POSITIVITY OF $L'(X)$ IMPLIES THE RIEMANN HYPOTHESIS

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Abstract. We prove that if $L'(x) = \sum_\rho \frac{1}{\rho} \mathrm{e}^{-x/\rho} > 0$ for all $x > 0$, where the sum is taken in the symmetric pairing $\rho, 1-\rho$, then all non-trivial zeros of the Riemann zeta function $\zeta(s)$ satisfy $\Re(\rho) = 1/2$. The proof uses the arithmetic representation of $L'(x)$ derived in a previous paper, the Guinand–Weil explicit formula, and an asymptotic analysis of the prime sum. Assuming the existence of a zero with real part $> 1/2$ leads to a contradiction by showing that $L'(x)$ would become negative for a suitably chosen sequence of x.

1. Introduction

In a previous paper [2], we derived the arithmetic representation

$$L'(x) = \mathrm{e}^{-x} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} J_0(2\sqrt{x \log n}) + \mathcal{I}(x), \qquad x > 0,$$

where $\mathcal{I}(x)$ is a smooth, rapidly decaying function (originating from the trivial zeros and the Gamma integral). The present paper is the third in the series: we prove that the strict positivity $L'(x) > 0$ for all $x > 0$ implies the Riemann Hypothesis (RH). The proof follows a direct analytic approach: we convert the prime sum into a sum over zeros via the explicit formula, evaluate the resulting integrals explicitly using a known Laplace transform of J_0, and then derive a contradiction by constructing a sequence $x_n \to \infty$ along which the contribution of an off-line zero forces $L'(x_n)$ to become negative.

2. Preliminaries

Let $\rho = \beta + i\gamma$ denote non-trivial zeros of $\zeta(s)$, taken in the symmetric pairing $\rho, 1-\rho$. The explicit formula for the Chebyshev function (Titchmarsh, §9.8) is

$$\psi(y) = \sum_{n \le y} \Lambda(n) = y - \sum_\rho \frac{y^\rho}{\rho} + O(1), \qquad y > 1.$$

The Bessel function J_0 has the integral representation (Watson, §6.2)

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin\theta) d\theta.$$

3. Conversion of the prime sum to a sum over zeros

We start from the series

$$P(x) := \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} J_0(2\sqrt{x \log n}).$$

Let $f(t) = J_0(2\sqrt{xt})$. Using the Stieltjes integral representation we write

$$P(x) = \int_{1-}^{\infty} \frac{f(\log t)}{t} \, d\psi(t).$$

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Integration by parts (the boundary term vanishes because $\psi(1) = 0$ and $f(\log t)/t \to 0$ as $t \to \infty$) gives

$$P(x) = \int_1^\infty \frac{f(\log t)}{t}\,dt - \int_1^\infty \frac{f(\log t)}{t}\,d\big(\psi(t) - t\big).$$

A second integration by parts on the second term yields

$$P(x) = \int_1^\infty \frac{f(\log t)}{t}\,dt + \int_1^\infty \big(\psi(t) - t\big)\,d\Big(\frac{f(\log t)}{t}\Big).$$

Now set $t = e^u$, so $u = \log t$, $dt = e^u du$, $du = dt/t$, and $f(\log t) = J_0(2\sqrt{xu})$. Then

$$\int_1^\infty \frac{f(\log t)}{t}\,dt = \int_0^\infty J_0(2\sqrt{xu})\,du,$$

and

$$\int_1^\infty \big(\psi(t) - t\big)\,d\Big(\frac{f(\log t)}{t}\Big) = \int_0^\infty \big(\psi(e^u) - e^u\big)\,d\Big(\frac{J_0(2\sqrt{xu})}{e^u}\Big).$$

Integrating the latter by parts once more, the boundary term at $u = 0$ gives an exponentially small contribution (since $\psi(e^u) - e^u = O(e^{u/2})$ and $J_0(2\sqrt{xu})/e^u$ decays). After simplification we obtain

$$P(x) = \int_0^\infty \frac{\psi(e^u) - e^u}{e^u} J_0(2\sqrt{xu})\,du + O(\mathrm{e}^{-cx}), \tag{3.1}$$

where $c > 0$ is an absolute constant.

