An Octonionic Foundation of Spacetime Geometry and Quantum Mechanics

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Abstract

We propose a framework in which spacetime geometry and quantum dynamics emerge from the non–associative algebra of octonions. A dynamical octonion-valued field is introduced whose interaction is governed by the octonionic associator. The induced G_2 three-form determines the metric, variation of the action produces Einstein equations, and projection onto quaternionic subalgebras yields quantum wavefunctions. The theory admits a compact master equation encoding gravitational and quantum dynamics simultaneously.

1 Octonionic Algebra

The mathematical foundation of the present framework is the algebra of octonions.

1.1 Definition

The octonions $\mathbb{O}$ form an eight–dimensional real algebra

$$\mathbb{O} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{O}) \tag{1}$$

with basis

$$\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}. \tag{2}$$

Every octonion can therefore be written as

$$x = x_0 + \sum_{i=1}^{7} x_i e_i. \tag{3}$$

1.2 Multiplication Rules

The multiplication of imaginary units is defined by

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k \tag{4}$$

where δ_{ij} is the Kronecker symbol and f_{ijk} are totally antisymmetric structure constants. These constants encode the multiplication structure of the algebra.

1.3 Fano Plane Representation

The multiplication rules of the imaginary units can be visualized by the Fano plane. Each oriented line in the diagram represents a quaternionic subalgebra.

For example

$$e_1e_2 = e_3 \tag{5}$$
$$e_2e_3 = e_1 \tag{6}$$
$$e_3e_1 = e_2. \tag{7}$$

Cyclic permutations preserve the sign, while reversing orientation introduces a minus sign.

1.4 Conjugation

The octonionic conjugation is defined as

$$\overline{x_0 + x_ie_i} = x_0 - x_ie_i. \tag{8}$$

This operation reverses the sign of all imaginary components.

1.5 Norm

The norm of an octonion is

$$N(x) = x\bar{x}. \tag{9}$$

Expanding explicitly yields

$$N(x) = x_0^2 + \sum_{i=1}^{7} x_i^2. \tag{10}$$

A fundamental property of the octonion algebra is the multiplicativity of the norm

$$N(xy) = N(x)N(y). \tag{11}$$

Thus $\mathbb{O}$ forms a normed division algebra.

1.6 Nonassociativity

Unlike real numbers, complex numbers and quaternions, the octonion algebra is nonassociative.

In general

$$(xy)z \neq x(yz). \tag{12}$$

The deviation from associativity is measured by the associator

$$A(x, y, z) = (xy)z - x(yz). \tag{13}$$

This property plays a central role in the physical model developed in this work.

2 G_2 Geometry from Octonions

The imaginary octonions naturally define a seven–dimensional vector space equipped with a special geometric structure.

2.1 Imaginary Octonion Space

Let

$$V = \mathrm{Im}(\mathbb{O}) \tag{14}$$

be the space of imaginary octonions.

This space has dimension

$$\dim V = 7. \tag{15}$$

A natural inner product on V is inherited from the octonion norm

$$\langle x, y \rangle = \mathrm{Re}(x\bar{y}). \tag{16}$$

2.2 Cross Product

The imaginary octonions also define a seven-dimensional cross product

$$x \times y = \frac{1}{2}(xy - yx). \tag{17}$$

This product satisfies

$$\langle x \times y, z \rangle = \varphi(x, y, z) \tag{18}$$

where φ is a three-form defined below.

2.3 Fundamental Three-Form

Using octonion multiplication we define

$$\varphi(x, y, z) = \langle x, yz \rangle. \tag{19}$$

In the basis of imaginary units this becomes

$$\varphi_{ijk} = f_{ijk}. \tag{20}$$

Thus the structure constants of the octonion algebra determine the components of the three-form.

2.4 Theorem: Stabilizer Group

Theorem.
The subgroup of $GL(7)$ preserving the three-form φ is the exceptional Lie group

$$G_2. \tag{21}$$

Proof (sketch).
The group G_2 is defined as the automorphism group of the octonion algebra.
Any automorphism preserves the octonion product and therefore the three-form

$$\varphi(x, y, z) = \langle x, yz \rangle.$$

Conversely, any transformation preserving φ also preserves the multiplication structure.
Thus the stabilizer of φ is precisely G_2.

□

2.5 G_2 Structure

The pair

$$(V, \varphi) \tag{22}$$

defines a G_2 structure on the seven-dimensional vector space.
This structure is central in many areas of differential geometry and mathematical physics.

3 Metric Reconstruction from the Octonionic 3-Form

A remarkable property of the octonion algebra is that its multiplication structure naturally defines a geometric structure on a seven-dimensional vector space. In particular the octonion product induces a special three-form whose stabilizer is the exceptional Lie group G_2. This three-form uniquely determines a Riemannian metric.

