Many-Body Filling Turns Soft Spectral Leakage into a Maintenance Floor

Exact Filled-SSH Sum Rules and an Interaction-Stable Obstruction

Lluis Eriksson

Independent Researcher

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Abstract

Soft spectral filtering has a more severe effect at finite filling than in a dilute edge-mode code. We consider two odd Su–Schrieffer–Heeger (SSH) rails, fill every negative-energy orbital, and encode one additional fermion in the zero mode of either rail. The code has fixed total number and supports physical coherence. For a boundary transfer at a site with zero-mode weight w_x, the desired logical Davies line has squared matrix element w_x^2. We prove the exact many-body leakage identity

$$W_{\text{leak}}^{\text{filled}}(x) = \frac{1 + 2w_x - 3w_x^2}{4}.$$

At the remote boundary, $w_x = \Theta(\zeta^{2\ell})$, so the logical line is $\Theta(\zeta^{4\ell})$ while the summed particle–hole leakage tends to $1/4$. This differs qualitatively from the dilute identity $w_x(1 - w_x) = \Theta(\zeta^{2\ell})$.

For a Davies filter whose off-target tail is $\epsilon_\ell = \exp[-q\ell + o(\ell)]$, the bounded-latency exact-refresh power therefore obeys the exponent law

$$\lim_{\ell\to\infty} -\frac{1}{\ell} \log P_{\ell,\tau} = \min\{4m, q\}, \qquad m = -\log\zeta,$$

under explicit uniform envelope and resource-ledger assumptions. A width-independent tail (a special case with $q = 0$) produces a nonzero maintenance floor, rather than merely halving the membrane exponent. More generally, $q = 0$ means only that the decay is subexponential. Every nonzero tail also makes the exact rapid-refresh limit logarithmically singular at each fixed width.

The floor is not a free-fermion accident. For arbitrary interacting number-conserving rails, we prove an exact static identity expressing leakage as a local occupation product minus the logical matrix element. Uniform finite filling and remote edge indistinguishability force a positive leakage floor. A new local spectral-window lemma then places a fixed fraction of that weight in a width-independent Bohr-frequency window using only a commutator norm. Consequently, any bath tail bounded below on that window yields an interaction-stable Davies leakage floor. Quasi-local spectral flow shows persistence in a neighborhood of a symmetry-preserving gapped SSH phase. Exact diagonalization of the interacting spinless SSH chain shows that repulsive and attractive interactions change the observed edge localization exponent while the leakage is already driven close to $1/4$ at accessible widths.

1 Result and novelty boundary

The distinction between a dilute code and a filled many-body code is easy to miss. In a one-particle sector, annihilating at the remote boundary first has to find the localized edge particle. This supplies one exponentially small factor before any final state is selected. At half filling, the same local operator can remove a sea particle and create a particle–hole excitation with order-one total weight. Selecting the desired zero mode on both rails remains exponentially unlikely, but leaking into the orthogonal many-body spectrum does not.

The paper establishes the following chain:

$$\begin{gathered}\text{chiral projector identity} \Longrightarrow \text{exact filled-SSH leakage sum rule} \Longrightarrow W_{\text{leak}} \to \tfrac{1}{4},\\ \text{soft Davies tail} \Longrightarrow \min\{4m, q\} \Longrightarrow \text{width-independent tail} \Longrightarrow \text{maintenance floor},\\ \text{finite filling + remote edge indistinguishability} \Longrightarrow \text{interacting static floor},\\ \text{interacting static floor + locality} \Longrightarrow \text{spectral-window Davies floor}.\end{gathered}$$

The nonanalyticity of entropy at rank-deficient states is standard and is treated systematically by Grace and Guha [3]. Davies generators, detailed balance, and Spohn entropy production are also classical [1, 2]. Half-filled SSH ladders and the operational entanglement of filled edge states have been studied by Monkman and Sirker [5]; we do not claim that finite filling or its edge physics is new. A companion paper converts first-order support leakage into a rapid-maintenance singularity and derives the dilute SSH law $\min\{4m, 2m + q\}$ [6]. Existing autonomous-QEC bounds concern logical error versus engineered correction rate [11]; they do not contain the filled-background line–continuum identity proved here. The new results are specifically the exact leakage sum rule, its Davies filter consequence, the exponent collapse to $\min\{4m, q\}$, and the local spectral-window obstruction. We do not claim a microscopic weak-coupling limit uniform in ℓ, autonomous work-only optimality, or a general proof that an arbitrary interacting SPT phase has a nonzero remote logical prefactor.

2 Odd SSH rails at half filling

2.1 Single-particle geometry

An odd SSH rail [4] has sites $A_0, \ldots, A_\ell$ and $B_0, \ldots, B_{\ell-1}$. Subtracting an irrelevant rail offset, its one-body Hamiltonian is

$$h_\ell = \sum_{j=0}^{\ell-1} \big(t_1 |A_j\rangle\langle B_j| + t_2 |A_{j+1}\rangle\langle B_j| + \text{h.c.}\big), \qquad 0 < t_1 < t_2.$$

Put $\zeta = t_1/t_2 < 1$ and $m = -\log\zeta$. Chiral symmetry is implemented by

$$\Gamma|A_j\rangle = |A_j\rangle, \qquad \Gamma|B_j\rangle = -|B_j\rangle, \qquad \Gamma h_\ell \Gamma = -h_\ell.$$

There is a unique zero mode

$$|\psi\rangle = \mathcal{N}_\ell \sum_{j=0}^{\ell} (-\zeta)^j |A_j\rangle, \qquad \mathcal{N}_\ell^2 = \frac{1-\zeta^2}{1-\zeta^{2\ell+2}}.$$

For a site x, write $w_x = |\langle x|\psi\rangle|^2$. Let $\Pi_0 = |\psi\rangle\langle\psi|$ and let $\Pi_\pm$ be the positive- and negative-energy spectral projectors of h_ℓ.

