Exact Stabilization by Additive GKLS Control Is Impossible at a Leaky Boundary

Order-Optimal Accuracy-Rate Bounds and an SSH Control Horizon

Lluis Eriksson

Independent Researcher

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Abstract

The tangent cone of quantum state space at a rank-deficient density matrix is known to have a positive exterior block, and recent work identifies all of its directions with Lindbladian velocities. We derive a quantitative stabilization consequence of this geometry. Let ρ be a finite-dimensional target with support projector P, put $Q = 1 - P$, and let an uncontrolled GKLS generator L have outward support-leakage rate $a = \mathrm{Tr}\big[QL(\rho)\big] > 0$. Every additive GKLS controller K, including a bounded time-dependent one, has its own nonnegative outward rate at ρ. It therefore cannot cancel a: no finite-rate additive Markovian controller can keep the system exactly at the target. The conclusion persists for arbitrary finite-dimensional autonomous ancillas governed jointly by additive Markovian dynamics, provided the adverse system generator remains additive and local to the system. If $\|L\|_{1\to1} \le M$ and $\|K_t\|_{1\to1} \le \Gamma$, we prove the dynamic corridor bound

$$\limsup_{t\to\infty} \|\rho_t - \rho\|_1 \ge 2a/(M + \Gamma).$$

The result applies to arbitrary prescribed time dependence and does not assume convergence to a steady state. An autonomous Poisson-reset generator supplies a matching inverse-rate upper bound: its unique stationary state is the exponential resolvent average of the uncontrolled orbit, its entire trajectory remains within $\|L(\rho)\|_1/\kappa$ of the target, and its stationary bulk population is $a/\kappa + O(\kappa^{-2})$. Thus the optimal high-rate Markovian accuracy has order $\Theta(\Gamma^{-1})$; no constant-optimality claim is made. For a soft-filter dual-rail SSH family, the microscopic leakage coefficient scales as $\exp\big[-(2m+q)\ell+o(\ell)\big]$, whereas the ideal logical disturbance scales as $\exp\big[-4m\ell+O(1)\big]$. We prove a sharp autonomous-control horizon: to retain trace accuracy at the ideal logical scale, a controller budget $\Gamma_\ell = \exp\big[g\ell + o(\ell)\big]$ is necessary and an explicit reset controller is sufficient precisely when

$$g \ge \max\{0, 2m - q\}.$$

Hence an imperfect filter with $q < 2m$ forces autonomous correction intensity to grow exponentially with membrane width. The filter threshold that changes the bounded-latency refresh exponent also governs whether autonomous control remains width independent.

1 Problem and contribution

Engineered dissipation can prepare and stabilize pure and mixed quantum states, and autonomous quantum error correction can suppress logical noise using always-on recovery. Those results naturally raise a different question. Suppose an uncontrolled Markovian generator already pushes a rank-deficient target out of its support at first order. Can another finite-rate additive Markovian generator cancel that motion exactly? The answer is no for additive unconditional GKLS control. The geometric sign itself is not new. The positive exterior-block characterization of the tangent cone appears in convex and quantum transport literature, and Cai, Govindarajan, and Junge recently proved that the entire tangent cone consists exactly of Lindbladian velocities. Our starting sign is an explicit jump-operator specialization of that known geometry. The contribution begins when the sign is combined with a fixed adverse generator, a quantitative norm budget, arbitrary time dependence, autonomous ancillas, and the SSH width limit. The contribution of this paper is not the construction of target-state Lindbladians or the renewal formula for stochastic resetting. It is the following composition.

- An additive no-cancellation consequence of the known support tangent cone, including finite-dimensional Markovian ancillas.
- A dynamic lower bound for arbitrary bounded time-dependent Markovian controllers, without a stationarity assumption.
- A matching autonomous construction proving the optimal inverse-rate accuracy law.
- A geometric exponent theorem for soft-filter SSH codes, with a necessary-and-sufficient controller-growth threshold.

The paper concerns unconditional additive Markovian evolution. Impulsive replacement, measurement-conditioned trajectories, non-Markovian memory, and controllers that alter the system-bath derivation rather than add a GKLS term are outside the theorem.

2 The support tangent cone

Let H be finite dimensional and let ρ be a density matrix. Write $P = \operatorname{supp}(\rho)$, $Q = 1 - P$. For a Hermiticity-preserving trace-preserving generator G, define its outward support flow at ρ by

$$a_G(\rho) = \operatorname{Tr}\big[QG(\rho)\big].$$

All induced trace norms below are taken on Hermitian operators:

$$\|G\|_{1\to 1} = \sup_{X=X^\dagger, X\neq 0} \|G(X)\|_1/\|X\|_1.$$

Lemma 2.1 (Explicit jump formula for the tangent-cone sign)**.** *Let*

$$G(X) = -i\big[H, X\big] + \sum_j \gamma_j \mathcal{D}\big[L_j\big](X),$$

where $\gamma_j \geq 0$ *and*

$$\mathcal{D}\big[L\big](X) = LXL^\dagger - \frac{1}{2}\{L^\dagger L, X\}.$$

Then

$$\begin{aligned} a_G(\rho) &= \sum_j \gamma_j \operatorname{Tr}\big[QL_j\rho L_j^\dagger\big] \\ &= \sum_j \gamma_j \|QL_j\sqrt{\rho}\|_2^2 \geq 0. \end{aligned}$$

If ρ is faithful on P, equality holds if and only if $QL_jP = 0$ for every retained jump.

