The Koide Angle as a Conformal Dimension: G_2 Geometry, $\mathrm{SU}(3)_3$ WZW Theory, and Fermion Mass Structure

Philippe Marcel Ndiaye*

Independent researcher

March 2026

Abstract

The charged lepton masses satisfy the Koide relation $Q = 2/3$ and are parametrized by a single Brannen phase $\delta_{\mathrm{exp}} = 0.22222(5) \approx 2/9$. We prove that the Brannen parametrization is the exact eigenvalue structure of a democratic element of the exceptional Jordan algebra $J_3(\mathbb{O})$, with $\cos(3\delta) = -\varphi(V)$ where φ is the G_2 3-form on the generation 3-plane.

The distinguished value 2/9 appears independently in five mathematical constructions: a Hessian ratio on $\mathrm{Gr}(3, \mathbb{R}^6)$, a Casimir quotient $C_2(\bar{3})/C_2(\mathrm{Sym}^3 3)$, the conformal dimension $h_\Box$ of $\mathrm{SU}(3)_3$ WZW theory, a crossing phase in conformal blocks, and the Knizhnik–Zamolodchikov singlet exponent. A Bridge Proposition proves these agree if and only if $N = 3$. A Master Identity $C_2(\mathrm{Sym}^N \Box) = k + h^\vee$ uniquely at $N = 3$ implies Casimir ratios equal conformal dimensions for all integrable representations at level $k = 3$.

We prove $Q = 1/3 + d_\Box/6$, making $Q = 2/3$ equivalent to the quantum dimension $d_\Box = 2$. From two hypotheses—the Sumino $\mathrm{SU}(3)_F$ family gauge symmetry (F) and the WZW identification of the Brannen parameters (W)—we derive $\delta = h_\Box = 2/9$ with zero free parameters: the democratic Brannen form follows from the weight geometry of the fundamental representation; the amplitude $A = \sqrt{d_\Box}$ is proven from (F) alone; the phase identification $\delta = h_\Box$ is the central conjecture (W), motivated by five independent characterizations of 2/9, spectral selection, and 0.002% experimental agreement (HFLAV 2025). The T^c identity selects $k = 3$ as the unique level within SU(3), matching the Sumino mechanism independently. Three selection mechanisms confirm uniqueness: spectral positivity, modular self-consistency, and WZW completeness. A Blindness Theorem shows calibrated geometry is exactly insensitive to δ; combined with CP, transcendence, gauge boson blindness, logarithmic sign, and KZ–circulant obstructions, this characterizes the class of mechanisms that the WZW identification bypasses.

Eighteen distinct conditions select $N = 3$ generations. For up-type quarks, $Q_{\mathrm{up}} = 8/9$ at 0.3σ. The neutrino extension is decisively falsified; $Q_\nu = 2/3$ is arithmetically unattainable for neutrino masses in either hierarchy. Eighty-six falsified approaches are cataloged.

Keywords: Koide formula, octonions, G_2 holonomy, exceptional Jordan algebra, Casimir invariants, WZW conformal field theory, neutrino masses, quark masses

*Email: mphilippe.ndiaye@gmail.com

Contents

1 Introduction

The Koide formula [1] relates the charged lepton masses through

$$Q \equiv \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} = 0.666661 \approx \frac{2}{3}, \tag{1}$$

an agreement at the 10^{-5} level using PDG 2024 values [4]. In the Brannen parametrization [2, 3],

$$\sqrt{m_k} = \mu\big(1 + \sqrt{2}\cos(\delta + 2\pi k/3)\big), \quad k = 0, 1, 2, \tag{2}$$

the $\mathbb{Z}_3$-symmetric structure enforces $Q = 2/3$ identically for any δ, while the single phase δ controls the entire mass hierarchy: $m_\mu/m_e \approx 207$ and $m_\tau/m_\mu \approx 16.8$.

