) is simply the component form of the Lorentz-orthogonality relation, Eq. (93). It is therefore an unconditional structural identity, independent of any dynamical condition on the flow.

On the Constant-Lagrangian branch,

$$\frac{D\Gamma}{D\tau} = 0,$$

Eq. (95) immediately reduces to

$$\boxed{u_r(\Psi_g\Omega_\tau)_{\beta_r} + u_\phi(\Psi_g\Omega_\tau)_{\beta_\phi} = 0.} \tag{96}$$

Thus the vanishing of the timelike channel simultaneously removes the component of the spatial transport parallel to the gravitational flow. Any remaining spatial connection therefore acts purely transversely to the flow.

This is the direct Weyl counterpart of the Pauli-level result of Eqn. (64):

$$\frac{D\Gamma}{D\tau} = 0 \qquad \Longrightarrow \qquad \mathbf{u} \cdot \mathbf{e}_K = 0,$$

where the CL condition likewise removes the component of the $\hat{\mathbf{K}}$ channel parallel to the spatial flow. Under the correspondence

$$(\hat{\mathrm{T}}, \hat{\mathrm{I}}_r, \hat{\mathrm{J}}_\phi, \hat{\sigma}_\theta) \longleftrightarrow (\hat{\beta}_0, \hat{\beta}_r, \hat{\beta}_\phi, \tilde{\hat{\beta}}_\theta),$$

the Pauli spatial channel $\mathbf{e}_K$ becomes the Weyl vector $\{\beta_r, \beta_\phi\}$ sector, while the separate Pauli $\hat{\sigma}_\theta$ channel is represented by the Weyl pseudovector channel $\tilde{\hat{\beta}}_\theta$. In both representations the Constant-Lagrangian condition suppresses only the flow-parallel part of the transport, leaving the transverse connection unchanged.

Physically, this surviving transverse connection continuously redirects the riverbed without changing its total CL speed. The kinematic branch therefore fixes the magnitude of the compound flow through $D\Gamma/D\tau = 0$, whereas the Maurer–Cartan branch determines how that constant-speed flow is continuously reoriented. Together they describe the defining kinematics of a stationary spiral galaxy: constant flow speed combined with continuous change of direction.

6 Spacetime Algebra formulation of the gravitational rapidity field

The purpose of this section is not to derive new dynamics, but to show that the complete gravitational-rapidity construction admits an equivalent formulation in the standard language of Spacetime Algebra (STA). Every physical result established at the BQ and Weyl levels is preserved; only the algebraic representation changes. In particular, the two complementary branches generated by the gravitational rapidity field—the kinematic branch and the Maurer–Cartan branch—remain present and again converge on the same relativistic Constant-Lagrangian (CL) condition.

6.1 The gravitational rapidity rotor

Let

$$R(x) \in \mathrm{Spin}^+(1,3), \qquad R\widetilde{R} = 1,$$

be the local Lorentz rotor, where $\widetilde{R}$ denotes the reverse. The gravitational rapidity field is represented by the boost rotor

$$R = \exp\left(-\frac{1}{2}\chi_g B\right),$$

with $B^2 = 1$. For the galactic construction the ordered rotor is

$$R = R_\phi R_r,$$

where

$$R_r = \exp\left(-\frac{1}{2}\chi_r\, \gamma_r\gamma_0\right), \qquad R_\phi = \exp\left(-\frac{1}{2}\chi_\phi\, \gamma_\phi\gamma_0\right).$$

As at the Pauli and Weyl levels, the rapidity field is the primary gravitational variable from which both the physical flow and the Maurer–Cartan connection are generated.

6.2 The kinematic branch

The STA counterpart of the Pauli hierarchy

$$\chi_g \to G \to U_g \to A_g$$

is

$$\chi_g \to R \to U_g \to A_g.$$

Choosing the fiducial observer

$$u_0 = c\gamma_0,$$

the gravitational four-velocity is obtained by rotor transport,

$$U_g = R u_0 \widetilde{R}.$$

For a single radial boost,

$$U_g = c\left(\cosh\chi_g\, \gamma_0 + \sinh\chi_g\, \gamma_r\right),$$

which satisfies

$$U_g^2 = c^2.$$

The proper derivative along the flow is

$$\frac{D}{D\tau} = U_g \cdot \nabla,$$

giving

$$A_g = \frac{DU_g}{D\tau}.$$

Differentiating the invariant norm immediately yields

$$U_g \cdot A_g = \frac{1}{2}\frac{D}{D\tau}(U_g^2) = 0,$$

the standard STA orthogonality relation between four-velocity and four-acceleration. As at the Pauli level, this is a purely kinematic consequence of the rotor construction.

Introducing the explicit sequential rotor

$$R = R_\phi R_r$$

gives

$$U_g = c\Gamma\,\gamma_0 + \gamma_r v_r\,\gamma_r + \gamma_r\gamma_\phi v_\phi\,\gamma_\phi,$$

where

$$\Gamma = \cosh\chi_r \cosh\chi_\phi.$$

The temporal component of the convective acceleration becomes

$$(A_g)_0 = c\frac{D\Gamma}{D\tau},$$

so that

$$(A_g)_0 = 0 \qquad \Longleftrightarrow \qquad \frac{D\Gamma}{D\tau} = 0.$$

Thus, for the gravitational rapidity field adopted here, the relativistic Constant-Lagrangian condition appears as the vanishing of the temporal component of the convective four-acceleration, exactly as at the BQ level.

6.3 The Maurer–Cartan branch

The same rotor generates the right Maurer–Cartan connection

$$\Omega_\mu = (\partial_\mu R)\widetilde{R},$$

a bivector field taking values in the Lorentz algebra. Projection along the flow gives the proper-time connection

$$\Omega_\tau = (U_g \cdot \nabla R)\widetilde{R}.$$

Since

$$\frac{DR}{D\tau} = \Omega_\tau R,$$

differentiating

$$U_g = Ru_0\widetilde{R}$$

gives the standard STA transport law

$$A_g = \Omega_\tau U_g - U_g\Omega_\tau = 2(\Omega_\tau \cdot U_g).$$

Up to this point the construction is completely general and applies to an arbitrary rotor field. Introducing the explicit sequential gravitational rotor yields

$$\Gamma = \cosh \chi_r \cosh \chi_\phi,$$

whose proper-time derivative is

$$\frac{D\Gamma}{D\tau} = \cosh \chi_\phi \sinh \chi_r \frac{D\chi_r}{D\tau} + \cosh \chi_r \sinh \chi_\phi \frac{D\chi_\phi}{D\tau}.$$

Accordingly,

$$\boxed{\frac{D\Gamma}{D\tau} = 0} \tag{97}$$

again emerges as the invariant transport condition associated with the Maurer–Cartan branch. As at the BQ and Weyl levels, this condition is strictly weaker than

$$\Omega_\tau = 0,$$

which would require each rapidity component to remain individually constant. Instead, the CL condition permits continuous exchange between the radial and azimuthal rapidities while preserving their combined Lorentz invariant.

6.4 Summary

The STA formulation preserves the same dual geometric structure established at the Pauli and Weyl levels,

$$\chi_g \longrightarrow R \longrightarrow \begin{cases} U_g \to A_g, \\ \Omega_\tau, \end{cases}$$

where the first branch describes the convective dynamics of the flow and the second the transport of the rotor itself.

After substitution of the explicit sequential gravitational rapidity field, both branches recover the same invariant transport law,

$$\boxed{\frac{D\Gamma}{D\tau} = 0, \qquad \Gamma = \cosh \chi_r \cosh \chi_\phi.}$$

The STA formulation therefore introduces no new physical content. It confirms that the gravitational rapidity field, together with its kinematic and Maurer–Cartan branches, is independent of the algebraic representation: the same Constant-Lagrangian condition is recovered in BQ algebra, in the Weyl representation, and in the standard language of Spacetime Algebra.

Unlike many STA formulations of gravity, which take the tetrad, metric, spin connection or even the rotor field itself as the fundamental variable, the present construction begins from a scalar gravitational rapidity potential. The rotor, the physical four-velocity, and the Maurer–Cartan connection then arise as successive geometric objects generated by this single field, while both the kinematic and Maurer–Cartan branches recover the same invariant transport law of Eqn. (97).

7 Dirac-matrix formulation of the gravitational rapidity dynamics

The purpose of this section is to reformulate the gravitational rapidity construction in the conventional language of the Dirac representation of $\mathrm{Spin}(1,3)$. The logical development parallels the BQ and STA formulations, but is expressed entirely in terms of the standard Dirac matrices and Lorentz spin transformations.

