Stochastic Derivation of the Dirac Equation:
Topological Spin Quantization via Covariant Current Regularity

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Abstract

We extend the stochastic coherence framework (Onsager–Machlup formalism) to the relativistic domain and derive spin-$\frac{1}{2}$ quantization as a geometric–topological consequence. In the non-relativistic setting, phase quantization was shown to follow from the combination of the Hamilton–Jacobi constraint and C^∞ regularity of the probability current [1]. Here, the configuration space $\mathbb{R}^n$ is replaced by Minkowski spacetime M^4, the scalar phase S is replaced by a Lorentz rotor field $R \in \mathrm{Spin}^+(1,3)$, and the physical observables become the four-current j^μ and the spin tensor $S^{\mu\nu}$, both bilinear in the rotor via the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

We prove that the C^∞ regularity of these bilinear observables over nodal surfaces, combined with the relativistic stochastic Hamilton–Jacobi equation, forces the internal holonomy of R to take values in $\pi\mathbb{Z}$ rather than $2\pi\mathbb{Z}$. This yields intrinsic angular momentum quantized in half-integer multiples of $\hbar$, recovering spin-$\frac{1}{2}$ without postulating spinor structure or the Dirac equation.

The complete variational derivation of the relativistic stochastic dynamics from the Onsager–Machlup action is presented, including the Euler–Poincaré reduction on $\mathrm{Spin}^+(1,3)$ and the chirality sector. The resulting hydrodynamic equations are shown to be equivalent, term by term, to the Takabayasi–Hestenes decomposition of the Dirac equation, completing the derivation: stochastic coherence plus observable regularity implies the Dirac equation with quantized spin.

The derivation is non-circular: the dynamical constraint and the observable regularity requirement are logically independent, and neither alone implies spin quantization.

1 Introduction and Motivation

In the framework of stochastic coherence [1], it was demonstrated that phase circulation quantization in the Schrödinger regime does not require postulating single-valuedness of a wave function. Instead, quantization emerges from the combination of the stochastic Hamilton–Jacobi equation and the requirement that the probability current $j = \rho\nabla S/m$ be a C^∞ vector field on all of physical space, including at nodal zeros.

The present work extends this principle to the relativistic domain, with the objective of demonstrating that spin-$\frac{1}{2}$ for elementary fermions is not an axiom—as assumed in the standard Dirac equation—but a geometric consequence: it is the unique rotational topology that permits a covariant stochastic fluid in Minkowski spacetime to possess C^∞ observable currents on its nodal surfaces.

The logical structure parallels the non-relativistic case:

- **Ingredient 1 (Dynamics):** The relativistic stochastic Hamilton–Jacobi equation constrains the vanishing rate of the density near nodal surfaces to the rotational frequency of the spin frame.

- **Ingredient 2 (Observable regularity):** C^∞ smoothness of the bilinear observables j^μ and $S^{\mu\nu}$ forces quantization of the internal rotation angle.

The key new feature is that observables are bilinear in the rotor R through the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$: a full 2π rotation of the observable requires only a π rotation of R, yielding half-integer rather than integer quantization.

A central contribution of this work is the *complete* variational derivation of the relativistic stochastic dy-

namics from the Onsager–Machlup action functional. In contrast to previous treatments that proceed from the Dirac equation to the fluid description, we invert the logic: the Dirac equation *emerges* from the stochastic variational principle combined with observable regularity.

Throughout, we work in the Spacetime Algebra (STA) formulation of Clifford algebra Cl(1, 3), following Hestenes [4, 5, 6] and Doran–Lasenby [7]. Natural units $c = \hbar = 1$ are used except where clarity requires explicit factors.

2 Mathematical Framework

2.1 Spacetime Algebra

Definition 2.1 (Spacetime Algebra). *The Spacetime Algebra* Cl(1, 3) *is the real Clifford algebra generated by an orthonormal frame* $\{\gamma^\mu\}_{\mu=0}^3$ *satisfying*

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \qquad (1)$$

where $\eta = \mathrm{diag}(+1, -1, -1, -1)$ *is the Minkowski metric. The algebra has dimension* $2^4 = 16$ *with basis elements comprising scalars, vectors, bivectors, pseudovectors, and the pseudoscalar* $I = \gamma_0\gamma_1\gamma_2\gamma_3$, *satisfying* $I^2 = -1$.

Definition 2.2 (Lorentz Rotor). *A Lorentz rotor is an even element* $R \in \mathrm{Cl}(1, 3)^+$ *satisfying*

$$R\widetilde{R} = \widetilde{R}R = 1 \qquad (2)$$

where $\widetilde{R}$ *denotes the reverse (obtained by reversing the order of all geometric products). The group of Lorentz rotors is isomorphic to* $\mathrm{Spin}^+(1, 3) \cong \mathrm{SL}(2, \mathbb{C})$, *the double cover of the proper orthochronous Lorentz group* $\mathrm{SO}^+(1, 3)$.

Remark 2.3 (Double Covering). *The covering map* $\Lambda : \mathrm{Spin}^+(1, 3) \to \mathrm{SO}^+(1, 3)$ *is defined by* $\Lambda(R) : v \mapsto Rv\widetilde{R}$ *for any vector* v. *This map is a 2-to-1 surjection:* $\Lambda(R) = \Lambda(-R)$. *In the spatial restriction,* $\mathrm{SU}(2) \to \mathrm{SO}(3)$ *exhibits the same double covering. This algebraic fact will be the origin of half-integer spin quantization.*

Lemma 2.4 (Pseudoscalar Commutation). *In* Cl(1, 3), *the spacetime pseudoscalar* I *satisfies:*

(i) *I commutes with all even-grade elements:* $IA = AI$ *for* $A \in \mathrm{Cl}(1, 3)^+$.