The experimental value $\delta_{\text{exp}} = 0.22222(5)$ is remarkably close to $2/9 = 0.22222\ldots$, suggesting an algebraic origin. This paper shows that 2/9 arises independently in multiple mathematical constructions and investigates whether the identification $\delta = 2/9$ can be given a structural explanation. The value 2/9 appears simultaneously as:

0. The exact eigenvalue structure of a democratic $J_3(\mathbb{O})$ element, with $\cos(3\delta) = -\varphi(V)$ (Theorem 2.4);
1. A geometric ratio from the Hessian of the G_2 3-form on $\text{Gr}(3, \mathbb{R}^6)$ (Theorem 3.6);
2. The Casimir quotient $C_2(\bar{3})/C_2(\text{Sym}^3 3)$ in $\text{SU}(3) \subset G_2$ (Observation 3.7);
3. The conformal dimension $h_\square = C_2(\square)/(k + h^\vee)$ of $\text{SU}(3)_3$ WZW theory (Theorem 4.1);
4. The crossing phase $\Delta h \times 2/3$ in $\text{SU}(3)_3$ conformal blocks (Theorem 4.10).

A Bridge Proposition (Proposition 5.1) shows that constructions 1 and 3 coincide if and only if $N = 3$, providing a structural explanation for the agreement rather than a numerical coincidence. The identification $\delta_{\text{phys}} = h_\square = 2/9$ remains a conjecture, supported by 0.002% agreement with experiment (HFLAV 2025) and the uniqueness results established below.

Crucially, we prove that the Brannen parametrization is not merely empirical: it is the exact eigenvalue structure of a democratic element of the exceptional Jordan algebra $J_3(\mathbb{O})$, with the Brannen phase δ identically equal to the angle of the G_2 3-form on the generation 3-plane (Theorem 2.4). The connection to $J_3(\mathbb{O})$ provides a natural algebraic home for the democratic mass matrix ansatz [21, 22, 23]. Only in $J_3(\mathbb{O})$ does the non-associativity of the octonions introduce the 3-form φ as an additional degree of freedom—and it is this degree of freedom that becomes the Brannen phase δ.

We prove six structural obstructions that sharply constrain the class of viable dynamical mechanisms, collectively characterizing δ as a quantity whose origin must be non-perturbative, CP-violating, and topological—consistent with the Chern–Simons interpretation developed in Section 7.

The paper is organized as follows: §2 establishes the mathematical framework including the $J_3(\mathbb{O})$ spectral theorem and the WZW number; §3 derives the geometric ratio 2/9; §4 develops the CFT structure including Q decomposition and KZ–circulant incompatibility; §5 proves the Hessian–WZW Bridge; §6 proves generation selection with eighteen conditions; §7 establishes structural obstructions; §8 proves spectral selection; §11 confronts the neutrino extension with data; §12 extends to the quark sector; §13 discusses the results including G_2 orbifold analysis.

2 Mathematical Framework

2.1 Octonions, G_2, and the exceptional Jordan algebra

The octonions $\mathbb{O}$ are the unique 8-dimensional normed division algebra. Their automorphism group $G_2 = \mathrm{Aut}(\mathbb{O})$ is a 14-dimensional exceptional Lie group acting on $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$. G_2 preserves the associative 3-form [8]

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} + e^{356}, \tag{3}$$

where $\{e_1, \ldots, e_7\}$ is the standard basis of $\mathrm{Im}(\mathbb{O})$ and $e^{ijk} = e^i \wedge e^j \wedge e^k$. For unit imaginary octonions u, v, w, one has $\mathrm{Re}(u \cdot v \cdot w) = -\varphi(u, v, w)$.

Three-generation fermion mass matrices embed naturally in the exceptional Jordan algebra $J_3(\mathbb{O})$, the 27-dimensional algebra of 3×3 Hermitian octonionic matrices [6, 7]. An element $X \in J_3(\mathbb{O})$ has determinant

$$\det(X) = x_1 x_2 x_3 - \sum_k x_k |a_k|^2 + 2\,\mathrm{Re}(a_1 a_2 a_3), \tag{4}$$

where the triple product $\mathrm{Re}(a_1 a_2 a_3)$ is the non-associative term controlled by G_2. Writing $a_k = r_k u_k$ with $u_k \in \mathrm{Im}(\mathbb{O})$ unit, this becomes $-r_1 r_2 r_3\, \varphi(u_1, u_2, u_3)$.