3.1 Imaginary Octonions

Let

$$V = \mathrm{Im}(\mathbb{O}) \tag{23}$$

be the seven-dimensional space of imaginary octonions.
Elements of V can be written as

$$x = x_i e_i \tag{24}$$

with basis $\{e_1, \dots, e_7\}$.
The scalar product induced by the octonion norm is

$$\langle x, y \rangle = \mathrm{Re}(x\bar{y}). \tag{25}$$

In coordinates this becomes

$$\langle x, y \rangle = x_i y_i. \tag{26}$$

3.2 Definition of the Fundamental 3-Form

Octonion multiplication defines a natural three-form

$$\varphi(x, y, z) = \langle x, yz \rangle. \tag{27}$$

Expanding in the basis of imaginary units gives

$$\varphi_{ijk} = \langle e_i, e_j e_k \rangle. \tag{28}$$

Using the multiplication rule

$$e_j e_k = -\delta_{jk} + f_{jkl} e_l \tag{29}$$

we obtain

$$\varphi_{ijk} = f_{ijk}. \tag{30}$$

Thus the structure constants of the octonion algebra define the components of the fundamental 3-form.

3.3 Stabilizer Group

The subgroup of $GL(7)$ preserving φ is the exceptional Lie group

$$G_2. \tag{31}$$

Hence the octonion algebra induces a G_2-structure on the seven-dimensional vector space V.

3.4 Metric from the 3-Form

A fundamental result of G_2 geometry is that the three-form φ uniquely determines a Riemannian metric.

Define the tensor

$$B_{ij} = \frac{1}{6} \varphi_{ikl} \varphi_{jkl}. \tag{32}$$

Substituting $\varphi_{ijk} = f_{ijk}$ gives

$$B_{ij} = \frac{1}{6} f_{ikl} f_{jkl}. \tag{33}$$

A known identity of the octonion structure constants is

$$f_{ikl} f_{jkl} = 6\delta_{ij}. \tag{34}$$

Therefore

$$B_{ij} = \delta_{ij}. \tag{35}$$

The induced metric is thus

$$g_{ij} = \delta_{ij}. \tag{36}$$

3.5 Volume Form

The metric determines a natural volume form on V.

Let

$$\text{vol} = \sqrt{\det(g)}\, dx^1 \wedge \cdots \wedge dx^7. \tag{37}$$

For the Euclidean metric obtained above we have

$$\text{vol} = dx^1 \wedge \cdots \wedge dx^7. \tag{38}$$

3.6 Hodge Dual

Using the metric we can define the Hodge dual of the 3-form

$$*\varphi. \tag{39}$$

The dual form is a 4-form whose components are

$$(*\varphi)_{ijkl} = \frac{1}{3!}\epsilon_{ijklmnp}\varphi_{mnp}. \tag{40}$$

The pair $(\varphi, *\varphi)$ completely characterizes the G_2 geometry.

3.7 Extension to Spacetime Geometry

Although the octonionic structure naturally lives in seven dimensions, it can induce geometric structures on lower-dimensional manifolds.

In the present framework the spacetime metric $g_{\mu\nu}$ appearing in the action is interpreted as an effective four-dimensional projection of the metric derived from the octonionic structure.

Thus the geometric structure of spacetime can be viewed as emerging from the algebraic properties of the octonions.

3.8 Geometric Interpretation

The chain of structures obtained from the octonion algebra can be summarized as follows:

$$\text{octonion multiplication} \to \text{structure constants } f_{ijk} \to G_2 \text{ 3-form } \varphi \to \text{metric } g_{ij}. \tag{41}$$

Hence the metric geometry of space is not introduced as a fundamental object but emerges from the algebraic structure of the octonions. Thus the metric emerges from octonion multiplication.

4 Fundamental Action

The dynamics of the theory are determined by a variational principle.

Let M be a four-dimensional spacetime manifold equipped with a metric $g_{\mu\nu}$.

We introduce an octonion valued field

$$\Psi(x) \in \mathbb{O}. \tag{42}$$

The action functional is defined as

$$S[\Psi, g] = \int_M d^4x \sqrt{-g}\, L. \tag{43}$$

The Lagrangian density is postulated to be

$$L = R + \alpha \langle \nabla_\mu \Psi, \nabla^\mu \Psi \rangle - \lambda |A(\Psi, \Psi, \Psi)|^2. \tag{44}$$

The three contributions correspond respectively to

- gravitational curvature
- kinetic energy of the octonion field
- a nonassociative interaction potential

4.1 Kinetic Term

The scalar product on $\mathbb{O}$ is

$$\langle x, y \rangle = \mathrm{Re}(x\bar{y}) \tag{45}$$

Thus the kinetic term can be written explicitly as

$$\langle \nabla_\mu \Psi, \nabla^\mu \Psi \rangle = g^{\mu\nu} \mathrm{Re}(\nabla_\mu \Psi \nabla_\nu \bar{\Psi}). \tag{46}$$

Expanding the field

$$\Psi = \psi_0 + \psi_i e_i \tag{47}$$

gives

$$\langle \nabla_\mu \Psi, \nabla^\mu \Psi \rangle = g^{\mu\nu} (\partial_\mu \psi_0 \partial_\nu \psi_0 + \partial_\mu \psi_i \partial_\nu \psi_i). \tag{48}$$

Hence the kinetic term corresponds to eight coupled scalar fields.

4.2 Associator Interaction

The octonionic associator is defined as

$$A(a, b, c) = (ab)c - a(bc). \tag{49}$$

For the field Ψ we define

$$A(\Psi, \Psi, \Psi) = (\Psi\Psi)\Psi - \Psi(\Psi\Psi). \tag{50}$$

The interaction potential is therefore

$$V(\Psi) = \lambda |A(\Psi, \Psi, \Psi)|^2 \tag{51}$$

where

$$|A|^2 = A\overline{A}. \tag{52}$$

Because the octonion algebra is nonassociative, this potential is generically nonzero.

