

Empirical Relations Between Quark Mass Ratios and CKM Mixing Angles

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Abstract

We document a set of empirical relations linking the three mixing angles of the CKM matrix to quark mass ratios in the form $\sin \theta_{ij} = \sqrt{m_i/m_j}$, extending the historical Gatto-Sartori-Tonin (1968) relation. Evaluated within the space of possible combinatorial configurations ($C(15, 3) = 455$ triplets of pairs), this structure stands out with a statistical significance of $p = 0.0022$ in reproducing the experimental hierarchy to better than 5%, and displays internal consistency at 0.90σ upon cross-extraction of the up quark mass. By algebraic construction, such a texture defined on positive real mass ratios is structurally incapable of generating a complex phase. The CP-violating phase is therefore treated separately via the conjecture $|\sin \delta_{CP}| = m_W/m_Z$, a testable numerical observation providing a sharp predictive target for the DUNE experiment ($\delta_{CP} \approx 298^\circ$). We emphasize that this work does not constitute a complete dynamical theory and presents an open anomaly: the paradoxical use of family 1–2 masses in the expression of the mixing angle θ_{23} (generations 2–3) contradicts standard texture models. These results are presented as a factual mapping of numerical regularities, laying down rigorous constraints for the development of future flavor symmetry models.

1 Introduction

Understanding the origin of quark masses and their mixings remains one of the major open problems in particle physics. In the Standard Model (SM), fermion masses arise from Yukawa couplings to the Higgs field, while the Cabibbo-Kobayashi-Maskawa (CKM) matrix describes flavor transitions mediated by the charged weak interaction. These two sectors — masses and mixings — are treated as independent, and the numerical values they take are free parameters determined by experiment.

Historically, several empirical relations have been proposed to connect these two sectors. The most famous is the Gatto-Sartori-Tonin (GST) relation [1]:

$$\sin \theta_C \approx \sqrt{\frac{m_d}{m_s}} \tag{1}$$

This relation, initially motivated by SU(3) flavor symmetry considerations, was extended by Fritzsch [2, 3] within the framework of zero-texture mass matrices, predicting:

$$|V_{cb}| \approx \sqrt{\frac{m_s}{m_b}}, \quad |V_{ub}| \approx \sqrt{\frac{m_d}{m_b}} \quad (2)$$

However, modern CKM angle measurements have shown that these Fritzsch predictions are in significant disagreement with experimental data, notably for $|V_{cb}|$ (deviation of approximately 257%).

In this work, we present an empirical generalization of the GST relation to all three CKM matrix angles. The proposed formula uses quark mass ratios belonging to the same electric charge sectors, but with a different index assignment from that of Fritzsch for the θ_{23} and θ_{13} angles.

2 The Formula

The proposed relation reads:

$$\boxed{\sin \theta_{ij} = \sqrt{\frac{m_i}{m_j}} \quad (m_i < m_j)} \quad (3)$$

Applied to the three CKM matrix angles with PDG 2024 quark masses [4] (MS-bar scale at 2 GeV for light quarks, pole mass for the top), we obtain:

Angle	Formula	Predicted	PDG 2024	Deviation
$\sin \theta_{12}$	$\sqrt{m_d/m_s}$	0.2236	0.2250 ± 0.0007	0.6%
$\sin \theta_{23}$	$\sqrt{m_u/m_c}$	0.0412	0.0418 ± 0.0007	1.4%
$\sin \theta_{13}$	$\sqrt{m_u/m_t}$	0.00354	0.00369 ± 0.0001	4.1%

Table 1: Predicted vs. measured CKM angles.

The masses used are: $m_u = 2.16$ MeV, $m_d = 4.67$ MeV, $m_s = 93.4$ MeV, $m_c = 1270$ MeV, $m_b = 4180$ MeV, $m_t = 172760$ MeV.