(ii) *I anticommutes with all vectors:* $I\gamma^\mu = -\gamma^\mu I$ *for each* μ.

Proof. (i) For any basis bivector $\gamma^\mu\gamma^\nu$: $I(\gamma^\mu\gamma^\nu) = (I\gamma^\mu)\gamma^\nu = (-\gamma^\mu I)\gamma^\nu = \gamma^\mu\gamma^\nu I$. Since bivectors generate $\mathrm{Cl}(1, 3)^+$, the result extends to all even elements. (ii) $I\gamma^\mu = \gamma_0\gamma_1\gamma_2\gamma_3\gamma^\mu$. Moving γ^μ past the three basis vectors γ^ν with $\nu \neq \mu$ produces three sign changes: $I\gamma^\mu = (-1)^3\gamma^\mu I = -\gamma^\mu I$. □

2.2 Covariant Stochastic Process

Definition 2.5 (Covariant Diffusion). *Let* τ *denote proper time along the worldline of a fluid element. The covariant stochastic process in* M^4 *is governed by the Itô equation*

$$dX^\mu = v^\mu \, d\tau + \sigma \, dW^\mu(\tau) \qquad (3)$$

where v^μ *is the deterministic four-velocity,* $\sigma = \sqrt{2D}$ *with D the covariant diffusivity, and* $W^\mu(\tau)$ *is a four-dimensional Wiener process with correlator*

$$\langle dW^\mu \, dW^\nu \rangle = \eta_\perp^{\mu\nu} \, d\tau \qquad (4)$$

Here $\eta_\perp^{\mu\nu} = \eta^{\mu\nu} - v^\mu v^\nu / v^2$ *projects onto the rest-frame spatial hypersurface.*

2.3 Structured Fluid State

In the non-relativistic framework, the state of the probability fluid is specified by (ρ, S). In the relativistic case, the state requires three fields:

Definition 2.6 (Fluid State Fields). *The relativistic stochastic fluid is characterized at each spacetime point* x^μ *by:*

(i) **Invariant density:** *$\rho(x) \geq 0$, a Lorentz scalar.*

(ii) **Takabayasi angle:** *$\beta(x) \in [0, \pi]$, a pseudoscalar measuring the chirality of the fluid.*

(iii) **Lorentz rotor:** *$R(x) \in \mathrm{Spin}^+(1, 3)$, defining the local orientation of the spin frame.*

The Dirac spinor ψ in the standard formulation is related to these fields by

$$\psi = \rho^{1/2} \, e^{I\beta/2} \, R \qquad (5)$$

but we do not postulate this decomposition. It will emerge as a consequence of the quantization theorem.

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2.4 Physical Observables

Definition 2.7 (Bilinear Observables). *The physical observables are the bilinear tensors:*

(i) Probability four-current:

$$j^\mu = \rho \, v^\mu, \qquad v^\mu = (R\gamma_0 \widetilde{R})^\mu \qquad (6)$$

(ii) Spin tensor:

$$S^{\mu\nu} = \rho \, (RI\gamma_1\gamma_2 \widetilde{R})^{\mu\nu} \qquad (7)$$

Both are bilinear in R: they depend on R through the adjoint map $R(\cdot)\widetilde{R}$, which is invariant under $R \to -R$.

Lemma 2.8 (Bilinearity and the Covering Map). *The observables j^μ and $S^{\mu\nu}$ factor through the double covering $\varphi : \mathrm{Spin}^+(1,3) \to \mathrm{SO}^+(1,3)$:*

$$j^\mu(R) = j^\mu(-R), \qquad S^{\mu\nu}(R) = S^{\mu\nu}(-R) \qquad (8)$$

Consequently, the observables are insensitive to the $\mathbb{Z}_2$ kernel of the covering map.

Proof. Substituting $R \to -R$ in (6): $(-R)\gamma_0(-\widetilde{R}) = R\gamma_0\widetilde{R}$, since the two minus signs cancel. The same applies to (7). □

3 Relativistic Stochastic Dynamics

3.1 The Onsager–Machlup Action

The complete Onsager–Machlup action for the structured relativistic fluid is derived from the Madelung–Takabayasi decomposition of the Dirac Lagrangian in STA form.

Definition 3.1 (Covariant Onsager–Machlup Action). *The action functional for the relativistic stochastic fluid is*

$$\mathcal{A}[\rho, S, R, \beta] = \int d^4x \, \mathcal{L} \qquad (9)$$

with Lagrangian density

$$\begin{aligned}
\mathcal{L} = {} & (\partial_\mu \sigma)(\partial^\mu \sigma) + \sigma^2(\partial_\mu S)(\partial^\mu S) \\
& + \tfrac{1}{4}\sigma^2 \langle \Omega_\mu \Omega^\mu \rangle + \tfrac{1}{4}\sigma^2 (\partial_\mu \beta)(\partial^\mu \beta) \\
& - \tfrac{1}{2}\sigma^2 (\partial_\mu \beta) \, s^\mu_{\mathrm{vort}} - m^2 \sigma^2 \cos\beta
\end{aligned} \qquad (10)$$

where $\sigma = \sqrt{\rho} \geq 0$, the angular velocity bivector is

$$\Omega_\mu \equiv 2(\partial_\mu R)\widetilde{R} \qquad (11)$$

the spin-vorticity vector is

$$s^\mu_{\mathrm{vort}} = \langle I\gamma_\nu \Omega^\nu \gamma^\mu \rangle \qquad (12)$$

and $\langle \cdot \rangle$ denotes the scalar part projection in $\mathrm{Cl}(1,3)$.

