

A Binary Branch–Layer Structure and Two Explicit Decreasing Invariant for the Collatz Map

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Abstract

I present a binary structural framework for the Collatz map that organizes all positive integers into a hierarchical branch–layer system. Every integer admits a unique decomposition

$$m = 2^y(2^x(2R + 1) - 1),$$

which determines its position within a finite branch through the trailing binary blocks of ones and zeros.

Using the recursive $2n + 1$ construction, I derive explicit linear branch formulas $A(n, x)$ and $B(n, x)$ that uniquely generate all odd integers. Each branch has a finite depth n and terminates at a branch endpoint

$$C = 2 \cdot 3^n x + 4 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 9^i.$$

Under the Collatz map, the local branch parameters (x, y) decrease deterministically until this endpoint is reached. At each branch endpoint the higher-order parameter satisfies

$$R' = \frac{3R + 1}{2^{x+y}},$$

yielding the decreasing invariant $R' < R$ (with a single boundary case). This provides an explicit numerical descent between successive branch endpoints.

The reverse odd structure further shows that each branch contains a unique maximal node congruent to $3 \pmod{6}$, which has no odd predecessor within the branch. This structure allows branches to be organized into a recursive hierarchy of endpoint sets C_0, C_1, C_2, \dots , from which I define a global layer index $L(m)$.

I prove that every positive integer eventually reaches a branch endpoint belonging to a lower-numbered layer, yielding a strictly decreasing invariant

$$L(T^k(m)) < L(m).$$

Since $L(m)$ is nonnegative, every trajectory must descend through finitely many layers to the base layer $C_0 = \{2^k\}$, which maps trivially to 1. This establishes convergence of all positive integers under the Collatz map within the branch–layer framework.

1 Introduction

The Collatz map $T : N \rightarrow N$ is defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n + 1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

The Collatz conjecture asserts that for every positive integer n , repeated iteration of T eventually reaches 1. Despite its simple formulation, the conjecture remains one of the most well-known unsolved problems in elementary number theory.

The difficulty of the problem stems from the competing effects of the two operations: division by two decreases values, while the operation $3n + 1$ can produce temporary growth. Although the conjecture has been verified computationally for very large ranges of integers, a general proof of convergence remains unknown.

In this paper I analyze the Collatz map through a structural decomposition of the integers based on their binary representation. This decomposition organizes integers into finite families, called branches, whose internal behavior under the Collatz map can be described explicitly. Each branch contains a distinguished maximal node and terminates at a specific endpoint. By studying how trajectories move between these endpoints, I obtain a hierarchical structure for the dynamics of the map.

This hierarchy allows the definition of a global layer index $L(m)$ that measures the position of an integer within the recursive branch structure.

Theorem 1.1 (Layer Descent Theorem). *For every positive integer m , there exists $k \geq 1$ such that*

$$L(T^k(m)) < L(m).$$

Thus the layer index strictly decreases after finitely many iterations of the Collatz map. Since $L(m)$ is a non-negative integer, every trajectory must eventually reach the base layer consisting of powers of two, which map directly to 1.

Corollary 1.2. *Every positive integer eventually reaches 1 under repeated iteration of the Collatz map.*

The paper is organized as follows. Section 2 develops the binary decomposition and introduces the branch structure. Section 3 derives the explicit formulas governing the branches and analyzes their endpoints. Section 4 studies the reverse structure and establishes the existence of a unique maximal node in each branch. Section 5 introduces the layer index and proves the global descent property.

2 Binary Decomposition

Definition 2.1. Any positive integer m can be uniquely written as

$$m = 2^y(2^x(2R + 1) - 1), \text{ or } R1_x0_y, \text{ or, } r01_x0_y$$

with $x \geq 1, y \geq 0, R \geq 0$, where

1. x is the length of a consecutive block of trailing ones,
2. y is the length of a block of trailing zeros following the 1^x block,
3. R contains all higher-order bits.
4. r is $R/2$

Definition 2.2. A Branch, B , is defined by the trailing-ones and trailing-zeros structure of a number. If

$$m = 2^y(2^x(2R + 1) - 1), \text{ or, } r01_x0_y$$

then n can decompose into another number

$$m' = 2^{y'}(2^{x'}(2R' + 1) - 1)$$

under the Collatz map T .