Lemma 2.1 (Chiral diagonal splitting)**.** *For every A-sublattice site x,*

$$\langle x|\Pi_-|x\rangle = \langle x|\Pi_+|x\rangle = \frac{1-w_x}{2}.$$

Proof. Chiral symmetry maps the positive spectral subspace unitarily onto the negative spectral subspace, hence $\Gamma\Pi_+\Gamma = \Pi_-$. Since $\Gamma|x\rangle = |x\rangle$ on the A sublattice, the two diagonal matrix elements coincide. Completeness gives $\Pi_- + \Pi_+ + \Pi_0 = 1$, so their sum is $1 - w_x$. □

At the far boundary $x = A_\ell$,

$$w_{\ell,\text{far}} = \frac{(1-\zeta^2)\zeta^{2\ell}}{1-\zeta^{2\ell+2}}, \qquad (1-\zeta^2)\zeta^{2\ell} \le w_{\ell,\text{far}} \le \zeta^{2\ell}. \tag{1}$$

2.2 Filled rails and the fixed-number code

Let $|g_\ell\rangle$ be the Slater determinant filling all ℓ negative-energy orbitals, and let

$$|e_\ell\rangle = f_0^\dagger |g_\ell\rangle$$

also occupy the zero mode. These states contain ℓ and $\ell+1$ particles. By Lemma 2.1, for x on the A sublattice,

$$\langle g_\ell|n_x|g_\ell\rangle = \frac{1-w_x}{2}, \qquad \langle e_\ell|n_x|e_\ell\rangle = \frac{1+w_x}{2}, \qquad \langle g_\ell|a_x|e_\ell\rangle = \psi_x. \tag{2}$$

Take two identical rails, with optional number offsets $\Delta_0 N_0$ and $\Delta_1 N_1$. The logical code is

$$|0_L\rangle = |e_\ell\rangle_0 \otimes |g_\ell\rangle_1, \qquad |1_L\rangle = |g_\ell\rangle_0 \otimes |e_\ell\rangle_1.$$

Both states have total number $2\ell+1$ and the same fermion parity. Their coherence is therefore physical. At corresponding boundary sites define the even, number-conserving transfer

$$S_x = a_{1,x}^\dagger a_{0,x}.$$

The desired logical Bohr frequency is $\delta = \Delta_1 - \Delta_0$.

3 Exact many-body leakage sum rule

Let $P_L = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ and $Q_L = 1 - P_L$.

Theorem 3.1 (Filled-SSH transfer identity)**.** *For a transfer from rail zero to rail one at an A-sublattice site x,*

$$W_{\log}(x) := |\langle 1_L|S_x|0_L\rangle|^2 = w_x^2, \tag{3}$$

$$W_{\mathrm{tot}}(x) := \|S_x|0_L\rangle\|^2 = \frac{(1+w_x)^2}{4}, \tag{4}$$

$$W_{\mathrm{leak}}(x) := \|Q_L S_x|0_L\rangle\|^2 = \frac{1+2w_x-3w_x^2}{4}. \tag{5}$$

The reverse transfer obeys the same identities.

Proof. Factorization across rails and Equation (2) give

$$\langle 1_L|S_x|0_L\rangle = \langle g_\ell|a_x|e_\ell\rangle\langle e_\ell|a_x^\dagger|g_\ell\rangle = |\psi_x|^2 = w_x.$$

The graded tensor-product signs are uniform and disappear after taking the modulus. Next,

$$S_x^\dagger S_x = n_{0,x}(1-n_{1,x})$$

on the two-rail Fock space. Hence

$$\|S_x|0_L\rangle\|^2 = \frac{1+w_x}{2}\left(1 - \frac{1-w_x}{2}\right) = \frac{(1+w_x)^2}{4}.$$

Orthogonally decomposing $S_x|0_L\rangle$ into its logical and nonlogical components yields Equation (5). □

The leakage can also be resolved into three mutually orthogonal processes:

process	summed squared weight
zero mode → positive band	$w_x(1-w_x)/2$
negative band → zero mode	$w_x(1-w_x)/2$
negative band → positive band	$(1-w_x)^2/4$

The last line is the finite-density channel absent in the dilute code.

Corollary 3.2 (Remote leakage floor). *At $x = A_\ell$,*

$$W_{\log} = \Theta(\zeta^{4\ell}), \qquad W_{\text{leak}} = \frac{1}{4} + O(\zeta^{2\ell}).$$

In particular, $W_{\text{leak}} \geq 1/4$ for all sufficiently large ℓ.

Remark 3.3 (Dilute versus filled). For the one-particle dual-rail code, completeness gives $W_{\text{leak}}^{\text{dilute}} = w_x(1-w_x)$. Filling the negative band adds the particle–hole term $(1-w_x)^2/4$ and changes a remote exponential tail into a constant. This is a change of asymptotic class, not a prefactor correction.

4 Davies rates under a soft filter

Couple the boundary operator $A_x = S_x + S_x^\dagger$ weakly to a thermal reservoir. At each fixed ℓ, the secular Davies generator has jump operators A_ω and nonnegative spectral weights $G_\ell(\omega)$. The logical line lies at $\pm\delta$. All free-fermion particle–hole leakage frequencies lie in a compact set independent of ℓ because the SSH bands have width bounded by $t_1 + t_2$.

