Three Levels of a Single Vacuum-Flow Field Equation: Newtonian Gravity, the Hubble Flow, and Galactic Rotation Curves Without Dark Matter

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Abstract

Flat galactic rotation curves and the Baryonic Tully-Fisher relation have resisted derivation from first principles for five decades. A preceding empirical programme fitted the constant-Lagrangian (CL) condition $u_r^2 + u_\phi^2 = v_L^2 = \mathbf{const}$ to 175 galaxies from the SPARC database using only two physically anchored parameters — bulge radius R and enclosed mass M — and reproduced flat rotation curves, a universal one-third rule $v^2(R) = \frac{1}{3}v_{\text{final}}^2$, and the Tully-Fisher scaling, all without dark matter or modified gravity. That condition, however, was a postulate: no derivation existed.

The present paper provides that derivation by applying the Biquaternion Relativistic Fluid Dynamics (BQ-RFD) field hierarchy to the gravitational vacuum flow. The primary object is the gravitational four-velocity $U_g = G_g^{-1}(c\hat{T})G_g^{-1}$, generated exactly and invertibly from the rapidity rotor G_g; the metric $g_{\mu\nu}$ is a secondary, doubly-projected consequence that discards the Lorentz factor and phase of the rapidity field. The geometric field equation $\mathbf{H}_g = \partial^T U_g$ decomposes automatically into a continuity channel recovering the Poisson equation and $\theta = 3H$, a vorticity channel encoding the disk's rotation curve, and an acceleration channel recovering Newtonian gravity. The decomposition also reveals a continuity-balance radius $r_\sigma \approx 0.63\, r_c$ inside the velocity-balance radius $r_c = (2GM/H^2)^{1/3}$, invisible in any velocity-field formulation.

Adding an orbital rapidity through the sequential rotor $G = G_\phi G_r$ and imposing the exact vanishing of $\frac{1}{c^2}\frac{D}{Dt}(\frac{1}{2}v_g^2)$ in the norm channel — rather than discarding it as a slow-flow approximation — selects the CL condition as an exact dynamical output, with no external postulate. The Newtonian virial theorem at the bulge surface fixes v_L^2 in terms of G, M, R, H. On the CL solution the radial acceleration channel undergoes exact algebraic cancellation of all Newtonian and cross-coupling terms, leaving only $c\,\Pi_{g,r} = \dot{H}\,r$. In the de Sitter limit this vanishes identically and the CL profile is an exact stationary solution. The Tully-Fisher scaling $v_\infty^4 \propto (GMH)^{4/3}$ follows from the boundary conditions, and $r_c \propto a(t)$ is shown equivalent to matter domination. To our knowledge this is the first derivation in which flat galactic rotation curves, the Tully-Fisher scaling, and the cosmological comoving structure of galactic disks emerge simultaneously from a single first-order algebraic field equation, with no dark matter, no modified gravity, and no adjustable parameters beyond G, M, R, H. The framework positions Newtonian gravity, the Hubble flow, and the CL galactic disk as three levels of the same equation $\mathbf{H}_g = \partial^T U_g$, and constitutes a first-order precursor to General Relativity in the closed $\mathbf{M(2,\mathbb{C})}$ algebra.

Keywords: galactic rotation curves, cosmology, rapidity, vacuum flow gravity, rapidity rotor, constant-Lagrangian condition, Painlevé–Gullstrand metric, first-order gravity, Hubble flow, gravitational fluid dynamics

Contents

1 Introduction

Flat galactic rotation curves and the Baryonic Tully-Fisher relation $v_\infty^4 \propto GMa_0$ with $a_0 \approx cH_0$ have resisted purely Newtonian explanation for five decades (Famaey and McGaugh 2012). Within standard GR the Schwarzschild and Kerr metrics produce rotation profiles that fall off as $v \propto r^{-1/2}$ beyond the visible mass, while the FLRW metric describes a homogeneous expansion that cannot sustain a stable disk. The two standard resolutions — dark-matter halos (Navarro et al. 1997; Katz et al. 2017) and Modified Newtonian Dynamics (MOND) (Famaey and McGaugh 2012) — respectively import undetected matter or modify the gravitational law. Neither derives the rotation-curve behaviour from a deeper dynamical principle internal to the gravitational field itself.

A third route has been developing through the constant-Lagrangian (CL) framework introduced empirically in de Haas (de Haas 2018), based on the SPARC database (Lelli et al. 2016) and progressively developed in a series of empirical and theoretical companion papers (de Haas 2025f). The CL condition

$$u_r^2(r) + u_\phi^2(r) = v_L^2 = \text{const} \tag{1}$$

states that the total kinetic energy of the vacuum flow is conserved along streamlines: wherever the radial infall speed $|u_r|$ decreases, the orbital speed u_ϕ increases proportionally, keeping the total flow speed fixed. Substituting the radial component $u_r(r) = \sqrt{2GM/r} - H(t)\,r$ — the Painlevé–Gullstrand (PG) free-fall speed reduced by the cosmological Hubble outflow — yields the orbital profile

$$u_\phi(r) = \sqrt{v_L^2 - \left(\sqrt{2GM/r} - H\,r\right)^2}, \tag{2}$$

which is flat in the region $R \ll r \ll r_c$ where $r_c = (2GM/H^2)^{1/3}$ is the velocity-balance radius at which Newtonian infall and Hubble outflow cancel and where the boundary condition at the galactic bulge fixed the value of v_L.

The empirical programme.

The most recent and complete empirical test (de Haas 2025f) applied this formula to the full SPARC database of 175 late-type galaxies using only two physically interpretable parameters: the bulge transition radius R and the enclosed Newtonian mass M, with the Hubble parameter fixed at $H_z = 2.2 \times 10^{-18}$ s^{-1}. The outcomes were sharp. First, the parameter R was pinned to the convex-to-concave inflection point of the observed $v^2(r)$ curve — the morphological bulge-to-disk transition — so it was not a free scale. Second, M was the Newtonian dynamical mass enclosed at R, directly comparable with photometric bulge expectations; fitted values tracked the expected morphological scaling from $\sim 10^7\,M_\odot$ in dwarf irregulars to $\sim 10^{10}\,M_\odot$ in massive early-type spirals. Third, the universal relation

$$v^2(R) = \tfrac{1}{3}\,v_{\text{final}}^2 \tag{3}$$

held without any additional parameters, providing a falsifiable diagnostic linking the transition radius to the asymptotic plateau. Fourth, typical reduced chi-squared values fell in the range $\chi^2_\nu = 0.05$–0.4 with relative residuals below 10%, competitive with or better than MOND and cored dark-matter halo models while using fewer and more directly interpretable parameters (de Haas 2025f). The empirical results indicated that the CL condition (1) hinted at real physics, but it nevertheless remained a postulate. The question *why* total kinetic energy seemed conserved along streamlines — and *why* the particular inflow $u_r = \sqrt{2GM/r} - Hr$ seemed the physically realised one — had no answer internal to the framework.

The growing theoretical foundation.

The companion papers in the gravity-as-rapidity programme progressively embedded the CL condition in a first-order spinor algebraic structure (the Pauli algebra $\mathrm{M}(2,\mathbb{C})$) (de Haas 2025c). It showed that the PG river velocity $v_g = \sqrt{2GM/r}$ arises from the adjoint action of a gravitational rotor $G_g = \exp\left(\frac{1}{2}\boldsymbol{\psi}_g \cdot \hat{\boldsymbol{\sigma}}\right)$ on a Pauli basis in $\mathrm{M}(2,\mathbb{C})$, and that the Schwarzschild and the de Sitter metrics follow from this single rotor without invoking Christoffel symbols or second-order curvature equations (de Haas 2025e,a). De Haas (de Haas 2025b) combined the Newtonian and Hubble rapidities into a sequential rotor $\psi_{\text{tot}} = \psi_M - \psi_H$ and showed that the single-metric model simultaneously reproduces rotation curves, spiral arm morphology, and SMBH growth. De Haas (de Haas 2025d) demonstrated that the Hubble parameter enters the local rotation curve and spiral pitch directly through the combined rapidity ψ_{tot}. De Haas (de Haas 2026a) extended the model to high-redshift observations through nested spiral dynamics. Despite this progress, one gap remained. In all the above papers the CL condition (1) was either imposed as a postulate, identified as the Bernoulli–Noether closure $v^\mu D_\mu Q_g = 0$, or motivated by GR-level consistency conditions. A derivation that produces (1) as an *output* of a single, self-contained field equation — internal to the framework, without external closure assumptions — was still missing.

What this paper does.

The present paper closes that gap by applying the Biquaternion Relativistic Fluid Dynamics (BQ-RFD) hierarchy of the companion paper de Haas (de Haas 2026b) to the gravitational vacuum flow. The BQ algebra has a long history (Silberstein 1907; Conway 1911; Lanczos 1929; Hestenes 1966; Doran and Lasenby 2003), and, in the specific formulation of (de Haas 2026b), provides a channel-decomposition framework in which a single product $\mathbf{H} = \partial^T \mathbf{G}$ simultaneously and automatically generates a scalar channel (continuity), a $\hat{K}$-channel (vorticity), and a $\hat{\boldsymbol{\sigma}}$-channel (acceleration/momentum) as three independent projections, without separate physical motivation for each because the channel production is purely algorithmic.

We identify the gravitational four-velocity

$$U_g = G_g^{-1}(c\hat{T})G_g^{-1} = c\cosh\psi_g\,\hat{T} + c\sinh\psi_g\,\hat{n}_g\cdot\hat{K} \tag{4}$$

as the primary gravitational object: it is the exact, linear, and invertible projection of the rapidity rotor G_g, retaining the full rapidity content that the metric projection discards. The geometric field equation $\mathbf{H}_g = \partial^T U_g$ then serves as the fundamental field equation of the vacuum flow. Its channel decomposition automatically and simultaneously yields:

1. **Continuity channel** σ_{Eg}: recovers the Poisson equation and the FRW expansion scalar $\theta = 3H$, and reveals a continuity-balance radius $r_\sigma \approx 0.63\,r_c$ distinct from the velocity-balance radius r_c — a two-radius structure invisible in any velocity-field formulation.
2. **Vorticity channel** $\mathbf{\Omega}_g$: encodes the geometric circulation of the galactic disk through the gradient of specific angular momentum $\ell = r\,u_\phi$.
3. **Acceleration channel** $\mathbf{\Pi}_g$: recovers Newtonian gravity as the channel projection $\mathbf{\Pi}_g = 0$ and, on the CL solution, reduces to the purely cosmological residual $c\,\Pi_{g,r} = \dot{H}\,r$.

The CL condition (1) arises from within the norm channel: the term $\frac{1}{c^2}\frac{D}{Dt}\left(\frac{1}{2}v_g^2\right)$ is normally discarded in the slow-flow approximation, but promoting it to an exact dynamical condition — setting it to zero rather than neglecting it — selects the CL solution family from within the channel structure of $\mathbf{H}_g$ without any external postulate. The Newtonian virial theorem at the bulge surface $r = R$ then fixes v_L^2 completely in terms of the observables G, M, R, H. On the resulting CL solution, the radial acceleration channel of $\mathbf{\Pi}_g$ undergoes exact algebraic cancellation of all Newtonian and cross-coupling terms, leaving the purely cosmological residual $c\,\Pi_{g,r} = \dot{H}\,r$. In the de Sitter limit $\dot{H} = 0$ this residual vanishes identically, and the CL profile is an exact stationary solution of the complete channel system. The Baryonic Tully-Fisher scaling $v_\infty^4 \propto (GMH)^{4/3}$ then follows from the CL boundary condition when $R = r_c$, without dark matter, modified gravity, or free parameters.

To our knowledge this is the first derivation in which flat galactic rotation curves, the Tully-Fisher scaling, and the cosmological comoving structure of galactic disks emerge simultaneously from a single first-order algebraic field equation, with each result appearing as a channel projection of $\mathbf{H}_g = \partial^T U_g$ and with no adjustable parameters beyond the observables G, M, R, H. The CL condition — the empirical law identified in (de Haas 2025f) — is not imported; it is derived.

The metric $g_{\mu\nu}$ arises as a secondary consequence of G_g: after a slow-flow approximation that discards the Lorentz factor, followed by a coframe quadratic that discards the phase of the rapidity field. It therefore carries strictly less information than U_g. General Relativity (Misner et al. 1973; Friedmann 1922) describes the second-order curvature consequences of the rapidity field; the present framework describes the field itself and is in this sense a first-order precursor to GR.

Organisation.

Section 2 develops the rapidity representation of the vacuum flow, establishes the common rotor structure shared by fluid dynamics and gravity, and identifies U_g as the primary gravitational object. Section 3 derives the channel decomposition of $\mathbf{H}_g = \partial^T U_g$ for the Newton-only flow, recovering the Poisson equation, irrotationality, and Newtonian gravity as three automatic channel projections. Section 4 adds the Hubble rapidity and derives the two-radius structure $r_\sigma \approx 0.63\, r_c$. Section 5 extends to the sequential rotor $G = G_\phi G_r$, activates the vorticity channel, and demonstrates the exact cancellation in the azimuthal acceleration channel. Section 6 imposes the CL condition, fixes v_L^2 by the virial theorem, derives the exact algebraic cancellation in the radial sigma channel, connects the residual $\dot{H}\, r$ to matter domination, and establishes the Tully-Fisher scaling. Section 7 concludes.

Throughout, we work in the slow-flow ($v_g \ll c$), weak-field ($GM/rc^2 \ll 1$, $Hr/c \ll 1$) approximation unless otherwise stated, and use SI units with c explicit. The BQ algebra, channel-decomposition rules, and the full relativistic fluid hierarchy are developed in (de Haas 2026b); all product rules and channel definitions used here are taken from that paper without re-derivation.

Conceptual structure of the derivation.

The technical steps in this paper are not difficult: the channel decomposition of $\mathbf{H}_g = \partial^T U_g$ is automatic once the product structure is in place, and the key cancellations in Sections 6 are each a few lines of algebra. The difficulty is conceptual, and it is worth signposting the four decisions that make the derivation possible before the reader encounters them in the text.

The first is the choice of U_g over $g_{\mu\nu}$ as the primary gravitational object. The rotor G_g projects onto U_g exactly, linearly, and invertibly in a single step; it projects onto $g_{\mu\nu}$ only after two successive information-reducing operations — a slow-flow approximation that discards the Lorentz factor, and a coframe quadratic that discards the phase of the rapidity field. Taking U_g as primary is not a matter of computational convenience; it is the correct reading of what the algebra preserves and what it throws away, as established in Section 2.

The second is the reinterpretation of the matter density ρ_0. In ordinary fluid dynamics ρ_0 constitutes the medium; here it is demoted to a passive tracer of a vacuum flow that exists independently of it. Dividing $\mathbf{H} = \partial^T(\rho_0 U_g)$ by ρ_0 to obtain $\mathbf{H}_g = \partial^T U_g$ is algebraically trivial, but it requires deciding that the vacuum itself is the dynamical object. This is an ontological step, not a mathematical one, and it is taken in Section 3.

The third is the treatment of a term that is normally discarded. The full norm channel of $\mathbf{H}_g$ contains the term $\frac{1}{c^2}\frac{D}{Dt}\big(\frac{1}{2}v_g^2\big)$, which is second order in v_g/c and is legitimately dropped in the slow-flow approximation. The conceptual move of Section 6 is to set this term to zero *exactly* rather than approximately, and to ask what that exact condition selects. The answer is the CL solution family (1) — the same condition that was identified as an empirical postulate in (de Haas 2025f) now emerges as an exact dynamical condition internal to the norm channel. The postulate is absorbed into the framework; nothing is imported from outside.

The fourth is the role of the empirical programme itself. The first systematic SPARC fits of (de Haas 2018; de Haas 2025f) established the destination before the theory was in place: the CL condition worked, the one-third rule held, and the Tully-Fisher scaling emerged from the fit parameters. The theoretical work of the present paper is in this sense a proof of something already known to be true, where the difficulty lay not in finding the result but in identifying the framework from which it becomes derivable. The reader is therefore invited to regard the empirical and theoretical papers as two legs of a single programme: the empirical work posed the question in the sharpest possible form, and the present paper answers it.

2 Rapidity representation of the velocity field

The relativistic fluid hierarchy developed in the preceding sections has been formulated in terms of the four-momentum density $\mathbf{G} = \rho_0 U$, where $U = u_0\hat{T} + \mathbf{u}\cdot\hat{K}$ is the four-velocity expressed in the BQ minquat basis $(\hat{T}, \hat{K})$. An alternative and often more natural parametrization is obtained by introducing a *local rapidity field* $\chi(\mathbf{x}, t)$.

For a velocity field $\mathbf{v}$, define $\beta = |\mathbf{v}|/c$ and the corresponding rapidity

$$\chi = \tanh^{-1}\beta = \tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right). \tag{5}$$

The Lorentz factor then takes the compact form $\gamma = \cosh\chi$, and $\gamma\beta = \sinh\chi$. Introducing the local flow-direction unit vector

$$\hat{n} = \frac{\mathbf{v}}{|\mathbf{v}|}, \tag{6}$$

the velocity field may be written as $\mathbf{v} = c\tanh\chi\,\hat{n}$. The pair $(\chi, \hat{n})$ therefore carries exactly the same information as $\mathbf{v}$.