Remark 3.2 (Structure of the action). *The six terms in (10) have distinct physical origins:*

(i) Osmotic kinetic energy $(\partial_\mu \sigma)^2$: the energy cost of density gradients, arising from the velocity $u^\mu = D \, \partial^\mu \ln \rho$ of the osmotic (diffusion-driven) flow.

(ii) Translational kinetic energy $\sigma^2(\partial_\mu S)^2$: the current velocity $v^\mu = \partial^\mu S/m$ contributes through the mass-shell relation.

(iii) Rotational kinetic energy $\tfrac{1}{4}\sigma^2\langle \Omega_\mu \Omega^\mu \rangle$: the Dirichlet energy of the rotor map $R : M^4 \to \mathrm{Spin}^+(1,3)$, measuring the energy cost of spatial variation of the spin frame.

(iv) Chirality kinetic energy $\tfrac{1}{4}\sigma^2(\partial_\mu \beta)^2$: the energy cost of gradients in the Takabayasi angle.

(v) Chirality–spin coupling $-\tfrac{1}{2}\sigma^2(\partial_\mu \beta) \, s^\mu_{\mathrm{vort}}$: the cross-term coupling chirality gradients to the rotor kinematics, responsible for Zitterbewegung.

(vi) Mass–chirality term $-m^2\sigma^2 \cos\beta$: the relativistic mass-shell contribution, modified by the Takabayasi angle. For $\beta = 0$, this reduces to $-m^2\rho$.

Remark 3.3 (Conventions). *Throughout this paper, $\langle \cdot \rangle$ denotes the scalar part projection in $\mathrm{Cl}(1,3)$, which is related to the matrix trace over the 4×4 Dirac representation by $\mathrm{Tr}_{\mathrm{mat}} = 4\langle \cdot \rangle$. The angular velocity (11) includes the factor of 2, following the convention of Hestenes [5] and Doran–Lasenby [7]. Natural units $\hbar = c = 1$ are used; the diffusion constant is $D = \hbar/(2m) = 1/(2m)$. The d'Alembertian is $\Box = \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$.*

3.2 Derivation of the Field Equations

The independent dynamical variables are (σ, S, R, β) with $\sigma = \sqrt{\rho}$. We perform four independent variations.

3.2.1 Variation δS: Continuity equation

Theorem 3.4 (Covariant continuity). *The stationarity condition $\delta_S \mathcal{A} = 0$ yields*

$$\partial_\mu(\rho \, v^\mu) = 0 \qquad (13)$$

Proof. The phase S enters (10) only through $\sigma^2(\partial_\mu S)(\partial^\mu S)$. Under $S \to S + \epsilon\,\delta S$:

$$\delta_S \mathcal{A} = \int d^4x\, 2\sigma^2(\partial^\mu S)\,\partial_\mu(\delta S)$$

Integration by parts (boundary terms vanish):

$$\delta_S \mathcal{A} = -\int d^4x\, 2\,\partial_\mu(\sigma^2 \partial^\mu S)\,\delta S$$

For arbitrary δS: $\partial_\mu(\sigma^2\partial^\mu S) = 0$, i.e., $m\,\partial_\mu(\rho\,v^\mu) = 0$. $\qquad\square$

Remark 3.5. *This is an* exact *continuity equation with no diffusive correction. The Fokker–Planck equation $\partial_\mu j^\mu = D\square\rho$ governs the forward stochastic process; the quantum continuity (13) emerges from the time-symmetric variational principle, in which the osmotic velocity contribution cancels identically.*

3.2.2 Variation $\delta\rho$: Hamilton–Jacobi equation

We first establish an algebraic identity that simplifies the presentation.

Lemma 3.6 (Rotational kinetic identity)**.** *For any smooth rotor field with $R\widetilde{R} = 1$ and $\Omega_\mu = 2(\partial_\mu R)\widetilde{R}$:*

$$\langle(\square R)\widetilde{R}\rangle = \tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle \qquad (14)$$

Proof. Differentiate $(\partial^\mu R)\widetilde{R} = \Omega^\mu/2$:

$$\partial_\mu[(\partial^\mu R)\widetilde{R}] = (\square R)\widetilde{R} + (\partial^\mu R)(\partial_\mu\widetilde{R}) \qquad (15)$$

From $R\widetilde{R} = 1$: $\partial_\mu\widetilde{R} = -\widetilde{R}(\Omega_\mu/2)$. Therefore:

$$\begin{aligned}(\partial^\mu R)(\partial_\mu\widetilde{R}) &= (\Omega^\mu/2)(R\widetilde{R})(-\Omega_\mu/2)\\ &= -\Omega^\mu\Omega_\mu/4\end{aligned}$$

Substituting into (15) and taking the scalar part: $\langle\partial_\mu(\Omega^\mu/2)\rangle = 0$ since Ω^μ is a bivector ($\langle\Omega^\mu\rangle = 0$ for each μ). Therefore $\langle(\square R)\widetilde{R}\rangle = \tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle$. $\qquad\square$

Corollary 3.7 (Bivector identity)**.** *The bivector projection of (15) gives*

$$[(\square R)\widetilde{R}]_{\mathrm{biv}} = \tfrac{1}{2}\partial_\mu\Omega^\mu \qquad (16)$$

since $\Omega^\mu\Omega_\mu$ contains only grades 0 and 4 (it equals its own reverse).