We impose the following explicit filter envelope. There are constants $g_\pm, c_\pm > 0$, independent of ℓ, and a sequence $0 \leq \epsilon_\ell \leq 1$ such that

$$g_- \leq G_\ell(\pm\delta) \leq g_+, \tag{6}$$

$$c_-\epsilon_\ell \leq G_\ell(\omega) \leq c_+\epsilon_\ell \quad \text{for every active leakage frequency } \omega. \tag{7}$$

KMS-related signs may have different constants; this does not affect the conclusions.

Theorem 4.1 (Davies line–continuum separation). *Let $a_{\ell,\log}$ and $a_{\ell,\text{leak}}$ denote the total desired logical and outward leakage coefficients generated by the boundary coupling, for a logical target whose two populations are bounded below by a constant. Then*

$$a_{\ell,\log} = \Theta(w_{\ell,\text{far}}^2), \qquad a_{\ell,\text{leak}} = \Theta(\epsilon_\ell).$$

The constants are uniform in ℓ.

Proof. The desired Bohr component contains the matrix element in Equation (3); Equation (6) therefore gives two-sided multiples of $w_{\ell,\text{far}}^2$. The remaining Bohr components are mutually orthogonal after summing over final eigenstates. Their unweighted total is exactly Equation (5). Applying Equation (7) term by term and using Corollary 3.2 gives two-sided multiples of ϵ_ℓ. The two transfer directions cover both logical populations. Coherences do not cancel the positive sum because the opposite rail-number changes land in orthogonal leakage sectors. □

Corollary 4.2 (Rate exponent collapse). *If*

$$\epsilon_\ell = \exp[-q\ell + o(\ell)], \qquad q \in [0, \infty],$$

then the total boundary dissipation coefficient has exponential rate

$$\lim_{\ell\to\infty} -\frac{1}{\ell}\log(a_{\ell,\log} + a_{\ell,\text{leak}}) = \min\{4m, q\}.$$

An exact filter is represented by $q = \infty$.

5 Maintenance consequences

5.1 Rapid exact refresh

Let $T_t^{(\ell)} = \exp(t\mathcal{L}_\ell)$ and let ρ_ℓ be faithful on the two-dimensional logical code and zero on its orthogonal complement. Set $P = P_L$, $Q = 1 - P$, and

$$a_\ell = \mathrm{Tr}[Q\mathcal{L}_\ell(\rho_\ell)].$$

Lemma 5.1 (Boundary free-energy loss)**.** *For a finite-dimensional Hamiltonian and $a_\ell > 0$,*

$$F(\rho_\ell) - F(T_t^{(\ell)}\rho_\ell) = k_\mathrm{B}T\, a_\ell t \log(1/t) + O_\ell(t) \qquad (t \downarrow 0).$$

Proof. Positivity gives $Q\mathcal{L}_\ell(\rho_\ell)Q \geq 0$ with trace a_ℓ. The new kernel eigenvalues are t times the positive eigenvalues of this block, up to $O(t^2)$. Their entropy contribution is $-a_\ell t \log t + O_\ell(t)$. Eigenvalues inside the original support and the energy expectation vary analytically at order t. Combining the terms gives the formula. This is the support-extending branch of standard entropy perturbation theory [3]. □

Under the explicit fresh-target-cell ledger, the optimal exact replacement cost is the free-energy loss. Thus any $\epsilon_\ell > 0$ gives $a_\ell > 0$ by Theorem 4.1 and the rapid power diverges logarithmically at every fixed width. This is not a statement about work-only autonomous control.

5.2 Fixed latency

Fix a refresh period $0 < \tau \leq \tau_0$. Let

$$P_{\ell,\tau} = \frac{F(\rho_\ell) - F(T_\tau^{(\ell)}\rho_\ell)}{\tau}.$$

We now make the uniformity assumptions precise. The code projector $P = P_{L,\ell}$ is assumed to be a sum of complete spectral projections of H_ℓ, separated from $Q = 1 - P$ by an isolated band. This spectral condition alone does *not* imply invariance of the $P \oplus Q$ block algebra: a single degenerate Bohr block can map two code energies into opposite blocks. We therefore impose the independent compatibility condition

$$\mathcal{E}_P \widetilde{\mathcal{L}}_\ell \mathcal{E}_P = \widetilde{\mathcal{L}}_\ell \mathcal{E}_P, \qquad \mathcal{E}_P(X) = PXP + QXQ. \tag{8}$$

Equivalently, the block-diagonal algebra $P\mathcal{B}P \oplus Q\mathcal{B}Q$ is invariant. A sufficient, directly checkable condition is that, for every ω, each source block is mapped wholly into one target block: $A_{\ell,\omega}P$ lies either in $P\mathcal{H}$ or in $Q\mathcal{H}$, and independently $A_{\ell,\omega}Q$ lies either in $P\mathcal{H}$ or in $Q\mathcal{H}$. The logical-line/leakage-frequency separation and rail-number rules verify the outgoing P condition in the filled-SSH model; the full condition on the reachable Q sector remains an explicit restriction on the Davies/filter construction. We do not infer it from $[P, H_\ell] = 0$.