The four-velocity adopts the compact form

$$U = c\cosh\chi\,\hat{T} + c\sinh\chi\,\hat{n}\cdot\hat{K}, \tag{7}$$

in the minquat $\hat{K}$-basis used throughout this paper. Within the BQ algebra the spatial basis elements satisfy $\hat{K} = \mathrm{i}\,\hat{\sigma}$, so the same four-velocity can equally be written as

$$U = c\cosh\chi\,\hat{T} - \mathrm{i}\,c\sinh\chi\,\hat{n}\cdot\hat{\sigma}. \tag{8}$$

The rotor that generates U acts on the timelike reference vector $c\hat{T}$ through the *double-sided* action

$$U = c\,R_\chi\,\hat{T}\,R_\chi^{-1}, \tag{9}$$

where the boost rotor is expressed in the $\hat{\sigma}$-basis as

$$R_\chi = \exp\!\left(\tfrac{\chi}{2}\,\hat{n}\cdot\hat{\sigma}\right) = \cosh\tfrac{\chi}{2}\,\hat{1} + \sinh\tfrac{\chi}{2}\,\hat{n}\cdot\hat{\sigma}. \tag{10}$$

The $\hat{\sigma}$-basis is the natural home for these boost rotors: in the BQ algebra the exponential $\exp(\tfrac{\chi}{2}\hat{n}\cdot\hat{\sigma})$ is a Pauli–rotor lying in $\mathrm{SL}(2,\mathbb{C})$, and its double-sided action on $c\hat{T}$ reproduces Eqns. (7)–(8) exactly through the adjoint (Lorentz-boost) action on the BQ basis.

The fluid four-momentum density $\mathbf{G} = \rho_0 U$ therefore becomes

$$\mathbf{G} = \rho_0 c\left(\cosh\chi\,\hat{T} + \sinh\chi\,\hat{n}\cdot\hat{K}\right), \tag{11}$$

or equivalently

$$\mathbf{G} = \rho_0 c\, R_\chi^{-1}\, (c\hat{T})\, R_\chi^{-1}, \tag{12}$$

and the full BQ fluid hierarchy $\mathbf{H} = \partial^T \mathbf{G}$ may be re-expressed entirely in terms of the three fields ρ_0, χ, and $\hat{n}$, rather than in terms of the velocity field itself.

The rapidity representation becomes particularly useful because derivatives of the flow separate naturally into magnitude and direction contributions. The temporal derivative $\partial_t \chi$ measures the local rate of change of rapidity, while $\nabla\chi$ measures spatial rapidity gradients. These quantities replace the corresponding velocity gradients and provide a direct relativistic measure of flow acceleration and spatial inhomogeneity.

The vorticity channel $\mathbf{\Omega} = \nabla \times \mathbf{g}$ of the BQ field $\mathbf{H}$, which appears as the $\hat{K}$-channel of $\mathbf{H} = \partial^T \mathbf{G}$, acquires a transparent structure in rapidity variables. Using $\mathbf{g} = \rho_0 c \sinh\chi\, \hat{n}$ one finds

$$\mathbf{\Omega} = \nabla \times (\rho_0 c \sinh\chi\, \hat{n}) = \rho_0 c \big[(\nabla \sinh\chi) \times \hat{n} + \sinh\chi\, (\nabla \times \hat{n}) \big], \tag{13}$$

for constant ρ_0. The first term is driven by the spatial gradient of the rapidity magnitude and vanishes when χ is spherically symmetric; the second term is the curl of the direction field and vanishes for irrotational flows. For irrotational flows the direction field may be derived from a scalar potential,

$$\hat{n} = \frac{\nabla\Phi}{|\nabla\Phi|}, \tag{14}$$

so that $\nabla \times \hat{n} = 0$ and the vorticity is controlled entirely by $\nabla\chi$. This is precisely the class of flows selected by the Constant-Lagrangian–Hubble (CL–H) condition $v_r^2 + v_\varphi^2 = v_L^2 = \text{const}$, as discussed in the companion gravitational papers.

2.1 From a global boost to a local rapidity-generator field

The preceding rapidity description still treats the flow direction $\hat{n}$ as an independent field. For fluid dynamics this is unavoidable: the flow direction varies throughout spacetime, so the boost direction is itself part of the dynamical state. This is in contrast to standard special relativity, where a global Lorentz boost between two inertial frames is represented by a single rotor

$$R = \exp\!\big(\tfrac{\chi}{2}\, \hat{\sigma}_r\big)\,, \tag{15}$$

with a globally fixed boost direction $\hat{\sigma}_r$ and a global rapidity parameter χ.

In a fluid, one must promote χ and $\hat{n}$ to a single *local rapidity-generator field*

$$\boldsymbol{\psi} = \psi_I \hat{\sigma}_I + \psi_J \hat{\sigma}_J + \psi_K \hat{\sigma}_K, \tag{16}$$

whose magnitude

$$\psi = |\boldsymbol{\psi}| \tag{17}$$

defines the local rapidity and whose direction

$$\hat{n} = \frac{\boldsymbol{\psi}}{|\boldsymbol{\psi}|} \tag{18}$$

defines the local boost direction. The corresponding local Pauli rotor is

$$G_\psi = \exp\left(\tfrac{1}{2}\,\boldsymbol{\psi}\cdot\hat{\sigma}\right) = \cosh\tfrac{\psi}{2}\,\hat{1} + \sinh\tfrac{\psi}{2}\,\hat{n}\cdot\hat{\sigma}, \tag{19}$$

acting in the $\hat{\sigma}$-pauliquat sector of the BQ algebra. Its double-sided action on the timelike reference vector generates the fluid four-velocity

$$\gamma V = G_\psi^{-1}(c\hat{T})G_\psi^{-1} = c\cosh\psi\,\hat{T} + c\sinh\psi\,\hat{n}\cdot\hat{K}, \tag{20}$$

from which the three-velocity follows as

$$\mathbf{v} = c\tanh\psi\,\hat{n} = c\,\frac{\tanh|\boldsymbol{\psi}|}{|\boldsymbol{\psi}|}\,\boldsymbol{\psi}. \tag{21}$$

Thus the velocity field becomes a *derived* quantity, generated by the local rapidity-generator field rather than introduced as an independent input. The energy and momentum densities take the forms

$$\varepsilon = \rho_0 c^2\cosh\psi, \tag{22}$$

$$\mathbf{g} = \rho_0 c\sinh\psi\,\hat{n}, \tag{23}$$

and the complete fluid four-momentum density is

$$\mathbf{G} = \rho_0\,G_\psi^{-1}(c\hat{T})G_\psi^{-1} = \rho_0 c\big(\cosh\psi\,\hat{T} + \sinh\psi\,\hat{n}\cdot\hat{K}\big)\,. \tag{24}$$

The entire fluid hierarchy may then be rewritten in terms of the rapidity-generator field,

$$\mathbf{H} = \partial^T\left[\rho_0\,G_\psi^{-1}(c\hat{T})G_\psi^{-1}\right], \tag{25}$$

with the fundamental dynamical variables being $(\rho_0,\,\boldsymbol{\psi})$ rather than $(\rho_0,\,\mathbf{v})$. This reformulation makes the Lorentz structure explicit at every level: $\mathbf{G}$ is generated from a single algebraic action of G_ψ on $c\hat{T}$, and the self-coupling $\mathbf{G} = \rho_0 U$ — the key structural feature that distinguishes the fluid hierarchy from the electromagnetic hierarchy in the BQ framework — is now entirely encoded in the rapidity-density pair (ρ_0, G_ψ).

2.2 A common rapidity rotor for fluid dynamics and gravity

The rapidity-generator field introduced above reveals a close structural relationship with the gravitational framework developed in the companion work. In the gravitational construction, one introduces a *gravitational* rapidity field $\boldsymbol{\psi}_g$ and the corresponding Pauli rotor

$$G_g = \exp\left(\tfrac{1}{2}\,\boldsymbol{\psi}_g \cdot \hat{\sigma}\right), \tag{26}$$

which acts directly on the spacetime line element,

$$dR_G = G_g^{-1}\, dR\, G_g^{-1}. \tag{27}$$

This deformed line element defines a Painlevé–Gullstrand-type coframe and hence a metric structure. In the spherically symmetric case, symmetry fixes the boost direction to be radial, $\boldsymbol{\psi}_g = -\psi_g(r)\hat{\sigma}_r$, and the rotor generates the PG river velocity

$$v_g(r) = c \tanh \psi_g(r). \tag{28}$$

For a Newtonian source the escape velocity is recovered:

$$v_g(r) = \sqrt{\frac{2GM}{r}}, \quad \Longrightarrow \quad \psi_g(r) = \tanh^{-1}\sqrt{\frac{2GM}{rc^2}}. \tag{29}$$

More generally, the rapidity generator need not be purely radial. Including azimuthal and polar components,

$$\boldsymbol{\psi}_g = \psi_r\hat{\sigma}_r + \psi_\theta\hat{\sigma}_\theta + \psi_\varphi\hat{\sigma}_\varphi, \tag{30}$$

allows radial inflow, azimuthal rotation, and polar deflection to appear simultaneously. This general form is precisely what is needed for the Constant-Lagrangian–Hubble (CL–H) galactic disk solution, where the non-zero radial component ψ_r generates the infall velocity $w(r) = \sqrt{2GM/r} - H(t)r$ and the azimuthal component ψ_φ generates the orbital support $v_\varphi(r)$, together satisfying the Bernoulli–Noether constant $v_r^2 + v_\varphi^2 = v_L^2$.

The comparison between fluid dynamics and gravity now becomes completely transparent. In the fluid framework the rapidity rotor $G_{\mathrm{FD}} = \exp(\tfrac{1}{2}\boldsymbol{\psi}_{\mathrm{FD}} \cdot \hat{\sigma})$ acts on the timelike reference vector,

$$\gamma V = G_{\mathrm{FD}}^{-1}(c\hat{T})G_{\mathrm{FD}}^{-1}, \tag{31}$$

generating the four-velocity

$$\gamma V = c \cosh \psi_{\mathrm{FD}}\, \hat{T} + c \sinh \psi_{\mathrm{FD}}\, \hat{n}_{\mathrm{FD}} \cdot \hat{K}. \tag{32}$$

In the gravitational framework the same algebraic object $G_g = \exp(\frac{1}{2}\boldsymbol{\psi}_g \cdot \hat{\sigma})$ acts instead on the spacetime line element, generating the deformed coframe dR_G and hence the metric. The two actions of the same rotor type may be summarised as

$$\boldsymbol{\psi} \longrightarrow G_\psi \longrightarrow \begin{cases} \gamma V = G_\psi^{-1}(c\hat{T})G_\psi^{-1}, & \text{fluid dynamics,} \\ dR_G = G_\psi^{-1}\, dR\, G_\psi^{-1}, & \text{gravity.} \end{cases} \tag{33}$$

The distinction between fluid dynamics and gravity is therefore not contained in the rapidity rotor itself, but in the geometric object upon which that rotor acts.

The fluid and gravitational descriptions share the same rotor when

$$G_{\mathrm{FD}} = G_g, \quad \text{i.e.} \quad \boldsymbol{\psi}_{\mathrm{FD}} = \boldsymbol{\psi}_g, \tag{34}$$

which implies $\mathbf{v}_{\mathrm{FD}} = \mathbf{v}_g$. For the spherically symmetric Newtonian case this becomes

$$\mathbf{v}_{\mathrm{FD}} = -\sqrt{\frac{2GM}{r}}\,\hat{r}, \tag{35}$$

and the single rapidity field simultaneously generates the fluid four-velocity through Eqn. (31) and the gravitational Painlevé–Gullstrand coframe through Eqn. (37).

The physical meaning of the identification (34) deserves comment. In ordinary fluid dynamics the rapidity field is a solution of the Euler–Lamb momentum equation together with the continuity equation, two dynamical conditions on $\boldsymbol{\psi}_{\mathrm{FD}}$. In the gravitational framework the rapidity field is determined by the PG boundary condition and the Bernoulli–Noether closure, two geometric conditions on $\boldsymbol{\psi}_g$. These are different determining equations, and there is no algebraic reason that the same field must satisfy both simultaneously. The identification $\boldsymbol{\psi}_{\mathrm{FD}} = \boldsymbol{\psi}_g$ is therefore not automatic: it holds when the fluid flow is itself in free fall within the gravitational geometry it generates, i.e. when the fluid streamlines coincide with the PG geodesics of the induced metric. In the vacuum-flow interpretation developed in Section 4, precisely this condition is realised: the matter is treated as a pressureless dust (tracer density ρ_0) moving geodesically within the rapidity-generated geometry, so the fluid and gravitational rapidity fields are identified by construction.

Basis note.

Throughout this section all boost rotors are expressed in the $\hat{\sigma}$-pauliquat basis of the BQ algebra. The relation to the $\hat{K}$-minquat basis used in the fluid-hierarchy equations of Sections 2–3 is the structural identity $\hat{K}_\mu = \mathrm{i}\,\hat{\sigma}_\mu$, so that the $\hat{K}$-sector and the $\hat{\sigma}$-sector are related by a factor of i at every component. Boost rotors lie in the $\hat{\sigma}$-sector because the exponential $\exp(\frac{\psi}{2}\hat{n}\cdot\hat{\sigma})$ is a Pauli rotor in $\mathrm{SL}(2,\mathbb{C})$, while the physical four-velocity and four-momentum density are expanded in the $\hat{K}$-sector as four-vectors in Minkowski spacetime. Both sectors are required and both appear automatically in any BQ product of two four-vectors through the relation $A^T B = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b})\hat{1} + (\mathbf{a} \times \mathbf{b}) \cdot \hat{K} + (a_0 \mathbf{b} - b_0 \mathbf{a}) \cdot \hat{\sigma}$, with the $\hat{K}$-channel carrying the spatial bivector content and the $\hat{\sigma}$-channel carrying the boost/spin-norm content.

The rapidity formulation therefore provides an alternative parametrization of the BQ fluid hierarchy in which the primary dynamical variables are the density field ρ_0 and the rapidity-generator field $\boldsymbol{\psi}$, with the velocity field, the four-momentum density, and the fluid hierarchy all following as derived quantities. This representation makes the Lorentz structure explicit at every level and — through the common rotor structure of Eqn. (33) — provides a natural bridge between relativistic fluid dynamics and rapidity-based gravitational vacuum-flow descriptions.

2.3 The asymmetry of the two rotor projections: U_g as the primary gravitational object

The common-rotor diagram of Eqn. (33) presents the fluid and gravitational projections side by side, as if they were symmetric alternatives of the same algebraic action. They are not. The fluid branch produces the four-velocity U through a single, exact, fully relativistic step. The gravitational branch reaches the metric $g_{\mu\nu}$ only after two successive operations, each of which discards information irreversibly. The asymmetry is structural and has a precise algebraic source.

The fluid projection is linear and exact.

The double-sided action of G_ψ on the timelike reference vector,

$$U = G_\psi^{-1}(c\hat{T})G_\psi^{-1} = c\cosh\psi\,\hat{T} + c\sinh\psi\,\hat{n} \cdot \hat{K}, \tag{36}$$

is *linear* in G_ψ. The full Lorentz factor $\gamma = \cosh\psi$ appears explicitly and undiluted in both the temporal component $u_0 = \gamma c$ and the spatial component $\mathbf{u} = \gamma\mathbf{v}$. No approximation is involved: the mapping $G_\psi \mapsto U$ is exact for all $|\mathbf{v}| < c$. Given U, the rotor G_ψ can be algebraically recovered up to an overall sign (corresponding to the physical choice of infall direction). The fluid projection is therefore both exact and *invertible*: U carries the complete first-order information of the rapidity field.

The gravitational branch: why the auto-quadratic gives flat spacetime.

The rotor G_g acts on the spacetime line element as

$$dR_G = G_g^{-1}\,dR\,G_g^{-1}, \tag{37}$$

producing an exact, fully relativistic deformed line element. A natural first attempt to extract a metric from dR_G would be to form its BQ auto-quadratic

$[(dR_G)^T dR_G]_{\hat{1}}$. The BQ calculus, however, shows that this route is blocked. For any minquat four-vector A that transforms as $A^L = U^{-1} A U^{-1}$ under a boost rotor U, the auto-quadratic is Lorentz invariant (BQ paper, Eq. 67):

$$(A^L)^T A^L = U(A^T A)U^{-1} = (c^2 a_\tau^2)\, U\hat{1}U^{-1} = c^2 a_\tau^2\, \hat{1} = A^T A. \tag{38}$$

Since dR_G is a minquat four-vector generated by the rotor action (37), this identity applies directly:

$$(dR_G)^T dR_G \;=\; dR^T dR \;=\; (c^2\, dt^2 - d\mathbf{r}^2)\, \hat{1}. \tag{39}$$

The gravitational rotor G_g cancels *exactly* in the auto-quadratic: the $\hat{1}$-channel of $(dR_G)^T dR_G$ is identically flat Minkowski spacetime for all values of ψ_g, without approximation. There is therefore *no algebraic path* from the auto-quadratic of dR_G to the gravitational metric. The rapidity information has been cancelled by the symmetric contraction of the two inverse rotors.

Step 1: the slow-flow approximation extracts the PG coframe.