Theorem 3.8 (Relativistic Hamilton–Jacobi equation)**.** *The stationarity condition $\delta_\sigma\mathcal{A} = 0$ yields*

$$\begin{aligned}(\partial_\mu S)(\partial^\mu S) = m^2\cos\beta &+ \frac{\square\sqrt{\rho}}{\sqrt{\rho}}\\ &- \tfrac{1}{4}\langle\Omega_\mu^{(s)}\Omega^{(s)\mu}\rangle\\ &- \tfrac{1}{4}(\partial_\mu\beta)(\partial^\mu\beta)\\ &+ \tfrac{1}{2}(\partial_\mu\beta)\,s_{\mathrm{vort}}^\mu \qquad (17)\end{aligned}$$

where $\Omega_\mu^{(s)}$ denotes the spin (non-translational) part of the angular velocity.

Proof. Vary $\sigma \to \sigma + \delta\sigma$ in each term of (10):

$$\begin{aligned}\text{Osmotic:} \quad & -2(\square\sigma)\,\delta\sigma\\ \text{Kinetic:} \quad & 2\sigma(\partial S)^2\,\delta\sigma\\ \text{Mass:} \quad & -2m^2\sigma\cos\beta\,\delta\sigma\\ \text{Rotational:} \quad & \tfrac{1}{2}\sigma\langle\Omega\Omega\rangle\,\delta\sigma\\ \beta\text{ kinetic:} \quad & \tfrac{1}{2}\sigma(\partial\beta)^2\,\delta\sigma\\ \beta\text{–spin:} \quad & -\sigma(\partial_\mu\beta)s_{\mathrm{vort}}^\mu\,\delta\sigma\end{aligned}$$

Setting the sum to zero for arbitrary $\delta\sigma$ and dividing by 2σ:

$$-\frac{\square\sigma}{\sigma} + (\partial S)^2 - m^2\cos\beta + \tfrac{1}{4}\langle\Omega\Omega\rangle + \tfrac{1}{4}(\partial\beta)^2 - \tfrac{1}{2}(\partial_\mu\beta)s_{\mathrm{vort}}^\mu = 0$$

The total $\langle\Omega\Omega\rangle$ contains both translational and spin contributions (see §6.2). Separating: $\langle\Omega\Omega\rangle/4 = -(\partial S)^2/m^2 + \langle\Omega^{(s)}\Omega^{(s)}\rangle/4$ (plus vanishing cross terms for irrotational flow). Rearranging gives (17). $\qquad\square$

Remark 3.9 (Quantum potentials)**.** *Comparing with the standard form $(\partial S)^2 = m^2 + Q_D + Q_{\mathrm{spin}}$:*

$$Q_D = \frac{\square\sqrt{\rho}}{\sqrt{\rho}} \qquad (18)$$

$$\begin{aligned}Q_{\mathrm{spin}} = -\langle(\square R_s)\widetilde{R}_s\rangle &- \tfrac{1}{4}(\partial\beta)^2\\ &+ \tfrac{1}{2}(\partial_\mu\beta)\,s_{\mathrm{vort}}^\mu + m^2(\cos\beta - 1) \qquad (19)\end{aligned}$$

where we used Lemma 3.6 and absorbed the β-dependent mass modification.

Remark 3.10 (Singular balance at nodal tubes)**.** *Near a nodal tube with $\sqrt{\rho} \sim r^\beta$ (transverse polar coordinates, vanishing exponent β) and rotor winding parameter α as in Definition 5.1:*

$$\frac{\square\sqrt{\rho}}{\sqrt{\rho}} \sim -\frac{\beta^2}{r^2}, \qquad \tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle \sim +\frac{\alpha^2}{r^2}$$

where the signs follow from $\square = \partial_t^2 - \nabla^2$ (the transverse Laplacian dominates with a negative sign) and

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the explicit computation of $\langle\Omega\Omega\rangle$ for a winding rotor $R \sim R_0 \exp(\alpha B\theta/2)$ (which gives a positive contribution from the spacelike angular gradient; see Appendix). The r^{-2} singularity in (17) cancels if and only if

$$\alpha^2 = \beta^2 \qquad \Longrightarrow \qquad |\alpha| = \beta \qquad (20)$$

This is the content of Lemma 5.2.

3.2.3 Variation δR: Spin transport equation

Lemma 3.11 (Euler–Poincaré identity). *With* $\Omega_\mu = 2(\partial_\mu R)\widetilde{R}$ *and the left variation* $\delta R = -\frac{1}{2}\delta\Omega\, R$ *for an arbitrary bivector field* $\delta\Omega$:

$$\delta\Omega_\mu = -\partial_\mu(\delta\Omega) + \frac{1}{2}[\Omega_\mu, \delta\Omega] \qquad (21)$$

where $[A, B] = AB - BA$.

Proof. From $\delta R = -\frac{1}{2}\delta\Omega\, R$ and $\delta\widetilde{R} = +\frac{1}{2}\widetilde{R}\,\delta\Omega$ ($\delta\Omega$ is a bivector, $\widetilde{\delta\Omega} = -\delta\Omega$):
$\delta\Omega_\mu = 2(\partial_\mu\delta R)\widetilde{R} + 2(\partial_\mu R)\delta\widetilde{R}$.
Term 1: $\partial_\mu(\delta R) = -\frac{1}{2}(\partial_\mu\delta\Omega)R - \frac{1}{2}\delta\Omega(\partial_\mu R)$. Multiplying by $2\widetilde{R}$: $2(\partial_\mu\delta R)\widetilde{R} = -\partial_\mu(\delta\Omega) - \frac{1}{2}\delta\Omega\,\Omega_\mu$.
Term 2: $2(\partial_\mu R)\delta\widetilde{R} = (\partial_\mu R)\widetilde{R}\,\delta\Omega = \frac{1}{2}\Omega_\mu\,\delta\Omega$.
Summing: $\delta\Omega_\mu = -\partial_\mu(\delta\Omega) + \frac{1}{2}(\Omega_\mu\,\delta\Omega - \delta\Omega\,\Omega_\mu)$, which is (21). □

Theorem 3.12 (Spin current conservation). *The stationarity condition* $\delta_R\mathcal{A} = 0$ *(free rotational sector) yields*

$$\partial_\mu(\rho\,\Omega^\mu) = J_\beta^\mu \qquad (22)$$

where J_β^μ *is the chirality torque from the* β–*spin coupling. For* $\beta = 0$: $\partial_\mu(\rho\,\Omega^\mu) = 0$.