We work in the interaction picture, where the Hamiltonian commutator drops out of both $q_\ell(t) = \mathrm{Tr}[Q\rho_\ell(t)]$ and the dissipative trace-norm estimates. Partition the Bohr frequencies into retained logical channels $\Omega_{\log}$ and tail channels Ω_{tail}. The exact-filter generator $\widetilde{\mathcal{L}}_\ell^{(0)}$ is defined by deleting the *complete Lindblad dissipator* $\mathcal{D}[J_{\ell,\omega}]$, including its anticommutator, for every $\omega \in \Omega_{\mathrm{tail}}$. It is therefore a trace-preserving GKLS generator, not a truncated jump term. We assume that it also obeys (8). Its logical restriction is

$$P\widetilde{\mathcal{L}}_\ell^{(0)}(PXP)P = w_{\ell,\mathrm{far}}^2 \mathcal{L}_{\log,0}(X),$$

and the faithful logical target has a normalized entropy-production coefficient bounded below independently of ℓ.

Why the block criterion is sufficient. The Davies relation and its zero-frequency consequence are

$$[H_\ell, A_{\ell,\omega}] = -\omega A_{\ell,\omega}, \qquad [H_\ell, A^\dagger_{\ell,\omega} A_{\ell,\omega}] = 0.$$

Since P is a sum of complete energy projections, the anticommutator with $A^\dagger_{\ell,\omega} A_{\ell,\omega}$ preserves the $P \oplus Q$ block algebra. Under the source-to-one-target condition above, the jump term $A_{\ell,\omega} X A^\dagger_{\ell,\omega}$ also sends each diagonal source block into a diagonal target block. Hence every complete dissipator leaves the block-diagonal algebra invariant, which is exactly (8). The criterion need not imply the stronger commutation relation $\mathcal{E}_P \widetilde{\mathcal{L}}_\ell = \widetilde{\mathcal{L}}_\ell \mathcal{E}_P$: an off-diagonal input may be sent into a diagonal block. No such input occurs along the trajectory considered here. This argument also explains why splitting or deleting only the jump term would be insufficient.

For the Davies jumps $J_{\ell,\omega} = G_\ell(\omega)^{1/2} A_{\ell,\omega}$ define the positive crossing maps

$$\Phi^{\mathrm{out}}_\ell(X) = \sum_\omega Q J_{\ell,\omega} P X P J^\dagger_{\ell,\omega} Q, \tag{9}$$

$$\Phi^{\mathrm{in}}_\ell(Y) = \sum_\omega P J_{\ell,\omega} Q Y Q J^\dagger_{\ell,\omega} P. \tag{10}$$

Lemma 5.2 (Uniform crossing norms)**.** *Under Equation* (7)*,*

$$\|\Phi^{\mathrm{out}}_\ell\|_{1\to1} + \|\Phi^{\mathrm{in}}_\ell\|_{1\to1} \le C_\times \epsilon_\ell.$$

The interaction-picture Davies dissipator also obeys

$$\|\widetilde{\mathcal{L}}_\ell\|_{1\to1} \le M_*$$

with constants independent of ℓ and of Bohr-frequency degeneracies.

Proof. For a positive map $\Phi(X) = \sum_j B_j X B_j^\dagger$, its induced trace norm is

$$\|\Phi\|_{1\to1} = \|\Phi^*(1)\|_\infty = \left\| \sum_j B_j^\dagger B_j \right\|_\infty .$$

Every $P \leftrightarrow Q$ component lies in the leakage envelope, hence

$$\sum_\omega P J^\dagger_{\ell,\omega} Q J_{\ell,\omega} P \le c_+ \epsilon_\ell \sum_\omega P A^\dagger_{\ell,\omega} A_{\ell,\omega} P.$$

The sum on the right is a sub-sum of the energy pinching of $A_x^\dagger A_x$ and is bounded by $\|A_x\|^2 P$. This proves the outward estimate; the incoming estimate is identical, using the KMS-related upper envelope. The same pinching identity, now without P, Q, bounds the completely positive and anticommutator pieces of the full dissipator by a constant multiple of $\sup_\omega G_\ell(\omega) \|A_x\|^2$. Combining degenerate transitions inside a single $A_{\ell,\omega}$ changes neither estimate, so no number-of-modes factor appears. □

Lemma 5.3 (Uniform fixed-time leakage)**.** *There exist $\tau_0, c, C > 0$, independent of ℓ, such that*

$$c\tau\epsilon_\ell \le q_\ell(\tau) := \mathrm{Tr}[Q T^{(\ell)}_\tau(\rho_\ell)] \le C\tau\epsilon_\ell$$

for $0 < \tau \le \tau_0$ and all sufficiently large ℓ.

Proof. Block preservation gives the exact balance equation

$$q'_\ell(t) = \mathrm{Tr}\, \Phi^{\mathrm{out}}_\ell(P\rho_\ell(t)P) - \mathrm{Tr}\, \Phi^{\mathrm{in}}_\ell(Q\rho_\ell(t)Q).$$

At $t = 0$, Theorem 4.1 gives

$$a_- \epsilon_\ell \le q'_\ell(0) \le a_+ \epsilon_\ell.$$

Contractivity and Lemma 5.2 imply

$$\left|\operatorname{Tr}\Phi_\ell^{\mathrm{out}}(P\rho_\ell(t)P) - q_\ell'(0)\right| \le C_\times \epsilon_\ell M_* t,$$
$$0 \le \operatorname{Tr}\Phi_\ell^{\mathrm{in}}(Q\rho_\ell(t)Q) \le C_\times \epsilon_\ell q_\ell(t).$$

