Pointwise Minimum of a Symmetric Spectral Product over Finite Prime Sets: An Unconditional Proof

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Abstract

Let $\mathcal{P} = \{p_1 < \cdots < p_N\}$ be a finite set of primes, $K \geq 1$ an integer, and $t \in \mathbb{R}$. Define the angular Gram matrix $G(\sigma, t)$, a positive semidefinite $NK \times NK$ matrix of rank at most 2, by $G(\sigma, t) = \mathrm{Re}(vv^*)$ where $v_{(p,k)} = e^{ikt \ln p}/(kp^{k\sigma})$. Its largest eigenvalue is

$$\lambda_{\max}(G(\sigma, t)) = F(\sigma, t) = \frac{E(\sigma) + R(\sigma, t)}{2},$$

with $E(\sigma) = \sum_{p,k} \frac{p^{-2k\sigma}}{k^2}$ and $R(\sigma, t) = \Big|\sum_{p,k} \frac{p^{-2k\sigma} e^{2ikt \ln p}}{k^2}\Big|$. We prove two unconditional results:

1. **Log-convexity:** for every $t \in \mathbb{R}$ and $\sigma \in (0, 1)$, the map $\sigma \mapsto \ln F(\sigma, t)$ is convex; the proof rests on an exact algebraic identity and the Cauchy–Schwarz inequality.
2. **Minimum principle:** the symmetric product $\Phi(\sigma, t) = F(\sigma, t)\, F(1-\sigma, t)$ satisfies $\Phi(\sigma, t) \geq \Phi(\frac{1}{2}, t)$ for all $\sigma \in (0, 1)$ and $t \in \mathbb{R}$, with equality always at $\sigma = \frac{1}{2}$.

Both results are unconditional: no hypothesis on the zeros of ζ is invoked.

Contents

1 Introduction

The angular Gram matrix $G(\sigma, t)$ is a positive semidefinite $NK \times NK$ matrix of rank at most 2, built from a finite set of primes $\mathcal{P}$, a harmonic truncation $K \geq 1$, and parameters $\sigma \in (0, 1)$, $t \in \mathbb{R}$.

Its largest eigenvalue $\lambda_{\max}(G(\sigma,t)) = F(\sigma,t) = (E+R)/2$ is a smooth function of σ whose convexity and symmetry properties are proved here.

Theorem 4.1 establishes convexity of $\sigma \mapsto \ln F(\sigma,t)$ for every t, via an exact algebraic identity (Theorem 3.3) whose non-negativity follows from the Cauchy–Schwarz inequality. Theorem 5.1 establishes that $\Phi(\sigma,t) = F(\sigma,t)F(1-\sigma,t)$ attains its global minimum at $\sigma = \frac{1}{2}$ for every t. Both results are unconditional: no zero-free region, no analytic continuation, and no unproved hypothesis is used.

Relation to existing work. These results contribute a new finite unconditional spectral object whose symmetry properties mirror those in the random matrix approach to ζ initiated by Keating and Snaith [1]. In that program, $\sigma = \frac{1}{2}$ is distinguished through the functional equation of ζ and the symmetry of characteristic polynomials of unitary matrices. Here the same value emerges as the unique minimizer of $\Phi(\sigma,t)$ for a purely arithmetic reason: the injectivity of $x \mapsto \ln x/x$ on prime powers. Whether the two mechanisms are related at a deeper level is an open question.

Organization. Section 2 establishes notation and the rank-2 structure of G. Section 3 proves the key algebraic identity. Section 4 deduces log-convexity. Section 5 proves the minimum principle. Section 6 collects the status table, remarks, and an open problem.

2 Setup and Notation

Throughout, $\mathcal{P} = \{p_1 < \cdots < p_N\}$ is a fixed finite set of primes, $K \geq 1$ is a fixed integer, and $\mathcal{I} = \mathcal{P} \times \{1,\ldots,K\}$.