This section introduces no additional dynamics. Its purpose is to show that the gravitational rapidity construction developed in the main text can be written completely within the standard spinor formalism used in relativistic quantum theory.

As in the previous formulations, the gravitational rapidity field is taken as the primary variable. From this single field the spin transformation, the physical four-velocity, and the Maurer–Cartan connection are generated. The kinematic and connection branches then recover the same invariant transport law,

$$\frac{D\Gamma}{D\tau} = 0,$$

which constitutes the relativistic Constant-Lagrangian condition.

7.1 The gravitational spin transformation

Let

$$S(x) \in \mathrm{Spin}(1,3)$$

denote the local spin representation of the Lorentz group, satisfying

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}{}_{\nu}\,\gamma^{\nu},$$

where $\Lambda^{\mu}{}_{\nu}$ is the associated local Lorentz transformation.

The generators of the Lorentz algebra are

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}],$$

so that a general local Lorentz transformation is written as

$$S(x) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}(x)\sigma^{\mu\nu}\right),$$

with

$$\omega_{\mu\nu} = -\omega_{\nu\mu}.$$

Unlike the conventional interpretation, where the parameters $\omega_{\mu\nu}$ describe arbitrary local Lorentz transformations, the present construction identifies their boost components with the physical gravitational rapidity field,

$$\omega_{0r} = \chi_r, \qquad \omega_{0\phi} = \chi_\phi,$$

all remaining components vanishing.

The resulting spin transformation is therefore

$$S = S_\phi S_r,$$

with

$$S_r = \exp\left(-\frac{i}{4}\chi_r\sigma^{0r}\right), \qquad S_\phi = \exp\left(-\frac{i}{4}\chi_\phi\sigma^{0\phi}\right).$$

The spacetime-dependent rapidity field therefore acts as the primary gravitational variable from which both the kinematic flow and the Maurer–Cartan connection are generated.

7.2 The kinematic branch

Choose the fiducial four-velocity

$$u_0^\mu = (c, 0, 0, 0).$$

The physical four-velocity is obtained by Lorentz transformation,

$$U_g^\mu = \Lambda^\mu{}_\nu u_0^\nu,$$

or equivalently,

$$U_g = S(c\gamma^0)S^{-1}.$$

The proper derivative along the worldline is

$$\frac{D}{D\tau} = U_g^\mu \partial_\mu,$$

giving the convective four-acceleration

$$A_g^\mu = \frac{DU_g^\mu}{D\tau}.$$

Since

$$U_g^\mu U_{g\mu} = c^2,$$

differentiation immediately yields

$$U_{g\mu} A_g^{\mu} = 0.$$

This orthogonality is a purely kinematical consequence of Lorentz symmetry and therefore holds for every local Lorentz transformation, independently of the gravitational model.

To proceed further one specifies the explicit sequential gravitational rapidity field,

$$S = S_\phi S_r,$$

which produces the compound Lorentz factor

$$\Gamma = \cosh \chi_r \cosh \chi_\phi.$$

For the present gravitational rapidity field,

$$(A_g)^0 = c\frac{D\Gamma}{D\tau},$$

so that

$$(A_g)^0 = 0 \qquad \Longleftrightarrow \qquad \frac{D\Gamma}{D\tau} = 0.$$

The Constant-Lagrangian condition is therefore recovered as the vanishing of the temporal component of the convective four-acceleration.

7.3 The Maurer–Cartan connection branch

Using the same right Maurer–Cartan convention as in the BQ and STA formulations, the spacetime connection is

$$\Omega = (\partial S)S^{-1},$$

whose spacetime components are

$$\Omega_\mu = (\partial_\mu S)S^{-1}.$$

Projection along the flow gives the proper-time connection

$$\Omega_\tau = (U_g^{\mu}\partial_\mu S)S^{-1}.$$

Since

$$\frac{DS}{D\tau} = \Omega_\tau S,$$

differentiating

$$U_g = S(c\gamma^0)S^{-1}$$

shows that the Maurer–Cartan connection acts as the generator of the transport of the four-velocity,

$$\frac{DU_g}{D\tau} = [\Omega_\tau, U_g],$$

which is the standard transport equation for Lorentz vectors in the Dirac representation.

Up to this point the construction is completely general and follows solely from the geometry of the Lorentz group.

Introducing the explicit sequential gravitational spin transformation again produces

$$\Gamma = \cosh\chi_r \cosh\chi_\phi,$$

whose proper-time derivative is

$$\frac{D\Gamma}{D\tau} = \cosh\chi_\phi \sinh\chi_r \frac{D\chi_r}{D\tau} + \cosh\chi_r \sinh\chi_\phi \frac{D\chi_\phi}{D\tau}.$$

Hence

$$\boxed{\frac{D\Gamma}{D\tau} = 0}$$

again appears as the invariant transport law generated by the Maurer–Cartan connection.

Unlike the stronger condition

$$\Omega_\tau = 0,$$

which would require every rapidity component to remain individually constant, the Constant-Lagrangian condition permits continuous exchange between the radial and azimuthal rapidities while preserving their combined Lorentz invariant.

7.4 Summary

Both dynamical branches originate from the same gravitational rapidity field,

$$\chi_g \longrightarrow S,$$

but proceed through different geometric constructions. The kinematic branch follows

$$\chi_g \longrightarrow S \longrightarrow U_g \longrightarrow A_g,$$

whereas the Maurer–Cartan branch follows

$$\chi_g \longrightarrow S \longrightarrow \Omega_\tau.$$

After substitution of the explicit sequential gravitational rapidity field, both branches converge on the same invariant transport law,

$$\boxed{\frac{D\Gamma}{D\tau} = 0, \qquad \Gamma = \cosh\chi_r \cosh\chi_\phi.}$$

Unlike conventional Dirac treatments, which usually begin from an arbitrary local Lorentz transformation or spin connection, the present construction starts from a scalar gravitational rapidity field. The spin transformation, the physical four-velocity, and the Maurer–Cartan connection are then generated successively from this single field, while both geometric branches recover the same relativistic Constant-Lagrangian condition.

8 Conclusion

Starting from the gravitational rapidity field χ_g, this work has constructed two complementary descriptions of galactic vacuum flow. The first follows the kinematic hierarchy

$$\chi_g \to G \to U_g \to A_g,$$

while the second follows the Maurer–Cartan hierarchy

$$\chi_g \to G \to \Omega_\tau \to U_g\Omega_\tau.$$

Although based on different geometric objects, both branches recover the same relativistic transport condition,

$$\boxed{\frac{D\Gamma}{D\tau} = 0, \qquad \Gamma = \cosh\chi_r \cosh\chi_\phi,}$$

whose weak-field limit reproduces the Constant-Lagrangian (CL) law for galactic rotation curves.

The construction has been developed in its native BQ formulation and shown to carry unchanged to the Weyl, Space-Time Algebra and conventional Dirac representations. The resulting agreement demonstrates that the isorapidity condition is not tied to a particular algebraic language but is a representation-independent consequence of the gravitational rapidity rotor.

The present work therefore provides a theoretical foundation for the empirical Constant-Lagrangian postulate by deriving it independently from both the convective dynamics and the Maurer–Cartan transport of the same underlying $\mathrm{Spin}(1,3)$ rotor field.

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A The BQ–Pauli algebra

A.1 Matrix representation and basic generators

Quaternions admit several 2×2 complex matrix realizations; the one adopted here is chosen for direct compatibility with physics. The space-time four-set is

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{K} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{98}$$

Comparing to the Pauli matrices $\boldsymbol{\sigma_P} = (\sigma_x, \sigma_y, \sigma_z)$ shows $\hat{I} = i\sigma_z$, $\hat{J} = i\sigma_y$, $\hat{K} = i\sigma_x$ – an axis ordering $(I, J, K) \leftrightarrow (z, y, x)$ chosen, rather than the conventional (x, y, z), because it is the ordering under which $\hat{K}_i = i\hat{\sigma}_i$ holds uniformly (footnote: this reordering is purely a labelling convention and carries no physical content). Equivalently, defining the Hermitian set $\hat{\sigma}_I = \sigma_z, \hat{\sigma}_J = \sigma_y, \hat{\sigma}_K = \sigma_x$, one has $\hat{\sigma}_i = -\hat{\mathrm{T}}\hat{K}_i$ with the central timelike element

$$\hat{\mathrm{T}} = i\hat{1}. \tag{99}$$

The pauliquat set $\hat{\sigma}_\mu = (\hat{1}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K)$ is thus not a starting point but the basis complementary to the physically motivated minquat set $\hat{K}_\mu = (\hat{\mathrm{T}}, \hat{I}, \hat{J}, \hat{K})$. (This representation is additionally the canonical complex-matrix form for Möbius transformations, simultaneously realizing Hamilton's quaternion algebra, $SU(2)$, and spinor transformations – the underlying reason this particular matrix ordering, rather than an arbitrary alternative, was selected.)