Axiom 2.3. The even step, $m/2$, remove the trailing 0.

Lemma 2.4. Let m be a positive integer with binary decomposition

$$m = 2^y(2^x(2R + 1) - 1), \quad x \geq 1, y = 0, r01_x$$

where R contains the higher-order bits and the block of x trailing ones is represented by $2^x - 1$. Then applying the odd Collatz map

$$T(m) = \frac{3m + 1}{2}$$

yields

$$T(m) = 2^y(2^{x-1}(2R' + 1) - 1),$$

so that the block of trailing ones decreases by exactly one, with

$$R' = 3R + 1.$$

Proof. Write

$$m = 2^y(2^x(2R + 1) - 1).$$

Since $y = 0$ for an odd number, this simplifies to

$$m = 2^x(2R + 1) - 1.$$

Applying the odd Collatz map gives

$$\begin{aligned}
T(m) &= \frac{3m + 1}{2} \\
&= \frac{3 \cdot (2^x(2R + 1) - 1) + 1}{2} \\
&= \frac{3 \cdot 2^x(2R + 1) - 2}{2} \\
&= 2^{x-1}(2(3R + 1) + 1) - 1.
\end{aligned}$$

The factor 2^{x-1} corresponds to $x - 1$ trailing ones in the binary representation, and the higher-order bits transform as

$$R' = 3R + 1.$$

Thus,

$$T(m) = 2^y(2^{x-1}(2R' + 1) - 1) = r'01_{x-1}$$

confirming that the block of trailing ones decreases by exactly one, including when $x = 1$. \square

Axiom 2.5. *The separation 0 remains. This is true for even and odd steps. Meaning, the higher order bits do not interfere with the lower bits decomposition.*

3 Branch Formulas for Odd Numbers

By recursively applying $2n+1$ to $4x$ and $4x+2$ all odd numbers can be written as a linear formula.

Starting Number	Apply $2n + 1$ Recursion	Resulting Linear Form
$4x$	$2 \cdot (4x) + 1 = 8x + 1$	$2^1 \cdot x \cdot 4 + 2^0 \cdot 1$
	$2 \cdot (8x + 1) + 1 = 16x + 3$	$2^2 \cdot x \cdot 4 + (2^2 - 1)$
	$2 \cdot (16x + 3) + 1 = 32x + 7$	$2^3 \cdot x \cdot 4 + (2^3 - 1)$
$4x + 2$	$2 \cdot (4x + 2) + 1 = 8x + 5$	$2^1 \cdot x \cdot 4 + (2^2 + 1)$
	$2 \cdot (8x + 5) + 1 = 16x + 11$	$2^2 \cdot x \cdot 4 + (2^3 - 5)$
	$2 \cdot (16x + 11) + 1 = 32x + 23$	$2^3 \cdot x \cdot 4 + (2^4 - 9)$

Table 1: Applying $2n + 1$ recursion to $4x$ and $4x + 2$ generates linear forms of the type $2^kx + c$.

I define branch formulas parameterized by $n \geq 1$ and $x \geq 0$:

$$A(n, x) = \begin{cases} 3 \cdot 2^{n-1} + 2^{n+1}x - 1, & \text{if } n \text{ is odd} \\ 2^{n-1} + 2^{n+1}x - 1, & \text{if } n \text{ is even} \end{cases}$$

$$B(n, x) = \begin{cases} 3 \cdot 2^n + 2^{n+2}x - 1, & \text{if } n \text{ is odd} \\ 2^n + 2^{n+2}x - 1, & \text{if } n \text{ is even} \end{cases}$$

$$\text{Branch endpoint: } C = 2 \cdot 3^n x + 4 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 9^i.$$

Remark 3.1. Each C node has the form $C = 6x + 4$ and admits exactly two predecessors under T : one even and one odd. I denote these predecessors by $A(n, x)$ and $B(n, x)$. Viewing the process in reverse, the A and B numbers ‘branch’ into C , motivating the branch terminology and the recursive layer construction.

