Canonical Classification of Framed Closures in a Three-Strand Braid Transfer Model

Kobie Janse van Rensburg
Kobievr@vokdit.com

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Abstract

We study a discrete dynamical system on the three-strand braid group B_3 in which generators act on an integer-valued framing vector through a local slot-based transfer rule. Terminal states are pairs (P, f) consisting of a permutation $P \in S_3$ and a framing vector $f \in \mathbb{Z}^3$. Under trace closure, strands belonging to the same permutation cycle form a single loop, and physical states are defined modulo cyclic relabeling of the starting point of each loop.

We prove a complete classification theorem: two terminal states are physically equivalent if and only if a finite sector-dependent invariant $\mathcal{I}$ agrees. The invariant decomposes according to the cycle type of P into three sectors (identity, transposition, three-cycle), and every equivalence class admits a unique canonical representative computable by a constant-time algorithm. A key structural finding is that the three-cycle sector carries a discrete cyclic-ordering obstruction not reducible to symmetric polynomial invariants of the framing differences. The analytic proofs are independently verified by exhaustive BFS enumeration of 631 terminal states within word length 10.

This work provides the classified-state foundation for a broader programme connecting combinatorial braid kinematics to RP^3 topology and Skyrmion physics within the Topological Inversion Model (TIM).

Contents

1 Introduction

Braid-based internal state spaces appear across mathematical physics, from knot invariants and topological quantum computing [7, 8] to models encoding particle quantum numbers in braided topological structures [4, 5]. In such systems braid generators act on internal variables, and physically meaningful states are equivalence classes under transformations that leave the closed configuration invariant.

This paper studies a minimal discrete instance of such a structure: a three-strand framed braid transfer model in which generators of B_3 update an integer-valued framing vector attached to the strands. The resulting terminal states carry both permutation data and framing memory, which decouple under closure in a precise sense established computationally in earlier work [9].

The central question addressed here is:

When do two distinct braid histories represent the same physical closed framed state?

Our main result is a complete classification theorem (Theorem 5.1) that answers this question by constructing the physical equivalence relation, identifying sector-wise residual memory invariants, and providing unique canonical representatives with an $O(1)$ normal-form algorithm.

A key structural finding (Proposition 7.1) is that the three-cycle sector carries a discrete obstruction class: the cyclic ordering of framing differences around the closure loop is a genuine invariant not captured by the natural symmetric polynomial invariants. This is the combinatorial counterpart of holonomy-like obstructions in continuum gauge and topological theories.

The paper is organised as follows. Section 2 defines the transfer model. Section 3 constructs the physical equivalence relation. Section 4 establishes the algebraic framework. Section 5 states and proves the classification theorem. Section 7 exhibits the polynomial-invariant obstruction. Section 8 provides the canonical normal-form algorithm. Section 9 presents the computational cross-check. Section 10 discusses implications and connections.

2 The framed braid transfer model

2.1 State space

We work with three ribbons labelled $0, 1, 2$ occupying three ordered slots. The instantaneous state consists of:

- a permutation shadow $P = [p_0, p_1, p_2] \in S_3$, specifying which ribbon occupies each slot;
- a slot-based framing vector $f = (f_0, f_1, f_2) \in \mathbb{Z}^3$.

The initial (vacuum) state is $P_0 = \mathrm{id}$, $f_0 = (0, 0, 0)$.

2.2 Transfer rule

For a generator σ_k^s with $k \in \{1, 2\}$ and sign $s = \pm 1$:

1. identify the active slots $k-1$ and k;
2. update slot framings: $f_{k-1} \leftarrow f_{k-1} + s, \quad f_k \leftarrow f_k - s$;
3. swap the entries of P in positions $k-1$ and k.

The four elementary framing steps are therefore $\pm(1, -1, 0)$ and $\pm(0, 1, -1)$.

Remark 2.1 (Design choice). The framing update acts on *slot positions*, not on ribbon labels. The rule is deliberately history-dependent: the framing output depends on the ordering of crossings, not only on the braid class. This generates the decoupling between coarse closure topology and framed memory that is central to the model [9].