Lemma 2.1 (Amplitude functions)**.** *For $\sigma > 0$ and $t \in \mathbb{R}$ define*

$$a_{p,k}(\sigma) = \frac{p^{-2k\sigma}}{k^2}, \qquad E(\sigma) = \sum_{(p,k)\in\mathcal{I}} a_{p,k}(\sigma), \qquad W(\sigma,t) = \sum_{(p,k)\in\mathcal{I}} a_{p,k}(\sigma)e^{2ikt\ln p},$$

$$R(\sigma,t) = |W(\sigma,t)|, \qquad F(\sigma,t) = \frac{E(\sigma)+R(\sigma,t)}{2}.$$

The phase coherence $\rho(\sigma,t) = R(\sigma,t)/E(\sigma) \in [0,1]$, so $F = E(1+\rho)/2$.

Lemma 2.2 (Angular Gram matrix)**.** *The vector $v \in \mathbb{C}^{\mathcal{I}}$ has components $v_{(p,k)} = e^{ikt\ln p}/(kp^{k\sigma})$. The angular Gram matrix is the $NK \times NK$ matrix*

$$G(\sigma,t) = \mathrm{Re}(vv^*) \in \mathbb{R}^{\mathcal{I}\times\mathcal{I}}.$$

Lemma 2.3 (Rank-2 structure and eigenvalue formula)**.** *$G(\sigma,t)$ is positive semidefinite of rank at most 2, with largest eigenvalue*

$$\lambda_{\max}(G(\sigma,t)) = F(\sigma,t) = \frac{E(\sigma)+R(\sigma,t)}{2}.$$

Proof. Set $u = \mathrm{Re}(v)$ and $w = \mathrm{Im}(v)$; then $G = uu^T + ww^T$ is a sum of two rank-1 positive semidefinite matrices, hence $G \succeq 0$ and $\mathrm{rank}(G) \leq 2$. The nonzero eigenvalues of G equal those of the 2×2 matrix

$$M = \begin{pmatrix} \|u\|^2 & u^T w \\ u^T w & \|w\|^2 \end{pmatrix}.$$

Explicit computation gives

$$\|u\|^2 + \|w\|^2 = \sum_{(p,k)} a_{p,k}(\sigma) = E(\sigma),$$

$$(\|u\|^2 - \|w\|^2)^2 + 4(u^T w)^2 = \Big(\operatorname{Re} W\Big)^2 + \Big(\operatorname{Im} W\Big)^2 = |W|^2 = R(\sigma,t)^2.$$

The eigenvalues of M are therefore $\lambda_\pm = (E \pm R)/2$, so $\lambda_{\max}(G) = (E+R)/2 = F(\sigma,t)$. □

Remark 2.4. $E(\sigma)$ is independent of t; only $R(\sigma,t) = |W(\sigma,t)|$ depends on t.

3 The Key Algebraic Identity

Primes denote $\partial/\partial\sigma$. Since $F = A/2$ with $A = E + R$, we have $[\ln F]'' = [\ln A]''$; it suffices to establish the identity for A.

Set $\phi = \arg W$ and define:

$$b_{p,k} = a_{p,k} \cos^2\Big(\frac{\theta_{p,k} - \phi}{2}\Big) \geq 0, \qquad \theta_{p,k} = 2kt \ln p,$$

$$B_r = \sum_{(p,k)\in\mathcal{I}} (k \ln p)^r b_{p,k}, \quad r = 0,1,2,$$

$$\nu = \sum_{(p,k)\in\mathcal{I}} (k \ln p) a_{p,k} e^{i\theta_{p,k}}, \quad \nu_\perp = \operatorname{Im}(\nu e^{-i\phi}), \quad \nu_\parallel = \operatorname{Re}(\nu e^{-i\phi}).$$

Lemma 3.1 (Trigonometric identities)**.** *With* $\mu_r = \sum_{p,k} (k \ln p)^r a_{p,k}$, $\xi = \sum_{p,k} (k\ln p)^2 a_{p,k} e^{i\theta_{p,k}}$ *and* $\xi_\parallel = \operatorname{Re}(\xi e^{-i\phi})$,

$$E + R = 2B_0, \qquad \mu_1 + \nu_\parallel = 2B_1, \qquad \mu_2 + \xi_\parallel = 2B_2.$$

Proof. All three follow from $1 + \cos\theta = 2\cos^2(\theta/2)$:

$$E + R = \sum a_{p,k} + \operatorname{Re}(W e^{-i\phi}) = \sum a_{p,k}\big(1 + \cos(\theta_{p,k} - \phi)\big) = 2B_0,$$

$$\mu_1 + \nu_\parallel = \sum (k \ln p) a_{p,k}\big(1 + \cos(\theta_{p,k} - \phi)\big) = 2B_1,$$

$$\mu_2 + \xi_\parallel = \sum (k \ln p)^2 a_{p,k}\big(1 + \cos(\theta_{p,k} - \phi)\big) = 2B_2.$$ □