Since the auto-quadratic forecloses the direct route, the standard Painlevé–Gullstrand metric is reached by a different path. The exact deformed line element is

$$dR_G = \gamma_f c\, dt\left(1 - \frac{v_f\, dr}{c^2\, dt}\right)\hat{T} + \gamma_f(dr - v_f\, dt)\hat{I}_r + r\, d\theta\, \hat{J}_\theta + r \sin\theta\, d\phi\, \hat{K}_\phi, \tag{40}$$

where $\gamma_f = \cosh\psi_g$ and $v_f = c\tanh\psi_g$. The components of dR_G in the minquat basis are the natural candidates for the orthonormal coframe one-forms. Setting $\gamma_f \approx 1$, valid in the gravitationally weak, slow-flow limit $v_f \ll c$, the components of dR_G reduce *directly* to the standard PG coframe:

$$\theta^0 = c\, dt, \quad \theta^r = dr - v_g\, dt, \quad \theta^\theta = r\, d\theta, \quad \theta^\phi = r\sin\theta\, d\phi. \tag{41}$$

The PG coframe one-forms θ^a are therefore the minquat-basis coordinates of dR_G after the slow-flow approximation has been applied. No further product is needed to identify them: they are read off directly from the basis expansion of $dR_G|_{\gamma_f\approx 1}$. This is the first and decisive information-reducing step: the γ_f factors that carried the exact Lorentz boost magnitude are discarded.

Step 2: the coframe quadratic yields the PG metric.

Given the PG coframe θ^a, the metric is assembled as the coframe auto-quadratic

$$g_{\mu\nu} = \eta_{ab}\, \theta^a_\mu\, \theta^b_\nu = \left[(dR_G\big|_{\gamma_g\approx 1})^T\, (dR_G\big|_{\gamma_g\approx 1})\right]_{\hat{1}\text{-channel}}, \tag{42}$$

where the slow-flow approximation is applied to dR_G *before* the quadratic is formed. This order is essential: as shown in Eqn. (39), applying the quadratic to the exact dR_G gives only flat spacetime, regardless of ψ_g. Expanding Eqn. (42) explicitly gives the PG line element

$$ds^2 = -(c^2 - v_g^2)\, dt^2 - 2v_g\, dr\, dt + dr^2 + r^2\, d\Omega^2. \tag{43}$$

The coframe quadratic in Step 2 takes the *symmetric scalar part* $\langle\tilde{\beta}_\mu\tilde{\beta}_\nu\rangle_S$ of the squared, slow-flow-projected rotor action. The antisymmetric (bivector) part of $\theta^a_\mu\theta^b_\nu$, which carries the *orientation and phase* of the gravitational flow, is discarded. The metric therefore encodes only even functions of ψ_g — the combinations $\cosh^2\psi_g$ and $\sinh^2\psi_g$ entering through $v_g^2 = c^2\tanh^2\psi_g$ — and contains neither the sign nor the direction of the rapidity field. Given $g_{\mu\nu}$ alone, one cannot reconstruct the direction of the vacuum flow, the sign of the infall, or the absolute rapidity ψ_g; General Relativity provides at best the derivatives $\nabla\psi_g$ and $\dot{\psi}_g$ through curvature, but never the state variable ψ_g itself.

The two information losses are independent and compounded.

The slow-flow approximation in Step 1 discards the Lorentz factor $\gamma_f = \cosh\psi_g$ from the temporal and radial coframe components — the information about the magnitude of the boost. The coframe quadratic in Step 2 discards the bivector orientation and phase from the metric — the information about the direction and sign of the flow. These losses occur at different stages and involve different parts of the rapidity content of G_g. Neither step can be omitted: without Step 1 the quadratic gives only flat spacetime (Eqn. 39); without Step 2 the coframe θ^a is not yet a metric. Together they produce a doubly compressed image of the rapidity field. Inverting this compression to recover G_g from $g_{\mu\nu}$ requires additional external information — a boundary condition, a reference rapidity at spatial infinity, or an explicit symmetry assumption.

The structural hierarchy.

The two branches of the common rotor stand in the following relation:

$$G_\psi \longrightarrow \begin{cases} U = G_\psi^{-1}(c\hat{T})G_\psi^{-1} & \text{linear in } G_\psi\text{, exact, invertible;} \\ dR_G \xrightarrow{\gamma_g\approx 1} \theta^a \xrightarrow{\eta_{ab}\theta^a\theta^b} g_{\mu\nu} & \text{two steps, both information-reducing.} \end{cases} \tag{44}$$

The four-velocity U is the primary, algebraically complete projection of the rapidity rotor. It is a first-order minquat four-vector, generated by a single exact rotor action that preserves all rapidity content. The metric $g_{\mu\nu}$ is reached only after a slow-flow approximation that eliminates the γ_g factors, followed by a coframe quadratic that eliminates the bivector phase. Both operations are irreversible. The asymmetry is not a matter of computational preference; it is a consequence of the BQ algebra, in which the auto-quadratic of any four-vector is Lorentz invariant and contains no information about the rotor that generated it.

Consequence for the primary gravitational object.

Within the BQ framework, this hierarchy determines which object should be regarded as primary in a fully relativistic description of gravity. The fundamental object is not the metric $g_{\mu\nu}$ but the gravitational four-velocity

$$U_g = G_g^{-1}(c\hat{T})G_g^{-1} = c\cosh\psi_g\,\hat{T} + c\sinh\psi_g\,\hat{n}_g\cdot\hat{K}, \tag{45}$$

which is the direct, exact, and invertible image of the rapidity field. The metric is then recovered as the doubly-projected, approximate consequence

$$g_{\mu\nu} = \eta_{ab}\,\theta^a_\mu\,\theta^b_\nu, \qquad \theta^a_\mu = \left(dR_G\right)^a_\mu\Big|_{\gamma_g\approx 1}, \tag{46}$$

valid in the slow-flow domain $v_g \ll c$ that encompasses all classical gravitational applications in this framework.

The vacuum-flow interpretation developed in Section 4 rests on this hierarchy. The primary geometric object there is the flow field $\mathbf{H}_g = \partial^T U_g$, generated directly from the first-order gravitational four-velocity. The metric and the Einstein tensor are secondary: they arise by projecting U_g through the two information-reducing steps of Eqn. (44). This ordering — U_g first, $g_{\mu\nu}$ second — is not a formalism choice but an algebraic necessity: the four-velocity projection retains what the metric projection discards, and no path leads from $g_{\mu\nu}$ back to the full rotor G_g without supplying the lost information from outside the metric framework. In this sense the present formulation positions itself relative to General Relativity not as an alternative but as a first-order precursor: GR describes the second-order curvature consequences of the rapidity field, while the BQ vacuum-flow hierarchy describes the field itself.

3 Vacuum Flow and Geometric Dynamics

Section 2.3 established that the gravitational four-velocity

$$U_g = G_g^{-1}(c\hat{T})G_g^{-1} = c\cosh\psi_g\,\hat{T} + c\sinh\psi_g\,\hat{n}_g\cdot\hat{K} \tag{47}$$

is the primary, algebraically complete projection of the rapidity rotor G_g: it is exact, linear, and invertible, and it retains the full rapidity content that the metric projection discards. The question that now arises is what field equation governs U_g.

The answer is supplied directly by the BQ relativistic fluid dynamics (BQ-RFD) hierarchy of (de Haas 2026b). There, the fundamental field equation

$$\mathcal{H} = \partial^T G, \qquad G = \rho_0 U, \tag{48}$$

is derived from the single substitution $J \to U$, $A \to G$ applied to the BQ electromagnetic hierarchy, where $G = \rho_0 U$ is the four-momentum density, ρ_0 is the rest-mass density, and U is the four-velocity. The field $\mathcal{H}$ decomposes into three independent channels — a scalar (continuity/energy), a $\hat{K}$-channel (vorticity), and a $\hat{\sigma}$-channel (momentum balance) — which together encode the full dynamics of the material flow (de Haas 2026b).

In ordinary fluid dynamics the density ρ_0 represents the material content of the fluid and U describes its motion through a fixed Minkowski background. The field $\mathcal{H}$ therefore characterizes the dynamics of the matter flow itself. The BQ-RFD result that $\mathcal{H} = \partial^T G$ holds for any four-momentum density $G = \rho_0 U$, however, does not require ρ_0 to be material: the equation is an algebraic identity in the BQ product structure, and its physical content is determined entirely by the interpretation assigned to ρ_0 and U.

This observation motivates a reinterpretation. The rapidity-generator field $\boldsymbol{\psi}_g$ of Section 2.1 simultaneously generates three structures:

$$G_\psi \longrightarrow \Big(U_g, \;\; dR_G, \;\; \partial_g\Big), \tag{49}$$

where U_g is the gravitational four-velocity of Eqn. (47), $dR_G = G_g^{-1}\, dR\, G_g^{-1}$ is the deformed spacetime line element, and

$$\partial_g = \frac{d}{dR_G} \tag{50}$$

is the geometry-adapted derivative generated by the same rotor. This triple generation differs fundamentally from the flat-space fluid hierarchy, where the derivative operator is fixed independently of the velocity field.

Suppose now that ordinary matter is treated as pressureless dust moving geodesically within the geometry generated by G_ψ. The dust does not generate the geometry; it merely responds to it. Its four-momentum density may be written $G_m = \rho_0 U_g$, where ρ_0 is a tracer density and U_g is the four-velocity field generated by the vacuum-flow geometry. Applying the BQ-RFD field equation (48) with $G \to G_m$ and $\partial \to \partial_g$ gives

$$\mathcal{H} = \partial_g^T G_m = \partial_g^T (\rho_0 U_g). \tag{51}$$

For pressureless dust moving along the flow the density is passively advected and does not contribute to the local geometric dynamics. One may therefore factor out the tracer density to obtain

$$\mathcal{H} = \rho_0\, \partial_g^T U_g, \tag{52}$$

and hence the fundamental geometric field quantity

$$\boxed{\mathcal{H}_g \equiv \frac{\mathcal{H}}{\rho_0} = \partial_g^T U_g.} \tag{53}$$

This is the gravitational counterpart of the BQ-RFD field equation $\mathcal{H} = \partial^T G$: the density ρ_0 has been divided out, leaving the intrinsic geometric field of the vacuum flow. In this interpretation $\partial_g^T U_g$ is no longer the field of a material fluid. It characterizes the dynamics of the vacuum flow itself — the rapidity-generated geometry acting on its own structure. The tracer density ρ_0 merely labels a collection of test particles moving geodesically within that flow, and the field $\mathcal{H}_g$ remains well-defined and meaningful in the limit $\rho_0 \to 0$.

The transition from the BQ-RFD material hierarchy to the geometric vacuum-flow hierarchy is therefore a single step: replace the material four-momentum density $G = \rho_0 U$ with the geometric four-momentum density $G_m = \rho_0 U_g$, use the geometry-adapted derivative ∂_g in place of the flat-space ∂, and divide by ρ_0. The algebraic structure — one field equation, three channels, the same BQ product — is identical. What changes is the ontological status of the density: from constitutive medium to passive tracer.

The channel decomposition of $\mathcal{H}_g = \partial_g^T U_g$ may therefore be interpreted geometrically. In the approximation $\partial_g \approx \partial$, which is adopted for the remainder of this section and holds in the slow-flow, weak-field regime, the three channels of $\mathcal{H}_g$ take the same algebraic form as the corresponding channels of the BQ-RFD field (de Haas 2026b): the $\hat{1}$-channel (σ_{Eg}) describes the local expansion or contraction of the vacuum flow, the $\hat{K}$-channel ($\mathbf{\Omega}_g$) describes geometric vorticity, and the $\hat{\sigma}$-channel ($\mathbf{\Pi}_g$) describes the acceleration structure of the flow. These quantities are properties of the rapidity-generated geometry itself. From this perspective, the fluid hierarchy and the gravitational hierarchy are two realizations of the same underlying rapidity algebra (de Haas 2026b): the difference lies in the interpretation of the density field, which in ordinary fluid dynamics constitutes the medium and in the vacuum-flow interpretation acts only as a passive tracer of a deeper geometric flow.

The approximation $\partial_g \approx \partial$ is adopted here as a matter of calculational transparency rather than physical necessity. In the exact formulation, $\partial_g = d/dR_G$ carries non-trivial connection terms of order v_g/c and v_g^2/c^2 that encode the self-coupling between the derivative operator and the rapidity field — the same self-coupling that makes the Einstein field equations non-linear. Setting $\partial_g \approx \partial$ linearises this coupling and is valid in exactly the regime where General Relativity itself recovers the Newtonian limit: $v_g \ll c$ and $GM/rc^2 \ll 1$. All gravitational applications developed in this paper lie within this regime, so the approximation introduces no physical loss for the purposes of the present analysis. The full $\partial_g \neq \partial$

extension, in which the derivative operator is itself generated by the same rotor as U_g and the field equation becomes genuinely non-linear in G_g, is the natural successor to the present construction and corresponds, in BQ language, to the passage from linearised to full gravitational dynamics.

3.1 Channel Decomposition of the Geometric Flow Field

With $\partial_g \approx \partial$ and the gravitational four-velocity

$$U_g = u_0 \hat{T} + \mathbf{u}_g \cdot \hat{K}, \tag{54}$$

the geometric field $\mathcal{H}_g = \partial^T U_g$ decomposes into three independent channels in direct analogy with the BQ-RFD field $\mathcal{H} = \partial^T G$ (de Haas 2026b):

$$\mathcal{H}_g = -\sigma_{Eg}\,\hat{1} + \mathbf{\Omega}_g \cdot \hat{K} - \frac{1}{c}\mathbf{\Pi}_g \cdot \hat{\sigma}, \tag{55}$$

with

$$\sigma_{Eg} = \frac{1}{c^2}\partial_t u_0 + \nabla \cdot \mathbf{u}_g, \tag{56}$$

$$\mathbf{\Omega}_g = \nabla \times \mathbf{u}_g, \tag{57}$$

$$\mathbf{\Pi}_g = \partial_t \mathbf{u}_g + c\nabla u_0. \tag{58}$$

The scalar channel σ_{Eg} measures the local expansion or contraction of the vacuum flow: $\sigma_{Eg} > 0$ corresponds to net geometric expansion (vacuum creation), $\sigma_{Eg} < 0$ to net geometric contraction (vacuum absorption), and $\sigma_{Eg} = 0$ to a local continuity equilibrium of the vacuum flow.

The $\hat{K}$-channel $\mathbf{\Omega}_g$ represents the geometric vorticity of the spacetime flow. For the spherically symmetric slow-flow Painlevé–Gullstrand case $\mathbf{u}_g \approx -v_{\text{esc}}(r)\hat{r}$ one obtains $\mathbf{\Omega}_g = 0$, since $\nabla \times (f(r)\hat{r}) = 0$ identically; rotating geometries such as Kerr-type river flows naturally generate non-vanishing geometric vorticity.

The $\hat{\sigma}$-channel $\mathbf{\Pi}_g$ measures the geometric acceleration structure of the vacuum flow: in the stationary, spherically symmetric case it reduces to the Newtonian gravitational acceleration $-\nabla(GM/r)$, as shown explicitly in Section 3.3.

The relation between the fluid and geometric channel decompositions follows directly from the density factorisation established in Eqn. (52). For constant tracer density, $\mathcal{H} = \rho_0 \partial^T U_g$, so that

$$\mathcal{H}_g = \frac{\mathcal{H}}{\rho_0} = \partial^T U_g. \tag{59}$$

Comparing the material fluid decomposition

$$\mathcal{H} = -\sigma_E\,\hat{1} + \mathbf{\Omega} \cdot \hat{K} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\sigma} \tag{60}$$

with Eqn. (90) yields the exact channel-by-channel normalisation relations

$$\sigma_{Eg} = \frac{\sigma_E}{\rho_0}, \qquad \mathbf{\Omega}_g = \frac{\mathbf{\Omega}}{\rho_0}, \qquad \mathbf{\Pi}_g = \frac{\mathbf{\Pi}}{\rho_0}. \tag{61}$$

The geometric channels σ_{Eg}, $\mathbf{\Omega}_g$, and $\mathbf{\Pi}_g$ are therefore the *intrinsic* continuity, vorticity, and acceleration channels of the rapidity-generated vacuum-flow geometry. The material channels σ_E, $\mathbf{\Omega}$, and $\mathbf{\Pi}$ of the BQ-RFD hierarchy (de Haas 2026b) are their density-weighted realizations, obtained when a tracer distribution ρ_0 is carried passively by the underlying geometric flow. In this sense the BQ-RFD fluid hierarchy is a density realization of the more primitive geometric-flow hierarchy generated by the rapidity field $\boldsymbol{\psi}_g$.

3.2 The continuity $\hat{\mathbf{1}}$-channel of $\mathcal{H}_g$ and an incompressible vacuum

In conventional fluid dynamics, incompressibility is identified with the condition $\nabla \cdot \mathbf{u} = 0$, which states that fluid volume is locally conserved. Within the present vacuum interpretation, however, the vacuum is an incompressible medium whose *density* remains constant while new vacuum *volume* may be created or absorbed: incompressibility is expressed by ρ_v = constant, not by the vanishing of the divergence. The consequence for the continuity channel is that $\sigma_{Eg} = 0$ does *not* correspond to $\nabla \cdot \mathbf{v} = 0$ in general, but rather to a balance between the material rate of change of specific kinetic energy in the interior and the flux of vacuum velocity through the boundary — as the derivation below makes precise. The definition $\sigma_{Eg} \equiv \sigma_E/\rho_0$ and the sign interpretation of the vacuum creation rate were established in Section 3.1; here we derive its explicit form and its integrated balance.

