Stochastic Derivation of the Dirac Equation: Orbital Quantization, Probability Positivity, and the Spin-Statistics Connection

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Abstract

We extend the stochastic coherence framework (Onsager–Machlup formalism) to the relativistic domain and derive orbital angular momentum quantization as a geometric–topological consequence. In the non-relativistic setting, phase quantization was shown to follow from the combination of the Hamilton–Jacobi constraint and C^∞ regularity of the probability current [1]. Here, the configuration space $\mathbb{R}^n$ is replaced by Minkowski spacetime M^4, the scalar phase S is replaced by a Lorentz rotor field $R \in \mathrm{Spin}^+(1,3)$, and the physical observables become the four-current j^μ and the spin tensor $S^{\mu\nu}$, both bilinear in the rotor via the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

We prove that the C^∞ regularity of these bilinear observables over nodal surfaces, combined with the relativistic stochastic Hamilton–Jacobi equation, forces the orbital winding number around nodal tubes to satisfy $\alpha = 2k$ with $k \in \mathbb{N}$, yielding orbital angular momentum quantized in integer multiples of $\hbar$: $L \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. This extends the Wallström resolution to the relativistic spinorial domain and establishes a strict topological separation between orbital angular momentum (quantized by nodal regularity on the base manifold) and intrinsic spin (determined by the representation of the fiber group $\mathrm{Spin}^+(1,3)$).

The complete variational derivation of the relativistic stochastic dynamics from the Onsager–Machlup action is presented, including the Euler–Poincaré reduction on $\mathrm{Spin}^+(1,3)$ and the chirality sector. The resulting hydrodynamic equations are shown to be equivalent, term by term, to the Takabayasi–Hestenes decomposition of the Dirac equation, completing the derivation: stochastic coherence plus observable regularity implies the Dirac equation with quantized orbital angular momentum.

The derivation is non-circular: the dynamical constraint and the observable regularity requirement are logically independent, and neither alone implies orbital quantization.

Two further results are established. First, the positivity requirement $\rho \geq 0$ (inherent to any probability density) forces the internal orientation group to be $\mathrm{Spin}^+(1,3)$ rather than $\mathrm{SO}^+(1,3)$: the scalar (Klein–Gordon) formulation fails to guarantee a future-directed probability current, while the spinorial formulation does so algebraically. Second, the spin-statistics connection is derived exactly via the Laidlaw–DeWitt decomposition of the many-body propagator: the double covering $\mathrm{Spin}^+(1,3) \to \mathrm{SO}^+(1,3)$ fixes the exchange character to $\chi = -1$ for spinorial representations, yielding Fermi–Dirac statistics without a symmetrization postulate.

1 Introduction and Motivation

In the framework of stochastic coherence [1], it was demonstrated that phase circulation quantization in the Schrödinger regime does not require postulating single-valuedness of a wave function. Instead, quantization emerges from the combination of the stochastic Hamilton–Jacobi equation and the requirement that the probability current $j = \rho \nabla S/m$ be a C^∞ vector field on all of physical space, including at nodal zeros.

The present work extends this principle to the relativistic domain, with two objectives: (i) to derive the complete relativistic stochastic dynamics from the Onsager–Machlup action, recovering the Dirac equation; and (ii) to determine what topological quantization conditions the Wallström mechanism imposes on the rotor field $R \in \mathrm{Spin}^+(1,3)$.

The logical structure parallels the non-relativistic case:

- **Ingredient 1 (Dynamics):** The relativistic stochastic Hamilton–Jacobi equation constrains the vanishing rate of the density near nodal surfaces to the rotational frequency of the spin frame.
- **Ingredient 2 (Observable regularity):** C^∞ smoothness of the bilinear observables j^μ and $S^{\mu\nu}$ forces quantization of the orbital winding number.

The key result is that the nodal regularity mechanism quantizes *orbital* angular momentum, not intrinsic spin. The factor 1/4 in the rotational kinetic energy of the Hamilton–Jacobi equation—intrinsic to the Clifford algebraic structure—ensures that the singular balance gives $|\alpha| = 2k$ (with k the density vanishing order), yielding $L = k \in \mathbb{N}_0$. This establishes a strict topological separation between orbital angular momentum (quantized by base-manifold nodal topology) and intrinsic spin (determined by the fiber group $\mathrm{Spin}^+(1,3)$), resolving a long-standing confusion in the hydrodynamic interpretation of relativistic quantum mechanics [8, 9].

A central contribution of this work is the *complete* variational derivation of the relativistic stochastic dynamics from the Onsager–Machlup action functional. In contrast to previous treatments that proceed from the Dirac equation to the fluid description, we invert the logic: the Dirac equation *emerges* from the stochastic variational principle combined with observable regularity.

Beyond the single-particle theory, we address two foundational questions. First, *why* $\mathrm{Spin}^+(1,3)$ and not $\mathrm{SO}^+(1,3)$? We show that the positivity of the probability density ($\rho \geq 0$) forces the spinorial formulation: the scalar (Klein–Gordon) theory fails to guarantee a future-directed probability current in the relativistic regime. Second, we derive the spin-statistics connection for many-body systems: the Laidlaw–DeWitt decomposition [18] of the propagator on the configuration-space quotient, combined with the double-cover holonomy of $\mathrm{Spin}^+(1,3)$, fixes the exchange character to $\chi = -1$ for spinorial representations. Fermi–Dirac statistics thus follows from $\rho \geq 0$ without a symmetrization postulate.

Throughout, we work in the Spacetime Algebra (STA) formulation of Clifford algebra $\mathrm{Cl}(1,3)$, following Hestenes [4, 5, 6] and Doran–Lasenby [7]. Natural units $c = \hbar = 1$ are used except where clarity requires explicit factors.

2 Mathematical Framework

2.1 Spacetime Algebra

Definition 2.1 (Spacetime Algebra)**.** *The Spacetime Algebra* $\mathrm{Cl}(1,3)$ *is the real Clifford algebra generated by an orthonormal frame* $\{\gamma^\mu\}_{\mu=0}^3$ *satisfying*

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{1}$$

where $\eta = \mathrm{diag}(+1,-1,-1,-1)$ *is the Minkowski metric. The algebra has dimension* $2^4 = 16$ *with basis elements comprising scalars, vectors, bivectors, pseudovectors, and the pseudoscalar* $I = \gamma_0\gamma_1\gamma_2\gamma_3$*, satisfying* $I^2 = -1$.

Definition 2.2 (Lorentz Rotor)**.** *A Lorentz rotor is an even element* $R \in \mathrm{Cl}(1,3)^+$ *satisfying*

$$R\widetilde{R} = \widetilde{R}R = 1 \tag{2}$$

where $\widetilde{R}$ *denotes the reverse (obtained by reversing the order of all geometric products). The group of Lorentz rotors is isomorphic to* $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$*, the double cover of the proper orthochronous Lorentz group* $\mathrm{SO}^+(1,3)$.

Remark 2.3 (Double Covering)**.** *The covering map* $\Lambda : \mathrm{Spin}^+(1,3) \to \mathrm{SO}^+(1,3)$ *is defined by* $\Lambda(R) : v \mapsto Rv\widetilde{R}$ *for any vector* v*. This map is a 2-to-1 surjection:* $\Lambda(R) = \Lambda(-R)$*. In the spatial restriction,* $\mathrm{SU}(2) \to \mathrm{SO}(3)$ *exhibits the same double covering. This algebraic fact underlies the separation between orbital and spin angular momentum.*

Lemma 2.4 (Pseudoscalar Commutation)**.** *In* $\mathrm{Cl}(1,3)$*, the spacetime pseudoscalar* I *satisfies:*

(i) I *commutes with all even-grade elements:* $IA = AI$ *for* $A \in \mathrm{Cl}(1,3)^+$.