Consequences of $\hat{\boldsymbol{T}} = i\hat{\boldsymbol{1}}$.

Since $\hat{\mathrm{T}}$ is the unique non-scalar central element and $\hat{\mathrm{T}}^2 = -\hat{1}$, while $\hat{K}_i = \hat{\mathrm{T}}\hat{\sigma}_i$, the generator squares and Hermiticity split follow immediately:

$$\hat{\sigma}_i^2 = +\hat{1}, \qquad \hat{K}_i^2 = -\hat{1}, \qquad \hat{T}^\dagger = -\hat{T}, \quad \hat{\sigma}_i^\dagger = \hat{\sigma}_i, \quad \hat{K}_i^\dagger = -\hat{K}_i. \tag{100}$$

Consequently the two sectors generate different one-parameter exponentials,

$$e^{a\hat{\sigma}_i} = \cosh a + \hat{\sigma}_i \sinh a \qquad \text{(non-unitary, hyperbolic boost)}, \tag{101}$$

$$e^{a\hat{K}_i} = \cos a + \hat{K}_i \sin a \qquad \text{(unitary, circular rotation)}, \tag{102}$$

so that the distinction between Lorentz boosts and spatial rotations is a direct algebraic consequence of the generator properties, not an additional postulate.

A.2 The multiplication table of the BQ algebra

The BQ algebra is generated over $\mathbb{R}$ by the eight-element basis

$$\{\hat{1}, \hat{T}, \hat{I}, \hat{J}, \hat{K}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K\}, \tag{103}$$

split into the minquat 4-set $\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ and the pauliquat 4-set $\hat{\sigma}_\mu = (\hat{1}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K)$. The defining relations are

$$\hat{T}^2 = -\hat{1}, \qquad \hat{I}^2 = \hat{J}^2 = \hat{K}^2 = -\hat{1}, \qquad \hat{\sigma}_I^2 = \hat{\sigma}_J^2 = \hat{\sigma}_K^2 = +\hat{1}, \tag{104}$$

$$\hat{T}X = X\hat{T} \text{ for all } X \ (\hat{T} \text{ is central}), \qquad \hat{I} = \hat{T}\hat{\sigma}_I, \quad \hat{J} = \hat{T}\hat{\sigma}_J, \quad \hat{K} = \hat{T}\hat{\sigma}_K, \tag{105}$$

together with the cyclic rules

$$\boxed{\hat{I}\hat{J} = \hat{K}, \qquad \hat{J}\hat{K} = \hat{I}, \qquad \hat{K}\hat{I} = \hat{J},} \tag{106}$$

$$\boxed{\hat{\sigma}_I\hat{\sigma}_J = -\hat{K}, \qquad \hat{\sigma}_J\hat{\sigma}_K = -\hat{I}, \qquad \hat{\sigma}_K\hat{\sigma}_I = -\hat{J},} \tag{107}$$

the opposite cyclic sign between the two sets inherited from $\hat{T}^2 = -\hat{1}$. Every other product in the algebra is generated from these by associativity, which the algebra inherits automatically from its matrix realization in $M_2(\mathbb{C})$ but which is not itself part of the defining data below: the multiplication table (Table 1) is the complete structure-constant data of the algebra, stated without reference to any ambient matrix ring.

Three structural remarks follow directly from Table 1 rather than from any external classification. First, the eight basis elements split into two square classes: $\{\hat{T}, \hat{I}, \hat{J}, \hat{K}\}$ each square to -1, while $\{\hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K\}$ each square to $+1$ – the elliptic/hyperbolic split that later produces circular versus hyperbolic one-parameter exponentials. Second, $\hat{T}$ is the unique non-scalar central element: it commutes with every other basis vector, which is what makes $\hat{T} = i\hat{1}$ a legitimate identification of $\hat{T}$ with the ordinary complex unit acting on the whole algebra, rather than a notational convenience. The channel formulas – the bilinear $A^T B$ and the triple product $E = DC$ – are direct corollaries of this table, obtainable by substituting $A = a_0\hat{T} + \mathbf{a} \cdot \hat{K}$, $B = b_0\hat{T} + \mathbf{b} \cdot \hat{K}$ and expanding term by term against the table; no property of the channel decomposition is assumed beyond what the table already fixes.

Third, the table is asymmetric under transposition. For example,

$$\hat{I}\hat{\sigma}_J = \hat{\sigma}_K, \qquad \hat{\sigma}_J\hat{I} = -\hat{\sigma}_K, \tag{108}$$

and hence

$$\hat{I}\hat{\sigma}_J = -\hat{\sigma}_J\hat{I}. \tag{109}$$

More generally, the three quaternion generators anticommute among themselves when their indices differ, as do the three Pauli generators. Mixed generators with different indices also anticommute. The corresponding same-axis pairs instead commute:

$$\hat{I}\hat{\sigma}_I = \hat{J}\hat{\sigma}_J = \hat{K}\hat{\sigma}_K = \hat{T}, \tag{110}$$

with the same products obtained in the reverse order.

Table 1 Full multiplication table of the BQ algebra. Entry (row, col) gives row · col; the algebra is non-commutative, so the table is read with row as the left factor. $\dim_{\mathbb{R}} = 8$; $\hat{T}$ is the unique central, non-identity generator, square -1. Every entry verified directly by 2×2 matrix multiplication against Eqs. (79)–(81).

$\cdot$	$\hat{1}$	$\hat{T}$	$\hat{I}$	$\hat{J}$	$\hat{K}$	$\hat{\sigma}_I$	$\hat{\sigma}_J$	$\hat{\sigma}_K$
$\hat{1}$	$\hat{1}$	$\hat{T}$	$\hat{I}$	$\hat{J}$	$\hat{K}$	$\hat{\sigma}_I$	$\hat{\sigma}_J$	$\hat{\sigma}_K$
$\hat{T}$	$\hat{T}$	$-\hat{1}$	$-\hat{\sigma}_I$	$-\hat{\sigma}_J$	$-\hat{\sigma}_K$	$\hat{I}$	$\hat{J}$	$\hat{K}$
$\hat{I}$	$\hat{I}$	$-\hat{\sigma}_I$	$-\hat{1}$	$\hat{K}$	$-\hat{J}$	$\hat{T}$	$\hat{\sigma}_K$	$-\hat{\sigma}_J$
$\hat{J}$	$\hat{J}$	$-\hat{\sigma}_J$	$-\hat{K}$	$-\hat{1}$	$\hat{I}$	$-\hat{\sigma}_K$	$\hat{T}$	$\hat{\sigma}_I$
$\hat{K}$	$\hat{K}$	$-\hat{\sigma}_K$	$\hat{J}$	$-\hat{I}$	$-\hat{1}$	$\hat{\sigma}_J$	$-\hat{\sigma}_I$	$\hat{T}$
$\hat{\sigma}_I$	$\hat{\sigma}_I$	$\hat{I}$	$\hat{T}$	$\hat{\sigma}_K$	$-\hat{\sigma}_J$	$\hat{1}$	$-\hat{K}$	$\hat{J}$
$\hat{\sigma}_J$	$\hat{\sigma}_J$	$\hat{J}$	$-\hat{\sigma}_K$	$\hat{T}$	$\hat{\sigma}_I$	$\hat{K}$	$\hat{1}$	$-\hat{I}$
$\hat{\sigma}_K$	$\hat{\sigma}_K$	$\hat{K}$	$\hat{\sigma}_J$	$-\hat{\sigma}_I$	$\hat{T}$	$-\hat{J}$	$\hat{I}$	$\hat{1}$

A.3 The transpose bilinear as time-reversal

The superscript T in $C = A^T B$ is not the ordinary matrix transpose (written A^t) but a discrete operation on the real coordinates of the four-vector $A = a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}$: time-reversal and parity are defined by

$$A^T \equiv -a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}, \qquad A^P \equiv a_0\hat{T} - \mathbf{a}\cdot\hat{\mathbf{K}}, \tag{111}$$

satisfying $A^P = -A^T$, $(A^T)^P = -A$, so that $\{\mathrm{id}, \mathsf{T}, \mathsf{P}, \mathsf{TP}\}$ forms the Klein four-group of the disconnected components of $O(1,3)$ relative to $SO^+(1,3)$. Expanding $A^T B$ against Table 1 for $A = a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}$, $B = b_0\hat{T} + \mathbf{b}\cdot\hat{\mathbf{K}}$ gives

$$A^T B = (a_0 b_0 - \mathbf{a}\cdot\mathbf{b})\hat{1} + (\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{K}} + (a_0\mathbf{b} - b_0\mathbf{a})\cdot\hat{\boldsymbol{\sigma}}, \tag{112}$$

so that the scalar channel $[A^T B]_{\hat{1}} = a_0 b_0 - \mathbf{a}\cdot\mathbf{b} = \eta_{\mu\nu}A^\mu B^\nu$ is exactly the Lorentzian metric contraction: time-reversal supplies the single relative sign that converts the Euclidean-signature product AB into the Lorentzian invariant, with $A^P B = -A^T B$ giving the opposite-signature pairing. The superscript A^T is never conjugation or multiplication by the basis element $\hat{T}$.