Starting from the BQ continuity channel

$$\sigma_E = \frac{1}{c^2}\partial_t \varepsilon + \nabla \cdot \mathbf{g}, \tag{62}$$

with $\varepsilon = \gamma\rho_0 c^2$ and $\mathbf{g} = \gamma\rho_0\mathbf{v}$, and assuming a constant vacuum density ρ_0, the specific vacuum creation rate becomes

$$\sigma_{Eg} = \partial_t \gamma + \nabla \cdot (\gamma \mathbf{v}). \tag{63}$$

Applying the product rule gives

$$\sigma_{Eg} = \partial_t \gamma + \mathbf{v} \cdot \nabla\gamma + \gamma\nabla \cdot \mathbf{v} = \frac{D\gamma}{Dt} + \gamma\nabla \cdot \mathbf{v}, \tag{64}$$

where D/Dt denotes the material derivative. In the weak-field limit $\gamma \simeq 1 + v^2/(2c^2)$, this reduces to

$$\sigma_{Eg} \simeq \frac{1}{c^2}\frac{D}{Dt}\left(\frac{1}{2}v^2\right) + \nabla \cdot \mathbf{v}. \tag{65}$$

The quantity σ_{Eg} has units of inverse time and measures the local fractional rate at which vacuum volume is created or absorbed.[1]

Integrating Eqn. (65) over a volume V in the weak-field approximation gives

$$\int_V \sigma_{Eg}\, dV = \frac{1}{c^2}\int_V \frac{D}{Dt}\left(\frac{1}{2}v^2\right) dV + \int_V \nabla\cdot\mathbf{v}\, dV. \tag{66}$$

The first integral measures the material change of specific kinetic energy in the volume. Applying Gauss's theorem to the second gives

$$\int_V \sigma_{Eg}\, dV = \frac{1}{c^2}\int_V \frac{D}{Dt}\left(\frac{1}{2}v^2\right) dV + \oint_{\partial V} \mathbf{v}\cdot d\mathbf{A}. \tag{67}$$

For a spherically symmetric flow $\mathbf{v} = v_r(r)\hat{r}$ in the stationary, weak-field approximation the kinetic-energy term vanishes and the volume integral of the specific vacuum creation rate equals the net radial vacuum flux through the enclosing spherical surface:

$$\int_V \sigma_{Eg}\, dV = 4\pi r^2 v_r = \Phi_v. \tag{68}$$

The condition $\sigma_{Eg} = 0$ expresses vacuum-flow conservation. For the vacuum-incompressible medium it does *not* imply $\nabla\cdot\mathbf{v} = 0$, but instead the integral balance

$$\frac{1}{c^2}\int_V \frac{D}{Dt}\left(\frac{1}{2}v^2\right) dV = -\oint_{\partial V} \mathbf{v}\cdot d\mathbf{A} : \tag{69}$$

the material rate of change of specific kinetic energy in the interior is exactly compensated by the net vacuum flux through the boundary. This is the precise sense in which the vacuum-incompressible continuity differs from both the Newtonian condition $\nabla\cdot\mathbf{v} = 0$ and the GR trace-free condition: neither kinetic-energy change alone nor divergence alone determines the balance, but their combined integral relation.

A physically distinguished subclass arises when the specific kinetic energy is constant along the streamlines of the flow:

$$\frac{D}{Dt}\left(\frac{1}{2}v^2\right) = 0. \tag{70}$$

[1]The term $D\gamma/Dt$ has no counterpart in Newtonian gravity and disappears in the standard Newtonian limit of GR. Although it is implicitly present in the relativistic continuity equation, its contribution is normally neglected in weak-field treatments. In the present formulation it appears explicitly as part of the continuity channel, where it acquires the interpretation of a kinetic-energy transport term.

In this case the continuity channel reduces to a purely spatial contribution and the integrated balance of Eqn. (67) becomes the pure flux relation of Eqn. (68). The condition $\sigma_{Eg} = 0$ combined with Eqn. (70) therefore selects exactly this class of vacuum flows: those in which there are no net sources or sinks of vacuum flux within the volume and the flow is kinematically conservative along its own streamlines.

3.3 The momentum balance sigma-channel of $\mathcal{H}_g$

The $\boldsymbol{\sigma}$-channel of the geometric flow field is the acceleration structure of the vacuum flow,

$$\mathbf{\Pi}_g = \partial_t \mathbf{u}_g + c\nabla u_0 = \partial_t(\gamma \mathbf{v}_g) + c^2\,\nabla\gamma = (\partial_t\gamma)\,\mathbf{v}_g + \gamma\,\partial_t\mathbf{v}_g + c^2\,\nabla\gamma. \qquad (71)$$

It is the direct geometric counterpart of the force-density (Euler) channel in relativistic fluid dynamics, but here it characterises the acceleration structure of the vacuum flow itself rather than the force per unit volume acting on a material medium.

Term-by-term interpretation.

The three contributions on the right-hand side of Eq. (71) have distinct physical meanings within the vacuum-flow picture.

The term $\gamma\,\partial_t\mathbf{v}_g$ is the inertial acceleration of the flow, weighted by the local Lorentz factor of the rapidity field. It measures how rapidly the spatial velocity of the vacuum flow changes at a fixed coordinate location. In the weak-field limit $\gamma \approx 1$ this reduces to $\partial_t\mathbf{v}_g$, which is the familiar acceleration field of Newtonian hydrodynamics.

The term $c^2\,\nabla\gamma$ is the gradient of the local Lorentz factor scaled by c. Since $\gamma = \cosh\psi_g$, where ψ_g is the magnitude of the rapidity field, this contribution encodes the spatial variation of the rapidity across the flow. In the weak-field approximation $\gamma \approx 1 + v^2/2c^2$, so that

$$c^2\,\nabla\gamma \;\approx\; \frac{1}{2}\,\nabla v^2 \;=\; (\mathbf{v}_g\cdot\nabla)\mathbf{v}_g + \mathbf{v}_g\times(\nabla\times\mathbf{v}_g), \qquad (72)$$

which separates into a convective acceleration and a contribution from the geometric vorticity $\mathbf{\Omega}_g$. In a purely irrotational flow $\mathbf{\Omega}_g = 0$ and this term reduces to the gradient of the specific kinetic energy, the familiar Bernoulli pressure term.

The term $(\partial_t\gamma)\,\mathbf{v}_g$ couples the time rate of change of the Lorentz factor to the local flow velocity. It has no counterpart in the Newtonian limit and vanishes identically for stationary flows. In the relativistic regime it represents the momentum carried by a time-varying rapidity field: as the local boost of the vacuum flow changes, an additional force-like contribution proportional to the flow velocity itself is generated. This term is therefore an intrinsically relativistic correction to the acceleration balance of the vacuum flow.

Relation to the Euler equation and gravitational acceleration.

In the stationary weak-field case, $\partial_t \gamma \approx 0$, $c^2 \nabla \gamma \approx \nabla v^2/2$ and $\gamma \approx 1$, so that $\mathbf{\Pi}_g$ reduces to

$$\mathbf{\Pi}_g \;\approx\; \partial_t \mathbf{v}_g + \nabla\left(\tfrac{1}{2}v^2\right). \tag{73}$$

This represents the kinematic acceleration of an irrotational flow. The condition $\mathbf{\Pi}_g = \mathbf{0}$, which states that the $\boldsymbol{\sigma}$-channel of the geometric field equation vanishes, then becomes

$$\partial_t \mathbf{v}_g = -\nabla\left(\tfrac{1}{2}v^2\right). \tag{74}$$

This is the pressure-free Euler equation or the irrotational inertial Burgers' equation for the vacuum flow, expressing local momentum balance without an explicit matter source. In the Painlevé–Gullstrand picture, the stationary solution $\partial_t \mathbf{v}_g = 0$ with $\mathbf{\Pi}_g = 0$ corresponds precisely to the condition that specific kinetic energy is constant along the flow, $D(\tfrac{1}{2}v^2)/Dt = 0$, which is the weak-field form of the Bernoulli–Noether constant derived from the constant-Lagrangian condition. The vanishing of $\mathbf{\Pi}_g$ therefore selects the same class of flows that the Constant-Lagrangian–Hubble (CL–H) framework identifies as physically realised galactic and gravitational streamlines.

Spherically symmetric Painlevé–Gullstrand flow.

For the spherically symmetric inflow $\mathbf{v}_g \approx -|v_{\rm esc}(r)|\,\hat{\mathbf{r}}$ with $v_{\rm esc}(r) = \sqrt{2GM/r}$, one obtains

$$\mathbf{\Pi}_g \;=\; \partial_t \mathbf{v}_g + \nabla\left(\tfrac{1}{2}v^2\right) \;\approx\; -\frac{GM}{r^2}\,\hat{\mathbf{r}}, \tag{75}$$

in the stationary, weak-field limit. The $\boldsymbol{\sigma}$-channel of the geometric field equation therefore recovers the Newtonian gravitational acceleration field directly, without any explicit matter source: the acceleration arises purely from the spatial gradient of the Lorentz factor of the rapidity-generated vacuum flow. Positive values of $|\mathbf{\Pi}_g|$ correspond to an inwardly accelerating flow (gravitational attraction), whereas a repulsive flow acceleration arises in the Hubble-outflow sector where $v_{\rm esc}(r)$ is replaced by the Hubble component $H(t)r$.

Connection to the density-weighted fluid channel.

As established in Section 3.1, $\mathbf{\Pi}_g = \frac{\mathbf{\Pi}}{\rho_0}$ is the density-normalised counterpart of the BQ-RFD momentum channel (de Haas 2026b). In ordinary relativistic fluid dynamics $\mathbf{\Pi}$ plays the role of the Euler force term and is balanced by pressure gradients. In the vacuum-flow interpretation there is no material pressure; instead,

$\boldsymbol{\Pi}_g = \mathbf{0}$ expresses the self-consistency of the rapidity-generated geometry, requiring that no net geometric force acts on the flow itself. A non-vanishing $\boldsymbol{\Pi}_g$ signals either a time-dependent rapidity field or a spatial mismatch between the flow velocity and the local Lorentz factor gradient, both of which correspond to departures from the stationary Bernoulli streamline condition.

3.4 Integrated channel interpretation and the Newtonian limit

The three channels of $\mathcal{H}_g = \partial^T U_g$ derived in Sections 3.2–3.3 are projections of a single algebraic object. In the weak-field, slow-flow approximation $\gamma_g \approx 1$, $\mathbf{u}_g \approx \mathbf{v}_g$, $u_0 \approx c + v^2/(2c)$, they reduce to

$$\sigma_{Eg} \approx \frac{1}{c^2}\frac{D}{Dt}\left(\tfrac{1}{2}v^2\right) + \nabla\cdot\mathbf{v}_g, \tag{76}$$

$$\boldsymbol{\Omega}_g = \nabla\times\mathbf{v}_g, \tag{77}$$

$$\boldsymbol{\Pi}_g \approx \partial_t\mathbf{v}_g + \nabla\left(\tfrac{1}{2}v^2\right). \tag{78}$$

These three equations are not independent: they are three channel projections of the same geometric field equation, and they form a closed kinematic description of a vacuum flow in the Newtonian limit. Their mutual consistency and collective content become clear when they are read as a system.

The $\hat{K}$-channel selects the kinematic class.

The vorticity $\boldsymbol{\Omega}_g = \nabla\times\mathbf{v}_g$ defines which class of vacuum flows the rapidity-generated geometry belongs to. For the spherically symmetric Painlevé–Gullstrand inflow $\mathbf{v}_g = -v_{\mathrm{esc}}(r)\hat{r}$, one has $\nabla\times(f(r)\hat{r}) = 0$ identically, so $\boldsymbol{\Omega}_g = 0$ without any dynamical condition being imposed. Irrotationality is a geometric property of the rapidity field in this case, not a constraint added by hand. Within the BQ-RFD hierarchy (de Haas 2026b), the condition $\mathcal{H}_g = 0$ at the field level yields exactly this irrotationality as its $\hat{K}$-channel projection — just as the conserved condition $\mathcal{H} = 0$ for an ordinary fluid gives $\nabla\times\mathbf{g} = 0$ (de Haas 2026b). Rotating geometries, such as the CL–H galactic disk with azimuthal rapidity component ψ_φ, naturally generate non-vanishing $\boldsymbol{\Omega}_g$ and thereby depart from this irrotational subclass.

The $\hat{\sigma}$-channel is the momentum equation of the flow.

For an irrotational flow $\boldsymbol{\Omega}_g = 0$, the $\hat{\sigma}$-channel reduces to

$$\boldsymbol{\Pi}_g \approx \partial_t\mathbf{v}_g + \nabla\left(\tfrac{1}{2}v^2\right), \tag{79}$$

because the vorticity term in the convective decomposition $\nabla(\frac{1}{2}v^2) = (\mathbf{v}_g\cdot\nabla)\mathbf{v}_g + \mathbf{v}_g\times\boldsymbol{\Omega}_g$ vanishes. This is the irrotational Euler equation — the kinematic acceleration of a potential flow. The condition $\mathbf{\Pi}_g = \mathbf{0}$ then becomes

$$\partial_t \mathbf{v}_g = -\nabla\left(\tfrac{1}{2}v^2\right), \tag{80}$$

which is the unsteady, pressureless Bernoulli condition: the acceleration of the vacuum flow is entirely determined by the spatial gradient of its own specific kinetic energy, with no external forcing. For a stationary flow this reduces further to $\nabla(\frac{1}{2}v^2) = 0$, i.e. $D(\frac{1}{2}v^2)/Dt = 0$, which is the Constant-Lagrangian (CL) condition identified in the companion gravitational papers (**?**de Haas 2026b) as the defining property of physically realised galactic and gravitational streamlines.

With a scalar flow potential ϕ defined by $\mathbf{v}_g = \nabla\phi$, Eqn. (80) integrates to the unsteady Bernoulli relation

$$\partial_t \phi + \tfrac{1}{2}(\nabla\phi)^2 = \text{constant}, \tag{81}$$

which is the standard result of classical potential flow theory.

The $\hat{1}$-channel is the source equation of the vacuum.

With the CL condition $D(\frac{1}{2}v^2)/Dt = 0$ satisfied, the $\hat{1}$-channel reduces to

$$\sigma_{Eg} \approx \nabla \cdot \mathbf{v}_g = \nabla^2\phi. \tag{82}$$

This is the Poisson equation for the flow potential. The specific vacuum creation rate σ_{Eg} acts as the *source density* of the potential flow: $\sigma_{Eg} = 0$ gives the Laplace equation $\nabla^2\phi = 0$ for a source-free vacuum, while $\sigma_{Eg} \neq 0$ identifies the location and strength of vacuum sources and sinks.

The $\mathcal{H}_g = 0$ case: source-free potential flow.

The condition $\mathcal{H}_g = 0$, in which all three channels vanish simultaneously, is the geometric vacuum analogue of the conserved condition $\mathcal{H} = 0$ in BQ-RFD (de Haas 2026b). In the weak-field, slow-flow, irrotational limit it imposes

$$\nabla^2\phi = 0, \qquad \nabla \times \mathbf{v}_g = 0, \qquad \partial_t\phi + \tfrac{1}{2}(\nabla\phi)^2 = \text{const.} \tag{83}$$

These are exactly the three equations of classical incompressible, irrotational, pressureless potential flow — Laplace's equation for the potential, the irrotationality condition, and the unsteady Bernoulli relation — all three emerging as channel projections of the single condition $\mathcal{H}_g = 0$ without any being assumed separately. This is the precise counterpart of the result in BQ-RFD (de Haas 2026b) that $\mathcal{H} = 0$ in the dust, slow-flow limit gives exactly classical potential flow: here the same structure appears for the geometric vacuum flow rather than a material fluid.

Crucially, as shown in Section 2.3, the $\hat{\sigma}$-channel at the field level is linear in the rapidity field and involves no convective $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term. The convective nonlinearity belongs to the force-level equation $U_g\mathcal{H}_g = F_g$, not to the field-level $\mathcal{H}_g = \partial^T U_g$ (de Haas 2026b). The field-level Bernoulli condition is therefore linear in ϕ and does not require the Lamb-vector decomposition — the $\mathbf{v}_g \times \mathbf{\Omega}_g$ cross term vanishes automatically because it belongs to the $\hat{K}$-channel, not the $\hat{\sigma}$-channel.

Newtonian gravity as source-bearing potential flow.

The $\mathcal{H}_g = 0$ case is source-free: the vacuum flow has no creation or absorption anywhere and satisfies Laplace's equation. Newtonian gravity corresponds to the case $\sigma_{Eg} \neq 0$ at isolated source points. For the spherically symmetric Schwarzschild–PG flow $\mathbf{v}_g = -\sqrt{2GM/r}\,\hat{r}$, the flow potential is $\phi = -2\sqrt{2GMr}$ and direct calculation gives $\nabla^2\phi \propto \delta^3(\mathbf{r})$, concentrated at the origin. The $\hat{1}$-channel then reads

$$\nabla^2\phi = \sigma_{Eg} = 4\pi GM\,\delta^3(\mathbf{r}), \tag{84}$$

which is the Newtonian Poisson equation $\nabla^2\Phi = 4\pi G\rho$ with the gravitational potential identified as $\Phi = \phi/(-2) = \sqrt{2GM/r}\cdot r$ and the point-mass density as the vacuum source density σ_{Eg}. The Newtonian gravitational potential is therefore the flow potential of the vacuum, the gravitational acceleration $-\nabla\Phi$ is the force-free Bernoulli condition $\mathbf{\Pi}_g = \mathbf{0}$, and the mass distribution is the vacuum source density σ_{Eg} — all three emerging from the same channel decomposition of $\mathcal{H}_g$ without any additional physical assumptions.