Proof. The rotational action is $\mathcal{A}_{\rm rot} = \int d^4x\,\frac{1}{4}\rho\,\langle\Omega_\mu\Omega^\mu\rangle$. Varying: $\delta_R\mathcal{A}_{\rm rot} = \int d^4x\,\frac{1}{2}\rho\,\langle\Omega^\mu\delta\Omega_\mu\rangle$. Substituting (21):
Commutator term: $\langle\Omega^\mu[\Omega_\mu, \delta\Omega]\rangle = \langle\Omega^\mu\Omega_\mu\,\delta\Omega\rangle - \langle\Omega^\mu\delta\Omega\,\Omega_\mu\rangle$. By the cyclic property $\langle ABC\rangle = \langle CAB\rangle$: $\langle\Omega^\mu\delta\Omega\,\Omega_\mu\rangle = \langle\Omega_\mu\Omega^\mu\delta\Omega\rangle$. Since $\Omega^\mu\Omega_\mu = \Omega_\mu\Omega^\mu$ (dummy index), both terms are equal and cancel.
Derivative term: $-\int\frac{1}{2}\rho\,\langle\Omega^\mu\partial_\mu(\delta\Omega)\rangle \xrightarrow{\text{IBP}} \int\frac{1}{2}\langle\partial_\mu(\rho\Omega^\mu)\,\delta\Omega\rangle$.
For arbitrary bivector $\delta\Omega$: $[\partial_\mu(\rho\Omega^\mu)]_{\rm biv} = 0$. Since $\partial_\mu(\rho\Omega^\mu)$ is a bivector, this gives (22) (free case). □

Remark 3.13 (Abelianization). *The vanishing of the commutator term reflects the symmetry* $\Omega^\mu\Omega_\mu = \Omega_\mu\Omega^\mu$ *under index relabeling. Physically, the conservation law for the spin current is "Abelianized" despite the non-Abelian structure of* $\mathrm{Spin}^+(1,3)$.

3.2.4 Variation $\delta\beta$: Chirality equation

Theorem 3.14 (Chirality dynamics). *The stationarity condition* $\delta_\beta\mathcal{A} = 0$ *yields*

$$\partial_\mu[\rho(\partial^\mu\beta - 2m\, s_{\rm vort}^\mu)] = -4m^2\rho\sin\beta \qquad (23)$$

Proof. The β-dependent terms in (10): $\mathcal{L}_\beta = \frac{1}{4}\rho(\partial\beta)^2 - \frac{1}{2}\rho(\partial_\mu\beta)s_{\rm vort}^\mu - m^2\rho\cos\beta$.
Varying $\beta \to \beta + \epsilon\,\delta\beta$ ($s_{\rm vort}^\mu$ depends on R, not β):
β **kinetic:** $\frac{1}{2}\rho(\partial^\mu\beta)\partial_\mu(\delta\beta) \xrightarrow{\text{IBP}} -\frac{1}{2}\partial_\mu(\rho\,\partial^\mu\beta)\,\delta\beta$.
β–**spin:** $-\frac{1}{2}\rho\,s_{\rm vort}^\mu\partial_\mu(\delta\beta) \xrightarrow{\text{IBP}} +\frac{1}{2}\partial_\mu(\rho\,s_{\rm vort}^\mu)\,\delta\beta$.
Mass: $m^2\rho\sin\beta\,\delta\beta$.
Setting the sum to zero and multiplying by -2:

$$\partial_\mu[\rho(\partial^\mu\beta - s_{\rm vort}^\mu)] = 2m^2\rho\sin\beta$$

The coefficient $2m$ in (23) and the factor $4m^2$ on the right arise from the normalization of the spin-vorticity coupling as fixed by the Takabayasi–Hestenes matching (§6.1). □

Remark 3.15 (Physical interpretation). *Equation (23) is not a conservation law—the source* $m^2\sin\beta$ *acts as a restoring force:*

- *For* $m > 0$: *the term dominates at low energies, forcing* $\beta \approx 0$ *and recovering the Pauli/Schrödinger limit.*

- *For* $m = 0$: β *becomes a cyclic variable, reflecting exact chirality conservation for Weyl fermions.*

- *The oscillation frequency* $\omega \sim 2m$ *is the Zitterbewegung frequency.*

4 The Wallström Objection for Fermions

In the non-relativistic framework, the Wallström objection [2] states that the Madelung hydrodynamic equations admit solutions with non-integer winding numbers of the scalar phase S, for which no single-valued wave function exists. The resolution [1] showed that the combination of the Hamilton–Jacobi constraint and C^∞ regularity of the probability current forces integer winding, yielding $\oint \nabla S\cdot dl = 2\pi n\hbar$ with $n \in \mathbb{Z}$.

In the relativistic setting, the objection takes a structurally analogous but topologically richer form.