Proof. The Hamiltonian contribution has zero Q trace because $Q\rho = \rho Q = 0$. Cyclicity and the same identities kill both anticommutator contributions. The remaining terms are positive squared Hilbert-Schmidt norms. Faithfulness makes $\sqrt{\rho}$ invertible on P. □

Theorem 2.2 (Exact Markovian no-cancellation)**.** *Let L be an uncontrolled GKLS generator with*

$$a = a_L(\rho) > 0.$$

For every additive GKLS controller K,

$$a_{L+K}(\rho) = a_L(\rho) + a_K(\rho) \geq a > 0.$$

Consequently:

- *ρ is not stationary for $L + K$;*
- *no time-homogeneous additive GKLS controller stabilizes ρ exactly;*
- *if K_t is any time-dependent family of GKLS generators, a solution of $\dot{\rho}_t = (L + K_t)(\rho_t)$ cannot equal ρ on a nonzero time interval.*

Proof. Linearity gives the equality and Lemma 2.1 gives $a_K(\rho) \geq 0$. A stationary target would require $(L+K)(\rho) = 0$, contradicting its positive Q trace. If a differentiable trajectory were identically ρ on an interval, its Q population would have derivative at least a almost everywhere on that interval, another contradiction. □

Remark 2.3 (Hamiltonian control). An arbitrarily strong Hamiltonian controller still has zero instantaneous outward support trace at ρ. It may rotate the support and change off-diagonal blocks, but it cannot cancel a positive dissipative support flow. This separates state-space direction from operator-norm magnitude.

Corollary 2.4 (Exact code-space invariance cannot be repaired additively)**.** *Let P project onto a code space and put $\rho_P = P/\mathrm{Tr}(P)$. If an uncontrolled GKLS jump satisfies*

$$(1 - P)L_j P \neq 0$$

with positive rate, then no additive GKLS controller can make the code space invariant for the combined dynamics.

Proof. The state ρ_P is faithful on the code space. Lemma 2.1 gives $a_L(\rho_P) > 0$, whereas invariance of the code space would require zero outward derivative at every code state, including ρ_P. Apply Theorem 2.2. □

This statement does not deny autonomous error correction: recovery may return population after an error. It says that finite-rate additive recovery does not turn a genuinely leaky code into an exactly invariant one.

Theorem 2.5 (Ancilla-assisted exact no-cancellation)**.** *Let A be an arbitrary finite-dimensional autonomous controller. Suppose the system noise acts as $L \otimes \mathrm{id}_A$, while an arbitrary joint GKLS generator $K_{SA}(t)$ is added. If the system marginal is ρ at some time, then for the joint state Ω,*

$$\mathrm{Tr}\big[(Q \otimes 1_A)((L \otimes \mathrm{id}_A) + K_{SA}(t))(\Omega)\big] \geq a.$$

Consequently no finite-dimensional joint Markovian controller can keep the system marginal identically equal to ρ on a nonzero interval.

Proof. Since $\mathrm{Tr}\big[(Q \otimes 1_A)\Omega\big] = \mathrm{Tr}(Q\rho) = 0$ and $\Omega \geq 0$, the joint state is supported inside $P \otimes H_A$. Thus $(Q \otimes 1_A)\Omega = \Omega(Q \otimes 1_A) = 0$. Repeating the proof of Lemma 2.1 with the projector $Q \otimes 1_A$ shows that the joint controller contributes a nonnegative outward system-support flow. The local noise contribution depends only on the system marginal and equals a. □

The theorem permits coherent system-ancilla coupling, controller dissipation, and time dependence. It does not cover a non-Markovian reduced model in which the nominal system noise itself is modified by memory or strong coupling.

3 A dynamic accuracy lower bound

Let $T_t = \exp(tL)$ be the uncontrolled semigroup and set

$$M = \|L\|_{1\to 1}.$$

An admissible rate-Γ Markovian controller is a measurable family K_t of GKLS generators satisfying

$$\operatorname*{ess\,sup}_t \|K_t\|_{1\to 1} \le \Gamma.$$

The controlled state is an absolutely continuous solution of

$$\dot{\sigma}_t = (L + K_t)(\sigma_t).$$

This class contains autonomous controllers and prescribed switching protocols. It also contains feedback laws whose unconditional closed-loop master equation has this additive GKLS form and budget. It does not include nonlinear conditional-state feedback equations.

Lemma 3.1 (Trace-zero projector bound)**.** *For every Hermitian trace-zero operator X and every projector Q,*

$$|\operatorname{Tr}(QX)| \le \frac{1}{2}\|X\|_1.$$

Proof. Write $X = X_+ - X_-$ with orthogonal positive and negative parts. Trace zero gives $\operatorname{Tr} X_+ = \operatorname{Tr} X_- = \frac{1}{2}\|X\|_1$. Since $0 \le Q \le 1$, both $\operatorname{Tr}(QX_+)$ and $\operatorname{Tr}(QX_-)$ lie in the interval from zero to this common trace. □

Theorem 3.2 (Universal dynamic corridor obstruction)**.** *Let $a = a_L(\rho) > 0$. Every admissible controlled trajectory satisfies*

$$\limsup_{t\to\infty} \|\sigma_t - \rho\|_1 \ge 2a/(M + \Gamma).$$

More locally, if $\|\sigma_t - \rho\|_1 \le \delta$ throughout a time interval, then almost everywhere on that interval

$$\frac{d}{dt}\operatorname{Tr}(Q\sigma_t) \ge a - \frac{1}{2}(M + \Gamma)\delta.$$