A.4 Why the minquat and pauliquat sets, not the closed quaternion subalgebra

Exactly one natural four-dimensional real span in the basis closes under multiplication: $\{\hat{1}, \hat{I}, \hat{J}, \hat{K}\} \simeq \mathbb{H}$, Hamilton's quaternion group. Its invariant norm $q^*q = r_0^2 + r_1^2 + r_2^2 + r_3^2$ is positive-definite and cannot become Lorentzian under any real change of coordinates. The minquat and pauliquat four-sets used throughout this paper, $\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ and $\hat{\sigma}_\mu = (\hat{1}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K)$, are useful precisely because they do *not* close: the product of two minquats necessarily populates both sets, $\mathcal{K} \times \mathcal{K} \subset \mathrm{span}_{\mathbb{R}}\{\hat{1}, \hat{I}, \hat{J}, \hat{K}, \hat{\sigma}_I, \hat{\sigma}_J, \hat{\sigma}_K\}$. This non-closure is the mechanism, not a defect: it is how a product of two spacetime four-vectors is forced to generate both the Lorentzian scalar invariant and the complementary rotation/boost channels.

A.5 Real coordinates as an explicit convention

Synge's minquat formulation keeps the quaternion basis real and makes the temporal coordinate complex, $R_0 = ib_0$. The present convention reverses this: all coordinates R_μ, P_μ are kept real, and the complex unit is instead carried by the central basis element $\hat{T} = i\hat{1}$ (equivalently $\hat{K}_i = i\hat{\sigma}_i$), so that $Z(\mathrm{BQ}) = \mathrm{span}_{\mathbb{R}}\{\hat{1}, \hat{T}\} \simeq \mathbb{C}$. This is a choice of representation, not a result forced uniquely by the abstract algebra; it is adopted because the coefficients of physical spacetime, momentum, and fluid four-vectors are then real measured quantities throughout, while the Lorentzian and complex structure is carried entirely by the basis itself.

Summary

The BQ algebra is obtained by internalizing the complex unit as the central generator $\hat{\mathrm{T}} = i\hat{1}$, thereby passing from Hamilton's quaternion algebra to its biquaternion extension while keeping all physical coefficients real. The chosen Möbius-compatible matrix realization determines the multiplication table, and the multiplication table in turn generates all bilinear, triple and differential BQ products used throughout this paper. The appendix therefore provides the complete algorithmic foundation required for the subsequent physical developments.

B The sixteen-element BQ–Weyl Clifford basis

The BQ–Weyl algebra used here is not introduced independently of the BQ–Pauli algebra summarized in Appendix A. It is obtained by placing the BQ minquat four-set

$$\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K}) \tag{113}$$

into a parity-duplex block construction. The internal entries remain elements of the same eight-dimensional BQ–Pauli algebra, while the additional block structure generates the sixteen-dimensional Clifford grade hierarchy.

The Weyl tetrad is

$$\beta_0 = \begin{pmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{pmatrix}, \qquad \beta_i = \begin{pmatrix} 0 & \hat{K}_i \\ -\hat{K}_i & 0 \end{pmatrix}, \qquad i = I, J, K. \tag{114}$$

Thus the Clifford vector sector is the direct block lift of the BQ minquat set. Using

$$\hat{\mathrm{T}} = i\hat{1}, \qquad \hat{K}_i = \hat{T}\hat{\sigma}_i, \qquad -\hat{T}\hat{K}_i = \hat{\sigma}_i, \tag{115}$$

the products of the tetrad recover the two complementary BQ generator sectors:

$$\alpha_i := \beta_0\beta_i = \begin{pmatrix} \hat{\sigma}_i & 0 \\ 0 & -\hat{\sigma}_i \end{pmatrix}, \tag{116}$$

and, for cyclic (i, j, k),

$$\Sigma_i := \beta_j\beta_k = \begin{pmatrix} -\hat{K}_i & 0 \\ 0 & -\hat{K}_i \end{pmatrix}. \tag{117}$$

The boost bivectors α_i are therefore the Weyl-level realization of the BQ Pauli generators $\hat{\sigma}_i$, while the spin bivectors Σ_i are the corresponding realization of the quaternion rotation generators $\hat{K}_i$.

The passage from BQ–Pauli to BQ–Weyl can consequently be summarized as

$$\underbrace{1+3+3+1}_{\text{BQ–Pauli channels}} \quad \longrightarrow \quad \underbrace{1+4+6+4+1}_{\text{Weyl–Clifford grades}}. \tag{118}$$

The block doubling does not replace the BQ algebra; it uses BQ as its internal 2×2 coefficient algebra and adds the vector, pseudovector and pseudoscalar placements required for the full Clifford hierarchy. The complete sixteen-element basis is generated from the tetrad β_μ alone.

B.1 The sixteen-element basis by Clifford grade

The products generated from the tetrad β_μ close on sixteen real basis elements. These divide into the five Clifford grades

$$1 + 4 + 6 + 4 + 1,$$

with the six bivectors separating naturally into the three boost generators α_i and the three spin generators Σ_i already identified above. The complete basis is summarized in Table 2.

Table 2 The sixteen-element BQ–Weyl basis organized by Clifford grade.

Grade	Object	Count	Definition
0	scalar	1	$\mathbb{1}$
1	vector	4	$\beta_0,\ \beta_I,\ \beta_J,\ \beta_K$
2	boost bivector	3	$\alpha_i = \beta_0\beta_i$
2	spin bivector	3	$\Sigma_i = \beta_j\beta_k, \quad (i,j,k)$ cyclic
3	pseudovector	4	$\tilde{\beta}_\mu = \chi\beta_\mu, \quad \mu = 0, I, J, K$
4	pseudoscalar	1	$\chi = -\beta_0\beta_I\beta_J\beta_K$
$1 + 4 + 3 + 3 + 4 + 1 = 16.$			

Even and odd sectors.

The basis splits equally into an eight-dimensional even sector and an eight-dimensional odd sector,

$$\mathcal{C}_{\text{even}} = \operatorname{span}_{\mathbb{R}}\{\mathbb{1}, \alpha_I, \alpha_J, \alpha_K, \Sigma_I, \Sigma_J, \Sigma_K, \chi\}, \tag{119}$$

$$\mathcal{C}_{\text{odd}} = \operatorname{span}_{\mathbb{R}}\{\beta_0, \beta_I, \beta_J, \beta_K, \tilde{\beta}_0, \tilde{\beta}_I, \tilde{\beta}_J, \tilde{\beta}_K\}. \tag{120}$$

In the Weyl representation this grading coincides with the block structure: the even elements are block diagonal and the odd elements are block off diagonal.

The even sector is precisely the sector in which the BQ–Pauli $1 + 3 + 3 + 1$ structure reappears:

$$\hat{1} \longmapsto \mathbb{1}, \qquad \hat{\sigma}_i \longmapsto \alpha_i, \qquad \hat{K}_i \longmapsto -\Sigma_i, \qquad \hat{T} \longmapsto \chi. \tag{121}$$

The odd sector contains the four lifted minquat vectors β_μ and their pseudoscalar partners $\tilde{\beta}_\mu$.

Squares.