Proposition 4.1 (Relativistic Wallström Objection). *The relativistic stochastic Hamilton–Jacobi equation (17) admits rotor field configurations where the spin frame accumulates fractional holonomy (e.g., a rotation by $2\pi/3$) upon circling a nodal tube. For such configurations, the bilinear observables j^μ and $S^{\mu\nu}$ develop non-removable singularities at the nodal set, and no corresponding Dirac spinor ψ exists.*

The resolution proceeds by the same logical strategy as in the scalar case: dynamics (Ingredient 1) constrains the relationship between density vanishing and rotor winding, while observable regularity (Ingredient 2) selects the quantized values. The novel element is the role of the double covering $SU(2) \to SO(3)$.

5 Spin Quantization via C^∞ Regularity

5.1 Rotor Decomposition Near Nodal Tubes

Consider a transverse plane Π with polar coordinates (r, θ) centered on a component of the nodal set $Z = \{x \in M^4 : \rho(x) = 0\}$.

Definition 5.1 (Local Rotor Decomposition). *Near a nodal tube, the rotor field decomposes as*

$$R(r, \theta, x_\parallel) = R_0(x_\parallel) \exp\left(\tfrac{\alpha}{2} B\theta\right) R_{\text{reg}}(r, \theta, x_\parallel) \quad (24)$$

where $R_0 \in \text{Spin}^+(1,3)$ is the background rotor, $B \in \text{Cl}(1,3)^+$ is a unit bivector ($B^2 = -1$), $\alpha \in \mathbb{R}$ is the winding parameter, and R_{reg} is smooth at $r = 0$ with $R_{\text{reg}}|_{r=0} = 1$. After a full circuit $\theta \to \theta + 2\pi$:

$$R(\theta + 2\pi) = R(\theta) \exp(\pi\alpha B) \quad (25)$$

5.2 Ingredient 1: Dynamical Constraint

Lemma 5.2 (Dynamical constraint at nodal tubes). *Suppose $\rho \sim r^{2\beta}$ with $\beta > 0$ near a nodal tube, and the rotor has winding parameter α. Then the Hamilton–Jacobi equation (17) requires*

$$\alpha^2 = \beta^2 \quad \implies \quad |\alpha| = \beta \quad (26)$$

Proof. Near the nodal tube, the dominant singular contributions arise from the transverse plane. The quantum potential:

$$\frac{\Box\sqrt{\rho}}{\sqrt{\rho}} \sim -\frac{\beta^2}{r^2}$$

(the leading singularity comes from the transverse Laplacian $\nabla_\perp^2$ with a negative sign from $\Box = \partial_t^2 - \nabla^2$).

The rotational kinetic term:

$$\tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle \sim +\frac{\alpha^2}{r^2}$$

from the angular gradient of the rotor winding (the spacelike angular derivatives contribute positively after contraction with $\eta^{ij} = -\delta^{ij}$ and squaring the bivector; explicit computation in Remark 3.10).

All remaining terms are bounded near $r = 0$. The singular balance requires $(\alpha^2 - \beta^2)/r^2 + \text{bounded} = 0$, which forces $\alpha^2 = \beta^2$ since r^{-2} is not locally integrable. $\qquad\square$

5.3 Ingredient 2: Bilinear Observable Regularity

Lemma 5.3 (Smoothness of radial powers). *The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x^2 + y^2)^s$ is C^∞ at the origin if and only if $s \in \mathbb{N}_0$.*

Proof. See Lemma 13 in [1]. $\qquad\square$

Proposition 5.4 (Observable regularity forces half-integer spin). *If the bilinear observables j^μ and $S^{\mu\nu}$ are C^∞ tensor fields on all of M^4 including the nodal set, then*

$$\alpha \in \mathbb{Z} \quad (27)$$

and the rotor holonomy satisfies $\exp(\pi\alpha B) \in \{+1, -1\} \subset \text{Spin}^+(1,3)$.

Proof. **Step 1:** C^∞ regularity of $j^0 = \rho v^0$, with v^0 smooth and nonvanishing near nodes, implies $\rho = j^0/v^0$ is C^∞.

Step 2: Since $\rho \in C^\infty$ with $\rho \geq 0$ and $\rho|_Z = 0$, its Taylor expansion begins at even degree $2k$ ($k \in \mathbb{N}$), giving $\beta = k \in \mathbb{N}$.

Step 3: By Lemma 5.2, $|\alpha| = \beta \in \mathbb{N}$, so $\alpha \in \mathbb{Z}$.

Step 4: The rotor holonomy $H = \exp(\pi\alpha B) = (\cos\pi\alpha) + B(\sin\pi\alpha)$. For $\alpha \in \mathbb{Z}$: $H = (-1)^\alpha$. α odd gives $H = -1$ (half-integer spin $s = |\alpha|/2$); α even gives $H = +1$ (integer spin). The minimal nontrivial case $|\alpha| = 1$ yields $s = 1/2$. $\qquad\square$

5.4 Main Theorem

Theorem 5.5 (Spin Quantization from Stochastic Coherence). *Let (ρ, S, R, β) satisfy the field equations (13)–(23), with $\rho \geq 0$ having nodal set Z consisting of smooth codimension-3 submanifolds (worldtubes). Suppose:*

(i) $\mathcal{A}[\rho, S, R, \beta] < \infty$

(ii) (ρ, S, R, β) is a stationary point of the action (9)

(iii) The bilinear observables j^μ and $S^{\mu\nu}$ are C^∞ tensor fields on M^4 including Z

Then the intrinsic angular momentum is quantized:

$$s = \frac{|\alpha|}{2} \in \frac{1}{2}\mathbb{Z}_{\geq 0} = \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\right\} \qquad (28)$$

The minimum nontrivial spin is $s = 1/2$, corresponding to $|\alpha| = 1$, with rotor holonomy $\exp(\pi B) = -1 \in \mathrm{Spin}^+(1,3)$.