Proof. Put $d_t = \|\sigma_t - \rho\|_1$ and $q_t = \operatorname{Tr}(Q\sigma_t)$. At almost every time,

$$\dot{q}_t = \operatorname{Tr}\big[Q(L + K_t)(\rho)\big] + \operatorname{Tr}\big[Q(L + K_t)(\sigma_t - \rho)\big].$$

The first term is at least a by Theorem 2.2. The second output is Hermitian and trace zero, so Lemma 3.1 and the induced-norm budget give

$$\dot{q}_t \ge a - \frac{1}{2}(M + \Gamma)d_t.$$

This proves the local statement. Suppose the displayed limsup were strictly smaller than $2a/(M + \Gamma)$. Choose δ strictly between them. Eventually $d_t \le \delta$, so $\dot{q}_t$ is bounded below by a positive constant. This is impossible because $0 \le q_t \le 1$. □

Corollary 3.3 (Stationary-state error)**.** *If an admissible autonomous controller K has a stationary state σ, then*

$$\|\sigma - \rho\|_1 \ge 2a/(M + \|K\|_{1\to 1}).$$

Proof. Insert the constant trajectory $\sigma_t = \sigma$ into Theorem 3.2, or set $\dot{q} = 0$ in its proof. □

Corollary 3.4 (Finite escape time from a forbidden corridor)**.** *Assume $\sigma_0 = \rho$ and choose $\delta < 2a/(M+\Gamma)$. The trajectory must leave the closed trace-distance ball of radius δ no later than*

$$t_{\text{exit}} \le \delta/\big[2a - (M+\Gamma)\delta\big].$$

Proof. As long as the trajectory remains in the ball, Theorem 3.2 gives $q_t \ge ct$, where $c = a - (M+\Gamma)\delta/2$. But the binary measurement $\{P, Q\}$ implies $q_t \le \frac{1}{2}\|\sigma_t - \rho\|_1 \le \delta/2$. Therefore $ct \le \delta/2$. □

The result is a corridor theorem for arbitrary bounded Markovian control, not merely a lower bound for a chosen periodic protocol.

4 Autonomous achievability by Poisson resetting

The lower bound has the optimal inverse-rate order. We prove this using a reset generator, while keeping its known status explicit. Define the replacement channel

$$\Phi_\rho(X) = \mathrm{Tr}(X)\rho$$

and the reset generator

$$R_\rho = \Phi_\rho - \mathrm{id}.$$

Lemma 4.1 (The reset generator is GKLS)**.** *R_ρ is a GKLS generator and*

$$\|R_\rho\|_{1\to 1} \le 2.$$

Proof. Let $\rho = \sum_i p_i |i\rangle\langle i|$ and introduce jumps

$$J_{ij} = \sqrt{p_i}|i\rangle\langle j|.$$

Then

$$\sum_{ij} J_{ij} X J_{ij}^\dagger = \mathrm{Tr}(X)\rho,$$

while $\sum_{ij} J_{ij}^\dagger J_{ij} = 1$. The sum of their dissipators is therefore $\Phi_\rho(X) - X$. The norm bound follows from $\|\Phi_\rho(X)\|_1 \le \|X\|_1$ on Hermitian operators. □

For a reset rate $\kappa > 0$, let $S_t^{(\kappa)}$ be generated by

$$L + \kappa R_\rho.$$

Theorem 4.2 (Renewal representation and uniform corridor)**.** *For every initial density matrix σ_0,*

$$S_t^{(\kappa)}(\sigma_0) = e^{-\kappa t} T_t(\sigma_0) + \int_0^t \kappa e^{-\kappa s} T_s(\rho) ds.$$

In particular, if $\sigma_0 = \rho$, then

$$\sup_{t\ge 0} \|S_t^{(\kappa)}(\rho) - \rho\|_1 \le \|L(\rho)\|_1/\kappa.$$

Proof. The displayed state is the mixture over the age since the most recent Poisson reset, with the no-reset event as the first term. Direct differentiation verifies the master equation and initial condition. Contractivity of T_t on Hermitian operators gives

$$\begin{aligned}&\|T_s(\rho)-\rho\|_1\\ \le&\int_0^s \|T_u L(\rho)\|_1 du\\ \le&\ s\|L(\rho)\|_1.\end{aligned}$$

Because the scalar weights in the renewal formula sum to one, subtraction of ρ and integration yield exactly

$$\begin{aligned}&\|S_t^{(\kappa)}(\rho)-\rho\|_1\\ \le&\ \|L(\rho)\|_1(1-e^{-\kappa t})/\kappa.\end{aligned}$$

□

Theorem 4.3 (Unique reset stationary state and exact leakage coefficient)**.** *The reset dynamics has the unique stationary state*

$$\sigma_\kappa=\int_0^\infty \kappa e^{-\kappa t}T_t(\rho)dt.$$

It satisfies

$$\|\sigma_\kappa-\rho\|_1\le\|L(\rho)\|_1/\kappa.$$

Moreover, as $\kappa\to\infty$,

$$\mathrm{Tr}(Q\sigma_\kappa)=a/\kappa+O(\kappa^{-2}).$$

Proof. For two initial states, the renewal formula makes their difference $e^{-\kappa t}T_t(\sigma-\sigma')$; its trace norm contracts at least as $e^{-\kappa t}$. Hence there is at most one stationary state, and the infinite-time renewal integral supplies it. The distance bound is the limit of Theorem 4.2. Analyticity in finite dimension gives

$$\mathrm{Tr}\big[QT_t(\rho)\big]=at+O(t^2).$$

Laplace integration gives

$$\int_0^\infty \kappa e^{-\kappa t}atdt=a/\kappa,$$

and the quadratic remainder contributes $O(\kappa^{-2})$. □

Theorem 4.4 (Order-optimal minimax inverse-rate law)**.** *Define*

$$E(\Gamma)=\inf_{K_t}\limsup_{t\to\infty}\|\sigma_t-\rho\|_1,$$

where the infimum ranges over admissible additive Markovian controllers with norm budget Γ and trajectories starting at ρ. Then

$$2a/(M+\Gamma)\le E(\Gamma)\le 2\|L(\rho)\|_1/\Gamma.$$

Consequently

$$E(\Gamma)=\Theta(\Gamma^{-1})$$

as $\Gamma\to\infty$.

