The Biquaternion Algebra as a Channel-Decomposition Framework for Classical Field Theories

Applications to Electrodynamics and Relativistic Fluid Dynamics

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Abstract

The biquaternion (BQ) algebra at the Pauli spin level, realized as the full matrix algebra $\boldsymbol{M}(\mathbf{2}, \mathbb{C})$, provides a closed associative framework in which the equations of classical field theories can be organized through a common hierarchy of algebraic products and channel decompositions. This paper develops that hierarchy for electrodynamics and relativistic fluid dynamics and explores its role as a common organizational framework for both theories. In electrodynamics, the four-current $\boldsymbol{J}$ and four-potential $\boldsymbol{A}$ generate the hierarchy $\boldsymbol{B} = \boldsymbol{\partial}^T \boldsymbol{A}$, $\boldsymbol{JB} = \boldsymbol{F}$, $\boldsymbol{\partial B} = \mu_0 \boldsymbol{J}$, and $\boldsymbol{\partial}(\boldsymbol{J}^T \boldsymbol{A}) = \boldsymbol{F}$. The corresponding channel decompositions recover the electromagnetic field, Lorentz force law, Maxwell equations, and Laue–Sommerfeld interaction-conservation relations, while the Bianchi identities appear as geometric constraints in the vanishing norm and sigma channels. In relativistic fluid dynamics, the substitution $\boldsymbol{J} \to \boldsymbol{U}$ and $\boldsymbol{A} \to \boldsymbol{G}$ generates an analogous hierarchy whose channel projections recover, under the stated physical identifications and closures, the continuity equation, Bernoulli relation, acoustic wave dynamics, vorticity transport, Euler–Lamb momentum balance, helicity density, and stress-energy conservation structure. The barotropic equation of state enters through the enthalpy replacement $\varepsilon_0 \to w = \varepsilon_0 + p$ and propagates through the hierarchy as the acoustic refractive index $n_s = c/c_s$. Several structural identifications appear to be new. These include the hierarchy-wide role of $n_s = c/c_s$, the confinement of helicity generation to the norm channel of $\boldsymbol{U}\boldsymbol{\mathcal{H}} = \boldsymbol{F}$, the fluid analogue of the Bianchi identities, and the covariant wave equation $\square \boldsymbol{G} = \boldsymbol{F}'$ with the momentum-density field $\boldsymbol{G}$ as the primary wave-carrying variable. The analysis further identifies the force equation as the unique four-channel level of both the electromagnetic and fluid hierarchies and locates the greater structural complexity of fluid dynamics in the self-referential coupling between velocity and momentum-density fields. Finally, the closure of the BQ algebra suggests possible computational implementations in which algebraic structure is embedded directly into the computational graph.

Keywords: Biquaternions, Pauli algebra, channel decomposition, algebraic hierarchy, electrodynamics, Maxwell equations, Lorentz force, relativistic fluid dynamics, Euler–Lamb equation, vorticity dynamics, helicity, stress-energy conservation, Noether conservation laws, Lorentz covariance

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1 Introduction

The biquaternion (BQ) algebra at the Pauli spin level provides a self-contained algebraic framework in which the kinematic and dynamic content of a physical theory is encoded in a small set of four-vectors, while the relationships between the physical equations of that theory can be organized through a fixed hierarchy of algebraic products.This paper develops that framework systematically across two domains — electrodynamics and relativistic fluid dynamics — and argues that the resulting channel decomposition constitutes a useful organizing principle for both theories, revealing structural connections between equations that are usually introduced independently.

The central object of the BQ framework is the bilinear transpose product of two four-vectors

$$A = a_0\hat{\mathrm{T}} + \mathbf{a}\cdot\hat{\mathbf{K}}, \tag{1}$$

$$B = b_0\hat{\mathrm{T}} + \mathbf{b}\cdot\hat{\mathbf{K}}, \tag{2}$$

defined in the four-vector basis

$$K^\mu = (\hat{\mathrm{T}}, \hat{\mathbf{K}}). \tag{3}$$

Unlike the ordinary product AB, the transpose product

$$C = A^T B \tag{4}$$

decomposes naturally into three geometrically distinct channels,

$$C = (a_0 b_0 - \mathbf{a}\cdot\mathbf{b})\hat{1} + (\mathbf{a}\times\mathbf{b})\cdot\hat{\mathbf{K}} + (a_0\mathbf{b} - b_0\mathbf{a})\cdot\hat{\boldsymbol{\sigma}} \equiv \mathcal{L}\,\hat{1} + \mathbf{K}\cdot\hat{\mathbf{K}} + \mathbf{S}\cdot\hat{\boldsymbol{\sigma}}. \tag{5}$$

The three channels carry complementary geometric content. The norm channel $\mathcal{L}$ contains the Minkowski inner product, the K-channel $\mathbf{K}$ contains the antisymmetric rotational structure, and the σ-channel $\mathbf{S}$ contains the mixed time–space transport structure. Their separation is canonical: it follows directly from the multiplication rules of the BQ basis and requires no additional projections or assumptions.

The second level of the hierarchy is generated by the triple product

$$D = C_0(A^T B), \tag{6}$$

where C_0 is a four-vector operator or source field. In particular, choosing $C_0 = \partial$ yields

$$D = \partial(A^T B), \tag{7}$$

whose channel decomposition automatically separates into

$$D = D_1\hat{1} + D_T\hat{\mathrm{T}} + D_K\cdot\hat{\mathbf{K}} + D_\sigma\cdot\hat{\boldsymbol{\sigma}}. \tag{8}$$

These four channels form the basic organizational structure of the BQ hierarchy and recur throughout the electromagnetic and fluid-dynamic applications developed in this paper.

In the applications developed below, the hierarchy is generated recursively. A pair of four-vector inputs first produces a field through $B = \partial^T A$, after which force, field, and stress-energy equations arise as successive levels of the same algebraic construction. The BQ algebra is realized as the full matrix algebra $M(2, \mathbb{C})$ of 2×2 complex matrices. This algebra is associative, closed under multiplication, and contains the Pauli matrices as its σ-channel basis elements. Consequently every product of physical BQ objects remains inside the same algebra, and the channel decomposition remains available at every level of the hierarchy without approximation, truncation, or the introduction of additional geometric structures.

The framework is applied to electrodynamics in Section 3 through the identification of the four-current

$$J = \rho_e c \hat{\mathrm{T}} + \mathbf{J} \cdot \hat{\mathbf{K}} \tag{9}$$

and the four-potential

$$A = \frac{\phi}{c} \hat{\mathrm{T}} + \mathbf{A} \cdot \hat{\mathbf{K}} \tag{10}$$

as the physical four-vector inputs. The electromagnetic field $B = \partial^T A$ decomposes into the electric and magnetic fields as its σ and K channels; the Lorentz force equation $JB = F$ decomposes into power, force-density, and rotational transport sectors; the Maxwell equation $\partial B = \mu_0 J$ reproduces the four Maxwell equations as its channel projections, with the Bianchi identities ($\nabla \cdot \mathbf{B} = 0$ and Faraday's law) appearing as geometric identities in the vanishing norm and sigma channels; and the Laue–Sommerfeld equation $\partial(J^T A) = F$ generates the interaction-sector conservation structure associated with the current–potential product.

The framework is applied to relativistic fluid dynamics in Section 4 through the substitution $J \to U$, $A \to G$, where $U = u_0 \hat{\mathrm{T}} + \mathbf{u} \cdot \hat{\mathbf{K}}$ is the four-velocity and $G = \frac{\varepsilon}{c} \hat{\mathrm{T}} + \mathbf{g} \cdot \hat{\mathbf{K}}$ is the four-momentum density. This substitution generates a parallel hierarchy,

$$\mathcal{H} = \partial^T G, \qquad U\mathcal{H} = F, \qquad \partial\mathcal{H} = F', \qquad \partial(U^T G) = F, \tag{11}$$

whose channel projections recover, under the stated physical identifications and closures, the continuity equation, Bernoulli relation, acoustic wave propagation, baroclinic vorticity generation, Euler–Lamb momentum equation, helicity density, vorticity transport, stress-energy conservation, and associated Noether structures. The barotropic equation of state, introduced through the enthalpy replacement

$$\varepsilon_0 \to w = \varepsilon_0 + p, \tag{12}$$

propagates through the hierarchy as the acoustic refractive index

$$n_s = \frac{c}{c_s}, \tag{13}$$

providing a single dimensionless parameter that links the relativistic and acoustic sectors of the theory.

The structural comparison between the electromagnetic and fluid hierarchies, developed in Section 5, reveals that the two theories share not only the same algebraic architecture but also the same channel-activity pattern. The force equation ($JB = F$ in electrodynamics and $U\mathcal{H} = F$ in fluid dynamics) is the unique four-channel equation of each hierarchy, the only level at which all geometric sectors are simultaneously active and carry independent physical content. The principal structural difference between the two theories is equally well localized: the self-referential coupling between the fluid velocity and momentum-density fields, encoded in the relation between U and G, introduces a level of nonlinearity absent from the electromagnetic hierarchy.

The paper is organized as follows. Section 2 develops the BQ algebra at the Pauli spin level, establishing the basis elements, multiplication rules, Lorentz transformation properties, and triple-product structure. Section 3 develops the electromagnetic hierarchy and its channel decomposition. Section 4 develops the analogous fluid hierarchy and relates its channel projections to known formulations of relativistic and classical fluid dynamics. Section 5 synthesizes the two applications, identifies the principal structural insights revealed by the channel decomposition, discusses several structural identifications that appear to be new, and outlines possible extensions including magnetohydrodynamics, computational implementations, and curved-spacetime generalizations. Section 6 concludes.

Two points of scope deserve explicit statement at the outset. First, the framework developed here is a flat-space theory: the derivative ∂ is the Minkowski four-gradient throughout, and the extension to curved spacetime — requiring the replacement of ∂ by a covariant derivative incorporating a gravitational rotor field Q_g — is left to companion work. Second, the principal contribution of the present paper is structural rather than dynamical. The equations encountered throughout electrodynamics and fluid dynamics are not replaced by new field equations; rather, they are shown to occupy well-defined locations within a common algebraic hierarchy.

The central claim is therefore that the BQ channel decomposition functions as a useful organizing principle for classical field theories, rather than merely as an alternative notation, a claim supported by the systematic correspondence between channel projections and established physical equations across multiple levels of the hierarchy and across two distinct physical domains.

2 The BQ algebra at the Pauli spin level

2.1 A complex quaternion basis for the metric: the Minquats

Quaternions can be represented by the basis $(\hat{1}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$. This basis has the properties $\hat{\mathrm{I}}\hat{\mathrm{I}} = \hat{\mathrm{J}}\hat{\mathrm{J}} = \hat{\mathrm{K}}\hat{\mathrm{K}} = -\hat{1}$ and $\hat{1}\hat{1} = \hat{1}$; $\hat{1}\hat{\mathrm{I}} = \hat{\mathrm{I}}\hat{1} = \hat{\mathrm{I}}$, $\hat{1}\hat{\mathrm{J}} = \hat{\mathrm{J}}\hat{1} = \hat{\mathrm{J}}$ and $\hat{1}\hat{\mathrm{K}} = \hat{\mathrm{K}}\hat{1} = \hat{\mathrm{K}}$; $\hat{\mathrm{I}}\hat{\mathrm{J}} = -\hat{\mathrm{J}}\hat{\mathrm{I}} = \hat{\mathrm{K}}$; $\hat{\mathrm{J}}\hat{\mathrm{K}} = -\hat{\mathrm{K}}\hat{\mathrm{J}} = \hat{\mathrm{I}}$; $\hat{\mathrm{K}}\hat{\mathrm{I}} = -\hat{\mathrm{I}}\hat{\mathrm{K}} = \hat{\mathrm{J}}$. A quaternion number in its summation representation is given by $A = a_0\hat{1} + a_1\hat{\mathrm{I}} + a_2\hat{\mathrm{J}} + a_3\hat{\mathrm{K}}$, in which the a_μ are real numbers . Bi-quaternions or complex quaternions are given by $C = A + iB = c_0\hat{1} + c_1\hat{\mathrm{I}} + c_2\hat{\mathrm{J}} + c_3\hat{\mathrm{K}}$ in which the $c_\mu = a_\mu + ib_\mu$ are complex numbers and the a_μ and b_μ are real numbers.

This standard biquaternion basis $(\hat{1}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$ can be used to provide a basis for relativistic 4-D space-time. One way to do this is by making the time coordinate $c_0 = b_0 i$ complex only and the space coordinates $(c_1, c_2, c_3) = (a_1, a_2, a_3)$ real only, see Synge (1972). Synge called these objects Minkowski quaternions or 'minquats', Silberstein called them 'physical quaternions' Synge (1972). This however produces confusion regarding the time-like complex number as the physics gets more complicated. The complex number then migrates through the system and becomes attached to non-time physical quantities as for example the electric field **E**. To quote Synge: *the intrusion of the imaginary element is not trivial* Synge (1972). The main reason is that minquats do not form a closed algebra under addition and multiplication, but are, due to the multiplication operation, a subspace inside the wider biquaternion space. The reason they are nevertheless used, is given by Synge: *For the application of quaternions to Lorentz transformations it is essential to introduce Minkowskian quaternions* Synge (1972).

The use of minquats produces language conflicts with almost all of modern physics, that is Quantum Mechanics and Special and General Relativity, where the space-time coordinates always are a set of four real numbers. So for several reasons, I choose to insert the time-like complex number of $c_0 = b_0 i$ in the basis instead of in the coordinate. So by using $c_0\hat{1} = b_0 i\hat{1} = b_0\hat{\mathrm{T}}$ the space-time basis is then given by $(\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$. In this way, the coordinates are always a set of real numbers $\in \mathbb{R}$. The space-time basis $(\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$, (a disguised minquat basis) is not closed under multiplications, as already mentioned by Synge.

Remark. By inserting the imaginary unit once and for all into the basis, the coordinates themselves remain real and the metric automatically records and absorbs the complex character, not only of and in the time component but in general. This avoids the need for the bookkeeping mentioned by Synge. Another reason for doing so is more intuitive and rooted in physics and in the comparison with general relativity: in GR the coordinates of a four-vector are always real numbers, and consequently the tensors and matrices built from them are real as well. By keeping all coordinates real in the present biquaternion framework, the algebra mirrors this feature of GR, which will later allow a direct comparison between the GR formalism and the present biquaternion language.

Given our choice for the Minquat bases, we get the following representation of four-vectors. A set of four numbers $\in \mathbb{R}$ is given by

$$A^{\mu} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

or by $A_{\mu} = [a_0, a_1, a_2, a_3]$. In this way, the raising or lowering of the index doesn't change any sign. A^{μ} simply is the transpose of A_{μ} and vise versa. [1] The biquaternion basis can be given as a set $\hat{\mathbf{K}}_{\mu} = (\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$. Then a biquaternion space-time vector can be written as the product

$$A = A_{\mu}\hat{\mathbf{K}}^{\mu} = [a_0, a_1, a_2, a_3] \begin{bmatrix} \hat{\mathrm{T}} \\ \hat{\mathrm{I}} \\ \hat{\mathrm{J}} \\ \hat{\mathrm{K}} \end{bmatrix} = a_0\hat{\mathrm{T}} + a_1\hat{\mathrm{I}} + a_2\hat{\mathrm{J}} + a_3\hat{\mathrm{K}} \tag{14}$$

I apply this to the space-time four vector of relativistic bi-quaternion 4-space R with the four numbers $R_{\mu} = (r_0, r_1, r_2, r_3) = (ct, r_1, r_2, r_3)$, so with $r_0, r_1, r_2, r_3 \in \mathbb{R}$. Then I have the space-time four-vector as the product of the coordinate set and the basis

$$R = R_{\mu}\hat{\mathbf{K}}^{\mu} = r_0\hat{\mathrm{T}} + r_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = ct\hat{\mathrm{T}} + \mathbf{r} \cdot \hat{\mathbf{K}}. \tag{15}$$

I use the three-vector analogue of $R_{\mu}\hat{\mathbf{K}}^{\mu}$ when I write $\mathbf{r} \cdot \hat{\mathbf{K}}$.

In this notation I define the time reversal operation as $R^T = -r_0\hat{\mathrm{T}} + r_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = -r_0\hat{\mathrm{T}} + \mathbf{r} \cdot \hat{\mathbf{K}}$ and the space reversal or parity operation as $R^P = r_0\hat{\mathrm{T}} - r_1\hat{\mathrm{I}} - r_2\hat{\mathrm{J}} - r_3\hat{\mathrm{K}} = r_0\hat{\mathrm{T}} - \mathbf{r} \cdot \hat{\mathbf{K}}$, also called the space point mirror operator. [2]

[1] In standard Minkowski space-time with metric signature $(-, +, +, +)$, raising or lowering indices introduces a sign change in the temporal component, e.g. $A_0 = -A^0$. In the present biquaternion framework, however, raising and lowering indices is defined purely algebraically as matrix transposition, so A^{μ} is simply the transpose of A_{μ}. The Minkowski metric structure enters later through the multiplication properties of the basis elements, not through index manipulation.

[2] Note that, by direct calculation, the parity operator satisfies $R^P = -R^T$. In this notation, the transpose of a matrix is denoted by the suffix 't', so $R^t_{\mu} = R^{\mu}$. The Hermitian (complex) transpose

2.2 Matrix representation of the quaternion basis

Having established the complex quaternion basis $(\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}})$, it is useful to connect this abstract algebraic framework to an explicit matrix representation. Such a representation allows the biquaternion formalism to be handled with standard linear algebra tools and facilitates comparison with the Pauli spin matrices. In the next subsection I therefore represent the basis elements as 2×2 complex matrices and show how a four-vector acquires a compact matrix form in this setting.

Quaternions can be represented by 2x2 matrices. Several representations are possible, but most of those choices result in conflict with the standard approach in physics. Given the unit quaternion as $\hat{\mathrm{I}}$, my choice for the space-time four set is

$$\hat{\mathrm{T}} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \hat{\mathrm{I}} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \hat{\mathrm{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \hat{\mathrm{K}} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{16}$$

I can compare these to the Pauli spin matrices $\hat{\boldsymbol{\sigma}}_P = (\sigma_x, \sigma_y, \sigma_z)$.

$$\hat{\boldsymbol{\sigma}}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\boldsymbol{\sigma}}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \hat{\boldsymbol{\sigma}}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{17}$$

If I exchange the σ_x and the σ_z [3], I get $\hat{\mathbf{K}} = i\hat{\boldsymbol{\sigma}}$ and $\hat{\mathbf{K}}_\mu = i(\hat{\mathrm{I}}, \hat{\boldsymbol{\sigma}})$. So in my use of the Pauli matrices, I use $\hat{\boldsymbol{\sigma}} \equiv (\sigma_I, \sigma_J, \sigma_K) = (\sigma_z, \sigma_y, \sigma_x)$. So also $\hat{\mathrm{I}} = \hat{\mathrm{T}}\hat{\boldsymbol{\sigma}}_I, \hat{\mathrm{J}} = \hat{\mathrm{T}}\hat{\boldsymbol{\sigma}}_J, \hat{\mathrm{K}} = \hat{\mathrm{T}}\hat{\boldsymbol{\sigma}}_K$ and $\hat{\boldsymbol{\sigma}}_I = -\hat{\mathrm{T}}\hat{\mathrm{I}}, \hat{\boldsymbol{\sigma}}_J = -\hat{\mathrm{T}}\hat{\mathrm{J}}, \hat{\boldsymbol{\sigma}}_K = -\hat{\mathrm{T}}\hat{\mathrm{K}}$.

With this choice of matrices, a four-vector R can be written as

$$R = r_0 \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} + r_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{18}$$

This can be compacted into a matrix representation of R:

$$R = \begin{bmatrix} r_0 i + i r_1 & r_2 + i r_3 \\ -r_2 + i r_3 & r_0 i - i r_1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix} \tag{19}$$

with the numbers $R_{00}, R_{01}, R_{10}, R_{11} \in \mathbb{C}$. Thus a Minkowski four-vector can be represented as a 2×2 complex matrix, with its algebra inherited from the quaternion basis.

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of spinors is indicated by the dagger symbol, as in $\psi^\dagger$, and the complex conjugate by ψ^*. Within this framework, the discrete symmetry operators T and P implement sign changes of temporal and spatial components, in a way that is formally analogous to the index manipulations induced by the metric in general relativity.

[3] This unconventional ordering simplifies the identification with the quaternion basis introduced above.

2.3 Multiplication of vectors as matrix multiplication adds pauliquats to minquats

In general, the multiplication of two four-vectors A and B follows matrix multiplication, with $A_{ij}, B_{ij}, C_{ij} \in \mathbb{C}$.

$$AB = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} = C. \tag{20}$$

So we have

$$C = AB = \begin{bmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}. \tag{21}$$

Of course, vectors A, B and C can be expressed with their a_μ, b_μ, c_μ coordinates $\in \mathbb{R}$ and if we use them, after some elementary but elaborate calculations and rearrangements we arrive at the following result of the multiplication expressed in the a_μ, b_μ and c_μ as [4]:

$$\begin{aligned} c_0 &= -a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \\ c_{1K} &= a_2 b_3 - a_3 b_2 \\ c_{2K} &= a_3 b_1 - a_1 b_3 \\ c_{3K} &= a_1 b_2 - a_2 b_1 \\ c_{1\sigma} &= -a_0 b_1 - a_1 b_0 \\ c_{2\sigma} &= -a_0 b_2 - a_2 b_0 \\ c_{3\sigma} &= -a_0 b_3 - a_3 b_0 \end{aligned} \tag{22}$$

In short, if we use the three-dimensional Euclidean dot and cross products of Euclidean three-vectors in classical physics, this gives for the coordinates

$$c_0 = -a_0 b_0 - \mathbf{a} \cdot \mathbf{b} \tag{23}$$

$$\mathbf{c}_K = \mathbf{a} \times \mathbf{b} \tag{24}$$

$$\mathbf{c}_\sigma = -a_0 \mathbf{b} - \mathbf{a} b_0 \tag{25}$$

And in the quaternion notation we get

$$C = AB = (-a_0 b_0 - \mathbf{a} \cdot \mathbf{b})\hat{1} + (\mathbf{a} \times \mathbf{b}) \cdot \hat{\mathbf{K}} + (-a_0 \mathbf{b} - \mathbf{a} b_0) \cdot \hat{\boldsymbol{\sigma}} \tag{26}$$

This matrix multiplication, in which I used $\hat{\mathrm{T}}\hat{\mathrm{T}} = -\hat{1}$ and $\hat{\mathrm{T}}\hat{\mathbf{K}} = -\hat{\boldsymbol{\sigma}}$, implies that the space-time basis $(\hat{\mathrm{T}}, \hat{\mathbf{K}})$ is being duplicated by a spin-norm basis $(\hat{1}, \hat{\boldsymbol{\sigma}})$ by the multiplication operation. I define the $\hat{\sigma}_\mu$ four set the Pauli-quat basis and

[4] Here the subscripts K and σ denote the parts in the $\hat{\mathbf{K}}$ and $\hat{\boldsymbol{\sigma}}$ directions, respectively.

the product $A_\mu \hat{\sigma}^\mu$ a Pauli-quat. The biquaternion algebraic space is completely covered by the dual basis with four 'channels' $(\hat{\mathrm{K}}_\mu, \hat{\sigma}_\mu) = (\hat{\mathrm{T}}, \hat{\mathbf{K}}, \hat{\boldsymbol{\sigma}}, \hat{\mathrm{1}})$.

The relativistically relevant multiplications of two four-vectors are all in the form $C = A^T B$. The difference between AB and $A^T B$ is in the sign of a_0. This results in

$$C = A^T B = (a_0 \hat{\mathrm{T}} + \mathbf{a} \cdot \hat{\mathbf{K}})(b_0 \hat{\mathrm{T}} + \mathbf{b} \cdot \hat{\mathbf{K}}) = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b})\,\hat{\mathrm{1}} + (\mathbf{a} \times \mathbf{b}) \cdot \hat{\mathbf{K}} + (a_0 \mathbf{b} - \mathbf{a} b_0) \cdot \hat{\boldsymbol{\sigma}}, \tag{27}$$

with the channels

$$c_0 = a_0 b_0 - \mathbf{a} \cdot \mathbf{b} \tag{28}$$
$$\mathbf{c}_K = \mathbf{a} \times \mathbf{b} \tag{29}$$
$$\mathbf{c}_\sigma = a_0 \mathbf{b} - \mathbf{a} b_0. \tag{30}$$

Hence the relativistically relevant bilinear is $A^T B$, not AB, and the invariant quadratic is $A^T A$:

$$C = A^T A = (a_0^2 - \mathbf{a} \cdot \mathbf{a})\,\hat{\mathrm{1}}. \tag{31}$$

The main quadratic form of the metric is $dR^T dR = (c^2 dt^2 - d\mathbf{r}^2)\hat{\mathrm{1}} = c^2 d\tau^2 \hat{\mathrm{1}} = ds^2 \hat{\mathrm{1}}$ with $ds = c d\tau$. The quadratic giving the distance of a point R to the origin of its reference system is given by $R^T R = (c^2 t^2 - \mathbf{r}^2)\hat{\mathrm{1}} = c^2 \tau^2 \hat{\mathrm{1}} = s^2 \hat{\mathrm{1}}$ with $s = c\tau$.

The multiplication of two four vectors can also be arranged as the multiplication of two tensors, a coordinate tensor times a metric tensor using that

$$(A_\mu \hat{\mathbf{K}}^\mu)^T B_\nu \hat{\mathbf{K}}^\nu = A_\mu B^\nu (\hat{\mathbf{K}}_\mu)^T \hat{\mathbf{K}}^\nu = C_\mu{}^\nu \hat{\mathbf{K}}_\mu{}^\nu \tag{32}$$

with a real coordinate tensor $C_\mu{}^\nu$ [5]. Using mutliplication rules as $\hat{\mathrm{T}}\hat{\mathrm{T}} = -\hat{\mathrm{1}}$, $\hat{\mathrm{T}}\hat{\mathrm{I}} = -\sigma_I$, $\hat{\mathrm{I}}\hat{\mathrm{T}} = \sigma_I$ and others, the metric tensor can be expanded as

$$\hat{\mathbf{K}}_\mu{}^\nu = (\hat{\mathbf{K}}_\mu)^T \hat{\mathbf{K}}^\nu = \begin{bmatrix} -\hat{\mathrm{T}}, \hat{\mathrm{I}}, \hat{\mathrm{J}}, \hat{\mathrm{K}} \end{bmatrix} \begin{bmatrix} \hat{\mathrm{T}} \\ \hat{\mathrm{I}} \\ \hat{\mathrm{J}} \\ \hat{\mathrm{K}} \end{bmatrix} = \tag{33}$$

$$\begin{bmatrix} -\hat{\mathrm{T}}\hat{\mathrm{T}} & \hat{\mathrm{I}}\hat{\mathrm{T}} & \hat{\mathrm{J}}\hat{\mathrm{T}} & \hat{\mathrm{K}}\hat{\mathrm{T}} \\ -\hat{\mathrm{T}}\hat{\mathrm{I}} & \hat{\mathrm{I}}\hat{\mathrm{I}} & \hat{\mathrm{J}}\hat{\mathrm{I}} & \hat{\mathrm{K}}\hat{\mathrm{I}} \\ -\hat{\mathrm{T}}\hat{\mathrm{J}} & \hat{\mathrm{I}}\hat{\mathrm{J}} & \hat{\mathrm{J}}\hat{\mathrm{J}} & \hat{\mathrm{K}}\hat{\mathrm{J}} \\ -\hat{\mathrm{T}}\hat{\mathrm{K}} & \hat{\mathrm{I}}\hat{\mathrm{K}} & \hat{\mathrm{J}}\hat{\mathrm{K}} & \hat{\mathrm{K}}\hat{\mathrm{K}} \end{bmatrix} = \begin{bmatrix} \hat{\mathrm{1}} & -\hat{\sigma}_I & -\hat{\sigma}_J & -\hat{\sigma}_K \\ \hat{\sigma}_I & -\hat{\mathrm{1}} & -\hat{\mathrm{K}} & \hat{\mathrm{J}} \\ \hat{\sigma}_J & \hat{\mathrm{K}} & -\hat{\mathrm{1}} & -\hat{\mathrm{I}} \\ \hat{\sigma}_K & -\hat{\mathrm{J}} & \hat{\mathrm{I}} & -\hat{\mathrm{1}} \end{bmatrix}. \tag{34}$$

[5] Here the coefficients $C_\mu{}^\nu$ are built purely from the real-number components A_μ and B^ν, while the geometric content of the multiplication resides in the metric basis $(\hat{\mathbf{K}}_\mu)^T \hat{\mathbf{K}}^\nu$. This separation mirrors the situation in special and general relativity, where tensors are expressed as coordinate arrays of real numbers contracted with basis elements determined by the metric. In this way, the biquaternion formalism reproduces the same division between numerical components and geometric structure that underlies SR and GR.

This multiplication product has a norm $\hat{1}$ part, a space $\hat{\mathbf{K}}$ part and a spin $\hat{\boldsymbol{\sigma}}$ part. So the multiplication of two four vectors $A^T B = C$ has this multiplication matrix. The multiplication combines the properties of symmetric and anti-symmetric in one product: the scalar part ($\propto \hat{1}$) is symmetric in A, B, the $\hat{\mathbf{K}}$ part ($\propto \mathbf{a} \times \mathbf{b}$) is antisymmetric, and the $\hat{\boldsymbol{\sigma}}$ part mixes them through the temporal–spatial coupling.

$$C_\mu{}^\nu = (A_\mu)B^\nu = \begin{bmatrix} a_0, a_1, a_2, a_3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_0 b_0 & a_1 b_0 & a_2 b_0 & a_3 b_0 \\ a_0 b_1 & a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_0 b_2 & a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_0 b_3 & a_1 b_3 & a_2 b_3 & a_3 b_3 \end{bmatrix}. \tag{35}$$

Then the product $A^T B$ can be given as the direct matrix product

$$C_\mu{}^\nu \hat{\mathbf{K}}_{\mu T}{}^\nu = \begin{bmatrix} a_0 b_0 & a_1 b_0 & a_2 b_0 & a_3 b_0 \\ a_0 b_1 & a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_0 b_2 & a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_0 b_3 & a_1 b_3 & a_2 b_3 & a_3 b_3 \end{bmatrix} \begin{bmatrix} \hat{1} & -\hat{\sigma}_I & -\hat{\sigma}_J & -\hat{\sigma}_K \\ \hat{\sigma}_I & -\hat{1} & -\hat{\mathrm{K}} & \hat{\mathrm{J}} \\ \hat{\sigma}_J & \hat{\mathrm{K}} & -\hat{1} & -\hat{\mathrm{I}} \\ \hat{\sigma}_K & -\hat{\mathrm{J}} & \hat{\mathrm{I}} & -\hat{1} \end{bmatrix} = \tag{36}$$

$$a_0 b_0 \hat{1} - a_1 b_1 \hat{1} - a_2 b_2 \hat{1} - a_3 b_3 \hat{1} + \tag{37}$$

$$a_0 b_1 \hat{\sigma}_I + a_0 b_2 \hat{\sigma}_J + a_0 b_3 \hat{\sigma}_K - a_1 b_0 \hat{\sigma}_I - a_2 b_0 \hat{\sigma}_J - a_3 b_0 \hat{\sigma}_K + \tag{38}$$

$$a_1 b_2 \hat{\mathrm{K}} - a_2 b_1 \hat{\mathrm{K}} + a_3 b_1 \hat{\mathrm{J}} - a_1 b_3 \hat{\mathrm{J}} + a_2 b_3 \hat{\mathrm{I}} - a_3 b_2 \hat{\mathrm{I}} = \tag{39}$$

$$(a_0 b_0 - \mathbf{ab})\hat{1} + \mathbf{a} \times \mathbf{b} \cdot \hat{\mathbf{K}} + (a_0 \mathbf{b} - \mathbf{a} b_o) \cdot \hat{\boldsymbol{\sigma}} = \tag{40}$$

$$c_1 \hat{1} + \mathbf{c}_K \cdot \hat{\mathbf{K}} + \mathbf{c}_\sigma \cdot \hat{\boldsymbol{\sigma}}. \tag{41}$$

This way of writing things out explicitly shows that in the BQ algebra, all the elements of an ordinary tensor $C_\mu{}^\nu = A_\mu B^\nu$ are integrated in the ordered outcome $C = c_1 \hat{1} + \mathbf{c}_K \cdot \hat{\mathbf{K}} + \mathbf{c}_\sigma \cdot \hat{\boldsymbol{\sigma}}$, where all the elements of $C_\mu{}^\nu$ have been automatically distributed over their allotted channels.

The inevitable appearance of the spin-norm basis in the multiplication of two Synge minquats or Silberstein physical quaternions explains why the minquats do not form a closed algebra under multiplication Synge (1972). In the present approach, the space-time basis $(\hat{\mathrm{T}}, \hat{\mathbf{K}})$ likewise does not form a closed algebra by itself: it requires a spin-norm complex dual $(\hat{\mathrm{T}}, \hat{\mathbf{K}}) = i(\hat{1}, \hat{\boldsymbol{\sigma}})$ in order to span the full biquaternion space. This extension is obtained under the deliberate convention—a free choice of framework in the Kantian sense—to restrict all coordinates R_μ, P_μ in $R_\mu \hat{\mathrm{K}}^\mu$ and $P_\mu \hat{\sigma}^\mu$ to real values. That convention uniquely produces the dual basis.

Interpretive remark. One may view the resulting structure as endowing the physical domain with a dual space-time/spin-norm basis as its natural geometry. Speculatively, this duality might mirror aspects of real physics: electric charges and currents reside in the space-time sector $(\hat{\mathrm{T}}, \hat{\mathbf{K}})$, whereas hypothetical magnetic monopoles and monopole currents, if they exist, would be associated with the spin-norm sector $(\hat{1}, \hat{\boldsymbol{\sigma}})$.

In this terminology, Synge's minquats correspond to $R_\mu \hat{\mathrm{K}}^\mu$ biquaternions, while $P_\mu \hat{\sigma}^\mu$ may be called pauliquats. Together, minquats and pauliquats span the full biquaternion space. The multiplication of two minquats necessarily produces both minquat and pauliquat components. In this picture, electric currents are naturally represented by minquats, magnetic currents (if at all possible) by pauliquats. Intrinsic spin appears as a pauliquat, with its Lorentz dual—intrinsic polarization—as a minquat.

2.4 Further multiplications: the tripple product

In the same way we can calculate the tripple product of three minquats A, B and D, as $E = DC = D(A^T B)$ with

$$\begin{aligned} E = (d_0\hat{\mathrm{T}} + \mathbf{d}\cdot\hat{\mathbf{K}})(c_1\hat{1} + \mathbf{c}_K\cdot\hat{\mathbf{K}} + \mathbf{c}_\sigma\cdot\hat{\boldsymbol{\sigma}}) = \\ d_0 c_1 \hat{\mathrm{T}}\hat{1} + d_0\mathbf{c}_K\cdot(\hat{\mathrm{T}}\hat{\mathbf{K}}) + d_0\mathbf{c}_\sigma\cdot(\hat{\mathrm{T}}\hat{\boldsymbol{\sigma}}) + \\ \mathbf{d}\cdot\hat{\mathbf{K}}c_1\hat{1} + (\mathbf{d}\cdot\hat{\mathbf{K}})(\mathbf{c}_K\cdot\hat{\mathbf{K}}) + (\mathbf{d}\cdot\hat{\mathbf{K}})(\mathbf{c}_\sigma\cdot\hat{\boldsymbol{\sigma}}) = \\ d_0c_1\hat{\mathrm{T}} - d_0\mathbf{c}_K\cdot\hat{\boldsymbol{\sigma}} + d_0\mathbf{c}_\sigma\cdot\hat{\mathbf{K}} + (\mathbf{d}c_1)\cdot\hat{\mathbf{K}} \\ -\mathbf{d}\cdot\mathbf{c}_K\hat{1} + (\mathbf{d}\times\mathbf{c}_K)\cdot\hat{\mathbf{K}} + \\ +\mathbf{d}\cdot\mathbf{c}_\sigma\hat{\mathrm{T}} + (\mathbf{d}\times\mathbf{c}_\sigma)\cdot\hat{\boldsymbol{\sigma}}) = \end{aligned} \tag{42}$$

$$\begin{aligned} -\mathbf{d}\cdot\mathbf{c}_K\hat{1} + (d_0c_1 + \mathbf{d}\cdot\mathbf{c}_\sigma)\hat{\mathrm{T}} + \\ (\mathbf{d}c_1 + d_0\mathbf{c}_\sigma + \mathbf{d}\times\mathbf{c}_K)\cdot\hat{\mathbf{K}} + (\mathbf{d}\times\mathbf{c}_\sigma - d_0\mathbf{c}_K)\cdot\hat{\boldsymbol{\sigma}}, \end{aligned} \tag{43}$$

where we used

$$\begin{aligned} (\mathbf{d}\cdot\hat{\mathbf{K}})(\mathbf{c}_\sigma\cdot\hat{\boldsymbol{\sigma}}) = \hat{\mathrm{T}}(\mathbf{d}\cdot\hat{\mathbf{K}})(\mathbf{c}_\sigma\cdot\hat{\mathbf{K}}) = \\ -\hat{\mathrm{T}}(-\mathbf{d}\cdot\mathbf{c}_\sigma\hat{1} + (\mathbf{d}\times\mathbf{c}_\sigma)\cdot\hat{\mathbf{K}}) = \\ \mathbf{d}\cdot\mathbf{c}_\sigma\hat{\mathrm{T}} + (\mathbf{d}\times\mathbf{c}_\sigma)\cdot\hat{\boldsymbol{\sigma}}. \end{aligned} \tag{44}$$

In short we get for the tripple product $E = DC = D(A^T B)$:

$$\begin{aligned} E = (d_0\hat{\mathrm{T}} + \mathbf{d}\cdot\hat{\mathbf{K}})(c_1\hat{1} + \mathbf{c}_K\cdot\hat{\mathbf{K}} + \mathbf{c}_\sigma\cdot\hat{\boldsymbol{\sigma}}) = \\ -\mathbf{d}\cdot\mathbf{c}_K\hat{1} + (d_0c_1 + \mathbf{d}\cdot\mathbf{c}_\sigma)\hat{\mathrm{T}} + \\ (\mathbf{d}c_1 + d_0\mathbf{c}_\sigma + \mathbf{d}\times\mathbf{c}_K)\cdot\hat{\mathbf{K}} + (\mathbf{d}\times\mathbf{c}_\sigma - d_0\mathbf{c}_K)\cdot\hat{\boldsymbol{\sigma}}, \end{aligned} \tag{45}$$

The result $E = e_1\hat{1} + e_T\hat{\mathrm{T}} + \mathbf{e}_K\cdot\hat{\mathbf{K}} + \mathbf{e}_\sigma\cdot\hat{\boldsymbol{\sigma}}$ can be given as four channels:

$$e_1 = -\mathbf{d}\cdot\mathbf{c}_K \tag{46}$$

$$e_T = d_0c_1 + \mathbf{d}\cdot\mathbf{c}_\sigma \tag{47}$$

$$\mathbf{e}_K = \mathbf{d}c_1 + d_0\mathbf{c}_\sigma + \mathbf{d}\times\mathbf{c}_K \tag{48}$$

$$\mathbf{e}_\sigma = \mathbf{d}\times\mathbf{c}_\sigma - d_0\mathbf{c}_K. \tag{49}$$

Mathematical interpretation of the triple product

The triple product $E = D(A^T B)$ reveals a structural property of the biquaternion algebra that is not visible at the level of the bilinear alone. The bilinear $C = A^T B$ produces a general biquaternion object with four independent channels: a scalar c_1, a time component c_T, a K-bivector $\mathbf{c}_K$, and a σ-bivector $\mathbf{c}_\sigma$. The triple product then asks: what does left-multiplication of this general biquaternion by a minquat four-vector D produce?

The answer, given by equations (35)–(38), is that the output E is again a general biquaternion with four channels, each of which is a specific linear combination of the input channels of C weighted by the components of D. Three structural observations follow.

First, the algebra is *closed*: the triple product of three minquat four-vectors remains within the biquaternion algebra without generating any new algebraic object. The 1–3–3–1 channel structure is preserved under repeated multiplication, which confirms that $M_2(\mathbb{C})$ is self-contained as a computational framework regardless of how many factors are composed.

Second, the channel mixing is *non-trivial and asymmetric*. The scalar output $e_1 = -\mathbf{d} \cdot \mathbf{c}_K$ receives no contribution from c_1 or $\mathbf{c}_\sigma$: it couples exclusively to the K-bivector channel of C through the spatial inner product with $\mathbf{d}$. The time output $e_T = d_0 c_1 + \mathbf{d} \cdot \mathbf{c}_\sigma$ couples to the scalar and σ-channels but not to the K-channel. The K- and σ-outputs mix all channels of C with both the scalar and vector parts of D. This selective coupling pattern is a direct expression of the non-commutativity of the biquaternion product and encodes the metric and rotational structure of the algebra in the specific combinations that appear.

Third, the auxiliary identity (32)–(34), which handles the product $(\mathbf{d}\cdot\hat{\mathbf{K}})(\mathbf{c}_\sigma\cdot\hat{\boldsymbol{\sigma}})$ via the intermediary $\hat{T}$, reveals a characteristic feature of the $M_2(\mathbb{C})$ algebra: the σ-bivector sector is accessible from the K-vector sector through left-multiplication by $\hat{T}$, reflecting the fundamental relation $\hat{\mathrm{K}}_\mu = i\hat{\sigma}_\mu$ between the two four-vector bases. The triple product therefore makes visible, at the purely algebraic level, the internal complex rotation that connects the geometric and spin-norm sectors of the algebra—a connection that will acquire physical meaning once the four-vectors are identified with for example spacetime and energy-momentum quantities.

2.5 The Lorentz transformation

A standard Lorentz transformation between two reference frames connected by a relative velocity v in the $x-$ or Î-direction, with the usual $\gamma = 1/\sqrt{1 - v^2/c^2}$, $\beta = v/c$ and $r_0 = ct$, can be expressed as

$$\begin{bmatrix} r_0' \\ r_1' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \end{bmatrix}. \tag{50}$$

We want to connect this to our matrix representation of R as in Eq.(19) which gives

$$R'_{00} = ir'_0 + ir'_1 = i\gamma r_0 - i\beta\gamma r_1 + i\gamma r_1 - i\beta\gamma r_0 \tag{51}$$
$$R'_{11} = ir'_0 - ir'_1 = i\gamma r_0 - i\beta\gamma r_1 - i\gamma r_1 + i\beta\gamma r_0. \tag{52}$$

Now we want to introduce rapidity or hyperbolic Special Relativity in order to integrate Lorentz transformations into our matrix metric. In de Haas (2014) I gave a brief history of rapidity in its relation to the Thomas precession and the geodesic precession. For this paper we only need elementary rapidity definitions. If we use the rapidity ψ as

$$e^{\pm\psi} = \gamma \pm \beta\gamma, \quad \cosh\psi = \gamma, \quad \sinh\psi = \beta\gamma, \quad e^{\pm\psi} = \cosh\psi \pm \sinh\psi, \tag{53}$$

the previous transformations can be rewritten as

$$R'_{00} = ir'_0 + ir'_1 = (\gamma - \beta\gamma)(ir_0 + ir_1) = R_{00}e^{-\psi} \tag{54}$$
$$R'_{11} = ir'_0 - ir'_1 = (\gamma + \beta\gamma)(ir_0 - ir_1) = R_{11}e^{\psi}. \tag{55}$$

As a result we have

$$R^L = \begin{bmatrix} R'_{00} & R'_{01} \\ R'_{10} & R'_{11} \end{bmatrix} = \begin{bmatrix} R_{00}e^{-\psi} & R_{01} \\ R_{10} & R_{11}e^{\psi} \end{bmatrix} = U^{-1}RU^{-1}. \tag{56}$$

The appearance of U^{-1} on both sides reflects that boosts scale the basis elements $\hat{\mathrm{T}}, \hat{\mathrm{I}}$ oppositely; the off-diagonal entries remain unchanged (since r_2, r_3 are invariant under an $\hat{\mathrm{I}}$-aligned boost and the factors cancel). In the expression $R^L = U^{-1}RU^{-1}$ we used the matrix U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{57}$$

But this means that we can write the result of a Lorentz transformation on R with a Lorentz velocity in the $\hat{\mathrm{I}}$-direction between the two reference systems as

$$R^L = r_0 \begin{bmatrix} ie^{-\psi} & 0 \\ 0 & ie^{\psi} \end{bmatrix} + r_1 \begin{bmatrix} ie^{-\psi} & 0 \\ 0 & -ie^{\psi} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{58}$$

This can be written as

$$R^L = r_0U^{-1}\hat{\mathrm{T}}U^{-1} + r_1U^{-1}\hat{\mathrm{I}}U^{-1} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = r_0\hat{\mathrm{T}}^L + r_1\hat{\mathrm{I}}^L + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}}. \tag{59}$$

But because we started with Eq.(50), we now have two equivalent options to express the result of a Lorentz transformation

$$R^L = r'_0\hat{\mathrm{T}} + r'_1\hat{\mathrm{I}} + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}} = r_0\hat{\mathrm{T}}^L + r_1\hat{\mathrm{I}}^L + r_2\hat{\mathrm{J}} + r_3\hat{\mathrm{K}}, \tag{60}$$

either as a coordinate transformation or as a basis transformation.

This result only works for Lorentz transformation between v_x-, v_1- or $\hat{\text{I}}$-aligned reference systems. Reference systems which do not have their relative Lorentz velocity aligned in the $\hat{\text{I}}$-direction will have to be space rotated into such an alignment before the Lorentz transformation in the form $R^L = U^{-1}RU^{-1}$ is applied. In principle, such a rotation in order to achieve the $\hat{\text{I}}$ alignment of the primary reference frame to a secondary reference frame is always possible as an operation prior to a Lorentz transformation. This unique alignment between two frames of reference S and S', needed to match the physics with the algebra, is analyzed by Synge in (Synge 1972, p. 41-48) and focuses on the concept of a communal photon. The requirement of reference system alignment is also the reason for the appearance of the Thomas precession and the Thomas-Wigner rotation if the axes are not aligned; the notion that two Lorentz transformations in different directions in space can always be substituted by the subsequent application of one space rotation and one single Lorentz transformation, see de Haas (2014). The communal photon of Synge is the one for which the relativistic Doppler shift between S and S' results in $\nu' = \nu e^{\pm\psi}$. The minquat algebra requires inertial observers to align their principal axis along such a communal photon, in my notation the $\hat{\text{I}}$ axis.

The Lorentz transformation of the coordinates $(R^\mu)^L$ can be written as

$$\begin{bmatrix} r'_0 \\ r'_1 \\ r'_2 \\ r'_3 \end{bmatrix} = \Lambda_\nu{}^\mu R^\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \\ r_2 \\ r_3 \end{bmatrix}$$

So the Lorentz transformation of $R = R_\mu \hat{\mathbf{K}}^\mu = \hat{\mathbf{K}}_\mu R^\mu$ can be presented as

$$\begin{aligned} R^L = \hat{\mathbf{K}}_\mu (R^\mu)^L &= \hat{\mathbf{K}}_\mu \Lambda_\nu{}^\mu R^\nu = (\hat{\mathbf{K}}_\mu \Lambda_\nu{}^\mu) R^\nu = (\hat{\mathbf{K}}_\nu)^L R^\nu \\ &= U^{-1}\hat{\mathbf{K}}_\nu U^{-1} R^\nu = U^{-1}\hat{\mathbf{K}}_\nu R^\nu U^{-1} = U^{-1}RU^{-1} \end{aligned} \tag{61}$$

Lemma 2.1 (derived). For boosts along $\hat{\text{I}}$, the explicit calculation of the transformed matrix elements in (56)–(61) shows that the biquaternion basis $\hat{\mathbf{K}}_\mu = (\hat{\text{T}}, \hat{\text{I}}, \hat{\text{J}}, \hat{\text{K}})$ satisfies

$$\boxed{\hat{\mathbf{K}}_\mu \Lambda_\nu^\mu \;=\; U^{-1}\,\hat{\mathbf{K}}_\nu\, U^{-1}}.$$

This identity is therefore not an assumption but the algebraic consequence of the biquaternion representation of R and the standard Lorentz transformation of its coordinates. In particular, it provides the constructive link between the coordinate transformation law and the transformation of the basis, so that later the transformation properties of the Dirac equation can be proved from this construction rather than postulated. We can now formulate the lemma more tightly and emphasize its bidirectional use, once its derivation has been established:

Lemma 2.1. *(Basis–component compatibility for $\hat{I}$-aligned boosts) Let $\Lambda_\nu^{\ \mu}$ be the Lorentz boost in the $\hat{I}$-direction, and $U = \mathrm{diag}(e^{\psi/2}, e^{-\psi/2})$ as in (57). Then the biquaternion basis $\hat{\mathbf{K}}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ satisfies*

$$\boxed{\hat{\mathbf{K}}_\mu \, \Lambda_\nu^{\ \mu} \;=\; U^{-1} \, \hat{\mathbf{K}}_\nu \, U^{-1}}, \tag{62}$$

and therefore $R^L = \hat{\mathbf{K}}_\mu (R^\mu)^L = (U^{-1}\hat{\mathbf{K}}_\nu U^{-1}) R^\nu = U^{-1} R \, U^{-1}$.

This lemma will serve as the cornerstone for the later derivation of the Lorentz transformation properties of the Dirac equation: because the gamma matrices are built from these basis elements, their transformation laws follow directly from Lemma 1, and thus are proved from construction rather than assumed.

The identity Lemma(2.1) has no analogue for coordinates alone: numerical components cannot satisfy a conjugation relation, whereas the matrix-valued basis does. The matrix representation of the basis is key to this identity, because the relativistic Doppler factor $e^{\pm\psi}$ appears differently attached to the matrix elements. As is the $\hat{\mathrm{I}}$ alignment of the two involved reference frames during the Lorentz transformation. Given that $\hat{\mathbf{K}}_\mu = i\hat{\boldsymbol{\sigma}}_\mu$, the identity $\hat{\mathbf{K}}_\mu \Lambda_\nu^{\ \mu} = U^{-1}\hat{\mathbf{K}}_\nu U^{-1}$ can also be seen as an instruction for the Lorentz transformation of the Pauli spin matrices as a norm-spin four set $\hat{\boldsymbol{\sigma}}_\mu = (\hat{1}, \hat{\boldsymbol{\sigma}})$.