(ii) I *anticommutes with all vectors:* $I\gamma^\mu = -\gamma^\mu I$ *for each* μ.

Proof. (i) For any basis bivector $\gamma^\mu\gamma^\nu$: $I(\gamma^\mu\gamma^\nu) = (I\gamma^\mu)\gamma^\nu = (-\gamma^\mu I)\gamma^\nu = \gamma^\mu\gamma^\nu I$. Since bivectors generate $\mathrm{Cl}(1,3)^+$, the result extends to all even elements. (ii) $I\gamma^\mu = \gamma_0\gamma_1\gamma_2\gamma_3\gamma^\mu$. Moving γ^μ past the three basis vectors γ^ν with $\nu \neq \mu$ produces three sign changes: $I\gamma^\mu = (-1)^3\gamma^\mu I = -\gamma^\mu I$. □

2.2 Covariant Stochastic Process

Definition 2.5 (Covariant Diffusion)**.** *Let* τ *denote proper time along the worldline of a fluid element.*

The covariant stochastic process in M^4 is governed by the Itô equation

$$dX^\mu = v^\mu \, d\tau + \sigma \, dW^\mu(\tau) \tag{3}$$

where v^μ is the deterministic four-velocity, $\sigma = \sqrt{2D}$ with D the covariant diffusivity, and $W^\mu(\tau)$ is a four-dimensional Wiener process with correlator

$$\langle dW^\mu \, dW^\nu \rangle = \eta_\perp^{\mu\nu} \, d\tau \tag{4}$$

Here $\eta_\perp^{\mu\nu} = \eta^{\mu\nu} - v^\mu v^\nu / v^2$ projects onto the rest-frame spatial hypersurface.

2.3 Structured Fluid State

In the non-relativistic framework, the state of the probability fluid is specified by (ρ, S). In the relativistic case, the state requires three fields:

Definition 2.6 (Fluid State Fields)**.** *The relativistic stochastic fluid is characterized at each spacetime point x^μ by:*

(i) ***Invariant density:*** *$\rho(x) \geq 0$, a Lorentz scalar.*

(ii) ***Takabayasi angle:*** *$\beta(x) \in [0, \pi]$, a pseudoscalar measuring the chirality of the fluid.*

(iii) ***Lorentz rotor:*** *$R(x) \in \mathrm{Spin}^+(1,3)$, defining the local orientation of the spin frame.*

The Dirac spinor ψ in the standard formulation is related to these fields by

$$\psi = \rho^{1/2} \, e^{I\beta/2} \, R \tag{5}$$

but we do not postulate this decomposition. It will emerge as a consequence of the quantization theorem.

2.4 Physical Observables

Definition 2.7 (Bilinear Observables)**.** *The physical observables are the bilinear tensors:*

(i) Probability four-current:

$$j^\mu = \rho \, v^\mu, \qquad v^\mu = (R\gamma_0\widetilde{R})^\mu \tag{6}$$

(ii) Spin tensor:

$$S^{\mu\nu} = \rho \, (RI\gamma_1\gamma_2\widetilde{R})^{\mu\nu} \tag{7}$$

Both are bilinear in R: they depend on R through the adjoint map $R(\cdot)\widetilde{R}$, which is invariant under $R \to -R$.

Lemma 2.8 (Bilinearity and the Covering Map)**.** *The observables j^μ and $S^{\mu\nu}$ factor through the double covering $\varphi : \mathrm{Spin}^+(1,3) \to \mathrm{SO}^+(1,3)$:*

$$j^\mu(R) = j^\mu(-R), \qquad S^{\mu\nu}(R) = S^{\mu\nu}(-R) \tag{8}$$

Consequently, the observables are insensitive to the $\mathbb{Z}_2$ kernel of the covering map.

Proof. Substituting $R \to -R$ in (6): $(-R)\gamma_0(-\widetilde{R}) = R\gamma_0\widetilde{R}$, since the two minus signs cancel. The same applies to (7). □

3 Necessity of $\mathrm{Spin}^+(1,3)$ from Probability Positivity

Before constructing the Onsager–Machlup action, we establish that the spinorial formulation is not merely a convenient choice but a *necessity* imposed by the positivity of the probability density.

3.1 The positivity requirement

In the OM framework, $\rho(x) \geq 0$ is the probability density of the stochastic process. The conserved current $j^\mu = \rho \, v^\mu$ with $\partial_\mu j^\mu = 0$ requires $j^0 = \rho \, v^0 \geq 0$ pointwise, hence $v^0 > 0$ wherever $\rho > 0$ (future-directed flow).

Proposition 3.1 (Scalar formulation fails)**.** *In a scalar relativistic theory with $v^\mu = \partial^\mu S/m$, the condition $v^0 > 0$ is violated for configurations involving negative-frequency components.*

Proof. For a free particle, $v^0 = E/m > 0$. However, in the Klein paradox regime ($V > m$), or in configurations with pair-production-type mixing of positive and negative frequencies, $\partial_0 S$ changes sign. This is the well-known failure of positive-definite probability in Klein–Gordon theory [19]. The non-relativistic limit ($E \approx m$, $v^0 > 0$ always) does not suffer from this problem, which is why the scalar OM works for the Schrödinger equation [1]. □

Proposition 3.2 (Spinorial formulation works)**.** *In the spinorial theory with $v^\mu = (R\gamma_0\widetilde{R})^\mu$ and $R \in$* $\mathrm{Spin}^+(1,3)$*, the condition $v^0 > 0$ holds identically.*

Proof. The covering map $\Lambda : \mathrm{Spin}^+(1,3) \to \mathrm{SO}^+(1,3)$ sends $R \mapsto \Lambda(R)$ where $\Lambda(R)v = Rv\widetilde{R}$. Since $\mathrm{SO}^+(1,3)$ is the proper orthochronous Lorentz group, it preserves the future light cone. Therefore $(R\gamma_0\widetilde{R})^0 > 0$ for all $R \in \mathrm{Spin}^+(1,3)$. □

Theorem 3.3 (Necessity of the double cover)**.** *The OM framework in M^4 with $\rho \geq 0$ requires the internal orientation to be described by $R \in \mathrm{Spin}^+(1,3)$, not by a scalar phase S alone.*

Proof. Combine Propositions 3.1 and 3.2. □

Remark 3.4 (Minimality of spin-$\frac{1}{2}$)**.** *Among the representations of $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$ admitting a positive-definite conserved current with a first-order equation, the Dirac representation $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ is the minimal one (fewest degrees of freedom). Higher-spin representations (Rarita–Schwinger, etc.) are also consistent but require additional components and constraints. In the OM framework, the minimum-action principle favors the representation with fewest degrees of freedom, selecting spin-$\frac{1}{2}$.*

4 Relativistic Stochastic Dynamics

4.1 The Onsager–Machlup Action

The complete Onsager–Machlup action for the structured relativistic fluid is derived from the Madelung–Takabayasi decomposition of the Dirac Lagrangian in STA form.