2.3 History dependence: the Yang–Baxter test

The Yang–Baxter relation asserts that $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ represent the same braid class. Under the slot-based rule, both produce $P = [2, 1, 0]$, but they yield different framings:

$$\sigma_1\sigma_2\sigma_1 \;\rightarrow\; f = (2, -1, -1), \qquad \sigma_2\sigma_1\sigma_2 \;\rightarrow\; f = (1, 1, -2).$$

Distinct orderings of topologically equivalent crossings produce distinct internal framed states.

3 Closure and physical equivalence

3.1 Closure sectors

Under trace closure, the cycle structure of P determines the number of closed components. Three cycle types arise:

Cycle type	Permutations	Closure topology
1+1+1	id	3 separate loops
2+1	(01), (02), (12)	1 loop + 1 loop
3	(012), (021)	1 loop

3.2 Physical equivalence

Definition 3.1 (Physical equivalence)**.** Two terminal states (P, f) and (P', f') are *physically equivalent*, written $(P, f) \sim (P', f')$, if and only if:

1. $P = P'$ (same permutation, hence same closure sector);
2. f' is obtained from f by the induced cyclic permutation of the framing entries along each cycle of P, with P itself held fixed as the geometric closure permutation.

Concretely, the symmetry group acting on framings in each sector is:

- **Sector** 1+1+1**:** $G_P = \{1\}$. No identification.
- **Sector** 2+1**:** $G_P = \mathbb{Z}_2$, swapping framings within the 2-cycle.
- **Sector** 3**:** $G_P = \mathbb{Z}_3$, cyclically rotating the three slot framings.

The following are ***not*** declared trivial: orientation reversal (chirality is physical) and global strand relabeling (slot positions are geometric).

Remark 3.2 (Design rationale). Quotienting only by loop-starting-point relabeling is the minimal closure equivalence: it removes exactly the artifact of choosing *where on a closed loop* to begin reading the framing. Keeping orientation physical preserves chiral structure. Keeping global relabeling physical preserves the geometric meaning of the three slot positions.

4 Framing algebra

Definition 4.1 (Total framing)**.** $Q = f_0 + f_1 + f_2$.

Definition 4.2 (Relative-difference triple)**.** For a terminal state (P, f) in the three-cycle sector with cycle $0 \rightarrow P(0) \rightarrow P^2(0) \rightarrow 0$, the *relative-difference triple* is

$$d = (d_1, d_2, d_3), \qquad d_i = f_{\text{next}(i)} - f_i \text{ along the cycle,}$$

so that $d_1 + d_2 + d_3 = 0$. For $P = (1, 2, 0)$: $d_1 = f_1 - f_0$, $d_2 = f_2 - f_1$, $d_3 = f_0 - f_2$.

Lemma 4.3 (Conservation of total framing)**.** *For every history h starting from vacuum, $Q(f(h)) = Q(f_0) = 0$.*

Proof. Each generator changes f by $\pm(1,-1,0)$ or $\pm(0,1,-1)$. In both cases $\sum_i \delta f_i = 0$. By induction on word length, Q is conserved. □

Lemma 4.4 (Framing reconstruction)**.** *The map $f \mapsto (Q, d_1, d_2)$ is injective on $\mathbb{Z}^3$. The inverse is*

$$f_0 = \frac{Q - 2d_1 - d_2}{3}, \quad f_1 = f_0 + d_1, \quad f_2 = f_1 + d_2.$$

Proof. Direct substitution: $f_0 + f_1 + f_2 = 3f_0 + 2d_1 + d_2 = Q$. □

Lemma 4.5 (Closure-sector invariance)**.** *Physical equivalence preserves the permutation P and hence the closure sector.*

Proof. By construction (Definition 3.1), equivalence acts within fixed P. □

Lemma 4.6 (Cycle-sum invariance)**.** *For each cycle C_a of P, the cycle sum $S_a = \sum_{i \in C_a} f_i$ is invariant under physical equivalence.*