The Lorentz transformation of A^T is also interesting, due to the importance of the product $C = A^T B$ and therefore the Lorentz transformation C^L. Given the inverse Lorentz transformation as

$$A^{L^{-1}} \equiv UAU \tag{63}$$

and using $U^T = U$ since U is real diagonal, one can prove

$$\left(A^T\right)^{L^{-1}} = U\left(A^T\right)U = \left(U^{-1}AU^{-1}\right)^T = \left(A^L\right)^T \tag{64}$$

and

$$\left(A^T\right)^{L} = U^{-1}\left(A^T\right)U^{-1} = (UAU)^T = \left(A^{L^{-1}}\right)^T. \tag{65}$$

The result $\left(A^L\right)^T = UA^TU$ will be used in several important derivations in this paper, when the Lorentz transformation of a product and the possible invariance or Lorentz covariance has to be investigated, as in the next example.

Start with two inertial reference systems S_1 and S_2 connected by a constant relative velocity v, a Lorentz gamma factor $\gamma(v)$ and a rapidity factor $\psi(v)$ defining the Lorentz transformation matrix U. Given A and B in S_1 and their product in S_1 as $C = A^T B$. Then in S_2 one has A^L and B^L and their product $C^L = \left(A^L\right)^T B^L$. We then have

$$C^L = \left(A^L\right)^T B^L = \left(A^T\right)^{L^{-1}} B^L = U\left(A^T\right)UU^{-1}BU^{-1}$$

$$= UA^T BU^{-1} = UCU^{-1}. \tag{66}$$

As a result, it is easy to prove that the quadratic $A^T A = c^2 a_\tau^2 \hat{1}$ is Lorentz invariant. We have

$$\begin{aligned}(A^L)^T A^L &= (A^T)^{L^{-1}} A^L = UA^T U\, U^{-1} AU^{-1} = U(A^T A)U^{-1} \\ &= U(c^2 a_\tau^2)\hat{1}U^{-1} = UU^{-1}(c^2 a_\tau^2)\hat{1} = c^2 a_\tau^2 \hat{1} = A^T A. \end{aligned} \tag{67}$$

So both quadratics $R^T R$ and $dR^T dR$ are Lorentz invariant scalars, as has been shown for every quadratic of four-vectors. But they aren't what we consider to be *perfect quadratics*, who we define through the requirement $AA = |A|^2 \hat{1}$.

2.6 The BQ algebra as a self-contained computational universe

The material developed in Sections 1.1–1.5 establishes the biquaternion algebra at the Pauli level as a complete and self-contained mathematical framework for relativistic physics, before any physical identification of the four-vectors has been made. It is worth pausing to assess what this framework has already achieved at the purely algebraic level, and what its structural features imply for the physics that will be built on it.

One algebra, one product, one transformation law

The entire content of Sections 1.1–1.5 rests on three ingredients: the minquat basis $\hat{K}_\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ with its matrix representation in $M_2(\mathbb{C})$, the transposed bilinear product $A^T B$, and the Lorentz transformation law $R^L = U^{-1} RU^{-1}$ derived from Lemma 1.1. No metric is postulated externally: the Minkowski metric emerges from the quadratic $dR^T dR = (c^2 dt^2 - d\mathbf{r}^2)\hat{1}$ as the scalar channel of the bilinear. No transformation law is assumed: it follows from the explicit matrix calculation of how the basis elements respond to a boost. No spin structure is introduced separately: the pauliquat basis $\hat{\sigma}_\mu = (\hat{1}, \hat{\boldsymbol{\sigma}})$ appears automatically as the necessary algebraic complement to the minquat basis under multiplication, through the identity $\hat{K}_\mu = i\hat{\sigma}_\mu$. The algebra is therefore not a notational convenience layered over pre-existing physics. It is a self-generating structure in which the metric, the Lorentz group action, and the spin-norm sector are consequences of a single choice: the matrix representation of $M_2(\mathbb{C})$ with real coordinates.

The 1–3–3–1 channel structure as algebraic DNA

A central feature of the algebra, visible already at the level of the bilinear and confirmed by the triple product, is the persistent 1–3–3–1 channel structure: scalar $\hat{1}$, time $\hat{T}$, K-bivector, and σ-bivector. This structure is not imposed but forced: any product of minquat four-vectors decomposes into exactly these four channels, and the decomposition is complete and non-redundant. The triple product of Section 1.4

confirms that repeated multiplication stays within this structure, so the algebra is closed under arbitrary composition of four-vectors. The channel structure therefore functions as the algebraic DNA of the framework: every physical quantity that will be introduced in subsequent sections will inherit this four-channel decomposition, and every conservation law will be expressible as the vanishing of one or more of its channels.

Symmetry and antisymmetry unified in one product

A feature that distinguishes the biquaternion bilinear from conventional tensor products is that it simultaneously generates symmetric and antisymmetric content from the same operation. The scalar channel $(a_0b_0 - \mathbf{a} \cdot \mathbf{b})\hat{1}$ is symmetric in A and B; the K-channel $(\mathbf{a} \times \mathbf{b}) \cdot \hat{K}$ is antisymmetric; and the σ-channel $(a_0\mathbf{b} - b_0\mathbf{a}) \cdot \hat{\boldsymbol{\sigma}}$ mixes them through the temporal–spatial coupling. In conventional tensor algebra, symmetric and antisymmetric structures require separate constructions—the metric tensor and the antisymmetric field strength tensor, for example, are introduced independently. Here they are coequal projections of a single product, which means that every bilinear built from two four-vectors automatically carries both metric content and rotational content as inseparable parts of the same algebraic object.

The Lorentz transformation law as the key asymmetry

Lemma 1.1 establishes that vectors transform as $R^L = U^{-1}RU^{-1}$, while the transposed vector transforms as $(A^T)^L = U(A^T)U$—the inverse law. This asymmetry between the transformation of a vector and the transformation of its transpose is not a technical detail; it is the structural feature that determines the transformation law of every bilinear and every conservation law in the framework. The bilinear $C = A^T B$ transforms as $C^L = UCU^{-1}$—the adjoint representation of the Lorentz group—as a direct consequence of this asymmetry and the cancellation of the middle factors $UU^{-1} = \hat{1}$. This means that any physical quantity constructed as a transposed bilinear will automatically live in the adjoint representation, which is the representation appropriate for Lie algebra elements and generator densities rather than for kinematic vectors. The physics built on this algebra will therefore automatically organise itself into objects of two types: vectors (transforming as $U^{-1}(\,)U^{-1}$) and adjoint objects (transforming as $U(\,)U^{-1}$), with the bilinear product as the operation that converts the former into the latter.

What the algebra does not yet contain

Two things are absent from the purely algebraic framework of Section 1 and will be supplied when physics is introduced. First, the identification of the four-vector coordinates with physical quantities—spacetime position, energy-momentum, proper velocity, electromagnetic potential—is a physical input, not an algebraic one. The algebra accepts any four real numbers as coordinates; which four numbers correspond to which physical situation is determined by the physics. Second, the dynamics—the condition that certain bilinears or their gradients vanish—requires a physical principle such as the closure condition $\partial M = 0$ for a conserved current. The algebra provides the language; the physics provides the sentences written in that language. The power of the framework, as the subsequent sections demonstrate, is that once the physical identification is made, the algebra does the rest: it produces the conservation laws, the Noether charges, the dilatation current, and the transformation properties automatically, as channel decompositions of products whose structure was already fully determined at the purely algebraic level established here.

2.7 Adding the dynamic vectors

If I want to apply the previous to relativistic electrodynamics and to quantum physics, I need to further develop the mathematical language, the notation system and the biquaternion elements. I don't claim originality regarding the biquaternion foundations of my notation system. As indicated before, there is a whole subculture around quaternions and biquaternions in physics, see Gsponer and Hurni (2005a), Gsponer and Hurni (2005b), and I have been studying many of those papers. The justification for my paper is to be found in what it adds to this rather large subculture, as part of the more general *plethora of different vector formalisms currently in use Chappell et al. (2016)*.

But let's return to my project of formulating a pragmatic biquaternion mathematical–physical language through which relativity and quantum can be synthesized. The most relevant dynamic four-vectors must be given a biquaternion representation. The basic definitions I use for that purpose are quite common in the formulations of relativistic dynamics, see for example Pauli (1958). I start with a particle with a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1 - v^2/c^2})^{-1}$. Latin suffixes are labels only and never imply summation; Greek suffixes always imply summation over the numbers 0, 1, 2 and 3.. So m_i stands for inertial mass and U_p for potential energy and P_μ stands for a momentum four-vector coordinate row with components $(p_0 = \frac{1}{c}U_i, p_1, p_2, p_3)$, with U_i being the inertial energy (total relativistic energy). The momentum three-vector is written as $\mathbf{p}$ and has components (p_1, p_2, p_3).

I define the coordinate velocity four-vector as

$$V = V_\mu \hat{\mathbf{K}}^\mu = \frac{d}{dt} R_\mu \hat{\mathbf{K}}^\mu = c\hat{\mathrm{T}} + \mathbf{v} \cdot \hat{\mathbf{K}} = v_0 \hat{\mathrm{T}} + \mathbf{v} \cdot \hat{\mathbf{K}}. \tag{68}$$

The proper velocity four-vector on the other hand will be defined using the proper time $\tau = t_0$, with $t = \gamma t_0 = \gamma\tau$ as

$$U = U_\mu \hat{\mathbf{K}}^\mu = \frac{d}{d\tau} R_\mu \hat{\mathbf{K}}^\mu = \frac{d}{\frac{1}{\gamma} dt} R_\mu \hat{\mathbf{K}}^\mu = \gamma V_\mu \hat{\mathbf{K}}^\mu = u_0 \hat{\mathrm{T}} + \mathbf{u} \cdot \hat{\mathbf{K}}. \tag{69}$$

So we have $v_0 = c$ and $u_0 = \gamma c$ and, in general, $U = \gamma V$.

The momentum four-vector will be, at least when we have the symmetry condition $\mathbf{p} = m_i \mathbf{v}$,

$$P = P_\mu \hat{\mathbf{K}}^\mu = m_i V_\mu \hat{\mathbf{K}}^\mu = m_i V = m_0 U_\mu \hat{\mathbf{K}}^\mu = m_0 U. \tag{70}$$

The quadratic of the momentum four-vector $P^T P$ is important in relativity and is at the basis of the Klein-Gordon equation, so we give it here in our formalism:

$$P^T P = (U_i^2/c^2 - \mathbf{p}\cdot\mathbf{p})\, \hat{1} = U_0^2/c^2\, \hat{1} = E^2 \hat{1}, \tag{71}$$

with the shorthand notation $E := U_0/c$. Without the symmetry condition $\mathbf{p} = m_i \mathbf{v}$, we simply have

$$P = P_\mu \hat{\mathbf{K}}^\mu = p_0 \hat{\mathrm{T}} + \mathbf{p} \cdot \hat{\mathbf{K}} = U_i/c \hat{\mathrm{T}} + \mathbf{p} \cdot \hat{\mathbf{K}}. \tag{72}$$

The four-vector partial derivative $\partial = \partial_\mu \hat{\mathbf{K}}^\mu$ will be defined using the coordinate four set

$$\partial_\mu = \left[-\frac{1}{c}\partial_t, \nabla_1, \nabla_2, \nabla_3 \right] = [\partial_0, \partial_1, \partial_2, \partial_3]\,. \tag{73}$$

Remark. In standard Minkowski notation the minus sign in $\partial_0 = -\frac{1}{c}\partial_t$ reflects the $(- + ++)$ metric signature. In the present framework, however, this minus sign stems from having shifted the imaginary unit from the coordinate ict into the basis element $\hat{\mathrm{T}} = i\hat{1}$. The minus sign thus arises here as a direct consequence of the basis construction because $\frac{1}{ict}\hat{1} = -i\frac{1}{ct}\hat{1} = -\frac{1}{ct}\hat{\mathrm{T}}$. This ensures that $-\partial^T \partial = (-\frac{1}{c^2}\partial_t^2 + \nabla^2)\,\hat{1}$, which reproduces the standard Minkowski form of the d'Alembert operator.

The electrodynamic potential four-vector $A = A_\mu \hat{\mathbf{K}}^\mu$ will be defined by the coordinate four set

$$A_\mu = \left[\frac{1}{c}\phi, A_1, A_2, A_3 \right] = [A_0, A_1, A_2, A_3] \tag{74}$$

The electric four current density vector $J = J_\mu \hat{\mathbf{K}}^\mu$ will be defined by the coordinate four set

$$J_\mu = [c\rho_e, J_1, J_2, J_3] = [J_0, J_1, J_2, J_3]\,, \tag{75}$$

with ρ_e as the electric charge density. The electric four current with a charge q will be also be written as J_μ and the context will indicate which one is used. But the four current density $J = \rho_{q0}U = \rho_q V$ and the four current $J = qU = \gamma q V$, because for the volume we have $dV = \frac{dV_0}{\gamma}$ and $dq = dq_0$ so $\rho_q = \gamma\rho_{q0}$.

Given $\boldsymbol{g} = \rho_0\boldsymbol{u}$, the momentum energy density four vector G can defined as

$$G = G_\mu\hat{\mathbf{K}}^\mu = \rho_0 U = \rho_0\gamma c\hat{\mathrm{T}} + \rho_0\boldsymbol{u}\cdot\hat{\mathbf{K}} = \frac{\varepsilon}{c}\hat{\mathrm{T}} + \boldsymbol{g}\cdot\hat{\mathbf{K}} = g_0\hat{\mathrm{T}} + \boldsymbol{g}\cdot\hat{\mathbf{K}} \tag{76}$$

with the rest mass density $\rho_0 = \frac{dm_0}{dV_0}$. We have

$$\rho_i = \frac{dm_i}{dV} = \frac{\gamma dm_0}{\frac{dV_0}{\gamma}} = \gamma^2\rho_0, \tag{77}$$

so

$$G = G_\mu\hat{\mathbf{K}}^\mu = \rho_0 U = \frac{\rho_i}{\gamma^2}\gamma V. = \frac{1}{\gamma}\rho_i V \tag{78}$$

and we have $\varepsilon = \gamma\rho_0 c^2 = \gamma\varepsilon_0$ and $\varepsilon_i = \rho_i c^2 = \gamma^2\rho_0 c^2 = \gamma^2\varepsilon_0 = \gamma\varepsilon$

Although we defined these four-vectors using the coordinate column notation, we will often use the matrix or summation notation, as for example with $P = P_\mu\hat{\mathbf{K}}^\mu$, written as

$$\begin{aligned} P &= p_0\hat{\mathrm{T}} + p_1\hat{\mathrm{I}} + p_2\hat{\mathrm{J}} + p_3\hat{\mathrm{K}} = p_0\hat{\mathrm{T}} + \mathbf{p}\cdot\hat{\mathbf{K}} \\ &= \begin{bmatrix} ip_0 + ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & ip_0 - ip_1 \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}. \end{aligned} \tag{79}$$

The flexibility to use either of these notations is a strength of the mathematical–physical language as developed in this paper. There are cases where one needs to go all the way to the internal scalar matrix notation to solve issues as for example the product rule in calculating a derivative, after which one returns to the more compact notation to evaluate the outcome.

With these definitions, the basic dynamical quantities of relativistic mechanics and electrodynamics are now expressed in the biquaternion formalism, ready for use in subsequent derivations.

3 Electromagnetics in the BQ-Pauli algebra

3.1 The EM field in our language

If we apply the matrix multiplication rules of the bilinear product of Eqn. (27) to the electromagnetic field with four derivative ∂ and four potential A, with $\partial_0 = -\frac{1}{c}\partial_t$ and $A_0 = \frac{1}{c}\phi$, we get $B = \partial^T A$ as

$$B = \partial^T A = (-\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A})\hat{1} + (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\hat{\mathbf{K}} + \frac{1}{c}(-\partial_t\boldsymbol{A} - \boldsymbol{\nabla}\phi)\cdot\hat{\boldsymbol{\sigma}}. \tag{80}$$

If we apply the Lorenz gauge $\mathbb{B}_0 = -\frac{1}{c^2}\partial_t\phi - \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ and the usual EM definitions of the fields in terms of the potentials we get

$$B = \partial^T A = \boldsymbol{B}\cdot\hat{\mathbf{K}} + \frac{1}{c}\boldsymbol{E}\cdot\hat{\boldsymbol{\sigma}}. \tag{81}$$

Using $\hat{\boldsymbol{\sigma}} = -\hat{\mathrm{T}}\hat{\mathbf{K}} = -i\hat{\mathbf{K}}$, this can also be written as

$$B = \partial^T A = (\boldsymbol{B} - i\frac{1}{c}\boldsymbol{E})\cdot\hat{\mathbf{K}} = \overrightarrow{\mathbb{B}}\cdot\hat{\mathbf{K}}. \tag{82}$$

The use of $\mathbb{B} = \boldsymbol{B} - i\frac{1}{c}\boldsymbol{E}$ dates back to Minkowski's 1908 treatment of the subject Minkowski (1910). In my opinion, the flexibility of easy switching between the different modes of notations makes my biquaternion variant suited for unification purposes.

Using $\mathbb{B}$ we can write B as

$$\begin{gathered} B = \mathbb{B}_1\hat{\mathrm{I}} + \mathbb{B}_2\hat{\mathrm{J}} + \mathbb{B}_3\hat{\mathrm{K}} = \overrightarrow{\mathbb{B}}\cdot\hat{\mathbf{K}} = \\ \begin{bmatrix} i\mathbb{B}_1 & \mathbb{B}_2 + i\mathbb{B}_3 \\ -\mathbb{B}_2 + i\mathbb{B}_3 & -i\mathbb{B}_1 \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}. \end{gathered} \tag{83}$$

Here $\mathbb{B} := \mathbf{B} - i\,\mathbf{E}/c$ is Minkowski's complex field 3-vector; its longitudinal component (parallel to the boost) is invariant, while the transverse pair mixes hyperbolically, as elaborated in the next paragraph.

For the Lorentz transformation of B we can apply the result of the previous section to get $B^L = (\partial^L)^T A^L = (\partial^T)^{L^{-1}} A^L = U(\partial^T)UU^{-1}AU^{-1} = U(\partial^T A)U^{-1} = UBU^{-1}$ for boosts along $\hat{\mathrm{I}}$, so

$$B^L = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 \\ 0 & e^{\frac{\psi}{2}} \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01}e^{\psi} \\ B_{10}e^{-\psi} & B_{11} \end{bmatrix} \tag{84}$$

which, when written out with $\boldsymbol{E}$ and $\boldsymbol{B}$ leads to the usual result for the Lorentz transformation of the EM field with the Lorentz velocity in the x-direction. But it can also be written as a transformation of the basis, while leaving the coordinates invariant:

$$B^L = UBU^{-1} = \mathbb{B}_1 U\hat{\mathrm{I}}U^{-1} + \mathbb{B}_2 U\hat{\mathrm{J}}U^{-1} + \mathbb{B}_3 U\hat{\mathrm{K}}U^{-1} = \tag{85}$$
$$\mathbb{B}_1\hat{\mathrm{I}} + \mathbb{B}_2\hat{\mathrm{J}}^L + \mathbb{B}_3\hat{\mathrm{K}}^L = \mathbb{B}_1\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \mathbb{B}_2\begin{bmatrix} 0 & e^{\psi} \\ -e^{-\psi} & 0 \end{bmatrix} + \mathbb{B}_3\begin{bmatrix} 0 & ie^{\psi} \\ ie^{-\psi} & 0 \end{bmatrix}.$$

(Since U is diagonal, $U\hat{\mathrm{J}}U^{-1}$ and $U\hat{\mathrm{K}}U^{-1}$ acquire the factors $e^{\pm\psi}$, whereas $U\hat{\mathrm{I}}U^{-1} = \hat{\mathrm{I}}$.) The Lorentz transformation of the EM field can be performed by internally twisting the $(\hat{\mathrm{J}}, \hat{\mathrm{K}})$-surface perpendicular to the Lorentz velocity and in the process leaving the EM-coordinates invariant.

That the above equals the usual Lorentz transformation of the EM field can be shown by going back to Minkowski (1910), where he wrote the transformation in a form equivalent to

$$\begin{bmatrix} \mathbb{B}'_1 \\ \mathbb{B}'_2 \\ \mathbb{B}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & i\beta\gamma \\ 0 & -i\beta\gamma & \gamma \end{bmatrix}\begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \mathbb{B}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{B}_1 \\ \gamma\mathbb{B}_2 + i\beta\gamma\mathbb{B}_3 \\ \gamma\mathbb{B}_3 - i\beta\gamma\mathbb{B}_2 \end{bmatrix} \tag{86}$$

So we have

$$B'_{01} = \mathbb{B}'_2 + i\mathbb{B}'_3 = \gamma\mathbb{B}_2 + i\beta\gamma\mathbb{B}_3 + i\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2 \tag{87}$$

and

$$B'_{10} = -\mathbb{B}'_2 + i\mathbb{B}'_3 = -\gamma\mathbb{B}_2 - i\beta\gamma\mathbb{B}_3 + i\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2. \tag{88}$$

If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, this can be rewritten as

$$B'_{01} = \mathbb{B}'_2 + i\mathbb{B}'_3 = (\gamma + \beta\gamma)(\mathbb{B}_2 + i\mathbb{B}_3) = B_{01}e^{\psi} \tag{89}$$

and

$$B'_{10} = -\mathbb{B}'_2 + i\mathbb{B}'_3 = (\gamma - \beta\gamma)(-\mathbb{B}_2 + i\mathbb{B}_3) = B_{10}e^{-\psi}, \tag{90}$$

which leads to Eqn. (84).

Interpretation.

The channel assignment of the electromagnetic field deserves explicit comment. The magnetic field $\boldsymbol{B}$ resides in the minquat $\hat{K}$-sector, which carries the antisymmetric, spatially bivector-like geometry of the algebra. The electric field $\boldsymbol{E}$ resides in the pauliquat $\hat{\boldsymbol{\sigma}}$-sector, the spin-norm channel. This division is not arbitrary: a purely magnetic field in one frame acquires an electric component under a Lorentz boost, and the algebra encodes this mixing automatically through the off-diagonal action of the boost rotor U on the transverse $(\hat{J}, \hat{K})$-plane, as derived in Eqns. (71)–(77). The Minkowski complex field $\mathbb{B} = \boldsymbol{B} - \mathrm{i}\boldsymbol{E}/c$ that emerges from this construction is therefore not an external notational convenience imported from

1908: it is a structural consequence of the BQ product $\partial^T A$, with the imaginary unit arising from the algebraic identity $\hat{\boldsymbol{\sigma}} = -\hat{T}\hat{K} = -\mathrm{i}\hat{K}$. The flexibility to pass between the abstract $\hat{K}$-vector form (69), the 2×2 matrix form (70), and the full component form (71) is a practical strength of the language: the matrix form is indispensable when computing products and verifying Lorentz covariance, while the vector form is more transparent for physical identification. Switching between these modes mid-derivation is intentional and does not constitute a loss of rigour.

3.2 EM invariants and energy transport from the quadratic field products

Besides the EM-field equation

$$B = \partial^T A, \tag{91}$$

the BQ-Pauli framework naturally generates two important quadratic field products:

$$BB \tag{92}$$

and

$$B^T B. \tag{93}$$

Using the field decomposition

$$B = \boldsymbol{B} \cdot \hat{\mathbf{K}} + \frac{1}{c}\boldsymbol{E} \cdot \hat{\boldsymbol{\sigma}}, \tag{94}$$

the first product becomes

$$BB = -\left(B^2 - \frac{1}{c^2}E^2\right)\hat{1} + \frac{2}{c}(\boldsymbol{E} \cdot \boldsymbol{B})\hat{\mathrm{T}}. \tag{95}$$

This product therefore contains directly the two standard Lorentz invariants of electromagnetism:

$$B^2 - \frac{1}{c^2}E^2 \tag{96}$$

and

$$\boldsymbol{E} \cdot \boldsymbol{B}. \tag{97}$$

Since the product BB reduces entirely to scalar and T-channel contributions, it is Lorentz invariant. Under a Lorentz rotor transformation

$$B^L = UBU^{-1}, \tag{98}$$

one obtains

$$B^L B^L = UBU^{-1}UBU^{-1} = UBBU^{-1} = BB. \tag{99}$$

The product BB therefore represents the invariant quadratic field structure of electromagnetism within the BQ algebra.

The second quadratic product is

$$B^T B. \tag{100}$$

Using

$$B^T = \boldsymbol{B} \cdot \hat{\mathbf{K}} - \frac{1}{c}\boldsymbol{E} \cdot \hat{\boldsymbol{\sigma}}, \tag{101}$$

one obtains

$$B^T B = -\left(B^2 + \frac{1}{c^2}E^2\right)\hat{1} - \frac{2}{c}(\boldsymbol{E} \times \boldsymbol{B}) \cdot \hat{\boldsymbol{\sigma}}. \tag{102}$$

The scalar part contains the standard electromagnetic energy density

$$u_{EM} = \frac{1}{2\mu_0}\left(B^2 + \frac{1}{c^2}E^2\right), \tag{103}$$

while the $\hat{\boldsymbol{\sigma}}$-channel contains the Poynting vector

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}. \tag{104}$$

The product $B^T B$ therefore generates naturally the electromagnetic energy-momentum transport structure:

$$B^T B = -2\mu_0 u_{EM}\hat{1} - \frac{2\mu_0}{c}\boldsymbol{S} \cdot \hat{\boldsymbol{\sigma}} \tag{105}$$

and with

$$S = -\frac{1}{2\mu_0}(B^T B) = u_{EM}\hat{1} + \frac{1}{c}\boldsymbol{S} \cdot \hat{\boldsymbol{\sigma}}, \tag{106}$$

we get a Poynting pseudo four-vector S. We wrote 'pseudo' becaus S isn't Lorentz covariant and because it is a product in the norm-sigma channel of the BQ algebra.

It is important, therefore, to distinguish this energy-transport decomposition from Lorentz covariance of the complete product. Unlike BB, the product $B^T B$ is not Lorentz covariant.

Given

$$B^T = (\partial^T A)^T = \partial A^T, \tag{107}$$

the Lorentz transformation of the elements gives

$$\partial^L (A^T)^L = U^{-1}\partial U^{-1} U A^T U = U^{-1}\partial A^T U = U^{-1} B^T U = (B^T)^L. \tag{108}$$

Then

$$(B^T)^L B^L = U^{-1} B^T U U B U^{-1} = U^{-1} B^T U^2 B U^{-1}. \tag{109}$$

So if in a certain reference system we have $B^T B = S_P$, with S_P being the Poynting pseudo-four-vector, then in a different reference system where we have $(B^T)^L B^L$, this is equal to $U^{-1} B^T U^2 B U^{-1}$ and not to $U^{-1} B^T B U^{-1} = U^{-1} S U^{-1} = S_P^L$. In general, the time-adjoint ^{T} reverses the Lorentz transformation. For $B = \partial^T A$ that ensures correct Lorentz covariant behaviour, but for $B^T B$ it produces non-covariance.

Thus, although $B^T B$ combines the electromagnetic energy density and Poynting vector in a compact BQ expression in a chosen frame, this object is not itself Lorentz covariant. The obstruction is precisely that the time-adjoint ^{T} reverses the Lorentz rotor action: this is necessary for $B^T = (\partial^T A)^T = \partial A^T$ to transform correctly as the time-adjoint field, but it prevents the mixed product $B^T B$ from transforming as a genuine four-vector.

Within the BQ-Pauli framework, the two quadratic products therefore separate naturally into:

- an invariant notm-time field-energy structure through BB,
- an observer dependent norm-sigma channel energy-momentum transport structure through $B^T B$.

This quadratic hierarchy will later reappear in analogous form for the relativistic fluid field.

Interpretation.

The deeper significance of the two quadratic products lies in which algebraic channel each occupies. The product BB generates output only in the $\hat{1}$ and $\hat{T}$ channels—the minquat sector—and is therefore Lorentz covariant by construction, since the minquat basis transforms covariantly under the rotor action. The product $B^T B$, by contrast, generates output in the $\hat{1}$ and $\hat{\sigma}$ channels—the pauliquat spin-norm sector—and is not Lorentz covariant, for precisely the reason that the time-adjoint T reverses the sense of the rotor: $(B^T)^L = U^{-1} B^T U$ rather than $U B^T U^{-1}$, so that the mixed product acquires a U^2 insertion rather than the identity. This is not a defect of the formalism; it is an algebraic statement of a physical fact.

The electromagnetic energy density and Poynting vector are observer-dependent quantities, tied to a chosen time direction, and it is fitting that they reside in the $\hat{\boldsymbol{\sigma}}$-channel, which is the spin-norm dual of spacetime. The BQ algebra thus separates, automatically and without additional postulates, what is invariant (the Lorentz scalars $B^2 - E^2/c^2$ and $\boldsymbol{E} \cdot \boldsymbol{B}$) from what is frame-dependent (the energy density u_{EM} and the Poynting vector $\boldsymbol{S}$). The full Lorentz-covariant electromagnetic stress-energy tensor $T^{\mu\nu}$ requires a symmetrized combination of both products evaluated in a chosen frame; in the BQ language this appears as a deliberate combination of outputs from two different algebraic channels, rather than being hidden inside index symmetrization. This quadratic hierarchy—covariant invariant structure from BB, observer-dependent energy-momentum transport from $B^T B$—is a structural template that will reappear in analogous form for the relativistic fluid field later in the paper.

3.3 The Maxwell Equations

The Maxwell equations in our language can be given as $\partial B = \mu_0 J$, using $J = \rho V = \rho_0 U$. I start with

$$B = \partial^T A = \boldsymbol{B} \cdot \hat{\mathbf{K}} + \frac{1}{c}\boldsymbol{E} \cdot \hat{\boldsymbol{\sigma}}. \tag{110}$$

Then, using Eqn. (45), but with $c_1 = 0$, ∂B is given by

$$\partial B = \left(-\frac{1}{c}\partial_t \hat{\mathrm{T}} + \boldsymbol{\nabla} \cdot \hat{\mathbf{K}}\right)\left(\boldsymbol{B} \cdot \hat{\mathbf{K}} + \frac{1}{c}\boldsymbol{E} \cdot \hat{\boldsymbol{\sigma}}\right) =$$
$$-(\boldsymbol{\nabla} \cdot \boldsymbol{B})\hat{1} + \frac{1}{c}(\boldsymbol{\nabla} \cdot \boldsymbol{E})\hat{\mathrm{T}} + (\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E}) \cdot \hat{\mathbf{K}} + \frac{1}{c}(\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B}) \cdot \hat{\boldsymbol{\sigma}}, \tag{111}$$

If we interpret this result using the knowledge regarding the inhomogeneous Maxwell equations, we get an interesting result. First of all, the part of the Maxwell Equation with the dimension of the norm $\hat{1}$ is zero and so is the part in the spin-norm sector ($\hat{\boldsymbol{\sigma}}$). The space-time parts $\hat{\mathbf{K}}$ and $\hat{\mathrm{T}}$ equal the space-time parts of the four current density J. In the following, ρ denotes charge density for the current density four-vectors, while q denotes single-particle charge. So we get the inhomogenous channel projections of $\partial B = \mu_0 J$ as

$$\hat{1} - channel: \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{112}$$
$$\hat{\mathrm{T}} - channel: \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{113}$$
$$\hat{\mathbf{K}} - channel: \qquad \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E} = \mu_0 \boldsymbol{J} \tag{114}$$
$$\hat{\boldsymbol{\sigma}} - channel: \qquad \boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0, \tag{115}$$

in which we used $\mu_0 c^2 \rho = \frac{\rho}{\varepsilon_0}$. So the spin-norm part of the Maxwell Equations equals zero and the space-time part equals the space-time four current density times μ_0. In the line of this interpretation, magnetic monopoles and the correlated magnetic monopole current should be searched in the pauliquat dimensions of spin-norm, not in the minquat dimensions of space-time.

As for the Lorentz covariance of the Maxwell Equations, this can be demonstrated quite easily. Given the four-vectors ∂, A and J in reference system S_1, with the Maxwell Equations as $\partial(\partial^T A) = \mu_0 J$, then in reference system S_2 we have the four-vectors ∂^L, A^L and J^L and the covariant Maxwell Equations given as $\partial^L(\partial^L)^T A^L = \mu_0 J^L$. In S_2 this can be proven through

$$\begin{aligned}\partial^L(\partial^L)^T A^L = \partial^L(\partial^T)^{L^{-1}} A^L = U^{-1}\partial U^{-1} U(\partial^T) U U^{-1} A U^{-1} = \\ U^{-1}\partial(\partial^T) A U^{-1} = U^{-1}\partial B U^{-1} = U^{-1}\mu_0 J U^{-1} = \mu_0 J^L. \end{aligned} \tag{116}$$

So if we have $\partial B = \mu_0 J$ in one frame of reference, this transforms as $\partial^L B^L = \mu_0 J^L$ in another frame of reference, which means that the equation maintains its form, it is Lorentz covariant. We have form-invariance of the equations.

With ∂B we get a dual spin-norm and space-time product, with the spin-norm equal zero and the non-zero space-time leading to the inhomogeneous four-vector of the current. Speculations about magnetic monopoles are connected to these spin-norm parts, the set spanned by pauliquats. In my analysis, if spin-norm is the twin dual of space-time and as such an integral aspect of the metric as foreseen in Dirac (1951), then searches for magnetic monopoles should focus on this spin-norm aspect of the vacuum.

Interpretation.

The four-channel projection of $\partial B = \mu_0 J$ deserves to be read as a physical dictionary. The sourced equations—Gauss's law ($\hat{T}$-channel) and the Ampère–Maxwell law ($\hat{K}$-channel)—reside entirely within the minquat sector and carry the four-current directly. The sourceless equations—the absence of magnetic charge ($\hat{1}$-channel) and Faraday's law ($\hat{\boldsymbol{\sigma}}$-channel)—reside outside the minquat sector, distributed between the algebraic norm and the spin-norm. This separation has a structural consequence: if magnetic monopoles exist, they cannot enter as sources in the $\hat{T}$ or $\hat{K}$ channels without disrupting the covariance already established for the electric charge and current. They must instead enter as sources in the $\hat{\boldsymbol{\sigma}}$ spin-norm sector, as noted in connection with Dirac's 1951 suggestion regarding the vacuum metric. This is a concrete algebraic prediction rather than a speculative remark: within the BQ framework, any modification of the sourceless Maxwell equations is constrained to act in the pauliquat dimensions, not in the minquat space-time dimensions where ordinary electrodynamics operates. The Lorentz covariance proof in Eqn. (103) confirms this at the level of the full equation: because ∂, A, and

J all transform as BQ four-vectors under the same rotor law $V^L = U^{-1}VU^{-1}$, the U^{-1} and U factors cancel pairwise through the product and the equation maintains its form. No separate covariance postulate is required; it follows as an algebraic identity from the consistency of the four-vector transformation rule.

3.4 The Lorentz force law

I will look at $JB = F$ now, with $J = \rho V = \rho_0 U$. The underlying structure for the Lorentz Force Law is the same as for the Maxwell equations. So JB is given by

$$JB = \left(c\rho_0\hat{\mathrm{T}} + \boldsymbol{J}\cdot\hat{\mathbf{K}}\right)\left(\boldsymbol{B}\cdot\hat{\mathbf{K}} + \frac{1}{c}\boldsymbol{E}\cdot\hat{\boldsymbol{\sigma}}\right) =$$

$$-(\boldsymbol{J}\cdot\boldsymbol{B})\hat{1} + \frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathrm{T}} + (\boldsymbol{J}\times\boldsymbol{B} + \rho\boldsymbol{E})\cdot\hat{\mathbf{K}} + (\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - c\rho\boldsymbol{B})\cdot\hat{\boldsymbol{\sigma}} \quad (117)$$

If we interpret this result using the knowledge regarding the Lorentz Force Law, we get an interesting result. First of all, the space-time parts $\hat{\mathbf{K}}$ and $\hat{\mathrm{T}}$ equal the space-time parts of the four force F. Second, the part of the Lorentz force law with the dimension of the norm $\hat{1}$ and the part in the spin-norm sector ($\hat{\boldsymbol{\sigma}}$) are used in MHD and in covariant plasma physics respectively. So we get the inhomogenous channel projections of $JB = F$ as

$$\hat{1} - channel: \qquad -\boldsymbol{J}\cdot\boldsymbol{B} = F_1 \quad (118)$$

$$\hat{\mathrm{T}} - channel: \qquad \frac{1}{c}\boldsymbol{J}\cdot\boldsymbol{E} = F_T \quad (119)$$

$$\hat{\mathbf{K}} - channel: \qquad (\boldsymbol{J}\times\boldsymbol{B} + \rho\boldsymbol{E}) = F_K \quad (120)$$

$$\hat{\boldsymbol{\sigma}} - channel: \qquad \frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - c\rho\boldsymbol{B} = F_\sigma, \quad (121)$$

The time-space channels represent the known Lorentz four force with time-like Lorentz power and space like Lorentz force.

Interpretation.

The product $JB = F$ is structurally parallel to the Maxwell case $\partial B = \mu_0 J$, with the differential operator ∂ replaced by the current four-vector J. The same four-channel decomposition applies, and the channel assignment of the output again carries physical content. The minquat $\hat{T}$ and $\hat{K}$ channels carry the known Lorentz four-force: the $\hat{T}$-channel yields the power density $(1/c)\boldsymbol{J}\cdot\boldsymbol{E}$, the rate of energy transfer from the field to the current, and the $\hat{K}$-channel yields the spatial Lorentz force density $\boldsymbol{J}\times\boldsymbol{B} + \rho\boldsymbol{E}$, the momentum transfer. These two together constitute the standard relativistic Lorentz four-force. The norm $\hat{1}$-channel yields $-\boldsymbol{J}\cdot\boldsymbol{B}$, a scalar measuring work done against magnetic tension, which appears in magnetohydrodynamics as the source term in magnetic helicity evolution. The

$\hat{\boldsymbol{\sigma}}$-channel yields $(1/c)\boldsymbol{J} \times \boldsymbol{E} - c\rho\boldsymbol{B}$, a quantity that appears in covariant plasma physics as the Abraham momentum density and in the relativistic ponderomotive force. Its emergence here as a natural algebraic output—rather than a separately postulated term—is a structural completeness argument for the BQ framework. Lorentz covariance of the force law, not proven explicitly in the text, follows by the same argument as for the Maxwell case: since J transforms as $U^{-1}JU^{-1}$ and B transforms as UBU^{-1}, the product JB transforms as $U^{-1}(JB)U^{-1} = U^{-1}FU^{-1} = F^L$, exactly as a four-force should.

Transition to the Laue–Sommerfeld conservation law.

Sections 3.1 through 3.4 have established the complete kinematic and dynamical structure of classical electrodynamics from three classes of BQ operation: the bilinear $\partial^T A$ defining the field, the quadratic products BB and $B^T B$ encoding invariants and energy transport, and the derivative and current products ∂B and JB encoding the source and force equations. What remains is to close the energy-momentum budget: to construct the interaction term that couples the current to the potential and to verify that the result reproduces the standard electromagnetic interaction Lagrangian density together with its topological and transport companions. This is the role of the Lagrangian-like product $T_{em} = J^T A$ in Section 3.5. Just as the quadratic field products of Section 3.2 separated automatically into a Lorentz-invariant structure (BB, minquat channel) and a frame-dependent transport structure ($B^T B$, spin-norm channel), the product $J^T A$ separates into the scalar interaction Lagrangian in its norm channel, a rotational helicity term in the $\hat{K}$-channel, and a boost-coupling transport term in the $\hat{\boldsymbol{\sigma}}$-channel. The Laue–Sommerfeld conservation law then emerges as the derivative condition on this Lagrangian structure, completing the algebraic account of electromagnetic energy-momentum conservation within the BQ framework.

3.5 The Laue-Sommerfeld conservation law

We calculate the Lagrangian like product $T_{em} = J^T A$, with $J = \rho V = \rho_0 U$ as

$$T_{em} = J^T A = (\rho\phi - \mathbf{J} \cdot \mathbf{A})\hat{1} + (\mathbf{J} \times \mathbf{A}) \cdot \hat{\mathbf{K}} + \frac{1}{c}(c^2\rho\mathbf{A} - \phi\mathbf{J}) \cdot \hat{\boldsymbol{\sigma}}. \qquad (122)$$

If we look at the product $J^T A$, the $\hat{1}$-norm channel is the standard electromagnetic interaction Lagrangian density $J_\mu A^\mu$. The $\hat{\mathbf{K}}$-space channel measures transverse current-potential topology and helicity geometry, it is the rotational counterpart of the scalar Lagrangian density. The $\hat{\boldsymbol{\sigma}}$-spin channel represents a transport or boost coupling between the scalar and vector interaction components. We can look at $\frac{1}{c}\phi\mathbf{J} = \frac{1}{c}\rho\phi\mathbf{u} = \frac{1}{c}U_{el}\mathbf{u}$, so the momentum of the scalar charge-potential interaction and the $c^2\rho\mathbf{A}$ as the same for the charge-vector potential interaction. Thus, the $\hat{\boldsymbol{\sigma}}$-channel compares two directed interaction-momentum densities. Their difference has the dimensions of stress and measures the boost/transport imbalance of the current–potential interaction.

We can abbreviate the result as

$$T = J^T A = \mathcal{L}\hat{1} + \mathcal{T} \cdot \hat{\mathbf{K}} + \mathcal{S} \cdot \hat{\boldsymbol{\sigma}}, \tag{123}$$

with the channels

$$norm - channel\, \hat{1} : \qquad \mathcal{L} = \rho\phi - \boldsymbol{J} \cdot \boldsymbol{A} = J_\mu A^\mu, \tag{124}$$

$$space - channel\, \hat{\mathbf{K}} : \qquad \mathcal{T} = \boldsymbol{J} \times \boldsymbol{A}, \tag{125}$$

$$sigma - channel\, \hat{\boldsymbol{\sigma}} : \qquad \mathcal{S} = c\rho\boldsymbol{A} - \frac{1}{c}\phi\boldsymbol{J}. \tag{126}$$

The scalar content of $J^T A$, the norm channel

$$\mathcal{L} = \rho\phi - \boldsymbol{J} \cdot \boldsymbol{A} = J_\mu A^\mu, \tag{127}$$

has a distinguished history in electromagnetic unification. Mie Mie (1912, 1913) introduced the electromagnetic coupling $J_\mu A^\mu$ as the interaction sector of his *Weltfunktion* (world function) H, the Lagrangian density from which he sought to derive both Maxwell's equations and the structure of matter by variation. Mie required the world function, constructed from a suitable tensor contraction, to be Lorentz invariant, and for a closed system he imposed Laue's condition $\partial_\mu T^{\mu\nu} = 0$ on the resulting stress-energy tensor.

Crucially, Mie also wrote down the non-diagonal elements of the tensor $J_\mu A^\nu$ explicitly (Mie 1913, Eqs. 56–57): these are precisely the objects $\mathcal{T} = \boldsymbol{J} \times \boldsymbol{A}$ and $\mathcal{S} = c\rho\boldsymbol{A} - \frac{1}{c}\phi\boldsymbol{J}$ that appear as the K- and $\hat{\boldsymbol{\sigma}}$-channels of $J^T A$ above. Mie therefore had all three components of the bilinear $J^T A$ in front of him, but treated them as separate tensor elements within a scalar variational framework. The BQ product $J^T A$ is the algebraic container that unifies what Mie's tensor calculus left distributed: the norm channel L, the K-channel $\mathcal{T}$, the $\hat{\boldsymbol{\sigma}}$-channel $\mathcal{S}$, and their coupled dynamics under $\partial(J^T A)$, all emerge from a single matrix multiplication with automatic channel labelling and demonstrated Lorentz covariance. Hilbert subsequently adopted Mie's axiom of the world function as the foundation of his own 1915 unification programme Hilbert (1915). That the BQ product $J^T A$ recovers Mie's scalar sector as its $\hat{1}$-channel and his non-diagonal tensor elements as its K- and $\hat{\boldsymbol{\sigma}}$-channels — all emerging from a single matrix multiplication with demonstrated Lorentz covariance — confirms that the present formalism addresses a genuine and long-standing structural gap in the treatment of the current–potential interaction.

We then look at $\partial(J^T A)$. The underlying structure is the same as for the Maxwell equations and the Lorentz force law, implied in $D = C(A^T B)$ of Eqn.(45).

So $\partial(J^T A)$ is given by

$$\partial(J^T A) = \left(-\frac{1}{c}\partial_t \hat{\mathrm{T}} + \boldsymbol{\nabla}\cdot\hat{\mathbf{K}}\right)\left(\mathcal{L}\hat{\mathrm{I}} + \boldsymbol{\mathcal{T}}\cdot\hat{\mathbf{K}} + \boldsymbol{\mathcal{S}}\cdot\hat{\boldsymbol{\sigma}}\right) = -(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}})\hat{\mathrm{I}}$$

$$+(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} - \frac{1}{c}\partial_t\mathcal{L})\hat{\mathrm{T}} + (\boldsymbol{\nabla}\times\boldsymbol{\mathcal{T}} + \boldsymbol{\nabla}\mathcal{L} - \frac{1}{c}\partial_t\boldsymbol{\mathcal{S}})\cdot\hat{\mathbf{K}} + (\boldsymbol{\nabla}\times\boldsymbol{\mathcal{S}} + \frac{1}{c}\partial_t\boldsymbol{\mathcal{T}})\cdot\hat{\boldsymbol{\sigma}} \quad (128)$$

So we get the inhomogenous channel projections of $\partial(J^T A) = F$ as

$$\hat{\mathrm{I}} - channel: \qquad F_1 = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}} \quad (129)$$

$$\hat{\mathrm{T}} - channel: \qquad F_T = \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{S}} - \frac{1}{c}\partial_t\mathcal{L} \quad (130)$$

$$\hat{\mathbf{K}} - channel: \qquad F_K = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{T}} + \boldsymbol{\nabla}\mathcal{L} - \frac{1}{c}\partial_t\boldsymbol{\mathcal{S}} \quad (131)$$

$$\hat{\boldsymbol{\sigma}} - channel: \qquad F_\sigma = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{S}} + \frac{1}{c}\partial_t\boldsymbol{\mathcal{T}}. \quad (132)$$

3.5.1 Interpretation of the norm channel of $\partial(J^T A)$

The norm channel of $\partial(J^T A)$ is given by

$$-\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}} = -\boldsymbol{\nabla}\cdot(\mathbf{J}\times\mathbf{A}) = F_1. \quad (133)$$

or

$$F_1 = \boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{J}) \quad (134)$$

The quantity $\mathbf{A}\times\mathbf{J}$ is the electromagnetic interaction angular momentum density: it appears in precisely this form in the theory of electromagnetic angular momentum exchange between fields and currents, in the Röntgen interaction term of the minimal-coupling Hamiltonian, and in discussions of the orbital angular momentum of light. Its divergence $\nabla\cdot(\mathbf{A}\times\mathbf{J})$ therefore measures the local source or sink density of this rotational interaction flow — the rate at which electromagnetic angular momentum is being injected into or extracted from a volume element by the coupled current-potential system. Applying the standard vector identity $\nabla\cdot(\mathbf{X}\times\mathbf{Y}) = \mathbf{Y}\cdot(\nabla\times\mathbf{X}) - \mathbf{X}\cdot(\nabla\times\mathbf{Y})$ to $\mathbf{A}\times\mathbf{J}$ gives

$$F_1 = \mathbf{J}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{J}) = \mathbf{J}\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{J}), \quad (135)$$

which separates the norm channel into two physically distinct contributions. The first term, $\mathbf{J}\cdot\mathbf{B}$, measures the alignment of the current density with the local magnetic flux and is the standard magnetohydrodynamic work-against-tension scalar: it vanishes when $\mathbf{J}$ is perpendicular to $\mathbf{B}$ and is maximal in force-free configurations where $\mathbf{J} \parallel \mathbf{B}$. The second term, $\mathbf{A}\cdot(\nabla\times\mathbf{J})$, measures the coupling of the vector potential to the vorticity of the current distribution. The norm channel F_1 is therefore the imbalance between these two: a nonzero F_1 signals that electromagnetic angular momentum is neither conserved nor uniformly transported, but is being locally created or annihilated by the mismatch between current-flux alignment and current-vorticity coupling.

In a conducting medium, Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ allows the vorticity term to be evaluated directly. Taking the curl of Ohm's law and inserting Faraday's law $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ gives $\nabla \times \mathbf{J} = -\sigma\partial_t \mathbf{B}$, so that $\mathbf{A} \cdot (\nabla \times \mathbf{J}) = -\sigma\mathbf{A} \cdot \partial_t \mathbf{B}$, and the norm channel becomes

$$F_1 = \sigma\mathbf{A} \cdot \partial_t \mathbf{B} + \mathbf{J} \cdot \mathbf{B}. \tag{136}$$

Here the two terms have a transparent physical meaning. The term $\mathbf{J} \cdot \mathbf{B}$ is the MHD scalar measuring current-flux alignment, as before. The term $\sigma\mathbf{A} \cdot \partial_t \mathbf{B}$ is the coupling between the vector potential and the rate of change of magnetic flux through the medium — a term that connects directly to electromagnetic induction in conductors and, at the quantum level, to the Aharonov-Bohm phase accumulated by charge carriers moving through a region of time-varying flux. The norm channel therefore couples two physically distinct induction mechanisms: one geometric ($\mathbf{J} \cdot \mathbf{B}$, tied to field-line topology) and one dynamic ($\sigma\mathbf{A} \cdot \partial_t \mathbf{B}$, tied to flux change).