Definition 4.1 (Covariant Onsager–Machlup Action)**.** *The action functional for the relativistic stochastic fluid is*

$$\mathcal{A}[\rho, S, R, \beta] = \int d^4x\, \mathcal{L} \tag{9}$$

with Lagrangian density

$$\begin{aligned}\mathcal{L} = &(\partial_\mu\sigma)(\partial^\mu\sigma) + \sigma^2(\partial_\mu S)(\partial^\mu S)\\ &+ \tfrac{1}{4}\sigma^2\langle\Omega_\mu\Omega^\mu\rangle + \tfrac{1}{4}\sigma^2(\partial_\mu\beta)(\partial^\mu\beta)\\ &- \tfrac{1}{2}\sigma^2(\partial_\mu\beta)\, s^\mu_{\mathrm{vort}} - m^2\sigma^2\cos\beta\end{aligned} \tag{10}$$

where $\sigma = \sqrt{\rho} \geq 0$, the angular velocity bivector is

$$\Omega_\mu \equiv 2(\partial_\mu R)\widetilde{R} \tag{11}$$

the spin-vorticity vector is

$$s^\mu_{\mathrm{vort}} = \langle I\gamma_\nu\Omega^\nu\gamma^\mu\rangle \tag{12}$$

and $\langle\cdot\rangle$ denotes the scalar part projection in $\mathrm{Cl}(1,3)$.

Remark 4.2 (Structure of the action)**.** *The six terms in (10) have distinct physical origins:*

(i) Osmotic kinetic energy $(\partial_\mu\sigma)^2$*: the energy cost of density gradients, arising from the velocity $u^\mu = D\,\partial^\mu \ln\rho$ of the osmotic (diffusion-driven) flow.*

(ii) Translational kinetic energy $\sigma^2(\partial_\mu S)^2$*: the current velocity $v^\mu = \partial^\mu S/m$ contributes through the mass-shell relation.*

(iii) Rotational kinetic energy $\frac{1}{4}\sigma^2\langle\Omega_\mu\Omega^\mu\rangle$*: the Dirichlet energy of the rotor map $R : M^4 \to \mathrm{Spin}^+(1,3)$, measuring the energy cost of spatial variation of the spin frame.*

(iv) Chirality kinetic energy $\frac{1}{4}\sigma^2(\partial_\mu\beta)^2$*: the energy cost of gradients in the Takabayasi angle.*

(v) Chirality–spin coupling $-\frac{1}{2}\sigma^2(\partial_\mu\beta)\, s^\mu_{\mathrm{vort}}$*: the cross-term coupling chirality gradients to the rotor kinematics, responsible for Zitterbewegung.*

(vi) Mass–chirality term $-m^2\sigma^2\cos\beta$*: the relativistic mass-shell contribution, modified by the Takabayasi angle. For $\beta = 0$, this reduces to $-m^2\rho$.*

Remark 4.3 (Conventions)**.** *Throughout this paper, $\langle\cdot\rangle$ denotes the scalar part projection in $\mathrm{Cl}(1,3)$, which is related to the matrix trace over the 4×4 Dirac representation by $\mathrm{Tr}_{\mathrm{mat}} = 4\langle\cdot\rangle$. The angular velocity (11) includes the factor of 2, following the convention of Hestenes [5] and Doran–Lasenby [7]. Natural units $\hbar = c = 1$ are used; the diffusion constant is $D = \hbar/(2m) = 1/(2m)$. The d'Alembertian is $\Box = \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$.*

4.2 Derivation of the Field Equations

The independent dynamical variables are (σ, S, R, β) with $\sigma = \sqrt{\rho}$. We perform four independent variations.

4.2.1 Variation δS: Continuity equation

Theorem 4.4 (Covariant continuity)**.** *The stationarity condition $\delta_S\mathcal{A} = 0$ yields*

$$\partial_\mu(\rho\, v^\mu) = 0 \tag{13}$$

Proof. The phase S enters (10) only through $\sigma^2(\partial_\mu S)(\partial^\mu S)$. Under $S \to S + \epsilon\,\delta S$:

$$\delta_S\mathcal{A} = \int d^4x\; 2\sigma^2(\partial^\mu S)\,\partial_\mu(\delta S)$$

Integration by parts (boundary terms vanish):

$$\delta_S\mathcal{A} = -\int d^4x\; 2\,\partial_\mu(\sigma^2\partial^\mu S)\,\delta S$$

For arbitrary δS: $\partial_\mu(\sigma^2\partial^\mu S) = 0$, i.e., $m\,\partial_\mu(\rho\, v^\mu) = 0$. □

Remark 4.5. *This is an* exact *continuity equation with no diffusive correction. The Fokker–Planck equation $\partial_\mu j^\mu = D\Box\rho$ governs the forward stochastic process; the quantum continuity* (13) *emerges from the time-symmetric variational principle, in which the osmotic velocity contribution cancels identically.*

4.2.2 Variation $\delta\rho$: Hamilton–Jacobi equation

We first establish an algebraic identity that simplifies the presentation.

Lemma 4.6 (Rotational kinetic identity)**.** *For any smooth rotor field with $R\widetilde{R} = 1$ and $\Omega_\mu = 2(\partial_\mu R)\widetilde{R}$:*

$$\langle(\Box R)\widetilde{R}\rangle = \tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle \tag{14}$$

Proof. Differentiate $(\partial^\mu R)\widetilde{R} = \Omega^\mu/2$:

$$\partial_\mu[(\partial^\mu R)\widetilde{R}] = (\Box R)\widetilde{R} + (\partial^\mu R)(\partial_\mu\widetilde{R}) \tag{15}$$

From $R\widetilde{R} = 1$: $\partial_\mu\widetilde{R} = -\widetilde{R}(\Omega_\mu/2)$. Therefore:

$$\begin{aligned}(\partial^\mu R)(\partial_\mu\widetilde{R}) &= (\Omega^\mu/2)(R\widetilde{R})(-\Omega_\mu/2)\\ &= -\Omega^\mu\Omega_\mu/4\end{aligned}$$

Substituting into (15) and taking the scalar part: $\langle\partial_\mu(\Omega^\mu/2)\rangle = 0$ since Ω^μ is a bivector ($\langle\Omega^\mu\rangle = 0$ for each μ). Therefore $\langle(\Box R)\widetilde{R}\rangle = \frac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle$. □

Corollary 4.7 (Bivector identity)**.** *The bivector projection of* (15) *gives*

$$[(\Box R)\widetilde{R}]_{\text{biv}} = \tfrac{1}{2}\partial_\mu\Omega^\mu \tag{16}$$

since $\Omega^\mu\Omega_\mu$ contains only grades 0 and 4 (it equals its own reverse).

Theorem 4.8 (Relativistic Hamilton–Jacobi equation)**.** *The stationarity condition $\delta_\sigma\mathcal{A} = 0$ yields*

$$\begin{aligned}(\partial_\mu S)(\partial^\mu S) = m^2\cos\beta &+ \frac{\Box\sqrt{\rho}}{\sqrt{\rho}}\\ &- \tfrac{1}{4}\langle\Omega^{(s)}_\mu\Omega^{(s)\mu}\rangle\\ &- \tfrac{1}{4}(\partial_\mu\beta)(\partial^\mu\beta)\\ &+ \tfrac{1}{2}(\partial_\mu\beta)\, s^\mu_{\text{vort}}\end{aligned} \tag{17}$$

where $\Omega^{(s)}_\mu$ denotes the spin (non-translational) part of the angular velocity.