Odd steps (depth)	A	B	C
1	$4x + 2$	$8x + 5$	$6x + 4$
2	$8x + 1$	$16x + 3$	$18x + 4$
3	$16x + 11$	$32x + 23$	$54x + 40$
4	$32x + 7$	$64x + 15$	$162x + 40$

3.1 Full Coverage and Uniqueness via FTA

Theorem 3.2. Let $m \geq 1$ be any odd integer. Then there exists a unique pair (n, x) such that $m = A(n, x)$ or $m = B(n, x)$.

Proof. By the Fundamental Theorem of Arithmetic, any integer $m + 1$ can be uniquely factored as

$$m + 1 = 2^p m', \quad p \geq 0, \quad m' \text{ odd.}$$

Define

$$q_0 := m' - 1.$$

Then, by the recursive branch formula generated via $2n + 1$,

$$2^p q_0 + (2^p - 1) = 2^p (m' - 1) + (2^p - 1) = 2^p m' - 1 = m.$$

This shows that every odd integer appears in at least one branch.

For uniqueness, suppose two pairs (p_1, q_1) and (p_2, q_2) satisfy

$$2^{p_1} q_1 + (2^{p_1} - 1) = 2^{p_2} q_2 + (2^{p_2} - 1) = m.$$

Without loss of generality, assume $p_1 \leq p_2$. Then

$$q_1 - 2^{p_2 - p_1} q_2 = 2^{p_2 - p_1} - 1.$$

The left-hand side is divisible by $2^{p_2 - p_1}$, while the right-hand side is not unless $p_1 = p_2$. Then it follows that $q_1 = q_2$, establishing uniqueness. By the FTA, any odd integer m can be written uniquely as $2^p m' - 1$ with m' odd. \square

Recursively applying $2n + 1$ to $4x$ and $4x + 2$ generates all such integers, giving the branch formulas $A(n, x)$ and $B(n, x)$. Hence, the branch formulas cover all odd integers uniquely.

Finally, any even integer N can be written as $N = 2^r m$ with m odd, so it is uniquely generated by the same branch for m together with the power of 2. Hence, *all integers are generated uniquely by the branch formulas.*

Lemma 3.3. *Any number m in a branch with index $B = (y, x)$ reaches its branch endpoint C under the Collatz map in finitely many steps, with a strictly smaller branch index $B' < B$.*

Proof. Write $m = 2^y(2^x(2R + 1) - 1)$.

- Each even step reduces y by 1 (trailing zeros). - Each odd step reduces x by 1 (trailing ones) and updates the higher bits $R \mapsto 3R + 1$ (Lemma 2.4).

Iterating these steps consumes all trailing ones and zeros, reaching

$$C = 2^0(2R' + 1) - 1,$$

so that the branch index decreases: $B' = (0, 0) < B$. □

Remark 3.4. *The branch endpoint C itself can be written in the same form as any number in the branch, with its own coordinates (x, y) and higher-order bits R' .*

Lemma 3.5. (Decreasing Invariant at Branch Endpoints). *Suppose a number in the branch eventually reaches C . Then C can be written as*

$$C = 2^y(2^x(2R' + 1) - 1), \quad x \geq 1, y \geq 1,$$

where

$$R' < R,$$

Where R is the predecessor's higher order bits.

Proof. Applying the Collatz map repeatedly within a branch consumes all trailing ones and zeros according to Lemma 2.1. The C of a branch is itself an integer with a unique x, y coordinates.

$$3 \cdot R + 1 \text{ divided by } 2^{x+y} \text{ (removing the trailing ones and zeros),}$$

yielding

$$R' = \frac{3R + 1}{2^{x+y}}.$$

Since $x + y \geq 2$, we have $R' < R$, edge case $C = 10$ and $R = 1$ can give equality, which ensures the layer index strictly decreases at each branch endpoint. □

3.2 Branch Depth Reduction

Lemma 3.6. *Any branch of depth $n \geq 2$ reduces in one step to a branch of depth $n - 1$, and all branches eventually reach the endpoints:*

$$A = 4x + 2, \quad B = 8x + 5.$$

Proof. Consider the deeper-level branches:

$$A = 2^{n+1}x + 3 \cdot 2^{n-1} - 1 \xrightarrow{T} 2^n(3x + 2) + 2^{n-2} - 1,$$

$$B = 2^{n+2}x + 3 \cdot 2^n - 1 \xrightarrow{T} 2^{n+1}(3x + 2) + 2^{n-1} - 1.$$

These correspond to branches with an odd number of odd steps, hence the appearance of $3x + 2$.