Proof. Cyclic relabeling within a cycle permutes the summands of S_a without changing the sum. The fixed-point framing in sector 2+1 is not permuted. □

Lemma 4.7 (Well-definedness of the residual invariant)**.** *The residual invariant μ is well-defined on equivalence classes:*

- *Sector* 1+1+1*: $\mu = f$ (no quotient).*
- *Sector* 2+1*: $\mu = |f_a - f_b|$. Under $\mathbb{Z}_2$ swap, $|f_a - f_b| = |f_b - f_a|$.*
- *Sector* 3*: $\mu = [d_1, d_2, d_3]_{\mathbb{Z}_3}$. Under cyclic rotation, the $\mathbb{Z}_3$ orbit maps to itself.*

5 The classification theorem

Theorem 5.1 (Canonical framed-closure classification)**.** *Let (P, f) and (P', f') be terminal states with total framings $Q = f_0 + f_1 + f_2$ and $Q' = f'_0 + f'_1 + f'_2$. Then $(P, f) \sim (P', f')$ if and only if $\mathcal{I}(P, f) = \mathcal{I}(P', f')$, where the sector-dependent invariant $\mathcal{I}$ is:*

Sector* 1+1+1 *($P = \mathrm{id}$):

$$\mathcal{I} = f = (f_0, f_1, f_2).$$

Sector* 2+1 *(P a transposition with 2-cycle $\{a, b\}$ and fixed point c):

$$\mathcal{I} = \big(P,\ f_a{+}f_b,\ f_c,\ |f_a{-}f_b|\big).$$

Sector* 3 *(P a 3-cycle with orientation $\varepsilon \in \{+, -\}$):

$$\mathcal{I} = \big(\varepsilon,\ Q,\ [d_1, d_2, d_3]_{\mathbb{Z}_3}\big),$$

where $\varepsilon = +$ for $P = (012)$ and $\varepsilon = -$ for $P = (021)$, and $[\cdot]_{\mathbb{Z}_3}$ denotes the orbit under $(d_1, d_2, d_3) \mapsto (d_2, d_3, d_1)$.

Moreover, every equivalence class admits a unique canonical representative:

- *Sector* 2+1*: the unique element with $f_a \le f_b$.*
- *Sector* 3*: the unique cyclic rotation placing (d_1, d_2, d_3) in lexicographic-minimum position.*

The proof splits into invariance and completeness.

5.1 Invariance

Theorem 5.2 (Invariance). *If $(P, f) \sim (P, f')$, then $\mathcal{I}(P, f) = \mathcal{I}(P, f')$.*

Proof. Follows immediately from Lemmas 4.3–4.7. □

5.2 Completeness

Theorem 5.3 (Completeness). *If $\mathcal{I}(P, f) = \mathcal{I}(P, f')$, then $(P, f) \sim (P, f')$.*

Proof. We proceed sector by sector.

Sector 1+1+1. $\mathcal{I} = f$. If $f = f'$, the states are identical.

Sector 2+1. Suppose $f_c = f'_c$, $f_a + f_b = f'_a + f'_b$, and $|f_a - f_b| = |f'_a - f'_b|$. Either $f_a - f_b = f'_a - f'_b$ (states identical) or $f_a - f_b = -(f'_a - f'_b)$. In the latter case, combining with $f_a + f_b = f'_a + f'_b$ gives $f_a = f'_b$ and $f_b = f'_a$, so f' is the $\mathbb{Z}_2$ swap of f.

