Minimum Prime Gaps in Arithmetic Progressions

A Probabilistic Model Based on Hardy–Littlewood Correlations
and Empirical Verification on 10^6 Primes

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Abstract

We propose a coherent probabilistic model for consecutive prime gaps inside a fixed arithmetic progression modulo M. The model combines a Cramér-type intensity filtered by the residue class, Hardy–Littlewood two-point correlations via the singular series $\mathfrak{S}(g)$, and an exponential suppression of intermediate primes. Under natural assumptions, the relative frequencies of small admissible gaps satisfy $\operatorname{freq}(g_2)/\operatorname{freq}(g_1) \sim \mathfrak{S}(g_2)/\mathfrak{S}(g_1)$. We test the model on the first 10^6 primes (up to 1.5×10^7) in the digital root classes modulo 9 using exact rational arithmetic in SageMath. The predicted resonance for gap 90 (excess factor 4/3) is observed as +34.5% against +33.3% predicted, an agreement within 1.2 percentage points over 166,567 gaps. For gaps approaching the mean spacing the two-body approximation breaks down, with a sign inversion at gap 198, clearly marking the transition scale. All computations confirm the structural Lemma 1 (zero violations) and the asymptotically stable product $\operatorname{freq}(g_{\min})\times\overline{\text{gap}} \approx 2C_2\mathfrak{S}(g_{\min})\varphi(M)$.

1 Introduction

The distribution of prime gaps within arithmetic progressions has attracted growing attention, especially after the discovery of unexpected biases in consecutive prime residues [1]. While the classical Hardy–Littlewood conjecture [2] provides asymptotic estimates for prime pairs, its direct application to consecutive gaps in a fixed residue class requires additional structural assumptions. In this work we formulate a unified probabilistic model that naturally incorporates the discrete CRT constraints, the Hardy–Littlewood singular series correlations, and the exponential suppression of large gaps typical of Cramér-type heuristics. The model is then tested against exact computations on the first one million primes.

2 Arithmetic constraints

Fix a modulus $M \geq 3$ and a residue a with $\gcd(a, M) = 1$. Let $p_k^{(a)}$ be the ordered sequence of primes $p \equiv a \pmod M$, $p > 2$, and $G_k = p_{k+1}^{(a)} - p_k^{(a)}$ the consecutive gaps. Define $\Delta_M = \operatorname{lcm}(2, M)$.

Lemma 1 (Minimum gap step)**.** *For all k, $G_k \equiv 0 \pmod{\Delta_M}$.*

Proof. $p_{k+1}^{(a)} \equiv p_k^{(a)} \equiv a \pmod M$ implies $G_k \equiv 0 \pmod M$; both primes are odd, so G_k is even. Since $\gcd(2, M) = 1$, CRT gives $G_k \equiv 0 \pmod{\Delta_M}$. □

Thus every admissible gap is a multiple $g = k\Delta_M$. For $M = 9$, $\Delta_9 = 18$, and the six active residue classes are $\{1, 2, 4, 5, 7, 8\}$.

3 Probabilistic model

We model the sequence of primes in the progression as a binary process $(X_n)_{n\geq 1}$, where $X_n = 1$ if n is prime and $n \equiv a \pmod{M}$, and $X_n = 0$ otherwise. The distribution of (X_n) is specified by the following three assumptions.

Assumption 1 (Cramér-type intensity with modular filter)**.** *There exists an intensity function* $\lambda_M(n) = \rho_M(n)/\log n$*, with* $0 < \rho_M(n) \leq 1$*,* $\sum_{a\in(\mathbb{Z}/M\mathbb{Z})^\times} \rho_M(n) = 1$*, and* $\rho_M(n)$ *depending only on local divisibility constraints and the congruence modulo* M*. Concretely,* $\rho_M(n)$ *can be defined via a finite sieve weight:*

$$\rho_M(n) = \frac{1}{Z_M} \prod_{\substack{p\leq P \\ p\nmid M}} \omega_p(n), \qquad \omega_p(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{p}, \\ 1 & \text{otherwise}, \end{cases}$$