The upper differential inequality first gives $q_\ell(t) \le C\epsilon_\ell t$ on a width-independent interval by Gronwall. Substitution into the lower inequality gives

$$q_\ell'(t) \ge a_-\epsilon_\ell - C_0\epsilon_\ell t - C_1\epsilon_\ell^2 t.$$

Choose τ_0 so that the last two terms are at most $a_-\epsilon_\ell/2$, then integrate. The upper bound follows from the corresponding upper inequality. □

Lemma 5.4 (Free-energy sandwich)**.** *Assume in addition that all active boundary-transfer Bohr frequencies lie in a width-independent compact interval and that* $\log d_{Q,\ell} \le c_d\ell$*. For* $0 < \tau \le \tau_0$ *and sufficiently small* ϵ_ℓ*,*

$$c_1\tau w_{\ell,\mathrm{far}}^2 + k_B T\big[h_2(q_\ell(\tau)) - C_1 q_\ell(\tau)\big] \le \tau P_{\ell,\tau} \le C_2\tau w_{\ell,\mathrm{far}}^2 + k_B T\big[h_2(q_\ell(\tau)) + C_3\ell q_\ell(\tau)\big].$$

Here h_2 *is binary entropy and all constants are width independent.*

Proof. By the explicit compatibility condition (8), the secular evolution is block diagonal:

$$T_\tau^{(\ell)}(\rho_\ell) = (1-q)\sigma_\ell \oplus q\chi_\ell.$$

Consequently,

$$S((1-q)\sigma \oplus q\chi) = h_2(q) + (1-q)S(\sigma) + qS(\chi).$$

Let $\eta_\ell(\tau)$ be the evolution with the crossing jumps removed. Duhamel's formula and Lemma 5.2 give

$$\|PT_\tau^{(\ell)}(\rho_\ell)P - \eta_\ell(\tau)\|_1 \le C\tau\epsilon_\ell.$$

The target is faithful on the two-dimensional code, so for sufficiently small τ_0 both normalized logical blocks have eigenvalues bounded below by a width-independent constant. Entropy and energy are therefore Lipschitz on this block. Strictly positive normalized logical entropy production gives

$$c_1\tau w_{\ell,\mathrm{far}}^2 \le F(\rho_\ell) - F(\eta_\ell(\tau)) \le C_2\tau w_{\ell,\mathrm{far}}^2.$$

For the leakage correction, the compact Bohr window bounds the energy transported per jump. Equivalently, the Davies energy flux is a frequency-weighted positive sum, so its absolute crossing contribution is at most $Cq_\ell(\tau)$. The logical-block Duhamel error is of the same order by Lemma 5.3. In the entropy identity, $qS(\chi) \ge 0$ gives the lower bound, while

$$qS(\chi) \le q\log d_{Q,\ell} \le c_d\ell q$$

gives the upper bound. Collecting the analytic $O(q)$ terms proves the sandwich. □

Theorem 5.5 (Many-body exponent collapse)**.** *Suppose* $\epsilon_\ell = \exp[-q\ell + o(\ell)]$ *and the assumptions above hold. For every fixed* $0 < \tau \le \tau_0$*,*

$$\boxed{\lim_{\ell\to\infty} -\frac{1}{\ell}\log P_{\ell,\tau} = \min\{4m, q\}.}$$

If $q = 0$*, the displayed exponent is zero. A positive width-independent lower bound on* $P_{\ell,\tau}$ *requires the additional condition* $\liminf_{\ell\to\infty}\epsilon_\ell > 0$ *(and a sufficiently weak tail so that the uniform small-time estimates apply).*

Proof. By Equation (1), the logical term has rate $4m$. By Lemma 5.3, $q_\ell(\tau) = \exp[-q\ell + o(\ell)]$. If $q > 0$, then

$$h_2(q_\ell(\tau)) = q_\ell(\tau)\log(1/q_\ell(\tau)) + O(q_\ell(\tau))$$

has rate q; polynomial factors in ℓ do not alter it. The sandwich therefore gives the minimum of the two rates, including the equality case $q = 4m$. When $q = 0$, the same bounds give exponent zero but allow subexponential decay, for example $\epsilon_\ell = 1/\ell$. If in addition $\liminf_\ell \epsilon_\ell > 0$, choose τ_0 and the tail amplitude small enough that the positive binary-entropy term dominates the analytic $O(q_\ell(\tau))$ correction. Only under this additional condition is the lower bound width independent. □

The two limits again do not commute whenever the tail remains nonzero at each fixed width. At fixed ℓ, $\lim_{\tau\downarrow 0} P_{\ell,\tau} = \infty$. At fixed τ, $q > 0$ forces the wide-width power to vanish exponentially. When $q = 0$, the exponential rate is zero: the power may vanish subexponentially, and a positive floor follows only if $\liminf_\ell \epsilon_\ell > 0$.

6 An exact interacting leakage identity

We now discard the free-fermion structure. On one interacting rail, let $|g_\ell\rangle$ and $|e_\ell\rangle$ be normalized eigenstates in adjacent number sectors. Define at a boundary site x

$$\alpha_\ell = \langle g_\ell|a_x|e_\ell\rangle, \quad n_{e,\ell} = \langle e_\ell|n_x|e_\ell\rangle, \quad v_{g,\ell} = \langle g_\ell|(1-n_x)|g_\ell\rangle.$$

Use the same fixed-number two-rail code and transfer S_x as before.