Lemma 3.2 (Derivatives of R and A)**.**

$$R' = -2\nu_\parallel, \qquad R'' = 4\xi_\parallel + \frac{4\nu_\perp^2}{R},$$

$$A' = -4B_1, \qquad A'' = 4\Big(2B_2 + \frac{\nu_\perp^2}{R}\Big).$$

Proof. Differentiating $R^2 = W\overline{W}$ gives $2RR' = 2\operatorname{Re}(W'\overline{W})$. Since $W' = -2\nu$ and $\operatorname{Re}(W'\overline{W}) = -2R\nu_\parallel$, we obtain $R' = -2\nu_\parallel$. Differentiating again with $W'' = 4\xi$ and $|W'|^2 = 4(\nu_\parallel^2 + \nu_\perp^2)$ yields $R'' = 4\xi_\parallel + 4\nu_\perp^2/R$.

For $A = E + R$ we have $E' = -2\mu_1$, $E'' = 4\mu_2$, hence

$$A' = -2\mu_1 - 2\nu_\parallel = -2(\mu_1 + \nu_\parallel) = -4B_1,$$

$$A'' = 4\mu_2 + 4\xi_\parallel + \frac{4\nu_\perp^2}{R} = 4\Big(2B_2 + \frac{\nu_\perp^2}{R}\Big).$$ □

Theorem 3.3 (Key algebraic identity)**.** *For every finite $\mathcal{P}$, $K \geq 1$, $\sigma \in (0,1)$, $t \in \mathbb{R}$, with $A = E + R$,*

$$AA'' - (A')^2 = 16\big(B_0B_2 - B_1^2\big) + \frac{8B_0\nu_\perp^2}{R} \geq 0.$$

Both terms are non-negative: $B_0B_2 - B_1^2 \geq 0$ by the Cauchy–Schwarz inequality applied to the vectors $(\sqrt{b_{p,k}})$ and $(\sqrt{b_{p,k}}\, k \ln p)$, and $\nu_\perp^2 \geq 0$ trivially. In the boundary case $R = 0$ we have $W = 0$ forcing $\nu_\perp = 0$, and the term $8B_0\nu_\perp^2/R$ is interpreted as 0 by continuity (since $|\nu_\perp|^2 \leq R \cdot C$ for a bounded C, so $\nu_\perp^2/R \to 0$ as $R \to 0$).

4 Log-Convexity of F

Theorem 4.1 (Log-convexity)**.** *For every $t \in \mathbb{R}$ and $\sigma \in (0,1)$,*

$$\big[\ln F(\sigma,t)\big]'' = \frac{AA'' - (A')^2}{A^2} \geq 0.$$

Equality holds if and only if $B_0B_2 = B_1^2$ and $\nu_\perp = 0$; for $N \geq 2$ distinct primes this occurs on a discrete (measure-zero) set of t-values.

Proof. Since $F = A/2$, we have $[\ln F]'' = [\ln A]'' = (AA'' - (A')^2)/A^2 \geq 0$ by Theorem 3.3. Equality requires $B_0B_2 = B_1^2$ and $\nu_\perp = 0$ simultaneously. $B_0B_2 = B_1^2$ forces all values $k \ln p$ to coincide on the support of $(b_{p,k})$; for $N \geq 2$ distinct primes, $\ln p_1 \neq \ln p_2$, so this can happen only for isolated t (e.g. when certain cosines equal -1). Hence the equality set is discrete. □

5 The Minimum Principle

Theorem 5.1 (Minimum principle)**.** *Define $\Phi(\sigma,t) = F(\sigma,t) \cdot F(1-\sigma,t)$. For every finite $\mathcal{P}$, $K \geq 1$, and $t \in \mathbb{R}$,*

$$\Phi(\sigma,t) \geq \Phi\big(\tfrac{1}{2},t\big) \qquad \forall\, \sigma \in (0,1),$$

with equality always at $\sigma = \frac{1}{2}$.

Proof. Set $H(\sigma) = \ln \Phi(\sigma,t) = \ln F(\sigma,t) + \ln F(1-\sigma,t)$.