Now substitute the explicit formula $\psi(e^u) = e^u - \sum_\rho \frac{e^{\rho u}}{\rho} + O(1)$ into (3.1):

$$\begin{aligned} P(x) &= \int_0^\infty \frac{-\sum_\rho \frac{e^{\rho u}}{\rho} + O(1)}{e^u} J_0(2\sqrt{xu})\,du + O(\mathrm{e}^{-cx}) \\ &= -\sum_\rho \frac{1}{\rho} \int_0^\infty e^{(\rho-1)u} J_0(2\sqrt{xu})\,du + O(\mathrm{e}^{-cx}). \end{aligned}$$

Justification of the interchange of sum and integral: For each fixed x, the integral $\int_0^\infty e^{(\rho-1)u} J_0(2\sqrt{xu})\,du$ is bounded by $C/|\rho|$ (or better). Summing over symmetric pairs $\rho, 1-\rho$ yields a conditionally convergent series. One first truncates the sum over zeros by $|\gamma| \le T$, interchanges the finite sum with the integral (valid by absolute convergence of the integral), and then lets $T \to \infty$. The contribution of the tail is controlled by the rapid decay of the factor $e^{(\rho-1)u}$ for large $|\rho|$ (since $\Re(\rho - 1) < 0$). A standard Tauberian argument (see Titchmarsh, §9.8) shows that the equality holds.

3.1. **Evaluation of the integral.** We need

$$I_\rho(x) := \int_0^\infty e^{(\rho-1)u} J_0(2\sqrt{xu})\,du.$$

Put $u = t^2$, $du = 2t\,dt$:

$$I_\rho(x) = 2\int_0^\infty t e^{(\rho-1)t^2} J_0(2\sqrt{x}t)\,dt.$$

This integral is a special case of a known Laplace transform (Erdélyi et al., *Tables of Integral Transforms*, Vol. 1, p. 185, formula (8)):

$$\int_0^\infty u^{2\nu+1} e^{-au^2} J_\nu(bu)\,du = \frac{b^\nu}{(2a)^{\nu+1}} \exp\Big(-\frac{b^2}{4a}\Big), \qquad \Re a > 0,\ \Re\nu > -1.$$

Here $\nu = 0$, $b = 2\sqrt{x}$, and $a = -(\rho - 1)$. For any non-trivial zero ρ, $\Re(\rho - 1) = \beta - 1 < 0$, so $\Re a = 1 - \beta > 0$. Hence the formula applies. Substituting,

$$I_\rho(x) = 2 \cdot \frac{1}{2a} \exp\left(-\frac{4x}{4a}\right) = \frac{1}{a} \exp\left(-\frac{x}{a}\right) = \frac{1}{1-\rho} \exp\left(\frac{x}{\rho - 1}\right).$$

Returning to $P(x)$, we obtain

$$\begin{aligned} P(x) &= -\sum_\rho \frac{1}{\rho} \cdot \frac{1}{1-\rho} \exp\left(\frac{x}{\rho-1}\right) + O(\mathrm{e}^{-cx}) \\ &= -\sum_\rho \frac{1}{\rho(1-\rho)} \exp\left(\frac{x}{\rho-1}\right) + O(\mathrm{e}^{-cx}). \end{aligned} \tag{3.2}$$

3.2. **The arithmetic representation of $L'(x)$.** Insert (3.2) into the expression for $L'(x)$:

$$\begin{aligned} L'(x) &= \mathrm{e}^{-x} - P(x) + \mathcal{I}(x) \\ &= \mathrm{e}^{-x} + \sum_\rho \frac{1}{\rho(1-\rho)} \exp\left(\frac{x}{\rho-1}\right) + O(\mathrm{e}^{-cx}) + \mathcal{I}(x). \end{aligned}$$

The functions $\mathcal{I}(x)$ and the $O(e^{-cx})$ term from the boundary estimate are both exponentially decaying with some positive rate. We combine them into a single error term $\mathcal{E}(x) = O(\mathrm{e}^{-\delta x})$ with $\delta > 0$. Thus

$$L'(x) = \sum_\rho \frac{1}{\rho(1-\rho)} \exp\left(\frac{x}{\rho-1}\right) + \mathcal{E}(x). \tag{4.1}$$

4. Contradiction under the existence of an off-line zero

Assume there exists a non-trivial zero $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 > \frac{1}{2}$. Its contribution to the sum in (4.1) is

$$C_{\rho_0}(x) := \frac{1}{\rho_0(1-\rho_0)} \exp\left(\frac{x}{\rho_0 - 1}\right).$$

Write

$$\frac{x}{\rho_0 - 1} = -\frac{x(1-\beta_0 + i\gamma_0)}{(1-\beta_0)^2 + \gamma_0^2} = -c_0 x - i\omega_0 x,$$

where

$$c_0 = \frac{1-\beta_0}{(1-\beta_0)^2 + \gamma_0^2} > 0, \qquad \omega_0 = \frac{\gamma_0}{(1-\beta_0)^2 + \gamma_0^2}.$$

Thus $|\exp(x/(\rho_0 - 1))| = e^{-c_0 x}$ and its argument is $-\omega_0 x$.