Theorem 6.1 (Interacting occupation–leakage identity)**.** *For arbitrary number-conserving rail Hamiltonians,*

$$W^{\text{int}}_{\text{log}} = |\alpha_\ell|^4, \tag{11}$$

$$W^{\text{int}}_{\text{tot}} = n_{e,\ell}v_{g,\ell}, \tag{12}$$

$$W^{\text{int}}_{\text{leak}} = n_{e,\ell}v_{g,\ell} - |\alpha_\ell|^4. \tag{13}$$

Consequently, if $n_{e,\ell}, v_{g,\ell} \ge v > 0$ *and* $|\alpha_\ell| \to 0$*, then*

$$\liminf_{\ell\to\infty} W^{\text{int}}_{\text{leak}} \ge v^2.$$

Proof. The logical matrix element factorizes into $\alpha_\ell\overline{\alpha_\ell}$. The identity $S_x^\dagger S_x = n_{0,x}(1-n_{1,x})$ and the product structure of the two rails give the total weight. Subtracting the squared logical projection gives the leakage identity. The lower bound is immediate. □

If particle–hole symmetry maps $|g_\ell\rangle$ to $|e_\ell\rangle$, then $v_{g,\ell} = n_{e,\ell}$ and the total weight is $n_{e,\ell}^2$. The theorem shows precisely which hypothesis replaces free-fermion completeness: finite local filling supplies the floor, while remote edge indistinguishability makes the desired matrix element vanish.

7 A local spectral-window theorem

A static leakage floor is useful only if a fixed fraction lies where the bath tail is controlled. The next result supplies that fraction without counting many-body levels.

Lemma 7.1 (Local spectral window)**.** *Let* $H_\ell|0_L\rangle = E_{0,\ell}|0_L\rangle$ *and* $H_\ell|1_L\rangle = E_{1,\ell}|1_L\rangle$*, with* $\delta_\ell = E_{1,\ell} - E_{0,\ell}$*. Assume* P_L *is a spectral projector and put*

$$v_\ell = Q_L S_x|0_L\rangle, \qquad W_\ell = \|v_\ell\|^2.$$

If

$$\|([H_\ell, S_x] - \delta_\ell S_x)|0_L\rangle\| \le K$$

uniformly in ℓ, then for every $\Omega > 0$,

$$\langle v_\ell, \mathbf{1}_{\{|H_\ell - E_{0,\ell} - \delta_\ell| \le \Omega\}} v_\ell\rangle \ge W_\ell - \frac{K^2}{\Omega^2}.$$

In particular, if $W_\ell \ge c_0 > 0$, choosing $\Omega \ge \sqrt{2}K/\sqrt{c_0}$ places at least $c_0/2$ leakage weight in a width-independent frequency window.

Proof. Because P_L is spectral, Q_L commutes with H_ℓ. The centered second moment of the spectral measure of v_ℓ is

$$\begin{aligned}\|(H_\ell - E_{0,\ell} - \delta_\ell)v_\ell\|^2 &\le \|(H_\ell - E_{0,\ell} - \delta_\ell)S_x|0_L\rangle\|^2 \\ &= \|([H_\ell, S_x] - \delta_\ell S_x)|0_L\rangle\|^2 \le K^2.\end{aligned}$$

Chebyshev's inequality for this positive spectral measure yields the claim. □

For a finite-range Hamiltonian with uniformly bounded local interactions, S_x is supported on two boundary sites and $\|[H_\ell, S_x]\| = O(1)$. Hence the hypothesis is genuinely local and independent of the many-body bandwidth or Hilbert-space dimension.

Theorem 7.2 (Interaction-stable Davies floor)**.** *Assume the hypotheses of Theorem 6.1, with $n_{e,\ell}, v_{g,\ell} \ge \nu$ and $|\alpha_\ell| \to 0$, and the uniform commutator hypothesis of Lemma 7.1. Let the bath spectral weight obey*

$$G_\ell(\omega) \ge c_- \epsilon_\ell$$

on the nonlogical spectrum inside the window supplied by Lemma 7.1. Then

$$a_{\ell,\mathrm{leak}} \ge c\epsilon_\ell$$

for a width-independent $c > 0$. If a matching upper spectral envelope holds, then $a_{\ell,\mathrm{leak}} = \Theta(\epsilon_\ell)$.

Proof. By Theorem 6.1, $W_\ell \ge \nu^2/2$ for all sufficiently large ℓ. Apply Lemma 7.1 with $c_0 = \nu^2/2$. At least $c_0/2$ of the positive sum of squared transition amplitudes lies in the controlled window. Multiplication by the bath lower envelope proves the lower rate bound. The upper bound follows from $\sum_\omega \|A_\omega|0_L\rangle\|^2 \le \|A_x\|^2$. □

8 Persistence under symmetry-preserving interactions

Consider a differentiable finite-range single-rail path $h_\ell(s)$, $s \in [0, s_0]$, starting at the filled SSH model. We state every input used below:

- (G1) $h_\ell(s)$ conserves number and particle–hole symmetry, and its $N = \ell$ and $N = \ell + 1$ eigenstates $|g_\ell(s)\rangle, |e_\ell(s)\rangle$ form an isolated band separated from the rest of the spectrum by $\Delta_* > 0$ uniformly in s and ℓ.
- (G2) The spectral-flow unitary $U_\ell(s)$ commutes with total number, intertwines the two sector projections, and has the uniform quasi-locality estimate

$$\inf_{O^{(r)}:\,\mathrm{supp}\,O^{(r)} \subset B_r(X)} \|U_\ell(s)^\dagger O_X U_\ell(s) - O^{(r)}\| \le C_F \|O_X\| F(r),$$

where $F(r)$ decreases faster than every inverse power.