Proof. The lower bound is Theorem 3.2. For the upper bound choose the autonomous reset controller with $\kappa=\Gamma/2$. Lemma 4.1 makes its induced norm at most Γ, and Theorem 4.2 supplies the stated error. □

The theorem compares all bounded time-dependent additive GKLS controllers against one explicit autonomous controller. It does not say that reset is constant-optimal.

Remark 4.5 (The constant gap is genuine). The lower coefficient is controlled by the exterior flow a, while the reset upper bound is controlled by the full tangent speed $\|L(\rho)\|_1$. Their ratio can be large when the uncontrolled generator produces substantial motion inside the support or large off-diagonal motion but only weak exterior leakage. Reset corrects all of that motion indiscriminately. A controller tailored to the decomposition may improve the constant, but Theorem 4.4 proves only the optimal order Γ^{-1}. Determining the exact minimax constant is open.

5 Exact saturating models and sharp boundaries

Proposition 5.1 (Two-state leakage and recovery)**.** *Let $\rho = |0\rangle\langle 0|$. Let the uncontrolled classical GKLS dynamics have jumps $0 \to 1$ at rate $\lambda > 0$ and $1 \to 0$ at rate $\mu \geq 0$. Add a recovery jump $1 \to 0$ at rate κ. The unique stationary error population is*

$$q_\kappa = \lambda/(\lambda + \mu + \kappa).$$

Hence

$$\lim_{\kappa\to\infty} \kappa q_\kappa = \lambda = a_L(\rho)$$

and

$$\|\sigma_\kappa - \rho\|_1 = 2q_\kappa.$$

Proof. The scalar master equation is

$$\dot{q} = \lambda(1 - q) - (\mu + \kappa)q.$$

Set its right-hand side to zero. □

This model shows that the coefficient a in Theorem 4.3 is not an artifact of the reset representation.

Proposition 5.2 (Hamiltonian strength cannot repair leakage)**.** *In the same two-level system, add an arbitrary Hamiltonian H_c. At the target,*

$$\mathrm{Tr}\big[Q(-i\big[H_c, \rho\big])\big] = 0,$$

so the outward derivative remains λ. Exact stabilization is impossible at every finite Hamiltonian strength.

Remark 5.3 (How the theorem can be evaded). An instantaneous replacement map is discontinuous in the continuous-time norm budget and corresponds to $\Gamma = \infty$. Measurement-conditioned feedback follows a stochastic nonlinear equation before averaging. A non-Markovian controller may temporarily return information from a memory. None contradicts Theorem 3.2 because none belongs to its controller class.

6 Uniform families and exponent transfer

Consider a sequence indexed by width ℓ, with targets ρ_ℓ, uncontrolled generators L_ℓ, and leakage coefficients a_ℓ. Assume

$$\|L_\ell\|_{1\to 1} = \exp\big[o(\ell)\big].$$

Write $M_\ell := \|L_\ell\|_{1\to 1}$. Theorem 3.2 is pointwise in the generator and therefore applies separately at every ℓ, with this width-dependent M_ℓ; it never requires one common bound in ℓ. The subexponential hypothesis permits polynomial growth from the number of Bohr blocks, so uniform boundedness is not required. Let the controller budgets obey $\Gamma_\ell = \exp[g\ell + o(\ell)]$, $g \geq 0$, and suppose $a_\ell = \exp[-\alpha\ell + o(\ell)]$, $\alpha > 0$.

Theorem 6.1 (Universal exponent ceiling under bounded Markovian control)**.** *For every family of admissible controllers,*

$$\limsup_{\ell\to\infty}[-1/\ell]$$
$$\log(\limsup_{t\to\infty}\|\sigma_{\ell,t} - \rho_\ell\|_1)$$
$$\leq \alpha + g.$$

Proof. Applying Theorem 3.2 at each fixed ℓ, with $M = M_\ell$, gives an error at least

$$2a_\ell/(\|L_\ell\|_{1\to1} + \Gamma_\ell).$$

The denominator has exponential rate g when $g > 0$ and rate zero when $g = 0$. Taking logarithms proves the claim. □

Thus geometry and controller strength contribute additively to the best possible accuracy exponent. This conclusion is independent of the controller construction.

7 A self-contained SSH control horizon

We now derive the microscopic inputs rather than importing them from companion preprints.