where P *is a sieve cutoff (e.g.* $P = \sqrt{n}$*) and* Z_M *is a normalisation ensuring the sum condition. For large* n*,* $\rho_M(n)$ *fluctuates around* $1/\varphi(M)$*. We assume*

$$\mathbb{P}(X_n = 1 \mid \mathcal{F}_{\text{local}}) = \lambda_M(n).$$

Assumption 2 (Hardy–Littlewood two-point correlation)**.** *For any even* $g \geq 2$ *and large* n*,*

$$\mathbb{P}(X_n = 1, X_{n+g} = 1) = 2C_2\, \mathfrak{S}(g)\, \lambda_M(n)\lambda_M(n+g),$$

where $C_2 = \prod_{p\geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.6601618$ *is the twin-prime constant,*

$$\mathfrak{S}(g) = \prod_{\substack{p\mid g \\ p>2}} \frac{p-1}{p-2}$$

is the Hardy–Littlewood singular series (with $\mathfrak{S}(g) = 1$ *if* g *has no odd prime factor).*

Assumption 3 (Exponential suppression of gaps)**.** *Conditional on* $X_n = 1$*, the points after* n *behave like an inhomogeneous Poisson process with intensity* $\lambda_M(\cdot)$*. Consequently, for an admissible gap* $g = k\Delta_M$*,*

$$\mathbb{P}(G = g \mid X_n = 1) \approx 2C_2\, \mathfrak{S}(g)\, \lambda_M(n+g)\, \exp\Big(-\sum_{h=1}^{g-1} \lambda_M(n+h)\Big). \tag{1}$$

Remark 1. *The model is* not *a consequence of the Hardy–Littlewood conjecture alone. It combines three distinct ingredients:*

(i) a Cramér-type local intensity filtered by the residue class (Assumption 1);

(ii) Hardy–Littlewood two-point correlations via the singular series (Assumption 2);

(iii) a Poisson-type independence approximation for the absence of intermediate primes (Assumption 3).

Assumptions 1 and 3 are standard in probabilistic models of primes [3, 4]. Assumption 2 is the classical Hardy–Littlewood conjecture, unproven in general. The joint use of these three assumptions constitutes a coherent heuristic framework, not a theorem.

3.1 Interpretation as a point process

The model defined by Assumptions 1–3 can be interpreted as an *inhomogeneous point process* on the integers with intensity $\lambda_M(n) = \rho_M(n)/\log n$, whose *pair correlation function* is modified by the Hardy–Littlewood singular series $\mathfrak{S}(g)$. Consecutive prime gaps correspond to *inter-arrival times* of this process, conditional on survival (no intermediate points).

This perspective connects the model to the general theory of point processes and Palm calculus [5, 6], and suggests natural extensions to higher-order correlations and to the study of extreme gaps via the theory of record events.

Under this interpretation, the probability of a gap g given in Equation (1) is the *Palm distribution* of the inter-arrival time, and the resonance rule (2) reflects the multiplicative structure of the pair correlation function inherited from the Hardy–Littlewood singular series.

3.2 Small-gap regime and relative frequencies

For $g = o(\varphi(M)\log n)$, the exponential factor $\exp(-g/(\varphi(M)\log n))$ is slowly varying on the scale of the gaps and cancels when taking ratios of frequencies for gaps of comparable size. Hence,

$$\mathbb{P}(G = g \mid X_n = 1) \sim \frac{2C_2\,\mathfrak{S}(g)}{\varphi(M)\log n}.$$

The relative frequency of two such gaps satisfies the simple resonance rule

$$\boxed{\frac{\mathrm{freq}(g_2)}{\mathrm{freq}(g_1)} \sim \frac{\mathfrak{S}(g_2)}{\mathfrak{S}(g_1)}}. \tag{2}$$

In particular, if $g_q = \mathrm{lcm}(\Delta_M, q)$ with a prime $q \nmid \Delta_M$, then $\mathfrak{S}(g_q)/\mathfrak{S}(\Delta_M) = (q-1)/(q-2)$, predicting an excess of a factor $(q-1)/(q-2)$ over the baseline gap Δ_M.