The three channels of $\mathcal{H}_g$ in the Newtonian limit therefore constitute a complete, closed, and mutually consistent description of Newtonian gravity as source-bearing potential flow of the vacuum. The standard Newtonian results — the Poisson equation, the gravitational acceleration, and the irrotationality of the gravitational field — are channel projections of the single BQ-RFD field equation $\mathcal{H}_g = \partial^T U_g$ applied to the rapidity-generated vacuum flow. General Relativity corresponds to the exact version of this system, in which the approximations $\gamma_g \approx 1$ and $\partial_g \approx \partial$ are removed and the Poisson equation is replaced by the Einstein field equations. The BQ vacuum-flow hierarchy is the first-order precursor that makes the Newtonian content of the rapidity field explicit in a form directly comparable to classical potential theory.

Structural significance of the result.

The strength of the above derivation lies not in recovering known equations — the Poisson equation and the Bernoulli condition have of course been known for two centuries — but in the precise algebraic sense in which they are recovered here. In classical mechanics these three Newtonian results are introduced as separate foundational inputs: the Poisson equation from Newton's law of gravitation, irrotationality from the central-force character of gravity, and the Bernoulli condition from energy conservation or force balance. Each requires its own physical argument and its own mathematical derivation. Here all three emerge without separate assumption as channel projections of the single condition $\mathcal{H}_g = \partial^T U_g$, via one algebraic product applied to the rapidity-generated four-velocity U_g. None is imported; all are outputs.

This is the gravitational counterpart of the result established in the BQ-RFD paper (de Haas 2026b): that the conserved condition $\mathcal{H} = 0$ in the dust, slow-flow limit yields incompressibility, irrotationality, and the unsteady Bernoulli equation simultaneously from one algebraic condition, with each of the three classical potential-flow equations appearing as a distinct channel projection (de Haas 2026b). The present result is not an analogy with that fluid-dynamics result but its direct gravitational realisation: replacing the material four-momentum density $G = \rho_0 U$ with the geometric four-velocity U_g generated by the rapidity rotor, the same channel structure gives the same potential-flow equations with the same algebraic economy, but now the "fluid" is the vacuum itself and the "source" is the gravitational mass distribution.

The rapidity field $\boldsymbol{\psi}_g$ is the object that makes this connection possible. Earlier vacuum-flow or river-model approaches to Painlevé–Gullstrand gravity also identified a flow velocity with the escape velocity, but they defined that velocity geometrically from the metric rather than generating it algebraically from a rapidity rotor. Without the rotor there is no natural connection to the BQ-RFD field hierarchy, and without that hierarchy there is no channel decomposition that separates the Poisson equation, irrotationality, and the Bernoulli condition into three distinct algebraic projections of one object. The chain $\boldsymbol{\psi}_g \to G_g \to U_g \to \mathcal{H}_g = \partial^T U_g$ is a sequence of exact algebraic steps, and Newtonian gravity sits at its end as a consequence rather than an input. The result positions Newtonian gravity, the $\mathcal{H}_g = 0$ source-free potential flow, and General Relativity as three regimes of the same field equation — distinguished by the presence or absence of vacuum sources and by the approximation level applied to the rapidity-generated U_g — rather than as three separate theories connected only through limiting procedures.

4 Adding Hubble to the Gravitational Flow Field

The goal of this section is not merely to recover known FRW kinematic quantities within the BQ channel language — that recovery, while confirming internal consistency, follows directly from the velocity field fed into the framework and contains no structural surprise. The primary result is different: the channel decomposition of $\mathcal{H}_g = \partial^T U_g$ reveals a *continuity-balance radius* r_σ that is strictly distinct from, and lies inside, the velocity-balance radius r_c familiar from standard Newtonian+Hubble analyses. The radius r_c, defined by $v_H(r_c) = v_N(r_c)$, is the turnaround radius at which the net radial flow velocity vanishes; it has been studied extensively in the context of local group dynamics and the zero-velocity surface in cosmology. The radius r_σ, defined by $\sigma_{Eg}(r_\sigma) = 0$, is the radius at which the volumetric creation rate of the Hubble expansion exactly balances the volumetric absorption rate of the gravitational sink. These two conditions are independent, and the channel decomposition makes their difference explicit: substituting r_c into σ_{Eg} gives $\sigma_{Eg}(r_c) = \frac{3}{2}H \neq 0$. At the radius where all net transport of space stops, the vacuum is still being produced at a rate equal to half the full Hubble expansion rate. This distinction is inaccessible to any formulation that works exclusively with the velocity field $\mathbf{u}_g$, because it requires the divergence of that field — the continuity channel — as a first-class physical object. It is the channel decomposition that makes the question natural to ask, and the answer that constitutes the section's main contribution.

4.1 The Combined Rotor and Its Weak-Field Velocity

The Newton-only section established the geometric flow field from the Schwarzschild rapidity ψ_M, defined by

$$\tanh \psi_M(r) = \frac{v_N(r)}{c}, \qquad v_N(r) = \sqrt{\frac{2GM}{r}}. \tag{85}$$

The cosmological Hubble expansion contributes an independent radial rapidity ψ_H, defined by

$$\tanh \psi_H(r,t) = \frac{v_H(r,t)}{c}, \qquad v_H(r,t) = H(t)\, r, \tag{86}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter. Because both boosts act in the same (t, r)-plane, the gravitational rotor for the combined field is simply

$$Q_g(r,t) = \exp\Bigl(-\tfrac{1}{2}\, \psi_{\text{tot}}(r,t)\, \beta_0\beta_r\Bigr), \qquad \psi_{\text{tot}} = \psi_M(r) - \psi_H(r,t), \tag{87}$$

where the minus sign reflects the fact that ψ_M drives inflow and ψ_H drives outflow. The rotor's adjoint action on the coframe basis vectors produces the Painlevé–Gullstrand tetrad; the total physical radial flow speed is

$$w(r,t) = c \tanh \psi_{\rm tot}(r,t) = v_N(r) - v_H(r,t) = \sqrt{\frac{2GM}{r}} - H(t)\, r. \tag{88}$$

In the weak-field, slow-flow limit $|w| \ll c$ (so $\tanh \psi \approx \psi$) the four-velocity generated by the rotor reduces to

$$\mathbf{u}_g = \left(H(t)\, r - \sqrt{\frac{2GM}{r}} \right) \hat{r}, \tag{89}$$

with the inward Newtonian term entering with a minus sign (inflow convention: inward is negative). This is the velocity field used throughout the rest of this section. Its derivation from the combined rotor (87) is the necessary first step: the channel decomposition that follows is structurally a property of the rotor-generated field, not of the Newtonian velocity superposition taken as a free ansatz.

4.2 Channel Decomposition of $\mathcal{H}_g = \partial^T U_g$

The geometric field of the vacuum flow is

$$\frac{\mathcal{H}_g}{\rho_0} = -\sigma_{Eg}\, \hat{1} + \boldsymbol{\Omega}_g \cdot \hat{K} - \frac{1}{c}\, \boldsymbol{\Pi}_g \cdot \hat{\boldsymbol{\sigma}}, \tag{90}$$

where the three channels are computed from the stationary flow field $\partial_t \mathbf{u}_g = \dot{H}(t)\, r\, \hat{r}$ (non-zero only when H is time-dependent) and the spatial differential operators applied to (89).

4.2.1 The continuity channel

The continuity channel is $\sigma_{Eg} = \nabla \cdot \mathbf{u}_g$. Using the spherical divergence

$$\nabla \cdot \mathbf{u}_g = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r \right), \qquad u_r = H(t)\, r - \sqrt{\frac{2GM}{r}}, \tag{91}$$

one obtains immediately

$$\boxed{\sigma_{Eg} = 3H(t) - \frac{3}{2} \sqrt{\frac{2GM}{r^3}}.} \tag{92}$$

The factor $3H(t)$ is not a coincidence: it is the direct consequence of the three-dimensional nature of the Hubble expansion, since $\nabla \cdot (H\, r\, \hat{r}) = 3H$ by the spherical divergence. It coincides with the FRW expansion scalar $\theta = 3H$ of standard relativistic cosmology, which here emerges as an automatic output of the continuity channel without imposing any cosmological framework beyond the scale factor $a(t)$. The term $-\frac{3}{2}\sqrt{2GM/r^3}$ is the gravitational sink already derived in the Newton-only section; it enters here unchanged. The continuity channel therefore places the cosmological expansion and the gravitational absorption within a *single algebraic quantity*,

$$\sigma_{Eg} = \underbrace{3H}_{\text{creation}} - \underbrace{\tfrac{3}{2}\sqrt{2GM/r^3}}_{\text{absorption}}. \tag{93}$$

Positive values correspond to net volume creation; negative values to net absorption.

4.2.2 The vorticity channel

The vorticity channel is $\mathbf{\Omega}_g = \nabla \times \mathbf{u}_g$. Since $\mathbf{u}_g = f(r)\hat{r}$ is purely radial for all r,

$$\mathbf{\Omega}_g = 0. \tag{94}$$

The Hubble correction preserves irrotationality. The flow remains a potential flow; no rotational structure is introduced by the cosmological term.

4.2.3 The acceleration channel

The acceleration channel is $\mathbf{\Pi}_g = \nabla u_0$ with $u_0 = \gamma c$ and

$$\gamma = \left[1 - \frac{1}{c^2}\left(H(t)\, r - \sqrt{\frac{2GM}{r}}\right)^2\right]^{-1/2}. \tag{95}$$

For a time-dependent $H(t)$, the flow is no longer stationary: $\partial_t u_r = \dot{H}(t)\, r$, so the full acceleration channel receives both a spatial gradient term and a time-derivative term. In the weak-field limit the result is

$$c\,\mathbf{\Pi}_g \simeq \left[\frac{\ddot{a}}{a} r - \frac{GM}{r^2} - \frac{1}{2} H\sqrt{\frac{2GM}{r}}\right]\hat{r}, \tag{96}$$

where $\ddot{a}/a\, r$ is the cosmological acceleration term, $-GM/r^2$ is the familiar Newtonian inverse-square term, and the cross-term $-\frac{1}{2}H\sqrt{2GM/r}$ couples the two rapidities. The acceleration channel therefore separates the cosmological acceleration $\ddot{a}/a$ from the vacuum creation rate $3H$, distributing them into distinct algebraic sectors:

$$\sigma_{Eg} \longleftrightarrow 3H(t) = \text{vacuum creation rate (FRW expansion scalar)}, \tag{97}$$

$$\boldsymbol{\Pi}_g \longleftrightarrow \frac{\ddot{a}}{a} = \text{vacuum creation acceleration.} \tag{98}$$

This separation is not put in by hand; it follows automatically from the channel structure of $\mathcal{H}_g = \partial^T U_g$.

4.3 Two Radii: Velocity Balance and Continuity Balance

The channel decomposition reveals two distinct radii associated with the competition between inflow and outflow, which are *not* visible in the velocity field alone.

4.3.1 The velocity-balance radius r_c

The net radial flow velocity vanishes when $u_r = 0$, i.e. $v_H = v_N$:

$$H(t)\, r_c = \sqrt{\frac{2GM}{r_c}}, \quad \Longrightarrow \quad r_c(t) = \left(\frac{2GM}{H(t)^2}\right)^{1/3}. \tag{99}$$

At r_c the inward Newtonian flow and the outward Hubble flow cancel. Substituting r_c into the continuity channel (92) gives

$$\sigma_{Eg}(r_c) = 3H - \frac{3}{2}H = \frac{3}{2}\,H \;\neq\; 0. \tag{100}$$

Velocity balance does *not* imply continuity balance. At the radius where the radial velocities cancel, the vacuum flow still exhibits a net positive expansion rate of $\frac{3}{2}H$.

4.3.2 The continuity-balance radius r_σ

The continuity channel (92) vanishes when creation and absorption rates are equal:

$$3H(t) = \frac{3}{2}\sqrt{\frac{2GM}{r_\sigma^3}}, \quad \Longrightarrow \quad r_\sigma(t) = \left(\frac{\sqrt{2GM}}{2H(t)}\right)^{2/3}. \tag{101}$$

Using (99), this can be written as

$$r_\sigma(t) = \frac{r_c(t)}{2^{2/3}} \approx 0.63\, r_c(t). \tag{102}$$

The continuity-balance radius lies strictly *inside* the velocity-balance radius. These two radii characterise different physical conditions:

- At r_c: the vacuum *velocity* of space is zero (no net transport of space), but space is still being created at rate $\frac{3}{2}H$. Matter at rest here is in kinematic equilibrium but not in volumetric equilibrium.

- At r_σ: the *rate* of volume creation equals the rate of absorption. The vacuum continuity channel is locally balanced, meaning the enclosed vacuum volume is neither growing nor shrinking.

This distinction is invisible in the velocity-field formulation of the problem and emerges naturally and automatically from the channel decomposition of $\mathcal{H}_g$.

4.3.3 Significance of the two-radius structure

The two-radius result is the central new contribution of the channel decomposition in the presence of Hubble expansion. In any velocity-based treatment of the combined Newtonian+Hubble flow, only r_c is available: it is defined by $u_r = 0$ and is a kinematic concept. The continuity-balance radius r_σ requires access to the divergence of the flow field, which is the domain of the $\hat{1}$-channel of $\mathcal{H}_g$.

Physically, $r_\sigma < r_c$ means that the volumetric balance between creation and absorption is achieved significantly closer to the mass than the kinematic balance. Between r_σ and r_c the net vacuum velocity is outward (space is escaping) while net volume creation still exceeds absorption; outside r_c the net velocity is also outward but now the Hubble creation *rate* also dominates. Only inside r_σ does the gravitational absorption exceed the Hubble creation rate.

The result further shows that the continuity channel σ_{Eg} plays a role analogous to the expansion scalar in the theory of relativistic congruences. It measures the convergence or divergence of the flow itself, providing gravitational-focusing information that is absent from the Newtonian acceleration channel $\mathbf{\Pi}_g$ alone.

4.4 Time-Dependent Hubble Parameter

When H carries explicit time dependence through the scale factor, $H(t) = \dot{a}/a$, all results extend by direct substitution. The continuity-balance radius becomes

$$r_\sigma(t) = \left(\frac{\sqrt{2GM}}{2H(t)}\right)^{2/3}, \tag{103}$$

and the ratio $r_\sigma/r_c = 2^{-2/3}$ is time-independent: it is a fixed geometric ratio determined by the power-law structure of the two flow fields.

The additional time derivative $\partial_t u_r = \dot{H}(t)\, r$ contributes to the acceleration channel (96) through

$$\partial_t u_r = \dot{H}\, r = \left(\frac{\ddot{a}}{a} - H^2\right) r, \tag{104}$$

so that the acceleration channel in the weak-field limit contains the combination

$$\dot{H} + H^2 = \frac{\ddot{a}}{a}, \tag{105}$$

which is the standard cosmological deceleration parameter combination. The two FRW quantities therefore land in precisely their expected algebraic sectors:

$$\sigma_{Eg} \leftrightarrow 3H = 3\frac{\dot{a}}{a} \quad \text{(norm channel, FRW expansion scalar)}, \tag{106}$$

$$c\,\mathbf{\Pi}_g \leftrightarrow \frac{\ddot{a}}{a}\,r \quad \text{(acceleration channel, FRW deceleration parameter)}. \tag{107}$$

The channel decomposition of $\mathcal{H}_g$ simultaneously and automatically recovers both FRW kinematic quantities from the single rotor-generated flow field (87), separating them into the continuity sector and the acceleration sector without any additional cosmological input beyond the definition $H(t) = \dot{a}/a$.

4.5 Summary

The addition of the Hubble rapidity ψ_H to the gravitational rotor Q_g generates a combined flow field whose channel decomposition yields four results, each emerging from the algebra without additional physical input:

1. **Continuity channel:** $\sigma_{Eg} = 3H - \frac{3}{2}\sqrt{2GM/r^3}$ places cosmological expansion and gravitational absorption in a single algebraic balance equation, reproducing the FRW expansion scalar $\theta = 3H$ in the creation term.
2. **Vorticity channel:** $\mathbf{\Omega}_g = 0$. The combined flow remains irrotational; the Hubble correction introduces no rotational structure.
3. **Acceleration channel:** $c\,\mathbf{\Pi}_g$ separates the cosmological deceleration $\ddot{a}/a$ from the Newtonian $-GM/r^2$ inverse-square term, locating the two in the same algebraic sector.
4. **Two-radius structure:** The channel decomposition reveals a continuity-balance radius $r_\sigma \approx 0.63\,r_c$ inside the velocity-balance radius r_c. At r_c the net velocity vanishes but the vacuum creation rate remains $\frac{3}{2}H > 0$; at r_σ creation and absorption are volumetrically balanced. This distinction is invisible in any velocity-only treatment and is the primary new structural content produced by the BQ channel decomposition.

The chain $\psi_{\text{tot}} \to Q_g \to U_g \to \mathcal{H}_g = \partial^T U_g$ positions Newtonian gravity, the cosmological Hubble flow, and their combined channel structure as three levels of the same first-order rapidity-rotor framework, with no need to import the FRW expansion scalar or the deceleration parameter as external concepts.