The formula has **no continuous adjustable parameters**. Quark masses are experimental inputs, and predictions for the three angles follow directly. The functional form $\sqrt{m_i/m_j}$ is inherited from the GST relation and Fritzsch textures. The fundamental difference from Fritzsch lies in the choice of mass pairs for θ_{23} : we use $\sqrt{m_u/m_c}$ (generations 1–2 up) rather than $\sqrt{m_c/m_t}$ or $\sqrt{m_s/m_b}$. This assignment constitutes the θ_{23} **anomaly** (Section 5).

3 Statistical Test

To evaluate the statistical significance of this coincidence, we performed an exhaustive permutation test over all available mass ratios.

From the 6 quark masses, one can form $C(6, 2) = 15$ distinct pairs. For each pair, the ratio $\sqrt{m_{\min}/m_{\max}}$ is computed, yielding 15 numerical values available for assignment to the 3 CKM angles.

The number of ways to choose 3 distinct pairs among these 15 and assign them to the 3 angles (respecting the observed hierarchy $\theta_{12} > \theta_{23} > \theta_{13}$) is given by the binomial coefficient $C(15, 3) = 455$.

For each configuration, the total error is computed as:

$$E = \sum_{k=1}^3 \frac{|\sin \theta_k^{\text{pred}} - \sin \theta_k^{\text{PDG}}|}{\sin \theta_k^{\text{PDG}}} \quad (4)$$

Rank	θ_{12} pair	θ_{23} pair	Total Error
1 (Lichani)	d/s (0.2236)	u/c (0.0412)	6.14%
2	d/s (0.2236)	d/b (0.0334)	24.8%
3	c/s (0.2712)	u/c (0.0412)	26.0%

Table 2: Top 3 configurations among 455 tested.

The proposed configuration ranks **first** and is the **only** one within this error interval. The second-best has an error approximately 4 times larger.

$$p = \frac{1}{455} \approx 0.0022 \quad (5)$$

This result has been verified with two different metrics (relative deviations and χ^2 weighted by experimental uncertainties), both yielding the same ranking. A Monte Carlo test (10,000 samples) varying quark masses within their PDG error bars confirms that the proposed configuration remains the best in all cases.

4 Robustness and Internal Consistency

4.1 Uncertainty Propagation

A Monte Carlo calculation ($N = 100,000$) propagating experimental uncertainties on light quark masses yields the prediction intervals shown in Table 3.

Angle	Prediction (MC)	PDG 2024	Tension
$\sin \theta_{12}$	0.2238 ± 0.0107	0.2250 ± 0.0007	0.11σ
$\sin \theta_{23}$	0.0411 ± 0.0037	0.0418 ± 0.0007	0.19σ
$\sin \theta_{13}$	0.00352 ± 0.00031	0.00369 ± 0.0001	0.51σ

Table 3: Monte Carlo uncertainty propagation.

All three predictions are compatible with measurements to better than 1σ . However, the prediction uncertainty is dominant: for θ_{12} , it is 15 times wider than the experimental uncertainty. The formula is therefore not falsifiable by current CKM measurements — it constitutes a consistency check rather than a discriminating prediction.

4.2 Cross-Extraction of m_u

The formula allows extraction of the up quark mass in two independent ways:

$$m_u = m_c \times \sin^2 \theta_{23} = 2.22 \pm 0.07 \text{ MeV} \quad (6)$$

$$m_u = m_t \times \sin^2 \theta_{13} = 2.35 \pm 0.13 \text{ MeV} \quad (7)$$

These two estimates are consistent with each other at 0.90σ , and compatible with the PDG value (2.16 ± 0.49 MeV). This internal consistency was not imposed by construction.

4.3 Scale Invariance

Mass ratios of quarks with the same electric charge are quasi-invariant under QCD renormalization group evolution at leading order, since the running factor is identical for both quarks. The formula $\sin^2 \theta = m_i/m_j$ is therefore stable under scale evolution.