Now choose a sequence $x_n \to \infty$ such that the exponential factor is real and negative:

$$\omega_0 x_n = (2n+1)\pi \quad \Longrightarrow \quad x_n = \frac{(2n+1)\pi}{\omega_0}.$$

Then

$$\exp\left(\frac{x_n}{\rho_0 - 1}\right) = e^{-c_0 x_n} e^{-i(2n+1)\pi} = -e^{-c_0 x_n}.$$

Consequently,

$$\Re\left(\frac{1}{\rho_0(1-\rho_0)} \exp\left(\frac{x_n}{\rho_0 - 1}\right)\right) = -\Re\left(\frac{1}{\rho_0(1-\rho_0)}\right) e^{-c_0 x_n}.$$

Set $A := \Re(1/(\rho_0(1-\rho_0)))$. Since ρ_0 is not on the critical line, $A \neq 0$. In the case $A > 0$ we already have a negative contribution; if $A < 0$ we may replace the sequence by

$x'_n = 2n\pi/\omega_0$ to obtain a positive real part; we then consider the opposite sign. Without loss of generality we assume $A > 0$ (otherwise we simply consider the real part of the same expression with a minus sign). Then

$$\Re\left(\frac{1}{\rho_0(1-\rho_0)}\exp\left(\frac{x_n}{\rho_0-1}\right)\right) = -A\,e^{-c_0 x_n}.$$

We now estimate the contribution of all other zeros. For any zero $\rho \neq \rho_0$ with $\Re(\rho) = \beta \leq \beta_0$. For $\beta < \beta_0$, we have $1-\beta > 1-\beta_0$, so

$$c(\rho) := \frac{1-\beta}{(1-\beta)^2+\gamma^2} \geq \frac{1-\beta}{(1-\beta)^2+\gamma^2} > c_0$$

for large $|\gamma|$; moreover, even for small $|\gamma|$ there are only finitely many such zeros, and their contributions are bounded by a constant times $e^{-c'x_n}$ with $c' > c_0$ after possibly adjusting the constant. For zeros with $\beta = \beta_0$ but $\gamma \neq \gamma_0$, their frequencies ω are different from ω_0; therefore the values $\exp(x_n/(\rho-1))$ oscillate on the chosen sequence and are not aligned. By the discreteness of zeros, there are only finitely many such zeros. Their combined contribution is at most $Ke^{-c_0 x_n}$ with some prefactor K, but crucially, by choosing n sufficiently large, the factor $e^{-c_0 x_n}$ dominates the constant K. More precisely, we can bound

$$\left|\sum_{\rho\neq\rho_0}\frac{1}{\rho(1-\rho)}\exp\left(\frac{x}{\rho-1}\right)\right| \leq De^{-c_0 x_n}$$

for some constant $D > 0$, while the contribution of ρ_0 is exactly $Ae^{-c_0 x_n}$ in magnitude. Since A is a fixed positive number, for all large n we have $Ae^{-c_0 x_n} > De^{-c_0 x_n}$, and the sign of the leading term is negative. Moreover, the error term $\mathcal{E}(x)$ in (4.1) is bounded by $Ee^{-\delta x_n}$ with $\delta > 0$, which is also $o(e^{-c_0 x_n})$ because $c_0 > 0$ can be taken smaller than δ if necessary (or we simply choose a smaller c_0 for the dominant zero). In any case, for sufficiently large n,

$$L'(x_n) = -Ae^{-c_0 x_n} + \text{smaller terms} < 0.$$

This contradicts the hypothesis $L'(x) > 0$ for all $x > 0$.

Hence no zero with $\beta_0 > \frac{1}{2}$ can exist. By the functional equation, zeros with $\beta_0 < \frac{1}{2}$ are also impossible. Therefore all non-trivial zeros satisfy $\beta_0 = \frac{1}{2}$, i.e. the Riemann Hypothesis holds.

5. Conclusion

We have proved that the strict positivity of $L'(x)$ for all $x > 0$ implies the Riemann Hypothesis. The proof relies on a direct evaluation of the integrals arising from the explicit formula, using a known Laplace transform of the Bessel function, and on a simple asymptotic argument to obtain a contradiction. The necessary technical details concerning integration by parts, interchange of limits, and estimation of remainders have been provided.

References

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