The two-radius structure also has a precise relativistic counterpart that places the result in a broader geometric context. In general relativity, the Raychaudhuri equation governs the expansion scalar $\theta = \nabla_\mu u^\mu$ of a congruence of geodesics; its source term $-R_{\mu\nu}u^\mu u^\nu$ encodes the focusing effect of matter on the congruence through the Ricci curvature. The BQ continuity channel $\sigma_{Eg} = \nabla \cdot \mathbf{u}_g$ is the flat-space, non-relativistic analogue of θ: it measures the local rate of expansion or contraction of the vacuum flow congruence generated by the rotor Q_g. The Newtonian sink term $-\frac{3}{2}\sqrt{2GM/r^3}$ in σ_{Eg} is, in this reading, the flat-space analogue of the Ricci focusing term — the gravitational absorption of space playing the role that spacetime curvature plays in the full Raychaudhuri equation. The two-radius result $r_\sigma \neq r_c$ is therefore not a kinematic accident but a non-relativistic instance of the general principle that geodesic focusing (here measured by $\sigma_{Eg} = 0$) and kinematic equilibrium (here measured by $u_r = 0$) are governed by independent conditions and occur at different locations. The BQ channel decomposition makes this separation visible at the Newtonian level, in a context where the standard velocity-field formulation of the Hubble+Newtonian problem offers no natural language for the distinction.

5 Geometric Field with Radial, Hubble, and Orbital Rapidity

The preceding sections established the channel decomposition of $\mathcal{H}_g = \partial^T U_g$ for the combined radial flow generated by the Newton rapidity ψ_M and the Hubble rapidity ψ_H. In that case the flow was purely radial, the vorticity channel vanished identically, and the sigma-channel was purely radial. We now extend the framework to include an independent orbital rapidity ψ_ϕ, which activates the vorticity channel and introduces azimuthal structure into the acceleration channel.

The principal structural result of the extension is a clean separation of roles: the radial rapidity sector (ψ_M, ψ_H) controls vacuum creation and absorption through σ_{Eg} and radial acceleration through $\Pi_{g,r}$, while the orbital rapidity ψ_ϕ controls geometric circulation through $\mathbf{\Omega}_g$ and azimuthal acceleration through $\Pi_{g,\phi}$. This separation is exact at the continuity and vorticity levels. At the acceleration level the two sectors couple through the common Lorentz factor $u_0 = \gamma c$, which depends on the total flow speed $v_g^2 = u_r^2 + u_\phi^2$. An additional coupling in the azimuthal acceleration channel arises from cylindrical geometry: the radial inflow transports angular momentum through the $u_r u_\phi / r$ term, which survives axisymmetry and is the force-level mechanism by which the vacuum inflow modifies orbital motion.

5.1 Sequential Rotor Construction

The radial Newton–Hubble boost acts in the (t, r)-plane and is generated by the radial rotor

$$G_r = \exp\left(-\tfrac{1}{2}\,\psi_r\,\sigma_0\sigma_r\right), \qquad \psi_r = \psi_M(r) - \psi_H(r,t), \tag{108}$$

with

$$\tanh\psi_M = \frac{v_N}{c} = \frac{\sqrt{2GM/r}}{c}, \qquad \tanh\psi_H = \frac{H(t)r}{c}. \tag{109}$$

The orbital motion acts in the (t, ϕ)-plane and is generated independently by the orbital rotor

$$G_\phi = \exp\left(\tfrac{1}{2}\,\psi_\phi\,\sigma_0\sigma_\phi\right), \qquad \tanh\psi_\phi = \frac{u_\phi}{c}, \tag{110}$$

where $u_\phi(r,t)$ is the orbital flow speed, left unspecified at this stage. Because the two boosts act in orthogonal planes they commute at leading order in the weak-field limit, but to maintain a clean physical separation of the riverbed and orbital structures we write the total rotor as the sequential product

$$G = G_\phi\,G_r, \tag{111}$$

with G_r applied first to generate the PG–Hubble riverbed, and G_ϕ applied second to superpose the orbital motion. This ordering preserves the interpretation of ψ_r as the riverbed rapidity and ψ_ϕ as the orbital rapidity, and prevents the two generators from mixing at the rotor level.

In the slow-flow limit $u_r, u_\phi \ll c$, the four-velocity generated by the sequential rotor is

$$U_g = u_0 \hat{T} + u_r \hat{r} + u_\phi \hat{\boldsymbol{\phi}}, \tag{112}$$

with

$$u_0 = \gamma c, \qquad \gamma = \left(1 - \frac{u_r^2 + u_\phi^2}{c^2}\right)^{-1/2}, \tag{113}$$

and with the slow-flow identifications $u_r \approx v_r$, $u_\phi \approx v_\phi$. The total physical flow speed entering γ combines both rapidities:

$$v_g^2 = u_r^2 + u_\phi^2, \qquad u_r = H(t)r - \sqrt{\frac{2GM}{r}}. \tag{114}$$

5.2 Slow-Flow Weak-Field Approximation for the Three-Component Flow

We apply the same slow-flow ($u_r, u_\phi \ll c$) and weak-field ($GM/rc^2 \ll 1, Hr/c \ll 1,\ u_\phi/c \ll 1$) limit used in Eqns. (72)–(74). Under these conditions the four-velocity components reduce to

$$u_0 \approx c + \frac{u_r^2 + u_\phi^2}{2c}, \qquad \mathbf{u}_g \approx \mathbf{v}_g = u_r\hat{r} + u_\phi\hat{\boldsymbol{\phi}}. \tag{115}$$

The correction to u_0 is negligible as a multiplicative prefactor ($u_0 \to c$ wherever u_0 multiplies another quantity), but must *not* be applied before differentiation. Using $c\, u_0 \approx c^2 + \frac{1}{2}(u_r^2 + u_\phi^2)$, the correct prescription for the acceleration channel is

$$c\, \boldsymbol{\Pi}_g = \nabla(c\, u_0) \approx \nabla\left(\tfrac{1}{2}v_g^2\right) + c\, \partial_t \mathbf{v}_g, \tag{116}$$

where now $v_g^2 = u_r^2 + u_\phi^2$ contains contributions from both rapidities. This is the direct generalisation of Eqn. (74) to the two-component flow; the orbital rapidity enters $\boldsymbol{\Pi}_g$ through both $\nabla(\frac{1}{2}u_\phi^2)$ and $\partial_t u_\phi$.

The three $\mathcal{H}_g$ channels in the slow-flow weak-field limit therefore generalise Eqns. (72)–(74) as follows.

5.3 The Norm Channel: Vacuum Continuity

The continuity channel is

$$\sigma_{Eg} \approx \frac{1}{c^2}\frac{D}{Dt}\left(\tfrac{1}{2}v_g^2\right) + \nabla \cdot \mathbf{v}_g. \tag{117}$$

For an axisymmetric flow ($\partial_\phi = 0$) the divergence of the azimuthal component vanishes identically,

$$\nabla \cdot (u_\phi \hat{\boldsymbol{\phi}}) = \frac{1}{r}\partial_\phi u_\phi = 0, \tag{118}$$

so only the radial component contributes to $\nabla \cdot \mathbf{v}_g$. The material-derivative correction $\frac{1}{c^2}\frac{D}{Dt}(\frac{1}{2}v_g^2)$ is second order in v_g/c and is dropped. The continuity channel therefore retains exactly the form established for the Newton+Hubble radial flow:

$$\boxed{\sigma_{Eg} = 3H(t) - \frac{3}{2}\sqrt{\frac{2GM}{r^3}}.} \tag{119}$$

The orbital rapidity does not enter the norm channel.

A subtle but important point concerns the Hubble term. Although the orbital rapidity is introduced within the disk plane, the Hubble expansion is a three-dimensional isotropic flow $\mathbf{u}_H = H(t)\mathbf{r}$ for which $\nabla \cdot \mathbf{u}_H = 3H(t)$. The factor $3H$ is therefore the divergence of a three-dimensional spherical flow, not of a two-dimensional disk flow, and it is unchanged by the addition of an azimuthal component. The continuity-balance radius

$$r_\sigma(t) = \left(\frac{\sqrt{2GM}}{2H(t)}\right)^{2/3} \tag{120}$$

is accordingly identical to the radial-only result. The radial rapidity sector determines the creation–absorption balance of the vacuum flow; the orbital rapidity leaves it untouched.

5.4 The $\hat{K}$-Channel: Geometric Vorticity

The vorticity channel is

$$\boldsymbol{\Omega}_g = \nabla \times \mathbf{v}_g. \tag{121}$$

For the purely radial Newton+Hubble flow this vanished identically. With the orbital component $u_\phi \hat{\boldsymbol{\phi}}$ present, the curl acquires a non-zero vertical component.

For an axisymmetric flow ($\partial_\phi = 0$) in cylindrical coordinates (r, ϕ, z), the vertical component of the curl is

$$\Omega_{gz} = \frac{1}{r}\frac{\partial}{\partial r}\big(r\,u_\phi\big) - \frac{1}{r}\frac{\partial u_r}{\partial \phi} = \frac{1}{r}\frac{\partial}{\partial r}\big(r\,u_\phi\big), \tag{122}$$

where the second term vanishes by axisymmetry. The radial component u_r does not contribute to Ω_{gz} under axisymmetry. The vorticity is therefore

$$\boxed{\boldsymbol{\Omega}_g = \frac{1}{r}\frac{\partial}{\partial r}\big(r\,u_\phi\big)\,\hat{\mathbf{z}}.} \tag{123}$$

The vorticity channel is activated entirely by the orbital rapidity: $\psi_\phi \neq 0$ implies $\mathbf{\Omega}_g \neq 0$, and $\psi_\phi = 0$ recovers $\mathbf{\Omega}_g = 0$ of the radial-only case.

Note that $\mathbf{\Omega}_g$ is the curl of the total velocity field, not only of the azimuthal component. However, for an axisymmetric disk the radial flow is irrotational ($\nabla \times (u_r \hat{r}) = 0$ when $u_r = u_r(r,t)$), so the full vorticity is generated solely by the gradient of the azimuthal momentum:

$$\mathbf{\Omega}_g = \nabla \times (u_\phi \hat{\boldsymbol{\phi}}) = \frac{1}{r}\partial_r (r\, u_\phi)\, \hat{\mathbf{z}}. \tag{124}$$

This is the specific angular momentum gradient $\partial_r \ell$ where $\ell = r\, u_\phi$ is the angular momentum per unit mass, expressed as a vertical vorticity. Solid-body rotation ($u_\phi \propto r$) gives $\mathbf{\Omega}_g = 2\Omega\hat{\mathbf{z}}$ (constant), while a Keplerian profile ($u_\phi \propto r^{-1/2}$) gives $\mathbf{\Omega}_g \propto r^{-3/2}\hat{\mathbf{z}}$ (singular at the origin).

5.5 The $\hat{\sigma}$-Channel: Geometric Acceleration

The acceleration channel is, from the generalised Eqn. (74),

$$\mathbf{\Pi}_g \approx \partial_t \mathbf{v}_g + \nabla\left(\tfrac{1}{2} v_g^2\right), \tag{125}$$

with $v_g^2 = u_r^2 + u_\phi^2$. This decomposes into radial and azimuthal components.

5.5.1 Radial component $\mathbf{\Pi}_{g,r}$

The radial component of $\partial_t \mathbf{v}_g$ is $\partial_t u_r = \dot{H}(t) r$ (from the time-dependent Hubble term in u_r). The radial component of $\nabla(\frac{1}{2}v_g^2)$ is

$$\partial_r\left(\tfrac{1}{2} v_g^2\right) = u_r \partial_r u_r + u_\phi \partial_r u_\phi. \tag{126}$$

Therefore

$$\boxed{c\,\Pi_{g,r} \approx \dot{H}(t) r + u_r \partial_r u_r + u_\phi \partial_r u_\phi.} \tag{127}$$

The first term is the cosmological acceleration (from $\partial_t u_r$); the second is the PG–Hubble riverbed self-acceleration; the third is the coupling of the orbital flow gradient to the radial channel through the Lorentz factor. The last term is the first point at which ψ_ϕ enters the radial channel: even though the orbital rapidity does not directly generate radial velocity, its gradient $\partial_r u_\phi$ couples into $\Pi_{g,r}$ through $c\, u_0 \approx c^2 + \frac{1}{2}(u_r^2 + u_\phi^2)$.

Substituting the Newton+Hubble expression for u_r and using $\dot{H} + H^2 = \ddot{a}/a$, the riverbed contribution expands as before to give

$$c\,\Pi_{g,r} \approx \frac{\ddot{a}}{a} r - \frac{H}{2}\sqrt{\frac{2GM}{r}} - \frac{GM}{r^2} + u_\phi \partial_r u_\phi, \tag{128}$$

where the first three terms are the purely radial result and the last term is the orbital correction to the radial acceleration channel.

5.5.2 Azimuthal component $\Pi_{g,\phi}$

The azimuthal component of $\partial_t \mathbf{v}_g$ is $\partial_t u_\phi$. For the azimuthal component of $\nabla(\frac{1}{2}v_g^2)$ in cylindrical coordinates, the $\hat{\boldsymbol{\phi}}$ direction gives

$$\frac{1}{r}\partial_\phi\left(\tfrac{1}{2}v_g^2\right) = 0 \tag{129}$$

by axisymmetry ($\partial_\phi = 0$). However, the full material time derivative of the azimuthal velocity in cylindrical coordinates contains a geometric coupling term that is not captured by ∂_ϕ alone. Writing $\nabla(\frac{1}{2}v_g^2) = (\mathbf{v}_g \cdot \nabla)\mathbf{v}_g + \mathbf{v}_g \times (\nabla \times \mathbf{v}_g)$ and taking the azimuthal component,

$$\left[(\mathbf{v}_g \cdot \nabla)\mathbf{v}_g\right]_\phi = u_r \partial_r u_\phi + \frac{u_r u_\phi}{r}, \tag{130}$$

where the $u_r u_\phi / r$ term is the cylindrical geometric coupling between the radial inflow and the azimuthal flow. This term arises because $\hat{\boldsymbol{\phi}}$ rotates with r: the radial displacement of a fluid element by u_r carries it into a new $\hat{\boldsymbol{\phi}}$ direction, generating a ϕ-acceleration even without any ϕ-gradient. Using the vector identity $\nabla(\frac{1}{2}v_g^2) = (\mathbf{v}_g \cdot \nabla)\mathbf{v}_g + \mathbf{v}_g \times \boldsymbol{\Omega}_g$ and evaluating the $\hat{\boldsymbol{\phi}}$ component with $\boldsymbol{\Omega}_g = \Omega_{gz}\hat{\mathbf{z}}$, $\mathbf{v}_g = u_r\hat{r} + u_\phi\hat{\boldsymbol{\phi}}$, and the cross-products $\hat{r} \times \hat{\mathbf{z}} = -\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{r}$:

$$\left[\mathbf{v}_g \times \boldsymbol{\Omega}_g\right]_\phi = \left[(u_r\hat{r} + u_\phi\hat{\boldsymbol{\phi}}) \times \Omega_{gz}\hat{\mathbf{z}}\right]_\phi = -u_r\Omega_{gz}. \tag{131}$$

The azimuthal component of $\nabla(\frac{1}{2}v_g^2)$ is therefore

$$\left[\nabla\left(\tfrac{1}{2}v_g^2\right)\right]_\phi = \left[(\mathbf{v}_g \cdot \nabla)\mathbf{v}_g\right]_\phi - u_r\,\Omega_{gz} = u_r\partial_r u_\phi + \frac{u_r u_\phi}{r} - u_r\,\Omega_{gz}. \tag{132}$$

Substituting $\Omega_{gz} = \frac{1}{r}\partial_r(ru_\phi) = \partial_r u_\phi + u_\phi/r$, one finds

$$u_r\partial_r u_\phi + \frac{u_r u_\phi}{r} - u_r\left(\partial_r u_\phi + \frac{u_\phi}{r}\right) = 0. \tag{133}$$

This cancellation is exact: for an axisymmetric flow the azimuthal component of $\nabla(\frac{1}{2}v_g^2)$ vanishes identically. The result is consistent with axisymmetry: $\partial_\phi(\frac{1}{2}v_g^2) = 0$ implies $[\nabla(\frac{1}{2}v_g^2)]_\phi = 0$ in any coordinate system.

The azimuthal acceleration channel therefore reduces to

$$\boxed{c\,\Pi_{g,\phi} \approx c\,\partial_t u_\phi.} \tag{134}$$

For a stationary orbital profile ($\partial_t u_\phi = 0$), the azimuthal acceleration channel vanishes at the field level.

This result is exact within the slow-flow weak-field limit and consistent with axisymmetry. The cylindrical coupling term $u_r u_\phi/r$ and the convective term $u_r\partial_r u_\phi$ are both present in $(\mathbf{v}_g \cdot \nabla)\mathbf{v}_g$, but they are cancelled exactly by the $\mathbf{v}_g \times \mathbf{\Omega}_g$ contribution to $\nabla(\frac{1}{2}v_g^2)$. These terms therefore do *not* appear in the field-level acceleration channel $\mathbf{\Pi}_g$; they will reappear at the force level $F_g = U_g\mathcal{H}_g$ through the $\mathbf{u}_g \times \mathbf{\Omega}_g$ term in the $\hat{K}$-channel, which is the Lamb-vector contribution to the force balance.

5.6 Channel Summary and Separation of Roles

The complete slow-flow weak-field channel decomposition of $\mathcal{H}_g/\rho_0 = \partial^T U_g$ for the three-component flow is:

$$\sigma_{Eg} = 3H(t) - \frac{3}{2}\sqrt{\frac{2GM}{r^3}}, \tag{135}$$

$$\mathbf{\Omega}_g = \frac{1}{r}\partial_r(r\,u_\phi)\,\hat{\mathbf{z}}, \tag{136}$$

$$c\,\Pi_{g,r} \approx \frac{\ddot{a}}{a}r - \frac{H}{2}\sqrt{\frac{2GM}{r}} - \frac{GM}{r^2} + u_\phi\partial_r u_\phi, \tag{137}$$

$$c\,\Pi_{g,\phi} \approx c\,\partial_t u_\phi. \tag{138}$$

The separation of roles between the two rapidity sectors is summarised in Table 1.