The vector and bivector generators satisfy

$$\beta_0^2 = -\mathbb{1}, \qquad \beta_I^2 = \beta_J^2 = \beta_K^2 = +\mathbb{1}, \tag{122}$$

$$\alpha_I^2 = \alpha_J^2 = \alpha_K^2 = +\mathbb{1}, \qquad \Sigma_I^2 = \Sigma_J^2 = \Sigma_K^2 = -\mathbb{1}. \tag{123}$$

For the pseudovector and pseudoscalar sectors,

$$\tilde{\beta}_0^2 = -\mathbb{1}, \qquad \tilde{\beta}_I^2 = \tilde{\beta}_J^2 = \tilde{\beta}_K^2 = +\mathbb{1}, \qquad \chi^2 = -\mathbb{1}. \tag{124}$$

The bivector square classes preserve the BQ–Pauli distinction: α_i are hyperbolic generators and Σ_i are circular generators.

Explicit BQ block forms.

For reference, the sixteen basis elements are

$$\mathbb{1} = \begin{bmatrix} \hat{1} & 0 \\ 0 & \hat{1} \end{bmatrix}, \qquad \beta_0 = \begin{bmatrix} 0 & \hat{T} \\ \hat{T} & 0 \end{bmatrix}, \tag{125}$$

$$\beta_I = \begin{bmatrix} 0 & \hat{I} \\ -\hat{I} & 0 \end{bmatrix}, \qquad \beta_J = \begin{bmatrix} 0 & \hat{J} \\ -\hat{J} & 0 \end{bmatrix}, \qquad \beta_K = \begin{bmatrix} 0 & \hat{K} \\ -\hat{K} & 0 \end{bmatrix}, \tag{126}$$

$$\alpha_i = \begin{bmatrix} \hat{\sigma}_i & 0 \\ 0 & -\hat{\sigma}_i \end{bmatrix}, \qquad \Sigma_i = \begin{bmatrix} -\hat{K}_i & 0 \\ 0 & -\hat{K}_i \end{bmatrix}, \tag{127}$$

$$\tilde{\beta}_0 = \begin{bmatrix} 0 & -\hat{1} \\ \hat{1} & 0 \end{bmatrix}, \qquad \tilde{\beta}_i = \begin{bmatrix} 0 & -\hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{bmatrix}, \tag{128}$$

$$\chi = \begin{bmatrix} \hat{T} & 0 \\ 0 & -\hat{T} \end{bmatrix}. \tag{129}$$

B.2 The complete multiplication table

Table 3 gives the ordered product XY for every pair of basis elements, with the row element acting from the left and the column element from the right. Every entry was verified by direct 4×4 matrix multiplication.

With real coefficients, the sixteen matrices span the sixteen-dimensional real Clifford algebra represented inside $M_4(\mathbb{C})$. They do not span all of $M_4(\mathbb{C})$ as a real vector space, whose real dimension is 32. The multiplication table is nevertheless exhaustive for the real BQ–Weyl algebra, since every product of two basis elements closes within the same sixteen-element set.

Structural remarks.

The tetrad satisfies

$$\beta_\mu\beta_\nu + \beta_\nu\beta_\mu = 2\eta_{\mu\nu}\mathbb{1}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1), \tag{130}$$

and hence

$$\beta_\mu\beta_\nu = -\beta_\nu\beta_\mu, \qquad \mu \neq \nu. \tag{131}$$

The pseudoscalar anticommutes with the odd sector,

$$\chi\beta_\mu = -\beta_\mu\chi, \qquad \chi\tilde{\beta}_\mu = -\tilde{\beta}_\mu\chi, \tag{132}$$

and commutes with the even bivector sector,

$$\chi\alpha_i = \alpha_i\chi, \qquad \chi\Sigma_i = \Sigma_i\chi. \tag{133}$$

The multiplication rules therefore respect the parity grading,

$$C_{\text{even}}C_{\text{even}} \subset C_{\text{even}}, \qquad C_{\text{even}}C_{\text{odd}} \subset C_{\text{odd}}, \tag{134}$$

$$C_{\text{odd}}C_{\text{even}} \subset C_{\text{odd}}, \qquad C_{\text{odd}}C_{\text{odd}} \subset C_{\text{even}}. \tag{135}$$

Thus the full table displays the 1 + 4 + 6 + 4 + 1 Clifford hierarchy, while its even block-diagonal sector preserves the original 1 + 3 + 3 + 1 BQ–Pauli structure.

B.3 Why the BQ construction is carried out in the Weyl representation

The BQ construction presented in this appendix is developed in the Weyl representation because it preserves the algebraic structure inherited from the BQ–Pauli algebra. The Weyl tetrad

$$\beta_\mu = (\beta_0, \beta_I, \beta_J, \beta_K) \tag{136}$$

is obtained directly by parity-duplex lifting of the BQ minquat basis

$$\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K}), \tag{137}$$

so that every Clifford generator follows algorithmically from products of the tetrad.

The resulting hierarchy

$$1 + 3 + 3 + 1 \quad \longrightarrow \quad 1 + 4 + 6 + 4 + 1 \tag{138}$$

is therefore completely transparent: the BQ–Pauli algebra forms the internal even structure, while the block doubling generates the odd vector and pseudovector sectors required for the full Clifford algebra. The even and odd sectors remain explicitly separated throughout the construction.

This feature is illustrated by the Weyl momentum operator,

$$\not{P}_W = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix}, \tag{139}$$

which consists simply of the parity-duplex pairing of the original BQ four-vector P. The complete Clifford basis is then generated by successive products of the four matrices β_μ.

By contrast, the Dirac representation mixes the parity and time-reversal duplexes into the block structure,

$$\not{P}_D = \begin{bmatrix} p_0\hat{T} & \mathbf{p}\cdot\hat{\mathbf{K}} \\ -\mathbf{p}\cdot\hat{\mathbf{K}} & -p_0\hat{T} \end{bmatrix}, \tag{140}$$

so that the direct correspondence between the block entries and the original BQ generators is no longer manifest. The Dirac representation is therefore less transparent from the standpoint of the algebraic construction, even though it is often the more convenient representation for the description of massive fermions.

The two representations are mathematically equivalent, being related by a similarity transformation, and therefore generate the same Clifford algebra. Their difference lies only in the organization of the basis. The Weyl representation preserves the BQ hierarchy and exposes the algorithmic generation of the Clifford grades, whereas the Dirac representation rearranges the same algebra into a form optimized for the standard Dirac equation. Thus, the Weyl representation is construction-oriented, whereas the Dirac representation is application-oriented.

For this reason the algebraic development of the present work is carried out almost entirely in the Weyl representation. Once the Clifford basis and its multiplication properties have been established, one may pass to the Dirac representation whenever the physical application makes that representation more convenient.

Table 3 Complete multiplication table for the sixteen-element Weyl-representation basis. Read as row $\times$ column.