The condition $F_1 = 0$, that is

$$\sigma\mathbf{A} \cdot \partial_t \mathbf{B} + \mathbf{J} \cdot \mathbf{B} = 0, \tag{137}$$

defines a balance between these two mechanisms and selects a physically distinguished class of configurations. It is satisfied, for instance, when the magnetic field is stationary ($\partial_t \mathbf{B} = 0$) and simultaneously $\mathbf{J} \perp \mathbf{B}$, which is the standard transverse-current condition of classical MHD equilibrium. It is also satisfied when the two terms exactly balance, $\sigma\mathbf{A} \cdot \partial_t \mathbf{B} = -\mathbf{J} \cdot \mathbf{B}$, which describes a dynamic equilibrium in which inductive flux change and current-flux alignment compensate one another locally. In both cases, the electromagnetic angular momentum density $\mathbf{A} \times \mathbf{J}$ has vanishing divergence and is therefore purely solenoidal: the rotational interaction structure is conserved and redistributed without local creation or annihilation. This is the norm-channel analogue of the divergence-free condition familiar from magnetic helicity conservation in ideal MHD, and it confirms that the norm channel of $\partial(J^T A)$ is the natural algebraic home of electromagnetic angular momentum balance within the BQ framework.

Further interpretation of these results

The result $F_1 = \nabla \cdot (\mathbf{A} \times \mathbf{J})$ has a significance that extends beyond its immediate derivation. In standard four-vector electrodynamics, the quantities $\mathbf{J} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{J}$ have no natural common home. They appear in the literature in entirely separate contexts: $\mathbf{J} \cdot \mathbf{B}$ is the central scalar of magnetohydrodynamics, governing magnetic helicity injection, force-free field configurations, and the topology of field-line reconnection in plasma physics, while $\mathbf{A} \times \mathbf{J}$ is the electromagnetic interaction angular momentum density, appearing in the Röntgen term of the minimal-coupling

Hamiltonian, in the angular momentum of radiation fields, and in the theory of the orbital angular momentum of light. In the tensor and four-vector formalism these two quantities are assembled from separate constructions added on top of the core Maxwell–Lorentz framework, with separate physical justifications and no algebraic relationship between them. The norm channel of $\partial(J^T A)$ places both into a single divergence identity as two terms of one scalar expression, revealing their common origin in the rotational structure of the current–potential interaction. This cross-domain connection between plasma physics and electromagnetic angular momentum theory is not visible in standard formulations.

The significance of this result is amplified by the algebraic position in which it arises. In the Maxwell equations and in the Lorentz force law, the physically sourced content lives entirely in the $\hat{T}$ and $\hat{K}$ space-time channels, while the $\hat{1}$ norm channel and the $\hat{\boldsymbol{\sigma}}$ spin-norm channel carry secondary or zero content. The norm channel of $\partial(J^T A)$ breaks this pattern: it carries independent physical content that is not reducible to the space-time sector and that has no counterpart in the standard minquat four-vector structure. This demonstrates that the norm channel is not a formal receptacle for algebraic overflow but has genuine physical meaning in the context of the current–potential interaction. More broadly, it indicates that the BQ framework is not merely reorganising known physics into a compact notation but is revealing a structural layer of the current–potential interaction that standard four-vector methods systematically suppress by projecting exclusively onto the space-time sector.

The equilibrium condition $F_1 = 0$, derived in Eqn. (137), further illustrates this point. Force-free magnetic configurations, defined by $\mathbf{J} \parallel \mathbf{B}$ and the vanishing of the Lorentz body force, and inductive balance in a conducting medium, defined by the compensation of flux change and current-flux alignment, are normally treated as belonging to different physical regimes of MHD: the former is static or quasi-static, the latter is dynamic. The condition $\sigma\mathbf{A} \cdot \partial_t\mathbf{B} + \mathbf{J} \cdot \mathbf{B} = 0$ places both on the same footing as two limiting cases of one scalar balance equation. In both limits the divergence of $\mathbf{A} \times \mathbf{J}$ vanishes and the electromagnetic angular momentum density is purely solenoidal: the rotational interaction structure is conserved and redistributed without local creation or annihilation. This is the norm-channel analogue of the divergence-free condition familiar from magnetic helicity conservation in ideal MHD, and it connects force-free field theory to electromagnetic angular momentum conservation through a single algebraic condition rather than through separate variational arguments.

It should be noted that the norm channel result is not a prediction that standard methods cannot in principle reach: all terms in F_1 are derivable by other means. The added value is structural and organisational. The BQ product $\partial(J^T A)$ collects $\mathbf{J} \cdot \mathbf{B}$, $\mathbf{A} \times \mathbf{J}$, their divergence relation, the Ohm's-law specialisation, and the equilibrium condition into one algebraic object with automatic Lorentz covariance, whereas in standard treatments these are assembled from separate pieces belonging to separate subdisciplines. The question of whether this organisational clarity leads to genuinely new physics — new conserved quantities, new solutions, or new observable predictions — depends on the full coupled dynamics of $\partial(J^T A) = F$ treated across all four channels simultaneously. The standard formalism has no natural mechanism for tracking the $\hat{1}, \hat{T}, \hat{K}$, and $\hat{\boldsymbol{\sigma}}$ channels as simultaneous outputs of one product; the BQ framework does. The cross-channel dynamics of $\partial(J^T A)$ therefore represents the regime in which the present formalism may produce results that go beyond a compact re-derivation of known equations, and the norm channel is where that potential is most clearly on display.

3.5.2 Interpretation of the time channel of $\partial(J^T A)$

The time channel of $\partial(J^T A)$ is given by

$$F_T = \nabla \cdot \mathcal{S} - \frac{1}{c}\partial_t \mathcal{L}, \tag{138}$$

or

$$F_T = \nabla \cdot \left(c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}\right) - \frac{1}{c}\partial_t(\rho\phi - \mathbf{J}\cdot\mathbf{A}). \tag{139}$$

The first term, written as

$$c\nabla \cdot \left(\rho\mathbf{A} - \frac{1}{c^2}\phi\mathbf{J}\right), \tag{140}$$

measures the divergence of the directed interaction transport between the charge–vector-potential coupling and the transported scalar-potential interaction energy. Using

$$\mathbf{J} = \rho\mathbf{v}, \tag{141}$$

the second part can be rewritten as

$$\frac{1}{c^2}\phi\mathbf{J} = \frac{1}{c^2}\rho\phi\mathbf{v} = \frac{\varepsilon}{c^2}\mathbf{v}, \tag{142}$$

which is the scalar interaction energy density transported with the charge flow velocity $\mathbf{v}$. The quantity

$$\rho\mathbf{A}, \tag{143}$$

acts as the corresponding vector-potential interaction transport term. Both terms have the same dimensions and together define the transport balance of the current–potential interaction.

The second term,

$$-\frac{1}{c}\partial_t(\rho\phi - \mathbf{J}\cdot\mathbf{A}), \tag{144}$$

is the time variation of the electromagnetic interaction Lagrangian density

$$\mathcal{L} = \rho\phi - \mathbf{J} \cdot \mathbf{A}. \tag{145}$$

The BQ product thus naturally delivers $\mathcal{L} = \rho\phi - \mathbf{J} \cdot \mathbf{A}$ in the time channel, which corresponds to the classical interaction energy density convention of Jackson and Griffiths. The covariant interaction Lagrangian density $\mathcal{L}_{\text{int}} = \mathbf{J} \cdot \mathbf{A} - \rho\phi = -\mathcal{L}$ carries the opposite sign, arising from the metric contraction $J^\mu A_\mu$ in the $(-+++)$ signature, where $J^\mu = (c\rho, \mathbf{J})$ and $A_\mu = (-\phi/c, \mathbf{A})$ give $J^\mu A_\mu = -\rho\phi + \mathbf{J} \cdot \mathbf{A} = \mathcal{L}_{\text{int}}$. The continuity equation in the time channel therefore reads $\nabla \cdot \mathbf{S} - \frac{1}{c}\partial_t \mathcal{L} = 0$, or equivalently $\nabla \cdot \mathbf{S} + \frac{1}{c}\partial_t \mathcal{L}_{\text{int}} = 0$ upon substituting $\mathcal{L}_{\text{int}} = -\mathcal{L}$, confirming that both sign conventions describe the same physical conservation law. We conclude that the time channel therefore has the structure of a continuity equation with as the conserved quantity:

$$F_T = \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{S}} + \frac{1}{c}\partial_t \mathcal{L}_{\text{int}}, \tag{146}$$

with L as the interaction energy density and $\boldsymbol{\mathcal{S}}$ the associated interaction transport flow. The time channel defines the local conservation or redistribution law of the current–potential interaction structure. Vanishing when $F_T = 0$corresponds to conservative transport, while nonzero values indicate local exchange, injection, dissipation, or transport imbalance, giving

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{S}} + \frac{1}{c}\partial_t \mathcal{L}_{\text{int}} = 0. \tag{147}$$

Thus, $\boldsymbol{\mathcal{S}}$ is the transport current associated with the 'charge density' $\mathcal{L}_{\text{int}}$.

The time channel result carries a significance that extends beyond the immediate derivation. In standard electrodynamics, energy conservation is obtained by combining Maxwell's equations with the Lorentz force law in a single argument that produces Poynting's theorem for the total electromagnetic energy, field and interaction combined. The BQ framework separates these two contributions automatically and structurally: the free-field energy density and the Poynting vector emerge from the quadratic product $B^T B$ in Section 3.2, while the interaction energy conservation law emerges here from the $\hat{T}$-channel of $\partial(J^T A)$. This separation is not imposed by hand but is what the algebra produces when the two objects are computed independently. It is a structural feature of the BQ language that standard four-vector methods do not provide, since in those methods field energy and interaction energy appear together in one combined Poynting argument with no algebraic mechanism to separate them.

The transport vector $\mathbf{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}$ that emerges as the natural companion of $\mathcal{L}_{\text{int}}$ in the time channel is not a commonly discussed standalone object in the literature. It is distinct from the Poynting vector $\frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$, which governs free-field energy transport, and from the mechanical energy current $\rho\varepsilon\mathbf{v}$, which governs kinetic energy transport. The term $c\rho\mathbf{A}$ is the convective transport of the vector-potential coupling energy by the charge distribution, while the term $\frac{1}{c}\phi\mathbf{J} = \frac{\varepsilon}{c^2}\mathbf{v}$ is the scalar potential interaction energy carried along with the moving charges. Together they define an interaction energy current that occupies its own physical layer between the mechanical and field-energy sectors, and the BQ product $\partial(J^T A)$ identifies it automatically as the correct transport companion of $\mathcal{L}_{\text{int}}$ without requiring a separate construction or variational argument.

The conservation condition $F_T = 0$, giving

$$\nabla \cdot \mathbf{S} + \frac{1}{c}\partial_t \mathcal{L}_{\text{int}} = 0, \tag{148}$$

states that the interaction Lagrangian density can only change at a point through the divergence of its transport current $\mathbf{S}$: there is no local creation or destruction of coupling energy, only redistribution. This is the condition for conservative interaction energy transport, and it is the interaction-sector analogue of the divergence-free Poynting condition familiar from free-field energy conservation. When $F_T \neq 0$, the time channel provides a local diagnostic for where and how the current-potential coupling is being driven, damped, or redistributed — a quantity relevant for driven plasma systems, antenna near-field coupling, and any configuration in which the interaction energy budget is not in local equilibrium.

There is a further connection to the variational foundations of electrodynamics that deserves to be made explicit. The quantity $\mathcal{L}_{\text{int}} = \mathbf{J} \cdot \mathbf{A} - \rho\phi$ is precisely the interaction term in the classical electromagnetic action

$$S = \int (\mathcal{L}_{\text{field}} + \mathcal{L}_{\text{int}})\, d^4x, \tag{149}$$

with $\mathcal{L}_{\text{field}} = \frac{1}{2\varepsilon_0}(E^2 - c^2B^2)$ the free-field Lagrangian density. The time channel of $\partial(J^T A)$ is therefore not an arbitrary continuity equation but the local expression of the conservation law associated with the interaction part of the action, projected onto the $\hat{T}$-channel of the BQ algebra. This places §3.5.2 in direct contact with the variational foundations of classical electrodynamics in a way that the standard derivation of Poynting's theorem does not reveal, since that derivation combines field and interaction contributions before any channel separation is possible.

It should be acknowledged that the time channel result, like the norm channel result of §3.5.1, is not a prediction that standard methods cannot in principle reach: the conservation law for $\mathcal{L}_{\text{int}}$ can be derived conventionally. The added value is structural. The BQ product $\partial(J^T A)$ delivers the interaction Lagrangian density, its transport current, and their conservation law as a single channel output, automatically separated from the free-field sector, with no need for a combined

Poynting argument or a separate variational derivation. Taken together, the norm channel of §3.5.1 and the time channel of the present section establish that $\partial(J^T A)$ is a physically richer object than its standard treatment as a scalar Lagrangian density suggests. The norm channel carries the electromagnetic angular momentum balance of the current-potential system; the time channel carries its interaction energy conservation law. Both emerge from the same matrix multiplication, in adjacent algebraic slots, with automatic Lorentz covariance. This is the structural economy that the BQ framework is designed to provide, and §3.5 is where it is most concretely on display.

3.5.3 Interpretation of the space channel of $\partial(J^T A)$

The space channel of $\partial(J^T A)$ is given by

$$F_K = \nabla \times \mathcal{T} + \nabla \mathcal{L} - \frac{1}{c}\partial_t \mathcal{S}, \tag{150}$$

but it can be interpreted using $\mathcal{L}_{\text{int}} = -\mathcal{L}$ as

$$F_K = \nabla \times \mathcal{T} - \nabla \mathcal{L}_{\text{int}} - \frac{1}{c}\partial_t \mathcal{S}, \tag{151}$$

with

$$\mathcal{L}_{\text{int}} = \mathbf{J} \cdot \mathbf{A} - \rho\phi, \tag{152}$$

$$\mathcal{T} = \mathbf{J} \times \mathbf{A}, \tag{153}$$

$$\mathcal{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}. \tag{154}$$

The channel naturally separates into two parts,

$$\nabla \times \mathcal{T} \tag{155}$$

and

$$-(\nabla \mathcal{L}_{\text{int}} + \frac{1}{c}\partial_t \mathcal{S}). \tag{156}$$

The first term,

$$\nabla \times (\mathbf{J} \times \mathbf{A}), \tag{157}$$

is the rotational part of the interaction structure. Since

$$\mathbf{J} \times \mathbf{A} \tag{158}$$

has the structure of a torque-like or angular-momentum interaction density, its curl measures the local rotational redistribution of the current–potential coupling.

Using the standard vector identity

$$\begin{aligned}\boldsymbol{\nabla} \times (\mathbf{J} \times \mathbf{A}) = (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{J} - (\mathbf{J} \cdot \boldsymbol{\nabla})\mathbf{A} \\ +\mathbf{J}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{J}),\end{aligned} \tag{159}$$

one observes that the channel contains contributions from:

- transport of current along the vector-potential structure,
- deformation of the vector-potential structure along the current flow,
- gauge-compression terms through $\boldsymbol{\nabla} \cdot \mathbf{A}$,
- and local charge accumulation through $\boldsymbol{\nabla} \cdot \mathbf{J}$.

The second contribution,

$$\boldsymbol{\nabla}\mathcal{L}_{\text{int}} + \frac{1}{c}\partial_t \boldsymbol{S}, \tag{160}$$

has the structure of a momentum-balance or force-density term. The gradient term

$$\boldsymbol{\nabla}\mathcal{L}_{\text{int}} \tag{161}$$

acts as the spatial interaction-energy gradient, while

$$\frac{1}{c}\partial_t \boldsymbol{S} \tag{162}$$

measures the temporal variation of the interaction transport channel.

The complete space channel therefore combines rotational interaction transport with interaction-energy and transport-balance dynamics. In this interpretation, the K channel of $\partial(J^T A)$ measures the local force-density or momentum-balance structure generated by the current–potential interaction.

The space channel result carries a significance that becomes fully visible only when it is read in conjunction with the norm and time channels of the preceding subsections. We address each component in turn before assembling the complete physical picture.

The second contribution $-(\nabla\mathcal{L}_{\text{int}} + \frac{1}{c}\partial_t \mathbf{S})$ is more important than its description as a generic momentum-balance term suggests. Recall from §3.5.2 that $\mathbf{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}$ is the interaction transport current and $\mathcal{L}_{\text{int}} = \mathbf{J} \cdot \mathbf{A} - \rho\phi$ is the interaction Lagrangian density. Their combination

$$-\nabla\mathcal{L}_{\text{int}} - \frac{1}{c}\partial_t \mathbf{S} \tag{163}$$

is structurally identical to the Euler–Lagrange momentum equation for the current–potential system. The gradient term $-\nabla\mathcal{L}_{\text{int}} = \nabla(\rho\phi - \mathbf{J} \cdot \mathbf{A})$ is the spatial interaction-energy gradient that drives charge redistribution, while $-\frac{1}{c}\partial_t \mathbf{S}$ is the inertial resistance of the interaction transport current to temporal change. Together they form the interaction-sector analogue of the canonical momentum equation $\dot{\mathbf{p}} = -\nabla V$ familiar from mechanics: the space channel is the local momentum balance of the current–potential interaction, not merely an object that resembles one.

This identification connects directly to the variational foundations of the theory. The combination $-\nabla \mathcal{L}_{\text{int}} - \frac{1}{c}\partial_t \mathbf{S}$ is the spatial Euler–Lagrange derivative of the interaction action with respect to the field coordinates. Setting $F_K = 0$ is therefore a stationarity condition on the interaction part of the action with respect to spatial variations, the space-channel counterpart of the time-channel stationarity condition identified in §3.5.2. Taken together, the $\hat{T}$ and $\hat{K}$ channels of $\partial(J^T A)$ encode the full four-dimensional Euler–Lagrange dynamics of the current–potential interaction: the time channel gives the temporal stationarity condition, and the space channel gives the spatial stationarity condition. The BQ product $\partial(J^T A)$ therefore does not merely contain the interaction Lagrangian density as one of its outputs — it generates the complete Euler–Lagrange equation of motion for the current–potential system across its space-time channels, automatically and without a separate variational argument.

The rotational contribution $\nabla \times (\mathbf{J} \times \mathbf{A})$ has an equally precise identification. It is the vorticity source of the angular momentum interaction density: it measures how the rotational structure of $\mathbf{J} \times \mathbf{A}$ is being generated or destroyed at each point in space. In plasma physics and MHD this term appears in the evolution equation for magnetic helicity and in the analysis of current-driven instabilities. Its presence here as the rotational complement of the momentum-balance term is not accidental. Comparing §3.5.1 and the present section, the norm channel of $\partial(J^T A)$ delivered the divergence $\nabla \cdot (\mathbf{A} \times \mathbf{J})$, and the $\hat{K}$-channel delivers the curl $\nabla \times (\mathbf{J} \times \mathbf{A})$. By the Helmholtz decomposition theorem, any sufficiently smooth vector field is completely determined by its divergence and its curl. The BQ product $\partial(J^T A)$ therefore places the complete Helmholtz decomposition of the angular momentum interaction density $\mathbf{J} \times \mathbf{A}$ across two algebraic channels: the divergence-free rotational part in the $\hat{K}$-channel, and the curl-free irrotational part in the $\hat{1}$-channel. No standard four-vector method produces this decomposition as a natural output of a single product; in the standard formalism these two objects belong to different parts of the theory with no algebraic relationship between them.

The equilibrium condition $F_K = 0$ gives

$$\nabla \times (\mathbf{J} \times \mathbf{A}) = \nabla \mathcal{L}_{\text{int}} + \frac{1}{c}\partial_t \mathbf{S}, \tag{164}$$

which is a balance between the rotational redistribution of angular momentum interaction density on the left and the momentum-energy gradient driving force on the right. This is the interaction-sector analogue of the vorticity-pressure balance in continuum mechanics, where vorticity generation is driven by pressure gradients. In the static limit $\partial_t \mathbf{S} = 0$ this reduces to

$$\nabla \times (\mathbf{J} \times \mathbf{A}) = \nabla \mathcal{L}_{\text{int}}, \tag{165}$$

a non-trivial constraint relating the spatial vorticity structure of the angular momentum density to the gradient of the interaction energy. Configurations satisfying this condition are interaction-sector analogues of Beltrami flows in fluid mechanics, where vorticity and velocity gradient are in local balance.

The complete structure of $\partial(J^T A) = F$ across all four channels can now be read as a coherent system. The $\hat{1}$-channel gives the divergence of the angular momentum interaction density and its MHD equilibrium condition. The $\hat{T}$-channel gives the conservation law for the interaction Lagrangian density and its transport current. The $\hat{K}$-channel gives the full Euler–Lagrange momentum equation of the current–potential system and the curl of the angular momentum density. The $\hat{\boldsymbol{\sigma}}$-channel, addressed in §3.5.4, will complete the picture. Together the $\hat{1}$ and $\hat{K}$ channels provide the Helmholtz decomposition of $\mathbf{J} \times \mathbf{A}$; together the $\hat{T}$ and $\hat{K}$ channels provide the four-dimensional Euler–Lagrange equation of the interaction action. These are not four separate results assembled after the fact: they are four simultaneous outputs of one matrix multiplication, delivered with automatic Lorentz covariance and automatic channel separation. This is the structural achievement of the BQ framework in §3.5 , and the space channel is where its full scope becomes visible.

It should be noted that, as with the preceding channels, the individual results in F_K are derivable by standard methods. The added value lies in the simultaneous and automatic delivery of the Euler–Lagrange structure, the Helmholtz decomposition, and the vorticity-pressure balance across the same algebraic product, with no need to invoke separate variational, vector-analytic, or fluid-mechanical arguments. The cross-channel coherence of $\partial(J^T A)$ — the fact that adjacent channels carry complementary aspects of the same underlying physics — is a property of the BQ algebra that has no counterpart in the standard tensor formalism, and §3.5 as a whole is the clearest demonstration of this in the present paper.

3.5.4 Interpretation of the σ-channel of $\partial(J^T A)$

The σ-channel of $\partial(J^T A)$ is given by

$$F_\sigma = \boldsymbol{\nabla} \times \boldsymbol{\mathcal{S}} + \frac{1}{c}\partial_t \boldsymbol{\mathcal{T}}, \tag{166}$$

with

$$\boldsymbol{\mathcal{T}} = \mathbf{J} \times \mathbf{A}, \tag{167}$$

and

$$\mathcal{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}. \tag{168}$$

This channel has the same structural form as a curl–time-evolution equation. It compares the circulation of the interaction-transport channel $\mathcal{S}$ with the time variation of the torque-like channel $\mathcal{T}$.

Written out explicitly, one has

$$F_\sigma = \boldsymbol{\nabla} \times \left(c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}\right) + \frac{1}{c}\partial_t(\mathbf{J} \times \mathbf{A}). \tag{169}$$

The σ-channel therefore combines the circulation of the boost-like transport channel with the time evolution of the torque-like interaction channel. In this sense it measures whether the transport imbalance of the current–potential coupling produces a rotational response in the spin-norm sector.

This is analogous in structure to the homogeneous Maxwell channel

$$\boldsymbol{\nabla} \times \mathbf{E} + \partial_t\mathbf{B} = 0, \tag{170}$$

but now applied to the interaction quantities $\mathcal{S}$ and $\mathcal{T}$. Thus the spin channel can be interpreted as the Faraday-like or rotational closure condition of the current–potential interaction structure.

If this channel is set to zero, one obtains

$$\boldsymbol{\nabla} \times \mathcal{S} + \frac{1}{c}\partial_t\mathcal{T} = 0, \tag{171}$$

which states that the circulation of the interaction-transport channel is exactly balanced by the time variation of the torque-like current–potential channel. If it is nonzero, the current–potential coupling contains an open spin-norm response, indicating a rotational imbalance or dual-channel source in the interaction structure.

The $\hat{\boldsymbol{\sigma}}$-channel result deserves careful interpretation before the complete four-channel structure of $\partial(J^T A)$ is assessed. The channel delivers

$$F_\sigma = \nabla \times \mathbf{S} + \frac{1}{c}\partial_t\mathbf{T} = 0, \tag{172}$$

with $\mathbf{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}$ the interaction transport current and $\mathbf{T} = \mathbf{J} \times \mathbf{A}$ the angular momentum interaction density. The text notes the structural parallel with Faraday's law $\nabla \times \mathbf{E} + \partial_t\mathbf{B} = 0$, but the analogy is exact rather than merely structural and deserves to be stated as such.

In the Maxwell equations, Faraday's law occupies the $\hat{\boldsymbol{\sigma}}$-channel of $\partial B = \mu_0 J$ and expresses the induction geometry of the free electromagnetic field: a time-varying magnetic field drives a circulating electric field, and vice versa. The

$\hat{\sigma}$-channel of $\partial(J^T A)$ produces the same algebraic form with $\mathbf{S}$ in the role of $\mathbf{E}$ and $\mathbf{T}$ in the role of $\mathbf{B}$. The correspondence is term by term and sign by sign. This means the $\hat{\sigma}$-channel of $\partial(J^T A)$ is not merely analogous to Faraday's law: it is the interaction-sector Faraday law, governing how a time-varying angular momentum interaction density $\mathbf{T} = \mathbf{J} \times \mathbf{A}$ induces a circulation in the interaction transport current $\mathbf{S}$, and how a spatially circulating $\mathbf{S}$ is balanced by the time evolution of $\mathbf{T}$. The BQ algebra reproduces the same induction geometry in the interaction sector that it produces in the free-field sector, with $(\mathbf{S}, \mathbf{T})$ playing the roles of $(\mathbf{E}, \mathbf{B})$ respectively.

The algebraic position of this result reinforces its significance. As established in §3.3, the $\hat{\sigma}$-channel is the 3D part of the 4D pauliquat spin-norm sector of the BQ algebra, the sector that has no counterpart in standard Minkowski four-vector geometry. In the Maxwell equations the $\hat{\sigma}$-channel carries Faraday's law, the homogeneous and sourceless closure condition of the free field. Here in $\partial(J^T A)$ the $\hat{\sigma}$-channel carries the homogeneous closure condition of the interaction sector. This is consistent with the general pattern established throughout the paper: the spin-norm sector carries the geometric, topological, and induction structure, while the space-time sector carries the source-driven dynamics. The $\hat{\sigma}$-channel of $\partial(J^T A)$ confirms that this pattern holds for the interaction sector as well as for the free-field sector.

The $\hat{\sigma}$-channel also completes the Helmholtz decomposition of $\mathbf{S}$ that was begun in §3.5.2. The time channel of $\partial(J^T A)$ gave the divergence $\nabla \cdot \mathbf{S}$ through the continuity equation for $\mathcal{L}_{\text{int}}$. The $\hat{\sigma}$-channel now gives the curl $\nabla \times \mathbf{S}$ through the interaction-sector Faraday law. By the Helmholtz decomposition theorem, the divergence and curl together completely determine a vector field up to boundary conditions. The $\hat{T}$ and $\hat{\sigma}$ channels of $\partial(J^T A)$ therefore provide the complete Helmholtz decomposition of the interaction transport current $\mathbf{S}$, just as the $\hat{1}$ and $\hat{K}$ channels of §3.5.1 and §3.5.3 provided the complete Helmholtz decomposition of the angular momentum interaction density $\mathbf{T} = \mathbf{J} \times \mathbf{A}$. The four channels of $\partial(J^T A)$ collectively deliver the Helmholtz decompositions of both fundamental interaction quantities $\mathbf{S}$ and $\mathbf{T}$ as simultaneous outputs of one matrix multiplication.

The condition $F_\sigma \neq 0$ deserves attention alongside the equilibrium case. When the $\hat{\sigma}$-channel is nonzero, the circulation of $\mathbf{S}$ is not closed by the time variation of $\mathbf{T}$, meaning the current-potential system is exchanging rotational content with the spin-norm sector in a way that cannot be accounted for by the interaction quantities alone. In the context of the broader BQ framework, the spin-norm sector is where magnetic monopole sources would appear, as noted in §3.3. A nonzero F_σ in the interaction sector would therefore signal the presence of a topological source in the current-potential coupling — a defect in the induction structure of the interaction field analogous to a magnetic monopole in the free-field sector. This connection between $F_\sigma \neq 0$ and topological defects in the interaction sector is a direction for future investigation, but it is a structurally natural one given the channel assignments established throughout the paper.

It should be noted that, as with the preceding channels, the individual result in F_σ is derivable by standard methods. The added value is that the interaction-sector Faraday law, the Helmholtz decomposition of **S**, and the topological closure condition all emerge from the same algebraic slot, without a separate induction argument, and in precise correspondence with the free-field $\hat{\boldsymbol{\sigma}}$-channel that produces the physical Faraday law. The parallel between the free-field sector and the interaction sector is exact at the level of channel assignments, and the $\hat{\boldsymbol{\sigma}}$-channel of $\partial(J^T A)$ is where that parallel is most transparently on display.

3.5.5 The complete interaction system $F = \partial(J^T A)$ and its equilibrium conditions

Looking at all four channels together, the product $\partial(J^T A)$ is not a Lagrangian scalar with three appendages. It is a complete Maxwell-like system for the interaction sector, with the same four-channel architecture that governs the free electromagnetic field. The sourced, space-time dynamics live in the $\hat{T}$ and $\hat{K}$ channels; the unsourced, topological, induction structure lives in the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels. This mirrors exactly the channel split in the Maxwell equations of §3.3, where the $\hat{T}$ and $\hat{K}$ channels carry Gauss's law and the Ampère–Maxwell law, and the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels carry the divergence-free magnetic condition and Faraday's law. The product $\partial(J^T A)$ reproduces this architecture in the interaction sector, with $(\mathbf{S}, \mathbf{T})$ playing the roles of $(\mathbf{E}, \mathbf{B})$ and $\mathcal{L}_{\text{int}}$ playing the role of the scalar source density. This parallel is not imposed by hand: it is what the BQ algebra produces when the product is computed, and it holds at the level of channel assignments, sign conventions, and algebraic structure simultaneously.

First condition: $\hat{1}$ ***and*** $\hat{\sigma}$ ***channels zero,*** $\hat{T}$ ***and*** $\hat{K}$ ***channels nonzero.***

This is the physically generic condition for a driven, sourced interaction system. Setting $F_1 = 0$ and $F_\sigma = 0$ while allowing $F_T \neq 0$ and $F_K \neq 0$ assigns a precise role to each vanishing channel.

The condition $F_1 = 0$ requires $\nabla \cdot (\mathbf{A} \times \mathbf{J}) = 0$, meaning the electromagnetic angular momentum interaction density $\mathbf{T} = \mathbf{J} \times \mathbf{A}$ is purely solenoidal: there is no local creation or destruction of interaction torque, and the angular momentum density is being redistributed without net sources or sinks.

The condition $F_\sigma = 0$ requires $\nabla \times \mathbf{S} + \frac{1}{c}\partial_t \mathbf{T} = 0$, the interaction-sector Faraday law holding exactly. The induction geometry of the interaction sector is closed: the time variation of **T** is balanced by the circulation of **S** at every point, with no topological sources or defects in the spin-norm sector.

Together, $F_1 = 0$ and $F_\sigma = 0$ impose the homogeneous conditions on both the divergence and the curl of **T**, and on both the curl and the divergence of **S** through its coupling to **T**. By the Helmholtz decomposition theorem, **T** and **S** then have well-defined, source-free spatial structures globally determined by their boundary conditions rather than by local sources.

The nonzero $\hat{T}$ and $\hat{K}$ channels carry the active physics. $F_T \neq 0$ means the interaction Lagrangian density is not locally conserved: there is active injection, dissipation, or redistribution of coupling energy at the point in question. $F_K \neq 0$ means the Euler–Lagrange momentum balance is not satisfied: there is a net force density driving the current-potential system away from its variational equilibrium.

This condition — homogeneous spin-norm channels, nonzero space-time channels — is therefore the interaction-sector analogue of the standard electromagnetic situation in which the free field satisfies the homogeneous Maxwell equations while the sourced equations carry the charge and current dynamics. It describes a current-potential system in which the interaction angular momentum and induction geometry are topologically clean while energy and momentum are actively exchanged. This is the natural condition for a driven MHD system, an antenna operating in the near-field regime, or any configuration in which interaction energy is being injected or extracted while the rotational and topological structure of the coupling remains intact.

Second condition: $F = 0$ ***in all four channels.***

Setting all four channels to zero simultaneously gives the coupled system

$$\nabla \cdot (\mathbf{A} \times \mathbf{J}) = 0, \tag{173}$$

$$\nabla \cdot \mathbf{S} + \frac{1}{c}\partial_t \mathcal{L}_{\text{int}} = 0, \tag{174}$$

$$\nabla \times (\mathbf{J} \times \mathbf{A}) - \nabla \mathcal{L}_{\text{int}} - \frac{1}{c}\partial_t \mathbf{S} = 0, \tag{175}$$

$$\nabla \times \mathbf{S} + \frac{1}{c}\partial_t \mathbf{T} = 0. \tag{176}$$

This is the condition for a completely self-consistent, closed, and conservative current-potential interaction system, with each channel contributing a different aspect of the closure.

The vanishing norm channel means the angular momentum interaction density has no sources or sinks. The vanishing time channel means the interaction Lagrangian density is exactly conserved by its transport current $\mathbf{S}$: no coupling energy is created or destroyed locally. The vanishing space channel means the Euler–Lagrange momentum balance is exactly satisfied: the rotational interaction structure is in precise balance with the interaction energy gradient and the transport inertia. The vanishing $\hat{\boldsymbol{\sigma}}$-channel means the interaction-sector Faraday law holds exactly: the induction geometry of the interaction sector is fully closed with no topological defects.

There is a deeper reading of this condition. In the Maxwell equations, the condition $\partial B = 0$ — the full four-channel system set to zero — would imply no sources and no fields: a trivial vacuum. The condition $\partial(J^T A) = 0$ is different in character. It does not mean no interaction; it means the interaction is in a state of complete internal self-consistency, in which every channel of the current-potential coupling is in balance with every other. This is more analogous to a force-free or Beltrami configuration in MHD than to a vacuum: the fields and currents are present and coupled, but the coupling is in a state of global equilibrium with no net flow of energy, momentum, angular momentum, or inductive imbalance in any channel.

This condition has a natural connection to the variational principle. Since the $\hat{T}$ and $\hat{K}$ channels carry the four-dimensional Euler–Lagrange equations of the interaction action, setting both to zero is the full stationarity condition for the interaction part of the action $S = \int(\mathcal{L}_{\text{field}} + \mathcal{L}_{\text{int}})d^4x$. Adding the requirement that the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels also vanish imposes the additional topological closure conditions on top of the variational equilibrium. The complete condition $F = 0$ is therefore the statement that the current-potential system simultaneously satisfies its equations of motion and its topological closure conditions — a constrained variational equilibrium in the fullest sense.

Whether non-trivial physical configurations satisfying $\partial(J^T A) = 0$ exist beyond the trivial case is a well-posed question. In the static limit the condition reduces to a coupled system relating the spatial structure of **J** and **A** through their divergences, curls, and their Lagrangian gradient, which admits non-trivial solutions in the presence of appropriate boundary conditions. These configurations would be the interaction-sector analogues of force-free fields in MHD, and their classification — which fields **J** and potentials **A** simultaneously satisfy all four channel conditions — is a natural continuation of the present analysis.

The two conditions in context.

The two conditions bracket the physically relevant range of the interaction system. The first — homogeneous spin-norm channels, nonzero space-time channels — describes the generic driven case in which active energy and momentum exchange occur within a topologically clean interaction structure. The second — all channels zero — describes the fully equilibrated, self-consistent case in which the interaction is closed in every sense simultaneously. Between these two extremes lie the mixed cases in which individual channels are nonzero, each corresponding to a specific type of imbalance in the angular momentum, energy, momentum, or inductive structure of the current-potential coupling. The BQ channel decomposition of $\partial(J^T A)$ provides the natural language for classifying all such cases within a single algebraic framework. No standard four-vector method produces this classification, because no standard method has access to all four channels as simultaneous outputs of one product. This is the structural contribution of §3.5 as a whole: not the derivation of any single new equation, but the demonstration that the current-potential interaction carries a complete four-channel Maxwell-like architecture that is invisible in the scalar Lagrangian treatment and becomes visible only when the full BQ product $\partial(J^T A)$ is computed and decomposed.

3.5.6 Gauge invariance of $\partial(J^T A)$ under charge conservation and the Lorenz gauge

The gauge properties of $\partial(J^T A)$ deserve explicit treatment, both because gauge invariance is a foundational requirement of any electromagnetic framework and because the four-channel structure of $\partial(J^T A)$ allows the conditions for gauge invariance to be stated with a precision that the scalar Lagrangian treatment does not provide.

The gauge transformation and its remainder.

Under a gauge transformation $A \to A + \partial\xi$, where ξ is a scalar gauge function, the product $J^T A$ transforms as

$$J^T(A + \partial\xi) = J^T A + J^T \partial\xi, \tag{177}$$

so that

$$\partial(J^T A) \to \partial(J^T A) + \partial(J^T \partial\xi). \tag{178}$$

Gauge invariance of $\partial(J^T A)$ therefore requires $\partial(J^T \partial\xi) = 0$. The object $J^T \partial\xi$ has the same BQ channel structure as $J^T A$ with A replaced by $\partial\xi$, so its four-channel decomposition follows directly from the general result of §3.5 , with the substitutions

$$\mathcal{L} \to \mathcal{L}_\xi = -J^\mu \partial_\mu \xi = \rho\partial_t\xi/c - \mathbf{J}\cdot\nabla\xi, \tag{179}$$
$$\mathbf{T} \to \mathbf{T}_\xi = \mathbf{J}\times\nabla\xi, \tag{180}$$
$$\mathbf{S} \to \mathbf{S}_\xi = c\rho\nabla\xi - \frac{1}{c}(\partial_t\xi)\mathbf{J}. \tag{181}$$

The condition $\partial(J^T \partial\xi) = 0$ requires all four channels of this object to vanish simultaneously.

The Lorenz gauge is already built into the framework.

A preliminary observation is important. The Lorenz gauge condition

$$\partial_\mu A^\mu = 0, \quad \text{that is,} \quad -\frac{1}{c}\partial_t\phi - \nabla\cdot\mathbf{A} = 0, \tag{182}$$

is precisely the condition $B_0 = 0$ imposed in 3.1 to pass from the general bilinear $\partial^T A$ to the physical field $B = \mathbf{B}\cdot\hat{K} + \frac{1}{c}\mathbf{E}\cdot\hat{\boldsymbol{\sigma}}$. The BQ electromagnetic framework is therefore already operating in the Lorenz gauge from 3.1 onwards; the gauge is not an external restriction imposed on §3.5 but an integral part of the algebraic foundations of the EM chapter. A residual gauge transformation $A \to A + \partial\xi$ preserves the Lorenz gauge if and only if ξ satisfies the wave equation

$$\Box\xi = \partial^{\mu}\partial_{\mu}\xi = 0. \tag{183}$$

The residual gauge freedom within the Lorenz gauge is therefore restricted to wave-equation solutions, not arbitrary scalar functions.

Time channel: gauge invariance from charge conservation alone.

The time channel of $\partial(J^T\partial\xi)$ involves $\nabla\cdot\mathbf{S}_{\xi}-\frac{1}{c}\partial_t\mathcal{L}_{\xi}$. Expanding using the product rule, terms of the form $(\partial_t\rho + \nabla\cdot\mathbf{J})\xi$ appear, which combine into the continuity expression $\partial_{\mu}J^{\mu}$ acting on ξ plus terms involving ξ acting on derivatives of J. The time channel of $\partial(J^T\partial\xi)$ vanishes for arbitrary ξ if and only if

$$\partial_{\mu}J^{\mu} = \partial_t\rho + \nabla\cdot\mathbf{J} = 0, \tag{184}$$

that is, charge conservation. This is the standard result: the interaction term $J^{\mu}A_{\mu}$ is gauge invariant precisely when the current four-vector is conserved, and the BQ time channel makes this condition explicit and channel-specific. The Lorenz gauge restriction $\Box\xi = 0$ adds nothing further here; charge conservation alone is necessary and sufficient for time-channel gauge invariance.

Norm channel: transverse currents in the Lorenz gauge.

The norm channel of $\partial(J^T\partial\xi)$ gives

$$\nabla\cdot(\mathbf{J}\times\nabla\xi) = \nabla\xi\cdot(\nabla\times\mathbf{J}) - \mathbf{J}\cdot(\nabla\times\nabla\xi) = \nabla\xi\cdot(\nabla\times\mathbf{J}), \tag{185}$$

since $\nabla\times\nabla\xi = 0$ identically. This vanishes for arbitrary ξ only if $\nabla\times\mathbf{J} = 0$, which is a stronger condition than charge conservation and is not satisfied in general. Within the Lorenz gauge, however, the residual gauge function ξ satisfying $\Box\xi = 0$ generates only longitudinal shifts in $\mathbf{A}$, so $\nabla\xi$ is longitudinal. For transverse currents — the physically relevant case for radiation and wave propagation, where the current is decomposed into transverse and longitudinal parts and the transverse part carries the radiative physics — the curl $\nabla\times\mathbf{J}$ is purely transverse and therefore orthogonal to the longitudinal $\nabla\xi$. The inner product $\nabla\xi\cdot(\nabla\times\mathbf{J})$ vanishes by orthogonality, and the norm channel is gauge invariant for transverse currents within the Lorenz gauge. For longitudinal or mixed current distributions the norm channel retains a residual gauge dependence, and results derived from it are properly stated within the Lorenz gauge with this qualification understood.

$\hat{\sigma}$-channel: transverse currents and the wave equation.

The $\hat{\boldsymbol{\sigma}}$-channel of $\partial(J^T\partial\xi)$ gives

$$\nabla\times\mathbf{S}_\xi+\frac{1}{c}\partial_t\mathbf{T}_\xi=\nabla\rho\times\nabla\xi+\frac{1}{c}(\partial_t\mathbf{J})\times\nabla\xi+\frac{1}{c}\mathbf{J}\times\partial_t\nabla\xi. \tag{186}$$

With $\Box\xi=0$, the temporal and spatial derivatives of ξ are related by the wave equation, so $\partial_t\nabla\xi=\nabla\partial_t\xi$ and the time derivative is constrained by the spatial structure of ξ. Each term in the $\hat{\boldsymbol{\sigma}}$-channel remainder involves a cross product of $\mathbf{J}$ or $\nabla\rho$ with $\nabla\xi$. Since $\nabla\xi$ is longitudinal within the Lorenz gauge, these cross products vanish when $\mathbf{J}$ and $\nabla\rho$ are also longitudinal or when the current is transverse and $\nabla\rho$ is orthogonal to $\nabla\xi$. For conserved transverse currents in a charge-neutral radiation zone, both conditions are satisfied and the $\hat{\boldsymbol{\sigma}}$-channel is gauge invariant. Outside the radiation zone or for non-neutral charge distributions with significant longitudinal components, a residual gauge dependence survives.

$\hat{K}$-channel: orthogonality of transverse and longitudinal modes.

The $\hat{K}$-channel gauge remainder is the most involved, containing the full Euler–Lagrange structure of $\partial(J^T\partial\xi)$. Under charge conservation, all terms involving $\partial_\mu J^\mu$ vanish. Under $\Box\xi=0$, all terms involving $\Box\xi$ vanish. What remains are cross terms mixing spatial derivatives of $\mathbf{J}$ with spatial derivatives of ξ. Within the Lorenz gauge, $\nabla\xi$ is longitudinal and the transverse components of $\mathbf{J}$ are orthogonal to it, so these cross terms vanish by the orthogonality of transverse and longitudinal modes. The $\hat{K}$-channel is therefore gauge invariant for transverse currents under the combined conditions of charge conservation and the Lorenz gauge.

Complete gauge invariance: the physical domain.

The combined conditions of charge conservation $\partial_\mu J^\mu=0$ and the Lorenz gauge $\partial_\mu A^\mu=0$ with residual freedom restricted to $\Box\xi=0$ give the following channel-by-channel result. The time channel is fully gauge invariant under charge conservation alone. The norm, $\hat{\boldsymbol{\sigma}}$, and $\hat{K}$ channels are gauge invariant for transverse, conserved currents within the Lorenz gauge, by the orthogonality of transverse currents and longitudinal gauge shifts. This covers the physically central cases of radiation fields, electromagnetic waves, and the standard coupling of matter currents to the electromagnetic field in the wave zone. The complete gauge invariance of $\partial(J^TA)$ under these conditions can be summarised as

$$\partial_\mu J^\mu=0,\quad \partial_\mu A^\mu=0,\quad \Box\xi=0,\quad \mathbf{J}=\mathbf{J}_\perp\quad\Rightarrow\quad\partial(J^T\partial\xi)=0. \tag{187}$$

For longitudinal currents or mixed current distributions, residual gauge dependence survives in the norm and $\hat{\boldsymbol{\sigma}}$ channels. This is not a deficiency of the BQ framework but a precise algebraic statement of a known physical fact: the electromagnetic angular momentum density $\mathbf{A}\times\mathbf{J}$ and the interaction transport current $\mathbf{S}$

are gauge-dependent objects in general, and their physical interpretation is properly tied to a fixed gauge. The norm channel result of §3.5.1 and the $\hat{\boldsymbol{\sigma}}$-channel result of §3.5.4 are therefore physically meaningful and gauge-invariant within the Lorenz gauge for transverse currents, which is the natural physical setting of the BQ electromagnetic framework established in 3.1. The gauge qualification does not undermine these results; it defines their precise domain of validity, which coincides with the domain in which the framework is designed to operate.

It is worth noting that the four-channel structure of $\partial(J^T A)$ provides a more refined gauge analysis than is possible in the scalar Lagrangian treatment. In the standard formulation, gauge invariance of $J^\mu A_\mu$ is established in one step by charge conservation, and no channel-specific information is available. The BQ decomposition reveals that the time channel is the gauge-invariant core of the interaction, that the space-time channels achieve gauge invariance under the standard physical conditions, and that the spin-norm channels carry the residual gauge sensitivity of the angular momentum and induction structure of the interaction. This channel-specific gauge analysis is a further example of the structural information that the BQ product makes visible and that the scalar treatment leaves undifferentiated.

3.5.7 Physical content of $\partial(J^T A)$ under gauge invariant conditions

A natural question following the gauge analysis of §3.5.6 is whether the combined conditions of charge conservation, the Lorenz gauge, and transverse currents leave sufficient physical content in $\partial(J^T A)$ to justify the four-channel analysis of §3.5.1–§3.5.5. The answer is affirmative, and the reason is instructive: gauge fixing removes precisely the longitudinal, non-radiative, and physically unobservable sector of the interaction structure, leaving intact the transverse, radiative, and topologically meaningful content that constitutes the physically richer part of $\partial(J^T A)$.

The time channel is fully gauge invariant.

The time channel conservation law

$$\nabla \cdot \mathbf{S} + \frac{1}{c}\partial_t \mathcal{L}_{\text{int}} = 0 \tag{188}$$

is gauge invariant under charge conservation alone, independently of the transversality of the current and independently of the Lorenz gauge condition. It loses nothing under gauge fixing. The interaction-energy conservation law of §3.5.2, the identification of $\mathbf{S}$ as the interaction transport current, the connection to the variational foundations of the action, and the separation from the free-field Poynting sector established in §3.5.2 all survive completely and without qualification.

The norm channel retains its most physically significant content.

Under charge conservation and the Lorenz gauge with transverse currents, the part of $\mathbf{A} \times \mathbf{J}$ that survives gauge fixing is the transverse part — precisely the radiative angular momentum interaction density. This is the part that is physically observable: it appears in the angular momentum exchange between radiation fields and matter, in the orbital angular momentum of light, and in the theory of radiation reaction. The longitudinal, Coulomb-like part of $\mathbf{A} \times \mathbf{J}$ that is removed by gauge fixing carries no radiation and no observable angular momentum exchange. The norm channel result of §3.5.1 therefore retains its physically most important content after gauge fixing, and the cross-domain connection between MHD angular momentum balance and the orbital angular momentum of light — identified in §3.5.1 as the central result of the norm channel — is a statement about the transverse, gauge-invariant sector and survives intact. In MHD the fields are solenoidal and the currents are divergence-free, which is precisely the transverse, conserved configuration where gauge invariance holds, so the connection to magnetic helicity conservation, force-free configurations, and the Ohm's-law specialisation of §3.5.1 is unaffected.

The $\hat{K}$-channel Euler–Lagrange structure is preserved.

The momentum balance equation of §3.5.3,

$$\nabla \times (\mathbf{J} \times \mathbf{A}) = \nabla \mathcal{L}_{\text{int}} + \frac{1}{c} \partial_t \mathbf{S}, \tag{189}$$

is gauge invariant for transverse currents in the Lorenz gauge. The full Euler–Lagrange content of the $\hat{K}$-channel — the spatial stationarity condition of the interaction action, the vorticity-pressure balance, and the connection to Beltrami-type interaction configurations — survives gauge fixing without modification. The variational interpretation of §3.5.3, which identified the $\hat{T}$ and $\hat{K}$ channels together as encoding the four-dimensional Euler–Lagrange equations of the interaction action, is therefore fully retained in the gauge-invariant domain.

The $\hat{\sigma}$-channel induction geometry survives in the radiation zone.

The interaction-sector Faraday law

$$\nabla \times \mathbf{S} + \frac{1}{c} \partial_t \mathbf{T} = 0 \tag{190}$$

is gauge invariant for conserved transverse currents in the radiation zone, where the charge distribution is neutral and the current is purely transverse. This is the natural physical setting for radiation fields and electromagnetic wave propagation — precisely the regime where the $\hat{\boldsymbol{\sigma}}$-channel result is most relevant. The induction geometry of the interaction sector, the exact structural parallel with the free-field

Faraday law, and the Helmholtz decomposition of $\mathbf{S}$ across the $\hat{T}$ and $\hat{\boldsymbol{\sigma}}$ channels all survive in this domain. Outside the radiation zone, where longitudinal current components and non-neutral charge distributions are present, a residual gauge dependence in the $\hat{\boldsymbol{\sigma}}$-channel survives; this is a precise statement of the known gauge dependence of the interaction transport current $\mathbf{S}$ in the near-field regime, not a deficiency of the BQ framework.

The equilibrium conditions of §3.5.5 are fully retained.

The two equilibrium conditions identified in §3.5.5 survive gauge fixing intact. The first condition — homogeneous spin-norm channels, nonzero space-time channels — describes driven transverse current systems operating in the radiation and wave propagation regime, which is exactly the domain where gauge invariance holds for all four channels. The second condition — all channels zero, $\partial(J^T A) = 0$ — is a gauge-invariant statement for transverse currents in the Lorenz gauge, since all four channels are gauge invariant there simultaneously. The classification of interaction equilibria in §3.5.5, including the identification of $\partial(J^T A) = 0$ as a constrained variational equilibrium and the analogy with force-free MHD configurations, therefore retains its full physical content under gauge-invariant conditions.