2.2 SU(3) embedding and the 3-form decomposition

The stabilizer of a unit imaginary octonion under G_2 is SU(3). Under this embedding $\mathrm{SU}(3) \subset G_2$, the fundamental representation decomposes as $\mathbf{7} \to \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$, and G_2's 3-form splits as $\varphi = \mathrm{Re}(\Omega) + \omega \wedge e^7$, where $\Omega = dz_1 \wedge dz_2 \wedge dz_3$ is the holomorphic volume form and ω the Kähler form on $\mathbb{R}^6 \cong \mathbb{C}^3$. Restricting to $\mathbb{R}^6 \perp e_7$:

$$\varphi|_{\mathbb{R}^6} = \mathrm{Re}(\Omega) = e^{123} - e^{156} + e^{246} - e^{345}. \tag{5}$$

2.3 Brannen parametrization and $Q = 2/3$

The parametrization (2) is a special case of the generalized form $\sqrt{m_k} = \mu(\psi + \sqrt{2}\,\varepsilon \cos(\delta + 2\pi k/3))$. The $\mathbb{Z}_3$ identities yield

$$Q = \frac{1}{3} + \frac{\varepsilon^2}{3\psi^2}, \tag{6}$$

so $Q = 2/3$ if and only if $\varepsilon = \psi$ (Koide normalization).

Remark 2.1 (Weight projection). The weights of $\square$ of SU(3) lie at the vertices of an equilateral triangle in the Cartan subalgebra $\mathfrak{h}$. Projecting onto a unit direction $\hat{n}(\delta)$ gives $\langle w_k, \hat{n} \rangle = \cos(\delta + 2\pi k/3)$. Thus δ is the orientation of flavor symmetry breaking in the Cartan plane.

2.4 $\mathrm{SU}(3)_3$ WZW model

The SU(3) WZW model at level $k = 3$ has central charge $c = 4$, dual Coxeter number $h^\vee = 3$, and 10 integrable representations [13]. The choice $k = 3$ arises from the Sumino mechanism [5]: integrating out three lepton generations generates $k_{\text{eff}} = 3$ (see §13.7). The conformal dimension is

$$h(R) = \frac{C_2(R)}{k + h^\vee}. \tag{7}$$

For $\square = (1,0)$: $C_2(\square) = 4/3$ and $h_\square = 2/9$. The quantum dimension is $d_\square = \sin(\pi/2)/\sin(\pi/6) = 2$.

Theorem 2.2 (WZW Brannen formula)**.** *With general amplitude A: $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$, the Koide quotient is*

$$Q = \frac{1}{3} + \frac{A^2}{6}. \tag{8}$$

Consequently $Q = 2/3$ iff $A^2 = 2 = d_\square(\mathrm{SU}(3)_3)$, motivating the WZW Brannen formula

$$\sqrt{m_k} = \mu\Big(1 + \sqrt{d_\square}\cos\big(h_\square + 2\pi k/3\big)\Big). \tag{9}$$

Proof. The $\mathbb{Z}_3$ sum rules give $\sum_k m_k = \mu^2(3+3A^2/2)$ and $(\sum_k \sqrt{m_k})^2 = 9\mu^2$. Their ratio gives (8). □

Remark 2.3. $Q = 2/3$ forces $A = \sqrt{2}$, which forces $d_\square = 2$, which (by Proposition 6.5) forces $N = 3$. The WZW Brannen formula encodes both the Koide quotient and the generation number through WZW data of a single model.

2.5 The $J_3(\mathbb{O})$ spectral theorem

Theorem 2.4 ($J_3(\mathbb{O})$ spectral theorem)**.** *Let $X \in J_3(\mathbb{O})$ be a democratic element:*

$$X = \begin{pmatrix} \psi & \bar{a}_3 & a_2 \\ a_3 & \psi & \bar{a}_1 \\ \bar{a}_2 & a_1 & \psi \end{pmatrix}, \quad |a_1| = |a_2| = |a_3| = r, \tag{10}$$

with $a_k = r\,u_k$ for unit imaginary octonions u_k. Let $V = \mathrm{span}(u_1, u_2, u_3) \in \mathrm{Gr}(3, \mathrm{Im}\,\mathbb{O})$. Then:

(a) The characteristic polynomial reduces to $s^3 - 3s + 2\varphi(V) = 0$, $s = (\lambda - \psi)/r$.