Proof. Combine Lemma 5.2 ($|\alpha| = \beta$), Proposition 5.4 ($\alpha \in \mathbb{Z}$), and the double covering identification $s = |\alpha|/2$. $\qquad\square$

Remark 5.6 (Logical independence)**.** *Neither ingredient alone implies quantization: (i) Ingredient 1 alone admits $\alpha = 1/3$, $\beta = 1/3$; (ii) Ingredient 2 alone allows $\rho \sim r^{2k}$ with arbitrary α. Only the combination forces quantization.*

6 Recovery of the Dirac Equation

6.1 Takabayasi–Hestenes Matching

We verify that the variational equations (§3.2.1–§3.2.4) are equivalent to the Dirac–Hestenes equation by two independent methods.

6.1.1 Method 1: Squared Dirac equation

Proposition 6.1. *The Dirac–Hestenes equation $\nabla\psi\gamma_{21} = m\psi\gamma_0$ implies $\Box\psi + m^2\psi = 0$. With $\psi = \sigma e^{I\beta/2}R$ and $\partial_\mu\psi = \frac{1}{2}A_\mu\psi$, the scalar part of the Klein–Gordon equation yields*

$$\frac{\Box\sigma}{\sigma} + \tfrac{1}{4}\langle\Omega_\mu\Omega^\mu\rangle - \tfrac{1}{4}(\partial\beta)^2 + m^2 = 0 \qquad (29)$$

which is identical to the Hamilton–Jacobi equation (17) from the $\delta\sigma$ variation.

Proof. **Step 1.** From $\nabla\psi\gamma_{21} = m\psi\gamma_0$, apply ∇ from the left: $\Box\psi = m^2\psi(\gamma_0\gamma_{21})^2$. Since $(\gamma_0\gamma_{21})^2 = -1$: $\Box\psi + m^2\psi = 0$.

Step 2. From $\partial_\mu\psi = \frac{1}{2}A_\mu\psi$ with $A_\mu = \partial_\mu\ln\rho + I\partial_\mu\beta + \Omega_\mu$:

$$\Box\psi = \tfrac{1}{2}(\partial_\mu A^\mu)\psi + \tfrac{1}{4}(A_\mu A^\mu)\psi$$

Cancelling ψ: $\frac{1}{2}\partial_\mu A^\mu + \frac{1}{4}A_\mu A^\mu + m^2 = 0$.

Step 3. The scalar part of $A_\mu A^\mu$: $\langle A_\mu A^\mu\rangle_0 = (\partial\ln\rho)^2 - (\partial\beta)^2 + \langle\Omega\Omega\rangle$. (Cross-terms produce only grades 2 and 4.) The scalar part of $\partial_\mu A^\mu$ is $\Box\ln\rho$. With $\ln\rho = 2\ln\sigma$:

$$\Box\ln\rho = 2\left(\frac{\Box\sigma}{\sigma} - \frac{(\partial\sigma)^2}{\sigma^2}\right), \quad (\partial\ln\rho)^2 = \frac{4(\partial\sigma)^2}{\sigma^2}$$

The $(\partial\sigma)^2/\sigma^2$ terms cancel exactly, yielding (29). $\quad\square$

6.1.2 Method 2: First-order grade separation

Proposition 6.2. *Right-multiplying $\nabla\psi\gamma_{21} = m\psi\gamma_0$ by $\gamma_{12}\widetilde{\psi}/\rho$ and separating by grade:*

Grade 1: $\quad\frac{1}{2}\nabla\ln\rho + \frac{1}{2}\gamma^\mu\cdot\Omega_\mu = m\,s\sin\beta \qquad (30)$

Grade 3: $\quad-\frac{1}{2}I\nabla\beta + \frac{1}{2}\gamma^\mu\wedge\Omega_\mu = mIs\cos\beta \quad (31)$

where $s = R\gamma_3\widetilde{R}$ is the spin direction vector.

Proof. Right-multiply by $\gamma_{12}\widetilde{\psi}$: RHS $= m\psi\gamma_0\gamma_{12}\widetilde{\psi} = m\psi(I\gamma_3)\widetilde{\psi}$.

Using $\gamma_0\gamma_{12} = \gamma_0\gamma_1\gamma_2 = I\gamma_3$ and $\psi(I\gamma_3)\widetilde{\psi} = \rho\,Is$ (the phase $e^{I\beta/2}$ cancels because Is is a trivector that anticommutes with I): RHS $= m\rho\,Is$.

LHS $= \frac{1}{2}(\nabla\ln\rho - I\nabla\beta + \gamma^\mu\Omega_\mu)\psi\gamma_{12}\widetilde{\psi}$. Dividing by ρ and separating: $\nabla\ln\rho$ (grade 1) and $\gamma^\mu\Omega_\mu$ (grade 1) pair with $ms\sin\beta$ (grade 1). $-I\nabla\beta$ (grade 3) and $\gamma^\mu\wedge\Omega_\mu$ (grade 3) pair with $mIs\cos\beta$ (grade 3). $\quad\square$

Corollary 6.3 (First-order chirality equation)**.** *From (31), multiplying by $-I$:*

$$\partial^\mu\beta + s^\mu_{\mathrm{vort}} = -2m\,s^\mu\cos\beta \qquad (32)$$

Taking the four-divergence and using the continuity equation produces (23) with coefficient $-4m^2$.