7.1 Dual-rail Hamiltonian and exact zero mode

Take two identical rails $r = 0, 1$, with A sites $j = 0, \ldots, \ell$ and B sites $j = 0, \ldots, \ell - 1$. In the one-particle sector,

$$H_\ell = \sum_{r,j}\big[t_1 a^\dagger_{r,j} b_{r,j}$$
$$+t_2 a^\dagger_{r,j+1} b_{r,j} + \text{h.c.}\big],$$

where $0 < t_1 < t_2$. Put $\zeta = t_1/t_2 \in (0, 1)$, $m = -\log\zeta$. Each rail has the exact normalized zero mode

$$f^\dagger_{0,r} = N_\ell \sum_{j=0}^{\ell}(-\zeta)^j a^\dagger_{r,j},$$
$$N_\ell^2 = (1 - \zeta^2)/(1 - \zeta^{2\ell+2}).$$

Its site weight is

$$w_x = |\langle vac|a_{r,x} f^\dagger_{0,r}|vac\rangle|^2.$$

At the remote site $x = \ell$,

$$w_{\text{far}} = (1 - \zeta^2)\zeta^{2\ell}/(1 - \zeta^{2\ell+2}).$$

The fixed-parity logical code is $|0_L\rangle = f^\dagger_{0,0}|vac\rangle$, $|1_L\rangle = f^\dagger_{0,1}|vac\rangle$. Both words contain one fermion. Let ρ_ℓ be any fixed logical density matrix faithful on this two-dimensional code, including physical coherence between the rails. At boundary x, use the even number-conserving transfer

$$S_x = a^\dagger_{1,x} a_{0,x}$$

and its adjoint.

Proposition 7.1 (Exact logical and bulk sum rules). *For a transfer from rail zero to rail one,*

$$|\langle 1_L|S_x|0_L\rangle|^2 = w_x^2,$$

while, summed over any orthonormal bulk eigenbasis $\{|k,1\rangle\}$ *of the destination rail,*

$$\sum_k |\langle k,1|S_x|0_L\rangle|^2 = w_x(1-w_x).$$

The reverse transfer obeys the same identities.

Proof. Expand the boundary field on either rail as

$$a_{r,x} = \psi_x f_{0,r} + \sum_k u_{xk} f_{k,r},$$

where $|\psi_x|^2 = w_x$. Acting on $|0_L\rangle$ gives

$$\begin{aligned} S_x|0_L\rangle = \psi_x \big[&\overline{\psi_x}|1_L\rangle \\ &+ \sum_k \overline{u_{xk}}|k,1\rangle\big]. \end{aligned}$$

The logical squared amplitude is w_x^2. Completeness at the site gives

$$\sum_k |u_{xk}|^2 = 1 - |\psi_x|^2 = 1 - w_x,$$

so the bulk norm is $w_x(1-w_x)$. If bulk eigenvalues are degenerate, a basis change inside the degenerate subspace leaves its spectral-projector norm and hence the sum unchanged. □

At the remote boundary this proves directly

$$\begin{aligned} w_{\mathrm{far}}^2 &= \exp\big[-4m\ell + O(1)\big], \\ w_{\mathrm{far}}(1-w_{\mathrm{far}}) &= \exp\big[-2m\ell + O(1)\big]. \end{aligned}$$

7.2 Davies generator and spectral hypotheses

Let the secular Davies generator for the boundary transfer be

$$\begin{aligned} L_\ell = \sum_\omega \big[&G_\ell(\omega)\mathcal{D}[A_\omega] \\ &+ G_\ell(-\omega)\mathcal{D}[A_\omega^\dagger]\big], \end{aligned}$$

where

$$A_\omega = \sum_{E'-E=\omega} \Pi_E S_x \Pi'_E.$$

The two rail directions may be listed separately; writing one plus its adjoint is only notation. Assume:

- the desired zero-mode-to-zero-mode coefficients lie between fixed positive constants d_- and d_+;
- every zero-mode-to-bulk coefficient lies between $g_-\varepsilon_\ell$ and $g_+\varepsilon_\ell$, including its KMS reverse, with $0 < g_- \le g_+$ independent of width;
- bath coefficients on every retained one-particle Bohr block are bounded above independently of width;
- the symmetric logical Lamb shift commutes with ρ_ℓ, or has target velocity bounded by the same $w_{\mathrm{far}}^2 + \varepsilon_\ell w_{\mathrm{far}}(1-w_{\mathrm{far}})$ scale.

Let $\varepsilon_\ell = \exp\big[-q\ell + o(\ell)\big]$, $q \ge 0$. Exact filtering is the separate case $\varepsilon_\ell = 0$.

Proposition 7.2 (Microscopic leakage bounds)**.** *For the remote-boundary coupling,*

$$g_{-}\varepsilon_\ell w_{\mathrm{far}}(1-w_{\mathrm{far}}) \le a_\ell$$
$$\le g_{+}\varepsilon_\ell w_{\mathrm{far}}(1-w_{\mathrm{far}}).$$

Consequently

$$a_\ell = \exp\bigl[-(2m+q)\ell + o(\ell)\bigr],$$

while the retained logical dissipator has scale

$$s_\ell = \exp\bigl[-4m\ell + O(1)\bigr].$$

Proof. Lemma 2.1 expresses a_ℓ as the sum of squared support-crossing amplitudes. For the forward logical population the unweighted sum is $p_0 w_{\mathrm{far}}(1-w_{\mathrm{far}})$; for the reverse population it is $p_1 w_{\mathrm{far}}(1-w_{\mathrm{far}})$. Since $p_0+p_1=1$, the two-sided spectral envelope and Proposition 7.1 give the bound. Logical coherence does not contribute to the outward trace because distinct secular jump outputs belong to orthogonal bulk sectors. Degenerate or coincident Bohr frequencies do not change the result: $A_\omega|0_L\rangle$ is a vector in the corresponding spectral subspace, and its squared norm is basis independent. Summing those orthogonal spectral-projector norms over ω recovers Proposition 7.1. □