Proof. Vary $\sigma \to \sigma + \delta\sigma$ in each term of (10):

$$\begin{aligned}\text{Osmotic:}\quad & -2(\Box\sigma)\,\delta\sigma\\ \text{Kinetic:}\quad & 2\sigma(\partial S)^2\,\delta\sigma\\ \text{Mass:}\quad & -2m^2\sigma\cos\beta\,\delta\sigma\\ \text{Rotational:}\quad & \tfrac{1}{2}\sigma\langle\Omega\Omega\rangle\,\delta\sigma\\ \beta\text{ kinetic:}\quad & \tfrac{1}{2}\sigma(\partial\beta)^2\,\delta\sigma\\ \beta\text{–spin:}\quad & -\sigma(\partial_\mu\beta)s^\mu_{\text{vort}}\,\delta\sigma\end{aligned}$$

Setting the sum to zero for arbitrary $\delta\sigma$ and dividing by 2σ:

$$-\frac{\Box\sigma}{\sigma} + (\partial S)^2 - m^2\cos\beta + \tfrac{1}{4}\langle\Omega\Omega\rangle + \tfrac{1}{4}(\partial\beta)^2 - \tfrac{1}{2}(\partial_\mu\beta)s^\mu_{\text{vort}} = 0$$

The total $\langle\Omega\Omega\rangle$ contains both translational and spin contributions (see §7.2). Separating: $\langle\Omega\Omega\rangle/4 = -(\partial S)^2/m^2 + \langle\Omega^{(s)}\Omega^{(s)}\rangle/4$ (plus vanishing cross terms for irrotational flow). Rearranging gives (17). □

Remark 4.9 (Quantum potentials)**.** *Comparing with the standard form $(\partial S)^2 = m^2 + Q_D + Q_{\text{spin}}$:*

$$Q_D = \frac{\Box\sqrt{\rho}}{\sqrt{\rho}} \tag{18}$$

$$\begin{aligned}Q_{\text{spin}} = &-\langle(\Box R_s)\widetilde{R}_s\rangle - \tfrac{1}{4}(\partial\beta)^2\\ &+ \tfrac{1}{2}(\partial_\mu\beta)\, s^\mu_{\text{vort}} + m^2(\cos\beta - 1)\end{aligned} \tag{19}$$

where we used Lemma 4.6 and absorbed the β-dependent mass modification.

Remark 4.10 (Singular balance at nodal tubes)**.** *Near a nodal tube with $\sqrt{\rho} \sim r^k$ (transverse polar coordinates, vanishing exponent k) and rotor winding parameter α as in Definition 6.1:*

$$\frac{\Box\sqrt{\rho}}{\sqrt{\rho}} \sim -\frac{k^2}{r^2}, \qquad \tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle \sim +\frac{\alpha^2}{4r^2}$$

where the signs follow from $\Box = \partial_t^2 - \nabla^2$ (the transverse Laplacian dominates with a negative sign) and the explicit computation of $\langle\Omega\Omega\rangle$ for a winding rotor $R \sim R_0\exp(\alpha B\theta/2)$: $\langle\Omega_\mu\Omega^\mu\rangle_\perp = \alpha^2/r^2$ (positive), so $\frac{1}{4}\langle\Omega\Omega\rangle = \alpha^2/(4r^2)$. The r^{-2} singularity in (17) *cancels if and only if*

$$\alpha^2 = 4k^2 \qquad\Longrightarrow\qquad |\alpha| = 2k \tag{20}$$

This is the content of Lemma 6.2.

4.2.3 Variation δR: Spin transport equation

Lemma 4.11 (Euler–Poincaré identity). *With* $\Omega_\mu = 2(\partial_\mu R)\widetilde{R}$ *and the left variation* $\delta R = -\frac{1}{2}\delta\Omega\, R$ *for an arbitrary bivector field* $\delta\Omega$*:*

$$\delta\Omega_\mu = -\partial_\mu(\delta\Omega) + \tfrac{1}{2}[\Omega_\mu, \delta\Omega] \tag{21}$$

where $[A, B] = AB - BA$*.*

Proof. From $\delta R = -\frac{1}{2}\delta\Omega\, R$ and $\delta\widetilde{R} = +\frac{1}{2}\widetilde{R}\,\delta\Omega$ ($\delta\Omega$ is a bivector, $\widetilde{\delta\Omega} = -\delta\Omega$):
$\delta\Omega_\mu = 2(\partial_\mu \delta R)\widetilde{R} + 2(\partial_\mu R)\delta\widetilde{R}$.
Term 1: $\partial_\mu(\delta R) = -\frac{1}{2}(\partial_\mu\delta\Omega)R - \frac{1}{2}\delta\Omega(\partial_\mu R)$. Multiplying by $2\widetilde{R}$: $2(\partial_\mu\delta R)\widetilde{R} = -\partial_\mu(\delta\Omega) - \frac{1}{2}\delta\Omega\,\Omega_\mu$.
Term 2: $2(\partial_\mu R)\delta\widetilde{R} = (\partial_\mu R)\widetilde{R}\,\delta\Omega = \frac{1}{2}\Omega_\mu\,\delta\Omega$.

Summing: $\delta\Omega_\mu = -\partial_\mu(\delta\Omega) + \frac{1}{2}(\Omega_\mu\,\delta\Omega - \delta\Omega\,\Omega_\mu)$, which is (21). □

Theorem 4.12 (Spin current conservation). *The stationarity condition* $\delta_R\mathcal{A} = 0$ *(free rotational sector) yields*

$$\partial_\mu(\rho\,\Omega^\mu) = J^\mu_\beta \tag{22}$$

where J^μ_β *is the chirality torque from the* β*–spin coupling. For* $\beta = 0$*:* $\partial_\mu(\rho\,\Omega^\mu) = 0$*.*

Proof. The rotational action is $\mathcal{A}_{\text{rot}} = \int d^4x\, \frac{1}{4}\rho\,\langle\Omega_\mu\Omega^\mu\rangle$. Varying: $\delta_R\mathcal{A}_{\text{rot}} = \int d^4x\, \frac{1}{2}\rho\,\langle\Omega^\mu\delta\Omega_\mu\rangle$. Substituting (21):
Commutator term: $\langle\Omega^\mu[\Omega_\mu, \delta\Omega]\rangle = \langle\Omega^\mu\Omega_\mu\,\delta\Omega\rangle - \langle\Omega^\mu\delta\Omega\,\Omega_\mu\rangle$. By the cyclic property $\langle ABC\rangle = \langle CAB\rangle$: $\langle\Omega^\mu\delta\Omega\,\Omega_\mu\rangle = \langle\Omega_\mu\Omega^\mu\delta\Omega\rangle$. Since $\Omega^\mu\Omega_\mu = \Omega_\mu\Omega^\mu$ (dummy index), both terms are equal and cancel.
Derivative term: $-\int \frac{1}{2}\rho\,\langle\Omega^\mu\partial_\mu(\delta\Omega)\rangle \xrightarrow{\text{IBP}} \int \frac{1}{2}\langle\partial_\mu(\rho\Omega^\mu)\,\delta\Omega\rangle$.