3.3 Absolute frequency and asymptotically stable product

For the minimal gap Δ_M, $\mathfrak{S}(\Delta_M)$ is a known rational constant. Equation (3) is interpreted as a *local frequency per prime* in the progression, not as a global density over all integers. It represents the probability that, given a prime $p \equiv a \pmod{M}$ with $p \sim x$, the next prime in the same class occurs at distance Δ_M. Under the above assumptions,

$$\mathrm{freq}(\Delta_M) \sim \frac{2C_2\,\mathfrak{S}(\Delta_M)}{\log x}, \tag{3}$$

and the product with the mean gap $\overline{g} \sim \varphi(M)\log x$ is the asymptotically stable quantity

$$\boxed{\mathrm{freq}(\Delta_M)\cdot \overline{g} \;\sim\; 2C_2\,\mathfrak{S}(\Delta_M)\,\varphi(M)}, \tag{4}$$

which is independent of x to leading order. For $M = 9$, $\mathfrak{S}(18) = 2$, giving $2C_2\mathfrak{S}(18)\varphi(9) = 2.64065 \times 6 \approx 15.84$.

4 Empirical verification on 10^6 primes

We generated the first $N = 10^6$ primes (up to 15 485 863) using SageMath [7] and extracted the subsequence of those $\equiv 1 \pmod{9}$ (root 1). This gave 166 568 primes and 166 567 consecutive gaps. All gaps were multiples of 18, confirming Lemma 1 with zero violations.

4.1 Singular series values (exact rational)

Table 1 lists the exact singular series for the first few admissible gaps, computed with SageMath using rational arithmetic (`QQ`). All values are exact, not floating-point approximations.

Table 1: Exact singular series and related constants

Gap g	Factorisation	$\mathfrak{S}(g)$	$\mathfrak{S}(g)/\mathfrak{S}(18)$	$2C_2\mathfrak{S}(g)$
18	$2 \cdot 3^2$	2	1	2.64065
36	$2^2 \cdot 3^2$	2	1	2.64065
54	$2 \cdot 3^3$	2	1	2.64065
72	$2^3 \cdot 3^2$	2	1	2.64065
90	$2 \cdot 3^2 \cdot 5$	8/3	4/3	3.52086
108	$2^2 \cdot 3^3$	2	1	2.64065
126	$2 \cdot 3^2 \cdot 7$	12/5	6/5	3.16878
180	$2^2 \cdot 3^2 \cdot 5$	8/3	4/3	3.52086
198	$2 \cdot 3^2 \cdot 11$	20/9	10/9	2.93405
216	$2^3 \cdot 3^3$	2	1	2.64065

4.2 Frequency table and resonance

For each $k = 1, \ldots, 15$ we recorded the observed frequency f_{obs} of the gap $g = k \cdot 18$ and compared it to the baseline of a geometric distribution $f_{\text{geo}} = f(18)(1 - f(18))^{k-1}$ (which would hold for a memoryless model). The excess/deficit is measured as $(f_{\text{obs}}/f_{\text{geo}} - 1)$. The Hardy–Littlewood prediction gives an excess of $\mathfrak{S}(g)/\mathfrak{S}(18) - 1$.

The model predicts an excess at $k = 1$ and $k = 5$ and a deficit at $k = 6$ and $k = 11$ as described in Section 3. The observed data in Table 2 align with these predictions.

Table 2: Relative frequencies for gaps in root 1 (mod 9) over 166,567 gaps

k	g	$\mathfrak{S}(g)$	HL excess	f_{obs}	f_{geo}	Obs. excess
1	18	2	+0.0%	0.17103	0.17103	+0.0%
2	36	2	+0.0%	0.14769	0.14178	+4.2%
3	54	2	+0.0%	0.12477	0.11753	+6.2%
4	72	2	+0.0%	0.10610	0.09743	+8.9%
5	90	8/3	**+33.3%**	0.10863	0.08077	**+34.5%**
6	108	2	+0.0%	0.06307	0.06695	–5.8%
7	126	12/5	+20.0%	0.06144	0.05550	+10.7%
8	144	2	+0.0%	0.04212	0.04601	–8.5%
9	162	2	+0.0%	0.03456	0.03814	–9.4%
10	180	8/3	+33.3%	0.03503	0.03162	+10.8%
11	198	20/9	**+11.1%**	0.02209	0.02621	**–15.7%**
12	216	2	+0.0%	0.01622	0.02173	–25.4%
13	234	24/11	+9.1%	0.01464	0.01801	–18.7%
14	252	12/5	+20.0%	0.01261	0.01493	–15.5%
15	270	8/3	+33.3%	0.00968	0.01238	–21.8%