5 The θ_{23} Anomaly

The angle θ_{23} physically corresponds to $|V_{cb}|$, describing mixing between the second generation (charm) and the third generation (bottom). The generationally natural formulas would be:

$$\sin \theta_{23} = \sqrt{\frac{m_c}{m_t}} \approx 0.0858 \quad (105\% \text{ deviation}) \quad (8)$$

$$\sin \theta_{23} = \sqrt{\frac{m_s}{m_b}} \approx 0.1495 \quad (257\% \text{ deviation}) \quad (9)$$

Our formula uses $\sqrt{m_u/m_c} = 0.0412$, i.e., first and second generation up masses to describe second–third generation mixing. This “generation shift” constitutes the major structural anomaly of this work.

Ten explanatory hypotheses have been tested without success:

1. Relative up/down hierarchy — contradicted by data
2. Closest-to-unity ratio criterion — contradicted
3. Cascade effect $\theta_{12} \times \theta_{13}$ — failure (103% error)
4. Global spectral ordering — u and c not adjacent
5. Alternative CKM convention — does not resolve the discrepancy
6. QCD renormalization running — insufficient (105% \rightarrow 43%)
7. Bi-sectorial formula — failure (1218% deviation)
8. Minimization of $S = \sum(m_i/m_j)$ — circular reformulation
9. Minimization of $||[M_u, M_d]||$ — not basis-invariant
10. Composite algebraic candidates — none approach 1.4%

This anomaly remains unexplained to date. We document it as an open question.

6 The φ Parametrization

Defining the dimensionless variable:

$$\boxed{\varphi = \frac{\sqrt{m}}{k} \quad \text{with} \quad k = \frac{\sqrt{m_Z}}{64} = 0.1492 \text{ GeV}^{1/2}} \quad (10)$$

one observes that the masses of the gauge bosons, the Higgs, and the top quark project onto values remarkably close to integers:

Particle	Mass (MeV)	φ	Integer	Deviation
f_π	92.1	2.04	2	+0.04
W	80,377	60.09	60	+0.09
Z	91,188	64.00	64	0 (def.)
H	125,200	74.96	75	-0.04
t	172,760	88.07	88	+0.07

Table 4: φ values for the heavy sector.

Two arithmetic constraints emerge from these integers:

$$\boxed{\Sigma\varphi = 2 + 60 + 64 + 75 + 88 = 289 = 17^2} \quad (11)$$

$$\boxed{\varphi_Z^2 - \varphi_W^2 = 64^2 - 60^2 = 496} \quad (12)$$

The number 496 is the third perfect number and the dimension of the $\text{SO}(32)$ and $E_8 \times E_8$ gauge groups in heterotic string theory. The number 289 is 17^2 , a Fermat square. The relation $(\Sigma e_i + r)/r = (128 + 8)/8 = 17$ connects E_8 exponents to $\sqrt{\Sigma\varphi}$, though this remains a numerical observation rather than a demonstrated mechanism.

Extension to lighter generations shows that the strange quark ($\varphi_s \approx 2.05$) and tau lepton ($\varphi_\tau \approx 8.95$) are also close to integers (2 and 9), while the u , d quarks and the electron have fractional φ (< 1). The integer structure appears to terminate at the QCD confinement threshold ($\varphi < 2$).

7 Conjecture for the δ_{CP} Phase

The CP-violating phase δ_{CP} is structurally absent from the proposed texture. A mixing matrix constructed from positive real mass ratios belongs to the orthogonal group $\text{SO}(3)$ (up to rephasing), for which the Jarlskog invariant is identically zero: $J_{CP} = 0$. CP violation cannot therefore be generated by the main texture.

We propose a separate conjecture for the phase:

$$\boxed{|\sin \delta_{CP}| = \frac{m_W}{m_Z}} \quad (13)$$

With PDG 2024 masses, this yields:

$$\delta_{CP} = 360^\circ - \arcsin\left(\frac{80,377}{91,188}\right) = 298.18^\circ \quad (\text{Quadrant IV}) \quad (14)$$

This value is compatible with current experimental constraints (T2K, NOvA) which suggest a phase in Quadrant IV. The DUNE experiment [5], with first results expected for 2028–2030, should measure δ_{CP} with a precision of $\pm 10\text{--}15^\circ$, providing a definitive test of this conjecture. The experimental uncertainty on the prediction is $\pm 0.02^\circ$ (dominated by m_W), but the theoretical uncertainty (radiative corrections, validity of the conjecture) is not calculated.