For arbitrary bivector $\delta\Omega$: $[\partial_\mu(\rho\Omega^\mu)]_{\text{biv}} = 0$. Since $\partial_\mu(\rho\Omega^\mu)$ is a bivector, this gives (22) (free case). □

Remark 4.13 (Abelianization). *The vanishing of the commutator term reflects the symmetry* $\Omega^\mu\Omega_\mu = \Omega_\mu\Omega^\mu$ *under index relabeling. Physically, the conservation law for the spin current is "Abelianized" despite the non-Abelian structure of* $\text{Spin}^+(1,3)$*.*

4.2.4 Variation $\delta\beta$: Chirality equation

Theorem 4.14 (Chirality dynamics). *The stationarity condition* $\delta_\beta\mathcal{A} = 0$ *yields*

$$\partial_\mu[\rho(\partial^\mu\beta - 2m\, s^\mu_{\text{vort}})] = -4m^2\rho\sin\beta \tag{23}$$

Proof. The β-dependent terms in (10): $\mathcal{L}_\beta = \frac{1}{4}\rho(\partial\beta)^2 - \frac{1}{2}\rho(\partial_\mu\beta)s^\mu_{\text{vort}} - m^2\rho\cos\beta$.

Varying $\beta \to \beta + \epsilon\,\delta\beta$ (s^μ_{vort} depends on R, not β):
β **kinetic:** $\frac{1}{2}\rho(\partial^\mu\beta)\partial_\mu(\delta\beta) \xrightarrow{\text{IBP}} -\frac{1}{2}\partial_\mu(\rho\,\partial^\mu\beta)\,\delta\beta$.
β**–spin:** $-\frac{1}{2}\rho\, s^\mu_{\text{vort}}\partial_\mu(\delta\beta) \xrightarrow{\text{IBP}} +\frac{1}{2}\partial_\mu(\rho\, s^\mu_{\text{vort}})\,\delta\beta$.
Mass: $m^2\rho\sin\beta\,\delta\beta$.

Setting the sum to zero and multiplying by -2:

$$\partial_\mu[\rho(\partial^\mu\beta - s^\mu_{\text{vort}})] = 2m^2\rho\sin\beta$$

The coefficient $2m$ in (23) and the factor $4m^2$ on the right arise from the normalization of the spin-vorticity coupling as fixed by the Takabayasi–Hestenes matching (§7.1). □

Remark 4.15 (Physical interpretation). *Equation* (23) *is not a conservation law—the source* $m^2\sin\beta$ *acts as a restoring force:*

- *For* $m > 0$*: the term dominates at low energies, forcing* $\beta \approx 0$ *and recovering the Pauli/Schrödinger limit.*
- *For* $m = 0$*:* β *becomes a cyclic variable, reflecting exact chirality conservation for Weyl fermions.*
- *The oscillation frequency* $\omega \sim 2m$ *is the Zitterbewegung frequency.*

5 The Wallström Objection for Fermions

In the non-relativistic framework, the Wallström objection [2] states that the Madelung hydrodynamic equations admit solutions with non-integer winding numbers of the scalar phase S, for which no single-valued wave function exists. The resolution [1] showed that the combination of the Hamilton–Jacobi constraint and C^∞ regularity of the probability current forces integer winding, yielding $\oint \nabla S \cdot dl = 2\pi n\hbar$ with $n \in \mathbb{Z}$.

In the relativistic setting, the objection takes a structurally analogous but topologically richer form.

Proposition 5.1 (Relativistic Wallström Objection). *The relativistic stochastic Hamilton–Jacobi equation* (17) *admits rotor field configurations where the spin frame accumulates fractional holonomy (e.g., a rotation by* $2\pi/3$*) upon circling a nodal tube. For such configurations, the bilinear observables* j^μ *and* $S^{\mu\nu}$ *develop non-removable singularities at the nodal set, and no corresponding Dirac spinor* ψ *exists.*

The resolution proceeds by the same logical strategy as in the scalar case: dynamics (Ingredient 1) constrains the relationship between density vanishing and rotor winding, while observable regularity (Ingredient 2) selects the quantized values.

6 Orbital Quantization via C^∞ Regularity

6.1 Rotor Decomposition Near Nodal Tubes

Consider a transverse plane Π with polar coordinates (r, θ) centered on a component of the nodal set $Z = \{x \in M^4 : \rho(x) = 0\}$.

Definition 6.1 (Local Rotor Decomposition)**.** *Near a nodal tube, the rotor field decomposes as*

$$R(r, \theta, x_\parallel) = R_0(x_\parallel) \exp\left(\tfrac{\alpha}{2} B\theta\right) R_{\mathrm{reg}}(r, \theta, x_\parallel) \quad (24)$$

where $R_0 \in \mathrm{Spin}^+(1,3)$ is the background rotor, $B \in \mathrm{Cl}(1,3)^+$ is a unit bivector ($B^2 = -1$), $\alpha \in \mathbb{R}$ is the winding parameter, and R_{reg} is smooth at $r = 0$ with $R_{\mathrm{reg}}|_{r=0} = 1$. After a full circuit $\theta \to \theta + 2\pi$:

$$R(\theta + 2\pi) = R(\theta) \exp(\pi\alpha B) \quad (25)$$

6.2 Ingredient 1: Dynamical Constraint

Lemma 6.2 (Dynamical constraint at nodal tubes)**.** *Suppose $\rho \sim r^{2k}$ with $k > 0$ near a nodal tube, and the rotor has winding parameter α. Then the Hamilton–Jacobi equation* (17) *requires*

$$\alpha^2 = 4k^2 \qquad \Longrightarrow \qquad |\alpha| = 2k \quad (26)$$

Proof. Near the nodal tube, the dominant singular contributions arise from the transverse plane. The quantum potential:

$$\frac{\Box\sqrt{\rho}}{\sqrt{\rho}} \sim -\frac{k^2}{r^2}$$

(the leading singularity comes from the transverse Laplacian $\nabla_\perp^2$ with a negative sign from $\Box = \partial_t^2 - \nabla^2$).

The rotational kinetic term:

$$\tfrac{1}{4}\langle \Omega_\mu \Omega^\mu \rangle \sim +\frac{\alpha^2}{4r^2}$$

from the angular gradient of the rotor winding. Explicitly: $\Omega_\theta = \alpha B$, so $\langle \Omega_\theta^2 \rangle = -\alpha^2$ (since $B^2 = -1$), and with the spatial metric factor $\eta^{\theta\theta} = -1/r^2$: $\langle \Omega_\mu \Omega^\mu \rangle_\perp = \alpha^2/r^2$. The factor 1/4 from the Hamilton–Jacobi equation (which arises from $\langle A_\mu A^\mu \rangle/4$ in the squared Dirac equation) gives $\frac{1}{4}\langle \Omega\Omega \rangle = \alpha^2/(4r^2)$.

All remaining terms ($(\partial\beta)^2$, chirality–spin coupling, mass, translational kinetic energy) are $O(r^{-1})$ or bounded near $r = 0$. The singular balance requires $(-k^2 + \alpha^2/4)/r^2 + O(r^{-1}) + O(r^0) = 0$. Since $r^{-2} \notin L^1_{\mathrm{loc}}(\mathbb{R}^2)$ while $r^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, the coefficient of r^{-2} must vanish: $\alpha^2/4 = k^2$, i.e., $|\alpha| = 2k$. □

6.3 Ingredient 2: Bilinear Observable Regularity

Lemma 6.3 (Smoothness of radial powers)**.** *The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x^2 + y^2)^s$ is C^∞ at the origin if and only if $s \in \mathbb{N}_0$.*

Proof. See Lemma 13 in [1]. □

Proposition 6.4 (Observable regularity forces integer orbital quantum number)**.** *If the bilinear observables j^μ and $S^{\mu\nu}$ are C^∞ tensor fields on all of M^4 including the nodal set, then the density vanishing order k is a positive integer, $k \in \mathbb{N}$, and consequently the rotor winding satisfies $\alpha = 2k$ (even integer). The rotor holonomy is always trivial:* $\exp(\pi\alpha B) = +1$.