The gauge analysis is constructive, not restrictive.

The gauge discussion of §3.5.6 does something more constructive than simply qualifying the results of §3.5.1–§3.5.5. It identifies the transverse, radiative sector as the gauge-invariant physical core of $\partial(J^T A)$, which is a positive result: the BQ product $\partial(J^T A)$ is the natural algebraic container for the physics of radiative angular momentum exchange, interaction energy conservation, transverse current-potential coupling, and inductive closure — all of which are gauge-invariant and physically observable quantities. The longitudinal sector that carries residual gauge dependence is precisely the sector that carries no radiation, no angular momentum exchange, and no inductive response: it is the least physically interesting part of the interaction structure, and its removal through gauge fixing leaves the physically relevant content undisturbed.

This conclusion can be stated compactly. Under the natural physical conditions of the BQ electromagnetic framework — charge conservation, the Lorenz gauge already imposed in 3.1, transverse currents for the radiative and MHD applications that motivate §3.5 — the product $\partial(J^T A)$ is gauge invariant in all four channels, and the complete four-channel analysis of §3.5.1–§3.5.5 is physically well-founded. The gauge analysis of §3.5.6 does not restrict the domain of §3.5 ; it confirms that the domain of §3.5 is the gauge-invariant domain, and that the structural economy of the BQ framework in producing all four channels simultaneously from one matrix multiplication is operative precisely where the physics is observable, radiative, and topologically meaningful.

3.5.8 The Aharonov–Bohm effect and the physical content of $\partial(J^T A)$ beyond the field level

The gauge analysis of §3.5.6 and the physical content assessment of §3.5.7 together identify the transverse, radiative sector of $\partial(J^T A)$ as its gauge-invariant physical core. A natural question then arises: what physical content survives in $\partial(J^T A)$ when the electromagnetic field itself vanishes identically — that is, when $B = \partial^T A = 0$ but $A \neq 0$? This is precisely the setting of the Aharonov–Bohm (AB) effect Aharonov and Bohm (1959a), and the four-channel structure of $\partial(J^T A)$ provides a natural algebraic framework for its analysis.

The AB setting in BQ language.

The standard magnetic AB setup consists of a charged particle moving in a region where $\mathbf{B} = \nabla\times\mathbf{A} = 0$ and $\mathbf{E} = 0$, so that the electromagnetic field $B = \partial^T A$ vanishes identically in the accessible region. At the local field-strength level $B = \partial^T A$, there is no physics: the field channels $\mathbf{\Omega}_{EM} = \mathbf{B} = 0$ and $\mathbf{\Pi}_{EM} = \mathbf{E}/c = 0$ both vanish, and the Maxwell equations reduce to source-free equations in the field-free region. Yet the particle acquires a measurable phase shift $\Delta\phi = \frac{q}{\hbar}\oint \mathbf{A}\cdot d\mathbf{l}$ when it traverses a closed path encircling a flux tube — a physical effect with no representation at the local field-strength level of the BQ hierarchy.

The stress-energy level $\partial(J^T A) = F$, by contrast, contains A directly through the bilinear $J^T A$, not through its curl. The four-channel decomposition of $J^T A$ established in §3.5 retains non-trivial content even when $B = \partial^T A = 0$, and it is at this level that the AB effect has its natural algebraic home in the BQ hierarchy.

The scalar interaction action as the home of the AB phase.

The magnetic AB phase originates in the interaction action

$$S_{\text{int}} = \int \mathcal{L}_{\text{int}}\, d^4x = \int (\rho_e\phi - \mathbf{J}\cdot\mathbf{A})\, d^4x, \tag{191}$$

where $\mathcal{L}_{\text{int}} = \rho_e\phi - \mathbf{J}\cdot\mathbf{A}$ is the scalar interaction Lagrangian density — the $\hat{1}$/T-channel content of the bilinear $J^T A$ identified in §3.5.2. For a point particle with charge q moving along a path C, this reduces to $\int q(\phi - \mathbf{v}\cdot\mathbf{A})\, dt$, and in the field-free region where $\mathbf{E} = 0$ and $\mathbf{B} = 0$ but $\mathbf{A} \neq 0$, the phase accumulated around a closed path is

$$\Delta\phi = \frac{q}{\hbar}\oint \mathbf{A}\cdot d\mathbf{l} = \frac{q}{\hbar}\int \mathbf{B}\cdot d\mathbf{\Sigma} = \frac{q\Phi}{\hbar}, \tag{192}$$

where Φ is the magnetic flux enclosed. This phase comes directly from the $-\mathbf{J}\cdot\mathbf{A}$ contribution to $\mathcal{L}_{\text{int}}$, i.e. from the scalar interaction action at the stress-energy level, not from any of the vector channels.

The electric AB effect — the phase $\Delta\phi_E = -\frac{q}{\hbar}\int \phi\, dt$ acquired by a particle shielded from the electric field but exposed to a time-varying scalar potential $\phi(t)$ — similarly originates in the $\rho_e\phi$ term of the same $\mathcal{L}_{\text{int}}$. The magnetic and electric AB effects are therefore two limits of the same scalar interaction phase: the magnetic AB phase comes from the $-\mathbf{J}\cdot\mathbf{A}$ term (particle moving in $\mathbf{A}\neq 0$ with $\mathbf{B}=0$) and the electric AB phase from the $\rho_e\phi$ term (particle in $\phi\neq 0$ with $\mathbf{E}=0$). Both are $\hat{1}$/T -channel effects of the stress-energy level, not assignments to separate vector channels. The AB family of topological phases therefore lives in the scalar interaction action $\mathcal{L}_{\text{int}} = \rho_e\phi - \mathbf{J}\cdot\mathbf{A}$, which is the $\hat{1}$-sector of the bilinear $J^T A$ established in §3.5.

The gauge invariance of the AB phases follows directly from the gauge analysis of §3.5.6: the T-channel of $\partial(J^T A)$ is gauge-invariant under charge conservation alone, independently of the Lorenz gauge or transversality conditions. Since both the magnetic and electric AB phases originate in $\mathcal{L}_{\text{int}}$ — the T-channel content — their gauge invariance is guaranteed by charge conservation alone, without requiring the additional transversality conditions needed for the vector channels.

The vector channels as companions to the AB phase.

The vector channels of $J^T A$ carry physical content that is distinct from the AB phases but related to the same potential A in the field-free region.

The K-channel $\mathbf{S} = \mathbf{J}\times\mathbf{A}$ is the electromagnetic angular momentum density identified in §3.5.1. It is non-zero when $\mathbf{A}\neq 0$ even though $\mathbf{B}=0$. Its line integral $\oint(\mathbf{J}\times\mathbf{A})\cdot d\mathbf{l}$ may be interpreted as an angular-momentum transport companion to the AB phase, associated with the path encircling the flux tube, though its precise physical interpretation in the AB setting requires further analysis beyond what is established here. Its gauge invariance for transverse currents in the Lorenz gauge follows from §3.5.6.

The $\hat{\boldsymbol{\sigma}}$-channel content $\mathbf{P}_{\text{int}} = \rho_e c\mathbf{A} - \frac{\phi}{c}\mathbf{J}$ carries the relativistic energy-transport structure of the interaction established in §3.5.4. In the field-free AB region with a static particle, this reduces to $\rho_e c\mathbf{A}$ — the electromagnetic canonical momentum density, the quantity that appears in the minimal coupling of quantum mechanics through $\mathbf{p}_{\text{can}} = m\mathbf{v} + q\mathbf{A}$. The $\hat{\boldsymbol{\sigma}}$-channel is therefore the BQ container for the classical canonical momentum, whose non-trivial value in the field-free region ($q\mathbf{A}\neq 0$ even though $\mathbf{B}=0$) is what makes the AB effect possible upon quantization. It is not itself the AB phase but the classical precursor from which the AB phase emerges when the particle is treated as a quantum wave.

The Aharonov–Casher effect as a dual extension.

The Aharonov–Casher (AC) effect Aharonov and Casher (1984) — the geometric phase acquired by a neutral magnetic dipole $\boldsymbol{\mu}$ moving in an electric field $\mathbf{E}$, with phase $\Delta\phi_{AC} = \frac{1}{\hbar c^2}\oint(\boldsymbol{\mu}\times\mathbf{E})\cdot d\mathbf{l}$ — is related to the AB effect by electromagnetic duality. In the BQ framework, the AC effect is not a direct channel projection of $\partial(J^T A)$ with the standard electromagnetic current J^μ, because the dipole coupling requires replacing J^μ by an effective current J^μ_{eff} encoding the magnetic dipole moment. Under this substitution, the AC phase appears in the scalar interaction action of the resulting $J^T_{\text{eff}}A$ through the same $\hat{1}$/T-channel mechanism as the standard AB phases. The AC effect is therefore best understood as a dual extension of the AB framework to effective dipole currents, rather than as a direct channel projection of $\partial(J^T A)$ with the standard current. Its character as an action-level topological geometric phase — not an energy accumulation — is consistent with the identification of $\mathcal{L}_{\text{int}}$ as the action-level content of the $\hat{1}$-sector of $J^T A$.

Channel classification of AB-type effects.

The four-channel classification of AB-type topological phases within the BQ framework is:

Channel	Physical content in AB setting	Gauge invariance
$\hat{1}$/T	Scalar interaction action $\mathcal{L}_{\text{int}} = \rho_e\phi - \mathbf{J}\cdot\mathbf{A}$; magnetic AB ($-\mathbf{J}\cdot\mathbf{A}$ term) and electric AB ($\rho_e\phi$ term) phases	Charge conservation alone
$\hat{K}$	Angular momentum density $\mathbf{J}\times\mathbf{A}$; angular-momentum transport companion to AB	Lorenz gauge, transverse J
$\hat{\boldsymbol{\sigma}}$	Canonical momentum density $\rho_e c\mathbf{A} - \frac{\phi}{c}\mathbf{J}$; classical precursor of quantum AB coupling	Lorenz gauge, transverse J
AC effect	Dual extension via effective dipole coupling; not a direct $J^T A$ channel projection	Same as $\hat{1}$/T under duality

The key organizational result is that both the magnetic and electric AB phases live in the same $\hat{1}$/T-channel of the stress-energy level — the scalar interaction action — and are gauge-invariant under charge conservation alone. The vector channels carry related but distinct physics: the K-channel carries an angular-momentum companion and the $\hat{\boldsymbol{\sigma}}$-channel carries the canonical momentum structure, both of which are classical precursors of quantum phenomena that become observable only through the scalar interaction action at the $\hat{1}$/T level.

The central observation.

The physical content of this subsection can be stated in one sentence:

$$\boxed{\text{AB phases appear because } J^T A \text{ retains potential information that } \partial^T A \text{ discards.}} \tag{193}$$

The field level $B = \partial^T A$ depends only on derivatives of A — it encodes the local field strength and discards the scalar interaction content of A through differentiation. The stress-energy level $J^T A$ depends on A itself — it retains the full potential, including its global holonomy structure $\oint A_\mu dx^\mu$, through the bilinear coupling to J. The AB effect is the experimental proof that this retained information is physically real: the local field strength vanishes, but the global holonomy — which lives at the stress-energy level, not the local field-strength level — produces a measurable phase. The four levels of the BQ hierarchy are therefore not redundant: each level retains physical information that the others discard, and the AB effect is the most direct experimental evidence that the stress-energy level and the local field-strength level carry genuinely independent physical content.

The AB effect and the BQ hierarchy.

The AB effect has been interpreted since its discovery as evidence that the four-potential A^μ carries more physical information than the field tensor $F^{\mu\nu}$ alone. In the BQ hierarchy, this interpretation is refined: it is not that A^μ is more fundamental than $F^{\mu\nu}$, but that the stress-energy level $\partial(J^T A)$, which retains A directly through the bilinear, and the field level $B = \partial^T A$, which retains only local derivatives of A, carry independent physical information. The AB effect demonstrates the independence of the stress-energy level; it does not establish its primacy.

The BQ four-channel decomposition does not explain the AB effect — that requires quantum mechanics and the wave nature of the particle — but it identifies the precise algebraic level and channel at which the classical precursor of the AB effect resides. The precursor is the gauge-invariant, charge-conservation -protected scalar interaction action $\mathcal{L}_{\text{int}} = \rho_e \phi - \mathbf{J} \cdot \mathbf{A}$, the $\hat{1}$-sector of $J^T A$, which retains the global holonomy of A even when the local field strength $\partial^T A$ vanishes. This is consistent with the conclusion of §3.5.7 that gauge fixing removes the unphysical longitudinal sector while leaving the transverse, topological, and radiative content intact: the AB phases are precisely the topological content of the scalar sector of $J^T A$ that survives gauge fixing when $\partial^T A = 0$, and that has no representation at the local field-strength level of the hierarchy. The BQ hierarchy makes this separation between local field-strength information and global potential information algebraically explicit through the distinction between the field level $B = \partial^T A$ and the stress-energy level $\partial(J^T A)$ — a distinction that is invisible in the scalar Lagrangian treatment but is a structural feature of the four-level BQ hierarchy.

3.6 The three products as a complete algebraic formulation of classical electrodynamics

The results established across §3.3–§3.5 allow a collective assessment of what the three fundamental BQ products

$$\partial B = \mu_0 J, \qquad JB = F, \qquad \partial(J^T A) = F \tag{194}$$

achieve together. Each product is significant individually, but their collective structure says something that none of them says alone, and it is this collective structure that constitutes the principal claim of the BQ electromagnetic framework.

The logical structure of the triad.

The three products are not independent constructions assembled after the fact. They form a closed and self-consistent triad with a precise logical architecture. The product $\partial B = \mu_0 J$ is the field-source equation: it takes the derivative of the electromagnetic field and returns the current four-vector as its source. The product $JB = F$ is the field-matter equation: it takes the current and the field and returns the four-force as the back-action of the field on the matter. The product $\partial(J^T A) = F$ is the interaction equation: it takes the derivative of the source-potential coupling and returns the complete internal structure of the current-potential interaction as its output.

The three products share the same algebraic objects B, J, A, ∂, and F, and they are related through the foundational identities of the framework: $B = \partial^T A$ connects the field to the potential, $J = \rho V$ connects the source to the current four-vector, and the force F that appears as the output of JB is the same F that governs the driven dynamics of $\partial(J^T A)$. The triad is algebraically coherent in a way that is invisible when the three results are derived separately by standard methods, because the standard methods assemble them from different variational, differential-geometric, and mechanical arguments with no common algebraic ancestor.

The first confident claim: algebraic closure of classical electrodynamics.

The three products cover, between them, every physically significant object in classical electrodynamics. From $\partial B = \mu_0 J$ emerge the Maxwell equations in all four channels, the Lorentz covariance of the field equations, and the topological constraints on the free field. From $JB = F$ emerge the Lorentz four-force, the Lorentz power density, the MHD helicity scalar, and the Abraham momentum density. From $\partial(J^T A) = F$ emerge the interaction Lagrangian density and its conservation law, the interaction transport current, the electromagnetic angular momentum balance, the Euler–Lagrange equations of the interaction action, and the inductive closure conditions of the current-potential coupling. No external construction is required beyond the BQ multiplication rule: no separate variational argument, no additional tensor structure, no differential forms, no explicit index gymnastics. It can be claimed with confidence that the BQ-Pauli algebra $M_2(\mathbb{C})$ closes the classical electromagnetic system completely at the level of one algebraic language and three products.

The second confident claim: universal four-channel architecture.

In every one of the three products the same four-channel split holds without exception. The $\hat{T}$ and $\hat{K}$ channels carry the sourced, space-time dynamics; the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels carry the unsourced, topological, closure conditions. In $\partial B = \mu_0 J$ the $\hat{T}$ and $\hat{K}$ channels give Gauss's law and the Ampère–Maxwell law, while the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels give the divergence-free magnetic condition and Faraday's law. In $JB = F$ the $\hat{T}$ and $\hat{K}$ channels give the Lorentz power density and the Lorentz force density, while the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels give the MHD scalar and the Abraham momentum term. In $\partial(J^T A) = F$ the $\hat{T}$ and $\hat{K}$ channels give the interaction energy conservation law and the Euler–Lagrange momentum balance, while the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels give the angular momentum divergence and the interaction-sector Faraday law.

This four-channel split between sourced space-time content and unsourced spin-norm content is not a coincidence or a notational artifact. It is a structural property of the BQ algebra that holds universally across all three products. The minquat $(\hat{T}, \hat{K})$ sector systematically carries the dynamical, source-driven physics; the pauliquat $(\hat{1}, \hat{\boldsymbol{\sigma}})$ sector systematically carries the geometric, topological, and inductive physics. This universality is a confident and non-trivial structural result of the framework.

The third confident claim: the physical indispensability of the spin-norm sector.

The pauliquat $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels — the spin-norm sector that has no counterpart in standard Minkowski four-vector geometry — carry physically significant and non-trivial content in all three products. In $\partial B = \mu_0 J$ the spin-norm sector carries Faraday's law and the magnetic divergence condition: the topological, geometric constraints on the free electromagnetic field. In $JB = F$ it carries the MHD helicity scalar and the Abraham momentum: the plasma-physical and relativistic extensions of the Lorentz force. In $\partial(J^T A) = F$ it carries the electromagnetic angular momentum balance and the interaction-sector Faraday law: the topological closure conditions of the current-potential coupling.

The spin-norm sector is not empty, not formal, and not secondary in any of the three products. It is the algebraic home of the geometric and topological layer of classical electrodynamics — the layer that governs field-line topology, magnetic helicity, angular momentum exchange, and inductive closure. The claim that the pauliquat dimensions of the BQ algebra are physically indispensable, and not merely algebraic scaffolding, is supported by all three products simultaneously and can be made with confidence.

The fourth confident claim: cross-domain unification invisible in the standard treatment.

The channel decompositions of the three products reveal cross-domain connections between results that belong to separate subdisciplines in the standard treatment and have no algebraic relationship there. The norm channel of $\partial(J^T A)$ places $\mathbf{J} \cdot \mathbf{B}$ from MHD and $\mathbf{A} \times \mathbf{J}$ from photon angular momentum theory into the same divergence identity as two terms of one scalar expression. The $\hat{\boldsymbol{\sigma}}$-channel of $JB = F$ places the Abraham momentum density from relativistic electrodynamics alongside the MHD plasma force in the same algebraic slot. The $\hat{\boldsymbol{\sigma}}$-channel of $\partial(J^T A)$ produces an interaction-sector Faraday law that is structurally identical to the free-field Faraday law of $\partial B = \mu_0 J$, with $(\mathbf{S}, \mathbf{T})$ playing the roles of $(\mathbf{E}, \mathbf{B})$. The $\hat{T}$ and $\hat{K}$ channels of $\partial(J^T A)$ together encode the full four-dimensional Euler–Lagrange equations of the interaction action, without a variational argument and with automatic Lorentz covariance.

None of these connections is visible in the standard tensor formulation, because the standard formulation has no mechanism for separating the four channels simultaneously across all three products. The BQ framework produces these separations automatically, and the connections they reveal are a confident result of the analysis.

The fifth confident claim: structural completeness beyond notational economy.

The collective result of the three products is strictly more informative than the standard tensor formulation, in the precise sense that it reveals structure that the standard formulation systematically suppresses. The standard formulation assembles the Maxwell equations, the Lorentz force, and the interaction Lagrangian from separate arguments in separate subdisciplines, with no common algebraic ancestor and no mechanism for comparing their internal structures. The BQ framework produces all three from the same multiplication rule, with the same four-channel architecture, the same Lorentz covariance proof, and the same gauge-invariance conditions. The channel assignments are the same across all three products, the spin-norm sector plays the same structural role in all three, and the Helmholtz decompositions of the interaction quantities $\mathbf{T}$ and $\mathbf{S}$ are delivered automatically across the four channels of $\partial(J^T A)$ without external vector-analytic machinery.

This structural completeness goes beyond notational economy. A more compact notation that re-derives known equations is a convenience. A framework that reveals the four-channel anatomy of every product, the universal role of the spin-norm sector, the cross-domain connections between separated subdisciplines, and the complete Helmholtz decomposition of the interaction quantities as automatic outputs of matrix multiplication is a structural insight. The distinction matters for the assessment of the BQ program: the claim is not that the algebra is shorter, but that it is deeper.

What cannot yet be claimed.

Intellectual honesty requires acknowledging the frontier of the result. It cannot yet be claimed that the three products produce predictions that differ from standard electrodynamics in any measurable regime. Every individual equation derivable from the three products can in principle be derived by standard methods. The passage from structural insight to new prediction depends on the cross-channel dynamics — on treating all four outputs of each product simultaneously as a coupled system rather than projecting onto individual channels — and this coupled dynamics has not yet been fully developed. The mixed-channel conditions identified in §3.5.5, the gauge-invariant domain established in §3.5.6–§3.5.7, and the interaction-sector Maxwell architecture revealed in §3.5.4 are the natural starting points for that development. The present analysis establishes the structural foundation; the predictive extensions remain open.

Summary.

What can be claimed with confidence is the following. The three products $\partial B = \mu_0 J$, $JB = F$, and $\partial(J^T A) = F$, computed within the BQ-Pauli algebra $M_2(\mathbb{C})$, constitute a complete, closed, and self-consistent algebraic formulation of classical electrodynamics in which every physically significant quantity in the theory — field, source, force, interaction energy, interaction momentum, angular momentum, inductive geometry, and their conservation laws — emerges as a channel output of one of three matrix multiplications, with universal four-channel architecture, automatic Lorentz covariance, gauge invariance in the physical domain of transverse conserved currents within the Lorenz gauge, and cross-domain connections between results that are invisible in the standard treatment. The internal anatomy of the classical electromagnetic interaction — its sourced space-time dynamics, its unsourced topological closure, its angular momentum geometry, its variational structure, and its inductive hierarchy — is fully visible in the BQ channel decomposition and systematically hidden in the standard scalar and tensor treatment. This is the principal structural claim of the BQ electromagnetic framework, and the analysis of §3.3–§3.5 establishes it with confidence.

3.7 Relation to Prior Biquaternion Formulations and to Geometric Algebra

3.7.1 Engagement with the Prior Biquaternion Literature

The biquaternion formulation of electrodynamics has a long history, and the present approach must be positioned precisely within it. The most important predecessors are Silberstein Silberstein (1907), Conway Conway (1911), and Lanczos Lanczos (1929), together with the comprehensive bibliographic surveys of Gsponer and Hurni Gsponer and Hurni (2005a,b).

Relation to Silberstein

Silberstein Silberstein (1907) introduced the complex quaternion $\mathbf{r} = ict\,\hat{\mathbf{1}} + x\hat{I} + y\hat{J} + z\hat{K}$, placing the imaginary unit i into the *time coordinate*. As a consequence, after every quaternion multiplication the factor i migrates into the spatial components and must be tracked explicitly. To illustrate this, consider the quadratic $\mathbf{r}^T\mathbf{r}$ in Silberstein's convention. The time-component contributes $(ict)^2 = -c^2t^2$, so the correct Minkowski interval emerges only after carefully following the sign produced by $i^2 = -1$ through the calculation.

In the present framework the imaginary unit is absorbed once and for all into the basis element $\hat{\mathrm{T}} = i\,\hat{1}$, so all coordinates remain real: $R^\mu = (ct, r_1, r_2, r_3) \in \mathbb{R}^4$. After any matrix multiplication the real/imaginary character of each output term is determined by the basis element it multiplies, not by a coordinate factor that must be traced. The quadratic then reads directly

$$R^T R = (c^2t^2 - \mathbf{r}^2)\,\hat{1}, \tag{195}$$

with no bookkeeping of a migrating imaginary unit required. This is not a cosmetic difference: in multi-step products such as $\partial(\partial^T A)$ or $\partial(J^T A)$ the coordinate-imaginary convention of Silberstein would require tracking i through two successive multiplications, whereas the basis-imaginary convention of the present paper reduces each step to straightforward real-matrix arithmetic. The channel decomposition of those products is then readable directly from which basis element ($\hat{1}$, $\hat{\mathrm{T}}$, $\hat{\mathbf{K}}$, $\hat{\boldsymbol{\sigma}}$) multiplies each term, without any intermediate re-identification step.

Relation to Conway

Conway Conway (1911) was the first to write Maxwell's equations as a single quaternion equation, but his formulation required *two* separate quaternion equations to capture all four Maxwell equations: one for the inhomogeneous pair ($\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ and $\nabla \times \mathbf{B} - \partial_t \mathbf{E}/c^2 = \mu_0 \mathbf{J}$) and one for the homogeneous pair ($\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$). In the present framework all four Maxwell equations are contained in the single product $\partial B = \mu_0 J$, expanded as

$$\begin{aligned}\partial B &= -(\nabla \cdot \boldsymbol{B})\,\hat{1} + \frac{1}{c}(\nabla \cdot \boldsymbol{E})\,\hat{\mathrm{T}} + \left(\nabla \times \boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E}\right) \cdot \hat{\mathbf{K}} + \frac{1}{c}(\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B}) \cdot \hat{\boldsymbol{\sigma}} \\ &= 0 \cdot \hat{1} + \mu_0 c \rho_0\,\hat{\mathrm{T}} + \mu_0 \boldsymbol{J} \cdot \hat{\mathbf{K}} + 0 \cdot \hat{\boldsymbol{\sigma}} = \mu_0 J,\end{aligned} \tag{196}$$

where the homogeneous Maxwell equations occupy the spin channel ($\hat{\boldsymbol{\sigma}}$) and the inhomogeneous equations occupy the time–space channels ($\hat{\mathrm{T}}$, $\hat{\mathbf{K}}$), all as automatic outputs of one algebraic product. The splitting of Conway's two equations into one is therefore not a notational compression but a structural consequence of the dual-basis architecture: the minquat sector $(\hat{\mathrm{T}}, \hat{\mathbf{K}})$ and the pauliquat sector $(\hat{1}, \hat{\boldsymbol{\sigma}})$ together span the full biquaternion space, and the product ∂B populates both sectors simultaneously.

Relation to Lanczos and the Riemann–Silberstein Tradition

Lanczos reformulated the Dirac equation in biquaternion language by working backward from the known matrix equation Lanczos (1929). The present approach proceeds in the opposite direction: the Pauli-level biquaternion structure is built first, from which the Dirac framework emerges as a natural $M_4(\mathbb{C})$ doubling, see Part II. The Pauli-level EM construction of the present paper therefore serves as the foundation from which the Lanczos-type reformulation will follow, rather than as a re-derivation of it.

The Riemann–Silberstein field vector $\mathbf{B} := \mathbf{B} - i\mathbf{E}/c$, credited to Minkowski (1910), appears in the present paper as Eq. (82). In the Riemann–Silberstein literature this object is typically *postulated* as a convenient complex combination of fields. In the present framework it is *derived* as the natural output of the product $\partial^T A$ under the Lorenz gauge, with no separate postulation required. The derivational status of $\mathbf{B}$ within the BQ algebra is therefore a structural result, not a notational choice.

3.7.2 Automatic Channel Production versus Grade Bookkeeping in Geometric Algebra

The Spacetime Algebra (STA) of Hestenes, Hestenes (1966), and the geometric algebra formulation of Doran and Lasenby, Doran and Lasenby (2003a), provide the most technically developed modern parallel to the present approach. The electromagnetic field in STA is the grade-2 multivector $\mathbf{F} = \mathbf{E} + I\mathbf{B}$, where I is the unit pseudoscalar of $Cl(1,3)$. Maxwell's equations are written compactly as

$$\partial\mathbf{F} = \mathbf{J}, \tag{197}$$

which encodes a sum of two grade-distinct objects,

$$\partial\mathbf{F} = \partial \cdot \mathbf{F} + \partial \wedge \mathbf{F}, \tag{198}$$

where $\partial \cdot \mathbf{F}$ is grade-1 and $\partial \wedge \mathbf{F}$ is grade-3. To extract the physics the user must apply grade projection operators explicitly:

$$\langle\partial\mathbf{F}\rangle_1 = \text{inhomogeneous Maxwell equations}, \tag{199}$$

$$\langle\partial\mathbf{F}\rangle_3 = I^{-1} \cdot (\text{homogeneous Maxwell equations}). \tag{200}$$

The grade-3 part must be dualized by multiplication with I^{-1} before Faraday's law and $\nabla \cdot \mathbf{B} = 0$ can be read off. The workflow is therefore: *multiply, then grade-select, then dualize*. The physical content is not directly visible in the product; it requires two post-hoc operations.

In the present BQ framework the analogous product ∂B is

$$\partial B = -(\nabla \cdot \boldsymbol{B})\,\hat{1} + \frac{1}{c}(\nabla \cdot \boldsymbol{E})\,\hat{\mathrm{T}} + \left(\nabla \times \boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E}\right) \cdot \hat{\mathbf{K}} + \frac{1}{c}(\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B}) \cdot \hat{\boldsymbol{\sigma}}.$$

The four Maxwell equations appear as coefficients of the four basis elements $\hat{1}$, $\hat{\mathrm{T}}$, $\hat{\mathbf{K}}$, $\hat{\boldsymbol{\sigma}}$. No grade projector is applied. No duality transformation is required. The user reads off the physics by inspecting which basis element multiplies each term. The channel decomposition and the algebraic product are *the same operation*.

This structural difference persists for the Lorentz force. In STA the inner product $\mathbf{J} \cdot \mathbf{F}$ yields the force as a grade-1 object, but the MHD pressure scalar $\mathbf{J} \cdot \mathbf{B}$ and the relativistic plasma term proportional to $\mathbf{J} \times \mathbf{E}$ are not naturally separated without further grade analysis. In the present framework the product JB yields

$$JB = -(\boldsymbol{J} \cdot \boldsymbol{B})\,\hat{1} + \frac{1}{c}(\boldsymbol{J} \cdot \boldsymbol{E})\,\hat{\mathrm{T}} + (\boldsymbol{J} \times \boldsymbol{B} + \rho_0 \boldsymbol{E}) \cdot \hat{\mathbf{K}} + \left(\frac{1}{c}\boldsymbol{J} \times \boldsymbol{E} - c\rho_0 \boldsymbol{B}\right) \cdot \hat{\boldsymbol{\sigma}},$$

so the MHD scalar occupies the $\hat{1}$ channel and the plasma term occupies the $\hat{\boldsymbol{\sigma}}$ channel as immediate, labeled outputs of the single product, without any further projection.

The deeper reason for this behaviour is the non-closure of the space-time basis $(\hat{\mathrm{T}}, \hat{\mathbf{K}})$ under multiplication. As noted in Section 2.3, the product of two minquats generates both minquat components and pauliquat components. In STA terms, this is the appearance of grade-0 and grade-2 terms alongside grade-1 and grade-3 terms in the same product. In the present framework this non-closure is a *feature*: the leakage of multiplication into the complementary basis sector $(\hat{1}, \hat{\boldsymbol{\sigma}})$ is exactly what distributes the physics across all four channels automatically, with each channel labelled by a distinct basis element.

The trade-off should be stated honestly. STA's grade structure is representation-independent in a strong sense: the pseudoscalar I, the grade projectors $\langle \cdot \rangle_k$, and the inner and outer products are defined independently of any matrix realisation of $Cl(1,3)$. The BQ channel decomposition is tied to the specific matrix representation established in Section 2.2 and to the convention $\hat{\mathrm{T}} = i\,\hat{1}$. The price of automatic channel production is therefore a more representation-dependent formalism.

For the specific program of this paper — unifying the three electromagnetic equations $\partial B = \mu_0 J$, $JB = F$, and $\partial(J^T A)$ within a single algebraic hierarchy, the automatic channel structure pays its way. The minquat sector $(\hat{\mathrm{T}}, \hat{\mathbf{K}})$ and the pauliquat sector $(\hat{1}, \hat{\boldsymbol{\sigma}})$ correspond directly to physically interpretable content (space-time versus spin-norm), whereas STA grades do not carry this physical labelling intrinsically but must acquire it through interpretation after grade selection. In this sense the BQ channel architecture trades formal basis-independence for direct physical readability of algebraic products.

3.7.3 Literature Context for the $\partial(J^T A)$ Equation and its Channel Structure

The product $J^T A$ and its derivative $\partial(J^T A)$ introduced in Section 3.5 have no direct precedent as a unified four-channel object in the prior electromagnetic literature. This subsection traces the partial analogues that exist for each channel individually, establishes precisely what the present formulation contributes beyond those analogues, and connects the results of §3.5.1–§3.5.7 to the relevant prior work in electromagnetic helicity theory, angular momentum conservation, interaction Lagrangian theory, and magnetohydrodynamics.

The whole product $J^T A$ as a biquaternion bilinear

The scalar part of $J^T A$, namely the norm channel

$$L = \rho\phi - \mathbf{J} \cdot \mathbf{A}, \tag{201}$$

is the standard electromagnetic interaction Lagrangian density $J_\mu A^\mu$, which appears in every relativistic treatment of electrodynamics as the coupling term added to the free-field Lagrangian Pauli (1958); Jackson (1999). In the standard treatment this scalar is the entire object of interest: the remaining content of $J^T A$ is not extracted, and the product is treated as a single number rather than as a structured algebraic object.

In the BQ framework, $J^T A$ is a bilinear of the same algebraic type as $\partial^T A$, and therefore automatically generates four channels $(\hat{1}, \hat{T}, \hat{K}, \hat{\boldsymbol{\sigma}})$ as labelled outputs of a single matrix product. The $\hat{K}$-channel $\mathbf{T} = \mathbf{J}\times\mathbf{A}$ and the $\hat{\boldsymbol{\sigma}}$-channel $\mathbf{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}$ appear alongside L as co-equal components of the same object. No standard tensor or differential-form treatment extracts these three alongside L from a single product operation. The four-channel presentation of $J^T A$ is a structural feature specific to the BQ algebra, and the demonstration in §3.5 that $\partial(J^T A)$ generates the complete Helmholtz decompositions of both $\mathbf{T}$ and $\mathbf{S}$, the full four-dimensional Euler–Lagrange equations of the interaction action, and an interaction-sector Maxwell architecture — all from one matrix multiplication — has no counterpart in the prior literature.

Norm channel: $\nabla \cdot (\mathbf{A} \times \mathbf{J})$

The quantity $\mathbf{A} \times \mathbf{J}$ has the dimensions of an electromagnetic angular momentum density and appears in the angular-momentum exchange literature in precisely this form. It is the interaction term in the electromagnetic angular momentum density, appearing in the Röntgen interaction Hamiltonian Sonnleitner and Barnett (2017) and in the theory of orbital angular momentum transfer between radiation fields and matter Allen et al. (1992); van Enk and Nienhuis (1994). The divergence $\nabla \cdot (\mathbf{A} \times \mathbf{J})$ measures the local source or sink of this rotational interaction flow.

The closest published parallel for the norm channel content is in the electromagnetic helicity literature. Bliokh, Bekshaev, and Nori Bliokh et al. (2013) derive a helicity continuity equation of the form $\partial_t(\mathbf{A} \cdot \mathbf{B}) + \nabla \cdot (\phi\mathbf{B} + \mathbf{E} \times \mathbf{A}) = 0$ in ideal source-free electromagnetism. Khosravi et al. Khosravi et al. (2024) derive local angular momentum conservation laws from the full QED Lagrangian using the Noether theorem and introduce a helicity current density tensor with components containing $\mathbf{A}_\perp \cdot \mathbf{B}$ and $\mathbf{J} \times \mathbf{A}$-type terms. Their analysis is the closest specific precedent for the norm-channel content, but it arises in a QED context from the Dirac–Maxwell Lagrangian, not as one labelled output of a BQ product of classical four-vectors.

The present paper adds three elements beyond these precedents. First, the vector identity decomposition $\nabla \cdot (\mathbf{A} \times \mathbf{J}) = \mathbf{J} \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{J})$ places $\mathbf{J} \cdot \mathbf{B}$ from magnetohydrodynamics and $\mathbf{A} \times \mathbf{J}$ from photon angular momentum theory into the same divergence identity as two terms of one channel expression — a cross-domain connection not present in either the MHD or the photon angular momentum literature. Second, the Ohm's-law specialisation $F_1 = \sigma\mathbf{A} \cdot \partial_t\mathbf{B} + \mathbf{J} \cdot \mathbf{B}$ connects the norm channel directly to the tearing mode drive in resistive MHD Furth et al. (1963), providing a single scalar that captures both the resistive and force-free contributions to current-driven instability. Third, the identification of $F_1 = 0$ with the Taylor relaxation condition Taylor (1974) derives the force-free equilibrium as a channel condition rather than as a minimum-energy postulate. None of these identifications appears in the prior helicity or angular momentum literature.

Time channel: $\nabla \cdot \mathbf{S} - \frac{1}{c}\partial_t L$

The time channel

$$\nabla \cdot \left(c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}\right) - \frac{1}{c}\partial_t(\rho\phi - \mathbf{J} \cdot \mathbf{A}) \tag{202}$$

has the form of a continuity equation for the interaction Lagrangian density $L = \rho\phi - \mathbf{J} \cdot \mathbf{A}$, with transport vector $\mathbf{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}$.

The two terms comprising $\mathbf{S}$ appear separately in the literature. The density $\rho\mathbf{A}$ — charge density times vector potential — is related to the canonical momentum density and has been discussed since Abraham and Minkowski in the context of electromagnetic momentum in media Griffiths (2012). The term $\frac{1}{c^2}\phi\mathbf{J}$ is a current weighted by the scalar potential, which appears in discussions of electromagnetic energy transport in driven antenna systems. However, their combination $\mathbf{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}$ — the interaction transport current that measures the transport imbalance between the vector-potential and scalar-potential coupling energies — does not appear as a named or studied object in the standard literature.

The present paper identifies $\mathbf{S}$ as the natural transport companion of $\mathcal{L}_{\text{int}}$ in the time channel, shows that it is the interaction-energy analogue of the Poynting vector for the current-potential sector, and demonstrates that the complete time-channel structure $\nabla\cdot\mathbf{S}+\frac{1}{c}\partial_t\mathcal{L}_{\text{int}} = 0$ is the interaction-sector analogue of Poynting's theorem — separating interaction energy conservation from free-field energy conservation as outputs of two different algebraic products ($B^T B$ and $\partial(J^T A)$ respectively) rather than combining them in a single Poynting argument. This separation, and the identification of $\mathbf{S}$ as its interaction-sector transport vector, appear to be without direct precedent.

The connection to the variational foundations of electrodynamics is also new. The identification of the time channel as the local expression of the conservation law associated with the interaction part of the action $S = \int(\mathcal{L}_{\text{field}} + \mathcal{L}_{\text{int}})d^4x$ — with the $\hat{T}$ and $\hat{K}$ channels together encoding the full four-dimensional Euler–Lagrange equations of the interaction Lagrangian — provides a derivation of the Euler–Lagrange structure from algebraic channel decomposition rather than from a separate variational argument. This connection is specific to the BQ product and has no standard-treatment counterpart.

Space channel: $\nabla\times\mathbf{T}+\nabla L-\frac{1}{c}\partial_t\mathbf{S}$

The space channel combines three contributions: the curl of the angular momentum interaction density $\mathbf{T} = \mathbf{J}\times\mathbf{A}$, the gradient of the interaction Lagrangian density L, and the time derivative of the transport vector $\mathbf{S}$. As established in §3.5.3, the combination $-\nabla\mathcal{L}_{\text{int}} - \frac{1}{c}\partial_t\mathbf{S}$ is the spatial Euler–Lagrange derivative of the interaction action, making the space channel the spatial stationarity condition of the interaction Lagrangian. Together with the time channel, the $\hat{T}$ and $\hat{K}$ channels of $\partial(J^T A)$ encode the complete four-dimensional Euler–Lagrange equations of the current-potential interaction.

The first term $\nabla\times(\mathbf{J}\times\mathbf{A})$ appears sporadically in vector-identity manipulations in MHD Biskamp (2000) and in the vorticity analysis of current-driven instabilities, but not as part of a four-component conservation structure. The identification of the space channel as the interaction-sector analogue of the vorticity-pressure balance in continuum mechanics — with the Beltrami-type equilibrium condition $\nabla\times(\mathbf{J}\times\mathbf{A}) = \nabla\mathcal{L}_{\text{int}}$ as its static limit — connects the BQ space channel to the classical theory of force-free fields Chandrasekhar and Kendall (1957) and to the intrinsic rotation problem in tokamak plasmas DeGrassie (2009); Parra et al. (2011), but this connection is not present in any of those sources. The combination of all three contributions as the $\hat{K}$-component of a single BQ derivative, with its identification as the Euler–Lagrange spatial equation and its connection to the

Helmholtz decomposition of $\mathbf{T}$ across the $\hat{1}$ and $\hat{K}$ channels, does not appear in the prior literature.

Spin channel: $\nabla \times \mathbf{S} + \frac{1}{c}\partial_t \mathbf{T}$

The spin channel

$$\nabla \times \left(c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J} \right) + \frac{1}{c}\partial_t(\mathbf{J} \times \mathbf{A}) \tag{203}$$

has the structural form of a Faraday-like induction equation for the interaction quantities $\mathbf{S}$ and $\mathbf{T}$, in exact structural analogy with $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$. The correspondence is term by term and sign by sign: $\mathbf{S}$ plays the role of $\mathbf{E}$ and $\mathbf{T}$ plays the role of $\mathbf{B}$.

The transport vector $\mathbf{S} = c\rho\mathbf{A} - \frac{1}{c}\phi\mathbf{J}$ is structurally related to the Röntgen interaction term Sonnleitner and Barnett (2017), which appears in the multipolar coupling of atoms to electromagnetic fields in the electric dipole approximation and involves cross-couplings between the current density and vector potential of precisely the kind found in $\mathbf{S}$. However, the Röntgen term is treated in the context of atom-field coupling, not as one channel of a four-channel derivative of a four-potential bilinear. The connection is structural rather than direct.

The present paper adds three elements beyond this precedent. First, the exact identification of the $\hat{\sigma}$-channel as the interaction-sector Faraday law — with the same channel assignment as the free-field Faraday law in $\partial B = \mu_0 J$ — establishes a precise algebraic parallel between the free-field and interaction sectors that is not present in the Röntgen-term literature. Second, the identification of the $\hat{\sigma}$-channel irrotationality condition $\nabla \times \mathbf{S} = 0$ as the local algebraic expression of the global topological non-triviality of the Aharonov–Bohm effect Aharonov and Bohm (1959b) — and its violation $F_\sigma \neq 0$ as a precursor to magnetic reconnection in plasma physics Biskamp (2000) — connects the spin channel to two physically significant phenomena that have no prior algebraic relationship. Third, together with the time channel, the $\hat{T}$ and $\hat{\sigma}$ channels provide the complete Helmholtz decomposition of $\mathbf{S}$ — divergence from the time channel, curl from the $\hat{\sigma}$-channel — as simultaneous outputs of one product, a result that has no counterpart in the prior literature.

The three-product triad and its collective significance

Beyond the individual channel results, the present paper establishes the three-product triad

$$\partial B = \mu_0 J, \qquad JB = F, \qquad \partial(J^T A) = F \tag{204}$$

as a complete and closed algebraic formulation of classical electrodynamics within $M_2(\mathbb{C})$. The claim that these three products, computed from the same four-vectors under the same multiplication rule, cover every physically significant object in classical electrodynamics — and do so with universal four-channel architecture, automatic Lorentz covariance, and gauge invariance in the physical domain — has no direct precedent in the prior BQ or quaternion electromagnetic literature.

Earlier BQ and quaternion treatments of electrodynamics Girard (1984); Adler (1995); Gsponer and Hurni (2005a); Demir and Tanışlı (2013) establish the Maxwell equations and Lorentz force in quaternionic form and demonstrate Lorentz covariance, but do not extract the $\partial(J^T A)$ product or its four-channel content. The Clifford algebra treatment of Hestenes Hestenes (1966, 2003) and the geometric algebra formulation of Doran and Lasenby Doran and Lasenby (2003b) achieve comparable compactness for the Maxwell and Lorentz equations but do not produce the interaction-sector Maxwell architecture of $\partial(J^T A)$ or the Helmholtz decompositions of $\mathbf{T}$ and $\mathbf{S}$. The present formulation is therefore distinguished not by the use of a quaternionic or Clifford algebra per se, but by the specific extraction and physical interpretation of all four channels of $\partial(J^T A)$ and by the demonstration that the resulting channel structure constitutes an interaction-sector Maxwell system with the same four-channel architecture as the free-field Maxwell equations.

Summary

Table 1 collects the channel structure, the physical identification established in the present paper, and the closest known precedents.

Table 1 Channel structure of $\partial(J^T A)$, physical identification, and closest prior literature.

Channel	Physical identification	Closest precedent
Full product $J^T A$	Four-channel bilinear of J and A	Standard as scalar $J_\mu A^\mu$ only Pauli (1958); Jackson (1999)
Norm: $\nabla \cdot (\mathbf{A} \times \mathbf{J})$	Angular momentum divergence; MHD–photon AM cross-domain connection; Taylor relaxation condition	Helicity conservation Bliokh et al. (2013); QED angular momentum Khosravi et al. (2024)
Time: $\nabla \cdot \mathbf{S} - \frac{1}{c}\partial_t L$	Continuity equation for $\mathcal{L}_{\text{int}}$; interaction-sector Poynting theorem; Euler–Lagrange temporal equation	None identified
Space: $\nabla \times \mathbf{T} + \nabla L - \frac{1}{c}\partial_t \mathbf{S}$	Euler–Lagrange spatial equation; vorticity-pressure balance; Helmholtz decomposition of $\mathbf{T}$	Sporadically in MHD Biskamp (2000); force-free fields Chandrasekhar and Kendall (1957)
Spin: $\nabla \times \mathbf{S} + \frac{1}{c}\partial_t \mathbf{T}$	Interaction-sector Faraday law; Helmholtz decomposition of $\mathbf{S}$; Aharonov–Bohm topology; reconnection precursor	Röntgen term Sonnleitner and Barnett (2017)
Triad ∂B, JB, $\partial(J^T A)$	Complete algebraic closure of classical EM; universal four-channel architecture	Partial: Girard (1984); Hestenes (2003); Doran and Lasenby (2003b)

The conclusion is that no prior source treats $\partial(J^T A)$ as a unified four-channel object. The norm and spin channels have structural analogues in the electromagnetic helicity literature Bliokh et al. (2013); Khosravi et al. (2024) and in the

Röntgen-interaction literature Sonnleitner and Barnett (2017) respectively, but neither is framed as a channel of a single algebraic product. The time and space channels appear to be without direct precedent in the literature. The three-product triad and its collective claim of algebraic closure have partial analogues in prior BQ and Clifford algebra treatments but are not established in those treatments at the level of channel-by-channel physical identification and gauge-invariance analysis achieved here. The contribution of the present paper is the unified derivation of all four channels from the single BQ product $\partial(J^T A)$, the identification of their physical content across electromagnetic angular momentum theory, interaction energy conservation, variational mechanics, and inductive topology, together with the demonstration that the complete object is Lorentz covariant under the same rotor structure that governs the Maxwell and Lorentz equations, and gauge-invariant under charge conservation and the Lorenz gauge for the physically relevant domain of transverse conserved currents.

4 Relativistic Fluid Dynamics in the BQ Framework

4.1 The substitution $J \to U$, $A \to G$ and the momentum-vorticity field $\mathcal{H}$

The three-equation BQ hierarchy of electrodynamics — Maxwell, Lorentz and Laue/Sommerfeld — maps onto a corresponding hierarchy in relativistic fluid dynamics (RFD) under a single algebraic substitution. In electrodynamics the four-current satisfies $J = \rho_e V$, so the four-velocity V is already present inside J as the charge-density-weighted velocity; going from J to V removes that weighting and promotes the four-velocity itself to the role of the source four-vector. The four-potential A carries the electromagnetic canonical momentum per unit charge; its mechanical analogue is the four-momentum density

$$G = \rho_0 U = \gamma \rho_0 V = G_\mu K^\mu = \rho_0 u_0 \hat{\mathrm{T}} + \rho_0 \boldsymbol{u} \cdot \hat{\mathbf{K}} = \frac{\varepsilon}{c}\hat{\mathrm{T}} + \boldsymbol{g} \cdot \hat{\mathbf{K}}, \tag{205}$$

where ε is the energy density and $\boldsymbol{g}$ is the momentum density. The last expression on the right is the most general one, because is valid for contexts where $\boldsymbol{g} \neq \rho_0 \boldsymbol{u}$. In all cases we have

$$\varepsilon \equiv \rho_0 c u_0 = \gamma \rho_0 c^2 = \frac{1}{\gamma}\rho_i c^2 = \frac{1}{\gamma}\varepsilon_i. \tag{206}$$

Since both V and G are four-vectors in the BQ basis $K^\mu = (\hat{\mathrm{T}}, \hat{I}, \hat{J}, \hat{K})$, the substitution $J \longrightarrow V$, $A \longrightarrow G$ preserves the algebraic type of every product, and all channel decompositions carry over from the EM case without modification.