(G3) At the free endpoint, every normalized number-lowering observable supported within distance r of the far boundary satisfies

$$|\langle g_\ell(0)|O^{(r)}|e_\ell(0)\rangle| \leq C_0\zeta^{\ell-r}\|O^{(r)}\|.$$

For the filled SSH Slater pair this follows by reducing the transition functional to the zero-mode weight in the support of $O^{(r)}$.

(G4) The spectral-flow generator has a uniformly summable interaction norm. In particular,

$$\sup_{\ell,s}\|[D_\ell(s),n_x]\| \leq C_{\rm occ} < \infty,$$

where $\partial_s U_\ell(s) = -iD_\ell(s)U_\ell(s)$.

Conditions (G1), (G2), and (G4) are the standard outputs of a symmetry-preserving uniformly gapped spectral flow [8, 9]; (G3) is the model-specific endpoint input. Gap-stability theorems provide sufficient conditions for (G1) in classes of weakly perturbed one-dimensional spin and fermion chains [10]. We do not assert that those sufficient conditions cover every value of the interacting SSH coupling used in the numerical section.

Theorem 8.1 (Conditional gapped-path persistence)**.** *Under (G1)–(G4), there exists $s_* > 0$, independent of ℓ, such that for $0 \leq s \leq s_*$:*

1. *the transported edge matrix element $\alpha_\ell(s)$ tends to zero faster than any inverse power of ℓ;*
2. *$n_{e,\ell}(s), v_{g,\ell}(s) \geq 1/4$ for all sufficiently large ℓ;*
3. *$\liminf_{\ell\to\infty} W^{\rm int}_{\rm leak}(s) \geq 1/16$;*
4. *the local spectral-window constant can be chosen uniformly in s and ℓ.*

Thus a width-independent soft tail produces a maintenance floor throughout this interaction neighborhood.

Proof. Number conservation allows phases to be chosen so that

$$|g_\ell(s)\rangle = U_\ell(s)|g_\ell(0)\rangle, \qquad |e_\ell(s)\rangle = U_\ell(s)|e_\ell(0)\rangle.$$

Let $a_x^{(s,r)}$ be the radius-r approximant from (G2). Conditions (G2)–(G3) give the explicit bound

$$\begin{aligned} |\alpha_\ell(s)| &= |\langle g_\ell(0)|U_\ell(s)^\dagger a_x U_\ell(s)|e_\ell(0)\rangle| && (14)\\ &\leq C_0\zeta^{\ell-r}\|a_x^{(s,r)}\| + C_F F(r) && (15)\\ &\leq C_0(1+C_F F(r))\zeta^{\ell-r} + C_F F(r). && (16)\end{aligned}$$

Choose $r = \lfloor \ell/2\rfloor$. The first term is exponentially small and the second decreases faster than every inverse power, proving item 1. The same construction is performed on each rail; their product gives the two-rail logical matrix element in Theorem 6.1.

For the occupation, differentiation along the flow yields

$$\left|\frac{d}{ds}\langle e_\ell(s)|n_x|e_\ell(s)\rangle\right| = |\langle e_\ell(s)|i[D_\ell(s),n_x]|e_\ell(s)\rangle| \leq C_{\rm occ}.$$

The same estimate holds for $\langle g_\ell(s)|(1-n_x)|g_\ell(s)\rangle$. Both quantities converge to $1/2$ at $s = 0$. After increasing the minimum-width threshold if needed, they are at least $1/2 - o(1) - C_{\rm occ}s$. Taking

$$s_* < \frac{1}{4C_{\rm occ}}$$

proves item 2. Item 3 now follows from the exact interacting identity and item 1.

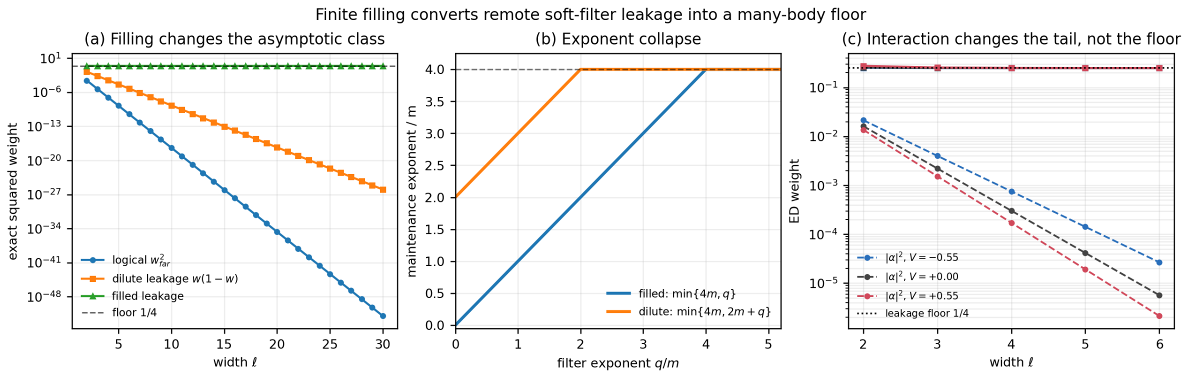

Figure 1: Left: exact remote logical and leakage weights in the dilute and half-filled SSH codes. Center: filter-exponent phase diagrams; finite filling changes the leakage branch from $2m+q$ to q. Right: exact diagonalization of the interacting spinless SSH rail. Interactions change the edge localization slope while the leakage retains its $1/4$ floor.