4.3 Physical Interpretation

The Lagrangian contains three geometric structures:

- the Ricci scalar R describing spacetime curvature
- the kinetic propagation of the octonionic field
- a self-interaction governed by the associator

Thus the dynamics of the theory are governed directly by the nonassociativity of the octonion algebra.

5 Variation of the Action and Einstein Equations

We now derive the gravitational field equations by varying the action with respect to the metric.

$$\delta S = 0 \tag{53}$$

with respect to $g^{\mu\nu}$.

5.1 Variation of the Einstein-Hilbert Term

The gravitational part of the action is

$$S_G = \int d^4x \sqrt{-g} R. \tag{54}$$

A standard result from differential geometry gives

$$\delta(\sqrt{-g}R) = \sqrt{-g}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)\delta g^{\mu\nu}. \tag{55}$$

5.2 Variation of the Kinetic Term

Consider the kinetic contribution

$$S_K = \alpha \int d^4x \sqrt{-g} g^{\mu\nu} \mathrm{Re}(\partial_\mu \Psi \partial_\nu \bar{\Psi}). \tag{56}$$

The variation contains two contributions:

$$\delta S_K = \alpha \int d^4x \left[\delta(\sqrt{-g}) L_K + \sqrt{-g} \delta(g^{\mu\nu}) K_{\mu\nu} \right] \tag{57}$$

where

$$K_{\mu\nu} = \mathrm{Re}(\partial_\mu \Psi \partial_\nu \bar{\Psi}). \tag{58}$$

Using the identity

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{59}$$

we obtain

$$\delta S_K = \int d^4x \sqrt{-g} \left[\partial_\mu \Psi \partial_\nu \bar{\Psi} - \frac{1}{2} g_{\mu\nu} |\nabla\Psi|^2 \right] \delta g^{\mu\nu}. \tag{60}$$

5.3 Energy Momentum Tensor

The energy momentum tensor is therefore

$$T_{\mu\nu} = \nabla_\mu \Psi \nabla_\nu \bar{\Psi} - \frac{1}{2} g_{\mu\nu} |\nabla \Psi|^2 + g_{\mu\nu} V(\Psi). \tag{61}$$

5.4 Einstein Field Equations

Combining gravitational and matter variations gives

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}. \tag{62}$$

5.5 Interpretation

The curvature of spacetime is sourced by the octonionic field energy.

The stress tensor contains contributions from

- kinetic propagation of the octonion field
- nonassociative interaction energy

Therefore spacetime curvature emerges as a dynamical response to the nonassociative structure of the octonionic field.

6 Field Equation of the Octonionic Field

We now derive the dynamical equation governing the octonionic field Ψ by varying the action with respect to Ψ.

The action is

$$S = \int d^4x \sqrt{-g} \left(R + \alpha \langle \nabla_\mu \Psi, \nabla^\mu \Psi \rangle - \lambda |A(\Psi, \Psi, \Psi)|^2 \right). \tag{63}$$

Since the Ricci scalar does not depend on Ψ, only the kinetic and potential terms contribute.

6.1 Euler-Lagrange Equation

For a field Ψ, the Euler-Lagrange equation reads

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \frac{\partial L}{\partial(\partial_\mu \Psi)} \right) - \frac{\partial L}{\partial \Psi} = 0. \tag{64}$$

We evaluate the two contributions separately.

6.2 Variation of the Kinetic Term

The kinetic term of the Lagrangian is

$$L_K = \alpha g^{\mu\nu} \mathrm{Re}(\partial_\mu \Psi \partial_\nu \bar{\Psi}). \tag{65}$$

Taking the derivative with respect to $\partial_\mu \Psi$ gives

$$\frac{\partial L_K}{\partial(\partial_\mu \Psi)} = \alpha g^{\mu\nu} \partial_\nu \bar{\Psi}. \tag{66}$$

Substituting into the Euler-Lagrange equation yields

$$\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi \right). \tag{67}$$

Dividing by $\sqrt{-g}$ gives the covariant Laplacian

$$\nabla^2 \Psi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi \right). \tag{68}$$

Thus the kinetic contribution produces the Laplace–Beltrami operator.

6.3 Variation of the Associator Potential

The interaction potential is

$$V(\Psi) = \lambda |A(\Psi, \Psi, \Psi)|^2 \tag{69}$$

where

$$A(\Psi, \Psi, \Psi) = (\Psi\Psi)\Psi - \Psi(\Psi\Psi). \tag{70}$$

Let

$$A = A(\Psi, \Psi, \Psi). \tag{71}$$

Then

$$V = \lambda A \bar{A}. \tag{72}$$

The variation gives

$$\delta V = \lambda(\delta A)\bar{A} + \lambda A(\delta \bar{A}). \tag{73}$$

We now compute the variation of the associator.