The *momentum vorticity field* $\mathcal{H}$, the BQ analogue of the EM field $B = \partial^T A$, is defined as

$$\mathcal{H} = \partial^T G. \tag{207}$$

This is the product

$$\partial^T G = \left(\frac{1}{c}\partial_t \hat{\mathrm{T}} + \nabla \cdot \hat{\mathbf{K}}\right)\left(\frac{\varepsilon}{c}\hat{\mathrm{T}} + \mathbf{g} \cdot \hat{\mathbf{K}}\right), \tag{208}$$

with the channels $\mathcal{H} = h_1 \hat{1} + \mathbf{h}_K \cdot \hat{\mathbf{K}} + \mathbf{h}_\sigma \cdot \hat{\boldsymbol{\sigma}}$ given by

$$\mathcal{H} = -\left(\frac{1}{c^2}\partial_t \varepsilon + \nabla \cdot \boldsymbol{g}\right)\hat{1} + (\nabla \times \boldsymbol{g}) \cdot \hat{\mathbf{K}} - \frac{1}{c}\left(\partial_t \boldsymbol{g} + \nabla \varepsilon\right) \cdot \hat{\boldsymbol{\sigma}}, \tag{209}$$

reflecting the bilinear product formula of Eqn. (27). The channel decomposition then is given by

$$norm - channel\ \hat{1}: \qquad h_{\hat{1}} = -\left(\frac{1}{c^2}\partial_t \varepsilon + \nabla \cdot \boldsymbol{g}\right) = -\sigma_E, \tag{210}$$

$$space - channel\,\hat{\mathbf{K}}: \qquad \mathbf{h}_K = \nabla \times \boldsymbol{g} = \boldsymbol{\Omega}, \tag{211}$$

$$sigma - channel\,\hat{\boldsymbol{\sigma}}: \qquad \mathbf{h}_\sigma = -\frac{1}{c}\left(\partial_t \boldsymbol{g} + \nabla \varepsilon\right) = -\frac{1}{c}\boldsymbol{\Pi}. \tag{212}$$

This can be abbreviated as

$$\mathcal{H} = -\sigma_E \hat{1} + \boldsymbol{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c}\boldsymbol{\Pi} \cdot \hat{\boldsymbol{\sigma}}, \tag{213}$$

where

$$\sigma_E = \nabla \cdot \boldsymbol{g} + \frac{1}{c^2}\partial_t \varepsilon \tag{214}$$

is the conserved mass density term,

$$\boldsymbol{\Omega} = \nabla \times \boldsymbol{g} \tag{215}$$

is the vorticity of the momentum density, playing the role of the magnetic field $\boldsymbol{B}$, and

$$\boldsymbol{\Pi} = \partial_t \boldsymbol{g} + \nabla \varepsilon \tag{216}$$

is the momentum–energy gradient, playing the role of the electric field $\boldsymbol{E}/c$.

4.1.1 The conserved condition

Now we can consider the limiting case $\mathcal{H} = 0$ or $\partial^T G = 0$, corresponding to a conserved vorticity-energymomentum. This implies

$$\frac{1}{c^2}\partial_t \varepsilon + \nabla \cdot \boldsymbol{g} = 0, \tag{217}$$

$$\nabla \times \boldsymbol{g} = 0, \tag{218}$$

and

$$\partial_t \boldsymbol{g} = -\nabla \varepsilon. \tag{219}$$

Thus, $\mathcal{H} = 0$ describes a closed, irrotational and force-free fluid state: local energy conservation holds, the momentum-density flow has no vorticity, and the time variation of momentum density is exactly balanced by the gradient of the energy density. The strongest claim is that $\mathcal{H} = 0$ implies an conservative, irrotational potential flow regime.

Restrict to situations with $\boldsymbol{g} = \rho_0 \boldsymbol{u}$

For situations where $\boldsymbol{g} = \rho_0 \boldsymbol{u}$, the space-channel being zero can be expanded as

$$\nabla \times \rho_0 \boldsymbol{u} = \rho_0 \nabla \times \boldsymbol{u} + (\nabla \rho_0) \times \boldsymbol{u} = \rho_0 \boldsymbol{\omega} + (\nabla \rho_0) \times \boldsymbol{u} = 0, \tag{220}$$

the sigma-channel can be given as

$$\rho_0 \partial_t \boldsymbol{u} + (\partial_t \rho_0) \boldsymbol{u} = -\rho_0 c \nabla u_0 - c(\nabla \rho_0) u_0, \tag{221}$$

and the norm channel has

$$\rho_0 \nabla \cdot \boldsymbol{u} + (\nabla \rho_0) \cdot \boldsymbol{u} = -\frac{\rho_0}{c} \partial_t u_0 - \frac{u_0}{c} \partial_t \rho_0 \tag{222}$$

Restrict further to situations with constant ρ_0

If the matter density ρ_0 is space-time independent, thus constant in space and time, then the equations reduce to

$$\nabla \cdot \boldsymbol{u} = -\frac{1}{c} \partial_t u_0, \tag{223}$$

$$\nabla \times \boldsymbol{u} = 0, \tag{224}$$

and

$$\partial_t \boldsymbol{u} = -c \nabla u_0. \tag{225}$$

Restrict further to low velocity approximations

For low velocity approximations we have

$$\boldsymbol{u} = \gamma \boldsymbol{v} \approx (1 + \frac{v^2}{2c^2}) \boldsymbol{v} \approx \boldsymbol{v} \tag{226}$$

$$\frac{1}{c} u_0 = \gamma \approx \left(1 + \frac{v^2}{2c^2}\right) \approx 1 \tag{227}$$

$$u_0 c = \gamma c^2 \approx (1 + \frac{v^2}{2c^2}) c^2 \approx c^2 + \frac{v^2}{2}, \tag{228}$$

$$\tag{229}$$

so we get incompressibility with

$$\nabla \cdot \boldsymbol{v} = 0, \tag{230}$$

irrotationallity with

$$\nabla \times \boldsymbol{v} = 0, \tag{231}$$

and the pressureless Euler condition with

$$\partial_t \boldsymbol{v} = -\nabla \left(\frac{v^2}{2} \right), \tag{232}$$

These last three equations together are exactly the equations of classical incompressible, irrotational, pressureless potential flow. With $\mathbf{v} = \nabla\chi$, the first equation gives the Laplace equation $\nabla^2\chi = 0$ and the third equation gives

$$\partial_t \nabla\chi + \nabla \left(\frac{v^2}{2} \right) = \nabla\partial_t\chi + \nabla \left(\frac{v^2}{2} \right) = \nabla \left(\partial_t\chi + \frac{v^2}{2} \right) = 0, \tag{233}$$

leading to

$$\partial_t\chi + \frac{v^2}{2} = constant. \tag{234}$$

The last equation is the unsteady Bernoulli relation. The result is the standard mathematical structure of classical potential flow theory.

Restrict to low velocity approximations but not ρ_0 = constant

If, given the above set of restriction, we relax the ρ_0 = constant one, then we get For situations where $\boldsymbol{g} = \rho_0 \boldsymbol{u}$, the norm channel has

$$\rho_0 \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot (\nabla\rho_0) = -\rho_0 \partial_t \left(\frac{v^2}{2c^2} \right) - \left(1 + \frac{v^2}{2c^2} \right) \partial_t \rho_0 = \tag{235}$$

$$-\rho_0 \partial_t \left(\frac{v^2}{2c^2} \right) - \partial_t\rho_0 - \frac{v^2}{2c^2} \partial_t\rho_0, \tag{236}$$

which can be rearranged into

$$\partial_t\rho_0 + \boldsymbol{v} \cdot (\nabla\rho_0) = -\rho_0 \nabla \cdot \boldsymbol{v} - \rho_0 \partial_t \left(\frac{v^2}{2c^2} \right) - \frac{v^2}{2c^2} \partial_t\rho_0, \tag{237}$$

so

$$\frac{D}{Dt}\rho_0 = -\rho_0 \nabla \cdot \boldsymbol{v} - \rho_0 \partial_t \left(\frac{v^2}{2c^2} \right) - \frac{v^2}{2c^2} \partial_t\rho_0, \tag{238}$$

and

$$c^2 \frac{D}{Dt}\rho_0 = -\rho_0 c^2 \nabla \cdot \boldsymbol{v} - \rho_0 \partial_t \left(\frac{v^2}{2} \right) - \frac{v^2}{2} \partial_t\rho_0, \tag{239}$$

leading to

$$\frac{D}{Dt}\varepsilon_0 = -\rho_0 c^2 \nabla \cdot \boldsymbol{v} - \partial_t \left(\frac{\rho_0 v^2}{2}\right), \tag{240}$$

thus, with $\mathcal{K} = \frac{\rho_0 v^2}{2}$,

$$\frac{D}{Dt}\varepsilon_0 = -\rho_0 c^2 \nabla \cdot \boldsymbol{v} - \partial_t \mathcal{K}, \tag{241}$$

The space-channel being zero can be expanded as

$$\rho_0 \nabla \times \boldsymbol{v} + (\nabla \rho_0) \times \boldsymbol{v} = 0. \tag{242}$$

or

$$\rho_0 \nabla \times \boldsymbol{v} = \boldsymbol{v} \times (\nabla \rho_0). \tag{243}$$

The sigma-channel can be given as

$$\rho_0 \partial_t \boldsymbol{u} + (\partial_t \rho_0) \boldsymbol{u} = -\rho_0 c \nabla u_0 - c (\nabla \rho_0) u_0, \tag{244}$$

which in the slow flow, variable ρ_0 gives

$$\rho_0 \partial_t \boldsymbol{v} + (\partial_t \rho_0) \boldsymbol{v} = -\rho_0 \nabla \left(c^2 + \frac{v^2}{2}\right) - (\nabla \rho_0) \left(c^2 + \frac{v^2}{2}\right), \tag{245}$$

so

$$\rho_0 \partial_t \boldsymbol{v} + (\partial_t \rho_0) \boldsymbol{v} = -\rho_0 \nabla \left(\frac{v^2}{2}\right) - \left(\frac{v^2}{2}\right) \nabla \rho_0 - c^2 \nabla \rho_0, \tag{246}$$

thus

$$\rho_0 \partial_t \boldsymbol{v} + (\partial_t \rho_0) \boldsymbol{v} = -\rho_0 \nabla \left(\frac{v^2}{2}\right) - \left(\frac{v^2}{2}\right) \nabla \rho_0 - c^2 \nabla \rho_0, \tag{247}$$

or, with $\mathcal{K} = \frac{\rho_0 v^2}{2}$,

$$\rho_0 \partial_t \boldsymbol{v} + \boldsymbol{v} \partial_t \rho_0 + c^2 \nabla \rho_0 = -\nabla \mathcal{K}, \tag{248}$$

and with $\boldsymbol{g}_0 = \rho_0 \boldsymbol{v}$

$$\partial_t \boldsymbol{g}_0 + \nabla \varepsilon_0 = -\nabla \mathcal{K}. \tag{249}$$

We conclude that in the situation described, we have the three channels as

$$\frac{D}{Dt}\varepsilon_0 = -\rho_0 c^2 \nabla \cdot \boldsymbol{v} - \partial_t \mathcal{K}, \tag{250}$$

for the norm channel,

$$\rho_0 \nabla \times \boldsymbol{v} = \boldsymbol{v} \times (\nabla \rho_0). \tag{251}$$

for the space channel, and

$$\partial_t \boldsymbol{g}_0 + \nabla \varepsilon_0 = -\nabla \mathcal{K}. \tag{252}$$

for the sigma channel.

The space channel equation (251) states that irrotationality $\nabla \times \boldsymbol{v} = 0$ no longer holds when ρ_0 varies spatially: the right-hand side $\boldsymbol{v} \times (\nabla \rho_0)$ is the baroclinic source term, which in classical fluid dynamics drives vorticity generation in stratified or compressible flows through the misalignment of velocity and density-gradient directions. This term vanishes identically when $\boldsymbol{v} \parallel \nabla \rho_0$ everywhere, recovering irrotational flow, or when ρ_0 is uniform, recovering $\nabla \times \boldsymbol{v} = 0$. Here it emerges automatically as the space-channel projection of $\mathcal{H} = 0$ once the constant-density restriction is relaxed, with no separate appeal to the vorticity equation or to the Bjerknes–Silberstein–Baroclinic theorem: baroclinic vorticity generation is encoded in the algebraic structure of the field condition itself.

Adding the relativistic definition of the speed of sound in a fluid medium

If we add the barotropic/adiabatic sound-speed relation in a fluid medium as

$$c_s^2 = c^2 \left(\frac{\partial p}{\partial \varepsilon_0}\right)_s, \tag{253}$$

and use it as

$$\frac{\partial p}{\partial \varepsilon_0} = \frac{c_s^2}{c^2}, \tag{254}$$

we get

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial \varepsilon_0}\frac{D\varepsilon_0}{Dt} = \frac{c_s^2}{c^2}\frac{D\varepsilon_0}{Dt}, \tag{255}$$

but also

$$\nabla p = \frac{\partial p}{\partial \varepsilon_0}\nabla \varepsilon_0 = \frac{c_s^2}{c^2}\nabla \varepsilon_0, \tag{256}$$

Inserted in the norm channel we get

$$\frac{Dp}{Dt} = -\rho_0 c_s^2 \nabla \cdot \boldsymbol{v} - \frac{c_s^2}{c^2}\partial_t \mathcal{K}, \tag{257}$$

and because we are already assuming slow flow conditions, we may assume $c_s << c$ and this reduces to

$$\frac{Dp}{Dt} = -\rho_0 c_s^2 \nabla \cdot \boldsymbol{v}, \tag{258}$$

the classical acoustic relation.

Inserted in the sigma channel we get

$$\partial_t \boldsymbol{g}_0 + \frac{c^2}{c_s^2}\nabla p = -\nabla \mathcal{K}. \tag{259}$$

or

$$\partial_t \boldsymbol{g}_0 = -\frac{c^2}{c_s^2}\nabla p - \nabla \mathcal{K}. \tag{260}$$

In the low-sound-speed limit, pressure gradients correspond to much larger rest-energy-density gradients, since $\nabla \varepsilon_0 = \frac{c^2}{c_s^2}\nabla p$.

Interpretation of result

The recovery of the unsteady Bernoulli relation (234) is a meaningful consistency check, but its scope must be stated precisely. It rests on $G = \rho_0 U$, the dust four-momentum density, for which $\varepsilon = \rho_0 c u_0$ contains only rest and kinetic energy, with no pressure contribution. Hence $\nabla\varepsilon \to \rho_0 \nabla(v^2/2)$ in the low-velocity limit, and the $\hat{\boldsymbol{\sigma}}$-channel condition $\mathbf{\Pi} = \partial_t \mathbf{g} + \nabla\varepsilon = 0$ yields Eqn. (232) with no pressure-gradient term. This is not an additional approximation but a direct consequence of describing a pressureless medium ($T^{00} = \varepsilon$, $T^{0i} = g^i$, no spatial stress). For a fluid with pressure p, replacing ε by the enthalpy density $w = \varepsilon + p$ would produce the term $-\frac{1}{\rho_0}\nabla p$ in Eqn. (232), recovering the full barotropic Euler equation. Once a barotropic equation of state $p = p(\varepsilon)$ is supplied, the relativistic sound speed $c_s^2 = c^2(\partial p/\partial\varepsilon)_s$ enters naturally: the $\hat{1}$-channel condition $\sigma_E = 0$ with variable ρ_0 then yields the classical acoustic relation $\frac{Dp}{Dt} = -\rho_0 c_s^2 \nabla \cdot \mathbf{v}$ as its time-channel projection (see §4.1.1 below), while the space channel encodes baroclinic vorticity generation through $\rho_0 \nabla \times \mathbf{v} = \mathbf{v} \times (\nabla\rho_0)$, and the sigma channel produces the relativistically amplified momentum equation $\partial_t \mathbf{g}_0 = -\frac{c^2}{c_s^2}\nabla p - \nabla\mathcal{K}$. The pressure physics is therefore not an external addition but is latent in the field-level condition $\mathcal{H} = 0$ from the outset, emerging as soon as the dust restriction is relaxed to a barotropic equation of state.

A second point concerns the absence of the convective term $(\mathbf{v} \cdot \nabla)\mathbf{v}$, so that Eqn. (232) contains $\partial_t \mathbf{v}$ rather than $D\mathbf{v}/Dt$. This is expected: $\mathcal{H} = \partial^T G$ is linear in G, so all its terms are at most first order in derivatives of $\mathbf{v}$ and ε, whereas the convective acceleration is quadratic in $\mathbf{v}$. It would arise instead from a fluid analogue of the Lorentz force law $JB = F$, contracting $\mathbf{v}$ a second time with $\mathcal{H}$, to be developed below. There, the T-channel of $V\mathcal{H} = F$ reproduces the kinetic power equation $\mathcal{P} = -\frac{D\mathcal{K}}{Dt}$ in the incompressible limit and the exact barotropic pressure-dilatation work relation $\frac{D\mathcal{K}}{Dt} = \frac{1}{\rho}\frac{Dp}{Dt}$ in the compressible case — confirming that the pressure structure visible at the field level in $\mathcal{H} = 0$ propagates consistently through to the force level in $V\mathcal{H} = F$.

With these qualifications, $\mathcal{H} = 0$ in the dust and low-velocity limits reduces exactly to incompressibility, irrotationality, and the unsteady Bernoulli relation (234) — a non-trivial consistency check, since none of these was assumed in advance; all three emerge as channel projections of $\partial^T G = 0$. The pressureless and linear character of the result identifies the two directions for extension: an equation of state via $w = \varepsilon + p$, and a fluid analogue of $JB = F$ for the convective nonlinearity. Both are structurally anticipated by the electromagnetic hierarchy of Section 3 and are developed in the remainder of the present section and in §4.3 respectively.

Compactness assessment

The classical route to potential flow combines three independently motivated ingredients: continuity $\nabla \cdot \mathbf{v} = 0$ from mass conservation under an incompressibility assumption; irrotationality $\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0$ as a separate kinematic postulate; and the force balance $\partial_t \mathbf{v} = -\nabla(v^2/2)$ from Newton's second law via the identity $(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla(v^2/2) - \mathbf{v} \times \boldsymbol{\omega}$, with the second term discarded using the irrotationality assumption introduced separately. These come from three different areas of physics and are combined by hand.

In the BQ derivation all three arise as channel projections of $\mathcal{H} = \partial^T G = 0$: the $\hat{1}$-channel gives continuity $\sigma_E = \nabla \cdot \mathbf{g} + \frac{1}{c^2}\partial_t \varepsilon = 0$; the $\hat{K}$-channel gives irrotationality $\boldsymbol{\Omega} = \nabla \times \mathbf{g} = 0$; and the $\hat{\boldsymbol{\sigma}}$-channel gives the force balance $\boldsymbol{\Pi} = \partial_t \mathbf{g} + \nabla \varepsilon = 0$. No separate kinematic postulate is needed — irrotationality is simply the $\hat{K}$-channel of the same condition — and no separate vector identity is needed, since $\partial_t \mathbf{v} + \nabla(v^2/2)$ already has the post-identity form, with the $\mathbf{v} \times \boldsymbol{\omega}$ term absent from the outset because it belongs to a different channel ($\hat{K}$). When the constant-density restriction is relaxed, the same three channels extend naturally to encode the classical acoustic relation, baroclinic vorticity generation, and the relativistically amplified momentum equation — all as projections of the single condition $\mathcal{H} = 0$ with a barotropic equation of state, and none assumed in advance.

This matters because continuity, irrotationality, and the force balance — three results the classical treatment derives from three independently motivated principles — are here three projections of one algebraic object under one condition.

Their mutual consistency is established once, by the channel structure, rather than checked by hand three times: $\mathcal{H} = 0$ is not a compact notation for the three classical equations but a statement that they are a single output, and their simultaneous validity is a structural consequence of the algebra rather than a coincidence. The same structural logic extends to the force level: the four channels of $V\mathcal{H} = F$ reproduce helicity transport, kinetic power exchange, the Euler–Lamb momentum equation, and rotational vorticity balance as projections of one product, again without separate appeal to independent physical principles.

Prior art in physics

The principle that physics "lives in the channels" of a single algebraic product has clear prior art. In geometric algebra, Hestenes' reformulation of electrodynamics as $\nabla F = J$, with F the field as one bivector combining $\mathbf{E}$ and $\mathbf{B}$, recovers the four Maxwell equations as the grade-0 through grade-3 projections of one statement Hestenes (2003); Doran and Lasenby (2003b) — structurally the same operation as the channel decomposition of $\partial B = \mu_0 J$ in §3.3. In general relativity, the Bianchi identity $\nabla_\mu G^{\mu\nu} \equiv 0$, once Einstein's equations are imposed, becomes $\nabla_\mu T^{\mu\nu} = 0$: local energy-momentum conservation, combining mass conservation and Newton's second law as a single output of one identity. Both precedents show that algebraic or geometric identities can yield, as automatic projections, results a less unified treatment derives from several independent principles. The present contribution is not this principle itself but its extension to a wider class of products than GA or GR have addressed: the interaction bilinear $\partial(J^T A) = F$, and, via $J \to V$, $A \to G$, the fluid hierarchy $\mathcal{H} = \partial^T G$, whose $\mathcal{H} = 0$ limit reproduces continuity, irrotationality, and the Bernoulli relation as three channels of one condition, and whose variable-density extension reproduces the acoustic relation, baroclinic vorticity generation, and the relativistic momentum equation as three further channels of the same condition with a barotropic equation of state supplied.

4.2 Fluid invariants and transport structure from the quadratic field products

For $\mathcal{H} \neq 0$ vorticity and transport dynamics appear through the non-vanishing channels $\mathbf{\Omega}$ and $\mathbf{\Pi}$. The transition from $\mathcal{H} = 0$ to $\mathcal{H} \neq 0$ therefore marks the transition from a purely conservative scalar-potential regime to a full relativistic transport and vortex dynamics described by the fluid four-potential. For the relativistic fluid field $\mathcal{H} = \partial^T G$, with

$$\mathcal{H} = -\sigma_E \hat{1} + \mathbf{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}, \tag{261}$$

the BQ-Pauli framework again generates two natural quadratic field products: $\mathcal{H}\mathcal{H}$ and $\mathcal{H}^T\mathcal{H}$.

4.2.1 Quadratic $\mathcal{H}$-field structure for locally conserved energy

For locally conserved fluids with $\sigma_E = 0$, the field quadratic $\mathcal{H}\mathcal{H}$ reduces to

$$\mathcal{H}\mathcal{H} = \left(\mathbf{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}\right)\left(\mathbf{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}\right), \tag{262}$$

resulting in

$$\mathcal{H}\mathcal{H} = \left(-\Omega^2 + \frac{1}{c^2}\Pi^2\right)\hat{1} - \frac{2}{c}(\mathbf{\Omega} \cdot \mathbf{\Pi})\hat{\mathrm{T}}. \tag{263}$$

This product therefore contains the invariant-like quadratic fluid field structures: $\frac{1}{c^2}\Pi^2 - \Omega^2$ and $\mathbf{\Omega} \cdot \mathbf{\Pi}$. The first term combines the vorticity magnitude and the momentum-pressure field magnitude into a Lorentz-like quadratic field norm, while the second measures the coupling between rotational and transport-pressure structure. Since $\mathcal{H}\mathcal{H}$ reduces entirely to scalar and T-channel contributions, it behaves as the invariant quadratic field structure of the relativistic fluid field.

The second quadratic product $\mathcal{H}^T\mathcal{H}$ reads

$$\mathcal{H}^T\mathcal{H} = \left(\mathbf{\Omega} \cdot \hat{\mathbf{K}} + \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}\right)\left(\mathbf{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}\right). \tag{264}$$

and one obtains

$$\mathcal{H}^T\mathcal{H} = \left(-\Omega^2 - \frac{1}{c^2}\Pi^2\right)\hat{1} + \frac{2}{c}(\mathbf{\Pi} \times \mathbf{\Omega}) \cdot \hat{\boldsymbol{\sigma}}. \tag{265}$$

The scalar contribution defines a quadratic fluid-field energy density

$$u_{\mathcal{H}} = -\Omega^2 - \frac{1}{c^2}\Pi^2, \tag{266}$$

while the $\hat{\boldsymbol{\sigma}}$-channel contains the Poynting-like rotational transport structure

$$\boldsymbol{S}_{\mathcal{H}} = \mathbf{\Pi} \times \mathbf{\Omega}. \tag{267}$$

The product $\mathcal{H}^T\mathcal{H}$ therefore generates naturally a fluid-dynamical analogue of the electromagnetic energy-transport structure:

$$\mathcal{H}^T\mathcal{H} = u_{\mathcal{H}}\hat{1} + \frac{2}{c}\boldsymbol{S}_{\mathcal{H}} \cdot \hat{\boldsymbol{\sigma}}. \tag{268}$$

The quantity

$$\boldsymbol{S}_{\mathcal{H}} = \mathbf{\Pi} \times \mathbf{\Omega} \tag{269}$$

plays the role of a rotational transport-energy flux, directly analogous to the electromagnetic Poynting vector

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}. \tag{270}$$

As in electromagnetism, however, the mixed product $\mathcal{H}^T\mathcal{H}$ is not Lorentz covariant. The obstruction again originates from the fact that the time-adjoint operation reverses the Lorentz rotor structure. For

$$\mathcal{H}^L = U\mathcal{H}U^{-1}, \tag{271}$$

the time-adjoint transforms as

$$(\mathcal{H}^T)^L = U^{-1}\mathcal{H}^T U, \tag{272}$$

so that

$$(\mathcal{H}^T)^L\mathcal{H}^L = U^{-1}\mathcal{H}^T UU\mathcal{H}U^{-1} = U^{-1}\mathcal{H}^T U^2\mathcal{H}U^{-1}. \tag{273}$$

As a result,

$$(\mathcal{H}^T)^L\mathcal{H}^L \neq U^{-1}(\mathcal{H}^T\mathcal{H})U^{-1}, \tag{274}$$

showing that $\mathcal{H}^T\mathcal{H}$ does not transform as a genuine Lorentz four-vector object. Thus, although $\mathcal{H}^T\mathcal{H}$ combines the quadratic fluid-field energy density and rotational transport flux into a compact BQ expression in a chosen frame, this object is not itself Lorentz covariant.

Within the BQ-Pauli framework, the two quadratic fluid products therefore separate naturally into:

- an invariant quadratic fluid-field structure through $\mathcal{H}\mathcal{H}$,
- and a rotational transport-energy structure through $\mathcal{H}^T\mathcal{H}$.

4.2.2 Quadratic fluid-field structure for non-conservative fluids

For the general relativistic fluid field

$$\mathcal{H} = -\sigma_E\hat{1} + \boldsymbol{\Omega}\cdot\hat{\mathbf{K}} - \frac{1}{c}\boldsymbol{\Pi}\cdot\hat{\sigma}, \tag{275}$$

the quadratic field products contain additional coupling terms associated with local energy non-conservation and transport forcing.

The first quadratic product becomes

$$)\mathcal{H}\mathcal{H} = \left(\sigma_E^2 - \Omega^2 + \frac{1}{c^2}\Pi^2\right)\hat{1}$$

$$-\frac{2}{c}(\mathbf{\Omega}\cdot\mathbf{\Pi})\hat{\mathrm{T}} - 2\sigma_E\mathbf{\Omega}\cdot\hat{\mathbf{K}} + \frac{2\sigma_E}{c}\mathbf{\Pi}\cdot\hat{\boldsymbol{\sigma}}. \tag{276}$$

The scalar contribution now contains an additional positive source-density term σ_E^2, which measures the magnitude of the local energy exchange structure of the fluid. The T-channel $\frac{2}{c}(\mathbf{\Omega}\cdot\mathbf{\Pi})$ continues to measure the coupling between vorticity and momentum-pressure transport. The K-channel contribution $-2\sigma_E\mathbf{\Omega}$ couples local energy non-conservation directly to the rotational vorticity structure, while the $\hat{\boldsymbol{\sigma}}$-channel contribution $\frac{2\sigma_E}{c}\mathbf{\Pi}$ couples it to the momentum-pressure transport field. The product $\mathcal{H}\mathcal{H}$ therefore describes the complete quadratic interaction structure between:

- local energy exchange,
- vorticity,
- and momentum-pressure transport.

The second quadratic product gives

$$\mathcal{H}^T\mathcal{H} = \left(-\sigma_E\hat{1} + \mathbf{\Omega}\cdot\hat{\mathbf{K}} + \frac{1}{c}\mathbf{\Pi}\cdot\hat{\boldsymbol{\sigma}}\right)\left(-\sigma_E\hat{1} + \mathbf{\Omega}\cdot\hat{\mathbf{K}} - \frac{1}{c}\mathbf{\Pi}\cdot\hat{\boldsymbol{\sigma}}\right), \tag{277}$$

and becomes

$$\begin{aligned}\mathcal{H}^T\mathcal{H} = {} & \left(\sigma_E^2 - \Omega^2 - \frac{1}{c^2}\Pi^2\right)\hat{1} \\ & -2\sigma_E\mathbf{\Omega}\cdot\hat{\mathbf{K}} + \frac{2}{c}(\mathbf{\Pi}\times\mathbf{\Omega})\cdot\hat{\boldsymbol{\sigma}}.\end{aligned} \tag{278}$$

The scalar contribution now defines the generalized quadratic fluid-field energy density

$$u_{\mathcal{H}} = \sigma_E^2 - \Omega^2 - \frac{1}{c^2}\Pi^2 \tag{279}$$

Besides the vorticity and momentum-pressure contributions, this energy structure now also contains the local dissipation or forcing density through σ_E^2. The K-channel term $-2\sigma_E\mathbf{\Omega}$ shows that energy non-conservation couples directly to the rotational vortex structure of the flow. The $\hat{\boldsymbol{\sigma}}$-channel again contains the rotational transport-energy flux $\boldsymbol{S}_{\mathcal{H}} = \mathbf{\Pi}\times\mathbf{\Omega}$, which acts as the fluid analogue of the electromagnetic Poynting vector. The quadratic product $\mathcal{H}^T\mathcal{H}$ therefore combines:

- fluid-field energy density,
- rotational transport-energy flux,
- dissipation,
- forcing,

- and vortex-transport coupling

within a single BQ field product. For conservative fluids with $\sigma_E = 0$, the expressions reduce to the simpler invariant and Poynting-like transport structures discussed previously.

The non-covariance of the product $\mathcal{H}^T\mathcal{H}$ is physically significant rather than problematic. Unlike the invariant product $\mathcal{H}\mathcal{H}$, which describes the intrinsic quadratic structure of the relativistic fluid field, $\mathcal{H}^T\mathcal{H}$ represents a frame-dependent decomposition into energy density and transport flux. The quantities $u_{\mathcal{H}} = \sigma_E - \Omega^2 - \frac{1}{c^2}\Pi^2$ and $\boldsymbol{S}_{\mathcal{H}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega}$ therefore behave analogously to the electromagnetic energy density and Poynting vector: they depend on the observer decomposition of the underlying relativistic field into transport and rotational sectors. The non-covariance of $\mathcal{H}^T\mathcal{H}$ thus reflects the fact that transport-energy flow is not an invariant scalar property of the fluid field, but an observer-dependent projection of the full relativistic transport structure.

4.2.3 Significance and interpretation of the quadratic fluid invariants

The structure of $u_{\mathcal{H}}$ as a production-dissipation balance

The general result of §4.2.2,

$$u_{\mathcal{H}} = \sigma_E^2 - \Omega^2 - \frac{1}{c^2}\Pi^2, \tag{280}$$

has a structure that deserves interpretation in its own right, independently of any analogy with the electromagnetic energy density u_{EM}. Both σ_E^2 and $\Omega^2 + \frac{1}{c^2}\Pi^2$ are individually non-negative, so $u_{\mathcal{H}}$ is the *difference* of two non-negative quantities and is therefore sign-indefinite: $u_{\mathcal{H}} > 0$ when local non-conservation dominates the combined vorticity-transport content, $u_{\mathcal{H}} < 0$ when vorticity and momentum-energy transport gradients dominate, and $u_{\mathcal{H}} = 0$ marks the local balance between the two.

This is not a defect relative to the positive-definite electromagnetic energy density $u_{EM} = \frac{1}{2\mu_0}(B^2 + E^2/c^2)$; it is a different and, for a fluid, more appropriate kind of object. u_{EM} is a static quantity: it measures how much field energy is present. $u_{\mathcal{H}}$, by contrast, is naturally read as a *rate* quantity: it measures whether the local fluid state is net-producing or net-dissipating the quadratic field structure carried by $\mathcal{H}$. The term σ_E^2 acts as a production term, sourced by local energy or mass non-conservation, while $\Omega^2 + \frac{1}{c^2}\Pi^2$ acts as a dissipation term, sourced by vorticity and by the momentum-energy transport gradient. A signed $u_{\mathcal{H}}$ that distinguishes net-production from net-dissipation regions is closer in spirit to the local entropy production rate of irreversible thermodynamics, or to the local energy transfer rate in turbulence cascade theory, than to a static energy density.

The relativistic case: $\sigma_E \neq 0$

In the general relativistic case, $\sigma_E = \nabla \cdot \mathbf{g} + \frac{1}{c^2}\partial_t \varepsilon$ measures the local departure from the energy-momentum continuity condition. A nonzero σ_E signals a local source or sink of mass-energy — physically, regions where matter is being created, destroyed, converted, or exchanged with another sector of the theory (for instance, a region exchanging energy with an electromagnetic or gravitational field through the corresponding coupling terms of the full theory). In such regions, σ_E^2 contributes positively to $u_{\mathcal{H}}$ regardless of the sign of σ_E itself, since only its magnitude enters. This means that *any* local non-conservation, whether a net creation or a net destruction of mass-energy, acts as a production term for $u_{\mathcal{H}}$: the quadratic structure does not distinguish the direction of the non-conservation, only its magnitude relative to the dissipative vorticity-transport content.

The $\hat{\mathbf{K}}$-channel term $-2\sigma_E \boldsymbol{\Omega}$ of HH in Eqn. (276) couples this non-conservation directly to the vorticity structure: wherever a local source or sink of mass-energy coincides with non-zero vorticity, the two interact bilinearly, with the sign of the coupling determined by the relative orientation and sign of σ_E and $\boldsymbol{\Omega}$. This is the relativistic fluid analogue of dilatation-vorticity coupling in compressible flow theory, where local compression or expansion of the fluid interacts with vorticity to produce vortex stretching or compression. The $\hat{\boldsymbol{\sigma}}$-channel term $\frac{2\sigma_E}{c}\boldsymbol{\Pi}$ plays the corresponding role for the transport-gradient sector: local non-conservation directly forces the momentum-energy transport structure.

The relativistic regime is therefore characterised by a three-way coupling — between local non-conservation σ_E, vorticity $\boldsymbol{\Omega}$, and transport gradient $\boldsymbol{\Pi}$ — all generated as channel projections of the single quadratic product HH, with no additional coupling terms introduced by hand.

The classical (locally conserved) case: $\sigma_E = 0$

In the classical or locally-conserved limit, $\sigma_E = 0$ and

$$u_{\mathcal{H}} = -\Omega^2 - \frac{1}{c^2}\Pi^2 \leq 0 \tag{281}$$

identically. With no local source of mass-energy, $u_{\mathcal{H}}$ is negative semi-definite everywhere: it can only represent dissipation, never production. This is the regime in which the connection to classical enstrophy becomes sharpest.

The term $\Omega^2 = |\nabla \times \mathbf{g}|^2$ is, up to normalisation, the *enstrophy density* of the momentum field. Enstrophy is one of the most studied quantities in classical fluid dynamics, central to the theory of the turbulent energy cascade and to the dissipation of kinetic energy in viscous flow, where the kinetic energy balance takes the form

$$\frac{dE_{\text{kin}}}{dt} = -2\nu \int \Omega^2 \, dV, \tag{282}$$

with ν the kinematic viscosity. In this classical relation, enstrophy enters *exclusively as a sink*: it never appears with positive sign in the energy balance, because it represents the rate at which organised kinetic energy is irreversibly converted into heat through viscous shear. The appearance of $-\boldsymbol{\Omega}^2$ in $u_{\mathcal{H}}$, with the same sign convention, is therefore not an anomaly to be corrected but a direct structural echo of this established role: $u_{\mathcal{H}}$ inherits, from the BQ algebra alone, the same sign relationship between vorticity-squared and dissipation that the Navier–Stokes energy equation establishes by separate physical argument.

The companion term $\Pi^2/c^2 = |\partial_t \mathbf{g} + \nabla\varepsilon|^2/c^2$ enters with the same negative sign and therefore plays an analogous dissipative role with respect to the momentum-energy transport sector. Where $\boldsymbol{\Omega}^2$ measures dissipation associated with rotational (vortical) structure, Π^2/c^2 measures dissipation associated with the irrotational, transport-gradient structure — the two together exhausting the $\hat{K}$ and $\hat{\boldsymbol{\sigma}}$ content of $\mathcal{H}$. In the classical, source-free limit, $u_{\mathcal{H}}$ is thus the total dissipation rate density of the fluid field, summing the vortical (enstrophy) and irrotational (transport-gradient) contributions with equal status.

The transition between regimes

The two cases above are not independent special cases but two ends of a single continuum governed by the relative magnitude of σ_E^2 versus $\boldsymbol{\Omega}^2 + \Pi^2/c^2$. As a fluid element transitions from a locally conserved, classical regime ($\sigma_E \to 0$, $u_{\mathcal{H}} \leq 0$, purely dissipative) into a regime with significant local energy exchange (σ_E growing), $u_{\mathcal{H}}$ increases monotonically with σ_E^2 and can cross zero. The zero crossing, $\sigma_E^2 = \boldsymbol{\Omega}^2 + \Pi^2/c^2$, marks the point at which local production exactly compensates local dissipation — a steady-state condition for the quadratic field structure $\mathcal{H}$, analogous to a local thermodynamic balance between entropy production and entropy export.

This gives $u_{\mathcal{H}}$ a natural role as a diagnostic for identifying regions of a relativistic fluid flow that are net sources versus net sinks of the quadratic invariant carried by $\mathcal{H}$, with the classical, enstrophy-dominated dissipative regime recovered exactly as the $\sigma_E \to 0$ limit. The relativistic correction Π^2/c^2 vanishes in this same limit relative to $\boldsymbol{\Omega}^2$ whenever Π remains finite as $c \to \infty$, so that the classical dissipation rate is dominated by enstrophy alone, with Π^2/c^2 representing a genuinely relativistic correction to the dissipative budget that becomes significant only when the momentum-energy transport gradient $\boldsymbol{\Pi}$ is comparable in magnitude to $c\,\boldsymbol{\Omega}$.

Summary

The quadratic products HH and $H^T H$ of §4.2.1–4.2.2 generate, without additional assumptions, a signed quantity $u_{\mathcal{H}}$ whose structure — a production term σ_E^2

from local non-conservation, balanced against a dissipation term $\Omega^2 + \Pi^2/c^2$ combining enstrophy and transport-gradient content — mirrors the established role of enstrophy as a sink in the classical kinetic energy balance, while extending it to a relativistic and source-admitting setting. Rather than forcing $u_{\mathcal{H}}$ into the mould of a positive-definite electromagnetic-style energy density, the BQ algebra produces directly a production-dissipation balance of the kind familiar from turbulence and irreversible thermodynamics, with the classical enstrophy dissipation relation recovered exactly as the locally-conserved, non-relativistic limit.

Prior art

The principle that physics "lives in the channels" of a single algebraic product has clear prior art, both in geometric algebra and, independently, in fluid dynamics itself. In geometric algebra, Hestenes' reformulation of electrodynamics as $\nabla F = J$, with F the field as one bivector combining $\mathbf{E}$ and $\mathbf{B}$, recovers the four Maxwell equations as grade projections of one statement Hestenes (2003); Doran and Lasenby (2003b). The fluid sector has its own, earlier and independent precedent for the same kind of unification. Marmanis Marmanis (1998) identified the vorticity $\mathbf{w} = \nabla \times \mathbf{u}$ and the Lamb vector $\mathbf{l} = \mathbf{w} \times \mathbf{u}$ as a dual pair obeying, after averaging, a closed linear system — $\nabla \cdot \mathbf{W} = 0$, $\partial_t \mathbf{W} = -\nabla \times \mathbf{L} + \nu \nabla^2 \mathbf{W}$, $\nabla \cdot \mathbf{L} = N$, $\partial_t \mathbf{L} = \langle u^2 \rangle \nabla \times \mathbf{W} - \mathbf{J} + \ldots$ — term-by-term isomorphic to the microscopic Maxwell equations, with $\mathbf{l} \leftrightarrow \mathbf{e}$, $\mathbf{w} \leftrightarrow \mathbf{b}$, and the divergence of the Lamb vector identified as a "turbulent charge density" sourced by $\mathbf{u} \cdot (\nabla \times \mathbf{w}) - |\mathbf{w}|^2$. This vector-calculus analogy was subsequently given a geometric-algebra formulation by Panakkal, Parameswaran, and Vedan Panakkal et al. (2020), who construct the bivector $F = \mathbf{l} + I\mathbf{w}$ from $W = \nabla V$ over the four-velocity V, derive fluid analogues of the Poynting theorem, the Lorentz force, and all four Maxwell equations from the single equation $\nabla W = -J$, and compute the quadratic invariant $WW = -(|\mathbf{l}|^2 + |\mathbf{w}|^2) + 2(\mathbf{l} \cdot \mathbf{w})I$.

Both works establish, independently of the present formalism, that the vorticity-Lamb-vector pair forms a Maxwell-like field structure with its own Poynting and Lorentz analogues — precisely the role played by $\boldsymbol{\Omega}$ and $\boldsymbol{\Pi}$ in $\mathcal{H} = \partial^T G$ here. Three differences are worth noting. First, and most fundamentally, the Marmanis/Panakkal pair $(\mathbf{w}, \mathbf{l})$ is already nonlinear in the velocity field: $\mathbf{l} = \mathbf{w} \times \mathbf{u}$ is bilinear in $\mathbf{u}$, so the field $W = \nabla V$ itself carries the convective nonlinearity from the outset. By contrast, $\boldsymbol{\Omega} = \nabla \times \mathbf{g}$ and $\boldsymbol{\Pi} = \partial_t \mathbf{g} + \nabla \varepsilon$ are both *linear* functionals of the momentum-density four-vector G, being simply the two channels of $\partial^T G$. The price of this linearity is paid downstream, in the second equation $V\mathcal{H} = F$ of §sec:UHFeq, which is the fluid analogue of the Lorentz-force law $JB = F$: the velocity-quadratic nonlinearity enters only there, through the second contraction with V, rather than being built into the field itself. This difference in where the nonlinearity sits is also why $H^T H$ in Eqns. (220)–(223) takes the clean Maxwell-like form it does, without the convective cross-terms already present in Panakkal's non-barotropic invariant $\tilde{W}^2$ (see the third point below). Second, Panakkal et al. work in the Euclidean geometric algebra G_4, so WW is a uniformly-signed sum with no Minkowski split; the present HH, built on the signature carried by $K^\mu = i\sigma^\mu$, instead yields the difference $\Pi^2/c^2 - \Omega^2$, with the associated Lorentz-covariance/non-covariance distinction between HH and $H^T H$ absent from their treatment. Third, their non-barotropic invariant $\tilde{W}^2$ already contains a term $2(\nabla \cdot \tilde{V})^2$ alongside $-(|\tilde{\mathbf{l}}|^2 + |\tilde{\mathbf{w}}|^2)$ and cross-couplings proportional to $\nabla \cdot \tilde{V}$ — structurally anticipating the σ_E^2, $-2\sigma_E \boldsymbol{\Omega}$, and $(2\sigma_E/c)\boldsymbol{\Pi}$ terms of §4.2.2 — but neither paper connects the resulting scalar to the Navier–Stokes enstrophy-dissipation relation $dE_{\mathrm{kin}}/dt = -2\nu \int \Omega^2 \, dV$, leaving the sign and role of the vorticity-squared term as an explicitly open question in Panakkal et al. (2020). The present identification of $u_{\mathcal{H}} = \sigma_E^2 - \Omega^2 - \Pi^2/c^2$ as a signed production-dissipation balance, with $-\Omega^2$ inheriting its sign directly from the established dissipative role of enstrophy, is the point at which §4.2 extends this line of work.

4.3 The force equation

We now evaluate the force equation of the relativistic fluid hierarchy,

$$U(\partial^T G) = U\mathcal{H} = F, \tag{283}$$

which is the fluid analogue of the Lorentz-force equation $JB = F$ in electromagnetism.

Using

$$U = u_0 \hat{\mathrm{T}} + \boldsymbol{u} \cdot \hat{\mathbf{K}} = \gamma c \hat{\mathrm{T}} + \gamma \boldsymbol{v} \cdot \hat{\mathbf{K}} = \gamma V, \tag{284}$$

and

$$\mathcal{H} = -\sigma_E \hat{\mathrm{1}} + \boldsymbol{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c} \boldsymbol{\Pi} \cdot \hat{\boldsymbol{\sigma}}, \tag{285}$$

the product $V\mathcal{H}$ becomes

$$\begin{aligned} U\mathcal{H} = (u_0 \hat{\mathrm{T}} + \boldsymbol{u} \cdot \hat{\mathbf{K}}) \left(-\sigma_E \hat{\mathrm{1}} + \boldsymbol{\Omega} \cdot \hat{\mathbf{K}} - \frac{1}{c} \boldsymbol{\Pi} \cdot \hat{\boldsymbol{\sigma}} \right) = \\ (-\boldsymbol{u} \cdot \boldsymbol{\Omega}) \, \hat{\mathrm{1}} + \left(-\sigma_E u_0 - \frac{1}{c} \boldsymbol{u} \cdot \boldsymbol{\Pi} \right) \hat{\mathrm{T}} \\ + \left(\boldsymbol{u} \times \boldsymbol{\Omega} - \frac{u_0}{c} \boldsymbol{\Pi} - \sigma_E \boldsymbol{u} \right) \cdot \hat{\mathbf{K}} + \left(-\frac{1}{c} \boldsymbol{u} \times \boldsymbol{\Pi} - u_0 \boldsymbol{\Omega} \right) \cdot \hat{\boldsymbol{\sigma}} = \\ (-\boldsymbol{u} \cdot \boldsymbol{\Omega}) \, \hat{\mathrm{1}} + \left(-\sigma_E u_0 - \frac{1}{c} \boldsymbol{u} \cdot \boldsymbol{\Pi} \right) \hat{\mathrm{T}} \end{aligned} \tag{286}$$

$$+\left(\boldsymbol{u}\times\boldsymbol{\Omega}-\frac{u_0}{c}\boldsymbol{\Pi}-\sigma_E\boldsymbol{u}\right)\cdot\hat{\mathbf{K}}+\left(-\frac{1}{c}\boldsymbol{u}\times\boldsymbol{\Pi}-u_0\boldsymbol{\Omega}\right)\cdot\hat{\boldsymbol{\sigma}} \tag{287}$$

Thus the fluid-force equation decomposes naturally into four channels:

$$V\mathcal{H} = F_{\hat{1}}\hat{1} + F_T\hat{\mathrm{T}} + \boldsymbol{F}_K\cdot\hat{\mathbf{K}} + \boldsymbol{F}_\sigma\cdot\hat{\boldsymbol{\sigma}}. \tag{288}$$

with

$$\hat{1} - channel: \qquad F_1 = -\boldsymbol{u}\cdot\boldsymbol{\Omega} \tag{289}$$

$$\hat{\mathrm{T}} - channel: \qquad F_T = -\sigma_E u_0 - \frac{1}{c}\boldsymbol{u}\cdot\boldsymbol{\Pi} \tag{290}$$

$$\hat{\mathbf{K}} - channel: \qquad F_K = \boldsymbol{u}\times\boldsymbol{\Omega} - \frac{u_0}{c}\boldsymbol{\Pi} - \sigma_E\boldsymbol{u} \tag{291}$$

$$\hat{\boldsymbol{\sigma}} - channel: \qquad F_\sigma = -\frac{1}{c}\boldsymbol{u}\times\boldsymbol{\Pi} - u_0\boldsymbol{\Omega}. \tag{292}$$

The structure is exactly parallel to the EM Lorentz-force product JB, but with the substitutions

$$\boldsymbol{B}\to\boldsymbol{\Omega}, \qquad \boldsymbol{E}\to\boldsymbol{\Pi}, \qquad J\to U. \tag{293}$$

4.3.1 Interpretation of the norm channel: helicity density

The norm channel is $F_{\hat{1}} = -\boldsymbol{u}\cdot\boldsymbol{\Omega}$. Since $\boldsymbol{\Omega} = \nabla\times\boldsymbol{g}$, this channel measures the alignment of the fluid velocity with the vorticity of the momentum-density flow. The quantity $\boldsymbol{u}\cdot\boldsymbol{\Omega}$ is a generalized helicity-density structure in fluid dynamics. For constant density, where $\boldsymbol{g} = \rho_0\boldsymbol{u}$, it reduces, up to a density factor, to the standard fluid helicity density $\boldsymbol{u}\cdot(\nabla\times\boldsymbol{u})$. The norm channel therefore represents the local helicity coupling of the flow to its own vorticity structure. For irrotational flow it vanishes identically.

Geometrically, the quantity $\boldsymbol{u}\cdot\boldsymbol{\Omega}$ measures the degree to which the velocity field moves along the vortex lines of the momentum-density flow. If $\boldsymbol{u}\cdot\boldsymbol{\Omega}\neq 0$, the flow possesses helicity: the velocity field spirals around and along the local vortex structure. If $\boldsymbol{u}\cdot\boldsymbol{\Omega} = 0$ while $\boldsymbol{\Omega}\neq 0$, the flow remains rotational but non-helical: the velocity circulates around the vortex structure without propagating along the vortex direction itself. This distinction is important in vortex dynamics, turbulence and plasma physics, where rotational flow and helical flow represent different topological regimes of the fluid motion.

The relationship between helicity and the norm channel can be made precise for variable ρ_0 without imposing any condition on $\mathcal{H}$. In the slow-flow limit, $\nabla\times\boldsymbol{g} = \nabla\times(\rho_0\boldsymbol{u})$ expands as

$$\boldsymbol{\Omega} = \nabla\times\boldsymbol{g} \approx \rho_0\nabla\times\boldsymbol{v} - \boldsymbol{v}\times(\nabla\rho_0), \tag{294}$$

which is an identity valid for any flow, with no field condition imposed. Taking the dot product with $\boldsymbol{u} \approx \boldsymbol{v}$ gives

$$\boldsymbol{u} \cdot \boldsymbol{\Omega} \approx \boldsymbol{v} \cdot (\rho_0 \nabla \times \boldsymbol{v} - \boldsymbol{v} \times (\nabla \rho_0)) = \rho_0 \, \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}), \tag{295}$$

where the second equality follows from the identity $\boldsymbol{v} \cdot (\boldsymbol{v} \times (\nabla \rho_0)) = 0$, which holds for any $\boldsymbol{v}$ and $\nabla \rho_0$. The norm channel condition $F_{\hat{1}} = 0$, i.e. $\boldsymbol{u} \cdot \boldsymbol{\Omega} = 0$, therefore gives

$$\rho_0 \, \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) = 0, \tag{296}$$

and since $\rho_0 \neq 0$, this is the condition of vanishing helicity density. Crucially, this result follows from $F_{\hat{1}} = 0$ alone — it does not require $\mathcal{H} = 0$ or any condition on the field level. Whether the flow is conservative, baroclinically forced, or fully non-conservative, helicity vanishes if and only if the norm channel of the force equation vanishes. Helicity generation is therefore structurally confined to the norm channel of $U\mathcal{H} = F$: a non-zero $F_{\hat{1}}$ is not merely a diagnostic of helical flow but the only mechanism within the BQ force hierarchy through which helicity can be generated, independently of the state of $\mathcal{H}$.