Proposition 7.3 (No hidden exponential mode factor)**.** *In the interaction picture described above,*

$$\|L_\ell(\rho_\ell)\|_1$$
$$\le C_d w_{\mathrm{far}}^2 + C_c\varepsilon_\ell w_{\mathrm{far}}(1-w_{\mathrm{far}})$$

with width-independent constants. One may take $C_d = 8d_+$ and $C_c = 8g_+$ when the two rail directions and their KMS reverses are listed as four jump families. Moreover,

$$\|L_\ell\|_{1\to1} = \exp\bigl[o(\ell)\bigr].$$

Proof. For every jump J,

$$\|\mathcal{D}\bigl[J\bigr]\|_{1\to1} \le 2\|J\|^2.$$

Compress every secular jump to its logical source block before estimating it. Energy-block orthogonality makes bulk-source pieces annihilate ρ_ℓ and removes mixed terms. For the remaining compression, $\|A_\omega P\|^2 \le \mathrm{Tr}(PA_\omega^\dagger A_\omega P)$, and summing these traces over orthogonal Bohr projectors gives exactly the logical and crossing sums of Proposition 7.1. Applying $\|\mathcal{D}\bigl[J\bigr]\|_{1\to1} \le 2\|J\|^2$ and the upper spectral envelopes gives the displayed constants. Bulk-to-bulk jumps and their anticommutators annihilate the initially logical target. This proves the target-velocity bound without multiplying by the number of bulk modes. For the full induced norm, a crude sum over one-particle Bohr blocks may introduce a polynomial factor because their number is polynomial in ℓ. Each block operator and bath coefficient is uniformly bounded, so this estimate is still $\exp\bigl[o(\ell)\bigr]$, exactly the hypothesis required in Section 6. □

The secular generator preserves a direct sum of logical and bulk density blocks: its Hamiltonian part is energy-block diagonal, jump terms map energy eigenspaces to energy eigenspaces, and anticommutators do not create logical-bulk coherence from a logical input. No logical rate is inherited by assumption; both scales were computed from boundary matrix elements.

Theorem 7.4 (Necessary autonomous controller exponent)**.** *Suppose a controller family has budget*

$$\Gamma_\ell = \exp\bigl[g\ell + o(\ell)\bigr].$$

If its long-time trace error is at most the ideal logical scale,

$$\limsup_{t\to\infty}\|\sigma_{\ell,t} - \rho_\ell\|_1$$
$$\le \exp\bigl[-4m\ell + o(\ell)\bigr],$$

then necessarily

$$g \ge \max\{0, 2m-q\}.$$

Proof. Theorem 6.1 and Proposition 7.2 say that the achievable error exponent cannot exceed $2m+q+g$. Reaching exponent $4m$ requires $g \geq 2m - q$; combine this with $g \geq 0$. □

Theorem 7.5 (Global-reset achievability and exact exponent threshold)**.** *Use the autonomous reset controller with $\kappa_\ell = \Gamma_\ell/2$. Then*

$$\limsup_{t\to\infty} \|\sigma_{\ell,t} - \rho_\ell\|_1 \leq \exp\big[-(\min\{4m, 2m+q\} + g)\ell + o(\ell)\big].$$

Consequently trace accuracy at the ideal logical scale is achievable whenever

$$g \geq \max\{0, 2m - q\}.$$

Combined with Theorem 7.4, the minimum controller-growth exponent is exactly

$$g_{\min} = \max\{0, 2m - q\}.$$

Proof. Theorem 4.2 and Proposition 7.3 give

$$\mathit{error} \leq (2/\Gamma_\ell)\big[C_d w_{\mathrm{far}}^2 + C_c \varepsilon_\ell w_{\mathrm{far}}(1 - w_{\mathrm{far}})\big].$$

The bracket has exponential rate $\min\{4m, 2m+q\}$. Division by the controller budget adds g. If $q < 2m$, choosing $g \geq 2m - q$ raises the exponent to at least $4m$; if $q \geq 2m$, a width-independent positive reset rate already suffices. Necessity is Theorem 7.4. □

The achievability witness is global replacement and is not claimed to be quasi-local.

Corollary 7.6 (Subexponential total controller rate fails below threshold)**.** *Suppose the total controller satisfies*

$$\sup_t \|K_\ell(t)\|_{1\to1} = \exp\big[o(\ell)\big].$$

If $q < 2m$, it cannot retain long-time trace accuracy at the ideal $\exp\big[-4m\ell + o(\ell)\big]$ *scale. In particular, this applies when $K_\ell(t) = \sum_{j=1}^{N_\ell} K_{\ell,j}(t)$, where N_ℓ and every induced norm $\|K_{\ell,j}(t)\|_{1\to1}$ grow at most polynomially with ℓ. The terms may be local or nonlocal; locality is not used in the proof.*

Proof. The assumed total induced norm has exponential rate zero, whereas Theorem 7.4 requires $g \geq 2m - q > 0$, a contradiction. For the stated polynomial decomposition, the triangle inequality verifies the total-rate hypothesis. □

This is a global norm-budget obstruction, not a geometric locality theorem: it uses no interaction range, light cone, or Lieb-Robinson estimate. Bounded-density local controllers are one physically relevant subclass, but the same conclusion holds for polynomially many nonlocal terms of polynomial strength. The corollary does not provide a quasi-local matching controller above the threshold.