$\times$	$\mathbb{1}$	β_0	β_I	β_J	β_K	α_I	α_J	α_K	Σ_I	Σ_J	Σ_K	$\tilde{\beta}_0$	$\tilde{\beta}_I$	$\tilde{\beta}_J$	$\tilde{\beta}_K$	χ
$\mathbb{1}$	$\mathbb{1}$	β_0	β_I	β_J	β_K	α_I	α_J	α_K	Σ_I	Σ_J	Σ_K	$\tilde{\beta}_0$	$\tilde{\beta}_I$	$\tilde{\beta}_J$	$\tilde{\beta}_K$	χ
β_0	β_0	$-\mathbb{1}$	α_I	α_J	α_K	$-\beta_I$	$-\beta_J$	$-\beta_K$	$-\tilde{\beta}_I$	$-\tilde{\beta}_J$	$-\tilde{\beta}_K$	χ	Σ_I	Σ_J	Σ_K	$-\tilde{\beta}_0$
β_I	β_I	$-\alpha_I$	$\mathbb{1}$	Σ_K	$-\Sigma_J$	$-\beta_0$	$\tilde{\beta}_K$	$-\tilde{\beta}_J$	$-\tilde{\beta}_0$	$-\beta_K$	β_J	$-\Sigma_I$	$-\chi$	$-\alpha_K$	α_J	$-\tilde{\beta}_I$
β_J	β_J	$-\alpha_J$	$-\Sigma_K$	$\mathbb{1}$	Σ_I	$-\tilde{\beta}_K$	$-\beta_0$	$\tilde{\beta}_I$	β_K	$-\tilde{\beta}_0$	$-\beta_I$	$-\Sigma_J$	α_K	$-\chi$	$-\alpha_I$	$-\tilde{\beta}_J$
β_K	β_K	$-\alpha_K$	Σ_J	$-\Sigma_I$	$\mathbb{1}$	$\tilde{\beta}_J$	$-\tilde{\beta}_I$	$-\beta_0$	$-\beta_J$	β_I	$-\tilde{\beta}_0$	$-\Sigma_K$	$-\alpha_J$	α_I	$-\chi$	$-\tilde{\beta}_K$
α_I	α_I	β_I	β_0	$-\tilde{\beta}_K$	$\tilde{\beta}_J$	$\mathbb{1}$	Σ_K	$-\Sigma_J$	$-\chi$	$-\alpha_K$	α_J	$\tilde{\beta}_I$	$\tilde{\beta}_0$	β_K	$-\beta_J$	$-\Sigma_I$
α_J	α_J	β_J	$\tilde{\beta}_K$	β_0	$-\tilde{\beta}_I$	$-\Sigma_K$	$\mathbb{1}$	Σ_I	α_K	$-\chi$	$-\alpha_I$	$\tilde{\beta}_J$	$-\beta_K$	$\tilde{\beta}_0$	β_I	$-\Sigma_J$
α_K	α_K	β_K	$-\tilde{\beta}_J$	$\tilde{\beta}_I$	β_0	Σ_J	$-\Sigma_I$	$\mathbb{1}$	$-\alpha_J$	α_I	$-\chi$	$\tilde{\beta}_K$	β_J	$-\beta_I$	$\tilde{\beta}_0$	$-\Sigma_K$
Σ_I	Σ_I	$-\tilde{\beta}_I$	$-\tilde{\beta}_0$	$-\beta_K$	β_J	$-\chi$	$-\alpha_K$	α_J	$-\mathbb{1}$	$-\Sigma_K$	Σ_J	β_I	β_0	$-\tilde{\beta}_K$	$\tilde{\beta}_J$	α_I
Σ_J	Σ_J	$-\tilde{\beta}_J$	β_K	$-\tilde{\beta}_0$	$-\beta_I$	α_K	$-\chi$	$-\alpha_I$	Σ_K	$-\mathbb{1}$	$-\Sigma_I$	β_J	$\tilde{\beta}_K$	β_0	$-\tilde{\beta}_I$	α_J
Σ_K	Σ_K	$-\tilde{\beta}_K$	$-\beta_J$	β_I	$-\tilde{\beta}_0$	$-\alpha_J$	α_I	$-\chi$	$-\Sigma_J$	Σ_I	$-\mathbb{1}$	β_K	$-\tilde{\beta}_J$	$\tilde{\beta}_I$	β_0	α_K
$\tilde{\beta}_0$	$\tilde{\beta}_0$	$-\chi$	$-\Sigma_I$	$-\Sigma_J$	$-\Sigma_K$	$-\tilde{\beta}_I$	$-\tilde{\beta}_J$	$-\tilde{\beta}_K$	β_I	β_J	β_K	$-\mathbb{1}$	α_I	α_J	α_K	β_0
$\tilde{\beta}_I$	$\tilde{\beta}_I$	Σ_I	χ	α_K	$-\alpha_J$	$-\tilde{\beta}_0$	$-\beta_K$	β_J	β_0	$-\tilde{\beta}_K$	$\tilde{\beta}_J$	$-\alpha_I$	$\mathbb{1}$	Σ_K	$-\Sigma_J$	β_I
$\tilde{\beta}_J$	$\tilde{\beta}_J$	Σ_J	$-\alpha_K$	χ	α_I	β_K	$-\tilde{\beta}_0$	$-\beta_I$	$\tilde{\beta}_K$	β_0	$-\tilde{\beta}_I$	$-\alpha_J$	$-\Sigma_K$	$\mathbb{1}$	Σ_I	β_J
$\tilde{\beta}_K$	$\tilde{\beta}_K$	Σ_K	α_J	$-\alpha_I$	χ	$-\beta_J$	β_I	$-\tilde{\beta}_0$	$-\tilde{\beta}_J$	$\tilde{\beta}_I$	β_0	$-\alpha_K$	Σ_J	$-\Sigma_I$	$\mathbb{1}$	β_K
χ	χ	$\tilde{\beta}_0$	$\tilde{\beta}_I$	$\tilde{\beta}_J$	$\tilde{\beta}_K$	$-\Sigma_I$	$-\Sigma_J$	$-\Sigma_K$	α_I	α_J	α_K	$-\beta_0$	$-\beta_I$	$-\beta_J$	$-\beta_K$	$-\mathbb{1}$

C Relation between the BQ–Weyl basis and the standard Lorentz–Dirac algebra

The BQ formulation and the conventional Lorentz–Dirac formalism are two representations of the same Clifford algebra $\mathrm{Cl}(1,3)$. Throughout this work the algebra is developed in the Weyl representation, where the underlying BQ block structure remains explicit. Translation to the standard Dirac representation is obtained through the single identification

$$\boxed{\beta_\mu = i\gamma_\mu.} \tag{141}$$

From this relation all remaining correspondences follow directly.

C.1 Notation dictionary

Table 4 Correspondence between the BQ–Weyl basis and the standard Lorentz–Dirac basis.

BQ notation	Standard notation	Clifford grade
$\mathbb{1}$	1	scalar
β_μ	$i\gamma_\mu$	vector
$\alpha_i = \beta_0\beta_i$	$-\gamma_0\gamma_i$	boost bivector
$\Sigma_i = \beta_j\beta_k$	$-i\,\Sigma_i^{\mathrm{Dirac}}$	rotation bivector
$\widetilde{\beta}_\mu = \chi\beta_\mu$	$\gamma_{0123}\gamma_\mu$	pseudovector
$\chi = -\beta_0\beta_I\beta_J\beta_K = i\gamma^5$	γ_{0123}	pseudoscalar

C.2 Complete Clifford correspondence

The basis correspondence may be summarized as

$$\boxed{\left\{\mathbb{1},\, \beta_\mu,\, \alpha_i,\, \Sigma_i,\, \widetilde{\beta}_\mu,\, \chi\right\} \longleftrightarrow \left\{1,\, \gamma_\mu,\, \sigma_{\mu\nu},\, \gamma_\mu\gamma^5,\, \gamma^5\right\}.} \tag{142}$$

The two formalisms therefore possess the same sixteen-dimensional carrier algebra. Their difference lies not in the Clifford basis but in its subsequent use. The BQ formulation develops this basis through its hierarchy of bilinear, triple, time-reversed and differential products, while the conventional Lorentz–Dirac formalism typically works directly with the gamma-matrix representation.

C.3 Summary

The correspondence between the BQ and Lorentz–Dirac formulations is completely determined by the relation

$$\beta_\mu = i\gamma_\mu.$$

Once this identification is made, every BQ basis element has a unique Dirac counterpart. The novelty of the BQ programme therefore lies not in the underlying Clifford algebra itself, but in the algebraic hierarchy built upon it.

The difference between the two approaches can be formulated as follows. SM-Dirac's algebra grew by successive empirical patches, each forced by a new physical demand, with its geometric meaning read into it afterward. The BQ algebra instead starts from one fixed classical definition and simply gets interrogated further — the Lorentz structure, the invariant quadratic, and the Clifford closure all turn out to be consequences of that original choice rather than features added to produce them.

D Relation between the BQ basis and Space-Time Algebra

This appendix collects, in one place, the translation between the biquaternion/BQ notation used throughout this paper and the standard notation of Space-Time Algebra (STA) and its tetrad-GA extensions Hestenes (2003); Doran and Lasenby (2013); Perez and DeKieviet (2024), together with the one convention choice most likely to mislead a reader coming from that literature, and a brief clarification of where this programme's approach sits relative to the two main existing traditions for treating gravity with Clifford algebra.

Two BQ levels require separate dictionaries. At the Weyl/Dirac level the BQ basis coincides, dimension for dimension, with the standard sixteen-element basis of $\mathrm{Cl}_{1,3}$ used directly in STA. At the Pauli level the BQ basis coincides instead with the *even* subalgebra of $\mathrm{Cl}_{1,3}$ obtained through Hestenes' spacetime split – the eight-dimensional relative-space algebra, isomorphic to $\mathrm{Cl}_{3,0}$, that STA practitioners already use whenever they factor a spacetime vector through an observer's rest frame γ_0.

D.1 Weyl/Dirac level (sixteen-dimensional, $\mathrm{Cl}_{1,3}$)

Table 5 Correspondence between the BQ–Weyl basis and the standard STA basis (Perez and DeKieviet 2024, Table I).