6.4 Variation of the Associator

Using the definition

$$A = (\Psi\Psi)\Psi - \Psi(\Psi\Psi) \tag{74}$$

the variation becomes

$$\delta A = (\delta\Psi\Psi)\Psi + (\Psi\delta\Psi)\Psi + (\Psi\Psi)\delta\Psi - \delta\Psi(\Psi\Psi) - \Psi(\delta\Psi\Psi) - \Psi(\Psi\delta\Psi). \tag{75}$$

Each term corresponds to a possible insertion of $\delta\Psi$ in the nonassociative product. The derivative of the potential therefore takes the schematic form

$$\frac{\partial V}{\partial \Psi} = \lambda \frac{\partial}{\partial \Psi} |A(\Psi, \Psi, \Psi)|^2. \tag{76}$$

6.5 Octonionic Field Equation

Combining the kinetic and interaction contributions yields the field equation

$$\nabla^2\Psi + \lambda\frac{\partial}{\partial\Psi}|A(\Psi,\Psi,\Psi)|^2 = 0. \tag{77}$$

6.6 Expanded Form

Substituting the explicit expression for the associator gives

$$\nabla^2\Psi + \lambda\frac{\partial}{\partial\Psi}\left|(\Psi\Psi)\Psi - \Psi(\Psi\Psi)\right|^2 = 0. \tag{78}$$

This equation is nonlinear due to the cubic structure of the associator.

6.7 Interpretation

The field equation contains two fundamentally different contributions:

- the Laplace–Beltrami operator describing propagation of the octonionic field in curved spacetime
- a nonlinear self-interaction generated by the nonassociativity of the octonion algebra

Thus the nonassociative structure of $\mathbb{O}$ directly produces interaction dynamics.

6.8 Relation to the Master Equation

Including the curvature coupling discussed later, the complete dynamical equation can be written as

$$\nabla^2\Psi + \lambda A(\Psi,\Psi,\Psi) + \kappa R\Psi = 0. \tag{79}$$

This equation summarizes the dynamics of the octonionic theory.

7 Dirac Structure from Octonionic Subalgebras

One of the crucial properties of the octonion algebra is the existence of quaternionic subalgebras. These subalgebras allow the construction of spinorial degrees of freedom and lead naturally to the Dirac equation.

7.1 Quaternionic Subalgebras

A fundamental property of octonions is the following lemma.

Lemma.

For any pair of imaginary octonion units e_i, e_j with $i \neq j$, the set

$$\{1, e_i, e_j, e_ie_j\} \tag{80}$$

forms a quaternionic subalgebra

$$\mathbb{H} \subset \mathbb{O}. \tag{81}$$

Proof.
Using the multiplication rule

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k \tag{82}$$

one finds that the elements

$$1, \quad e_i, \quad e_j, \quad e_i e_j$$

close under multiplication and satisfy the quaternion relations

$$e_i^2 = -1 \tag{83}$$

$$e_j^2 = -1 \tag{84}$$

$$e_i e_j = -e_j e_i. \tag{85}$$

Thus they generate a copy of $\mathbb{H}$. □

7.2 Complex Structure

Quaternion algebras contain complex subspaces.

Choose one imaginary unit e_i and define

$$i \equiv e_i. \tag{86}$$

Then the subset

$$\{a + be_i\} \tag{87}$$

is isomorphic to the complex numbers

$$\mathbb{C} \subset \mathbb{H} \subset \mathbb{O}. \tag{88}$$

This allows the definition of complex spinor fields.

7.3 Relation to Clifford Algebras

Quaternion algebras are closely related to Clifford algebras.

In particular

$$\mathbb{H} \cong \mathrm{Cl}(0, 2). \tag{89}$$

Spinors can therefore be constructed by representing the octonionic field in a suitable quaternionic basis.

7.4 Spinor Decomposition

We decompose the octonion field as

$$\Psi = \psi_1 + \psi_2 e_k \tag{90}$$

where ψ_1, ψ_2 lie in a complex subspace.

This pair can be written as a two-component spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{91}$$

7.5 Gamma Matrices

The Clifford algebra of spacetime is defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{92}$$

A representation can be constructed using Pauli matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{93}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{94}$$

The Pauli matrices can be represented using quaternion units.

Thus the Dirac operator can be written as

$$D = i\gamma^\mu \partial_\mu. \tag{95}$$

7.6 Dirac Equation

Projecting the octonion field onto the spinor representation yields

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{96}$$

Thus fermionic dynamics arise from the internal structure of the octonion field.

7.7 Interpretation

The appearance of spinors follows from the existence of quaternionic subalgebras inside $\mathbb{O}$.

The Dirac equation therefore emerges as the natural relativistic propagation equation of the projected octonionic field.

8 Nonrelativistic Limit and the Schrödinger Equation

We now show that the Schrödinger equation arises as the nonrelativistic limit of the octonionic field dynamics.

8.1 Rest Energy Factorization

We separate the rapidly oscillating rest energy term by writing

$$\Psi(x,t) = e^{-imc^2t/\hbar}\psi(x,t). \tag{97}$$

The function ψ varies slowly compared to the phase factor.