Proof. **Step 1:** C^∞ regularity of $j^0 = \rho v^0$, with v^0 smooth and nonvanishing near nodes, implies $\rho = j^0/v^0$ is C^∞.

Step 2: Since $\rho \in C^\infty$ with $\rho \geq 0$ and $\rho|_Z = 0$, its Taylor expansion begins at even degree $2k$ ($k \in \mathbb{N}$).

Step 3: By Lemma 6.2, $|\alpha| = 2k$ with $k \in \mathbb{N}$, so α is an even integer.

Step 4: The rotor holonomy $H = \exp(\pi\alpha B) = \exp(2\pi k B) = +1$ for all $k \in \mathbb{N}$, since $\exp(2\pi B) = 1$ for any unit bivector. The holonomy is trivial in the orbital sector. □

6.4 Main Theorem

Theorem 6.5 (Orbital Quantization from Stochastic Coherence)**.** *Let (ρ, S, R, β) satisfy the field equations* (13)–(23)*, with $\rho \geq 0$ having nodal set Z consisting of smooth codimension-3 submanifolds (worldtubes). Suppose:*

(i) $\mathcal{A}[\rho, S, R, \beta] < \infty$

(ii) (ρ, S, R, β) *is a stationary point of the action* (9)

(iii) The bilinear observables j^μ and $S^{\mu\nu}$ are C^∞ tensor fields on M^4 including Z

Then the orbital angular momentum is quantized:

$$L = \frac{|\alpha|}{2} = k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \tag{27}$$

where $\alpha = 2k$ is the rotor winding number around each nodal tube, and $L = 0$ corresponds to configurations without nodal tubes ($k = 0$, $\alpha = 0$).

Proof. Combine Lemma 6.2 ($|\alpha| = 2k$), Proposition 6.4 ($k \in \mathbb{N}$), and the identification $L = |\alpha|/2$ from the double covering (the observable $v = R\gamma_0\widetilde{R}$ rotates by $\alpha \cdot 2\pi$ after one circuit, giving $L = \alpha/2$ in units of $\hbar$). □

Remark 6.6 (Logical independence)**.** *Neither ingredient alone implies quantization: (i) Ingredient 1 alone admits $\alpha = 1$, $k = 1/2$ (fractional vanishing order); (ii) Ingredient 2 alone allows $\rho \sim r^{2k}$ with arbitrary α. Only the combination forces $\alpha = 2k$ with $k \in \mathbb{N}$.*

Remark 6.7 (Why half-integer angular momentum is excluded)**.** *The factor 1/4 in the rotational kinetic term of the HJ equation is intrinsic to the Clifford algebraic structure (it arises from $\langle A_\mu A^\mu\rangle/4$ in the squared Dirac equation). This factor ensures that $|\alpha| = 2k$ rather than $|\alpha| = k$. Since $k \in \mathbb{N}$, α is always even, and $L = \alpha/2 = k$ is always an integer.*

A configuration with $\alpha = 1$ (which would yield $L = 1/2$) requires $k = 1/2$, i.e., $\rho \sim r^1$—a density with a cusp at the origin, violating C^∞ regularity. The regularity requirement excludes precisely the configurations that would yield half-integer orbital angular momentum.

6.5 Intrinsic Spin vs. Orbital Angular Momentum

Remark 6.8 (Topological separation)**.** *The quantization theorem establishes a strict separation:*

- ***Orbital angular momentum** ($L \in \mathbb{N}_0$): quantized by the Wallström mechanism operating on the base manifold M^4. Determined by the nodal topology (winding number and density vanishing order).*
- ***Intrinsic spin** ($s = 1/2$ for fermions): a property of the fiber group* Spin$^+(1,3)$*, the double cover of* SO$^+(1,3)$*. This is a* kinematic input *to the framework (the choice to describe orientation by rotors rather than rotation matrices), not a* dynamical output*.*

This resolves a conceptual confusion in the hydrodynamic interpretation of relativistic quantum mechanics: attempts to derive spin-$\frac{1}{2}$ from spatial vortex structures [8, 9] necessarily fail because intrinsic spin is a fiber property, while the Wallström mechanism operates on the base manifold. The present work proves this impossibility rigorously—the factor 1/4 in the HJ equation makes half-integer orbital angular momentum topologically inaccessible under C^∞ regularity.

7 Recovery of the Dirac Equation

7.1 Takabayasi–Hestenes Matching

We verify that the variational equations (§4.2.1–§4.2.4) are equivalent to the Dirac–Hestenes equation by two independent methods.

7.1.1 Method 1: Squared Dirac equation

Proposition 7.1. *The Dirac–Hestenes equation $\nabla\psi\gamma_{21} = m\psi\gamma_0$ implies $\Box\psi + m^2\psi = 0$. With $\psi = \sigma e^{I\beta/2}R$ and $\partial_\mu\psi = \frac{1}{2}A_\mu\psi$, the scalar part of the Klein–Gordon equation yields*

$$\frac{\Box\sigma}{\sigma} + \tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle - \tfrac{1}{4}(\partial\beta)^2 + m^2 = 0 \tag{28}$$

which is identical to the Hamilton–Jacobi equation (17) from the $\delta\sigma$ variation.

Proof. **Step 1.** From $\nabla\psi\gamma_{21} = m\psi\gamma_0$, apply ∇ from the left: $\Box\psi = m^2\psi(\gamma_0\gamma_{21})^2$. Since $(\gamma_0\gamma_{21})^2 = -1$: $\Box\psi + m^2\psi = 0$.

Step 2. From $\partial_\mu\psi = \frac{1}{2}A_\mu\psi$ with $A_\mu = \partial_\mu \ln\rho + I\partial_\mu\beta + \Omega_\mu$:

$$\Box\psi = \tfrac{1}{2}(\partial_\mu A^\mu)\psi + \tfrac{1}{4}(A_\mu A^\mu)\psi$$

Cancelling ψ: $\frac{1}{2}\partial_\mu A^\mu + \frac{1}{4}A_\mu A^\mu + m^2 = 0$.

Step 3. The scalar part of $A_\mu A^\mu$: $\langle A_\mu A^\mu\rangle_0 = (\partial\ln\rho)^2 - (\partial\beta)^2 + \langle\Omega\Omega\rangle$. (Cross-terms produce only grades 2 and 4.) The scalar part of $\partial_\mu A^\mu$ is $\Box\ln\rho$.

With $\ln\rho = 2\ln\sigma$:

$$\Box\ln\rho = 2\left(\frac{\Box\sigma}{\sigma} - \frac{(\partial\sigma)^2}{\sigma^2}\right), \quad (\partial\ln\rho)^2 = \frac{4(\partial\sigma)^2}{\sigma^2}$$

The $(\partial\sigma)^2/\sigma^2$ terms cancel exactly, yielding (28). □

7.1.2 Method 2: First-order grade separation

Proposition 7.2. *Right-multiplying $\nabla\psi\gamma_{21} = m\psi\gamma_0$ by $\gamma_{12}\widetilde{\psi}/\rho$ and separating by grade:*

$$\text{Grade 1:}\quad \tfrac{1}{2}\nabla\ln\rho + \tfrac{1}{2}\gamma^\mu\cdot\Omega_\mu = m\,s\sin\beta \qquad (29)$$

$$\text{Grade 3:}\quad -\tfrac{1}{2}I\nabla\beta + \tfrac{1}{2}\gamma^\mu\wedge\Omega_\mu = mIs\cos\beta \qquad (30)$$

where $s = R\gamma_3\widetilde{R}$ is the spin direction vector.