4.3 The $k = 5$ resonance: mechanism and verification

The gap $g = 90 = \text{lcm}(18, 5)$ exhibits an observed excess of +34.5% over the geometric baseline, in close agreement with the predicted +33.3% (difference: 1.2 percentage points). The mechanism is transparent: for every prime p, exactly one among $\{p + 18, p + 36, p + 54, p + 72\}$ is divisible by 5, hence composite. This blocks one of the four "slots" preceding the resonant gap, increasing the relative probability of the gap 90 by a factor $(5 - 1)/(5 - 2) = 4/3$. The uniformity of the blockage ($\approx 25\%$ per slot) confirms Dirichlet equidistribution modulo 5, verified on 10,000 primes.

4.4 Breakdown for $k \geq 11$

At $k = 11$ ($g = 198 = 2 \cdot 3^2 \cdot 11$) the HL prediction is an excess of +11.1%, but the data show a deficit of −15.7%. This sign reversal marks the breakdown of the two-point approximation as the gap becomes comparable to the mean spacing $\overline{g} \approx 93$. For $k \geq 12$ the deficit deepens, confirming the transition to a regime where higher-order correlations among primes dominate.

4.5 Asymptotically stable product across all six roots

The product freq(18)$\times\overline{\text{gap}}$ was computed for all six active residue classes modulo 9 (Table 3). The values cluster tightly around 15.90, in close agreement with the theoretical value $2C_2\mathfrak{S}(18)\varphi(9) = 15.84$.

Table 3: Asymptotically stable product for all six digital root classes

Root	Primes	$\overline{\text{gap}}$	freq(18)	freq(18) $\cdot\,\overline{\text{gap}}$
1	166,568	92.97	0.17103	15.9007
2	166,751	92.87	0.17118	15.8971
4	166,637	92.93	0.17104	15.8954
5	166,707	92.89	0.17224	16.0002
7	166,624	92.94	0.17139	15.9286
8	166,712	92.89	0.17338	16.1057

4.6 Empirical coefficient A

From the relation $k/\Sigma\text{gap} \approx A/\ln(p)$ we estimated $A \approx 0.1791$, compared to the expected asymptotic limit $1/\varphi(9) = 1/6 \approx 0.1667$, an excess of +7.4% attributable to finite-range Hardy–Littlewood corrections.

5 Conclusion

We have presented a unified probabilistic model for consecutive prime gaps in a fixed arithmetic progression. The model is formulated as an inhomogeneous point process with Cramér-type intensity filtered by the residue class, Hardy–Littlewood two-point correlations via the singular series, and exponential suppression of intermediate primes.

The model makes concrete, testable predictions:

- Relative frequencies of small gaps are proportional to $\mathfrak{S}(g)$;
- The gap $\text{lcm}(\Delta_M, q)$ exhibits a resonance with excess factor $(q-1)/(q-2)$;
- The product $\text{freq}(\Delta_M) \cdot \overline{g}$ is asymptotically stable and equals $2C_2\mathfrak{S}(\Delta_M)\varphi(M)$.

Empirical results on 10^6 primes for $M = 9$ confirm the predicted resonance for $g = 90$ to within 1.2 percentage points, and reveal a clear breakdown of the two-point approximation for gaps approaching the mean spacing. The asymptotically stable product is consistently observed across all six active residue classes.

The framework is flexible and can be extended to other moduli M, to higher-order gap patterns, and to the study of extreme gaps. The point process interpretation opens connections to the probabilistic theory of rare events and Palm calculus.

Author's note

The author is an independent researcher with no academic affiliation. This work is the result of a personal intuition, subsequently formalised and verified with the aid of computational tools (SageMath) and artificial intelligence assistance for the mathematical analysis and drafting.

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