8.2 Time Derivatives

The first time derivative becomes

$$\partial_t\Psi = e^{-imc^2t/\hbar}\left(\partial_t\psi - \frac{imc^2}{\hbar}\psi\right). \tag{98}$$

The second derivative gives

$$\partial_t^2\Psi = e^{-imc^2t/\hbar}\left(\partial_t^2\psi - \frac{2imc^2}{\hbar}\partial_t\psi - \frac{m^2c^4}{\hbar^2}\psi\right). \tag{99}$$

8.3 Substitution into the Field Equation

The relativistic equation for Ψ has the schematic form

$$\Box\Psi + m^2c^2\Psi = 0 \tag{100}$$

where

$$\Box = \frac{1}{c^2}\partial_t^2 - \nabla^2. \tag{101}$$

Substituting the ansatz yields

$$\frac{1}{c^2}\partial_t^2\Psi - \nabla^2\Psi + m^2c^2\Psi = 0. \tag{102}$$

8.4 Expansion

Inserting the derivatives and cancelling the dominant terms of order m^2c^4 gives

$$-\frac{2im}{\hbar}\partial_t\psi - \nabla^2\psi = 0. \tag{103}$$

Multiplying by $\frac{\hbar^2}{2m}$ yields

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi. \tag{104}$$

8.5 Result

We recover the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi. \tag{105}$$

8.6 Physical Interpretation

The Schrödinger equation appears as the low-energy limit of the relativistic octonionic field equation.

The complex wavefunction arises from the projection

$$\mathbb{O} \to \mathbb{H} \to \mathbb{C}. \tag{106}$$

Thus quantum wave mechanics emerges naturally from the algebraic structure of the octonions.

9 Probability Structure and the Born Rule

A fundamental requirement of quantum theory is the existence of a probabilistic interpretation of the wavefunction. In the present framework this interpretation arises naturally from the norm structure of the octonion algebra.

9.1 Octonionic Norm

For an octonion

$$x = x_0 + x_i e_i \tag{107}$$

the conjugate element is

$$\bar{x} = x_0 - x_i e_i. \tag{108}$$

The norm is defined as

$$N(x) = x\bar{x}. \tag{109}$$

Expanding explicitly gives

$$N(x) = x_0^2 + \sum_{i=1}^{7} x_i^2. \tag{110}$$

The norm satisfies the multiplicative property

$$N(xy) = N(x)N(y). \tag{111}$$

This property plays a central role in the probabilistic interpretation.

9.2 Norm of the Wavefunction

After projection onto a complex subspace $\mathbb{O} \to \mathbb{C}$ the field becomes a complex wavefunction

$$\psi(x, t). \tag{112}$$

The norm becomes

$$N(\psi) = \psi\bar{\psi}. \tag{113}$$

In complex notation this reduces to

$$N(\psi) = |\psi|^2. \tag{114}$$

9.3 Expansion in Basis States

Consider an expansion in orthonormal states

$$\psi = \sum_i c_i \phi_i. \tag{115}$$

Using orthonormality

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} \tag{116}$$

we obtain

$$\begin{aligned} N(\psi) &= \langle \psi, \psi \rangle & (117) \\ &= \left\langle \sum_i c_i \phi_i, \sum_j c_j \phi_j \right\rangle. & (118) \end{aligned}$$

Expanding gives

$$N(\psi) = \sum_{ij} c_i \bar{c}_j \langle \phi_i, \phi_j \rangle. \tag{119}$$

Using orthonormality

$$N(\psi) = \sum_i |c_i|^2. \tag{120}$$

9.4 Born Rule

The probability of measuring the state ϕ_i is therefore

$$P_i = |c_i|^2. \tag{121}$$

Thus the Born rule follows directly from the norm structure of the octonion algebra after projection to the complex subspace.

9.5 Interpretation

The probabilistic interpretation of quantum mechanics arises from three algebraic ingredients:

- the normed division structure of the octonions
- the projection $\mathbb{O} \to \mathbb{H} \to \mathbb{C}$
- the Hilbert space expansion of the projected field

Hence the Born rule is not an independent postulate but follows from the underlying algebraic structure.

10 Quantum Superposition from Nonassociativity

One of the most distinctive properties of the octonion algebra is its nonassociativity. In this section we investigate how this algebraic property can lead naturally to quantum superposition.

10.1 Associativity and Nonassociativity

For associative algebras the multiplication satisfies

$$(xy)z = x(yz). \tag{122}$$

However, the octonion algebra violates this property. The deviation from associativity is measured by the associator

$$A(x, y, z) = (xy)z - x(yz). \tag{123}$$

For general octonions we have

$$A(x, y, z) \neq 0. \tag{124}$$

This implies that different multiplication orderings correspond to distinct algebraic operations.

10.2 Multiple Dynamical Paths

Consider a dynamical evolution in which three field components interact.

The octonionic multiplication allows two distinct compositions:

$$\Psi_A = (\Psi_1\Psi_2)\Psi_3 \tag{125}$$

$$\Psi_B = \Psi_1(\Psi_2\Psi_3). \tag{126}$$

Because the algebra is nonassociative these two expressions are generally different.

Thus the evolution of the field may proceed along multiple algebraic paths.

10.3 Lemma: Associator as Path Difference

Lemma.

The difference between the two evolution paths is precisely the associator

$$\Psi_A - \Psi_B = A(\Psi_1, \Psi_2, \Psi_3). \tag{127}$$

Proof.

Using the definition

$$A(x, y, z) = (xy)z - x(yz)$$

the statement follows immediately.

□

10.4 Linear Combination of Paths

In a dynamical system governed by the master equation

$$\nabla^2\Psi + \lambda A(\Psi, \Psi, \Psi) + \kappa R\Psi = 0 \tag{128}$$

both multiplication paths appear simultaneously.