Sector 3. Same P (hence same ε) and same Q. By hypothesis, $[d]_{\mathbb{Z}_3} = [d']_{\mathbb{Z}_3}$, so $\exists\, k \in \{0, 1, 2\}$ with $d' = R^k(d)$ where $R\colon (d_1, d_2, d_3) \mapsto (d_2, d_3, d_1)$. By Lemma 4.4, f is uniquely determined by (Q, d_1, d_2). The framing corresponding to $R^k(d)$ is the cyclic rotation (f_k, f_{k+1}, f_{k+2}) (indices mod 3), which is precisely the $\mathbb{Z}_3$ starting-point relabeling. Therefore $(P, f) \sim (P, f')$. □

6 The memory hierarchy

Corollary 6.1 (Structured coarse-graining). *Closure acts as a structured coarse-graining on framed memory: increasing cycle connectivity reduces the number of independent residual framing degrees of freedom, but does not generically annihilate them.*

Sector	*Memory space*	*Degrees of freedom*
1+1+1	$\mathbb{Z}^2$ *(at fixed Q)*	*2 integers (maximal)*
2+1	$\mathbb{Z}_{\geq 0} \times \mathbb{Z}$	*1 non-negative + 1 integer*
3	$\mathbb{Z}^2/\mathbb{Z}_3$	*1 element of $\mathbb{Z}^2/\mathbb{Z}_3$ (minimal nontrivial)*

The residual memory vanishes only at the origin $f = (0, 0, 0)$.

7 The polynomial-invariant obstruction

The $\mathbb{Z}_3$ action on relative differences $(d_1, d_2, d_3) \mapsto (d_2, d_3, d_1)$ is a 120° rotation on the plane $d_1 + d_2 + d_3 = 0$. Its natural polynomial invariants are the Eisenstein norm $p = d_1^2 + d_1 d_2 + d_2^2$ (quadratic) and the symmetrised cubic $s_3 = -3\, d_1\, d_2\, (d_1 + d_2)$.

Proposition 7.1 (Polynomial invariants do not classify). *The pair (p, s_3) does not separate $\mathbb{Z}_3$ orbits.*

Proof. Consider $d = (-3, 0, 3)$ and $d' = (-3, 3, 0)$. Both satisfy $p = 9$ and $s_3 = 0$. However, no cyclic rotation maps one to the other: the cyclic orders $(-3 \to 0 \to 3)$ and $(-3 \to 3 \to 0)$ are distinct. Both triples are realised by length-2 braid histories from vacuum:

Difference triple	Framing	History
$(-3, 0, 3)$ via rotation	$(-1, 2, -1)$	$\sigma_1^{-1}\sigma_2^{+1}$
$(-3, 3, 0)$ via rotation	$(1, -2, 1)$	$\sigma_1^{+1}\sigma_2^{-1}$

Both yield $P = (1, 2, 0)$ with $Q = 0$, confirming the obstruction is realised by the transfer rule. □

Remark 7.2. The pair (p, s_3) classifies the *unordered multiset* $\{d_1, d_2, d_3\}$ (the S_3 orbit) but not the *cyclic ordering* (the $\mathbb{Z}_3$ orbit). Two orbits sharing the same (p, s_3) differ only in the sense of cyclic traversal, which is a discrete orientation obstruction. This parallels the distinction between symmetric polynomial link invariants and refined closed-braid invariants in knot theory, and provides a natural combinatorial counterpart to holonomy-like classes in continuum topological theories.

8 Canonical normal-form algorithm

The following algorithm reduces any terminal state (P, f) to its unique canonical representative $\hat{f}$ in $O(1)$ operations.

Input: Terminal state $(P, f) \in S_3 \times \mathbb{Z}^3$.
Output: Canonical representative $(P, \hat{f})$ with $(P, \hat{f}) \sim (P, f)$.

1. Compute the cycle decomposition of P.
2. **If** $P = \text{id}$ **(sector** 1+1+1**):** Set $\hat{f} = f$.
3. **If** P **is a transposition with** 2**-cycle** $\{a, b\}$ **(sector** 2+1**):** If $f_a > f_b$, swap $f_a \leftrightarrow f_b$; otherwise keep f.
4. **If** P **is a** 3**-cycle (sector** 3**):** Compute the relative-difference triple $d = (d_1, d_2, d_3)$ along the cycle of P. Let $\hat{d} = \min\{(d_1, d_2, d_3), (d_2, d_3, d_1), (d_3, d_1, d_2)\}$ (lexicographic). Reconstruct $\hat{f}$ from $(Q, \hat{d}_1, \hat{d}_2)$ using Lemma 4.4.