Channel	Radial sector (ψ_M, ψ_H)	Orbital sector (ψ_ϕ)
σ_{Eg} (norm/$\hat{1}$)	$3H - \frac{3}{2}\sqrt{2GM/r^3}$	0
$\mathbf{\Omega}_g$ (vorticity/$\hat{K}$)	0	$\frac{1}{r}\partial_r(ru_\phi)\hat{\mathbf{z}}$
$c\,\Pi_{g,r}$ ($\hat{\sigma}$, radial)	$\frac{\ddot{a}}{a}r - \frac{H}{2}v_N - \frac{GM}{r^2}$	$u_\phi\partial_r u_\phi$
$c\,\Pi_{g,\phi}$ ($\hat{\sigma}$, azimuthal)	0	$c\,\partial_t u_\phi$

Table 1 Slow-flow weak-field channel decomposition of $\mathcal{H}_g/\rho_0 = \partial^T U_g$ for the combined Newton+Hubble+orbital flow. The radial and orbital sectors are orthogonal at the continuity and vorticity levels; they couple only in the radial component of the acceleration channel through $u_\phi\partial_r u_\phi$.

Four structural points follow directly from this table.

The norm channel is purely radial.

The orbital rapidity contributes nothing to σ_{Eg} under axisymmetry, because the divergence of an azimuthal field vanishes. The continuity-balance radius r_σ, the velocity-balance radius r_c, and the acceleration-balance radius $r_a = (GM/H^2)^{1/3}$ are all determined by the radial sector alone and are unaffected by the orbital flow.

The vorticity channel is purely orbital.

The radial Newton+Hubble flow is irrotational and contributes nothing to $\mathbf{\Omega}_g$. The vorticity is generated entirely by the gradient of the specific angular momentum $\ell = ru_\phi$: $\mathbf{\Omega}_g = \partial_r \ell / r\, \hat{\mathbf{z}}$. The vorticity channel is therefore the channel through which the disk's rotation curve enters the geometric field.

The acceleration channel couples the two sectors.

The radial acceleration $\Pi_{g,r}$ receives a correction $u_\phi \partial_r u_\phi$ from the orbital sector through the common Lorentz factor. This term is the response of the radial acceleration to the gradient of the orbital kinetic energy density $\frac{1}{2}u_\phi^2$: where the orbital speed increases outward ($\partial_r u_\phi > 0$), this coupling pushes the radial channel outward; where it decreases ($\partial_r u_\phi < 0$, as in a Keplerian profile), the coupling is inward. The azimuthal acceleration $\Pi_{g,\phi}$, by contrast, reduces to $\partial_t u_\phi$ alone: the cylindrical coupling terms $u_r \partial_r u_\phi$ and $u_r u_\phi / r$ that appear in the material derivative Du_ϕ / Dt cancel exactly against the $\mathbf{v}_g \times \mathbf{\Omega}_g$ contribution to $\nabla(\frac{1}{2}v_g^2)$. These terms are not absent from the physics — they reappear at the force level through the Lamb-vector term $\mathbf{u}_g \times \mathbf{\Omega}_g$ in the $\hat{K}$-channel of $F_g = U_g \mathcal{H}_g$.

The $\hat{\boldsymbol{\sigma}}$-channel label refers to Π_g, not to σ_{Eg}.

In the BQ decomposition $\mathcal{H}_g / \rho_0 = -\sigma_{Eg}\, \hat{1} + \mathbf{\Omega}_g \cdot \hat{K} - \frac{1}{c}\mathbf{\Pi}_g \cdot \hat{\boldsymbol{\sigma}}$, the expansion scalar σ_{Eg} lives in the norm ($\hat{1}$) channel, while the acceleration field $\mathbf{\Pi}_g$ lives in the spin ($\hat{\boldsymbol{\sigma}}$) channel. The labels norm, $\hat{K}$, and $\hat{\boldsymbol{\sigma}}$ refer to the BQ basis elements carrying each contribution, not to the physical quantity denoted by σ.

6 The Constant-Lagrangian Solution

The preceding section derived the slow-flow weak-field channel decomposition of $\mathcal{H}_g/\rho_0 = \partial^T U_g$ for the three-component vacuum flow generated by the sequential rotor $G = G_\phi G_r$. The channels are determined by the radial and orbital rapidities but leave the orbital profile $u_\phi(r)$ unspecified. The present section imposes two conditions that together select a unique orbital profile, fixes the single free parameter by a physical boundary condition at the bulge surface, and shows that the resulting solution satisfies both the norm channel and the sigma channel exactly, with the radial sigma channel reducing to a purely cosmological residual. This solution is the constant-Lagrangian galactic disk profile, and its derivation from the BQ channel structure is the central result of the paper.

6.1 The Two Conditions

6.1.1 Condition I: Stationarity of the total kinetic energy

The full norm channel in the slow-flow limit is (Eqn. (72)):

$$\sigma_{Eg} = \frac{1}{c^2}\frac{D}{Dt}\left(\tfrac{1}{2}v_g^2\right) + \nabla \cdot \mathbf{v}_g. \tag{139}$$

The first term is second order in v_g/c and was discarded in the slow-flow approximation, leaving $\sigma_{Eg} = \nabla \cdot \mathbf{v}_g$. We now impose this term as a dynamical condition rather than discarding it. Setting

$$\frac{D}{Dt}\left(\tfrac{1}{2}v_g^2\right) = 0 \tag{140}$$

states that the total kinetic energy of the vacuum flow is conserved along streamlines: the flow speed v_g is constant following a fluid element. For the combined radial and orbital flow, $v_g^2 = u_r^2 + u_\phi^2$, this gives

$$\boxed{u_r^2(r) + u_\phi^2(r) = v_L^2 = \text{const},} \tag{141}$$

the constant-Lagrangian (CL) condition. What is traded between the two components along a streamline is kinetic energy: where the radial inflow speed $|u_r|$ decreases, the orbital speed u_ϕ increases, and vice versa, keeping the total v_g^2 fixed. Imposing Eqn. (140) leaves the slow-flow norm channel unchanged,

$$\sigma_{Eg} = 3H(t) - \frac{3}{2}\sqrt{\frac{2GM}{r^3}}, \tag{142}$$

because the CL condition sets the discarded term to zero exactly rather than approximately. The continuity-balance radius r_σ and all radii derived from σ_{Eg} are therefore unaffected.

Condition (141) determines the orbital profile at every radius $r \geq R$ once v_L^2 is known:

$$u_\phi(r) = \sqrt{v_L^2 - u_r^2(r)} = \sqrt{v_L^2 - \left(H(t)r - \sqrt{\frac{2GM}{r}}\right)^2}. \tag{143}$$

This is a closed-form, parameter-free expression for the rotation curve given v_L^2.

6.1.2 Condition II: Stationarity of the azimuthal acceleration channel

For a stationary disk ($\partial_t u_\phi = 0$) the azimuthal sigma channel gives (Eqn. (138)):

$$c\,\Pi_{g,\phi} = 0. \tag{144}$$

This is automatically satisfied by any time-independent orbital profile. No additional constraint on $u_\phi(r)$ is imposed by this channel beyond stationarity.

6.2 The Boundary Condition: Virial Theorem at $r = R$

Condition (141) contains a single free parameter v_L^2. It is fixed by a physical boundary condition at the bulge surface $r = R$.

The bulge of mass M and radius R is treated as a solid rotating disk composed of orbiting stars. At the surface $r = R$, the orbiting stars satisfy the Newtonian virial theorem: the orbital kinetic energy equals half the gravitational potential energy,

$$\frac{1}{2}u_\phi^2(R) = \frac{1}{2}\frac{GM}{R}, \qquad \Longrightarrow \qquad u_\phi^2(R) = \frac{GM}{R}. \tag{145}$$

Evaluating the CL condition (141) at $r = R$ and substituting Eqn. (145):

$$v_L^2 = u_r^2(R) + u_\phi^2(R) = \left(HR - \sqrt{\frac{2GM}{R}}\right)^2 + \frac{GM}{R}. \tag{146}$$

Expanding:

$$v_L^2 = H^2R^2 - 2H\sqrt{2GMR} + \frac{3GM}{R}. \tag{147}$$

This expression is exact within the slow-flow limit and is completely determined by the four observables G, M, R, and $H(t)$. No free parameters remain.

The three terms in Eqn. (147) have distinct physical origins:

- H^2R^2: the kinetic energy of the Hubble flow at the bulge surface, contributed by $u_r^2(R)$.

- $-2H\sqrt{2GMR}$: the cross-coupling between the Hubble expansion and the Newtonian inflow at R, the same $H\,v_N$ interference term that appears in the K-channel of $F_g = U_g\mathcal{H}_g$.
- $3GM/R$: the total gravitational contribution, consisting of $2GM/R$ from the Newtonian inflow kinetic energy $u_r^2(R)\big|_{H=0} = 2GM/R$ and GM/R from the virial orbital speed.

6.3 The Complete CL Orbital Profile

With v_L^2 fixed by Eqn. (147), the complete orbital profile for $r \geq R$ is

$$\boxed{u_\phi(r) = \sqrt{H^2R^2 - 2H\sqrt{2GMR} + \frac{3GM}{R} - \left(Hr - \sqrt{\frac{2GM}{r}}\right)^2}.} \tag{148}$$

This is the constant-Lagrangian rotation curve. It is a closed-form expression depending only on G, M, R, H, and the radial coordinate r.

The profile has the following limiting behaviours. At $r = R$: $u_\phi(R) = \sqrt{GM/R}$, recovering the virial speed by construction. At large r where $u_r \to Hr$ (Hubble-dominated): $u_\phi(r) \to \sqrt{v_L^2 - H^2r^2}$, which falls to zero at the velocity-balance radius r_c where $u_r^2(r_c) = v_L^2$. Between R and r_c the profile is approximately flat, giving the observed flat galactic rotation curve, because u_r^2 changes slowly in that region and $u_\phi \approx v_L$ when $|u_r| \ll v_L$.

The vorticity channel activated by this profile is

$$\mathbf{\Omega}_g = \frac{1}{r}\partial_r\big(r\,u_\phi(r)\big)\,\hat{\mathbf{z}}, \tag{149}$$

which is non-zero and encodes the geometric circulation of the galactic disk. Its gradient structure determines the spiral arm geometry through the free-fall spirals of the constant-Lagrangian model.

6.4 Reduction of the Sigma Channel on the CL Solution

We now show that the CL profile (143) is an exact solution of the radial sigma channel, and establish what that channel reduces to when the CL condition is imposed.

The radial sigma channel is (Eqn. (137)):

$$c\,\Pi_{g,r} \approx \frac{\ddot{a}}{a}r - \frac{H}{2}\sqrt{\frac{2GM}{r}} - \frac{GM}{r^2} + u_\phi\partial_r u_\phi. \tag{150}$$

The CL condition $u_r^2 + u_\phi^2 = v_L^2$ gives, on differentiation with respect to r:

$$u_r\partial_r u_r + u_\phi\partial_r u_\phi = 0, \qquad \Longrightarrow \qquad u_\phi\partial_r u_\phi = -u_r\partial_r u_r. \tag{151}$$

Substituting into Eqn. (150):

$$c\,\Pi_{g,r} \approx \frac{\ddot{a}}{a}r - \frac{H}{2}\sqrt{\frac{2GM}{r}} - \frac{GM}{r^2} - u_r\partial_r u_r. \tag{152}$$

We now compute $u_r\partial_r u_r$ explicitly. With $u_r = Hr - \sqrt{2GM/r}$:

$$\partial_r u_r = H + \frac{1}{2}\sqrt{\frac{2GM}{r^3}}, \tag{153}$$

so

$$\begin{aligned} u_r\partial_r u_r &= \left(Hr - \sqrt{\frac{2GM}{r}}\right)\left(H + \frac{1}{2}\sqrt{\frac{2GM}{r^3}}\right) \\ &= H^2 r + \frac{H}{2}\sqrt{\frac{2GM}{r}} - H\sqrt{\frac{2GM}{r}} - \frac{GM}{r^2} \\ &= H^2 r - \frac{H}{2}\sqrt{\frac{2GM}{r}} - \frac{GM}{r^2}. \end{aligned} \tag{154}$$

Substituting Eqn. (154) into Eqn. (152):

$$c\,\Pi_{g,r} \approx \frac{\ddot{a}}{a}r - \frac{H}{2}\sqrt{\frac{2GM}{r}} - \frac{GM}{r^2} - H^2 r + \frac{H}{2}\sqrt{\frac{2GM}{r}} + \frac{GM}{r^2}. \tag{155}$$

The Newtonian term $-GM/r^2$ and its counterpart cancel exactly. The Hubble cross terms $\mp\frac{H}{2}\sqrt{2GM/r}$ cancel exactly. The result is:

$$\boxed{c\,\Pi_{g,r} = \left(\frac{\ddot{a}}{a} - H^2\right) r = \dot{H}\,r,} \tag{156}$$

where $\ddot{a}/a - H^2 = \dot{H}$ has been used.

6.5 Interpretation of the Reduced Sigma Channel

The cancellation in Eqn. (155) is exact and complete. On the CL solution, the radial acceleration channel loses all memory of local gravity: the Newtonian inverse-square term, the Hubble cross-coupling term, and the orbital correction term all cancel pairwise. What remains is the single cosmological quantity $\dot{H}\,r$ — the local rate of change of the Hubble flow, evaluated at position r.

The physical interpretation is precise. The CL condition $u_r^2 + u_\phi^2 = v_L^2$ states that the vacuum flow is in local kinetic equilibrium: the total flow speed is conserved along streamlines, with radial and orbital kinetic energy trading continuously. This equilibrium is exactly the condition under which the local gravitational dynamics cancel in the radial acceleration channel. The vacuum flow on the CL manifold is in a state of complete local gravitational equilibrium; the only residual dynamics in the radial sigma channel is the background cosmological deceleration or acceleration of the Hubble flow itself.

The cosmological limits are:

Matter-dominated universe.
$\dot{H} = -\frac{3}{2}H^2$, giving

$$c\,\Pi_{g,r}^{(\mathrm{mat})} = -\frac{3}{2}H^2 r. \tag{157}$$

The radial sigma channel is a gentle inward deceleration proportional to $H^2 r$, driven by the slowing of the cosmological expansion.

De Sitter (dark energy dominated) universe.
$\dot{H} = 0$, $H = H_\Lambda = \mathrm{const}$, giving

$$c\,\Pi_{g,r}^{(\mathrm{dS})} = 0. \tag{158}$$

The radial sigma channel vanishes identically. The CL orbital profile is an *exact stationary solution* of both the norm channel and the sigma channel in the de Sitter limit: no residual radial acceleration acts on the constant-Lagrangian vacuum flow.

General case.
The residual $\dot{H}\,r$ is non-zero but of cosmological magnitude — suppressed by H^2/c^2 relative to the Newtonian terms that cancelled. For galactic scales $r \lesssim 100$ kpc, $H_0 \approx 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, the residual acceleration is $|\dot{H}|r \lesssim H_0^2 r/c \sim 10^{-12}\ \mathrm{m\,s^{-2}}$, far below the observational threshold for rotation curve measurements. The CL profile is therefore observationally indistinguishable from an exact stationary solution for all practical purposes.

6.6 Summary of the Channel Structure on the CL Solution

The complete slow-flow weak-field channel decomposition of $\mathcal{H}_g/\rho_0 = \partial^T U_g$ evaluated on the constant-Lagrangian solution is:

$$\sigma_{Eg} = 3H(t) - \frac{3}{2}\sqrt{\frac{2GM}{r^3}}, \tag{159}$$

$$\boldsymbol{\Omega}_g = \frac{1}{r}\partial_r\left(r\,u_\phi^{\mathrm{CL}}(r)\right)\hat{\mathbf{z}}, \tag{160}$$

$$c\,\Pi_{g,r} = \dot{H}(t)\,r, \tag{161}$$

$$c\,\Pi_{g,\phi} = 0. \tag{162}$$

These four equations constitute the geometric field $\mathcal{H}_g$ of the constant-Lagrangian galactic disk.

The structure is strikingly clean. The norm channel retains the full creation–absorption balance of the radial sector, unmodified by the orbital flow. The vorticity channel encodes the rotation curve through the gradient of the specific angular momentum of the CL profile. The radial sigma channel is reduced to the cosmological deceleration $\dot{H}\,r$ — all local gravitational and orbital content has cancelled exactly. The azimuthal sigma channel vanishes.

In the de Sitter limit the channel structure simplifies to:

$$\sigma_{Eg} = 3H_\Lambda - \frac{3}{2}\sqrt{\frac{2GM}{r^3}}, \tag{163}$$

$$\mathbf{\Omega}_g = \frac{1}{r}\partial_r\left(r\,u_\phi^{\mathrm{CL}}(r)\right)\hat{\mathbf{z}}, \tag{164}$$

$$c\,\Pi_{g,r} = 0, \qquad c\,\Pi_{g,\phi} = 0, \tag{165}$$

with H_Λ constant. The CL galactic disk in a de Sitter background is therefore a solution of $\mathcal{H}_g = \partial^T U_g$ in which only the norm and vorticity channels are active.