Therefore the general solution can be written as

$$\Psi = a\Psi_A + b\Psi_B \tag{129}$$

with coefficients a, b determined by boundary conditions.

10.5 Theorem: Emergence of Superposition

Theorem.

The nonassociativity of the octonion algebra naturally produces linear superpositions of dynamical states.

Proof.

Because octonion multiplication admits multiple orderings, the dynamical evolution generates multiple algebraic outcomes Ψ_A and Ψ_B.

The general solution of the field equation must therefore contain linear combinations of these outcomes

$$\Psi = a\Psi_A + b\Psi_B.$$

This is precisely the structure of quantum superposition.

□

10.6 Relation to Quantum Interference

When the octonionic field is projected onto a complex subspace

$$\mathbb{O} \to \mathbb{H} \to \mathbb{C} \tag{130}$$

the state becomes a complex wavefunction

$$\psi = a\psi_A + b\psi_B. \tag{131}$$

The probability density becomes

$$|\psi|^2 = |a|^2|\psi_A|^2 + |b|^2|\psi_B|^2 + 2\mathrm{Re}(a^* b \psi_A^* \psi_B). \tag{132}$$

The cross term represents quantum interference.

Thus interference arises from the coexistence of multiple nonassociative multiplication paths.

10.7 Generalization to Multi-Path Systems

The same mechanism extends naturally to larger systems.

For n interacting fields there exist many different multiplication orderings.

Each ordering defines a distinct algebraic path

$$\Psi_i. \tag{133}$$

The general solution becomes

$$\Psi = \sum_i c_i \Psi_i. \tag{134}$$

This reproduces the standard Hilbert space structure of quantum mechanics.

10.8 Conceptual Interpretation

In this framework quantum superposition does not arise as a fundamental postulate.

Instead it emerges from the algebraic structure of the octonion multiplication.

The chain of concepts can therefore be summarized as

$$\text{nonassociativity} \rightarrow \text{multiple algebraic paths} \rightarrow \text{linear combinations} \rightarrow \text{quantum superposition.} \tag{135}$$

This perspective suggests that the origin of quantum behavior may lie in the nonassociative nature of the underlying algebra.

10.9 Implications

If this interpretation is correct, the characteristic features of quantum theory — including superposition and interference — would ultimately arise from the algebraic structure of the octonions.

This provides a possible explanation for why quantum mechanics possesses a linear Hilbert space structure despite being based on nonlinear underlying dynamics.

10.10 Decoherence from Nonassociative Interactions

In realistic physical systems the idealized superposition of states is often destroyed by interactions with the surrounding environment. This process is known as decoherence.

Within the octonionic framework decoherence can be interpreted as a dynamical consequence of the nonassociative interaction structure.

10.11 Environment Coupling

Consider a system field Ψ_S interacting with an environment field Ψ_E.

The total field can be written as

$$\Psi = \Psi_S + \Psi_E. \tag{136}$$

Because of the nonassociative multiplication of octonions, the associator generates additional interaction terms

$$A(\Psi, \Psi, \Psi) = A(\Psi_S, \Psi_S, \Psi_S) + A(\Psi_S, \Psi_S, \Psi_E) + A(\Psi_S, \Psi_E, \Psi_E) + A(\Psi_E, \Psi_E, \Psi_E). \tag{137}$$

The mixed associator terms couple the system to the environment.

10.12 Suppression of Interference

Consider a superposition state

$$\Psi_S = a\Psi_A + b\Psi_B. \tag{138}$$

The probability density contains an interference term

$$I = 2\mathrm{Re}(a^* b \Psi_A^* \Psi_B). \tag{139}$$

When the system interacts with the environment, the mixed associator contributions introduce fluctuating phases.

The interference term becomes

$$I \to 2\mathrm{Re}\left(a^* b \Psi_A^* \Psi_B e^{i\theta_E}\right) \tag{140}$$

where θ_E depends on the environmental degrees of freedom.

Averaging over many environmental states leads to

$$\langle e^{i\theta_E} \rangle \approx 0. \tag{141}$$

Thus the interference term vanishes:

$$I \to 0. \tag{142}$$

10.13 Emergence of Classical Probabilities

After decoherence the probability density reduces to

$$|\psi|^2 = |a|^2 |\psi_A|^2 + |b|^2 |\psi_B|^2. \tag{143}$$

The cross terms disappear, leaving a classical statistical mixture.

10.14 Physical Interpretation

In the octonionic model decoherence arises from the coupling between the system and environmental degrees of freedom through the nonassociative interaction term.

The sequence of processes can therefore be summarized as

$$\text{nonassociativity} \to \text{multiple paths} \to \text{superposition} \to \text{environment coupling} \to \text{decoherence}. \tag{144}$$

Thus the same algebraic mechanism that generates quantum superposition also provides a natural explanation for the loss of coherence in macroscopic systems.

10.15 Macroscopic Limit

In macroscopic systems the number of environmental degrees of freedom is extremely large.

The mixed associator terms therefore produce rapidly fluctuating phases which effectively suppress all interference effects.

As a result the system behaves classically, despite being governed by an underlying nonassociative quantum dynamics.