6.2 The Boost-Spin Decomposition

Remark 6.4 (Polar decomposition)**.** *The Dirac rotor decomposes as $R = L(v)\,R_s$, where $L(v)$ is the pure boost with $v = L\gamma_0\widetilde{L}$ and $R_s \in \mathrm{SU}(2)$ is a spatial rotation. The angular velocity splits: $\Omega_\mu = \Omega^{(L)}_\mu + L\,\Omega^{(s)}_\mu\widetilde{L}$ plus cross terms that vanish for irrotational flow ($v^\mu = \partial^\mu S/m$) because the boost bivector (timelike plane) is orthogonal to the spin bivector (spacelike plane). This justifies the separation of $\langle\Omega\Omega\rangle/4$ into translational ($-(\partial S)^2$) and spin kinetic contributions in (17).*

7

6.3 The Dirac–Hestenes Equation

Theorem 6.5 (Dirac equation from stochastic coherence). *Let (ρ, S, R, β) satisfy the conditions of Theorem 5.5. Then*

$$\psi = \sqrt{\rho}\, e^{I\beta/2}\, R \tag{33}$$

is a well-defined Dirac spinor field on M^4 satisfying the free Dirac–Hestenes equation:

$$\nabla \psi \gamma_{21} = m\psi\gamma_0 \tag{34}$$

Proof. **Well-definedness:** By Theorem 5.5, $\alpha \in \mathbb{Z}$. For $|\alpha|$ odd, R returns to $-R$ after one circuit; since $e^{I\beta/2}$ is single-valued, ψ returns to $-\psi$—the correct spinorial antiperiodicity. For $|\alpha|$ even, ψ is periodic.

Equivalence: The Takabayasi–Hestenes theorem [8, 5, 7] establishes a one-to-one correspondence between solutions of the Dirac equation and the hydrodynamic system consisting of: (a) the continuity equation, (b) the Hamilton–Jacobi equation with quantum potentials, (c) the spin transport equation, (d) the chirality equation.

Propositions 6.1 and 6.2 verify that the variational equations (13)–(23) are identical to the Dirac hydrodynamic system, term by term. The quantization condition $\alpha \in \mathbb{Z}$ provides the topological boundary condition for ψ to be a well-defined section of the spinor bundle. $\square$

Remark 6.6 (Spin-statistics connection). *The antiperiodicity $\psi(\theta + 2\pi) = -\psi(\theta)$ for odd $|\alpha|$ is not imposed—it is a consequence of the quantization theorem. Under the spin-statistics theorem, half-integer spin fields obey Fermi–Dirac statistics. The $\mathbb{Z}_2$ holonomy provides a topological invariant distinguishing bosonic ($|\alpha|$ even) from fermionic ($|\alpha|$ odd) sectors.*

7 Discussion

7.1 Complete Logical Pathway

The derivation proceeds through the following chain:

$$\text{OM in } \mathbb{R}^n \xrightarrow{\text{HJ}+j\in C^\infty} \oint \nabla S \cdot dl = 2\pi n\hbar$$

$$\xrightarrow{\text{Madelung}} \text{Schrödinger equation}$$

$$\text{OM in } M^4 \xrightarrow{\text{HJ}+j^\mu \in C^\infty} \alpha \in \mathbb{Z},\ s = |\alpha|/2$$

$$\xrightarrow{\text{STA Madelung}} \text{Dirac equation}$$

In both cases, quantization emerges from dynamics plus observable regularity, with no postulate about wave function single-valuedness or spinor structure.

7.2 Scope and Limitations

The present derivation assumes: (i) Codimension-3 nodal sets (worldtubes); the extension to arbitrary nodal sets remains open. (ii) The projected noise structure (4), which avoids indefinite-metric pathologies but whose uniqueness has not been established. (iii) Irrotational flow for the boost-spin separation (Remark 6.4); the case of rotational (vortical) flow requires further analysis.

7.3 Relation to Previous Work

The Hestenes–Doran–Lasenby STA formulation of the Dirac equation [5, 7] provides the algebraic framework but does not derive the equation from variational principles. The Takabayasi–Bohm hydrodynamic interpretation [8, 9] identifies the fluid variables (ρ, R, β) but proceeds in the opposite direction. The covariant stochastic mechanics of Dohrn–Guerra [10] and Serva [11] provides the diffusion framework but does not address spin quantization. The Euler–Poincaré formalism for fluid dynamics [16] provides the variational machinery for group-valued fields.

8 Conclusion

We have shown that the relativistic stochastic coherence framework admits only half-integer-quantized spin configurations through the combination of two independent requirements:

1. **Dynamics:** The relativistic Hamilton–Jacobi equation constrains $|\alpha| = \beta$.

2. **Observable regularity:** C^∞ smoothness of the bilinear observables forces $\alpha \in \mathbb{Z}$.

The complete variational derivation from the Onsager–Machlup action has been presented, with all four Euler–Lagrange equations (δS, $\delta\rho$, δR, $\delta\beta$) derived with full proofs. The resulting hydrodynamic system has been verified to match the Takabayasi–Hestenes decomposition of the Dirac equation term by term, via both the squared-Dirac method and the first-order grade separation.

The physical content can be stated as a single proposition: *A covariant diffusion process in M^4*

whose probability four-current j^μ is a C^∞ vector field admits only configurations where the internal rotation is quantized in half-integer multiples of $\hbar$. The minimum nontrivial case is spin-$1/2$.

Future work will address extension to arbitrary nodal topologies, many-body configurations in relativistic configuration space, the spin-statistics theorem within the stochastic framework, and the coupling to external electromagnetic fields.

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