8 Underlying Algebraic Structure

Although this work does not propose a complete dynamical mechanism, several algebraic properties emerge from the observed structure:

Weinberg blocks. The $\sqrt{m_i/m_j}$ form for mixing angles is the signature of 2×2 Weinberg-type submatrices (zeros on the diagonal, non-zero off-diagonal terms). For such a matrix, eigenvalues are approximately $m_1 \approx A^2/m_2$, and the rotation angle satisfies $\tan \theta \approx A/m_2$, hence $\sin \theta \approx \sqrt{m_1/m_2}$.

Collinearity constraint in M_{up} . The θ_{23} anomaly implies that the coupling element between the second and third generation in the up mass matrix is proportional to the coupling element between the first and second: $M_{up}(2,3) \propto M_{up}(1,2) \propto \sqrt{m_u}$. This collinearity constraint distinguishes this structure from standard Fritzsch textures.

LO/NLO separation. The positive real texture constrains the CKM matrix to $SO(3)$ at leading order (LO), imposing $J_{CP} = 0$. CP violation is necessarily a higher-order effect (NLO). A minimal perturbative framework would consist of an imaginary antisymmetric perturbation $i\varepsilon Y^{(1)}$ localized on the 2–3 transition block, generating δ_{CP} at $\mathcal{O}(\varepsilon)$ while correcting masses and angles only at $\mathcal{O}(\varepsilon^2)$, thereby preserving the sub-5% agreement.

9 Limitations

This work presents several limitations that we wish to state explicitly:

1. **Absence of a fundamental mechanism.** The formula is an empirical regularity, not derived from a principle or symmetry. It is analogous to the Balmer formula (1885) for hydrogen spectral lines — a precise description without an underlying explanation.
2. **Unresolved θ_{23} anomaly.** The generation shift in the θ_{23} expression remains unexplained after the failure of ten explanatory hypotheses.
3. **Non-independent test.** The formula was discovered and tested on the same CKM data. The only truly independent prediction is δ_{CP} , whose verification will not occur before 2028–2030.
4. **Dominant prediction uncertainty.** For θ_{12} and θ_{23} , the prediction uncertainty is 3 to 15 times wider than the experimental uncertainty, due to the limited precision on light quark masses (notably m_u at $\pm 23\%$).
5. **Non-universality.** The formula does not apply to the charged lepton sector, and its extension to neutrinos would require an additional mechanism (such as the seesaw).
6. **Underived δ_{CP} conjecture.** The choice of the m_W/m_Z ratio among the various possible boson mass ratios is not theoretically justified.

10 Conclusion

We have documented a statistically significant empirical regularity ($p = 0.0022$) linking the three CKM matrix mixing angles to quark mass ratios in the form $\sin \theta_{ij} = \sqrt{m_i/m_j}$. This formula, extending the historical Gatto-Sartori-Tonin (1968) relation, reproduces the CKM angles with deviations of 0.6%, 1.4%, and 4.1% without adjustable parameters.

The $\varphi = \sqrt{m}/k$ parametrization reveals an underlying integer structure in the heavy sector (W, Z, H, t), with two exact arithmetic constraints: $\Sigma\varphi = 289 = 17^2$ and $\varphi_Z^2 - \varphi_W^2 = 496$. The conjecture $|\sin \delta_{CP}| = m_W/m_Z$ ($\delta_{CP} \approx 298^\circ$) provides a falsifiable target for the DUNE experiment.

The θ_{23} anomaly — the use of generation 1–2 masses to describe generation 2–3 mixing — resists ten explanatory hypotheses and constitutes the major challenge for theoretical interpretation.

These results are presented not as a complete theory, but as a factual mapping of numerical regularities, laying down rigorous constraints for the development of future flavor symmetry models.

References

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