Corollary 7.7 (Three regimes)**.**

- ***Exact filter or*** $q > 2m$***.*** *A width-independent global reset retains ideal-scale accuracy.*
- ***Threshold filter*** $q = 2m$***.*** *A width-independent global reset remains sufficient, with constants deciding the prefactor.*
- ***Soft tail*** $q < 2m$***.*** *The minimum total controller intensity grows as* $\exp\big[(2m - q)\ell + o(\ell)\big]$.

For a fixed nonzero tail, $q = 0$, widening the membrane while retaining ideal accuracy requires total controller intensity $\exp\big[2m\ell + o(\ell)\big] = \zeta^{-2\ell + o(\ell)}$*. This is the autonomous control horizon. The necessity is valid for every norm-bounded additive GKLS controller; the sufficient construction is global, and quasi-local achievability remains open.*

8 Physical boundary of the control model

8.1 When additivity is justified

The equation

$$\dot{\sigma}_t = (L + K_t)(\sigma_t)$$

is appropriate when the adverse environment and engineered controller admit separate Markovian weak-coupling descriptions, or when an unconditional feedback reduction produces an additional GKLS term without changing the original noise channel. In that regime the positive support flows add, and the theorem applies literally. The ancilla extension allows coherent joint system-controller Hamiltonians and joint dissipation, but assumes the adverse system noise remains $L \otimes \mathrm{id}_A$. The model does not cover strong coupling that dresses the system spectrum, coherent interference between adverse and engineered couplings to a shared reservoir, adiabatic elimination that changes the effective jump operators, or non-Markovian memory that feeds information back into the system. In such architectures the decomposition into a fixed L plus an independent GKLS controller may fail. This is a physical change of model, not a counterexample to the additive theorem. The distinction is experimentally material. An additive recovery channel waits for leaked population and pumps it back. Reservoir interference or spectral dressing may instead remove or reshape the original leakage channel itself. The present no-go applies to the former and deliberately makes no claim about the latter.

8.2 Rate is not thermodynamic power

The controller budget in this paper is an induced generator norm. It is not a work rate. No universal conversion is possible without specifying controller reservoirs, energy levels, reverse transitions, and a resource ledger. The two-state model makes the issue explicit. The same recovery rate κ may be implemented by a chemically driven transition, by coupling to a cold reservoir, or by a reset cell. These implementations can have different work and entropy-production costs. Conversely, a large static energy bias can suppress a stationary error with little steady dissipation after its preparation cost is ignored. Therefore an autonomous law proportional to $a \log(1/\delta)$ requires a declared local-detailed-balance architecture and cannot be inferred from the rate theorem alone. This negative statement prevents the accuracy-rate result from being misreported as a universal work-only theorem. A thermodynamic extension is a separate project.

9 Numerical verification

The following independent checks accompany the analytic proofs.

- **Boundary cone.** For 250 random four-dimensional controller jumps, the smallest outward controller flow was $2.70e-2$; no negative value occurred. Adding every controller to a noise generator with $a = 0.2257$ increased the outward flow.
- **Renewal identity.** Direct matrix exponentiation and the Poisson-age integral agreed in Euclidean vector norm to $6.3e - 17$.
- **Inverse-rate coefficient.** In the exact two-state model, κq_κ at $\kappa = 10^4$ differed from a by $7.6e - 6$.
- **SSH geometry.** Direct diagonalization through width 34 gave worst far-weight error $8.3e - 17$ and worst $w(1-w)$ sum-rule residual $2.8e-17$. Random unitary changes of bulk basis at representative widths preserved the summed weight within $1.1e - 16$, testing the degeneracy-safe formulation.

The numerical suite is diagnostic and does not replace the proofs.