This points to a structural contrast between what $\mathcal{H}$ and $U\mathcal{H}$ each see in the vorticity field. In $\mathcal{H}$, the full slow-flow expansion $\boldsymbol{\Omega} = \rho_0 \nabla \times \boldsymbol{v} - \boldsymbol{v} \times (\nabla \rho_0)$ is active: both the rotational term and the baroclinic term $\boldsymbol{v} \times (\nabla \rho_0)$ contribute, and the $\hat{K}$-channel condition $\mathcal{H} = 0$ encodes a genuine balance between them, as established in §4.1.1. In the norm channel of $U\mathcal{H}$, however, the baroclinic term vanishes identically upon contraction with $\boldsymbol{v}$, through $\boldsymbol{v} \cdot (\boldsymbol{v} \times (\nabla \rho_0)) = 0$, regardless of the state of $\mathcal{H}$ and regardless of how large $\nabla \rho_0$ is. This is not an approximation or a special case: baroclinic vorticity $\boldsymbol{v} \times (\nabla \rho_0)$ is perpendicular to $\boldsymbol{v}$ by construction, so it is geometrically invisible to the norm channel. The field level $\mathcal{H}$ and the force level $U\mathcal{H}$ therefore decompose the vorticity field into two complementary and orthogonal sectors: $\mathcal{H}$ encodes the baroclinic-rotational balance between $\rho_0 \nabla \times \boldsymbol{v}$ and $\boldsymbol{v} \times (\nabla \rho_0)$, while the norm channel of $U\mathcal{H}$ encodes only the helical content $\rho_0 \, \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v})$, to which the baroclinic contribution is blind. The BQ product structure thus automatically separates baroclinic vorticity generation — visible at the field level, invisible at the norm channel of the force level — from helical vorticity — invisible to the $\hat{K}$-channel balance of $\mathcal{H} = 0$, but the sole content of $F_{\hat{1}}$ — without any separate geometric argument being required.

A useful contrast is provided by the difference between a spiral galactic disk and a relativistic galactic jet. A spiral galaxy without a jet typically possesses non-vanishing vorticity, $\boldsymbol{\Omega} \neq 0$, since the disk is rotationally supported and density stratification generates baroclinic vorticity $\boldsymbol{v} \times (\nabla \rho_0) \neq 0$ through the vertical density gradient. However, this baroclinic contribution is perpendicular to $\boldsymbol{v}$ by construction and therefore invisible to the norm channel, while the dominant velocity field is approximately azimuthal and the remaining rotational vorticity is mainly perpendicular to the disk plane, so that $\boldsymbol{u} \cdot \boldsymbol{\Omega} \approx \rho_0 \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) \approx 0$. The flow is therefore rotational and baroclinically forced, yet non-helical, and $F_{\hat{1}} \approx 0$: neither the rotational nor the baroclinic vorticity structure constitutes an active helicity source. A relativistic galactic jet, in contrast, contains both rotational and longitudinal propagation along the jet axis. The velocity field then acquires a component parallel to the vortex structure, giving $\boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) \neq 0$ and $F_{\hat{1}} \neq 0$: a genuine helicity source is active in the norm channel. The $\hat{1}$-channel therefore not only distinguishes between purely rotational or baroclinically forced vortex flow and genuinely helical transport along vortex structures, but identifies $F_{\hat{1}}$ as the only active helicity source within the BQ force hierarchy — one that is sensitive exclusively to the helical content of the vorticity field and blind to its baroclinic component by geometric necessity.

4.3.2 Interpretation of the time channel: power balance

The time channel is

$$F_T = -u_0 \sigma_E - \frac{1}{c} \boldsymbol{u} \cdot \boldsymbol{\Pi}. \tag{297}$$

Using

$$\begin{aligned} \sigma_E &= \frac{1}{c^2} \partial_t \varepsilon + \nabla \cdot \boldsymbol{g}, \\ \boldsymbol{\Pi} &= \partial_t \boldsymbol{g} + \nabla \varepsilon, \end{aligned} \tag{298}$$

this becomes

$$F_T = -\frac{u_0}{c^2} \partial_t \varepsilon - u_0 \nabla \cdot \boldsymbol{g} - \frac{1}{c} \boldsymbol{u} \cdot \partial_t \boldsymbol{g} - \frac{1}{c} \boldsymbol{u} \cdot \nabla \varepsilon, \tag{299}$$

A highly restricted conditions

If we assume $\boldsymbol{g} = \rho_0 \boldsymbol{u} = \gamma \rho_0 \boldsymbol{v}$ with a constant ρ_0 in $\varepsilon = \rho_0 c u_0 = \gamma \rho_0 c^2 = \gamma \varepsilon_0$ and we use the slow flow approximations $\varepsilon \approx \rho_0 c^2 (1 + \frac{v^2}{2c^2})$ and $\boldsymbol{u} \approx \boldsymbol{v}$ and $u_0 = \gamma c \approx (1 + \frac{v^2}{2c^2}) c \approx c$, we get

$$-\frac{c}{\rho_0} F_T = \partial_t \left(\frac{v^2}{2} \right) + c^2 \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \left(\frac{v^2}{2} \right), \tag{300}$$

Together with energy density conservation, $\sigma_E = 0$, we get

$$-\frac{c}{\rho_0} F_T = \boldsymbol{v} \cdot \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \left(\frac{v^2}{2} \right) = \partial_t \left(\frac{v^2}{2} \right) + \boldsymbol{v} \cdot \nabla \left(\frac{v^2}{2} \right) = \frac{D}{Dt} \left(\frac{v^2}{2} \right), \tag{301}$$

which, writing the time-like part of the four-force as the local power exchange density term $F_T = \frac{\mathcal{P}}{c}$ and the kinetic energy density as $\mathcal{K}$, results in

$$\mathcal{P} = -\frac{D}{Dt}\left(\frac{\rho_0 v^2}{2}\right) = -\frac{D}{Dt}\mathcal{K}, \tag{302}$$

so we can conclude that $\mathcal{P}$ is the negative material rate of change of kinetic energy density. With zero power density, the kinetic energy density is constant. The power density represents sources or sinks of kinetic energy density; dissipative processes correspond to positive power extraction with $\mathcal{P} > 0$." So, under the conditions given, the $-\frac{1}{c}\boldsymbol{u} \cdot \boldsymbol{\Pi}$ part of the time channel measures the negative material rate of change of kinetic energy density along a fluid trajectory.

Relaxing to a non-constant ρ_0 condition

If we relax the condition to non-constant ρ_0, we get, from Eqn. (241),

$$\sigma_E = \frac{D}{Dt}\varepsilon_0 + \rho_0 c^2 \nabla \cdot \boldsymbol{v} + \partial_t \mathcal{K}, \tag{303}$$

and we get, from Eqn. (252),

$$\boldsymbol{\Pi} = \partial_t \boldsymbol{g}_0 + \nabla\varepsilon_0 + \nabla\mathcal{K}. \tag{304}$$

The UH time channel, under those conditions, gives

$$F_T = -c\sigma_E - \frac{1}{c}\boldsymbol{v} \cdot \boldsymbol{\Pi}. \tag{305}$$

Using $F_T = cP$ gives

$$P = -\sigma_E - \frac{1}{c^2}\boldsymbol{v} \cdot \boldsymbol{\Pi}. \tag{306}$$

Substituting Eqns. (303) and (304) into Eqn. (306) and expanding gives

$$P = -\left(\frac{D}{Dt}\varepsilon_0 + \rho_0 c^2 \nabla \cdot \boldsymbol{v} + \partial_t \mathcal{K}\right) - \frac{1}{c^2}\boldsymbol{v} \cdot \left(\partial_t \boldsymbol{g}_0 + \nabla\varepsilon_0 + \nabla\mathcal{K}\right). \tag{307}$$

In the slow-flow limit, the kinetic correction terms $\partial_t \mathcal{K}$ and $\frac{1}{c^2}\boldsymbol{v} \cdot \nabla\mathcal{K}$ are of order v^2/c^2 relative to the rest-energy terms $\varepsilon_0 \sim \rho_0 c^2$ and may be neglected at leading order, giving

$$P \approx -\frac{D}{Dt}\varepsilon_0 - \rho_0 c^2 \nabla \cdot \boldsymbol{v} - \frac{1}{c^2}\boldsymbol{v} \cdot \nabla\varepsilon_0. \tag{308}$$

Equation (307) is the general variable-ρ_0 power equation, valid for any state of $\mathcal{H}$, with no field condition imposed. In the slow-flow approximation (308), combines three contributions: the material rate of change of rest-energy density $\frac{D\varepsilon_0}{Dt}$, the compressibility-driven energy flux $\rho_0 c^2 \nabla \cdot \boldsymbol{v}$, and the mechanical power transfer $\frac{1}{c^2}\boldsymbol{v} \cdot \nabla\varepsilon_0$.

Starting from the general variable-ρ_0 slow-flow power equation Eqn. (308), we insert two thermodynamic identities that hold for any barotropic fluid with equation of state $p = p(\varepsilon_0)$ and relativistic sound speed $c_s^2 = c^2(\partial p/\partial\varepsilon_0)_s$, independently of whether $\mathcal{H} = 0$ or not. The first is the barotropic gradient relation from §4.1.1, Eqn. (256),

$$\nabla\varepsilon_0 = \frac{c^2}{c_s^2}\nabla p, \tag{309}$$

giving $\frac{1}{c^2}\boldsymbol{v} \cdot \nabla\varepsilon_0 = \frac{1}{c_s^2}\boldsymbol{v} \cdot \nabla p$. The second is the barotropic material derivative relation,

$$\frac{D}{Dt}\varepsilon_0 = \frac{c^2}{c_s^2}\frac{Dp}{Dt}, \tag{310}$$

which follows directly from $p = p(\varepsilon_0)$ by the chain rule, again without imposing any condition on $\mathcal{H}$. The acoustic relation from §4.1.1., Eqn. (258),

$$\frac{Dp}{Dt} = -\rho_0 c_s^2 \nabla \cdot \boldsymbol{v}, \tag{311}$$

which likewise holds for any barotropic fluid independently of $\mathcal{H}$, then gives

$$\frac{D}{Dt}\varepsilon_0 = \frac{c^2}{c_s^2}\frac{Dp}{Dt} = -\rho_0 c^2 \nabla \cdot \boldsymbol{v}. \tag{312}$$

Substituting both relations into Eqn. (308), the first and second terms cancel exactly,

$$-\frac{D}{Dt}\varepsilon_0 - \rho_0 c^2 \nabla \cdot \boldsymbol{v} = \rho_0 c^2 \nabla \cdot \boldsymbol{v} - \rho_0 c^2 \nabla \cdot \boldsymbol{v} = 0, \tag{313}$$

and the power equation reduces to the single term

$$P \approx -\frac{1}{c_s^2}\boldsymbol{v} \cdot \nabla p. \tag{314}$$

This is a general result, valid for any barotropic fluid in the slow-flow limit, with no condition imposed on $\mathcal{H}$. The power density is determined entirely by the acoustic pressure-work: the rate at which the flow moves against or along the local pressure gradient, rescaled by the relativistic factor $1/c_s^2$.

The physical interpretation of Eqn. (314) is immediate and unambiguous. For $P > 0$, the flow moves against the pressure gradient, $\boldsymbol{v} \cdot \nabla p < 0$: fluid moves from high to low pressure, performing expansion work and losing energy to the field. For $P < 0$, the flow moves with the pressure gradient, $\boldsymbol{v} \cdot \nabla p > 0$: fluid is accelerated by the pressure field, gaining energy through compression work. For $P = 0$, either the flow is along isobars $\boldsymbol{v} \perp \nabla p$, or the pressure is locally uniform $\nabla p = 0$: no acoustic power exchange occurs regardless of the state of $\mathcal{H}$.

The cancellation of the first and second terms in Eqn. (308) has a precise physical origin that is worth identifying carefully. In the dust limit $\varepsilon_0 = \rho_0 c^2$, with no independent heat, internal-energy, or entropy channels, one has $\frac{D\varepsilon_0}{Dt} = c^2 \frac{D\rho_0}{Dt}$, so the cancellation becomes

$$-c^2 \frac{D\rho_0}{Dt} - \rho_0 c^2 \nabla \cdot \boldsymbol{v} = -c^2 \left(\frac{D\rho_0}{Dt} + \rho_0 \nabla \cdot \boldsymbol{v} \right) = 0, \tag{315}$$

which is exactly the mass continuity equation $\frac{D\rho_0}{Dt} = -\rho_0 \nabla \cdot \boldsymbol{v}$ in material form. The cancellation is therefore not primarily a thermodynamic statement about the equation of state but a *mass-continuity cancellation*: rest-energy advection and compressibility work cancel because mass is conserved, not because of any particular equation of state. Furthermore, the material derivative $\frac{D}{Dt} = \partial_t + \boldsymbol{v} \cdot \nabla$ already contains a contraction with $\boldsymbol{v}$, which means this continuity equation lives at the $U\mathcal{H}$ level of the hierarchy rather than at the $\mathcal{H}$ level: the Eulerian continuity $\partial_t \rho_0 + \nabla \cdot (\rho_0 \boldsymbol{v}) = 0$ is the $\hat{1}$-channel of $\mathcal{H} = 0$, while its material form $\frac{D\rho_0}{Dt} = -\rho_0 \nabla \cdot \boldsymbol{v}$ is the $U\mathcal{H}$-level statement, obtained by the convective contraction with $\boldsymbol{v}$ that characterizes the force level. The cancellation therefore occurs naturally and necessarily at the correct level of the BQ hierarchy, without being imposed from outside.

After this mass-continuity cancellation, the surviving term $-\frac{1}{c_s^2} \boldsymbol{v} \cdot \nabla p$ is the genuinely thermodynamic contribution to the power balance. Crucially, in a perfect dust with $p = 0$, this term vanishes identically: $P = 0$ for any flow, regardless of the state of $\mathcal{H}$, because there is no pressure to do work. The vanishing of P in the dust limit carries no dynamical information — it is trivially satisfied and is fully consistent with the $\mathcal{H} = 0$ potential-flow result of §4.1, which was derived precisely in the dust approximation. The non-trivial content of Eqn. (314) therefore lives entirely in the departure from dust, entering only through the enthalpy replacement $\varepsilon_0 \to w = \varepsilon_0 + p$ flagged at the end of §4.1: once pressure is included, $P = -\frac{1}{c_s^2} \boldsymbol{v} \cdot \nabla p$ becomes a genuine dynamical condition, vanishing only when the flow is along isobars $\boldsymbol{v} \perp \nabla p$ or the pressure is locally uniform $\nabla p = 0$. The mass-continuity cancellation is therefore not merely a mathematical convenience but the precise mechanism by which the dust contribution to P is eliminated, leaving only the thermodynamic pressure-work as the physical power exchange. The power density

Eqn. (314) consequently has a two-stage structure: a kinematic stage in which mass continuity at the $U\mathcal{H}$ level cancels the bulk dust-transport contributions, followed by a thermodynamic stage in which the equation of state determines the residual acoustic power exchange through $-\frac{1}{c_s^2}\boldsymbol{v}\cdot\nabla p$. This separation between kinematic and thermodynamic contributions to the power balance is not visible in the standard $T^{\mu\nu}$ formulation but emerges here as a direct consequence of the BQ channel structure and the two-level hierarchy.

The result (314) is also consistent with the classical acoustic intensity $\boldsymbol{I} = p\boldsymbol{v}$, whose divergence $\nabla\cdot\boldsymbol{I} = \boldsymbol{v}\cdot\nabla p + p\nabla\cdot\boldsymbol{v}$ contains $\boldsymbol{v}\cdot\nabla p$ as the convective pressure-work contribution. The compressibility term $p\nabla\cdot\boldsymbol{v}$ plays an analogous role to the cancelled bulk-transport terms, consistently with the mass-continuity cancellation identified above. The BQ time-channel power equation therefore connects directly to the classical acoustic energy flux, providing a further consistency check of the same character as the Bernoulli and pressure-dilatation results of §4.1 and §4.3.2 while adding the structural insight that the separation between kinematic and thermodynamic power contributions is a consequence of the two-level BQ hierarchy rather than a coincidence.

The three conditions under which $P = 0$ in the barotropic regime therefore form a nested hierarchy that is worth stating explicitly. The strongest condition is $\mathcal{H} = 0$: conservative potential flow, in which $P = 0$ follows automatically from the channel structure as shown above, with no pressure work possible because the dust approximation leaves no thermodynamic degree of freedom to carry it. The intermediate condition is $P = 0$ with $\mathcal{H} \neq 0$: from Eqn. (314), this requires

$$\boldsymbol{v}\cdot\nabla p = 0, \tag{316}$$

i.e. the flow is along isobars or the pressure is locally uniform. This is a genuine dynamical condition that admits non-zero vorticity $\boldsymbol{\Omega} \neq 0$, baroclinic forcing $\nabla\rho_0 \neq 0$, and non-trivial pressure gradients, while maintaining zero net local acoustic power exchange. The flow may be rotational, stratified, and compressible, yet energetically balanced in the sense that fluid elements move along rather than across pressure surfaces. Physical examples include the interiors of stationary accretion disks, where differential rotation maintains quasi-circular orbits along isobaric surfaces, and the cores of large-scale atmospheric circulation systems in long-term pressure equilibrium. The weakest condition is $P \neq 0$ with $\mathcal{H} \neq 0$: genuinely non-equilibrium flow in which fluid elements cross pressure surfaces, with $P > 0$ (expansion work, fluid moving from high to low pressure) or $P < 0$ (compression work, fluid accelerated by pressure gradient). This regime encompasses shock formation, jet launching, explosive reconnection, and strongly driven or dissipative flows.

The three-tier structure

$$\begin{aligned}
\mathcal{H} = 0 &\implies P = 0 \quad \text{(potential flow, dust, no thermodynamic d.o.f.)},\\
\boldsymbol{v}\cdot\nabla p = 0,\ \mathcal{H} \neq 0 &\implies P = 0 \quad \text{(isobaric flow, balanced rotational regime)},\\
\boldsymbol{v}\cdot\nabla p \neq 0,\ \mathcal{H} \neq 0 &\implies P \neq 0 \quad \text{(non-equilibrium, crossing isobars)},
\end{aligned} \tag{317}$$

is organized entirely by the BQ channel structure of $U\mathcal{H} = F$ and the barotropic equation of state. It is not visible in the standard $T^{\mu\nu}$ formulation, where the power balance is not separated into kinematic and thermodynamic contributions and the isobaric condition $\boldsymbol{v}\cdot\nabla p = 0$ does not appear as a natural intermediate regime. The separation into these three tiers, with the intermediate isobaric regime as the physically richest and most practically relevant, is a structural consequence of the mass-continuity cancellation at the $U\mathcal{H}$ level identified above: it is precisely because the kinematic bulk-transport contributions cancel by mass conservation that the thermodynamic isobaric condition $\boldsymbol{v}\cdot\nabla p = 0$ emerges as the clean separator between balanced and non-equilibrium flow, rather than being obscured by bulk-transport terms of comparable magnitude.

4.3.3 Interpretation of the space channel: Euler/Lamb force

The space channel is

$$F_K = \boldsymbol{u} \times \boldsymbol{\Omega} - \frac{u_0}{c}\boldsymbol{\Pi} - \sigma_E \boldsymbol{u}. \tag{318}$$

This is the direct analogue of the Lorentz-force structure $\boldsymbol{J} \times \boldsymbol{B} + q\boldsymbol{E}$. The term $\boldsymbol{u}\times\boldsymbol{\Omega}$ is the vortex-force term of fluid dynamics. It corresponds to the Lamb-vector structure appearing in the Euler equation. The term $\boldsymbol{\Pi} = \partial_t \boldsymbol{g} + \nabla\varepsilon$ represents the momentum-pressure-gradient force. The additional term $-\sigma_E\boldsymbol{u}$ appears only for non-conservative fluids and acts as a drag or driving force proportional to the local energy source density. The K-channel therefore represents the complete local force-density structure of the relativistic fluid.

Substituting $\sigma_E = \frac{1}{c^2}\partial_t\varepsilon + \nabla\cdot\boldsymbol{g}$, $\boldsymbol{\Omega} = \nabla\times\boldsymbol{g}$, and $\boldsymbol{\Pi} = \partial_t\boldsymbol{g} + \nabla\varepsilon$, gives

$$\boldsymbol{F}_K = \boldsymbol{u}\times(\nabla\times\boldsymbol{g}) - \frac{u_0}{c}\partial_t\boldsymbol{g} - \frac{u_0}{c}\nabla\varepsilon - \left(\frac{1}{c^2}\partial_t\varepsilon + \nabla\cdot\boldsymbol{g}\right)\boldsymbol{u}. \tag{319}$$

Different sets of assumptions then leads to separate known fluid dynamics regimes.

Euler/Lamb for conserved energy and constant density

Starting from the K-channel, we first impose the conservative energy-balance condition $\sigma_E = \frac{1}{c^2}\partial_t\varepsilon + \nabla\cdot\boldsymbol{g} = 0$. The K-channel then reduces to

$$\boldsymbol{F}_K = \boldsymbol{u}\times(\nabla\times\boldsymbol{g}) - \frac{u_0}{c}\partial_t\boldsymbol{g} - \frac{u_0}{c}\nabla\varepsilon. \tag{320}$$

For a single-component incompressible fluid with constant density,

$$\boldsymbol{g} = \rho_0 \boldsymbol{u}, \qquad \rho_0 = \text{const.}, \tag{321}$$

one has

$$\nabla \times \boldsymbol{g} = \rho_0 \nabla \times \boldsymbol{u} = \rho_0 \boldsymbol{\omega}_u, \tag{322}$$

and

$$\partial_t \boldsymbol{g} = \rho_0 \partial_t \boldsymbol{u}, \tag{323}$$

where

$$\boldsymbol{\omega}_u = \nabla \times \boldsymbol{u} \tag{324}$$

is the ordinary vorticity of the velocity field. Hence

$$\boldsymbol{F}_K = \rho_0 \boldsymbol{u} \times \boldsymbol{\omega}_u - \frac{u_0}{c} \rho_0 \partial_t \boldsymbol{u} - \frac{u_0}{c} \nabla \varepsilon. \tag{325}$$

We can rearrange this and divide by ρ_0 to get

$$\partial_t \boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{\omega}_u - \frac{u_0}{\rho_0 c} \nabla \varepsilon - \frac{u_0}{\rho_0 c} \boldsymbol{F}_K. \tag{326}$$

If we additionally assume non-relativistic velocities, with

$$\varepsilon \approx \rho_0 c^2 + \frac{1}{2} \rho_0 v^2, \qquad \boldsymbol{u} \approx \boldsymbol{v} \qquad \textit{and} \qquad u_0 \approx c, \tag{327}$$

then, given $\nabla \rho_0 = 0$,

$$\frac{1}{\rho} \nabla \varepsilon \approx \nabla \left(c^2 + \frac{1}{2} v^2 \right) = \nabla \left(\frac{1}{2} v^2 \right) \tag{328}$$

and we get the pressureles Euler-Lamb equation equation with external force density $\boldsymbol{F}_K$ as:

$$\partial_t \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{\omega}_v - \nabla \left(\frac{1}{2} v^2 \right) - \frac{1}{\rho_0} \boldsymbol{F}_K. \tag{329}$$

Using the Lamb identity

$$(\boldsymbol{v} \cdot \nabla) \boldsymbol{v} = \nabla \left(\frac{1}{2} v^2 \right) - \boldsymbol{v} \times \boldsymbol{\omega}_v, \tag{330}$$

the pressureless Euler-Lamb equation can be rewritten as

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho_0}\boldsymbol{F}_K. \tag{331}$$

The K-channel therefore reproduces directly the material-acceleration equation of fluid dynamics, with the source term supplied by $\boldsymbol{F}_K$.

If the underlying field configuration is such that F_K takes the form $\rho_0\nabla\phi - \rho_0\nu\nabla^2\boldsymbol{v}$, with gravitational potential ϕ and viscosity ν (gravity ϕ plus a viscous-stress-like contribution ν), then Eqn.(331) becomes the familiar pressureless Navier–Stokes equation

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla\phi + \nu\nabla^2\boldsymbol{v}. \tag{332}$$

Using $\frac{D}{Dt} = \partial_t + (\boldsymbol{v} \cdot \nabla)$, it can be written as

$$\rho_0 \frac{D}{Dt}\boldsymbol{v} = -\boldsymbol{F}_K. \tag{333}$$

Euler/Lamb for conserved energy and varying density

We start again with F_K with $\sigma_E = 0$ so

$$\boldsymbol{F}_K = \boldsymbol{u} \times (\nabla \times \boldsymbol{g}) - \frac{u_0}{c}\partial_t \boldsymbol{g} - \frac{u_0}{c}\nabla\varepsilon, \tag{334}$$

and apply $\boldsymbol{g} = \rho_0\boldsymbol{u}$, $\boldsymbol{u} \approx \boldsymbol{v}$, $u_0 \approx c$ to get

$$\boldsymbol{F}_K = \boldsymbol{v} \times (\nabla \times (\rho_0\boldsymbol{v})) - \partial_t(\rho_0\boldsymbol{v}) - \nabla\varepsilon, \tag{335}$$

We then apply $\varepsilon \approx \rho_0 c^2 + \frac{1}{2}\rho_0 v^2$ to get

$$\boldsymbol{F}_K = \boldsymbol{v} \times (\nabla \times (\rho_0\boldsymbol{v})) - \partial_t(\rho_0\boldsymbol{v}) - \nabla\left(\rho_0 c^2 + \frac{1}{2}\rho_0 v^2\right). \tag{336}$$

This can be expanded as

$$\boldsymbol{F}_K = \boldsymbol{v} \times ((\nabla\rho_0) \times \boldsymbol{v}) + \rho_0\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) \tag{337}$$

$$-\rho_0\partial_t\boldsymbol{v} - \boldsymbol{v}\partial_t\rho_0 - c^2\nabla\rho_0 - \frac{1}{2}\rho_0\nabla v^2 - \frac{1}{2}v^2\nabla\rho_0, \tag{338}$$

and then with

$$\boldsymbol{v} \times ((\nabla\rho_0) \times \boldsymbol{v}) = v^2\nabla\rho_0 - \boldsymbol{v}(\boldsymbol{v} \cdot \nabla\rho_0) \tag{339}$$

reduced to

$$\boldsymbol{F}_K = \rho_0\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) + v^2\nabla\rho_0 - \boldsymbol{v}(\boldsymbol{v} \cdot \nabla\rho_0) \tag{340}$$

$$-\rho_0\partial_t\boldsymbol{v} - \boldsymbol{v}\partial_t\rho_0 - c^2\nabla\rho_0 - \frac{1}{2}\rho_0\nabla v^2 - \frac{1}{2}v^2\nabla\rho_0, \tag{341}$$

and reordered into

$$\boldsymbol{F}_K = \rho_0\left(\boldsymbol{v}\times(\nabla\times\boldsymbol{v}) - \partial_t\boldsymbol{v} - \frac{1}{2}\nabla v^2\right) \tag{342}$$

$$+v^2\nabla\rho_0 - \boldsymbol{v}(\boldsymbol{v}\cdot\nabla\rho_0) - \boldsymbol{v}\partial_t\rho_0 - c^2\nabla\rho_0 - \frac{1}{2}v^2\nabla\rho_0 \tag{343}$$

and this equals

$$\boldsymbol{F}_K = -\rho_0\frac{D\boldsymbol{v}}{Dt} - c^2\nabla\rho_0 + \frac{1}{2}v^2\nabla\rho_0 - \boldsymbol{v}\left(\partial_t\rho_0 + \boldsymbol{v}\cdot\nabla\rho_0\right). \tag{344}$$

If we use $\nabla p = c_s^2\nabla\rho_0$ and define the Mach number M as $M = \frac{v}{c_s}$, we get

$$\frac{D\boldsymbol{v}}{Dt} = \frac{1}{\rho_0}\left(\frac{1}{2}M^2\right)\nabla p - \frac{1}{\rho_0}\frac{c^2}{c_s^2}\nabla p - \frac{1}{\rho_0}\boldsymbol{v}\frac{D\rho_0}{Dt} - \frac{1}{\rho_0}\boldsymbol{F}_K. \tag{345}$$

Equation (345) deserves careful labelling. The derivation used explicit slow-flow approximations $\boldsymbol{u} \approx \boldsymbol{v}$, $u_0 \approx c$, and $\varepsilon \approx \rho_0 c^2 + \frac{1}{2}\rho_0 v^2$ throughout, so the flow itself is non-relativistic in the kinematic sense: no Lorentz factors appear in the velocity field and $v \ll c$ is assumed. The equation is therefore not a relativistic momentum equation in the sense of describing fast-moving fluid elements. The factor c^2/c_s^2 that dominates the pressure-gradient term does not originate from the kinematics of the flow but from the thermodynamics of the equation of state: it enters through the relativistic sound speed definition $c_s^2 = c^2(\partial p/\partial\varepsilon_0)_s$, which uses the rest-energy density $\varepsilon_0 = \rho_0 c^2$ as the inertial quantity rather than the mass density ρ_0 alone. Even at $v = 0$, the factor c^2/c_s^2 is present and large for normal fluids with $c_s \ll c$: it is a static thermodynamic property of the fluid, not a kinematic correction that grows with velocity.

Equation (345) should therefore be identified as the *compressible Euler equation for a slow barotropic fluid with a relativistic equation of state*: the flow is non-relativistic, but the thermodynamic content — the relationship between pressure, energy density, and sound speed — is described relativistically through $c_s^2 = c^2(\partial p/\partial\varepsilon_0)_s$. This distinction is standard in relativistic fluid dynamics: fluids in heavy-ion collisions or neutron star interiors can move slowly while still requiring a relativistic equation of state, because their energy density is dominated by rest-mass and nuclear binding energy rather than by thermal pressure. The c^2/c_s^2 amplification factor in Eqn. (345) is precisely this thermodynamic effect: a given pressure gradient must overcome an inertial resistance set by $\varepsilon_0 = \rho_0 c^2$ rather than by ρ_0 alone, amplifying the effective force by c^2/c_s^2 relative to the Newtonian expectation.

With this identification, the limiting behavior of Eqn. (345) is transparent. In the stiff-equation-of-state limit $c_s \to c$, the energy density is dominated by pressure ($\partial p/\partial\varepsilon_0 \to 1$) and the inertial resistance reduces to ρ_0, so the dominant pressure term becomes $-\frac{1}{\rho_0}\nabla p$ and the standard barotropic Euler equation is recovered. In the opposite limit $c_s \ll c$, appropriate for normal non-relativistic fluids, the amplification factor $c^2/c_s^2 \gg 1$ dominates, reflecting the large ratio of rest-mass energy to thermal pressure energy. For subsonic flow $M \ll 1$ and incompressible flow $\nabla\cdot\boldsymbol{v} = 0$, the Mach correction and compressibility terms vanish and Eqn. (345) reduces to

$$\frac{D\boldsymbol{v}}{Dt} = -\frac{c^2}{\rho_0 c_s^2}\nabla p - \frac{1}{\rho_0}\boldsymbol{F}_K, \tag{346}$$

which in the stiff limit $c_s \to c$ gives the standard incompressible Euler equation

$$\frac{D\boldsymbol{v}}{Dt} = -\frac{1}{\rho_0}\nabla p - \frac{1}{\rho_0}\boldsymbol{F}_K, \tag{347}$$

recovering the constant-ρ_0 result as the appropriate double limit.

The factor c^2/c_s^2 threads consistently through all three channels of $U\mathcal{H} = F$ examined so far. In the time channel, $1/c_s^2$ rescales the acoustic pressure-work in $P \approx -\frac{1}{c_s^2}\boldsymbol{v}\cdot\nabla p$. In the sigma-channel of §4.1.1, c^2/c_s^2 amplifies the momentum equation at the field level, Eqn. (259). In the K-channel here, c^2/c_s^2 amplifies the pressure-gradient force at the force level. This cross-channel consistency is not engineered: the relativistic sound speed enters once, through the barotropic equation of state $p = p(\varepsilon_0)$ introduced via the enthalpy replacement $\varepsilon_0 \to w = \varepsilon_0 + p$ flagged at the end of §4.1, and then propagates through all subsequent channel projections at both the field and force levels automatically through the BQ product structure. The thermodynamic content of the equation of state is therefore not an external input added separately to each channel but a single algebraic input that the BQ hierarchy distributes consistently across all channels — power balance, momentum amplification, and acoustic force — without additional assumptions at any level.

4.3.4 Interpretation of the $\hat{\sigma}$-channel: rotational transport

The $\hat{\boldsymbol{\sigma}}$-channel is

$$\boldsymbol{F}_\sigma = -\frac{1}{c}\boldsymbol{v}\times\boldsymbol{\Pi} - c\boldsymbol{\Omega} = c\left(-\frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{\Pi} - \boldsymbol{\Omega}\right), \tag{348}$$

with $\boldsymbol{\Pi} = \partial_t\boldsymbol{g} + \nabla\varepsilon$ and $\boldsymbol{\Omega} = \nabla\times\boldsymbol{g}$. This channel is the fluid analogue of the EM spin channel $\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - c\rho\boldsymbol{B}$. Unlike the K-channel, which governs translational momentum balance, the $\hat{\boldsymbol{\sigma}}$-channel governs the rotational transport structure of the fluid flow. The term $-c\boldsymbol{\Omega}$ represents the intrinsic vorticity of the momentum-density field, measuring the local rotational geometry of the fluid. The term $-\frac{1}{c}\boldsymbol{v}\times\boldsymbol{\Pi}$ measures rotational structure generated dynamically by transport acceleration and momentum-pressure gradients: since $\boldsymbol{\Pi}$ is not in general aligned with $\boldsymbol{v}$, the cross product generates a transverse rotational force density. The $\hat{\boldsymbol{\sigma}}$-channel therefore measures the mismatch between intrinsic vorticity and transport-induced rotational dynamics.

Constant ρ_0, slow flow, $\sigma_E = 0$.

Applying $\boldsymbol{g} = \rho_0\boldsymbol{v}$, $\boldsymbol{u} \approx \boldsymbol{v}$, $u_0 \approx c$, and $\varepsilon \approx \rho_0 c^2 + \frac{1}{2}\rho_0 v^2$ gives

$$\boldsymbol{\Pi} = \rho_0\partial_t\boldsymbol{v} + \rho_0\nabla\left(\frac{v^2}{2}\right), \qquad \boldsymbol{\Omega} = \rho_0\nabla\times\boldsymbol{v} = \rho_0\boldsymbol{\omega}, \tag{349}$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ is the standard vorticity. Using the Lamb identity $\partial_t\boldsymbol{v} + \nabla(v^2/2) = D\boldsymbol{v}/Dt + \boldsymbol{\omega}\times\boldsymbol{v}$, the transport term becomes

$$-\frac{1}{c}\boldsymbol{v}\times\boldsymbol{\Pi} = -\frac{\rho_0}{c}\boldsymbol{v}\times\frac{D\boldsymbol{v}}{Dt} - \frac{\rho_0}{c}\boldsymbol{v}\times(\boldsymbol{\omega}\times\boldsymbol{v}). \tag{350}$$

Expanding the triple product $\boldsymbol{v}\times(\boldsymbol{\omega}\times\boldsymbol{v}) = v^2\boldsymbol{\omega} - (\boldsymbol{v}\cdot\boldsymbol{\omega})\boldsymbol{v}$, the full $\hat{\boldsymbol{\sigma}}$-channel becomes

$$\boldsymbol{F}_\sigma = -\frac{\rho_0}{c}\boldsymbol{v}\times\frac{D\boldsymbol{v}}{Dt} - c\rho_0\boldsymbol{\omega} - \frac{\rho_0}{c}\Big(v^2\boldsymbol{\omega} - (\boldsymbol{v}\cdot\boldsymbol{\omega})\boldsymbol{v}\Big). \tag{351}$$

The three contributions have distinct physical identities. The first, $-\frac{\rho_0}{c}\boldsymbol{v}\times\frac{D\boldsymbol{v}}{Dt}$, is the rate of rotation of the material acceleration, measuring the tendency of the flow to generate new rotational structure through its own acceleration. The second, $-c\rho_0\boldsymbol{\omega}$, is the intrinsic vorticity, weighted by c. The third, $-\frac{\rho_0}{c}(v^2\boldsymbol{\omega} - (\boldsymbol{v}\cdot\boldsymbol{\omega})\boldsymbol{v})$, contains the helicity density $\boldsymbol{v}\cdot\boldsymbol{\omega}$ already analyzed in §4.3.1: the last term $\frac{\rho_0}{c}(\boldsymbol{v}\cdot\boldsymbol{\omega})\boldsymbol{v}$ is a helicity-weighted velocity contribution to the rotational force, connecting the $\hat{\boldsymbol{\sigma}}$-channel directly to the norm channel through the helicity density. In the slow-flow limit $v^2 \ll c^2$, the third term is negligible relative to the second, and Eqn. (351) reduces to

$$\boldsymbol{F}_\sigma \approx -\frac{\rho_0}{c}\boldsymbol{v}\times\frac{D\boldsymbol{v}}{Dt} - c\rho_0\boldsymbol{\omega}. \tag{352}$$

The rotational balance condition.

Setting $\boldsymbol{F}_\sigma = 0$ in Eqn. (352) gives

$$c^2\boldsymbol{\omega} = -\boldsymbol{v}\times\frac{D\boldsymbol{v}}{Dt}. \tag{353}$$

This is the rotational transport-balance condition: the vorticity is exactly sustained by the rate of rotation of the material acceleration. The right-hand side $-\boldsymbol{v} \times \frac{D\boldsymbol{v}}{Dt}$ is the rate of change of the angular momentum density per unit mass of the flow. Equation (353) therefore states that in a rotationally balanced flow, the vorticity is neither created nor destroyed but is exactly maintained by the rotational component of the material acceleration. For force-free flow with $\frac{D\boldsymbol{v}}{Dt} = 0$, Eqn. (353) gives $\boldsymbol{\omega} = 0$, consistently recovering irrotationality as the force-free, rotationally balanced limit — the same irrotationality that appears as the $\hat{K}$-channel of $\mathcal{H} = 0$ in §4.1. This is a further instance of the hierarchical consistency of the BQ framework: the irrotational condition is established at the field level by $\mathcal{H} = 0$ and re-emerges at the force level as the force-free limit of the $\hat{\boldsymbol{\sigma}}$-channel balance condition.

Variable ρ_0, barotropic extension.

For variable ρ_0 with barotropic equation of state $p = p(\varepsilon_0)$, the momentum-pressure field $\boldsymbol{\Pi}$ acquires additional contributions from $\nabla\varepsilon_0 = \frac{c^2}{c_s^2}\nabla p$ from §4.1.1, giving

$$\boldsymbol{\Pi} = \partial_t \boldsymbol{g}_0 + \nabla\varepsilon_0 + \nabla\mathcal{K} \approx \partial_t \boldsymbol{g}_0 + \frac{c^2}{c_s^2}\nabla p, \tag{354}$$

where the kinetic correction $\nabla\mathcal{K}$ is negligible in the slow-flow limit. The transport term in the $\hat{\boldsymbol{\sigma}}$-channel then contains

$$-\frac{1}{c}\boldsymbol{v} \times \boldsymbol{\Pi} \supset -\frac{c}{c_s^2}\boldsymbol{v} \times \nabla p, \tag{355}$$

which is the transverse pressure-gradient force: the tendency of the pressure gradient to generate rotational deflection of the flow perpendicular to $\boldsymbol{v}$. This is the force-level counterpart of the baroclinic vorticity generation identified in the $\hat{K}$-channel of $\mathcal{H} = 0$ in §4.1.1: where the field-level space channel encoded vorticity generation from the misalignment of $\boldsymbol{v}$ and $\nabla\rho_0$, the force-level $\hat{\boldsymbol{\sigma}}$-channel encodes the transverse rotational forcing from the misalignment of $\boldsymbol{v}$ and ∇p. The factor c/c_s^2 shows that this baroclinic rotational forcing is amplified by c^2/c_s^2 relative to the Newtonian expectation $-\frac{1}{c}\boldsymbol{v} \times \nabla p/\rho_0$, consistently with the same thermodynamic amplification factor that appeared in the T-channel and K-channel results of §4.3.2 and §4.3.3.

The rotational balance condition (353) in the barotropic case becomes

$$c^2\boldsymbol{\omega} + \frac{c^2}{c_s^2}\boldsymbol{v} \times \nabla p \approx -\boldsymbol{v} \times \partial_t \boldsymbol{g}_0, \tag{356}$$

which generalizes (353) to include baroclinic rotational forcing. The left-hand side balances intrinsic vorticity against pressure-driven rotational generation, while the right-hand side measures the rotational component of the momentum-density acceleration. In the incompressible limit $\nabla p = 0$, Eqn. (356) reduces to (353), and in the force-free limit $\partial_t \boldsymbol{g}_0 = 0$ it gives $\boldsymbol{\omega} = -\frac{1}{c_s^2}\boldsymbol{v} \times \nabla p$, a vorticity exactly balanced by baroclinic pressure-gradient forcing.

Three regimes and geometric role.

The full structure of the $\hat{\boldsymbol{\sigma}}$-channel organizes naturally into three regimes. For slow flows and weak gradients, $\frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{\Pi} \ll \boldsymbol{\Omega}$ and $\boldsymbol{F}_\sigma \approx -c\boldsymbol{\Omega}$: the rotational structure is dominated by pre-existing vorticity, corresponding to slowly evolving rotational flow and quasi-static circulation patterns. For strong pressure gradients, turbulent transport, or baroclinically forced stratified flows, $\frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{\Pi} \sim \boldsymbol{\Omega}$ and the transport contribution becomes comparable to the intrinsic vorticity: rotational structure is dynamically generated and modified by the flow itself. The balanced regime $\boldsymbol{F}_\sigma = 0$, Eqn. (353) or (356), defines a dynamically self-consistent rotational transport configuration in which vorticity is neither created nor destroyed but exactly sustained by the rotational component of the material acceleration and baroclinic pressure forcing.

The geometric role of the $\hat{\boldsymbol{\sigma}}$-channel is complementary to that of the $\hat{1}$-channel established in §4.3.1. The norm channel $\boldsymbol{v} \cdot \boldsymbol{\Omega}$ measures helical propagation *along* vortex lines — the component of vorticity parallel to the flow. The $\hat{\boldsymbol{\sigma}}$-channel $\boldsymbol{v} \times \boldsymbol{\Pi}$ measures transverse rotational generation *perpendicular* to the flow — the component of rotational forcing orthogonal to $\boldsymbol{v}$. Together, the two channels decompose the rotational dynamics of the fluid into parallel (helical) and perpendicular (baroclinic/transport) components, with the $\hat{1}$-channel measuring the former and the $\hat{\boldsymbol{\sigma}}$-channel the latter. This decomposition is not imposed by hand but follows automatically from the geometric structure of the BQ product: the dot product in the norm channel projects onto the parallel component, while the cross product in the $\hat{\boldsymbol{\sigma}}$-channel projects onto the perpendicular component, and the two together exhaust the rotational content of the fluid force equation. The $\hat{\boldsymbol{\sigma}}$-channel can therefore be interpreted as the rotational dual of the K-channel momentum equation within the BQ fluid hierarchy: where the K-channel governs translational force balance, the $\hat{\boldsymbol{\sigma}}$-channel governs the rotational balance between intrinsic vorticity, baroclinic pressure forcing, and transport-generated rotational structure.

4.3.5 Concluding assessment of the force equation

The equation

$$U(\partial^T G) = F \tag{357}$$

provides the relativistic force and transport equation of the BQ fluid hierarchy. Its channel decomposition separates the dynamics into four distinct but coupled geometric sectors, each of which has been shown to reproduce a standard, named result of fluid dynamics without that result being assumed in advance:

- a helicity and vortex-alignment channel through the $\hat{1}$-sector,
- a kinetic power exchange and acoustic pressure-work channel through the T-sector,
- a momentum-balance and Euler–Lamb channel through the K-sector,
- and a rotational transport and vorticity-balance channel through the $\hat{\boldsymbol{\sigma}}$-sector.

The $\hat{1}$-channel measures the degree to which the flow propagates along its own vortex structure, distinguishing purely rotational or baroclinically forced flow from genuinely helical transport. It was shown in §4.3.1 that the norm-channel condition $F_{\hat{1}} = 0$ is both necessary and sufficient for the vanishing of the helicity density $\boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) = 0$, independently of whether $\mathcal{H} = 0$ or not, and that baroclinic vorticity — which is visible to the field-level $\hat{K}$-channel of $\mathcal{H}$ — is geometrically invisible to the norm channel because it is perpendicular to $\boldsymbol{v}$ by construction. Helicity generation is therefore structurally confined to the $\hat{1}$-channel of $U\mathcal{H} = F$.

The T-channel acts as the local power-balance equation of the fluid force structure. Under constant ρ_0 and energy conservation $\sigma_E = 0$, it reproduces the kinetic power equation $\mathcal{P} = -\frac{D\mathcal{K}}{Dt}$: the time-like part of the four-force measures the negative material rate of change of kinetic energy density along fluid trajectories. Under the variable-ρ_0, barotropic extension, a mass-continuity cancellation at the $U\mathcal{H}$ level eliminates the bulk dust-transport contributions, leaving the acoustic pressure-work $P \approx -\frac{1}{c_s^2}\boldsymbol{v} \cdot \nabla p$ as the sole power-exchange term. This organizes the energetic regimes into a three-tier hierarchy: potential flow ($\mathcal{H} = 0$, $P = 0$ trivially), isobaric balanced flow ($\boldsymbol{v} \cdot \nabla p = 0$, $\mathcal{H} \neq 0$, $P = 0$ dynamically), and non-equilibrium flow ($\boldsymbol{v} \cdot \nabla p \neq 0$, $P \neq 0$), organized entirely by the BQ channel structure and the barotropic equation of state.

The K-channel reproduces the translational momentum structure of fluid dynamics. Under constant ρ_0, energy conservation, and slow flow, it yields the pressureless Euler–Lamb equation $\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho_0}\boldsymbol{F}_K$, with the Navier–Stokes equation following once non-conservative stress contributions are included in $\boldsymbol{F}_K$. Under the variable-ρ_0, barotropic extension, the K-channel produces the compressible Euler equation with a relativistic equation of state, Eqn. (345), in which the dominant pressure-gradient force $-\frac{c^2}{\rho_0 c_s^2}\nabla p$ is amplified by the thermodynamic factor c^2/c_s^2 — the same factor that appeared in the T-channel acoustic pressure-work and in the field-level sigma-channel of §4.1.1, confirming cross-channel thermodynamic consistency of the BQ hierarchy.

The $\hat{\boldsymbol{\sigma}}$-channel governs the rotational transport structure of the fluid flow, acting as the rotational dual of the K-channel. Under constant ρ_0 and slow flow, it produces the rotational balance condition $c^2\boldsymbol{\omega} = -\boldsymbol{v} \times \frac{D\boldsymbol{v}}{Dt}$: vorticity is exactly sustained by the rotational component of the material acceleration. The force-free limit $\frac{D\mathbf{v}}{Dt} = 0$ recovers irrotationality $\boldsymbol{\omega} = 0$, consistently with the $\hat{K}$-channel of $\mathcal{H} = 0$ established in §4.1. Under the variable-ρ_0, barotropic extension, the channel acquires a baroclinic pressure-forcing term $-\frac{c}{c_s^2}\mathbf{v} \times \nabla p$, which is the force-level counterpart of the baroclinic vorticity generation identified in the $\hat{K}$-channel of $\mathcal{H} = 0$ in §4.1.1, with the same c^2/c_s^2 amplification factor threading through consistently. The $\hat{\boldsymbol{\sigma}}$-channel and the $\hat{1}$-channel together decompose the rotational dynamics into complementary geometric sectors: the norm channel measures helical propagation along vortex lines, while the sigma channel measures transverse rotational generation perpendicular to the flow, the two exhausting the full rotational content of the fluid force equation.

Taken together, the four channels of $U\mathcal{H} = F$ reproduce, without separate assumption, the following standard results of fluid dynamics: the helicity density and its vanishing under non-helical baroclinic flow; the kinetic power equation and the acoustic pressure-work balance; the Euler–Lamb momentum equation and its compressible barotropic generalization; and the vorticity balance and baroclinic rotational forcing. These results span mechanics, thermodynamics, and rotational dynamics, and in the classical treatment each requires a separate derivation from independent physical principles. Here they emerge as channel projections of the single product $U\mathcal{H} = F$ under clearly stated approximations, with their mutual consistency established once by the channel structure rather than verified separately.

The thermodynamic content of the barotropic equation of state enters the hierarchy once, through the enthalpy replacement $\varepsilon_0 \longrightarrow w = \varepsilon_0 + p$ and the relativistic sound speed $c_s^2 = c^2(\partial p/\partial\varepsilon_0)_s$, and propagates consistently through all four channels at the force level and through all three channels at the field level of §4.1.1, without additional assumptions at any level. The c^2/c_s^2 amplification factor in the T, K, and $\hat{\boldsymbol{\sigma}}$ channels, and the mass-continuity cancellation at the $U\mathcal{H}$ level in the T-channel, are structural consequences of the BQ algebra rather than separately imposed physical inputs.

This cross-channel and cross-level consistency is the central result of §sec:UHFeq. It shows that $U\mathcal{H} = F$ is not merely a compact notation for the known equations of fluid dynamics but a statement that those equations are a single output of one algebraic product, whose channel projections distribute the mechanical, thermodynamic, and rotational content of the fluid force structure into four geometrically distinct but mutually consistent sectors. The BQ hierarchy does not introduce new fluid equations, but reveals that the known equations share a common algebraic origin that the standard $T^{\mu\nu}$ formulation does not make visible.

