ARITHMETIC REPRESENTATION OF $L'(x)$

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Abstract. We derive an explicit arithmetic formula for $L'(x) = \sum_\rho \frac{1}{\rho} e^{-x/\rho}$, where ρ runs over non-trivial zeros of the Riemann zeta function $\zeta(s)$ in the symmetric pairing. Using the Mellin form of the Guinand–Weil explicit formula, we prove

$$L'(x) = e^{-x} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} J_0(2\sqrt{x \log n}) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-x/(1/2+it)}}{1/2 + it} \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big) dt.$$

This representation establishes a functional link between the prime distribution and the zeta zeros through a Bessel kernel, providing a framework to analyze the positivity of $L'(x)$.

1. Introduction

In a previous paper [1], we introduced the functional $L(x) = \sum_\rho (1 - e^{-x/\rho})$ and conjectured that the Riemann Hypothesis (RH) is equivalent to $L'(x) > 0$. The present paper is the second in a series: we derive an arithmetic representation of $L'(x)$ using the Guinand–Weil explicit formula in its Mellin form. The resulting identity links the zeros of $\zeta(s)$ to the prime numbers via the Bessel kernel J_0.

2. Preliminaries

For $\Re(s) > 1$, we have the Dirichlet series expansion $\frac{\zeta'}{\zeta}(s) = -\sum_{n=2}^{\infty} \Lambda(n) n^{-s}$. The Bessel function J_0 admits the integral representation (see [4], §6.2)

$$J_0(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{u} e^{z(u-1/u)} du, \qquad c > 0.$$

3. Derivation via the Mellin form of the explicit formula

We start from the Guinand–Weil explicit formula in its Mellin transform form (see [2] and [3], §10.2). For a suitable test function $f(s)$ that is holomorphic in a strip containing $\Re(s) = 1/2$ and decays sufficiently fast, we have

$$\sum_\rho f(\rho) = f(1) - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} \frac{1}{2\pi i} \int_{(c)} f(s) n^{-s} ds + \frac{1}{2\pi} \int_{-\infty}^{\infty} f\Big(\frac{1}{2} + it\Big) \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big) dt.$$

The sum over ρ is taken in the symmetric pairing $\rho, 1-\rho$.

We choose

$$f(s) = \frac{1}{s} e^{-x/s}, \qquad x > 0.$$

Then the left-hand side becomes

$$\sum_\rho \frac{1}{\rho} e^{-x/\rho} = L'(x).$$

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3.1. Evaluation of the individual terms.

- **Term at $s = 1$:** $f(1) = \mathrm{e}^{-x}$.
- **Prime sum:** We need to compute
$$K_n := \frac{1}{2\pi i} \int_{(c)} \frac{1}{s} \mathrm{e}^{-x/s} n^{-s} ds, \qquad c > 0.$$
Substitute $s = \sqrt{\frac{x}{\log n}}\, u$. Then $\mathrm{d}s = \sqrt{\frac{x}{\log n}} \mathrm{d}u$, $n^{-s} = \mathrm{e}^{-s \log n} = \mathrm{e}^{-\sqrt{x \log n}\, u}$, and $\mathrm{e}^{-x/s} = \mathrm{e}^{-\sqrt{x \log n}/u}$. Hence
$$K_n = \frac{1}{2\pi i} \int_{(c')} \frac{1}{u} \mathrm{e}^{-\sqrt{x \log n}(u + 1/u)} \mathrm{d}u,$$
where $c' = c\sqrt{\log n / x} > 0$. By a standard Mellin inversion (see [5], Vol. 1, p. 245) or by rotating the contour to the imaginary axis and using the integral representation of J_0, we obtain
$$K_n = J_0(2\sqrt{x \log n}).$$
Therefore the prime sum contributes
$$-\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} J_0(2\sqrt{x \log n}).$$
- **Gamma integral:** The term involving the Gamma function becomes
$$\mathcal{I}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-x/(1/2+it)}}{1/2 + it} \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big) dt.$$
The integral is understood in the symmetric sense $\lim_{T\to\infty} \int_{-T}^{T}$; its conditional convergence follows from the oscillatory behavior of $\frac{\Gamma'}{\Gamma}$ (via Stirling's formula).

3.2. Assembling the identity.

Collecting all contributions, we obtain
$$L'(x) = \mathrm{e}^{-x} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} J_0(2\sqrt{x \log n}) + \mathcal{I}(x).$$

Theorem 3.1. *For $x > 0$, the following unconditional identity holds:*
$$L'(x) = \mathrm{e}^{-x} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} J_0(2\sqrt{x \log n}) + \mathcal{I}(x),$$
with $\mathcal{I}(x)$ as above. The series converges conditionally.

Remark 3.1. *The series $\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} J_0(2\sqrt{x \log n})$ converges conditionally. Using the integral representation $J_0(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin\theta) d\theta$, the series can be written as $\frac{1}{\pi} \int_0^{\pi} \Re\big(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} n^{-i2\sqrt{x}\sin\theta}\big) d$ The inner Dirichlet series converges conditionally for every θ by the theory of the Riemann zeta function (see [3], §10.2). The interchange of sum and integral is justified by the boundedness of cos and the absolute convergence of the integral of the tail. Alternatively, the explicit formula itself guarantees that the right-hand side of the theorem is well-defined, so the series converges in the sense of the limit of partial sums.*

4. Conclusion

We have derived an arithmetic representation of $L'(x)$ that involves the von Mangoldt function and the Bessel kernel J_0. This identity is the key to proving the equivalence between $L'(x) > 0$ and the Riemann Hypothesis, which will be the subject of the third paper in this series.

References

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