11 Master Equation of the Octonionic Theory

We now summarize the dynamics of the theory in a single fundamental equation.

11.1 Variational Origin

The action of the theory is

$$S = \int d^4x\sqrt{-g}\left(R + \alpha|\nabla\Psi|^2 - \lambda|A(\Psi,\Psi,\Psi)|^2\right). \tag{145}$$

Variation with respect to the octonion field yields

$$\nabla^2\Psi + \lambda\frac{\partial}{\partial\Psi}|A(\Psi,\Psi,\Psi)|^2 = 0. \tag{146}$$

In curved spacetime the coupling to curvature generates an additional term proportional to the Ricci scalar.

11.2 Master Equation

Including the curvature coupling gives the fundamental dynamical equation

$$\nabla^2\Psi + \lambda A(\Psi,\Psi,\Psi) + \kappa R\Psi = 0. \tag{147}$$

This equation will be referred to as the **octonionic master equation**.

11.3 Structure of the Equation

The equation contains three different contributions.

- propagation term

  $$\nabla^2\Psi \tag{148}$$

  which governs the kinetic dynamics of the field.

- nonassociative interaction

  $$A(\Psi,\Psi,\Psi) \tag{149}$$

  which originates directly from the octonion algebra.

- gravitational coupling

  $$R\Psi \tag{150}$$

  which links the field to spacetime curvature.

11.4 Emergent Physical Theories

Different limits of the master equation reproduce the known fundamental equations of physics.

Einstein Gravity Variation of the action with respect to the metric yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}. \tag{151}$$

Dirac Equation Projection of Ψ onto quaternionic subalgebras produces spinor fields obeying

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{152}$$

Schrödinger Equation The nonrelativistic limit leads to

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m}\nabla^2 \psi. \tag{153}$$

11.5 Superposition in the Master Equation

An important consequence of the octonionic master equation is the natural emergence of quantum superposition.

The master equation reads

$$\nabla^2 \Psi + \lambda A(\Psi, \Psi, \Psi) + \kappa R \Psi = 0. \tag{154}$$

The crucial term is the associator

$$A(\Psi, \Psi, \Psi) = (\Psi\Psi)\Psi - \Psi(\Psi\Psi). \tag{155}$$

Because the octonion algebra is nonassociative, the two multiplication orders appearing in the associator correspond to distinct dynamical contributions.

11.6 Multiple Evolution Channels

The associator generates two independent evolution channels

$$\Psi_A = (\Psi\Psi)\Psi \tag{156}$$

$$\Psi_B = \Psi(\Psi\Psi). \tag{157}$$

Inserting these terms into the master equation shows that the field dynamics involve both contributions simultaneously.

Therefore the general solution of the equation takes the form

$$\Psi = a\Psi_A + b\Psi_B. \tag{158}$$

This structure is mathematically identical to the superposition principle of quantum mechanics.

11.7 Projection to the Quantum Wavefunction

After projection onto the complex subspace

$$\mathbb{O} \to \mathbb{H} \to \mathbb{C} \tag{159}$$

the octonionic field becomes a complex wavefunction

$$\psi = a\psi_A + b\psi_B. \tag{160}$$

The corresponding probability density becomes

$$|\psi|^2 = |a|^2|\psi_A|^2 + |b|^2|\psi_B|^2 + 2\mathrm{Re}(a^* b \psi_A^* \psi_B). \tag{161}$$

The last term represents quantum interference.

11.8 Interpretation

The appearance of superposition therefore follows directly from the structure of the master equation.

The associator term introduces multiple algebraic evolution paths, and the general solution becomes a linear combination of these paths.

This suggests the following conceptual chain:

$$\text{octonion algebra} \rightarrow \text{nonassociativity} \rightarrow \text{multiple dynamical channels} \rightarrow \text{superposition}. \tag{162}$$

Thus the superposition principle of quantum mechanics may originate from the nonassociative nature of the underlying algebraic structure.

11.9 Conceptual Interpretation

The master equation suggests that the fundamental structure of physics is encoded in the nonassociative algebra of octonions.

From this algebraic principle emerge

- spacetime geometry
- gravitational dynamics
- quantum wave mechanics
- probabilistic interpretation
- superposition

Thus the octonionic master equation provides a unified mathematical framework connecting geometry and quantum theory.

12 Conclusion

In this work we have proposed a theoretical framework in which the structure of spacetime geometry and quantum dynamics emerges from the nonassociative algebra of octonions.

The central assumption of the model is the existence of a fundamental octonion-valued field

$$\Psi : M \rightarrow \mathbb{O} \tag{163}$$

defined on a spacetime manifold M. The dynamics of this field are governed by a variational principle whose Lagrangian contains three essential components:

- the Ricci scalar R describing spacetime curvature,
- a kinetic propagation term for the octonionic field,
- a nonlinear interaction determined by the octonionic associator

$$A(a, b, c) = (ab)c - a(bc). \tag{164}$$

The resulting theory can be summarized by the master equation

$$\nabla^2\Psi + \lambda A(\Psi, \Psi, \Psi) + \kappa R\Psi = 0. \tag{165}$$

This equation encodes the interaction between algebraic structure, field dynamics and spacetime curvature.

12.1 Emergence of Spacetime Geometry

A central result of the framework is that the geometry of spacetime arises from the algebraic structure of the octonions.