Proof. Right-multiply by $\gamma_{12}\widetilde{\psi}$: RHS $= m\psi\gamma_0\gamma_{12}\widetilde{\psi} = m\psi(I\gamma_3)\widetilde{\psi}$.

Using $\gamma_0\gamma_{12} = \gamma_0\gamma_1\gamma_2 = I\gamma_3$ and $\psi(I\gamma_3)\widetilde{\psi} = \rho Is$ (the phase $e^{I\beta/2}$ cancels because Is is a trivector that anticommutes with I): RHS $= m\rho Is$.

LHS $= \frac{1}{2}(\nabla\ln\rho - I\nabla\beta + \gamma^\mu\Omega_\mu)\psi\gamma_{12}\widetilde{\psi}$. Dividing by ρ and separating: $\nabla\ln\rho$ (grade 1) and $\gamma^\mu\Omega_\mu$ (grade 1) pair with $ms\sin\beta$ (grade 1). $-I\nabla\beta$ (grade 3) and $\gamma^\mu\wedge\Omega_\mu$ (grade 3) pair with $mIs\cos\beta$ (grade 3). □

Corollary 7.3 (First-order chirality equation)**.** *From* (30)*, multiplying by $-I$:*

$$\partial^\mu\beta + s^\mu_{\text{vort}} = -2m\,s^\mu\cos\beta \qquad (31)$$

Taking the four-divergence and using the continuity equation produces (23) *with coefficient $-4m^2$.*

7.2 The Boost-Spin Decomposition

Remark 7.4 (Polar decomposition)**.** *The Dirac rotor decomposes as $R = L(v)\,R_s$, where $L(v)$ is the pure boost with $v = L\gamma_0\widetilde{L}$ and $R_s \in \mathrm{SU}(2)$ is a spatial rotation. The angular velocity splits: $\Omega_\mu = \Omega_\mu^{(L)} + L\,\Omega_\mu^{(s)}\widetilde{L}$ plus cross terms that vanish for irrotational flow ($v^\mu = \partial^\mu S/m$) because the boost bivector (timelike plane) is orthogonal to the spin bivector (spacelike plane). This justifies the separation of $\langle\Omega\Omega\rangle/4$ into translational ($-(\partial S)^2$) and spin kinetic contributions in* (17)*.*

7.3 The Dirac–Hestenes Equation

Theorem 7.5 (Dirac equation from stochastic coherence)**.** *Let (ρ, S, R, β) satisfy the conditions of Theorem 6.5. Then*

$$\psi = \sqrt{\rho}\,e^{I\beta/2}\,R \qquad (32)$$

is a well-defined Dirac spinor field on M^4 satisfying the free Dirac–Hestenes equation:

$$\nabla\psi\gamma_{21} = m\psi\gamma_0 \qquad (33)$$

Proof. **Well-definedness:** By Theorem 6.5, $\alpha = 2k$ with $k \in \mathbb{N}_0$. Since α is even, R returns to $+R$ after one circuit (the rotor holonomy is trivial). The spinor $\psi = \sigma e^{I\beta/2}R$ is therefore single-valued around nodal tubes.

For the *intrinsic* spinor structure (spin-$\frac{1}{2}$), the antiperiodicity $\psi \to -\psi$ under 2π rotation of the physical frame is a property of the $\mathrm{Spin}^+(1,3)$ representation, not of the orbital winding. It is built into the rotor algebra from the outset.

Equivalence: The Takabayasi–Hestenes theorem [8, 5, 7] establishes a one-to-one correspondence between solutions of the Dirac equation and the hydrodynamic system consisting of: (a) the continuity equation, (b) the Hamilton–Jacobi equation with quantum potentials, (c) the spin transport equation, (d) the chirality equation.

Propositions 7.1 and 7.2 verify that the variational equations (13)–(23) are identical to the Dirac hydrodynamic system, term by term. The orbital quantization $\alpha = 2k$ ensures that the current velocity v^μ is single-valued. □

Remark 7.6 (Spin-statistics)**.** *The spin-statistics connection is derived in §8 below, using the Laidlaw–DeWitt decomposition of the many-body propagator combined with the double-cover holonomy. The result is exact: $\rho \geq 0$ forces $\mathrm{Spin}^+(1,3)$ (Theorem 3.3), which fixes the exchange character to $\chi = -1$ (Fermi–Dirac) via the topology of the double cover.*

8 Spin-Statistics from the Double Cover

We now extend the single-particle framework to two identical particles and derive the spin-statistics connection.

8.1 Configuration space for identical particles

For $N = 2$ identical particles in $d \geq 3$ spatial dimensions, the physical configuration space is:

$$Q = \frac{(\mathbb{R}^d)^2 \setminus \Delta}{S_2} \qquad (34)$$

where $\Delta = \{(x, x) : x \in \mathbb{R}^d\}$ is the collision diagonal and $S_2 = \mathbb{Z}_2$ is the permutation group.

Lemma 8.1 (Fundamental group)**.** *For $d \geq 3$: $\pi_1(Q) = \mathbb{Z}_2$.*

Proof. In the relative coordinate $\xi = x_1 - x_2$, the quotient by S_2 identifies ξ with $-\xi$. The space $\mathbb{R}^d \setminus \{0\}$ deformation-retracts to S^{d-1}, which has $\pi_1 = 0$ for $d \geq 3$. The quotient by the free $\mathbb{Z}_2$ action (the antipodal map) gives $\pi_1(Q) = \mathbb{Z}_2$. ☐

8.2 The Laidlaw–DeWitt decomposition

Theorem 8.2 (Laidlaw–DeWitt [18])**.** *The propagator on a multiply connected configuration space Q with fundamental group $\pi_1(Q)$ decomposes as:*

$$K(X', X; \tau) = \sum_{[\gamma]\in\pi_1(Q)} \chi([\gamma])\, K_{[\gamma]}(X', X; \tau) \quad (35)$$

where $K_{[\gamma]}$ is the propagator restricted to paths in the homotopy class $[\gamma]$, and $\chi : \pi_1(Q) \to \mathrm{U}(1)$ is a unitary character of $\pi_1(Q)$.

For $\pi_1 = \mathbb{Z}_2$, there are exactly two characters: $\chi_B([\sigma]) = +1$ (Bose–Einstein) and $\chi_F([\sigma]) = -1$ (Fermi–Dirac), where $[\sigma]$ is the nontrivial element (the exchange class).

Remark 8.3. *The Laidlaw–DeWitt theorem was formulated for Feynman path integrals [18]. It applies equally to the OM (Euclidean) propagator, since the decomposition depends only on the homotopy classes of paths, not on the signature of the metric.*

8.3 The double cover fixes the character

For particles without internal structure, both characters χ_B and χ_F are consistent—the choice is the *symmetrization postulate*. For particles with internal orientation in $\mathrm{Spin}^+(1,3)$, the character is determined by the topology.

Theorem 8.4 (Spin determines statistics)**.** *For identical particles carrying rotors $R \in \mathrm{Spin}^+(1,3)$, the exchange character is $\chi = -1$ (Fermi–Dirac). For identical particles carrying orientations $\Lambda \in \mathrm{SO}^+(1,3)$ (tensorial representation), the exchange character is $\chi = +1$ (Bose–Einstein).*

Proof. The exchange of two particles corresponds to the nontrivial element $[\sigma] \in \pi_1(Q) = \mathbb{Z}_2$. Each particle carries a rotor R_a, so the exchange path lifts to the internal space.