Finally, finite range and a uniform local interaction norm J_* imply

$$\|[H_\ell(s), S_x]\| \le 2\|S_x\| \sum_{Z:\, Z\cap \operatorname{supp} S_x \neq \varnothing} \|h_Z(s)\| \le K_*,$$

because only a width-independent number of terms meet the two-site support of S_x. The logical splitting $\delta_\ell(s)$ is bounded on the isolated compact path, so the centered commutator constant in Lemma 7.1 is uniform. This proves item 4. □

The theorem deliberately claims a perturbative neighborhood, not an entire interacting phase. Extending the lower filling bound and the remote matrix-element control along an arbitrary gapped path requires phase-specific input. The leakage mechanism itself, however, is independent of integrability and of a one-particle reduction.

9 Exact-diagonalization diagnostics

We test the interacting spinless SSH rail

$$H_\ell(V) = H_\ell(0) + V \sum_{j=0}^{2\ell-1} \left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right)$$

in the $N = \ell$ and $N = \ell + 1$ sectors. The code uses their respective ground states. The verification suite constructs the fermionic Fock basis exactly, diagonalizes both sectors, evaluates α_ℓ, the occupation product, and the projected leakage, and checks particle–hole conjugacy. It also verifies the free closed forms and exponent fits without fitting any theorem constant.

For $t_1/t_2 = 0.37$ and $V/t_2 \in \{-0.55, 0, 0.55\}$, the interaction changes the fitted decay of $|\alpha_\ell|^2$: attraction delocalizes the edge charge and repulsion localizes it further, consistent with interacting-SSH studies [7]. In all three cases the leakage rapidly approaches $1/4$; at $\ell = 6$ it equals 0.2500019, 0.2500028, and 0.2500560, respectively. These calculations are diagnostics, not a substitute for Theorems 6.1 and 8.1 and Lemma 7.1.

10 Scope, falsification, and implications

The exact free result is falsified by any violation of Equation (5) under the stated filling and boundary transfer. The interacting theorem is falsified if its occupation product stays bounded below and its logical matrix element vanishes, yet the projected leakage does not retain a floor. The spectral-window result is a direct moment inequality and can fail only if the commutator bound or the spectral-band hypothesis is removed.

Several limitations are explicit:

- The Davies generator is constructed after taking the weak-coupling limit at each fixed width; no uniform pre-Davies limit is proved.
- The exact exponent equality $\min\{4m, q\}$ is proved for the free filled SSH family. The interacting theorem proves a stable leakage floor and a Davies lower bound, but not a universal interacting logical exponent.
- The bath lower envelope on a compact nonlogical window is an assumption. The local spectral-window lemma makes that window width independent but does not derive the reservoir spectrum.
- The fixed-latency equality uses a fresh-target-cell resource ledger. It is not an autonomous work-only controller theorem.
- The result concerns a boundary transfer. A controller engineered to annihilate the filled background before transferring the edge charge implements a different system operator and may evade the floor at an additional resource cost.

The practical conclusion is sharper than in the dilute model. Perfect filtering was already the boundary between finite and singular rapid exact maintenance. At finite filling, imperfect filtering also determines whether widening the membrane helps at all. A constant tail does not merely reduce the exponent: it removes it.

A Fermionic tensor factors

The two rails are combined with the graded tensor product. The operator $S_x = a_{1,x}^\dagger a_{0,x}$ is even and preserves total number. Matrix elements between fixed-total-number code states may acquire a convention-dependent uniform sign from ordering the rails, but every weight used in the paper is a modulus square. Moreover,

$$S_x^\dagger S_x = a_{0,x}^\dagger a_{1,x} a_{1,x}^\dagger a_{0,x} = a_{0,x}^\dagger (1 - n_{1,x}) a_{0,x} = n_{0,x}(1 - n_{1,x}),$$

because the even operator $n_{1,x}$ commutes with the rail-zero fields. Thus no hidden fermionic sign enters the occupation identities.

B Free spectral decomposition of the leakage

Expand the boundary field as

$$a_x = \psi_x f_0 + \sum_k u_{k,-}(x) f_{k,-} + \sum_k u_{k,+}(x) f_{k,+}.$$

In $|e_\ell\rangle$, the zero and negative modes are occupied; in $|g_\ell\rangle$, only the negative modes are occupied. Therefore $S_x|0_L\rangle$ contains exactly four types of terms. The zero-to-zero term is logical. Orthogonality of the Slater determinants makes all other terms mutually orthogonal after summing over band labels. Lemma 2.1 gives

$$\sum_k |u_{k,-}(x)|^2 = \sum_k |u_{k,+}(x)|^2 = \frac{1 - w_x}{2},$$

which yields the three leakage rows displayed after Theorem 3.1. This derivation also shows that degeneracies of particle–hole Bohr frequencies cannot create a multiplicity factor: coherent recombination within a degenerate frequency block preserves the same total squared norm.

C Resource ledger

For completeness, the fixed-period achievability statement uses a cell whose Hamiltonian is isomorphic to the logical-plus-bulk system and whose initial state is a fresh copy of the target. A number-conserving graded SWAP exchanges system and cell. If the discarded system state after free evolution is σ, the minimum reversible work of re-preparing the cell is

$$F(\rho) - F(\sigma) = k_{\mathrm{B}}T\big[D(\rho\|\gamma) - D(\sigma\|\gamma)\big].$$

This ledger proves equality inside a permissive collision-model class. It consumes target-state resources and must not be conflated with ordinary work-only autonomous stabilization.

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