9 Computational verification

Theorem 5.1 is proved analytically above; the BFS enumeration below does not enter the proof but serves as an independent verification over a large finite subset of reachable states.

Starting from the vacuum $(\text{id}, (0, 0, 0))$, breadth-first search with the four generators $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}$ produces all terminal states reachable within bounded word length L. The canonicalization algorithm of Section 8 was applied to every state.

	$L \leq 8$		$L \leq 10$	
Sector	States	Classes	States	Classes
1+1+1	73	73	113	113
2+1	176	126	284	206
3	154	118	234	178
$\varepsilon = +$	77	59	117	89
$\varepsilon = -$	77	59	117	89
Total	403	317	631	497

Zero violations of Lemmas 4.3–4.7 were found at either depth. At $L \leq 12$, 54 polynomial-invariant collisions (same (p, s_3), different $\mathbb{Z}_3$ orbit) were found among reachable three-cycle states, confirming Proposition 7.1 over a substantial data set.

The identity sector has the same number of states and classes (no quotient), confirming that $G_P = \{1\}$ in this sector. The $\varepsilon = +$ and $\varepsilon = -$ sub-sectors have identical counts, consistent with the structural symmetry between the two three-cycle orientations.

10 Discussion

10.1 Summary of results

This paper proves a complete classification theorem for closed framed states in a three-strand braid transfer model. The classification is analytic, with sector-wise invariants, unique canonical representatives, and a constant-time reduction algorithm. The polynomial-invariant obstruction in the three-cycle sector is a genuinely new structural feature that distinguishes the model's residual memory from naive symmetric invariants.

10.2 The memory hierarchy

A striking feature of the classification is the memory hierarchy: closure acts as a structured coarse-graining that reduces but does not generically destroy framed memory. The identity sector retains maximal memory (2 integers), the transposition sector retains reduced memory (1 non-negative integer plus 1 integer), and the three-cycle sector retains minimal nontrivial memory (an element of $\mathbb{Z}^2/\mathbb{Z}_3$). This hierarchy parallels the physical expectation that more connected closure topologies carry less distinguishable internal structure.

10.3 The cyclic-ordering obstruction

The failure of polynomial invariants (p, s_3) to classify $\mathbb{Z}_3$ orbits means that the residual memory in the three-cycle sector is not reducible to the symmetric functions of the framing differences. The cyclic ordering itself is the invariant. This phenomenon parallels the role of refined closed-braid invariants in knot theory [2, 3], where symmetric polynomial summaries may fail to capture full topological information.

In the context of the broader TIM programme [9], this obstruction class provides a discrete combinatorial counterpart to continuum topological classes in Finkelstein–Rubinstein-type constructions on $\mathrm{RP}^3 = S^3/\mathbb{Z}_2$ [6].

10.4 Implications for particle-state models

The classification theorem upgrades the braid sector from a history-dependent construction to a canonically classified state theory. This has three immediate consequences for models that assign particle quantum numbers to braid states:

1. Particle assignments become functions on physical equivalence classes rather than on ad hoc history choices.
2. Mass functionals become genuine state functionals on canonical representatives, removing representative-dependence.
3. The explicit quotient by physically trivial deformations provides the mathematically correct setting for a gauge-like interpretation of the internal framing data.

10.5 Connections and open problems

The model studied here is the combinatorial layer of the Topological Inversion Model [9], in which RP^3 topology at the Planck scale generates the Standard Model particle spectrum via Skyrmion quantization. The present classification provides the rigorous state-space foundation on which the continuum connection is built. Open problems include: promoting the $\mathbb{Z}_3$ cyclic obstruction to a continuous gauge structure, deriving the vacuum tensor K from a dynamical principle, and connecting the braid-level mass relations to physical particle masses via renormalisation.

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