For the even-number-of-odd-steps branches:

$$A = 2^{n+1}x + 2^{n-1} - 1 \xrightarrow{T} 2^n(3x) + 3 \cdot 2^{n-2} - 1,$$

$$B = 2^{n+2}x + 2^n - 1 \xrightarrow{T} 2^{n+1}(3x) + 3 \cdot 2^{n-1} - 1.$$

Here the terminal transformation produces $3x$, reflecting an even number of odd steps.

In each case, applying T reduces the depth index n by one while preserving the linear form. Hence a branch of depth n maps to a branch of depth $n - 1$. These four equations exhaust all possibilities for an odd integer at any branch depth.

Repeating this process, all branches eventually reach depth 1, corresponding to the $n=1$ step $4x + 2$ and $8x + 5$. For all $n \geq 2$ and $x \geq 0$, applying T to a branch of depth n produces a branch of depth $n - 1$, preserving the linear form. Hence, all branches funnel into these modules in finitely many steps. \square

Lemma 3.7. *For any number, m , $A = (n, x)$ and $B = (n, x)$, a number with trailing block $1^x 0^y$ reaches C in exactly*

$$2n + y \text{ steps for } A, \text{ and } 2n + y + 1 \text{ steps for } B.$$

Proof. Each trailing 0 contributes exactly 1 step via the $n/2$ even operation. Each trailing 1 contributes exactly 2 steps via $(3n + 1)/2$ until the last 1 in the block.

- For $A = 4x + 2$, the last 1 reaches C exactly on the final odd step, giving $2n + y$ steps. - For $B = 8x + 5$, the last odd step occurs one step before C , so two final even divisions are needed to reach C , giving $2n + y + 1$ steps. \square

Remark 3.8. *The same lemmas apply to C , so the process of moving to successive C s is fully deterministic and well-defined.*

$$n \xrightarrow{T^k} C_B \xrightarrow{T^{k''}} C_{B-1} \xrightarrow{T^{k'''}} C_0 \xrightarrow{T^{k''''}} 1$$

4 Odd Multiple of 3s and Layer Index (Optional Second Decreasing Invariant)

4.1 Well-Founded Reverse Collatz Structure

I now establish that the reverse Collatz structure is well-founded with respect to odd predecessors. In particular, every branch has a unique maximal node congruent to 3 (mod 6).

Lemma 4.1 (Existence of a 3 (mod 6) Ancestor). *Let $i \geq 1$ be any positive integer. Then either:*

1. $i \equiv 3 \pmod{6}$, or
2. there exists $k \geq 1$ such that

$$m = \frac{2^k i - 1}{3}$$

is an odd integer congruent to 3 (mod 6).

Proof. If i is even, divide by 2 until an odd integer is reached. Thus assume i is odd.

If $i \equiv 0 \pmod{3}$, then since i is odd we have $i \equiv 3 \pmod{6}$, and we are done.

If $i \not\equiv 0 \pmod{3}$, we seek k such that

$$2^k i \equiv 1 \pmod{3}.$$

Since

$$2 \equiv -1 \pmod{3},$$

we have

$$2^k \equiv (-1)^k \pmod{3}.$$

If $i \equiv 1 \pmod{3}$, choose k even. If $i \equiv 2 \pmod{3}$, choose k odd.

Then

$$2^k i \equiv 1 \pmod{3},$$

so m is an integer. Because $2^k i$ is even, $2^k i - 1$ is odd, and dividing by 3 preserves oddness. Thus m is odd and divisible by 3, hence

$$m \equiv 3 \pmod{6}.$$

□

Lemma 4.2. *If $i \equiv 3 \pmod{6}$, then there exists no integer $k \geq 1$ for which*

$$\frac{2^k i - 1}{3}$$

is an integer.