The imaginary octonions define a seven-dimensional vector space equipped with a natural G_2 three-form

$$\varphi(x, y, z) = \langle x, yz\rangle. \tag{166}$$

This three-form uniquely determines a metric tensor, establishing a direct link between the octonion algebra and spacetime geometry.

In this sense the geometric structure of spacetime can be interpreted as emerging from the underlying algebra.

12.2 Emergence of Quantum Dynamics

Projection of the octonionic field onto quaternionic and complex subspaces leads naturally to the appearance of quantum wavefunctions.

Within this framework

- spinor fields arise from quaternionic subalgebras,
- the Dirac equation appears as the relativistic propagation equation of the projected field,
- the Schrödinger equation emerges in the nonrelativistic limit.

The probabilistic interpretation of quantum mechanics is not introduced as an independent postulate but follows from the norm structure of the octonion algebra, which directly leads to the Born rule.

12.3 Role of Nonassociativity

The most distinctive feature of the octonion algebra is its nonassociativity.

In the present theory this property is encoded in the associator term of the Lagrangian.

The associator introduces nonlinear self-interactions that are absent in associative algebras.

These interactions provide a natural algebraic origin for phenomena typically associated with quantum theory, including interference and state superposition.

Thus the framework suggests that the fundamental source of quantum behavior may lie in the nonassociative structure of the underlying algebra.

12.4 Outlook

Several important questions remain open and require further investigation.

Among the most important directions for future work are

- the emergence of gauge interactions from the symmetry structure of the octonions,
- the relation between the octonionic framework and the Standard Model gauge group,
- quantization of the octonionic field theory,
- cosmological solutions of the master equation.

In particular the exceptional Lie group G_2 and its relation to internal symmetry groups suggests that the octonionic structure may provide a natural mathematical setting for unified field theories.

12.5 Final Remarks

The framework proposed in this work suggests that several fundamental structures of modern physics — spacetime geometry, fermionic fields, quantum wave dynamics and probabilistic interpretation — may share a common mathematical origin in the nonassociative algebra of octonions.

If confirmed, this perspective would indicate that the deep unity of physical law is rooted not only in symmetry principles but also in the algebraic structure of the number systems underlying physical theory.

13 Possible Experimental Tests of the Octonionic Framework

A fundamental requirement of any physical theory is the existence of experimental predictions. The octonionic framework proposed in this work leads to several potential observable consequences.

In particular the theory predicts physical effects related to

- nonassociative corrections to quantum interference
- geometric couplings between internal algebra and spacetime curvature
- deviations from standard quantum dynamics at very high energies

We discuss several possible experimental tests.

13.1 Nonassociative Quantum Interference

The central algebraic property of octonions is nonassociativity:

$$(\Psi_1\Psi_2)\Psi_3 \neq \Psi_1(\Psi_2\Psi_3). \tag{167}$$

In the present framework this property leads to multiple dynamical evolution paths.

In quantum mechanics this could manifest as small corrections to interference amplitudes.

Experimental Setup A possible test is a generalized multi-path interferometer extending the standard double-slit experiment.

Instead of two paths, three coherent paths are used:

$$\Psi = a\Psi_1 + b\Psi_2 + c\Psi_3. \tag{168}$$

The octonionic structure predicts corrections to the interference term

$$I_{123} \neq 0. \tag{169}$$

Such third-order interference terms have already been experimentally tested in quantum optics. Detecting a nonzero value would signal deviations from standard quantum mechanics.

13.2 Spinor Structure and Internal Symmetry

The theory predicts that fermionic spinor fields originate from quaternionic subalgebras of the octonions.

This suggests that internal symmetry structures may be linked to the geometry of octonion multiplication.

Possible experimental signatures include

- relationships between fermion generations
- constraints on Yukawa couplings
- symmetry patterns in particle mass spectra

High precision measurements at particle colliders could test such relations.

13.3 Curvature Coupling Effects

The master equation contains a curvature coupling term

$$\kappa R\Psi. \tag{170}$$

This predicts that quantum wave dynamics may be slightly modified in strongly curved spacetime.

Possible environments include

- neutron star gravitational fields
- black hole accretion disks
- gravitational wave backgrounds

Observations of quantum systems in gravitational fields may therefore reveal small deviations from standard theory.

13.4 Cosmological Signatures

Because the octonionic interaction is nonlinear, it may contribute to early universe dynamics.
Possible consequences include

- modified inflation dynamics
- nonstandard primordial fluctuations
- additional scalar modes in cosmological perturbations

Such effects could be tested through measurements of the cosmic microwave background.

13.5 High-Energy Corrections

At energies approaching the fundamental scale of the theory, the nonassociative interaction term

$$A(\Psi, \Psi, \Psi) \tag{171}$$

may produce observable corrections to particle interactions.
Possible signatures include

- deviations from Standard Model scattering amplitudes
- modified dispersion relations
- new resonance structures

Future high-energy collider experiments may probe these regimes.

13.6 Summary of Experimental Tests

The octonionic theory can be tested through several complementary experimental directions:

- multi-path quantum interference experiments
- particle physics measurements
- quantum systems in gravitational fields
- cosmological observations

Confirmation of any of these signatures would provide strong evidence for an underlying octonionic structure of physical law.

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