6.7 The Tully-Fisher Scaling

The asymptotic flat rotation speed of the CL profile is $v_\infty = v_L$, reached in the region $R \ll r \ll r_c$ where $|u_r| \ll v_L$ and hence $u_\phi \approx v_L$. From Eqn. (147),

$$v_\infty^2 = v_L^2 = H^2R^2 - 2H\sqrt{2GMR} + \frac{3GM}{R}. \tag{166}$$

The dominant term for a massive bulge at small R is $3GM/R$; the Hubble corrections H^2R^2 and $2H\sqrt{2GMR}$ are suppressed by $HR/v_N \ll 1$ in the Newtonian regime. In the dominant Newtonian limit:

$$v_\infty^2 \approx \frac{3GM}{R}, \tag{167}$$

which gives $v_\infty^4 \approx 9G^2M^2/R^2$ — a relation between the flat rotation speed and the bulge parameters M and R.

To obtain the Tully-Fisher scaling one eliminates R using the bulge virial condition together with an observational relation between M and R for galactic bulges. If the bulge satisfies $GM/R \sim v_N^2(R)$ and the velocity-balance condition $H^2R^3 = 2GM$ (i.e. $R = r_c$, the identification of the bulge radius with the velocity-balance radius), then $R = (2GM/H^2)^{1/3}$ and:

$$v_\infty^2 \approx \frac{3GM}{R} = 3GM\left(\frac{H^2}{2GM}\right)^{1/3} = \frac{3}{2^{1/3}}(GMH)^{2/3}, \tag{168}$$

giving

$$\boxed{v_\infty^4 \approx \frac{9}{2^{2/3}}\,(GMH)^{4/3} \propto M^{4/3}\,H^{4/3}.} \tag{169}$$

This is the Baryonic Tully-Fisher relation $v_\infty^4 \propto GMa_0$ with the identification $a_0 \propto cH$ (up to a numerical factor), emerging from the BQ channel structure without any modification of gravity, without dark matter, and without free parameters beyond G, M, R, and H.

The numerical coefficient $9/2^{2/3} \approx 5.67$ is determined by the virial condition $u_\phi^2(R) = GM/R$ and the CL boundary condition (147): it is not a fit parameter.

6.8 Summary

The constant-Lagrangian galactic disk profile arises as an exact solution of the BQ geometric field equations through the following logical chain:

1. The **norm channel** of $\mathcal{H}_g = \partial^T U_g$, Eqn. (72), contains the term $\frac{1}{c^2}\frac{D}{Dt}(\frac{1}{2}v_g^2)$ which is discarded in the slow-flow approximation. Setting this term to zero rather than discarding it imposes the constant-Lagrangian condition $u_r^2 + u_\phi^2 = v_L^2 =$ const along streamlines.
2. The **boundary condition** at the bulge surface $r = R$ combines the Newtonian virial theorem $u_\phi^2(R) = GM/R$ with the CL condition to fix $v_L^2 = H^2R^2 - 2H\sqrt{2GMR} + 3GM/R$ completely in terms of the observables G, M, R, H.
3. The **orbital profile** at all $r \geq R$ is then determined without further input: $u_\phi(r) = \sqrt{v_L^2 - (Hr - \sqrt{2GM/r})^2}$.
4. The **radial sigma channel**, on substituting the CL condition $u_\phi\partial_r u_\phi = -u_r\partial_r u_r$, undergoes exact cancellation of all Newtonian and cross-coupling terms, leaving the purely cosmological residual $c\,\Pi_{g,r} = \dot{H}\,r$.
5. In the **de Sitter limit** $\dot{H} = 0$, both sigma channels vanish and the CL profile is an exact stationary solution of the full channel system.
6. The **Tully-Fisher relation** $v_\infty^4 \propto (GMH)^{4/3}$ emerges from the CL boundary condition when the bulge radius is identified with the velocity-balance radius $r_c = (2GM/H^2)^{1/3}$.

The CL galactic spiral is therefore not postulated but derived: it is the unique stationary orbital solution of the BQ vacuum-flow hierarchy consistent with the constant-Lagrangian condition in the norm channel, the virial theorem at the bulge surface, and the Newton+Hubble radial rapidity structure of the riverbed flow.

6.9 The Residual $\dot{H}r$ and the Cosmological Scaling of r_c

The residual $c\,\Pi_{g,r} = \dot{H}\,r$ on the CL solution and the empirical observation that the velocity-balance radius scales as $r_c(t) \propto a(t)$ are not merely compatible — they are two expressions of the same identity.

6.9.1 $r_c \propto a(t)$ implies matter domination

The velocity-balance radius is

$$r_c(t) = \left(\frac{2GM}{H(t)^2}\right)^{1/3}. \tag{170}$$

Differentiating,

$$\dot{r}_c = -\frac{2}{3}\frac{\dot{H}}{H}\,r_c. \tag{171}$$

The condition $r_c \propto a(t)$ requires $\dot{r}_c/r_c = \dot{a}/a = H$, which from Eqn. (171) gives

$$-\frac{2}{3}\frac{\dot{H}}{H} = H \qquad \Longrightarrow \qquad \dot{H} = -\frac{3}{2}H^2. \tag{172}$$

This is exactly the matter-dominated Friedmann relation, satisfied by $a(t) \propto t^{2/3}$, $H = 2/(3t)$. The empirical scaling $r_c \propto a(t)$ is therefore equivalent to the statement that the background cosmology is matter-dominated.

6.9.2 The residual as the comoving deceleration of r_c

Under the condition $r_c \propto a(t)$, the time derivatives of r_c are

$$\dot{r}_c = Hr_c, \qquad \ddot{r}_c = \dot{H}r_c + H\dot{r}_c = \left(\dot{H} + H^2\right) r_c = \frac{\ddot{a}}{a}\,r_c. \tag{173}$$

The residual sigma channel evaluated at $r = r_c$ is

$$c\,\Pi_{g,r}(r_c) = \dot{H}\,r_c = \ddot{r}_c - H^2 r_c = \ddot{r}_c - H\dot{r}_c. \tag{174}$$

This is the acceleration of r_c *relative to the local Hubble flow*: the difference between the absolute acceleration $\ddot{r}_c$ and the Hubble-flow acceleration $H\dot{r}_c = H^2 r_c$. In other words, $c\,\Pi_{g,r}(r_c)$ measures how much r_c deviates from perfect comoving motion.

Since $r_c \propto a(t)$ by assumption, r_c *is* exactly comoving — but its comoving acceleration is non-zero whenever the expansion is decelerating ($\ddot{a} < 0$) or accelerating ($\ddot{a} > 0$). The residual is therefore the second Friedmann equation evaluated at r_c:

$$\boxed{c\,\Pi_{g,r}(r_c) = \frac{\ddot{a}}{a}\,r_c - H^2 r_c = \dot{H}\,r_c.} \tag{175}$$

6.9.3 The two cosmological limits

Matter-dominated: $\dot{H} = -\frac{3}{2}H^2$.

$$c\,\Pi_{g,r}(r_c) = -\frac{3}{2}H^2 r_c = -\frac{3}{2}H^2\left(\frac{2GM}{H^2}\right)^{1/3} = -\frac{3}{2^{1/3}}(GMH)^{2/3}\,. \tag{176}$$

The residual is the same combination $(GMH)^{2/3}$ that appears in the flat rotation speed v_∞^2 (Eqn. (168)): the comoving deceleration of r_c is of the same order as the square of the galactic rotation speed. The CL disk is slowly decelerating at the cosmological rate, carried inward as the matter-dominated universe slows its expansion.

De Sitter: $\dot{H} = 0$, $H = H_\Lambda = \text{const}$.

$r_c = (2GM/H_\Lambda^2)^{1/3}$ is *constant in time*. The residual vanishes:

$$c\,\Pi_{g,r}(r_c) = 0, \tag{177}$$

and r_c is permanently comoving. The CL disk is an exact stationary solution of both the norm and sigma channels, and its characteristic scale r_c does not evolve.

6.9.4 Consistency of the three radii

Under matter domination all three characteristic radii defined by the channel structure scale identically with $a(t)$:

$$r_c(t) \propto H^{-2/3} \propto a(t), \qquad r_\sigma(t) = \frac{r_c(t)}{2^{2/3}} \propto a(t),$$
$$r_a(t) = \frac{r_c(t)}{2^{1/3}} \propto a(t). \tag{178}$$

All three radii are comoving: their mutual ratios $r_\sigma : r_a : r_c = 2^{-2/3} : 2^{-1/3} : 1$ are time-independent constants fixed by the geometry of the combined flow. The structure of the vacuum flow — the partition into creation-dominated, transition, and absorption-dominated regions — is frozen into the comoving frame and simply scales with the expansion of the universe.

6.9.5 Summary

The empirical observation $r_c \propto a(t)$, the analytical residual $c\,\Pi_{g,r} = \dot{H}\,r$, and the matter-dominated Friedmann relation $\dot{H} = -\frac{3}{2}H^2$ form a mutually consistent and closed set:

$$r_c \propto a(t) \iff \dot{H} = -\tfrac{3}{2}H^2 \iff c\,\Pi_{g,r}(r_c) = -\frac{3}{2^{1/3}}(GMH)^{2/3}. \tag{179}$$

Each implies the other two. The CL galactic disk sits on a comoving velocity-balance radius in a matter-dominated universe, its residual sigma channel is the Friedmann deceleration evaluated at r_c, and the magnitude of that residual is set by the same $(GMH)^{2/3}$ combination that determines the flat rotation speed. In the de Sitter limit all three statements collapse to $\dot{H} = 0$, $r_c = \text{const}$, $c\,\Pi_{g,r} = 0$: the CL disk becomes an exact, permanently stationary solution with a fixed characteristic scale.

6.10 Significance of the CL Solution

The constant-Lagrangian galactic disk is derived, not postulated. Every step from the sequential rotor $G = G_\phi G_r$ to the flat rotation curve, the Tully-Fisher scaling, and the cosmological comoving structure follows by algebra from a single first-order equation $\mathcal{H}_g = \partial^T U_g$, with no free parameters beyond the observables G, M, R, and H.

The derivation has five features that, taken together, are without precedent in the existing literature.

The CL condition is internal to the framework.

The constant-Lagrangian condition $u_r^2 + u_\phi^2 = v_L^2$ is not imposed from outside. It is the statement that the term $\frac{1}{c^2}\frac{D}{Dt}(\frac{1}{2}v_g^2)$ in the norm channel Eqn. (72) is set to zero. In the slow-flow approximation this term is discarded; promoted to an exact condition it selects the CL solution family from within the channel structure of $\mathcal{H}_g$.

Exact cancellation in the sigma channel.

On the CL solution, the Newtonian term $-GM/r^2$, the Hubble cross-term $-\frac{H}{2}\sqrt{2GM/r}$, and the orbital coupling $u_\phi \partial_r u_\phi$ cancel identically in the radial sigma channel. The cancellation is algebraically exact (Eqns. (155)–(156)), not perturbative. Its physical meaning is precise: the CL disk is in complete local gravitational equilibrium, and the sigma channel retains only the cosmological residual $c\,\Pi_{g,r} = \dot{H}\,r$.

Local and global are the same object.

The residual $\dot{H}\,r$ evaluated at r_c is the acceleration of r_c relative to the Hubble flow — the second Friedmann equation at the galactic scale (Eqn. (175)). A local galactic observable (r_c scaling) and a global cosmological statement (matter domination) are the same condition (Eqn. (179)).

The de Sitter limit is exact.

For $\dot{H} = 0$ the residual vanishes identically, r_c is constant, and the CL profile is an exact stationary solution of both the norm and sigma channels with no approximation. The CL galactic disk is self-consistent in the cosmological background the universe is currently approaching.

No dark matter, no modified gravity, no new fields.

Flat rotation curves and the Tully-Fisher scaling $v_\infty^4 \propto (GMH)^{4/3}$ (Eqn. (169)) emerge from the virial boundary condition and the CL norm channel condition alone. The MOND-like behaviour is a consequence of the Hubble rapidity entering the radial flow, not of any modification of the gravitational law.

To our knowledge this is the first derivation in which flat galactic rotation curves, Tully-Fisher scaling, and the cosmological comoving structure of galactic disks emerge simultaneously from a single first-order algebraic field equation without dark matter, modified gravity, or adjustable parameters.

7 Conclusion

This paper has developed the vacuum-flow interpretation of gravity as a BQ-RFD field hierarchy, starting from the rapidity rotor G_g and the geometric field $\mathcal{H}_g = \partial^T U_g$, and ending with the constant-Lagrangian galactic disk as an exact solution of that hierarchy. The derivation chain is:

$$\psi_g \longrightarrow G_g \longrightarrow U_g \longrightarrow \mathcal{H}_g = \partial^T U_g, \tag{180}$$

with each arrow an exact algebraic step and each level contributing physically distinct content. The individual steps are not technically difficult. What the derivation required was a sequence of conceptual decisions, each of which unlocked the next level of the chain, and it is worth summarising them here in retrospect.

The four conceptual steps.

The first was taking U_g rather than $g_{\mu\nu}$ as the primary gravitational object. The rotor G_g projects onto U_g exactly, linearly, and invertibly; it reaches $g_{\mu\nu}$ only after two successive information-reducing operations — a slow-flow approximation and a coframe quadratic. Recognising that asymmetry and following it to its conclusion placed the gravitational four-velocity, not the metric, at the centre of the framework and made the BQ-RFD channel decomposition directly applicable.

The second was the reinterpretation of ρ_0. Dividing $\mathcal{H} = \partial^T(\rho_0 U_g)$ by ρ_0 to obtain $\mathcal{H}_g = \partial^T U_g$ is algebraically trivial, but it requires deciding that the vacuum itself is the dynamical object and matter only a passive tracer of the flow. Without that ontological step the field equation remains attached to a material medium and the geometric interpretation of the channels — vacuum continuity, geometric vorticity, geometric acceleration — does not follow.

The third was the treatment of a normally discarded term. The full norm channel of $\mathcal{H}_g$ contains $\frac{1}{c^2}\frac{D}{Dt}(\frac{1}{2}v_g^2)$, legitimately neglected in the slow-flow approximation. Setting it to zero *exactly* rather than approximately selects the constant-Lagrangian solution family (1) from within the channel structure, without any external postulate. The empirical condition identified in (de Haas 2025f) is thereby absorbed into the framework as an exact dynamical condition of the norm channel.

The fourth was recognising the role of the empirical programme as a guide. The SPARC fits of (de Haas 2025f) established the destination — flat rotation curves, the one-third rule, Tully-Fisher scaling — before the theory was in place. The theoretical work of this paper is a proof of something already known to be true, where the difficulty lay not in finding the result but in identifying the framework from which it becomes derivable. The empirical and theoretical papers are two legs of a single programme: the former posed the question in its sharpest form; the present paper answers it.

Results at the field level.

The channel decomposition of $\mathcal{H}_g = \partial^T U_g$ separates vacuum continuity (σ_{Eg}), geometric vorticity ($\mathbf{\Omega}_g$), and geometric acceleration ($\mathbf{\Pi}_g$) as three automatic projections of one BQ product. Newtonian gravity, the Poisson equation, and irrotationality emerge as channel projections of $\mathcal{H}_g = 0$ without separate assumption. The addition of the Hubble rapidity reveals two distinct radii — the velocity-balance radius r_c and the continuity-balance radius $r_\sigma \approx 0.63\,r_c$ — whose separation is invisible to any velocity-field formulation and constitutes the primary new structural result at the field level.

The constant-Lagrangian exact solution.

The addition of an orbital rapidity ψ_ϕ through the sequential rotor $G = G_\phi G_r$ activates the vorticity channel while leaving the continuity channel unchanged. The CL condition $u_r^2 + u_\phi^2 = v_L^2$ arises from the norm channel as described above; combined with the Newtonian virial theorem at the bulge surface $r = R$, it fixes v_L^2 completely in terms of G, M, R, H and determines the rotation curve at all $r \geq R$ in closed form.

On the CL solution the radial sigma channel undergoes exact algebraic cancellation of all Newtonian and cross-coupling terms, leaving the purely cosmological residual $c\,\Pi_{g,r} = \dot{H}\,r$. In the de Sitter limit $\dot{H} = 0$ this residual vanishes identically and the CL profile is an exact stationary solution of the full channel system. The empirical observation $r_c \propto a(t)$, the matter-dominated Friedmann relation $\dot{H} = -\frac{3}{2}H^2$, and the residual evaluated at r_c form a closed equivalence: each implies the other two. The Baryonic Tully-Fisher scaling $v_\infty^4 \propto (GMH)^{4/3}$ follows from the CL boundary condition when $R = r_c$, without dark matter, modified gravity, or free parameters.

Position of the framework.

The BQ vacuum-flow hierarchy positions Newtonian gravity, the Hubble flow, and the CL galactic disk as three levels of the same first-order field equation $\mathcal{H}_g = \partial^T U_g$, distinguished by the rapidity content of G_g and the approximation level applied, rather than as three separate theories connected only through limiting procedures. The metric and the Einstein tensor are secondary: they arise by projecting U_g through two information-reducing steps, whereas U_g itself retains the full rapidity content of G_g exactly and invertibly. In this sense the present framework is a first-order precursor to General Relativity: GR describes the second-order curvature consequences of the rapidity field; the BQ vacuum-flow hierarchy describes the field itself, in the closed $\mathrm{M}(2, \mathbb{C})$ algebra.

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