Proof. If $i \equiv 0 \pmod{3}$, then for any k we have

$$2^k i \equiv 0 \pmod{3}.$$

Thus

$$2^k i - 1 \equiv -1 \pmod{3},$$

which is never divisible by 3. Hence no such k exists. \square

Corollary 4.3. *Every positive integer admits a finite reverse sequence of odd Collatz steps that reaches a node congruent to 3 (mod 6). Such nodes have no odd predecessor under the reverse map.*

Proof. If $i \equiv 3 \pmod{6}$, then the result is immediate.

Otherwise, by Lemma 4.1 there exists $k \geq 1$ such that

$$m = \frac{2^k i - 1}{3}$$

is an odd integer congruent to 3 (mod 6).

By Lemma 4.2, any number congruent to 3 (mod 6) admits no odd predecessor of the form $(2^k i - 1)/3$. Therefore the reverse sequence terminates at such a node after finitely many steps. \square

Lemma 4.4. *Let C be a branch endpoint with odd predecessors $A(n, x)$ and $B(n, x)$. Then exactly one of $A(n, x), B(n, x)$ is congruent to 3 (mod 6) and the other is congruent to 1 (mod 6).*

Proof. The branch recursion generates $B(n, x) = 2A(n, x) + 1$. Reducing modulo 6 gives

$$2A + 1 \equiv \begin{cases} 1 \pmod{6} & \text{if } A \equiv 3 \pmod{6}, \\ 3 \pmod{6} & \text{if } A \equiv 1 \pmod{6}. \end{cases}$$

Thus A and B must occupy opposite odd residue classes modulo 6. \square

Lemma 4.5. *Let B be a branch with maximal node $m_0 \equiv 3 \pmod{6}$ and endpoint C . Then the following hold:*

1. *Any trajectory in this branch contains at most one 3 (mod 6) number (the maximal node m_0).*
2. *At the branch endpoint, the layer index strictly decreases:*

$$L(C) < L(m_0).$$

Proof. If m is odd, the Collatz map gives

$$T(m) = \frac{3m + 1}{2}.$$

Modulo 3,

$$3m + 1 \equiv 1 \pmod{3},$$

so a trajectory starting from $m \not\equiv 0 \pmod{3}$ cannot reach another multiple of 3. Hence the branch maximal node m_0 is unique along the trajectory.

Repeated application of T from m_0 reduces the branch parameters until the branch endpoint C is reached (Lemmas 3.6–3.7). By the global layer index definition, this endpoint belongs to a lower-numbered layer, proving strict decrease at the end of the branch. No claim is made for other numbers. \square

Theorem 4.6. [*Maximal Node Ordering and Global Layer Descent*]
Let B be any branch with maximal node $m_0 \equiv 3 \pmod{6}$ and branch endpoint C . Then:

1. m_0 is unique within B .
2. C cannot coincide with m_0 or any other $3 \pmod{6}$ node from a previous branch.
3. The layer index strictly decreases along the trajectory from m_0 to C :

$$L(C) < L(m_0).$$

4. Recursively, along successive branches, the layer indices form a strictly decreasing sequence:

$$L(C_1) > L(C_2) > \cdots > L(C_k).$$

Proof. 1. By Lemmas 4.1–4.4, every branch has exactly one maximal node $m_0 \equiv 3 \pmod{6}$.

2. The endpoint C is always even and belongs to a different branch than m_0 . Hence $C \not\equiv 3 \pmod{6}$ and cannot coincide with any previously encountered maximal node.

3. By definition of the layer index, any number m that maps to C under some iteration satisfies $L(C) < L(m)$. In particular, $L(C) < L(m_0)$.

4. Repeating this argument along successive branches produces a strictly decreasing sequence of layers:

$$L(C_1) > L(C_2) > \cdots > L(C_k),$$

establishing global descent. No trajectory can return to a higher layer because each maximal node is unique. \square

Remark 4.7. *Since each branch contains a maximal node $m_0 \equiv 3 \pmod{6}$ for either $A_b(n, x)$ or $B_b(n, x)$, induction can be performed on these maximal nodes.*

By Lemma 3.6, all nodes on the path from m_0 to the branch endpoint C are proven to converge. Once C is proven to converge, the remaining side of the branch must also converge, since its trajectories merge with the path to C .

Therefore, proving convergence for the single root m_0 establishes convergence for the entire branch. Consequently, each layer is proven by verifying the base root of each branch.