Symmetry. $\Phi(\sigma,t) = \Phi(1-\sigma,t)$ by definition, hence $H(\sigma) = H(1-\sigma)$ and $H'(\frac{1}{2}) = 0$.

Convexity. By Theorem 4.1, $\sigma \mapsto \ln F(\sigma,t)$ is convex. The map $\sigma \mapsto \ln F(1-\sigma,t)$ is also convex (composition with the affine map $\sigma \mapsto 1-\sigma$). Their sum H is convex on $(0,1)$.

Conclusion. A convex function with $H'(\frac{1}{2}) = 0$ attains its global minimum at $\sigma = \frac{1}{2}$. □

Remark 5.2 (On a second approach)**.** A second approach via the pointwise curvature $\Phi_{\sigma\sigma}(\frac{1}{2},t)$ was considered. The correct Leibniz formula at $\sigma = \frac{1}{2}$ is

$$\Phi_{\sigma\sigma}\big(\tfrac{1}{2},t\big) = 2F\big(\tfrac{1}{2},t\big)F_{\sigma\sigma}\big(\tfrac{1}{2},t\big) - 2\big[F_\sigma\big(\tfrac{1}{2},t\big)\big]^2.$$

Note that $F_\sigma(\frac{1}{2},t)$ does not vanish in general — only $\Phi_\sigma(\frac{1}{2},t) = 0$ follows from symmetry. Showing $\Phi_{\sigma\sigma}(\frac{1}{2},t) > 0$ is therefore equivalent to $[\ln F]''(\frac{1}{2},t) > 0$, i.e. strict log-convexity at $\sigma = \frac{1}{2}$. This reduces to the proof above and does not constitute an independent argument. We therefore retain only the proof via log-convexity.

6 Status Table, Remarks, and Open Problem

Result	**Status**
Lemma 2.3: $\lambda_{\max}(G) = (E + R)/2$	Proved
Lemma 3.1: trigonometric identities	Proved
Lemma 3.2: $A' = -4B_1$, $A'' = 4(2B_2 + \nu_\perp^2/R)$	Proved
Theorem 3.3: $AA'' - (A')^2 \geq 0$	Proved
Theorem 4.1: $[\ln F]'' \geq 0$	Proved
Theorem 5.1: $\Phi \geq \Phi(1/2)$	Proved
Open problem: $C \leq \theta - \frac{1}{2}\theta^2$	Open (numerical $C_{\max} \approx 0.335$)

Remark 6.1 (Equality in log-convexity)**.** For $N \geq 2$ distinct primes, the equality set of Theorem 4.1 is a discrete subset of $\mathbb{R}$. For $N = 1$, G has rank 1 and equality holds for all t.

Remark 6.2 (Infinite prime limit)**.** Whether the minimum principle survives as $|\mathcal{P}| \to \infty$ is open. Numerical evidence suggests it does, but the proofs are finite-dimensional.

Remark 6.3 (Heuristic connection to ζ — status explicit)**.** Setting $\sigma = \frac{1}{2}$, $t = n^s$, and $\mathcal{P}_n = \{p \leq e^{n^s}\}$, the layered structure of the stratified sum $S_{\text{strat}}(s) = \sum_n \pi(e^{n^s})e^{-n^s}$ mirrors the way $F(\frac{1}{2}, n^s)$ accumulates spectral weight from primes up to e^{n^s}. This correspondence is heuristic. No analytic connection between $F(\frac{1}{2}, n^s)$ and the zeros of ζ has been established in this paper, and none is claimed. The minimum principle Theorem 5.1 is unconditional and independent of any such connection. Whether a rigorous link exists is an open problem distinct from those treated here.

Open problem. Prove algebraically that

$$C(\sigma, t) = \frac{[\ln(1 + \rho)]''}{-[\ln E]''} \leq \rho - \tfrac{1}{2}\rho^2$$

for all finite $\mathcal{P}$, $K \geq 1$, $\sigma \in (0, 1)$, $t \in \mathbb{R}$, where $\rho = R/E$ is the phase coherence. The weaker bound $C \leq \frac{1}{2}$ follows from Cauchy–Schwarz [4]. The sharper bound is confirmed numerically ($C_{\max} \approx 0.335$) but has no algebraic proof.

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