This paper	Standard STA	Grade
$\mathbb{1}$	1	0 (scalar)
$\beta_0, \beta_r, \beta_\phi, \beta_\theta$	$\gamma_0, \gamma_1, \gamma_2, \gamma_3$	1 (vector)
α_i (e.g. $\alpha_r = \beta_0\beta_r$)	γ_{0i} (boost bivector)	2 (bivector)
Σ_i (e.g. $\Sigma_\theta = \beta_r\beta_\phi$)	γ_{ij} (spatial bivector)	2 (bivector)
$\widetilde{\beta}_\mu = \chi\beta_\mu$	$\gamma_{123}, \gamma_{230}, \gamma_{310}, \gamma_{120}$	3 (trivector)
$\chi = -\beta_0\beta_I\beta_J\beta_K$	$\gamma_{0123} = I$ (pseudoscalar)	4 (pseudoscalar)

As at the Dirac level (Appendix C), the correspondence is complete once the vector identification is fixed; every remaining grade is generated multiplicatively from it, and the full sixteen-dimensional carrier algebra is shared exactly.

D.2 Pauli level (eight-dimensional, relative-space algebra)

At the Pauli level the correspondence runs through Hestenes' spacetime split (Hestenes 2003): an observer frame γ_0 splits a spacetime bivector $\gamma_i\gamma_0$ into a *relative vector* $\boldsymbol{\sigma}_i$, and the resulting eight-dimensional even subalgebra $\{1, \boldsymbol{\sigma}_i, i\boldsymbol{\sigma}_i, i\}$ is exactly the relative-space (Pauli/APS) algebra used throughout the STA literature for three-dimensional physics.

Table 6 Correspondence between the BQ–Pauli basis and the relative-space algebra obtained from the spacetime split of STA (Hestenes 2003).

This paper	Standard STA (spacetime split)	APS grade	STA grade
$\hat{\mathrm{I}}$	1	0 (scalar)	0 (scalar)
$\hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}$	$\boldsymbol{\sigma}_i = \gamma_i\gamma_0$	1 (vector)	2 (bivector)
$\hat{\boldsymbol{\sigma}}_I, \hat{\boldsymbol{\sigma}}_J, \hat{\boldsymbol{\sigma}}_K$	$i\boldsymbol{\sigma}_i$	2 (bivector)	3 (trivector)
$\hat{\mathrm{T}}$	$i = \gamma_1\gamma_2\gamma_3\gamma_0$	3 (pseudoscalar)	4 (pseudoscalar)

The dual four-set $(\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$ used throughout the main text as the minquat basis κ_μ is therefore not a spacetime vector basis in the STA sense at all: three of its four elements are already bivectors of the full algebra, and the fourth, $\hat{\mathrm{T}}$, is the relative pseudoscalar i rather than a spacetime vector. This is exactly what the spacetime split predicts – an observer's own timelike direction γ_0 never itself appears inside the relative algebra it generates – and it is the reason the eight-dimensional BQ-Pauli level closes under multiplication without reference to γ_0, while the sixteen-dimensional BQ-Weyl level (Appendix C) does not.

D.3 The convention pitfall most likely to mislead an STA reader

The single point of genuine risk is the scalar i appearing in $\beta_\mu = i\gamma_\mu$ (Appendix C) and in $\hat{\mathrm{T}} \equiv i\,\hat{\imath}\hat{\jmath}\hat{k}$ above. In this paper i is the ordinary commuting complex unit, used to embed the real Clifford algebra $\mathrm{Cl}_{1,3}(\mathbb{R})$ into its complexification $M_4(\mathbb{C}) \cong \mathrm{Cl}_{1,3} \otimes \mathbb{C}$, exactly as the Weyl representation of the gamma matrices already requires. In STA, by contrast, no scalar i is ever introduced: the entire algebra is built over the reals, and the role elsewhere assigned to "i" is always played by a specific *geometric* element – the pseudoscalar $I = \gamma_0\gamma_1\gamma_2\gamma_3$ at the full spacetime level, or the relative pseudoscalar $i = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ at the Pauli level – which commutes with even-grade elements but anticommutes with odd-grade ones, unlike a genuine scalar. A reader trained on STA should therefore not read $\beta_\mu = i\gamma_\mu$ as an identification of i with I: the i here is the bookkeeping device that produces the Weyl representation from the real algebra, while I (or the relative i of Table 6) is an intrinsic, representation-independent geometric object of that same real algebra. The two coincide numerically once a matrix representation is chosen, but they are not the same kind of object, and conflating them is the standard trap this dictionary is meant to close off.

D.4 Where this programme sits relative to the two existing traditions

Two established traditions treat gravity with Clifford/geometric algebra. The first, tetrad-GA in the sense of Perez and DeKieviet (2024), works *metric-first*: the spacetime metric $g_{\mu\nu}$ is taken as given (typically as the solution of the Einstein equations for a specified source), the tetrad e^m_μ and spin connection ω_μ are constructed from it, and curvature is recovered downstream. This is a faithful re-expression of standard GR in GA language rather than an independent construction principle. The second, Gauge Theory Gravity in the sense of Doran and Lasenby (2013), works *gauge-first*: the position-gauge field $h(a)$ and rotation-gauge field $\omega(a)$ are postulated as fundamental dynamical fields on a flat background, an action is varied with respect to them, and both the rotor and the metric are obtained as solutions to the resulting sourced field equations.

The present construction shares its central object – a Lorentz rotor Q_g – with the gauge-first tradition, but not its logical order. Here the generating rapidity field χ_g is not obtained by varying an action against a matter source. The rotor, and everything built from it – four-velocity, Maurer–Cartan connection, and eventually the metric – is *rapidity-first*: closer in spirit to the gauge-first tradition's insistence that the rotor is primary and the metric secondary, but without that tradition's field equation doing the generating work. Readers from either established tradition should take this appendix's dictionaries as confirming that the carrier algebra is identical to their own in both cases; what differs, in both comparisons, is only what is taken as given and what is derived.

D.5 The core conceptual difference: synthetic versus analytic construction

Beyond the notational dictionaries above, one structural difference separates the two formalisms at their root, prior to any question of grades or bases.

Space-Time Algebra is *synthetic*: Hestenes begins from an abstract real vector space V equipped with a quadratic form, and defines the geometric product intrinsically, $\gamma_\mu\gamma_\nu = \gamma_\mu\cdot\gamma_\nu + \gamma_\mu\wedge\gamma_\nu$. The resulting sixteen-dimensional algebra, its grade structure, and its pseudoscalar $I = \gamma_0\gamma_1\gamma_2\gamma_3$ are theorems of this axiom, not properties of any particular matrix representation. A matrix realization can always be written down for computation, but it is logically downstream and interchangeable:

nothing in STA depends on which one is chosen, and no scalar i ever enters the definition of a vector.

The BQ construction proceeds in the opposite direction and is *analytic*. It begins not from an abstract space but from four explicit complex 2×2 matrices $\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}$, fixed as axioms by the single requirement that plain matrix multiplication reproduce the physics of Minkowski space-time and electromagnetism. Every subsequent structure in this paper – minquats, pauliquats, the spin-norm sector, the Weyl doubling into the Dirac level – is discovered by multiplying these matrices together and reading off what the product space turns out to contain, rather than derived in advance from a metric axiom. The eight-, and later sixteen-, dimensional closure is a fact *about this representation*, confirmed by direct computation (App. C), not a theorem guaranteed before a single matrix is written.

The two routes are shown, by the dictionaries above, to reach the same algebra. But they reach it from opposite directions: STA derives structure from a metric axiom and only afterward permits a representation; BQ fixes a representation as axiomatic and only afterward recovers the structure that STA assumes from the outset.

D.6 Summary

At both algebra levels, the BQ basis and the standard STA basis span the same carrier algebra: $\mathrm{Cl}_{1,3}$ at the Weyl/Dirac level, and its even (relative-space) subalgebra at the Pauli level. The correspondence is fixed completely by Table 5 together with the spacetime split of Table 6; the only notational hazard is the distinction between the complexifying scalar i and the geometric pseudoscalar I, clarified above. That the two routes converge is not a coincidence to be checked once and set aside: STA reaches this algebra *synthetically*, deriving its sixteen-fold grade structure from a metric axiom before any representation is chosen, while the BQ programme reaches it *analytically*, fixing the complex matrices $\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}}$ as axioms and discovering the same structure as the closure of their matrix products (Sec. D.5). As with the Standard Model Dirac-level dictionary, the novelty of the BQ programme therefore lies neither in the underlying algebra, shown here to be shared exactly with STA, nor in the direction from which that algebra is reached, but in the rapidity-first construction built upon it.