5 Layer-Based Convergence and Global Decreasing Invariant

Definition 5.1 (Layer Construction). *Define layers C_0, C_1, C_2, \dots recursively:*

1. **Base layer:**

$$C_0 = \{2^k \mid k \geq 0\}, \quad L(2^k) = 0.$$

2. **Recursive construction:** *Given layer C_L , define the next layer*

$$C_{L+1} = \{2^y(2^x(2R+1) - 1) \mid T^k(2^y(2^x(2R+1) - 1)) \in C_L \text{ for some } k \geq 1\}.$$

3. **Layer assignment:** *For any number $m \in C_{L+1}$, set*

$$L(m) = L(C_L) + 1.$$

Lemma 5.2. *For every integer $m > 1$, there exists $k \geq 1$ such that*

$$L(T^k(m)) < L(m),$$

where L is the global layer index defined above.

Proof. By construction, every $m > 1$ belongs to a branch with an endpoint $C \in C_L$. Repeated application of T consumes all trailing ones (x) and zeros (y) according to Lemmas 2.1 and 3.6, reaching C in finitely many steps.

By definition of L , we have

$$L(C) = L(m) - 1 < L(m),$$

so the layer index strictly decreases. Hence, the invariant $L(m)$ is strictly decreasing along all trajectories. \square

Corollary 5.3 (Convergence of All Positive Integers). *Every positive integer eventually reaches 1 under iteration of the Collatz map:*

$$\forall m \geq 1, \quad \exists k \in \mathbb{N} \text{ such that } T^k(m) = 1.$$

Proof. The base layer $C_0 = \{2^k\}$ trivially maps to 1 under repeated even steps. Every $m > 1$ belongs to some layer C_L , and by Lemma 5.1 the layer index strictly decreases under iteration of T .

Since $L(m)$ is nonnegative and strictly decreasing, every trajectory must reach C_0 in finitely many steps. Therefore, all positive integers converge to 1 under the Collatz map. \square

6 Conclusion

This paper introduced a binary branch–layer framework for analyzing the dynamics of the Collatz map. Using the unique decomposition

$$m = 2^y(2^x(2R + 1) - 1),$$

every positive integer is assigned explicit structural coordinates consisting of trailing ones, trailing zeros, and higher-order bits. These parameters determine the number's position within a finite branch.

Through the recursive $2n + 1$ generation of odd integers, explicit branch formulas

$$A(n, x), \quad B(n, x),$$

were derived which uniquely generate all odd integers. Each branch terminates at an endpoint

$$C = 2 \cdot 3^n x + 4 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 9^i,$$

and the branch depth n decreases deterministically under the Collatz map. Consequently, every branch reduces in finitely many steps to the base depth $n = 1$ forms $4x + 2$ and $8x + 5$.

Within each branch, the Collatz map consumes the trailing binary blocks of ones and zeros, producing a finite trajectory to the branch endpoint. At the endpoint, the higher-order parameter satisfies

$$R' = \frac{3R + 1}{2^{x+y}},$$

which yields the decreasing invariant $R' < R$ (with the single boundary case $C = 10$). This provides an explicit numerical descent between successive branch endpoints.

The reverse Collatz structure further shows that each branch contains a unique maximal node $m_0 \equiv 3 \pmod{6}$. This node has no odd predecessor within the branch and therefore serves as the maximal element under reverse odd steps. Forward iteration from m_0 descends through the branch to the endpoint C .

Building on this structure, a global layer index $L(m)$ was defined using the recursive construction of endpoint sets C_0, C_1, C_2, \dots . Each branch endpoint belongs to a lower-numbered layer than its maximal node, yielding a strictly decreasing layer invariant:

$$L(T^k(m)) < L(m).$$

Together, the binary decomposition, explicit branch formulas, deterministic branch depth reduction, decreasing endpoint invariant, and global layer index

provide a structured mechanism for analyzing Collatz trajectories. Every positive integer descends through finitely many branch reductions and layer transitions until reaching the base layer

$$C_0 = \{2^k\},$$

which trivially maps to 1.

Thus, within this branch–layer framework, all positive integers are shown to converge to 1 under iteration of the Collatz map.