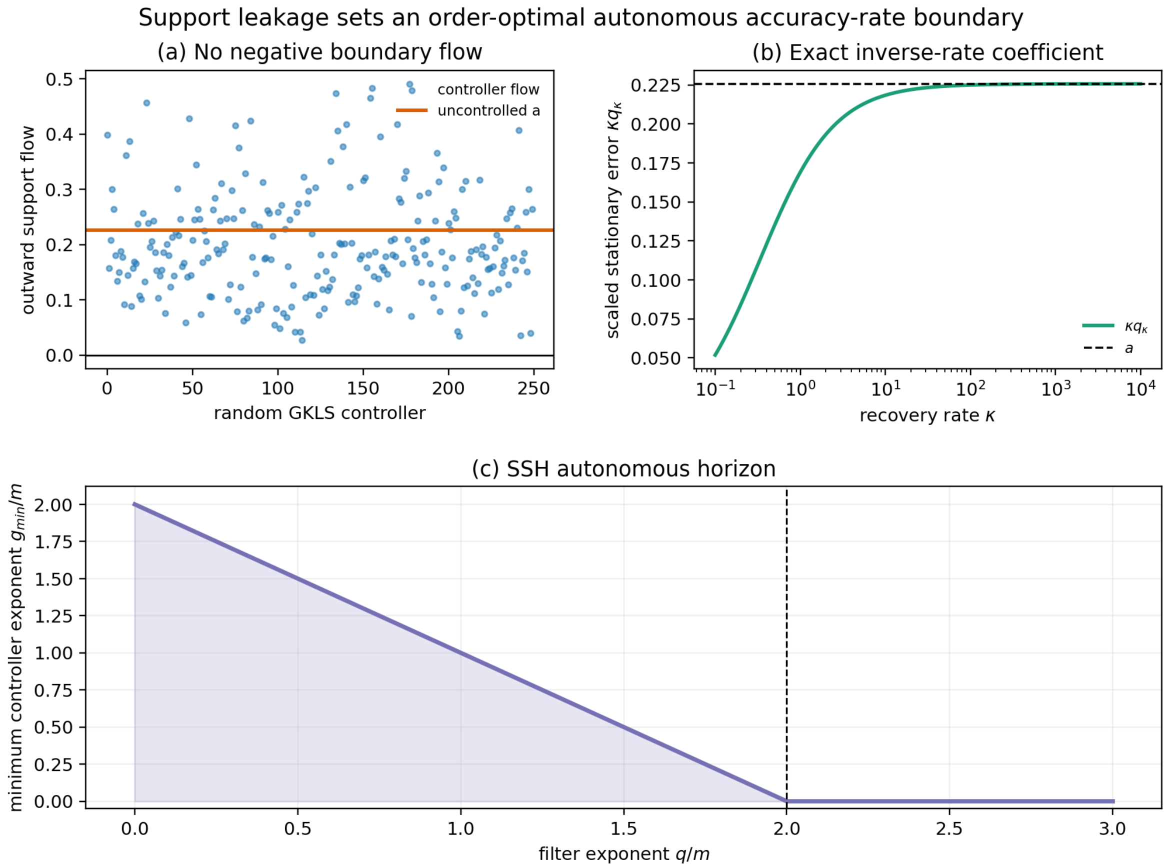

Figure 1: Left: a leaky boundary point cannot receive a negative outward current from an additive GKLS controller. Center: the two-state stationary error approaches a/κ. Right: the SSH controller-growth phase boundary $g_{\min} = \max\{0, 2m - q\}$.

10 Relation to prior work

The geometric engine must be attributed first. Carlen and Maas use the positive exterior-block tangent condition in their noncommutative transport framework. Cai, Govindarajan, and Junge state the boundary tangent cone as the Hermitian trace-zero directions X satisfying $QXQ \geq 0$ and prove that every such direction, and only such a direction, equals $G(\rho)$ for some Lindbladian G. Therefore the sign in Lemma 2.1 is an explicit jump-operator realization of known tangent-cone geometry, not a new characterization. The quantum-control literature also characterizes invariant and attractive subspaces and gives constructive Lindblad generators for prescribed steady states. Those results ask when engineered dynamics can stabilize a target. Our contribution fixes an adverse generator and converts the known one-sided geometry into an additive no-cancellation statement, a quantitative long-time corridor for arbitrary bounded time dependence, an ancilla extension, and a minimax inverse-rate law. Work on autonomous quantum error correction derives bounds on logical error in terms of correction and noise rates, including scaling requirements for many-body code families. In particular, Shtanko et al.'s global decoder has the algebraic form $\kappa(R - \mathrm{id})$ and already contains the recovery structure used by our reset witness. Their results concern correctable-error trajectories and logical memory rates. Our complementary lower bound concerns trace distance to an arbitrary rank-deficient target, assumes no code or decoder structure, covers every bounded time-dependent additive GKLS term, and is determined by the local support-flow coefficient. The SSH application then turns that coefficient into a membrane exponent. We do not replace their stronger code-specific bounds or claim the global-reset architecture as new. Stochastic resetting and quantum reset protocols already provide renewal stationary states and applications to state preparation. We use that established construction as the achievability witness in a minimax theorem. We do not claim the reset formula as new. The closest conceptual antecedents are tangent-cone geometry, invariant-subspace criteria, robust dissipative state preparation, quantum Zeno stabilization, and autonomous recovery-rate bounds. The new claim is the quantitative application: additive and ancilla no-cancellation, the dynamic corridor inequality, order-matching inverse-rate achievability, and the self-contained SSH controller exponent with its bounded-density local no-go.

11 Scope, falsification, and open extensions

Proved here:

- positive support flow for every GKLS contribution;
- impossibility of exact additive Markovian cancellation at a leaky target, including finite-dimensional joint Markovian ancillas;
- a long-time corridor lower bound for arbitrary bounded time-dependent GKLS controllers;
- an explicit autonomous reset protocol with uniform trajectory control;
- the minimax $\Theta(\Gamma^{-1})$ accuracy law;
- a microscopic SSH derivation with degeneracy-safe mode summation;
- a necessary-and-sufficient total controller-growth threshold for retaining ideal-scale accuracy;
- failure of polynomially many polynomial-rate local terms below the filter threshold.

Not proved:

- impossibility for non-Markovian or measurement-conditioned controllers;
- constant-optimality of Poisson resetting;
- a universal conversion from controller norm to work or entropy production;
- quasi-local achievability matching the total-rate exponent;
- uniform convergence from a microscopic system-bath model to the Davies controller family.

The universal theorem is falsified by a finite-dimensional additive GKLS controller with $a_L(\rho) > 0$ but $\mathrm{Tr}\big[Q(L+K)(\rho)\big] \le 0$. Lemma 2.1 rules this out under the stated assumptions. The SSH horizon is falsified if the microscopic crossing norm gains an unaccounted exponential volume factor, or if a norm-bounded controller attains ideal trace accuracy with growth exponent below $\max\{0, 2m - q\}$. The most valuable extension would replace the global reset witness by a quasi-local autonomous decoder with the same exponent. Corollary 7.6 proves only that subexponential total rate cannot succeed below threshold; bounded-density local control is a corollary, not the source of the obstruction. It does not construct a matching local controller when the total-rate condition is satisfied. A genuinely locality-dependent lower bound would require spatial input such as interaction range or propagation estimates. A separate thermodynamic extension could impose local detailed balance and price the stationary recovery current, but it must retain the distinction between rate, stored bias, and dissipated power.

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