Channel	Physical role	Key result
$\hat{1}$	helicity transport and vortex alignment	$\boldsymbol{v} \cdot (\nabla \times \boldsymbol{v}) = 0 \Leftrightarrow F_{\hat{1}} = 0$
T	kinetic power and acoustic balance	$P = -\frac{1}{c_s^2} \boldsymbol{v} \cdot \nabla p$
K	momentum transport and Euler–Lamb	$\frac{D\boldsymbol{v}}{Dt} = -\frac{c^2}{\rho_0 c_s^2} \nabla p - \frac{1}{\rho_0} \boldsymbol{F}_K$
$\hat{\boldsymbol{\sigma}}$	rotational balance and baroclinic forcing	$c^2 \boldsymbol{\omega} = -\boldsymbol{v} \times \frac{D\boldsymbol{v}}{Dt} - \frac{c^2}{c_s^2} \boldsymbol{v} \times \nabla p$

As in the electromagnetic case, the separate channels are not individually Lorentz covariant. The covariance applies to the complete relativistic force equation $U\mathcal{H} = F$, while the channel decomposition represents a frame-dependent projection of the underlying relativistic fluid interaction structure. The physical content of each channel is therefore observer-dependent in the sense that a Lorentz boost mixes the channels, but the complete equation $U\mathcal{H} = F$ and the full set of channel projections together constitute a Lorentz-covariant description of relativistic fluid dynamics.

The channel decomposition of $V\mathcal{H} = F$ also provides a precise answer to the question posed by Marmanis Marmanis (1998) and given geometric-algebra form by Panakkal et al. Panakkal et al. (2020): how does the vorticity-Lamb-vector pair form a Maxwell-like field structure with its own Poynting and Lorentz analogues? In the BQ framework, the answer is that the vorticity $\boldsymbol{\Omega}$ and the momentum-pressure field $\boldsymbol{\Pi}$ play the roles of the magnetic and electric analogues respectively, but with a structural difference identified in the prior-art discussion of §4.2: the Marmanis/Panakkal field already carries the convective nonlinearity inside the Lamb vector $\boldsymbol{l} = \boldsymbol{\omega} \times \boldsymbol{v}$, whereas in the BQ framework $\boldsymbol{\Omega}$ and $\boldsymbol{\Pi}$ are both linear functionals of G. The nonlinearity is not absent but relocated: it appears in the K-channel vortex force $\boldsymbol{v} \times \boldsymbol{\Omega}$ — which is precisely the Lamb-vector structure $\boldsymbol{\omega} \times \boldsymbol{v}$ recovered in §4.3.3 — and in the $\hat{\boldsymbol{\sigma}}$-channel transverse pressure-gradient forcing $\boldsymbol{v} \times \boldsymbol{\Pi}$, both arising from the second contraction with V in $V\mathcal{H} = F$ rather than from the field $\mathcal{H}$ itself. The BQ channel structure therefore does not contradict the Marmanis/Panakkal analogy but clarifies it: the Maxwell-like field structure lives in $\mathcal{H} = \partial^T G$, linear and clean, and the Lamb-vector nonlinearity emerges downstream in $V\mathcal{H} = F$, in the channel where fluid mechanics has always known it belongs — the momentum equation.

4.3.6 A principled relativistic fluid dynamics from two Lorentz-covariant four-vector equations

The results of §4.1–§sec:UHFeq establish, to our knowledge, the first construction of relativistic fluid dynamics from two Lorentz-covariant four-vector equations in which the fluid equations are not postulated or covariantized but emerge as channel projections of the BQ product structure. The construction proceeds from a single algebraic substitution $J \to U$, $A \to G$, where U is the proper four-velocity and $G = \rho_0 U$ is the four-momentum density — both genuine four-vectors in the BQ basis $K^\mu = (\hat{T}, \hat{I}, \hat{J}, \hat{K})$ — and forms two Lorentz-covariant equations:

$$\mathcal{H} = \partial^T G, \qquad U\mathcal{H} = F, \tag{358}$$

from which all fluid dynamical content is extracted by channel decomposition under clearly stated physical approximations (slow flow, barotropic equation of state), with no fluid equation assumed in advance at either level.

This distinguishes the present construction from the two standard approaches to relativistic fluid dynamics. The first standard approach covariantizes the known non-relativistic equations by promoting ∂_t to ∂_μ and inserting Lorentz factors: this is a relativistic *extension* of fluid dynamics, not a *derivation* of it, and it carries the non-relativistic equations as a hidden input. The second standard approach imposes $\nabla_\mu T^{\mu\nu} = 0$ on a postulated stress-energy tensor of perfect-fluid form $T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}$: this is correct and covariant, but the physical content of $T^{\mu\nu}$ is assumed rather than derived, and the channel structure that separates mechanical, thermodynamic, helical, and rotational sectors is not visible in the tensor-divergence formulation. The geometric algebra approaches of Hestenes Hestenes (2003) and Panakkal et al. Panakkal et al. (2020) derive Maxwell-like fluid equations from a single algebraic product, but work in Euclidean G_4 without a Minkowski split, start from the vorticity-Lamb-vector analogy (which is already fluid dynamics), and do not construct the fluid hierarchy from a covariant four-vector substitution applied to the electromagnetic product structure.

In the present construction, no fluid equation is an input. The continuity equation, irrotationality, and the unsteady Bernoulli relation emerge as the three channel projections of $\mathcal{H} = 0$ in the dust, slow-flow limit (§4.1). The classical acoustic relation, baroclinic vorticity generation, and the relativistically amplified momentum equation emerge as channel projections of $\mathcal{H} = 0$ under the variable-ρ_0, barotropic extension (§4.1.1). The helicity density and its geometric separation from baroclinic vorticity, the kinetic power equation and acoustic pressure-work balance, the Euler–Lamb momentum equation and its compressible barotropic generalization, and the vorticity balance and baroclinic rotational forcing emerge as the four channel projections of $U\mathcal{H} = F$ (§sec:UHFeq). In every case the result was not targeted but read off from the channel structure: the algebra produced it.

The thermodynamic content of the barotropic equation of state enters once, through the enthalpy replacement $\varepsilon_0 \to w = \varepsilon_0 + p$ and the relativistic sound speed $c_s^2 = c^2(\partial p/\partial \varepsilon_0)_s$, and propagates automatically through all channels at both levels without additional assumptions. The c^2/c_s^2 amplification factor threads through the field-level sigma-channel of §4.1.1, the T-channel acoustic pressure-work of §4.3.2 , the K-channel pressure-gradient force of §4.3.3, and the $\hat{\boldsymbol{\sigma}}$-channel baroclinic rotational forcing of §4.3.4 — not because it was inserted at each point, but because the BQ product structure distributes it automatically once the equation of state is supplied. The $U\mathcal{H}$-level mass-continuity cancellation in the T-channel, which eliminates the bulk dust-transport contributions and leaves the acoustic pressure-work as the sole power-exchange term, is a further example of the same phenomenon: a physical result (mass conservation in material form) appearing at the correct level of the hierarchy as an automatic consequence of the algebra, not as a separate input.

The construction is principled in the precise sense that it derives relativistic fluid dynamics from the same algebraic principles that give relativistic electrodynamics in the BQ framework — the same product structure, the same channel decomposition, the same Lorentz-covariant four-vector language — with fluid dynamics emerging from the substitution $J \to U$, $A \to G$ rather than from independent physical postulates. To our knowledge, no prior treatment of relativistic fluid dynamics derives the fluid equations in this way: as channel projections of two Lorentz-covariant four-vector products $\mathcal{H} = \partial^T G$ and $U\mathcal{H} = F$, with the mechanical, thermodynamic, helical, and rotational sectors of the fluid force structure separated automatically by the algebra into four geometrically distinct channels whose physical content is determined by the channel structure itself rather than by the physicist imposing a decomposition from outside.

Two directions for extension are structurally anticipated by the present construction and form its natural continuation. The first is the fully relativistic case without the slow-flow approximation: removing $\mathbf{u} \approx \mathbf{v}$ and $u_0 \approx c$ would retain the full $\gamma(\mathbf{r}, t)$ field in all channel projections, producing genuine kinematic relativistic corrections alongside the thermodynamic c^2/c_s^2 amplification identified here. The second is the third level of the BQ fluid hierarchy, $\partial\mathcal{H} = F$, which is the fluid analogue of the electromagnetic Maxwell equation $\partial B = \mu_0 J$ and governs acoustic and vortex-wave propagation in the source-free interior of a closed fluid (§4.4). Both extensions follow the same algebraic logic as the present construction: form the product, decompose into channels, read off the physics.

4.4 The fluid field equation $\partial\mathsf{H} = F$: channel decomposition

Starting from the field tensor

$$H = -\sigma_E \hat{1} + \mathbf{\Omega} \cdot \hat{K} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\sigma}, \tag{359}$$

with

$$\sigma_E = \frac{1}{c^2}\partial_t \varepsilon + \nabla \cdot \mathbf{g}, \tag{360}$$

$$\mathbf{\Omega} = \nabla \times \mathbf{g}, \tag{361}$$

and

$$\mathbf{\Pi} = \partial_t \mathbf{g} + \nabla \varepsilon, \tag{362}$$

and with the differential operator defined as

$$\partial = -\frac{1}{c}\partial_t \hat{T} + \nabla \cdot \hat{K}, \tag{363}$$

the field equation is $F = \partial H$.

Applying the BQ triple-product rules gives

$$F = F_1 \hat{1} + F_T \hat{\mathbf{T}} + \mathbf{F}_K \cdot \hat{\mathbf{K}} + \mathbf{F}_\sigma \cdot \hat{\boldsymbol{\sigma}}, \tag{364}$$

with channel decomposition

$$F_1 = -\nabla \cdot \mathbf{\Omega}, \tag{365}$$

$$F_T = \frac{1}{c}\partial_t \sigma_E - \frac{1}{c}\nabla \cdot \mathbf{\Pi}, \tag{366}$$

$$\mathbf{F}_K = \nabla \times \mathbf{\Omega} - \nabla \sigma_E + \frac{1}{c^2}\partial_t \mathbf{\Pi}, \tag{367}$$

and

$$\mathbf{F}_\sigma = \frac{1}{c}\left(\partial_t \mathbf{\Omega} - \nabla \times \mathbf{\Pi}\right). \tag{368}$$

Substituting the definitions of σ_E, $\mathbf{\Omega}$, and $\mathbf{\Pi}$, the norm channel becomes

$$F_1 = -\nabla \cdot (\nabla \times \mathbf{g}) = 0, \tag{369}$$

while the σ channel becomes

$$\mathbf{F}_\sigma = \frac{1}{c}\left(\partial_t (\nabla \times \mathbf{g}) - \nabla \times (\partial_t \mathbf{g} + \nabla \varepsilon)\right) \tag{370}$$

$$= 0. \tag{371}$$

The remaining channels reduce to

$$F_T = \frac{1}{c}\left[\frac{1}{c^2}\partial_t^2 \varepsilon + \partial_t (\nabla \cdot \mathbf{g}) - \nabla \cdot (\partial_t \mathbf{g}) - \nabla^2 \varepsilon\right] \tag{372}$$

$$= \frac{1}{c}\left(\frac{1}{c^2}\partial_t^2 \varepsilon - \nabla^2 \varepsilon\right), \tag{373}$$

and

$$\mathbf{F}_K = \nabla \times (\nabla \times \mathbf{g}) - \nabla\left(\frac{1}{c^2}\partial_t \varepsilon + \nabla \cdot \mathbf{g}\right) + \frac{1}{c^2}\partial_t (\partial_t \mathbf{g} + \nabla \varepsilon) \tag{374}$$

$$= \frac{1}{c^2}\partial_t^2 \mathbf{g} - \nabla^2 \mathbf{g}. \tag{375}$$

The field equation therefore becomes

$$\boxed{\partial H = \frac{1}{c}\left(\frac{1}{c^2}\partial_t^2\varepsilon - \nabla^2\varepsilon\right)\hat{T} + \left(\frac{1}{c^2}\partial_t^2\mathbf{g} - \nabla^2\mathbf{g}\right)\cdot\hat{K}} \tag{376}$$

with identically vanishing norm and σ channels,

$$F_1 = 0, \qquad \mathbf{F}_\sigma = 0. \tag{377}$$

The equation $F = \partial H$ therefore yields coupled wave equations for the energy density ε and momentum density $\mathbf{g}$, while the norm and σ channels reduce to geometric identities analogous to the homogeneous Maxwell equations.

Interpretation

The field equation $\partial\mathcal{H} = F'$ is the third level of the BQ fluid hierarchy, governing the spacetime propagation of the fluid field $\mathcal{H}$ itself. Its relationship to the force level $U\mathcal{H} = F$ requires precise statement. The two equations

$$U\mathcal{H} = F, \qquad \partial\mathcal{H} = F', \tag{378}$$

are not competing equations but complementary projections of the same non-trivial field $\mathcal{H} \neq 0$: $U\mathcal{H} = F$ describes how the field acts on matter through the force exerted on a fluid element with four-velocity U, while $\partial\mathcal{H} = F'$ describes how the field itself evolves in spacetime. The structural analogy with electromagnetism is exact: $JB = F$ and $\partial B = \mu_0 J$ both hold simultaneously for the same non-trivial field B, one governing the Lorentz force on the source and the other governing the field evolution, and their mutual consistency is enforced by the continuity equation $\partial_\mu J^\mu = 0$. In the fluid case the analogous consistency is carried by the BQ hierarchy itself: the three levels $G \to \mathcal{H} = \partial^T G \to \{U\mathcal{H}, \partial\mathcal{H}\}$ form a single algebraic chain in which $U\mathcal{H}$ and $\partial\mathcal{H}$ are two different projections of the same field.

The BQ Bianchi identities.

The most fundamental result of the channel decomposition is the automatic vanishing of the norm and $\hat{\boldsymbol{\sigma}}$ channels:

$$F_{\hat{1}} = -\nabla\cdot\boldsymbol{\Omega} = 0, \qquad \boldsymbol{F}_{\hat{\sigma}} = \frac{1}{c}\left(\partial_t\boldsymbol{\Omega} - \nabla\times\boldsymbol{\Pi}\right) = 0. \tag{379}$$

These are not dynamical equations — they are geometric identities, following from $\nabla\cdot(\nabla\times\mathbf{g}) = 0$ and $\nabla\times\nabla\varepsilon = 0$ combined with the commutativity of mixed partial derivatives, independently of any fluid condition or approximation. They are the *BQ fluid Bianchi identities*: structural constraints on the fluid field enforced by the algebra itself, exactly as the electromagnetic Bianchi identities $\nabla\cdot\mathbf{B} = 0$ and

$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ follow from $\partial F = 0$ as geometric identities in the homogeneous Maxwell equations, and as the GR Bianchi identity $\nabla_\mu G^{\mu\nu} \equiv 0$ follows from the contracted Bianchi identity of Riemannian geometry. In all three cases — EM, GR, and the present BQ fluid hierarchy — the Bianchi identities are not imposed as physical conditions but emerge automatically from the algebraic or geometric structure, enforcing topological and rotational compatibility as necessary consequences of the formalism. The first identity $\nabla \cdot \mathbf{\Omega} = 0$ is a topological constraint: the vorticity of the momentum density is a solenoidal field, exactly as $\nabla \cdot \mathbf{B} = 0$ says the magnetic field has no monopoles. The second identity $\partial_t \mathbf{\Omega} = \nabla \times \mathbf{\Pi}$ is a rotational compatibility condition: the time evolution of the vorticity field is exactly determined by the curl of the momentum-pressure field, exactly as Faraday's law $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ determines the time evolution of the magnetic field from the curl of the electric field.

Channel roles.

The four-channel decomposition of $\partial \mathcal{H} = F'$ separates naturally into two geometric and two dynamical sectors:

Channel	Role	Content
$\hat{1}$	topological constraint (Bianchi)	$\nabla \cdot \mathbf{\Omega} = 0$
$\hat{\boldsymbol{\sigma}}$	rotational compatibility (Bianchi)	$\partial_t \mathbf{\Omega} = \nabla \times \mathbf{\Pi}$
T	energy propagation	$\Box \varepsilon = cF'_T$
K	momentum propagation	$\Box \mathbf{g} = \mathbf{F}'_K$

The geometric channels ($\hat{1}$ and $\hat{\boldsymbol{\sigma}}$) carry no dynamical content — they are identically zero by the algebra. The dynamical channels (T and K) carry the wave propagation of energy density and momentum density respectively. This two-plus-two split mirrors the structure of the Maxwell equations, where $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$ are the geometric (Bianchi) channels and $\nabla \cdot \mathbf{E} = \rho_e/\varepsilon_0$ and $\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = \mu_0 \mathbf{J}$ are the dynamical (source) channels. The channel role division is also complementary to the $U\mathcal{H} = F$ decomposition of §sec:UHFeq: the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels play geometric roles at both the force and wave levels, while the T and K channels carry dynamical content at both levels — force balance and power exchange at the $U\mathcal{H}$ level, and energy and momentum wave propagation at the $\partial\mathcal{H}$ level. The BQ algebra assigns topological and compatibility roles to the same channels at every level of the hierarchy.

The d'Alembertian structure is exact.

The reduction of the T and K channels to d'Alembertian form

$$\Box \varepsilon = cF'_T, \qquad \Box \mathbf{g} = \mathbf{F}'_K, \tag{380}$$

is an exact algebraic result, following from the BQ product rules and the definitions of σ_E, $\mathbf{\Omega}$, and $\mathbf{\Pi}$ without approximation. In the source-free interior $F' = 0$, these reduce to the wave equations

$$\Box\varepsilon = 0, \qquad \Box\mathbf{g} = 0, \tag{381}$$

describing relativistic wave propagation of energy density and momentum density at speed c. It should be noted that while the d'Alembertian structure is exact, the wave equations (381) hold specifically in the source-free case $F' = 0$; in the presence of sources, the T and K channels give inhomogeneous wave equations with source terms $(F'_T, \mathbf{F}'_K)$ playing the roles of acoustic charge density and acoustic current density respectively, analogous to $(\rho_e, \mathbf{J})$ in the inhomogeneous Maxwell equations.

Source terms and acoustic charge.

The source four-vector $(F'_T, \mathbf{F}'_K)$ carries the acoustic source structure of the fluid. $F'_T \propto \Box\varepsilon$ measures local sources or sinks of acoustic energy — nonzero wherever energy density is being created, destroyed, or exchanged with an external sector. $\mathbf{F}'_K \propto \Box\mathbf{g}$ measures local sources or sinks of momentum density — nonzero wherever body forces or momentum sources act on the fluid. The fluid analogue of the electromagnetic charge-current four-vector $J^\mu = (\rho_e c, \mathbf{J})$ is therefore the acoustic source four-vector $(F'_T, \mathbf{F}'_K)$. This interpretation is presented as a physical analogy rather than a derived result: the precise relationship between the acoustic sources and the fluid's own dynamical variables requires connecting the third level back to the force level $U\mathcal{H} = F$, which is the natural direction for future development.

The $\mathcal{H} = 0$ regime.

The condition $\mathcal{H} = 0$, established in §4.1 as the potential-flow regime, is physically distinct from the wave regime $\partial\mathcal{H} = F'$. When $\mathcal{H} = 0$, the intrinsic BQ field dynamics vanish: $\mathbf{\Omega} = 0$, $\mathbf{\Pi} = 0$, and $\sigma_E = 0$, so that $\partial\mathcal{H} = 0$ trivially and no BQ field wave propagation occurs. This does not preclude all wave-like behaviour — the velocity potential χ satisfying $\nabla^2\chi = 0$ can still have time-dependent boundary conditions — but the intrinsic field structure of the BQ hierarchy carries no wave dynamics in this regime. The $\mathcal{H} = 0$ condition is therefore the boundary between the quiescent potential-flow regime and the dynamically active $\mathcal{H} \neq 0$ regime in which both force dynamics ($U\mathcal{H} = F$) and wave propagation ($\partial\mathcal{H} = F'$) are active.

Position in the hierarchy.

The three levels of the BQ fluid hierarchy organize the fluid dynamics into physically distinct regimes:

- $\mathcal{H} = \partial^T G$: the field definition level, mathematically the most fundamental. The vanishing of $\mathcal{H}$ defines the potential-flow regime; its non-vanishing is the prerequisite for all dynamical content.
- $U\mathcal{H} = F$: the force dynamics level. Governs helicity, power exchange, momentum transport, and rotational vorticity balance through the four channel projections of §sec:UHFeq. Requires $\mathcal{H} \neq 0$.
- $\partial\mathcal{H} = F'$: the most fundamental *dynamical* level. Governs wave propagation of energy and momentum density through the T and K channels, with the $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels enforcing the BQ Bianchi identities. Unlike the results of §4.1–§sec:UHFeq, the d'Alembertian structure of $\partial\mathcal{H} = F'$ is exact — it requires no slow-flow approximation, no constant-ρ_0 restriction, and no barotropic equation of state. It holds for any relativistic fluid described by $G = \rho_0 U$, making it the approximation-free core of the BQ fluid hierarchy.

The transition from $\mathcal{H} = 0$ to $\mathcal{H} \neq 0$ — the onset of vorticity, the departure from potential flow, the activation of acoustic wave propagation — is precisely the physically interesting regime of fluid dynamics. The BQ hierarchy provides a natural algebraic language for characterizing this transition: $\mathcal{H}$ measures the departure from the quiescent state, $U\mathcal{H}$ measures its force consequences, and $\partial\mathcal{H}$ measures its wave propagation. The field $\mathcal{H}$ acts as an order parameter for the activation of the fluid dynamics: $\mathcal{H} = 0$ corresponds to the potential-flow sector, while $\mathcal{H} \neq 0$ activates both force and wave dynamics. The fact that all three emerge from the same algebraic object $G = \rho_0 U$ through successive BQ products, without any fluid equation assumed at any level, is the central structural result of Section 4.

Classical acoustics as a sourced BQ wave regime.

The momentum-density wave equation $\Box\mathbf{g} = \mathbf{F}'_K$ clarifies precisely how classical acoustics relates to the source-free BQ wave regime. In the source-free case $F'_T = 0$, $\mathbf{F}'_K = 0$, both ε and $\mathbf{g}$ satisfy relativistic wave equations at speed c:

$$\Box\varepsilon = 0, \qquad \Box\mathbf{g} = 0. \tag{382}$$

Classical acoustics, by contrast, uses the linearized relations

$$\mathbf{g} = \rho_0\mathbf{v}, \qquad \partial_t\mathbf{g} = -\nabla p, \qquad \partial_t p = -c_s^2\nabla\cdot\mathbf{g}, \tag{383}$$

from which the acoustic pressure-wave equation follows:

$$\partial_t^2 p = -c_s^2\nabla\cdot\partial_t\mathbf{g} = c_s^2\nabla^2 p, \tag{384}$$

giving

$$\frac{1}{c_s^2}\partial_t^2 p - \nabla^2 p = 0. \tag{385}$$

For the momentum density, taking a further time derivative and using Eqns. (383):

$$\partial_t^2 \mathbf{g} = -\nabla \partial_t p = c_s^2 \nabla(\nabla \cdot \mathbf{g}). \tag{386}$$

For irrotational acoustic motion, $\nabla \times \mathbf{g} = 0$, so $\nabla(\nabla \cdot \mathbf{g}) = \nabla^2 \mathbf{g}$, giving

$$\frac{1}{c_s^2} \partial_t^2 \mathbf{g} - \nabla^2 \mathbf{g} = 0. \tag{387}$$

Comparing Eqn. (387) with the BQ momentum-density wave equation $\Box \mathbf{g} = \mathbf{F}'_K$, i.e. $\frac{1}{c^2} \partial_t^2 \mathbf{g} - \nabla^2 \mathbf{g} = \mathbf{F}'_K$, gives the effective medium source:

$$\mathbf{F}'_K = \left(\frac{1}{c^2} - \frac{1}{c_s^2} \right) \partial_t^2 \mathbf{g}. \tag{388}$$

This is the key result. Classical acoustic wave propagation at c_s does not correspond to the source-free BQ regime $\mathbf{F}'_K = 0$ but to a *sourced* regime in which the medium response enters through $\mathbf{F}'_K$ as an effective inertial source. Since $c_s < c$ for any normal fluid, $\frac{1}{c^2} - \frac{1}{c_s^2} < 0$, so $\mathbf{F}'_K$ opposes the acceleration $\partial_t^2 \mathbf{g}$: the fluid's compressibility acts as an effective inertial drag that reduces the propagation speed from c to c_s. The magnitude of the source is dominated by $\frac{1}{c_s^2}$ for $c_s \ll c$, consistently with the c^2/c_s^2 amplification factor that threads through the field-level sigma-channel of §4.1.1, the T-channel acoustic pressure-work of §4.3.2 , and the K-channel pressure-gradient force of §4.3.3: the same thermodynamic factor that amplifies force and power at the lower levels here appears as the medium source that controls wave propagation at the third level.

By the same argument applied to the energy channel, the effective energy source is

$$F'_T = \frac{1}{c} \left(\frac{1}{c^2} - \frac{1}{c_s^2} \right) \partial_t^2 \varepsilon, \tag{389}$$

giving a consistent picture for both dynamical channels: classical acoustic propagation at c_s corresponds to sourced BQ wave equations in both the T and K channels, with the medium response encoded in $(F'_T, \mathbf{F}'_K)$ as the acoustic source four-vector.

The irrotationality condition $\nabla \times \mathbf{g} = 0$ used in the derivation of Eqn. (387) is not an additional assumption but is consistent with the $\mathcal{H} = 0$ potential-flow regime of §4.1, where the $\hat{K}$-channel gives $\mathbf{\Omega} = \nabla \times \mathbf{g} = 0$. For rotational acoustic fields $\nabla \times \mathbf{g} \neq 0$, the momentum-density wave equation takes the more general form $\frac{1}{c_s^2} \partial_t^2 \mathbf{g} - \nabla(\nabla \cdot \mathbf{g}) = 0$, and the effective source acquires an additional rotational correction:

$$\mathbf{F}'_K = \left(\frac{1}{c^2} - \frac{1}{c_s^2} \right) \partial_t^2 \mathbf{g} + \nabla \times (\nabla \times \mathbf{g}), \tag{390}$$

where the second term vanishes in the irrotational limit, recovering Eqn. (388). The clean form of the effective source is therefore associated precisely with the irrotational, potential-flow regime — another instance of the hierarchical consistency of the BQ framework.

The three-step descent from the exact BQ wave equation to the classical acoustic equation can be summarized as:

$$\begin{aligned}
&\boxed{\Box\varepsilon = 0, \quad \Box\mathbf{g} = 0} \quad \text{source-free BQ waves at speed } c,\\
&\qquad\qquad\Downarrow \quad \text{medium response enters through}\\
&F'_T = \frac{1}{c}\left(\frac{1}{c^2} - \frac{1}{c_s^2}\right)\partial_t^2\varepsilon, \quad \mathbf{F}'_K = \left(\frac{1}{c^2} - \frac{1}{c_s^2}\right)\partial_t^2\mathbf{g},\\
&\qquad\qquad\Downarrow\\
&\boxed{\frac{1}{c_s^2}\partial_t^2\delta p - \nabla^2\delta p = 0,} \quad \boxed{\frac{1}{c_s^2}\partial_t^2\mathbf{g} - \nabla^2\mathbf{g} = 0,}
\end{aligned} \tag{391}$$

where δp denotes the acoustic pressure perturbation around a constant background pressure p_0, and the $\frac{1}{c}$ normalization in F'_T reflects the BQ channel structure of $\partial\mathcal{H} = F'$ established in §4.4. Source-free BQ waves propagate at c because $G = \rho_0 U$ is a relativistic four-vector and the BQ algebra is Lorentz covariant. Classical acoustic waves propagate at $c_s < c$ because the medium response — the fluid's own compressibility and inertial structure — enters the wave equation through the effective sources $(F'_T, \mathbf{F}'_K)$, which are not independent fields but the fluid's own inertial response to its acceleration, encoded in the acoustic refractive index $n_s = c/c_s$ as

$$F'_T = -\frac{n_s^2 - 1}{c^3}\partial_t^2\varepsilon, \qquad \mathbf{F}'_K = -\frac{n_s^2 - 1}{c^2}\partial_t^2\mathbf{g}. \tag{392}$$

Classical acoustics is therefore not the source-free BQ wave regime but a sourced regime in which the medium encodes its thermodynamic response through the acoustic source four-vector $(F'_T, \mathbf{F}'_K)$, with the wave speed $c_s = c/n_s$ determined by the equation of state rather than by the relativistic structure of the BQ algebra. The BQ hierarchy does not introduce a new acoustic theory but reveals that the standard acoustic wave equation is a sourced projection of the exact relativistic wave equation $\Box G = 0$, with the source strength set by the acoustic refractive index $n_s^2 - 1 = (c^2 - c_s^2)/c_s^2$ that measures the refractive strength of the acoustic medium.

The acoustic refractive index and fluid medium response.

The effective sources Eqns. (388) and (389) admit a natural interpretation in terms of an acoustic refractive index, structurally analogous to the role of the refractive index in electromagnetic wave propagation in a medium. In vacuum, the electromagnetic wave equation

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\mathbf{A} = 0 \tag{393}$$

propagates at c; inside a dielectric medium, polarization and magnetization modify the wave operator and the phase velocity becomes $v = c/n$, where $n = \sqrt{\varepsilon_r \mu_r}$ is the refractive index, giving

$$\left(\frac{n^2}{c^2}\partial_t^2 - \nabla^2\right)\mathbf{A} = 0. \tag{394}$$

The vacuum operator $\frac{1}{c^2}\partial_t^2 - \nabla^2$ is modified by the medium response, encoded in the factor $n^2 - 1$ measuring the departure from vacuum. The BQ fluid hierarchy has exactly the same mathematical structure. Define the *acoustic refractive index*

$$n_s \equiv \frac{c}{c_s}, \tag{395}$$

so that $\frac{1}{c_s^2} = \frac{n_s^2}{c^2}$. The classical acoustic wave equations Eqns. (385) and (387) then take the form

$$\left(\frac{n_s^2}{c^2}\partial_t^2 - \nabla^2\right)\delta p = 0, \qquad \left(\frac{n_s^2}{c^2}\partial_t^2 - \nabla^2\right)\mathbf{g} = 0, \tag{396}$$

where δp denotes the acoustic pressure perturbation around a constant background p_0. The effective medium sources become

$$F_T' = -\frac{n_s^2 - 1}{c^3}\partial_t^2 \varepsilon, \qquad \mathbf{F}_K' = -\frac{n_s^2 - 1}{c^2}\partial_t^2 \mathbf{g}, \tag{397}$$

where the factor $\frac{1}{c}$ in F_T' reflects the BQ channel normalization of $\partial\mathcal{H} = F'$ established in §4.4, and the factor $n_s^2 - 1$ is the same in both channels, measuring the departure of the fluid medium from the BQ vacuum state $c_s = c$ ($n_s = 1$), exactly as $\varepsilon_r - 1$ in electromagnetism measures the departure of a dielectric from vacuum. In the stiff-equation-of-state limit $c_s \to c$, $n_s \to 1$ and both sources vanish: the medium response disappears and the BQ source-free wave equations are recovered. For normal fluids with $c_s \ll c$, $n_s = c/c_s \gg 1$ and $n_s^2 - 1 \approx n_s^2 = c^2/c_s^2$, so

$$F_T' \approx -\frac{1}{c^3 c_s^{-2}}\partial_t^2 \varepsilon = -\frac{1}{c_s^2 c}\partial_t^2 \varepsilon, \qquad \mathbf{F}_K' \approx -\frac{1}{c_s^2}\partial_t^2 \mathbf{g}, \tag{398}$$

consistently with the c^2/c_s^2 amplification factor that threads through the field-level sigma-channel of §4.1.1, the T-channel power balance of §4.3.2 , and the K-channel momentum equation of §4.3.3. The acoustic refractive index n_s therefore provides a unified expression for the thermodynamic amplification factor that appears at every level of the BQ fluid hierarchy: it is the single dimensionless quantity encoding how far the fluid medium departs from the relativistic vacuum state, and its square $n_s^2 = c^2/c_s^2$ controls force, power, and wave propagation simultaneously across all three levels.

The interpretation is this: the exact BQ wave equations $\Box\varepsilon = cF'_T$ and $\Box\mathbf{g} = \mathbf{F}'_K$ describe propagation of the energy-density and momentum-density fields at the fundamental relativistic speed c, set by the Lorentz-covariant algebraic structure of the BQ product. The fluid equation of state — encoded in $c_s^2 = \partial p/\partial\rho_0$ or equivalently in $n_s = c/c_s$ — acts as an acoustic refractive index, modifying the propagation through the effective sources Eqns. (397) and slowing the observable wave speed from c to $c_s = c/n_s$. The fluid medium response plays the role of dielectric polarization in electromagnetism: it does not change the fundamental wave equation but sources it, converting BQ vacuum propagation at c into acoustic propagation at c_s.

This analogy is structural rather than mechanistic. Three qualifications should be noted. First, in electromagnetism the medium response is carried by independent polarization and magnetization fields $\mathbf{P}$ and $\mathbf{M}$ with their own dynamics; in the BQ fluid case, $(F'_T, \mathbf{F}'_K)$ are expressed directly in terms of $(\partial_t^2\varepsilon, \partial_t^2\mathbf{g})$ — the fluid's own inertial response to its own acceleration and energy change — and are therefore self-referential rather than independent fields. Second, the electromagnetic refractive index n can be frequency-dependent (dispersion), complex (absorption), and anisotropic (birefringence); the acoustic refractive index $n_s = c/c_s$ is real, non-dispersive, and isotropic in the linearized barotropic approximation used here. Third, the mechanism differs: electromagnetic medium response arises from bound charge oscillations, while acoustic medium response arises from compressibility and inertia. The analogy is one of mathematical structure and physical role, not of physical mechanism.

With these qualifications, the acoustic refractive index $n_s = c/c_s$ provides the cleanest single expression of the relationship between the BQ hierarchy and classical fluid dynamics. In the BQ hierarchy, c_s enters as the medium-response parameter that modifies the fundamental relativistic wave speed c, with $n_s = c/c_s$ encoding the departure of the fluid medium from source-free BQ propagation at c, exactly as optical phase velocities are c reduced by the electromagnetic medium response n. The BQ hierarchy reveals this relationship by placing c and c_s at their correct algebraic levels — c at the third level as the fundamental BQ wave speed, c_s as the medium-modified observable speed encoded in the source four-vector $(F'_T, \mathbf{F}'_K)$ — and connecting them through the single dimensionless ratio $n_s = c/c_s$, which measures the refractive strength of the acoustic medium.

The momentum-density wave equation and relativistic acoustic structure.

The four-vector structure of the source Eqn. (397) reveals a further result. Writing the BQ four-momentum density as

$$G = \frac{\varepsilon}{c}\hat{T} + \mathbf{g} \cdot \hat{K}, \tag{399}$$

the second time derivative acts channel by channel as

$$\partial_t^2 G = \frac{1}{c}\partial_t^2 \varepsilon \hat{T} + \partial_t^2 \mathbf{g} \cdot \hat{K}, \tag{400}$$

so that the two medium sources Eqn. (397) combine into the single four-vector source

$$F' = -\frac{n_s^2 - 1}{c^2}\partial_t^2 G. \tag{401}$$

The BQ wave equation $\Box G = F'$ then becomes

$$\Box G + \frac{n_s^2 - 1}{c^2}\partial_t^2 G = 0, \tag{402}$$

or equivalently

$$\left(\frac{n_s^2}{c^2}\partial_t^2 - \nabla^2\right) G = 0. \tag{403}$$

This is the four-dimensional acoustic wave equation for the four-momentum density G in the fluid rest frame, propagating at $c_s = c/n_s$. It contains the classical acoustic wave equations for both ε and $\mathbf{g}$ simultaneously as its T and K channel projections, and reduces to the BQ source-free wave equation $\Box G = 0$ in the vacuum limit $n_s \to 1$ $(c_s \to c)$.

The role of $\mathbf{g}$ as the primary wave-carrying field deserves emphasis. In standard fluid dynamics, the acoustic wave equation is formulated in terms of the pressure perturbation δp or the velocity potential χ, not the momentum density $\mathbf{g}$. This is historically natural: the Euler and Navier–Stokes equations are written in terms of $\mathbf{v}$, and $\mathbf{g} = \rho_0 \mathbf{v}$ offers no advantage over $\mathbf{v}$ itself in the non-relativistic, constant-density case. In the BQ framework, however, $\mathbf{g}$ is the spatial component of the four-vector $G = \rho_0 U$, and the wave equation $\Box \mathbf{g} = \mathbf{F}'_K$ is the spatial projection of the rest-frame equation $\Box G = F'$. The momentum density is therefore the natural primary variable at the relativistic level, in the same way that the four-potential A^μ is the natural primary variable for electromagnetic waves rather than the electric or magnetic field components separately.

The analogy with electromagnetism in a medium is structurally exact. In the Lorenz gauge, the electromagnetic wave equation is

$$\Box A = \mu_0 J, \tag{404}$$

where A is the four-potential and J is the four-current sourcing the field. The BQ fluid wave equation in the fluid rest frame is

$$\Box G = F' = -\frac{n_s^2 - 1}{c^2}\partial_t^2 G, \tag{405}$$

where G plays the role of A and F' plays the role of $\mu_0 J$. The structural difference is that F' is not an independent external source but is expressed directly in terms of G itself: the fluid medium sources its own wave equation through its inertial response to its own acceleration and energy change. The correspondence is:

Electromagnetism	BQ fluid acoustics
Four-potential A	Four-momentum density $G = \rho_0 U$
Four-current $\mu_0 J$	Acoustic source $F' = -\frac{n_s^2-1}{c^2}\partial_t^2 G$
Vacuum: $\Box A = 0$	BQ vacuum: $\Box G = 0$, speed c
Medium: $\Box A = \mu_0 J$, speed c/n	Acoustic medium: $\Box G = F'$, speed $c_s = c/n_s$
EM refractive index $n = \sqrt{\varepsilon_r \mu_r}$	Acoustic refractive index $n_s = c/c_s$

The vacuum limit $n_s = 1$ ($c_s = c$, stiff equation of state) gives $F' = 0$ and $\Box G = 0$: the four-momentum density propagates freely at c with no medium response. The acoustic limit $n_s \gg 1$ ($c_s \ll c$, normal fluid) gives a large effective source $F' \approx -\frac{n_s^2}{c^2}\partial_t^2 G$, strongly slowing the propagation from c to c_s.

A crucial qualification concerns the Lorentz covariance of this structure. The fundamental source-free BQ wave equation $\Box G = 0$ is fully Lorentz covariant: $\Box = \frac{1}{c^2}\partial_t^2 - \nabla^2$ is a Lorentz scalar operator and $G = \rho_0 U$ is a four-vector, so $\Box G = 0$ holds in all inertial frames. The sourced equation $\Box G = F'$, however, holds only in the fluid rest frame. The acoustic refractive index $n_s = c/c_s$ is defined by the equation of state $c_s^2 = \partial p/\partial \rho_0$ in the fluid rest frame and is not a Lorentz scalar: in a boosted frame, the apparent sound speed is direction-dependent through the relativistic velocity addition law, and n_s would need to be replaced by a direction- and velocity-dependent quantity, breaking the simple scalar form. The operator $\frac{n_s^2}{c^2}\partial_t^2 - \nabla^2$ in Eqn. (403) is therefore not a Lorentz scalar operator but a rest-frame acoustic operator, exactly as the wave operator $\frac{n^2}{c^2}\partial_t^2 - \nabla^2$ in electromagnetism in a medium holds in the rest frame of the dielectric. This is not a limitation of the BQ framework but a reflection of a genuine physical fact: acoustic wave propagation is intrinsically tied to the rest frame of the medium, just as electromagnetic wave propagation in a dielectric is tied to the rest frame of the polarizable material. The BQ hierarchy makes this explicit by separating the Lorentz-covariant level — the source-free equation $\Box G = 0$, exact and frame-independent — from the rest-frame acoustic level — the sourced equation $\Box G = F'$, valid in the fluid rest frame with medium response encoded in n_s.

The equation $\Box G = F'$ is therefore the most compact rest-frame statement of relativistic fluid acoustics available within the BQ framework. It describes the propagation of the four-momentum density as a single wave field in the fluid rest frame, with the equation of state entering through the acoustic refractive index $n_s = c/c_s$ as a medium response rather than as an external input. Standard fluid dynamics does not have access to this form because it does not treat $G = \rho_0 U$ as the primary field variable and does not derive the acoustic wave equation from a product structure rooted in the BQ algebra. The BQ hierarchy provides both: G as the fundamental four-vector field, $\mathcal{H} = \partial^T G$ as the fluid field tensor, and $\Box G = F'$ as the rest-frame relativistic acoustic wave equation, with the classical scalar pressure wave $\frac{1}{c_s^2}\partial_t^2 \delta p - \nabla^2 \delta p = 0$ recovered as the barotropic, linearized, irrotational, rest-frame projection of the single equation Eqn. (403).

The acoustic refractive index $n_s = c/c_s$, introduced at the wave level of §4.4 as the medium parameter connecting the fundamental BQ propagation speed c to the observable acoustic speed c_s, is not a new quantity at that level but the same dimensionless ratio $c^2/c_s^2 = n_s^2$ that has threaded through every channel and every level of the BQ fluid hierarchy from §4.1.1 onward: appearing as the relativistic momentum amplification factor in the field-level sigma-channel, as the acoustic pressure-work rescaling in the force-level T-channel, as the dominant pressure-gradient amplification in the force-level K-channel, and as the baroclinic rotational forcing factor in the force-level $\hat{\boldsymbol{\sigma}}$-channel — always from the same barotropic equation of state, introduced once through the enthalpy replacement $\varepsilon_0 \to w = \varepsilon_0 + p$ and propagated automatically through the BQ product structure without additional assumptions at any level. The acoustic refractive index is therefore the universal BQ fluid medium parameter: the single quantity that encodes the departure of the fluid from the relativistic vacuum state $\Box G = 0$ and that controls force, power, momentum, rotation, and wave propagation simultaneously across all three levels of the hierarchy.

$n_s = c/c_s$ as the dimensionless bridge between relativistic and practical fluid dynamics.

The acoustic refractive index $n_s = c/c_s$ is dimensionless, and this is not incidental but structurally significant. Dimensional quantities — c, c_s, ε_0, $\mathbf{g}$ — depend on unit conventions and carry no intrinsic physical content beyond their numerical values in a chosen system. Dimensionless ratios, by contrast, are unit-independent: they measure something intrinsic about the physical situation, independent of how the physicist chooses to represent it. The ratio $n_s = c/c_s$ is precisely such an intrinsic quantity: it measures the relationship between two fundamental speeds belonging to two different levels of physics — the speed of light c, which is the structural constant of the BQ algebra and of special relativity, and the speed of sound c_s, which is the structural constant of the fluid's acoustic response set by its barotropic equation of state. Their ratio n_s is therefore the natural dimensionless bridge between special-relativistic fluid dynamics and practical fluid dynamics: it cannot be made to take any particular value by a change of units, and its numerical value directly expresses how far the fluid's thermodynamic response departs from the relativistic structural limit $c_s = c$.

The range of n_s spans the full spectrum of fluid physics. For normal non-relativistic fluids — air, water, ordinary liquids — $n_s = c/c_s \sim 10^5$–10^6: enormous numbers reflecting the fact that acoustic wave speeds are a tiny fraction of c and the fluid medium is far from the BQ vacuum state $\Box G = 0$. For neutron star matter near nuclear saturation density, $n_s \sim 3$–10, approaching the relativistic regime. For a conformal quark-gluon plasma with equation of state $p = \varepsilon/3$, the sound speed reaches its conformal limit $c_s = c/\sqrt{3}$ and $n_s = \sqrt{3}$. For the maximally stiff equation of state $p = \varepsilon$, $c_s \to c$ and $n_s \to 1$: the BQ vacuum limit, in which the medium response vanishes and $\Box G = 0$ is recovered exactly. The entire range of fluid thermodynamics from deeply non-relativistic to ultra-relativistic is therefore parameterized by a single dimensionless number running from $n_s \gg 1$ to $n_s = 1$, with the BQ vacuum state as the natural upper bound.

This dimensionless character gives n_s a role analogous to the fine structure constant $\alpha = e^2/4\pi\varepsilon_0\hbar c \approx 1/137$ in quantum electrodynamics: just as α measures the strength of electromagnetic coupling as a pure dimensionless ratio connecting quantum mechanics ($\hbar$) to electromagnetism (e) and special relativity (c), so n_s measures the acoustic response of the fluid as a pure dimensionless ratio connecting special relativity (c) to fluid thermodynamics (c_s). In both cases, the dimensionless ratio is the natural bridge between two physical theories that would otherwise remain connected only through limiting procedures and dimensional arguments. The analogy is structural rather than quantitative — n_s is not a coupling constant in the quantum sense — but the role is the same: a single dimensionless number encoding the relationship between two levels of physics.

The Occam's razor economy of the BQ fluid hierarchy is sharpest when stated in these terms. The entire thermodynamic content of a barotropic relativistic fluid — its equation of state, its acoustic response, its departure from the relativistic vacuum, its force amplification, its power balance, its wave propagation — is encoded in a single dimensionless number $n_s = c/c_s$, derived once from $c_s^2 = c^2(\partial p/\partial\varepsilon_0)_s$ and propagated automatically through all channels at all three levels of the BQ hierarchy without re-introduction at any point. The field-level sigma-channel amplification $c^2/c_s^2 = n_s^2$, the force-level T-channel acoustic pressure-work $1/c_s^2 = n_s^2/c^2$, the force-level K-channel momentum amplification $c^2/c_s^2 = n_s^2$, the force-level $\hat{\boldsymbol{\sigma}}$-channel baroclinic forcing $c/c_s^2 = n_s^2/c$, and the wave-level medium source $(n_s^2 - 1)/c^2$ are all the same dimensionless ratio n_s^2 in different channel normalizations. One dimensionless number, one equation of state, one algebraic input: this is the sense in which $n_s = c/c_s$ is the key connecting special-relativistic BQ fluid dynamics to practical fluid dynamics — not through a limiting procedure or a separate derivation at each level, but through a single dimensionless bridge that the BQ algebra distributes automatically and consistently across the entire hierarchy.

4.5 The stress-energy equation $\partial(U^T G) = F$: channel decomposition

4.5.1 The bilinear product $U^T G$

The third equation of the relativistic fluid hierarchy,

$$\partial(U^T G) = F, \tag{406}$$

is the fluid analogue of the electromagnetic Laue–Sommerfeld equation $\partial(J^T A) = F$ from section 3.5 . It governs the conservation and transport of the combined stress-energy structure encoded in the bilinear $U^T G$. Substituting

$$U = u_0 \hat{\mathrm{T}} + \boldsymbol{u} \cdot \hat{\mathbf{K}} = \gamma c\, \hat{\mathrm{T}} + \boldsymbol{u} \cdot \hat{\mathbf{K}}, \qquad G = \frac{\varepsilon}{c} \hat{\mathrm{T}} + \boldsymbol{g} \cdot \hat{\mathbf{K}}, \tag{407}$$

into $T_{fd} = U^T G$ and using the BQ product rule $A^T B = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b})\hat{1} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{K} + (a_0 \mathbf{b} - b_0 \mathbf{a}) \cdot \boldsymbol{\sigma}$ with $a_0 = u_0$, $\mathbf{a} = \mathbf{u}$, $b_0 = \varepsilon/c$, $\mathbf{b} = \mathbf{g}$, one obtains

$$T_{fd} = U^T G = (\frac{u_0}{c}\varepsilon - \mathbf{u} \cdot \mathbf{g})\hat{1} + (\mathbf{u} \times \mathbf{g}) \cdot \hat{\mathbf{K}} + (u_0 \mathbf{g} - \frac{1}{c}\varepsilon \mathbf{u}) \cdot \hat{\boldsymbol{\sigma}}. \tag{408}$$

We can abbreviate the result as

$$T_{fd} = U^T G = \mathcal{T}_1 \hat{1} + \boldsymbol{\mathcal{T}}_K \cdot \hat{\mathbf{K}} + \boldsymbol{\mathcal{T}}_\sigma \cdot \hat{\boldsymbol{\sigma}}, \tag{409}$$

with the channels

$$norm - channel\, \hat{1} : \qquad \mathcal{T}_1 = \frac{u_0}{c}\varepsilon - \mathbf{u} \cdot \mathbf{g} = \mathcal{L}, \tag{410}$$

$$space - channel\, \hat{\mathbf{K}} : \qquad \boldsymbol{\mathcal{T}}_K = \boldsymbol{u} \times \boldsymbol{g} \tag{411}$$

$$sigma - channel\, \hat{\boldsymbol{\sigma}} : \qquad \boldsymbol{\mathcal{T}}_\sigma = u_0 \boldsymbol{g} - \frac{1}{c}\varepsilon \boldsymbol{u}. \tag{412}$$

The three sectors carry distinct physical content. The $\hat{1}$-channel scalar $\mathcal{T}_1 = \mathcal{L}$ is the local Lagrangian density. The $\mathbf{K}$-channel vector $\boldsymbol{\mathcal{T}}_K$ is the intrinsic spin density: the cross product of velocity and momentum density, this channel carries non-trivial content only for multi-component or spinning fluids. The $\boldsymbol{\sigma}$-channel vector $\boldsymbol{\mathcal{T}}_\sigma$ combines the momentum density scaled by c and the energy-flux density $(\varepsilon/c)\mathbf{u}$; it encodes the relativistic energy-transport structure.

The Lagrangian Lorentz scalar for $\mathbf{g} = \rho_0\mathbf{u}$

For cases with $\mathbf{g} = \rho_0\mathbf{u} = \gamma\rho_0\mathbf{v}$, and using $u_0 = \gamma c$ and $\varepsilon = \rho_0 c u_0 = \gamma\rho_0 c^2$ we get

$$\mathcal{T}_1 = \frac{u_0}{c}\varepsilon - \mathbf{u}\cdot\mathbf{g} = \gamma^2\rho_0 c^2 - \gamma^2\rho_0 v^2 = \rho_0 c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) = \rho_0 c^2 = \varepsilon_0, \quad (413)$$

$$\boldsymbol{\mathcal{T}}_K = \boldsymbol{u}\times\boldsymbol{g} = \rho_0\boldsymbol{u}\times\boldsymbol{u} = 0, \quad (414)$$

and

$$\boldsymbol{\mathcal{T}}_\sigma = u_0\boldsymbol{g} - \frac{1}{c}\varepsilon\boldsymbol{u} = \gamma c\rho_0\boldsymbol{u} - \gamma\rho_0 c\boldsymbol{u} = 0. \quad (415)$$

Under these conditions, $T_{fd} = U^T G = \varepsilon\hat{1}$, a Lorentz scalar. In the literature, (the negative of) this scalar is given the name Lagrangian density.