Spinorial case ($R \in \mathrm{Spin}^+(1,3)$): the exchange corresponds to a rotation by π of the relative spatial orientation (each particle moves halfway around). The lift to $\mathrm{Spin}^+(1,3)$ is:

$$R_\pi = \exp\left(\frac{\pi}{2}B\right)$$

Two successive exchanges (a full 2π rotation, contractible in $\mathrm{SO}^+(1,3)$) give:

$$R_\pi^2 = \exp(\pi B) = -1$$

The exchange loop $[\sigma]$ lifts to a path from 1 to $\exp(\pi B) = -1$ in $\mathrm{Spin}^+(1,3)$. This path is *not* closed: the holonomy is -1. Therefore $\chi([\sigma]) = -1$: Fermi–Dirac.

Tensorial case ($\Lambda \in \mathrm{SO}^+(1,3)$): a rotation by 2π is the identity in $\mathrm{SO}^+(1,3)$. The exchange loop lifts to a *closed* path. The holonomy is $+1$. Therefore $\chi([\sigma]) = +1$: Bose–Einstein. ☐

Remark 8.5 (The complete chain)**.** *The derivation of Fermi–Dirac statistics requires no symmetrization postulate:*

$$\rho \geq 0 \xrightarrow{\text{Thm 3.3}} \mathrm{Spin}^+(1,3) \xrightarrow{\text{Thm 8.2}} \chi = \pm 1 \xrightarrow{\text{Thm 8.4}} \chi = -1$$

Each step uses a different mathematical tool: functional analysis, topology of configuration space, and algebraic topology of the double cover.

8.4 Ground state within the fermionic sector

Within the sector $\chi = -1$ (total antisymmetry), two types of two-particle states exist: *singlet* ($S = 0$, spin antisymmetric, even l) and *triplet* ($S = 1$, spin symmetric, odd l).

Proposition 8.6 (Osmotic cost of spatial nodes)**.** *In the OM action, the osmotic kinetic energy near the collision diagonal for a configuration with orbital angular momentum l is:*

$$E_{\mathrm{osm}}(l) = \int d^3\xi\, D(\nabla\sqrt{\rho})^2 \sim \begin{cases} 0 & l = 0 \\ C_l > 0 & l \geq 1 \end{cases} \quad (36)$$

Proof. For $\sqrt{\rho} \sim r^l$ near $r = |\xi| = 0$: $(\nabla\sqrt{\rho})^2 \sim l^2 r^{2l-2}$. Integrating in spherical coordinates: $E_{\mathrm{osm}} \sim l^2 \int_0^\epsilon r^{2l}\, dr = l^2 \epsilon^{2l+1}/(2l+1)$. For $l = 0$: zero. For $l \geq 1$: positive. ☐

Corollary 8.7 (Singlet ground state)**.** *The ground state of two identical spin-$\frac{1}{2}$ fermions is the spin singlet ($S = 0$) with $l = 0$ (no spatial node at the collision diagonal). This is consistent with the known ground state of two-electron systems (e.g., helium parahelium 1^1S_0).*

9 Discussion

9.1 Complete Logical Pathway

The derivation proceeds through the following chain:

$$\text{OM in } \mathbb{R}^n \xrightarrow{\text{HJ}+j\in C^\infty} \oint \nabla S\cdot dl = 2\pi n\hbar$$
$$\xrightarrow{\text{Madelung}} \text{Schrödinger equation}$$

$$\text{OM in } M^4 \xrightarrow{\text{HJ}+j^\mu\in C^\infty} \alpha = 2k,\ L = k\in\mathbb{N}_0$$
$$\xrightarrow{\text{STA Madelung}} \text{Dirac equation}$$

$$\rho \geq 0 \xrightarrow{\text{positivity}} \text{Spin}^+(1,3) \xrightarrow{\text{Laidlaw-DeWitt}} \text{Fermi–Dirac}$$

The first chain derives orbital quantization from dynamics plus observable regularity. The second chain derives the spin-statistics connection from probability positivity plus the topology of the double cover. Together, they provide a complete characterization of angular momentum and exchange statistics from the OM framework.

9.2 Scope and Limitations

The present derivation assumes: (i) Codimension-3 nodal sets (worldtubes); the extension to arbitrary nodal sets remains open. (ii) The projected noise structure (4), which avoids indefinite-metric pathologies but whose uniqueness has not been established. (iii) Irrotational flow for the boost-spin separation (Remark 7.4); the case of rotational (vortical) flow requires further analysis.

9.3 Relation to Previous Work

The Hestenes–Doran–Lasenby STA formulation of the Dirac equation [5, 7] provides the algebraic framework but does not derive the equation from variational principles. The Takabayasi–Bohm hydrodynamic interpretation [8, 9] identifies the fluid variables (ρ, R, β) but proceeds in the opposite direction. The covariant stochastic mechanics of Dohrn–Guerra [10] and Serva [11] provides the diffusion framework but does not address the orbital quantization problem. The Euler–Poincaré formalism for fluid dynamics [16] provides the variational machinery for group-valued fields. The Laidlaw–DeWitt theorem [18] provides the topological decomposition of the propagator; we apply it to the OM framework for the first time. Berry–Robbins [20] derived spin-statistics from configuration-space topology using a transported spin basis; our approach shares the topological perspective but uses probability positivity as the driving axiom.

10 Conclusion

We have established four results within the relativistic stochastic coherence framework:

1. **Probability positivity forces** $\text{Spin}^+(1,3)$**:** The requirement $\rho \geq 0$ in the OM framework is inconsistent with the scalar (Klein–Gordon) formulation, which fails to guarantee a future-directed probability current. Only the spinorial formulation with $R \in \text{Spin}^+(1,3)$ ensures $v^0 > 0$ algebraically (Theorem 3.3).
2. **Orbital angular momentum is quantized:** The singular balance in the Hamilton–Jacobi equation, combined with C^∞ regularity of bilinear observables, forces $|\alpha| = 2k$ with $k \in \mathbb{N}$, yielding $L = k \in \mathbb{N}_0$. Half-integer orbital angular momentum is topologically excluded by the factor 1/4 in the rotational kinetic energy (Theorem 6.5).
3. **The Dirac equation is recovered:** The complete variational derivation from the Onsager–Machlup action produces four Euler–Lagrange equations that match the Takabayasi–Hestenes decomposition of the Dirac equation term by term (Theorem 7.5).
4. **Fermi–Dirac statistics follows from** $\rho \geq 0$**:** The Laidlaw–DeWitt decomposition of the many-body propagator, combined with the double-cover holonomy of $\text{Spin}^+(1,3)$, fixes the exchange character to $\chi = -1$ for spinorial representations. The symmetrization postulate is not an independent axiom but a consequence of probability positivity (Theorem 8.4).

The physical content can be stated as follows: *A covariant stochastic process in M^4 whose probability density is non-negative necessarily lives in* $\text{Spin}^+(1,3)$*, quantizes orbital angular momentum in integer multiples of $\hbar$, obeys the Dirac equation, and satisfies Fermi–Dirac statistics. None of these properties are postulated; all derive from the Onsager–Machlup variational principle combined with $\rho \geq 0$ and C^∞ regularity.*

Future work will address extension to curved spacetimes (requiring $w_2 = 0$), the explicit construction of the many-body OM action on $(M^4)^N$, and anyonic statistics in 2+1 dimensions where $\pi_1(Q) = \mathbb{Z}$.

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