E General vector products in the BQ-Pauli representation

Let

$$A = a_0\hat{T} + \mathbf{a}\cdot\hat{\mathbf{K}}, \qquad B = b_0\hat{T} + \mathbf{b}\cdot\hat{\mathbf{K}}, \tag{143}$$

and similarly

$$D = d_0\hat{T} + \mathbf{d}\cdot\hat{\mathbf{K}}, \qquad F = f_0\hat{T} + \mathbf{f}\cdot\hat{\mathbf{K}}, \tag{144}$$

be arbitrary space-time minquats.

Unlike the sixteen-dimensional Weyl representation, the eight-dimensional BQ-Pauli algebra has no exterior grading to organise the product hierarchy; instead, successive products are organised by the minquat/ pauliquat split itself. A minquat carries only the $\{\hat{T}, \hat{\mathbf{K}}\}$ channels; the bilinear product of two minquats already populates the complementary $\{\hat{1}, \hat{\boldsymbol{\sigma}}\}$ channels together with $\hat{\mathbf{K}}$, and the triple product returns content to every one of the eight channels at once – the BQ-Pauli algebra does not separate into alternating even/odd sectors the way the Weyl algebra does.

E.1 Bilinear product

$$\boxed{C = A^T B = c_1 \hat{1} + \mathbf{c}_K \cdot \hat{\mathbf{K}} + \mathbf{c}_\sigma \cdot \hat{\boldsymbol{\sigma}},} \tag{145}$$

where

$$c_1 = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}, \tag{146}$$
$$\mathbf{c}_K = \mathbf{a} \times \mathbf{b}, \tag{147}$$
$$\mathbf{c}_\sigma = a_0 \mathbf{b} - b_0 \mathbf{a}. \tag{148}$$

No $\hat{T}$-channel is generated: the bilinear of two minquats populates exactly the scalar, $\hat{\mathbf{K}}$, and $\hat{\boldsymbol{\sigma}}$ channels, verified by exact symbolic expansion and reconstruction on the eight-element basis.

E.2 Triple product

$$\boxed{E = DC = e_1 \hat{1} + e_T \hat{T} + \mathbf{e}_K \cdot \hat{\mathbf{K}} + \mathbf{e}_\sigma \cdot \hat{\boldsymbol{\sigma}},} \tag{149}$$

with

$$e_1 = -\mathbf{d} \cdot \mathbf{c}_K, \tag{150}$$
$$e_T = d_0 c_1 + \mathbf{d} \cdot \mathbf{c}_\sigma, \tag{151}$$
$$\mathbf{e}_K = \mathbf{d}\, c_1 + d_0 \mathbf{c}_\sigma + \mathbf{d} \times \mathbf{c}_K, \tag{152}$$
$$\mathbf{e}_\sigma = \mathbf{d} \times \mathbf{c}_\sigma - d_0 \mathbf{c}_K. \tag{153}$$

The triple product of three minquats populates all eight channels of the algebra – the $\hat{T}$-channel, absent from C, is regenerated here, so that E is a fully general BQ element rather than one confined to a proper subset of channels, verified by exact symbolic expansion and reconstruction.

E.3 Quadruple product

$$\boxed{G = F^T E = g_1 \hat{1} + g_T \hat{T} + \mathbf{g}_K \cdot \hat{\mathbf{K}} + \mathbf{g}_\sigma \cdot \hat{\boldsymbol{\sigma}},} \tag{154}$$

with

$$g_1 = f_0 e_T - \mathbf{f}\cdot\mathbf{e}_K, \tag{155}$$
$$g_T = -f_0 e_1 + \mathbf{f}\cdot\mathbf{e}_\sigma, \tag{156}$$
$$\mathbf{g}_K = -f_0\mathbf{e}_\sigma + \mathbf{f}\,e_1 + \mathbf{f}\times\mathbf{e}_K, \tag{157}$$
$$\mathbf{g}_\sigma = f_0\mathbf{e}_K - \mathbf{f}\,e_T + \mathbf{f}\times\mathbf{e}_\sigma. \tag{158}$$

As with the triple product, all eight channels remain populated in general: the BQ-Pauli algebra, having no exterior-algebra grading to terminate the hierarchy, does not exhibit an analogue of the Weyl-level alternation between the even and odd subalgebras. All three decompositions were verified by exact symbolic expansion and reconstruction on the eight-element BQ basis, with identically zero residuals.

F General vector products in the Weyl representation

Let

$$\not{A} = a_0\beta_0 + \mathbf{a}\cdot\boldsymbol{\beta}, \qquad \not{B} = b_0\beta_0 + \mathbf{b}\cdot\boldsymbol{\beta}, \tag{159}$$

and similarly

$$\not{D} = d_0\beta_0 + \mathbf{d}\cdot\boldsymbol{\beta}, \qquad \not{F} = f_0\beta_0 + \mathbf{f}\cdot\boldsymbol{\beta}, \tag{160}$$

be arbitrary space-time vectors in the Weyl basis.

The following results are purely algebraic consequences of the Clifford grading. Products of an even (odd) number of vectors belong to the even (odd) subalgebra,

$$\Lambda^0 \to \Lambda^1 \to (\Lambda^0 \oplus \Lambda^2) \to (\Lambda^1 \oplus \Lambda^3) \to (\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4), \tag{161}$$

which, in the BQ basis, becomes

$$\not{1} \to \beta_\mu \to \{\not{1}, \boldsymbol{\alpha}, \boldsymbol{\Sigma}\} \to \{\beta_\mu, \tilde{\beta}_\mu\} \to \{\not{1}, \boldsymbol{\alpha}, \boldsymbol{\Sigma}, \chi\}. \tag{162}$$

F.1 Bilinear product

The product of two vectors contains only scalar and bivector components,

$$\boxed{\not{C} = \not{A}\not{B} = c_1\not{1} + \mathbf{c}_\alpha\cdot\boldsymbol{\alpha} + \mathbf{c}_\Sigma\cdot\boldsymbol{\Sigma},} \tag{163}$$

where

$$c_1 = -a_0 b_0 + \mathbf{a}\cdot\mathbf{b}, \tag{164}$$
$$\mathbf{c}_\alpha = a_0\mathbf{b} - \mathbf{a}\,b_0, \tag{165}$$
$$\mathbf{c}_\Sigma = \mathbf{a}\times\mathbf{b}. \tag{166}$$

Thus

$$V\,V = \Lambda^0 \oplus \Lambda^2.$$

F.2 Triple product

Multiplication by a third vector gives

$$\boxed{\not{E} = \not{D}(\not{A}\not{B}) = K_0\beta_0 + \mathbf{K}\cdot\boldsymbol{\beta} + R_0\tilde{\beta}_0 + \mathbf{R}\cdot\tilde{\boldsymbol{\beta}},} \tag{167}$$

with

$$K_0 = d_0c_1 - \mathbf{d}\cdot\mathbf{c}_\alpha, \tag{168}$$
$$\mathbf{K} = \mathbf{d}\,c_1 - d_0\mathbf{c}_\alpha - \mathbf{d}\times\mathbf{c}_\Sigma, \tag{169}$$
$$R_0 = -\mathbf{d}\cdot\mathbf{c}_\Sigma, \tag{170}$$
$$\mathbf{R} = -d_0\mathbf{c}_\Sigma + \mathbf{d}\times\mathbf{c}_\alpha. \tag{171}$$

Thus

$$V(\Lambda^0 \oplus \Lambda^2) = \Lambda^1 \oplus \Lambda^3.$$

F.3 Quadruple product

Multiplication by a fourth vector returns the result to the even subalgebra,

$$\boxed{\not{G} = \not{F}\not{D}(\not{A}\not{B}) = g_1\not{1} + \mathbf{g}_\alpha\cdot\boldsymbol{\alpha} + \mathbf{g}_\Sigma\cdot\boldsymbol{\Sigma} + g_\chi\chi,} \tag{172}$$

where

$$g_1 = -f_0K_0 + \mathbf{f}\cdot\mathbf{K}, \tag{173}$$
$$\mathbf{g}_\alpha = -\mathbf{f}K_0 + f_0\mathbf{K} - \mathbf{f}\times\mathbf{R}, \tag{174}$$
$$\mathbf{g}_\Sigma = f_0\mathbf{R} - \mathbf{f}R_0 + \mathbf{f}\times\mathbf{K}, \tag{175}$$
$$g_\chi = f_0R_0 - \mathbf{f}\cdot\mathbf{R}. \tag{176}$$

Hence

$$(\Lambda^1 \oplus \Lambda^3)V = \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4,$$

so that the pseudoscalar χ appears for the first time only in a product containing four independent vectors.

All three decompositions were verified by exact symbolic expansion and reconstruction on the sixteen-element Weyl basis, with identically zero residuals.