4.5.2 The tripple product $\partial(U^T G)$

We then look at $\partial(U^T G)$. The underlying structure is the same as for the Maxwell equations and the Lorentz force law, implied in $D = C(A^T B)$ of Eqn.(45), and the outcome is of course mirroring section (3.5) with its $T_{em} = \partial(J^T A)$. So $\partial(U^T G)$ is given by

$$\begin{aligned}\partial(U^T G) = \left(-\frac{1}{c}\partial_t\hat{\mathrm{T}} + \boldsymbol{\nabla}\cdot\hat{\mathbf{K}}\right)\left(\mathcal{L}\hat{1} + \boldsymbol{\mathcal{T}}_K\cdot\hat{\mathbf{K}} + \boldsymbol{\mathcal{T}}_\sigma\cdot\hat{\boldsymbol{\sigma}}\right) = \\ -(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}}_K)\hat{1} + (\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}}_\sigma - \frac{1}{c}\partial_t\mathcal{L})\hat{\mathrm{T}} \\ +(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{T}}_K + \boldsymbol{\nabla}\mathcal{L} - \frac{1}{c}\partial_t\boldsymbol{\mathcal{T}}_\sigma)\cdot\hat{\mathbf{K}} + (\boldsymbol{\nabla}\times\boldsymbol{\mathcal{T}}_\sigma + \frac{1}{c}\partial_t\boldsymbol{\mathcal{T}}_K)\cdot\hat{\boldsymbol{\sigma}}. \quad (416)\end{aligned}$$

So we get the inhomogenous channel projections of $\partial(J^T A) = F$ as

$$\hat{1} - channel: \qquad F_1 = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}}_K \quad (417)$$

$$\hat{\mathrm{T}} - channel: \qquad F_T = \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{T}}_\sigma - \frac{1}{c}\partial_t\mathcal{L} \quad (418)$$

$$\hat{\mathbf{K}} - channel: \qquad F_K = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{T}}_K + \boldsymbol{\nabla}\mathcal{L} - \frac{1}{c}\partial_t\boldsymbol{\mathcal{T}}_\sigma \quad (419)$$

$$\hat{\boldsymbol{\sigma}} - channel: \qquad F_\sigma = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{T}}_\sigma + \frac{1}{c}\partial_t\boldsymbol{\mathcal{T}}_K. \quad (420)$$

Applying the four-gradient and collecting the resulting $\hat{1}$, $\hat{\mathrm{T}}$, $\hat{\mathbf{K}}$ and $\boldsymbol{\sigma}$ channels yields

$$F_{\hat{1}} = -\nabla\cdot(\boldsymbol{u}\times\boldsymbol{g}), \quad (421)$$

$$F_T = \nabla \cdot \left(u_0 \boldsymbol{g} - \frac{1}{c} \varepsilon \boldsymbol{u} \right) - \frac{1}{c} \partial_t \left(\frac{u_0}{c} \varepsilon - \mathbf{u} \cdot \mathbf{g} \right), \tag{422}$$

$$\boldsymbol{F}_K = \nabla \times (\boldsymbol{u} \times \boldsymbol{g}) + \nabla \left(\frac{u_0}{c} \varepsilon - \mathbf{u} \cdot \mathbf{g} \right) - \frac{1}{c} \partial_t \left(u_0 \boldsymbol{g} - \frac{1}{c} \varepsilon \boldsymbol{u} \right), \tag{423}$$

$$\boldsymbol{F}_\sigma = \nabla \times \left(u_0 \boldsymbol{g} - \frac{1}{c} \varepsilon \boldsymbol{u} \right) + \frac{1}{c} \partial_t (\boldsymbol{u} \times \boldsymbol{g}). \tag{424}$$

Single-component fluids

If $\mathbf{g} = \rho_0 \mathbf{u}$, we have $\mathcal{T}_\sigma = 0$ and $\mathcal{T}_K = 0$,. In that case $U^T G = \mathcal{L} \hat{\mathrm{I}}$ and

$$\partial (U^T G) = \partial \mathcal{L} = -\frac{1}{c} \partial_t \mathcal{L} \hat{\mathrm{T}} + \nabla \mathcal{L} \cdot \hat{\mathbf{K}} = F, \tag{425}$$

in short

$$F = \partial \mathcal{L}. \tag{426}$$

With a force-power density four vector $F = \frac{1}{c} \mathcal{P} \hat{\mathrm{T}} + \mathcal{F} \cdot \hat{\mathbf{K}}$ we get

$$\mathcal{P} = -\partial_t \mathcal{L} \tag{427}$$

and

$$\mathcal{F} = \nabla \mathcal{L} \tag{428}$$

for conservative fluids.

Because $\partial \mathcal{L}$ is composed of a genuine four vector and a Lorentz scalar, the four force $F = \partial \mathcal{L}$ is Lorentz covariant. In the absence of external forces, $F = 0$ and $\partial \mathcal{L} = 0$, so $\mathcal{L}$ is constant in time and in space.

Adding the local barotropic/adiabatic speed of sound relation

If we use the local barotropic/adiabatic speed of sound relation of Eqn. (253):

$$c_s^2 = c^2 \left(\frac{\partial p}{\partial \varepsilon_0} \right)_s, \tag{429}$$

and use it as

$$\partial \varepsilon_0 = \frac{c^2}{c_s^2} \partial p = n_s^2 \partial p, \tag{430}$$

it then can be inserted into Eqn. (428) to get

$$F = n_s^2 \partial p. \tag{431}$$

and with δp as the acoustic pressure perturbation around a constant background pressure p_0, we get

$$F = n_s^2\, \partial(\delta p) = n_s^2 \left(-\frac{1}{c} \partial_t(\delta p)\, \hat{T} + \nabla(\delta p) \cdot \hat{K} \right). \tag{432}$$

The channel decomposition of this four-force gives

$$\hat{T}\text{-channel}: \quad \mathcal{P} = -n_s^2\, \partial_t(\delta p), \tag{433}$$
$$\hat{K}\text{-channel}: \quad \mathbf{F} = n_s^2\, \nabla(\delta p). \tag{434}$$

The $\hat{T}$-channel gives the acoustic power density: the rate of energy exchange is proportional to the time derivative of the pressure perturbation, amplified by n_s^2. Using the linearized continuity relation $\partial_t(\delta p) = -\rho_0 c_s^2 \nabla \cdot \mathbf{v}$ from §4.1.1, this gives $\mathcal{P} = n_s^2 \rho_0 c_s^2 \nabla \cdot \mathbf{v} = \rho_0 c^2 \nabla \cdot \mathbf{v}$, consistently with the compressibility-driven power term identified in the T-channel of $U\mathcal{H} = F$ in §4.3.2 .

The $\hat{K}$-channel gives the acoustic force density $\mathbf{F} = n_s^2 \nabla(\delta p)$. This differs from the standard non-relativistic acoustic momentum equation $\partial_t \mathbf{g} = -\nabla(\delta p)$ by the factor $n_s^2 = c^2/c_s^2$. The factor is not new physics: in the relativistic Euler equation from the $T^{\mu\nu}$ formulation, the same factor appears through the enthalpy density as $(\varepsilon + p)\partial_t \mathbf{v} = -c_s^2 \nabla(\delta\rho_0) = -\frac{c_s^2}{c^2} \cdot c^2 \nabla(\delta\rho_0)$, which gives $\partial_t \mathbf{v} \approx -\frac{1}{\rho_0} \nabla(\delta p)$ in the non-relativistic limit. In the BQ treatment, the same n_s^2 factor is not hidden inside the enthalpy density but appears explicitly as the prefactor of $\nabla(\delta p)$, as a direct consequence of the bilinear structure $\partial(U^T G) = F$ and the barotropic relation $\partial\varepsilon_0 = n_s^2 \partial p$. The form $\mathbf{F} = n_s^2 \nabla(\delta p)$ as the K-channel of a covariant four-force equation does not appear in this explicit form in the standard fluid dynamics or relativistic hydrodynamics literature, where the n_s^2 factor is distributed across the enthalpy and equation-of-state terms rather than collected into a single prefactor.

In the source-free case $F = 0$, Eqn. (432) gives $\partial(\delta p) = 0$, i.e. $\partial_t(\delta p) = 0$ and $\nabla(\delta p) = 0$: the acoustic pressure perturbation is uniform in space and constant in time. This is the barotropic generalization of the Bernoulli condition established in §4.1: in the source-free, conservative regime, the pressure perturbation is frozen, consistently with $\mathcal{L} = \varepsilon_0 = \rho_0 c^2$ being constant in space and time when $F = \partial\mathcal{L} = 0$.

The result $F = n_s^2 \partial(\delta p)$ therefore provides a compact covariant expression for the acoustic four-force in the barotropic regime: the complete force-power structure of the acoustic field is encoded in the four-gradient of the pressure perturbation, amplified by the acoustic refractive index squared $n_s^2 = c^2/c_s^2$. This is the stress-energy level manifestation of the same n_s^2 factor that threads through the field level (§4.1.1), the force level (§sec:UHFeq), and the wave level (§4.4) of the BQ fluid hierarchy, confirming that the acoustic refractive index n_s is the universal medium parameter of the hierarchy at all four levels.

Comparison with the standard Euler–Lagrange approach.

In the standard field-theoretic treatment of a relativistic fluid, the equations of motion are obtained by postulating a Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$ depending on one or more field variables ϕ and their spacetime derivatives, and then demanding that the action $\int \mathcal{L}\, d^4x$ be stationary under variations $\delta\phi$ that vanish on the boundary. The resulting Euler–Lagrange equations,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial \mathcal{L}}{\partial\phi} = 0, \tag{435}$$

require a prior choice of field variable. For a perfect fluid this is non-trivial: the velocity field $\mathbf{v}$ is constrained by the continuity equation $\partial_t\rho + \nabla \cdot (\rho\mathbf{v}) = 0$, which must be enforced by a Lagrange multiplier, typically a velocity potential for irrotational flow or Clebsch potentials (ϕ, λ, μ) for the rotational case. The Lagrangian density itself must be separately identified — usually as the pressure or the negative of the energy density — and the passage from $\mathcal{L}$ to the energy and momentum conservation equations requires constructing the Noether stress-energy tensor $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}$ and then extracting $\partial_\mu T^{\mu\nu} = 0$ by index contraction. Each step — field-variable choice, constraint handling, Noether construction, index projection — is a separate operation.

The BQ result of Eqns. (425)–(426) bypasses this entire procedure. The Lagrangian density $\mathcal{L} = \varepsilon_0 = \rho_0 c^2$ is not postulated but emerges as the $\hat{1}$-sector of the bilinear $U^T G$, whose algebraic definition requires no choice of field variable and no constraint handling: it is the Minkowski scalar product of the kinematic four-vector U and the dynamic four-vector G, evaluated exactly for all $|v| < c$. The force equation $\mathbf{F} = \nabla\mathcal{L}$ and power equation $\mathcal{P} = -\partial_t\mathcal{L}$ are not derived by variational calculus but read off directly as the $\hat{K}$- and $\hat{T}$-channels of the single first-order equation $\partial(U^T G) = F$. These results are relativistically exact, since $\mathrm{T}_K = \mathbf{u}\times\mathbf{g} = 0$ and $\mathrm{T}_\sigma = u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u} = 0$ hold identically for $\mathbf{g} = \rho_0\mathbf{u}$ and $\varepsilon = \rho_0 c u_0$, without approximation.

The structural difference is significant. In the standard approach, $\mathbf{F} = -\nabla\mathcal{L}$ is a *conclusion* reached after assembling variational machinery; in the BQ approach it is a *direct channel readout* from a bilinear product. The Lagrangian density acquires a geometric identity — it is the $\hat{1}$-sector of the Minkowski bilinear of the kinematic and dynamic four-vectors of the fluid — rather than being a postulated scalar whose physical meaning must be separately argued. The source-free condition $F = 0$ gives $\partial\mathcal{L} = 0$, meaning $\mathcal{L} = \varepsilon_0 = \rho_0 c^2$ is constant in both space and time: its constancy in space ($\mathbf{F} = 0$) and in time ($\mathcal{P} = 0$) emerge simultaneously from the same four-vector equation rather than as two separate results requiring separate arguments. This is the relativistic Bernoulli theorem, obtained here as a spacetime conservation law for the Lagrangian density rather than as a first integral derived by integrating the momentum equation along a streamline. In the barotropic regime, the same source-free condition gives equivalently $\partial(\delta p) = 0$ through the relation $\partial\mathcal{L} = \partial\varepsilon_0 = n_s^2\partial(\delta p)$ of Eqn. (432): the acoustic pressure perturbation is uniform in space and constant in time, with the two forms of the Bernoulli condition — $\varepsilon_0 = \text{const}$ and $\delta p = \text{const}$ — connected by the acoustic refractive index n_s as the single medium parameter threading through the hierarchy.

4.5.3 Expansion of $\partial(U^T G)$ for the general case.

Assembled channel decomposition.

Collecting by basis element:

$$\partial(U^T G) = F_{\hat{1}}\,\hat{1} + F_T\,\hat{T} + \mathbf{F}_K \cdot \hat{K} + \mathbf{F}_\sigma \cdot \hat{\boldsymbol{\sigma}} = F. \tag{436}$$

$\hat{1}$-channel (norm/scalar): intrinsic spin density conservation

$$\boxed{F_{\hat{1}} = -\nabla \cdot (\mathbf{u} \times \mathbf{g}).} \tag{437}$$

For a single-component fluid, $\mathcal{T}_K = \mathbf{u} \times \mathbf{g} = \rho_0(\mathbf{u} \times \mathbf{u}) = \mathbf{0}$, so $F_{\hat{1}} = 0$ identically and the $\hat{1}$-channel is trivial. For a multi-component or spinning fluid, $\mathbf{g}$ and $\mathbf{u}$ are no longer parallel and $\mathcal{T}_K \neq \mathbf{0}$. The condition $F_{\hat{1}} = 0$ then states that the angular momentum density flux $\mathbf{u} \times \mathbf{g}$ is divergence-free: angular momentum is locally conserved. A non-vanishing $F_{\hat{1}}$ signals an external torque density acting on the fluid. This channel is structurally analogous to the $\hat{1}$-channel of $U\mathcal{H} = F$ in §4.3.1, where $F_{\hat{1}} = -\mathbf{u}\cdot\boldsymbol{\Omega}$ measured helicity; here the $\hat{1}$-channel of $\partial(U^T G) = F$ measures the divergence of the angular momentum density flux — a different rotational quantity at a different level of the hierarchy, but occupying the same geometric channel.

$\hat{T}$-channel (time/energy): energy conservation

$$\boxed{F_T = \frac{1}{c}\partial_t\mathcal{L} + \nabla \cdot \mathcal{T}_\sigma = \frac{1}{c}\partial_t(\mathbf{u} \cdot \mathbf{g} - \varepsilon) + \nabla \cdot \left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right).} \tag{438}$$

Expanding:

$$F_T = -\frac{1}{c}\partial_t\varepsilon + \frac{1}{c}\partial_t(\mathbf{u} \cdot \mathbf{g}) + u_0\,\nabla \cdot \mathbf{g} - \frac{1}{c}\nabla \cdot (\varepsilon\mathbf{u}). \tag{439}$$

For a closed conservative fluid ($F_T = 0$), the condition $\frac{1}{c}\partial_t(\varepsilon - \mathbf{u} \cdot \mathbf{g}) = \nabla \cdot (u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u})$ is the local energy-conservation equation, combining the time evolution of the Lagrangian density $\mathcal{L}$ with the divergence of the energy-flux vector $\mathcal{T}_\sigma$. The $\hat{T}$-channel is therefore the Noether energy-conservation equation of the fluid hierarchy, in exact parallel with charge conservation $\partial_\mu J^\mu = 0$ in electromagnetism, and consistently with the T-channel power balance established in §4.3.2 : the same channel carries energy transport at every level of the hierarchy.

In the single-component, slow-flow, barotropic regime, the T-channel acquires additional content through the barotropic relation $\partial\varepsilon_0 = n_s^2 \partial p$ of Eqn. (429). Setting $F_T = 0$ then gives

$$\frac{1}{c}\partial_t(\delta p) = -\frac{1}{n_s^2}\nabla \cdot \boldsymbol{\mathcal{T}}_\sigma, \tag{440}$$

connecting the time evolution of the acoustic pressure perturbation to the divergence of the energy-flux vector, with the acoustic refractive index n_s mediating the relationship. This is the stress-energy level analogue of the acoustic relation $\frac{Dp}{Dt} = -\rho_0 c_s^2 \nabla \cdot \mathbf{v}$ derived from the field level in §4.1.1, confirming that n_s threads through the T-channel at both levels.

$\hat{K}$-channel (space/momentum): momentum conservation and Euler structure

$$\boxed{\mathbf{F}_K = -\frac{1}{c}\partial_t \boldsymbol{\mathcal{T}}_\sigma - \nabla\mathcal{L} + \nabla \times \boldsymbol{\mathcal{T}}_K.} \tag{441}$$

Substituting the definitions of $\mathcal{L}$, $\boldsymbol{\mathcal{T}}_\sigma$, and $\boldsymbol{\mathcal{S}}$:

$$-\frac{1}{c}\partial_t \boldsymbol{\mathcal{T}}_\sigma = -\partial_t \mathbf{g} + \frac{1}{c^2}\partial_t(\varepsilon \mathbf{u}), \tag{442}$$

$$\nabla\mathcal{L} = \nabla(\mathbf{u}\cdot\mathbf{g}) - \nabla\varepsilon, \tag{443}$$

$$\nabla \times \boldsymbol{\mathcal{T}}_K = \nabla \times (\mathbf{u}\times\mathbf{g}). \tag{444}$$

The condition $\mathbf{F}_K = \mathbf{0}$ for a closed fluid is the local momentum-conservation equation. The term $\nabla \times (\mathbf{u}\times\mathbf{g})$ encodes vortex stretching, compressibility, and velocity gradient effects through the vector identity $\nabla \times (\mathbf{u}\times\mathbf{g}) = (\mathbf{g}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{g} + \mathbf{u}(\nabla\cdot\mathbf{g}) - \mathbf{g}(\nabla\cdot\mathbf{u})$. This is the Lamb-vector structure at the stress-energy level, complementing the Lamb-vector appearance in the K-channel of $U\mathcal{H} = F$ in §4.3.3.

For a single-component fluid ($\boldsymbol{\mathcal{T}}_K = \mathbf{0}$, $\boldsymbol{\mathcal{T}}_\sigma = \mathbf{0}$), the K-channel reduces to $\mathbf{F}_K = \nabla\mathcal{L} = \nabla\varepsilon_0$, which in the barotropic regime gives

$$\mathbf{F}_K = \nabla\varepsilon_0 = n_s^2 \nabla(\delta p), \tag{445}$$

consistently with the result $F = n_s^2 \partial(\delta p)$ of Eqn. (432). The $\hat{K}$-channel is therefore the Noether momentum-conservation equation of the fluid hierarchy, the analogue of Poynting momentum conservation in electromagnetism, and its barotropic form makes the n_s^2 amplification factor explicit at the stress-energy level.

$\hat{\sigma}$-channel (spin/rotation): angular momentum transport and energy-flux irrotationality

$$\boxed{\mathbf{F}_\sigma = \frac{1}{c}\partial_t \boldsymbol{\mathcal{T}}_K - \nabla \times \boldsymbol{\mathcal{T}}_\sigma = \frac{1}{c}\partial_t(\mathbf{u}\times\mathbf{g}) - \nabla \times \left(u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}\right).} \tag{446}$$

Expanding the curl:

$$\nabla \times \boldsymbol{\mathcal{T}}_\sigma = u_0\,\nabla \times \mathbf{g} - \frac{1}{c}(\nabla\varepsilon \times \mathbf{u} + \varepsilon\,\boldsymbol{\omega}) = u_0\boldsymbol{\Omega} - \frac{1}{c}(\nabla\varepsilon \times \mathbf{u} + \varepsilon\,\boldsymbol{\omega}), \tag{447}$$

where $\boldsymbol{\Omega} = \nabla \times \mathbf{g}$ is the momentum-density vorticity and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the velocity vorticity. The $\hat{\boldsymbol{\sigma}}$-channel condition $\mathbf{F}_\sigma = 0$ couples the time evolution of the angular momentum density $\mathbf{u}\times\mathbf{g}$ to the curl of the energy-transport vector $\boldsymbol{\mathcal{P}}$.

For a single-component fluid $\boldsymbol{\mathcal{T}}_K = \mathbf{0}$, so $\partial_t\boldsymbol{\mathcal{T}}_K = \mathbf{0}$ and the channel reduces to

$$\nabla \times \boldsymbol{\mathcal{T}}_\sigma = 0, \tag{448}$$

i.e. the energy-flux vector $\boldsymbol{\mathcal{T}}_\sigma = u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u}$ is irrotational. This is a constraint on the relative geometry of momentum density and energy-density transport with no direct standard textbook counterpart. It complements the $\hat{\boldsymbol{\sigma}}$-channel of $U\mathcal{H} = F$ in §4.3.4, which gave the rotational balance $c^2\boldsymbol{\omega} = -\mathbf{v}\times\frac{D\mathbf{v}}{Dt}$: where the force-level $\hat{\boldsymbol{\sigma}}$-channel constrains the relationship between vorticity and material acceleration, the stress-energy-level $\hat{\boldsymbol{\sigma}}$-channel constrains the irrotationality of the energy-flux vector. Both are rotational conditions emerging from the same geometric channel at different levels of the hierarchy, without classical precedent in this compact first-order form.

In the barotropic regime, $\nabla\times\boldsymbol{\mathcal{T}}_\sigma = 0$ acquires additional content through $\nabla\varepsilon_0 = n_s^2\nabla p$: the irrotationality condition on $\boldsymbol{\mathcal{T}}_\sigma$ becomes a constraint on the relative orientation of the pressure gradient and the momentum-density flow, mediated by n_s^2. For irrotational acoustic motion ($\boldsymbol{\Omega} = 0$), this reduces to a constraint on $\nabla\varepsilon\times\mathbf{u}$, consistently with the irrotational acoustic regime of §4.4.

Table 2 collects the channel decomposition of $\partial(U^T G) = F$, updated to reflect the barotropic extension and the connections to the lower levels of the hierarchy.

4.5.4 The Bernoulli equation as a multi-level consistency result.

An important structural observation is that the Bernoulli equation is not unique to a single level or channel of the BQ fluid hierarchy but appears at every level, in a form appropriate to that level. In standard fluid dynamics, the dual origin of the Bernoulli equation — derivable either from the momentum equation or from energy conservation (Hosking and Dewar 2015, p. 73–74) — has long been noted as a consistency property of ideal fluid theory. The BQ hierarchy makes this duality algebraically explicit and extends it to three levels:

Table 2 Channel decomposition of $\partial(U^T G) = F$. Source-free limit $F = 0$, single-component fluid, with barotropic extension $\partial\varepsilon_0 = n_s^2 \partial p$.

Channel	Source-free content	Physical interpretation	Connection to hierarchy
$\hat{\imath}$	$\nabla \cdot (\mathbf{u} \times \mathbf{g}) = 0$	Angular momentum conservation	Rotational content at stress-energy level
$\hat{T}$	$-\frac{1}{c}\partial_t \mathcal{L} = \nabla \cdot \mathcal{T}_\sigma$	Noether energy conservation	T-channel power balance of §4.3.2
$\hat{K}$	$\frac{1}{c}\partial_t \mathcal{T}_\sigma = -\nabla\mathcal{L} + \nabla \times \mathcal{T}_K$; barotropic: $\mathbf{F}_K = n_s^2 \nabla(\delta p)$	Momentum conservation / Euler	K-channel Euler-Lamb of §4.3.3
$\hat{\sigma}$	$\nabla \times \mathcal{T}_\sigma = 0$	Energy-flux irrotationality	$\hat{\sigma}$-channel of §4.3.4

At the field level (§4.1), the $\hat{\sigma}$-channel of $\mathcal{H} = \partial^T G = 0$ in the dust, slow-flow limit gives the unsteady Bernoulli relation $\partial_t \chi + \frac{v^2}{2} = \text{const}$, with no momentum equation or energy equation separately invoked.

At the force level (§4.3), the T-channel of $U\mathcal{H} = F$ gives $\frac{D\mathcal{K}}{Dt} = 0$ in the incompressible, $\sigma_E = 0$ limit — the energy-balance route to Bernoulli — while the K-channel gives the Euler–Lamb equation whose steady, irrotational limit yields the Bernoulli streamline condition — the momentum-balance route. These are the two classical derivations of Hosking and Dewar (2015), here appearing as T- and K-channel projections of the same force equation.

At the stress-energy level (§4.5), the source-free condition $F = \partial\mathcal{L} = 0$ gives $\mathcal{L} = \varepsilon_0 = \rho_0 c^2$ constant in both space and time simultaneously, with spatial constancy ($\mathbf{F}_K = \nabla\mathcal{L} = 0$) and temporal constancy ($\mathcal{P} = -\partial_t\mathcal{L} = 0$) emerging together from the single four-vector equation rather than as two separate results. In the barotropic regime, the same condition gives equivalently $\delta p = \text{const}$ through $\partial\mathcal{L} = n_s^2 \partial(\delta p) = 0$.

The three-level appearance of the Bernoulli equation is a structural consistency result of the BQ hierarchy: the same physical condition — conservative, source-free, irrotational flow — manifests as a channel projection at every level, in the form natural to that level (velocity potential, kinetic energy, Lagrangian density). The standard "two routes" of classical fluid dynamics correspond to the T- and K-channels of the force level, which is one level of a three-level structure. The BQ hierarchy does not introduce a new Bernoulli theorem but reveals that the known theorem is a single algebraic condition expressed at multiple levels of the same product hierarchy, with each level adding a more complete and more relativistic form of the same underlying conservation law.

4.6 General conclusions for the BQ fluid hierarchy

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The results of §4.1 through §4.5 establish a coherent and self-contained treatment of relativistic fluid dynamics within the BQ algebraic framework. Starting from the single substitution $J \to V$, $A \to G$ applied to the electromagnetic BQ hierarchy, four equations are constructed:

$$\mathcal{H} = \partial^T G, \qquad U\mathcal{H} = F, \qquad \partial\mathcal{H} = F', \qquad \partial(U^T G) = F, \tag{449}$$

from which the known results of fluid dynamics — continuity, irrotationality, Bernoulli, acoustic wave propagation, helicity, power balance, Euler–Lamb momentum transport, vorticity dynamics, energy conservation, and the Lagrangian structure — emerge as channel projections without being assumed at any point. No fluid equation is an input; all are outputs of the algebraic channel structure applied to the four-vector $G = \rho_0 U$.

The derivations are organized by level and channel: each equation contributes specific physical content to specific geometric sectors, and the mutual consistency of the results across levels and channels — confirmed repeatedly through the threading of the acoustic refractive index $n_s = c/c_s$ and the $U\mathcal{H}$-level continuity cancellation — is a structural property of the BQ algebra rather than a coincidence. The following subsection identifies the most fundamental organizational pattern of the hierarchy: the channel activity structure of the four equations under the single-component fluid condition $\mathbf{g} = \rho_0\mathbf{u}$.

4.6.1 Channel activity and the unique role of $U\mathcal{H} = F$

Under the single-component perfect fluid condition $\mathbf{g} = \rho_0\mathbf{u}$, the four-channel structure of each equation in the BQ fluid hierarchy takes a strikingly organized form. The field tensor is

$$\mathcal{H} = -\sigma_E \hat{1} + \mathbf{\Omega} \cdot \hat{K} - \frac{1}{c}\mathbf{\Pi} \cdot \hat{\boldsymbol{\sigma}}, \tag{450}$$

which has no $\hat{T}$-channel by the algebraic structure of ∂^T. The force equation $U\mathcal{H} = F$ produces all four channels:

$$F_{\hat{1}} = -\mathbf{u} \cdot \mathbf{\Omega}, \quad \text{(helicity)} \tag{451}$$

$$F_T = -u_0\sigma_E - \frac{1}{c}\mathbf{u} \cdot \mathbf{\Pi}, \quad \text{(power balance)} \tag{452}$$

$$\mathbf{F}_K = \mathbf{u} \times \mathbf{\Omega} - \frac{u_0}{c}\mathbf{\Pi} - \sigma_E\mathbf{u}, \quad \text{(Euler–Lamb force)} \tag{453}$$

$$\mathbf{F}_\sigma = -\frac{1}{c}\mathbf{u} \times \mathbf{\Pi} - u_0\mathbf{\Omega}. \quad \text{(rotational transport)} \tag{454}$$

For the wave equation $\partial\mathcal{H} = F'$, the BQ Bianchi identities of §4.4 give $F'_{\hat{1}} = 0$ and $\mathbf{F}'_\sigma = 0$ identically as geometric identities, leaving only the dynamical channels:

$$F'_T = \frac{1}{c}\Box\varepsilon, \qquad \mathbf{F}'_K = \Box\mathbf{g}. \tag{455}$$

For the stress-energy equation $\partial(U^T G) = F$, the single-component condition gives $\boldsymbol{\mathcal{S}} = \mathbf{u} \times \mathbf{g} = \rho_0(\mathbf{u} \times \mathbf{u}) = 0$ and $\boldsymbol{\mathcal{P}} = u_0\mathbf{g} - \frac{\varepsilon}{c}\mathbf{u} = 0$ exactly for all $|v| < c$, so that $U^T G = \mathcal{L}\hat{1}$ is a pure Lorentz scalar and

$$\partial(U^T G) = \partial\mathcal{L} = \frac{1}{c}\partial_t\mathcal{L}\hat{T} + \nabla\mathcal{L} \cdot \hat{K} = F, \tag{456}$$

with $\hat{1}$- and $\hat{\boldsymbol{\sigma}}$-channels vanishing exactly.

The channel activity pattern across all four equations is therefore:

Equation	$\hat{1}$	$\hat{T}$	$\hat{K}$	$\hat{\boldsymbol{\sigma}}$
$\mathcal{H} = \partial^T G$	$\sigma_E \neq 0$	absent	$\boldsymbol{\Omega} \neq 0$	$\boldsymbol{\Pi} \neq 0$
$U\mathcal{H} = F$	$\neq 0$ (helicity)	$\neq 0$ (power)	$\neq 0$ (force)	$\neq 0$ (rotation)
$\partial\mathcal{H} = F'$	$= 0$ (Bianchi)	$\neq 0$ (wave ε)	$\neq 0$ (wave $\mathbf{g}$)	$= 0$ (Bianchi)
$\partial(U^T G) = F$	$= 0$ (exact)	$\neq 0$ (energy)	$\neq 0$ (momentum)	$= 0$ (exact)

The pattern is structurally significant. Three of the four equations have either one channel absent by algebraic structure or two channels vanishing exactly: $\mathcal{H} = \partial^T G$ has no $\hat{T}$-channel; $\partial\mathcal{H} = F'$ and $\partial(U^T G) = F$ both have $\hat{1} = \hat{\boldsymbol{\sigma}} = 0$. The two equations with vanishing norm and sigma channels reduce to purely relativistic time-space structures: $\partial\mathcal{L} = F$ is the four-gradient of a Lorentz scalar — the most elementary covariant equation possible — and $\partial\mathcal{H} = F'$ gives the relativistic wave operator $\Box$ acting on the T and K components of G.

The force equation $U\mathcal{H} = F$ is the unique exception: it is the only equation in the hierarchy where all four channels are simultaneously non-trivial. The reason is algebraic: the four-velocity $U = u_0\hat{T} + \mathbf{u} \cdot \hat{K}$ has both T and K components, and its contraction with $\mathcal{H} = -\sigma_E\hat{1} + \boldsymbol{\Omega} \cdot \hat{K} - \frac{1}{c}\boldsymbol{\Pi} \cdot \hat{\boldsymbol{\sigma}}$ through the BQ product rules mixes the three input channels of $\mathcal{H}$ into all four output channels of $U\mathcal{H}$. The force equation is therefore the channel-mixing equation of the hierarchy: the kinematic four-vector U couples to the field $\mathcal{H}$ and distributes its content across all four geometric sectors simultaneously.

This explains why $U\mathcal{H} = F$ is the physically richest equation of the hierarchy. Helicity, power balance, Euler–Lamb momentum transport, and rotational vorticity dynamics appear together as distinct, coupled projections of a single product — not because they are forced together by physical argument, but because the contraction with U algebraically generates all four channels from the three-channel field $\mathcal{H}$. The other equations are relativistically cleaner — $\partial\mathcal{H} = F'$ and $\partial(U^T G) = F$ reduce to purely T-K structures — but carry less coupled physical content per equation.

The hierarchy therefore has a natural algebraic organization under $\mathbf{g} = \rho_0\mathbf{u}$:

- $\mathcal{H} = \partial^T G$: the field definition, three active channels ($\hat{1}$, $\hat{K}$, $\hat{\boldsymbol{\sigma}}$), no $\hat{T}$-channel, encoding the vorticity, momentum-pressure, and energy-conservation structure of the fluid field.
- $U\mathcal{H} = F$: the force equation, all four channels active, the unique channel-mixing equation encoding helicity, power, momentum, and rotation as four coupled projections of one product.
- $\partial\mathcal{H} = F'$: the wave equation, two active channels ($\hat{T}$, $\hat{K}$), two vanishing by Bianchi identities, encoding relativistic wave propagation of ε and $\mathbf{g}$ at speed c.
- $\partial(U^T G) = F$: the stress-energy equation, two active channels ($\hat{T}$, $\hat{K}$), two vanishing exactly, encoding Noether energy-momentum conservation and the Lagrangian structure as the four-gradient of a Lorentz scalar.

The fact that $\partial\mathcal{H} = F'$ and $\partial(U^T G) = F$ share the same channel activity pattern — $\hat{T}$ and $\hat{K}$ active, $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ vanishing — is itself a structural result: both are equations for the four-gradient of a relativistic object ($\mathcal{H}$ and $\mathcal{L}$ respectively), and in both cases the single-component fluid condition eliminates the rotational and helical content, leaving a clean relativistic time-space structure. The $\hat{1}$ and $\hat{\boldsymbol{\sigma}}$ channels carry the rotational and topological content of the fluid, and this content is active only when the kinematic four-vector U is contracted into the hierarchy through $U\mathcal{H} = F$ — the single equation where the fluid's motion couples fully to its own field structure.

The algebraic uniqueness of $U\mathcal{H} = F$ as the sole four-channel equation of the hierarchy has a direct counterpart in the perspective of the practicing fluid dynamicist. The four active channels of $U\mathcal{H} = F$ correspond precisely to the four physical sectors that fluid dynamics experts already organize their thinking around, even without the BQ algebraic language:

- The $\hat{1}$-channel carries *topological* content: the helicity density $\mathbf{v} \cdot (\nabla \times \mathbf{v})$, which measures the linking and knotting of vortex lines and is central to turbulence theory, vortex dynamics, and magnetohydrodynamics. Helicity is recognized by FD experts as a topological invariant of ideal flow, conserved under the Euler equation and broken by viscosity or baroclinic effects.
- The $\hat{T}$-channel carries *energetic* content: the kinetic power equation $\mathcal{P} = -\frac{D\mathcal{K}}{Dt}$ in the incompressible limit and the acoustic pressure-work $P \approx -\frac{1}{c_s^2}\mathbf{v} \cdot \nabla p$ in the compressible case. These are standard results in fluid energetics and acoustics, governing how kinetic energy is exchanged, dissipated, and transported along fluid trajectories.
- The $\hat{K}$-channel carries *momentum* content: the Euler–Lamb equation $\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p$, the central equation of fluid mechanics. This is the equation every fluid dynamicist encounters first and returns to most often: it governs

force balance, pressure-driven flow, boundary layers, shock formation, and the transition to turbulence.

- The $\hat{\boldsymbol{\sigma}}$-channel carries *rotational* content: the vorticity balance $c^2\boldsymbol{\omega} = -\mathbf{v} \times \frac{D\mathbf{v}}{Dt}$ and the baroclinic rotational forcing $-\frac{c}{c_s^2}\mathbf{v}\times\nabla p$. These govern vortex generation, vortex stretching, baroclinic instability, and angular momentum transport — the physical mechanisms that control the rotational organization of geophysical, astrophysical, and engineering flows.

These four sectors — topology, energetics, momentum, and rotation — are not imposed by the BQ algebra but are the natural organizing categories that fluid dynamics experts already use when analyzing complex flows. Turbulence researchers think in terms of helicity and enstrophy; acousticians think in terms of acoustic power and pressure work; engineers think in terms of force balance and the Navier–Stokes equation; geophysicists think in terms of vorticity generation and baroclinic instability. The BQ algebra does not reorganize fluid dynamics into unfamiliar categories: it reveals that the four categories the fluid dynamics community already uses correspond exactly to the four geometric channels of the single product $U\mathcal{H} = F$.

The other three equations of the hierarchy produce results that, while correct and important, map less directly onto FD practice. The field equation $\mathcal{H} = \partial^T G$ defines the fluid field in terms of $\boldsymbol{\Omega}$, $\boldsymbol{\Pi}$, and σ_E — objects that contain familiar quantities (vorticity, pressure gradients, energy conservation) but are not the primary variables of practical FD. The wave equation $\partial\mathcal{H} = F'$ produces relativistic wave equations and the acoustic refractive index — recognizable to specialists in relativistic acoustics but not to the broader community. The stress-energy equation $\partial(U^T G) = F$ produces Lagrangian structure and Noether conservation laws — familiar to theorists but not the language of everyday fluid dynamics practice.

$U\mathcal{H} = F$ is therefore the equation where the BQ hierarchy speaks most directly to fluid dynamics as it is actually practised: its four channels are the four physical sectors FD experts already think about separately, and the BQ framework shows for the first time that these four sectors are not independent physical principles but four projections of a single algebraic product. This is the sense in which $U\mathcal{H} = F$ stands out not only algebraically — as the unique four-channel equation of the hierarchy — but also physically, as the equation that makes the BQ fluid hierarchy most immediately recognizable and useful to the practicing fluid dynamicist.

5 Synthesis

5.1 A common hierarchy for electrodynamics and fluid dynamics

The principal result of this paper is that electrodynamics and relativistic fluid dynamics can be organized within the same BQ algebraic hierarchy. In both theories, a pair of four-vector inputs generates a sequence of field, force, wave, and stress-energy constructions,

$$\begin{aligned} \text{field:} \quad & Y = \partial^T X, \\ \text{force:} \quad & AY = F, \\ \text{wave:} \quad & \partial Y = F', \\ \text{stress-energy:} \quad & \partial(A^T X) = F. \end{aligned} \tag{457}$$

For electrodynamics the inputs are the four-current and four-potential,

$$(A, X) = (J, A), \tag{458}$$

giving the hierarchy

$$B = \partial^T A, \qquad JB = F, \qquad \partial B = \mu_0 J, \qquad \partial(J^T A) = F. \tag{459}$$

For fluid dynamics the corresponding inputs are the four-velocity and four-momentum density,

$$(A, X) = (U, G), \tag{460}$$

giving

$$\mathcal{H} = \partial^T G, \qquad U\mathcal{H} = F, \qquad \partial\mathcal{H} = F', \qquad \partial(U^T G) = F. \tag{461}$$

Although the physical interpretations differ, the algebraic structure is identical. In both cases the hierarchy is constructed from the same operations, uses the same channel decomposition, and distributes physical content across the same geometric sectors. The comparison therefore suggests that the BQ channel decomposition is not tied to a specific physical theory but captures a more general organizational structure shared by multiple classical field theories.

At the level of channel activity, the strongest parallel appears in the force equations $JB = F$ and $U\mathcal{H} = F$. These are the unique four-channel equations of their respective hierarchies: the only level at which all geometric sectors are simultaneously active and carry independent physical content. The field, wave, and stress-energy levels contain inactive or identically vanishing channels under the corresponding geometric constraints, whereas the force level couples all channels at once. This identifies the force equation as the central interaction level of both hierarchies.

The comparison also reveals the principal structural difference between the two theories. In electrodynamics, the current J and potential A are independent input fields. In fluid dynamics, the momentum-density field is related to the velocity field through

$$G = \rho_0 U, \tag{462}$$

so that the field $\mathcal{H} = \partial^T G$ is generated from a quantity that already contains the dynamical variable appearing in the force equation $U\mathcal{H} = F$. The fluid hierarchy is therefore self-coupled in a way that has no direct analogue in the electromagnetic case. Within the BQ framework, this self-referential relation provides a natural explanation for the greater nonlinearity and structural complexity of fluid dynamics relative to electrodynamics.

The shared hierarchy and the distinct role of the relation $G = \rho_0 U$ together suggest that the BQ framework does more than reformulate existing equations. It provides a common algebraic language in which similarities and differences between physical theories can be identified, compared, and located precisely within the same channel-decomposition structure.

5.2 What the channel decomposition reveals

The comparison developed in the previous sections suggests that the principal value of the BQ framework lies not in the introduction of new dynamical laws but in the explicit organization of existing ones. The channel decomposition separates each equation into geometrically distinct sectors, allowing common structural features of electrodynamics and fluid dynamics to be identified directly.

A particularly striking result is the role of the force equation. In both theories,

$$JB = F, \qquad U\mathcal{H} = F, \tag{463}$$

are the unique four-channel equations of their respective hierarchies. At this level the norm, time, space, and sigma channels are simultaneously active and carry independent physical information. The force equations therefore occupy a distinguished position within the hierarchy: they are the only level at which topology, energetics, momentum transport, and rotational dynamics appear together in a single algebraic object.

The channel decomposition also clarifies the relationship between electrodynamics and fluid dynamics. Under the substitution

$$J \to U, \qquad A \to G, \tag{464}$$

the electromagnetic hierarchy maps onto a fluid hierarchy with the same algebraic structure but different physical interpretation. Quantities that appear unrelated in their traditional formulations — electric and fluid fields, Lorentz and Euler–Lamb forces, electromagnetic and fluid stress-energy relations — occupy corresponding locations within the same channel-decomposition framework.

At the same time, the comparison makes the principal difference between the two theories explicit. In electrodynamics the source and field variables are independent inputs. In fluid dynamics, however, the momentum-density field satisfies

$$G = \rho_0 U, \tag{465}$$

so that the field-generating quantity already contains the velocity field appearing in the force equation. The fluid hierarchy is therefore self-coupled: the field is generated from a quantity that depends on the dynamical variable on which the field subsequently acts. Within the present framework this self-referential structure provides a natural explanation for the greater nonlinearity and structural complexity of fluid dynamics relative to electrodynamics.

Viewed from this perspective, the channel decomposition acts as a common organizational language. Rather than replacing established formulations, it provides a means of locating equations, conservation laws, and geometric constraints within a single hierarchy. Similarities and differences between theories become visible as differences in channel activity and coupling structure rather than as differences in notation or mathematical formalism.

5.3 New structural identifications

Beyond providing a common organizational framework, the BQ hierarchy appears to reveal several structural features that are not evident in conventional formulations of either electrodynamics or fluid dynamics. While each of these results requires further investigation, together they illustrate the kind of information that becomes visible when the theories are analyzed through their channel structure.

A first example is the role of the acoustic refractive index

$$n_s = \frac{c}{c_s}, \tag{466}$$

which enters through the barotropic equation of state and subsequently propagates through multiple levels of the fluid hierarchy. In the present formulation, n_s appears not merely as a wave-speed ratio but as a hierarchy-wide medium parameter linking the relativistic and acoustic sectors of the theory. Rather than being reintroduced independently at different stages of the analysis, it emerges from a single constitutive relation and reappears throughout the field, force, and wave levels.

A second result is the existence of fluid analogues of the electromagnetic Bianchi identities. In the electromagnetic hierarchy, the vanishing norm and sigma channels of the wave equation encode the geometric identities

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{467}$$

The fluid hierarchy exhibits an analogous structure, where corresponding channel constraints arise as identities associated with the geometry of the fluid field rather than as independently imposed dynamical equations. This parallel further strengthens the structural correspondence between the two theories.

A third observation concerns helicity generation. In the BQ formulation, helicity appears naturally in the norm channel of the force equation

$$U\mathcal{H} = F, \tag{468}$$

through the quantity

$$h = \mathbf{u} \cdot \mathbf{\Omega}. \tag{469}$$

The confinement of helicity generation to a specific channel suggests a geometric interpretation in which topological information occupies a well-defined sector of the hierarchy, distinct from energetic, momentum, and rotational transport effects.

The wave level yields a further result in the form of the covariant equation

$$\Box G = F', \tag{470}$$

with the momentum-density field G acting as the primary wave-carrying variable. In contrast to formulations that focus separately on density, pressure, or velocity perturbations, the BQ hierarchy places the wave dynamics directly at the level of the four-momentum density. This provides a compact covariant description in which relativistic and acoustic effects appear within a common framework.

Finally, the channel decomposition itself gives rise to a characteristic activity pattern across the hierarchy. Some equations occupy only a subset of the available channels, while others activate all four. Viewed in this way, each hierarchy possesses a distinct channel signature. The force equations $JB = F$ and $U\mathcal{H} = F$ stand out as the unique fully active levels, while the field, wave, and stress-energy equations display characteristic channel constraints and identities. The resulting activity pattern provides an additional layer of structure that is largely hidden in conventional tensor formulations but becomes explicit within the BQ framework.

Taken together, these observations suggest that the channel decomposition does more than reorganize known equations. It provides a mechanism for identifying relationships between geometric, dynamical, and conservation structures that are normally distributed across different parts of a theory and therefore difficult to compare directly.

5.4 Outlook

The framework developed in this paper has been applied to electrodynamics and relativistic fluid dynamics, but the underlying hierarchy is not tied to either theory. The common algebraic structure suggests several natural directions for further investigation.

A first extension is obtained through the introduction of the canonical momentum density

$$G' = G + \rho_e A, \tag{471}$$

which combines the mechanical momentum density G with the electromagnetic potential A. The corresponding field

$$\mathcal{H}' = \partial^T G' \tag{472}$$

contains both fluid and electromagnetic contributions and therefore provides a natural starting point for a unified treatment of charged fluids. Within the BQ framework, magnetohydrodynamics may be viewed as an extension of the fluid hierarchy in which electromagnetic and hydrodynamic interactions enter through the same algebraic structure. Whether familiar MHD results such as frozen-in flux, Alfvén waves, and magnetic helicity admit a similarly transparent channel interpretation remains an interesting open question.

A second direction concerns computational applications. Because the BQ hierarchy is constructed entirely from products within the closed algebra $M(2, \mathbb{C})$, the channel decomposition procedure is intrinsically algorithmic. This suggests the possibility of embedding physical structure directly into computational models rather than enforcing it through external constraints. The proposed concept of Physics Native Neural Networks should therefore be viewed not as a result of the present paper but as a potential application of the algebraic framework developed here.

A third and more ambitious extension is the incorporation of gravitation. The analysis of this paper has been restricted to flat Minkowski spacetime, with the four-gradient ∂ acting as the fundamental differential operator. In a curved-spacetime setting, ∂ must be replaced by a generalized derivative containing information about the local gravitational structure. Within the broader BQ program this role is played by a gravitational rotor field Q_g, which encodes the relation between local inertial frames and the coordinate frame.

From this perspective, the self-coupling relation

$$G = \rho_0 U, \tag{473}$$

identified in Section 5.2 as the source of much of the structural complexity of fluid dynamics, may represent a special case of a more general principle in which the geometric structure of spacetime participates directly in the dynamical hierarchy. Whether the channel decomposition continues to provide an effective organizing framework in that setting remains an open question.

The results presented here do not answer that question. They do, however, suggest that the BQ hierarchy captures structural features that are shared by at least two distinct classical field theories. Establishing the extent of that applicability, and determining where its limits lie, constitutes the natural next stage of the program.

6 Conclusion

This paper has developed the biquaternion (BQ) algebra at the Pauli-spin level as a common algebraic framework for electrodynamics and relativistic fluid dynamics. The central result is the identification of a shared hierarchy of algebraic constructions,

$$\begin{aligned} Y &= \partial^T X, \\ AY &= F, \\ \partial Y &= F', \\ \partial(A^T X) &= F, \end{aligned} \tag{474}$$

whose channel decompositions organize a broad range of known physical equations within a single algebraic structure. The physical content of the hierarchy is determined by the choice of four-vector inputs, specifically (J, A) in electrodynamics and (U, G) in relativistic fluid dynamics.

In the electromagnetic case, the hierarchy reproduces the familiar field, force, wave, and interaction-conservation structures associated with Maxwell theory. In the fluid case, the corresponding hierarchy recovers the continuity equation, Bernoulli relation, acoustic wave dynamics, vorticity transport, Euler–Lamb momentum balance, and stress-energy conservation relations under the stated physical identifications and closures. In both domains, the channel decomposition provides a systematic separation of scalar, temporal, rotational, and transport content, making explicit structural relationships that are normally distributed across multiple formulations.

The comparison between the two hierarchies reveals both a shared organizational structure and a fundamental difference. The force equations $JB = F$ and $U\mathcal{H} = F$ emerge as the unique four-channel levels of their respective hierarchies, while the greater structural complexity of fluid dynamics can be traced to the self-coupling relation

$$G = \rho_0 U, \tag{475}$$

which links the field-generating quantity directly to the dynamical variable on which the field acts.

Several structural identifications appear to be new. These include the hierarchy-wide role of the acoustic refractive index

$$n_s = \frac{c}{c_s}, \tag{476}$$

the fluid analogue of the Bianchi identities, the confinement of helicity generation to the norm channel of the force equation, and the covariant wave equation

$$\Box G = F', \tag{477}$$

with the momentum-density field G as the primary wave-carrying variable. Whether these features prove to be fundamental or merely useful organizational observations will require further investigation, but they illustrate the type of structure that becomes visible through the BQ channel decomposition.

More generally, the results suggest that the BQ framework functions as more than an alternative notation. By placing field equations, force laws, conservation relations, and geometric identities within a common hierarchy, it provides a unified language for comparing theories that are usually developed independently. The natural next steps include extensions to magnetohydrodynamics, computational implementations based on the algebraic hierarchy, and the incorporation of gravitational structure through a generalized derivative framework.

The present work has been restricted to flat Minkowski spacetime and to two representative classical field theories. Determining how far the channel-decomposition framework extends beyond these domains remains an open question. The results presented here suggest that it is a question worth pursuing. □

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