(b) The eigenvalues are $\lambda_k = \psi + 2r\cos(\delta + 2\pi k/3)$ where $\cos(3\delta) = -\varphi(V)$.

(c) Under Koide normalization $r = \psi/\sqrt{2}$: $\lambda_k = \psi(1+\sqrt{2}\cos(\delta+2\pi k/3))$, identically the Brannen parametrization.

Proof. Step 1. Using (4) with $x_i = \psi$, $|a_k| = r$, $\mathrm{Re}(a_1a_2a_3) = -r^3\varphi(V)$: $\det(X) = \psi^3 - 3\psi r^2 - 2r^3\varphi(V)$. The trace is 3ψ and $S_2 = 3\psi^2 - 3r^2$. Under $\lambda = \psi + rs$, the characteristic equation reduces to $s^3 - 3s + 2\varphi(V) = 0$.

Step 2. Substituting $s = 2\cos\alpha$: $2\cos(3\alpha) + 2\varphi(V) = 0$, so $\cos(3\alpha) = -\varphi(V) \equiv \cos(3\delta)$.

Step 3. Setting $r = \psi/\sqrt{2}$ gives $\lambda_k = \psi(1 + \sqrt{2}\cos(\delta + 2\pi k/3))$. □

Remark 2.5 (Interpretation). The Brannen parametrization is the exact eigenvalue structure of a democratic $J_3(\mathbb{O})$ element. The phase δ is determined by $\cos(3\delta) = -\varphi(V)$. Positivity requires $\delta < \pi/12$; for $\delta = 2/9$, $\varphi(V) = -\cos(2/3) \approx -0.786 < -1/\sqrt{2}$, confirming the physical generation 3-plane lies in the anti-associative region.

Corollary 2.6. *If the lepton mass matrix corresponds to a democratic $J_3(\mathbb{O})$ element with Koide normalization, then the mass ratios are determined by a single geometric datum: $\varphi(V)$, via $\cos(3\delta) = -\varphi(V)$.*

Remark 2.7 (Non-associativity as mass splitting origin). For associative Jordan algebras $J_3(\mathbb{R})$, $J_3(\mathbb{C})$, $J_3(\mathbb{H})$, the democratic case gives degenerate eigenvalues—no mass hierarchy. Only for $J_3(\mathbb{O})$ does non-associativity introduce the independent invariant $\varphi(u_1, u_2, u_3)$ that becomes δ.

2.6 The WZW number

Definition 2.8 (WZW number)**.** For a representation R of $\mathrm{SU}(N)_k$, define

$$z_R = \sqrt{d_R} \cdot e^{ih_R}, \tag{11}$$

where d_R is the quantum dimension and $h_R = C_2(R)/(k + h^\vee)$ the conformal dimension. For $\square$ of $\mathrm{SU}(3)_3$: $z_\square = \sqrt{2}\, e^{i\cdot 2/9}$.

Theorem 2.9 (Circulant mass formula)**.** *The WZW Brannen formula* (9) *is equivalent to*

$$\frac{\sqrt{M}}{\mu} = I + \frac{z}{2} P + \frac{\bar{z}}{2} P^\dagger, \tag{12}$$

where P is the 3×3 cyclic permutation matrix $P_{ij} = \delta_{i,j+1 \bmod 3}$ and $z = z_\square$.

Proof. The eigenvalues of P are $\omega^k = e^{2\pi i k/3}$. Writing $z = |z|e^{i\delta}$: $\sqrt{m_k}/\mu = 1+\mathrm{Re}(z\omega^k) = 1+|z|\cos(\delta+2\pi k/3)$. With $|z| = \sqrt{2}$ and $\delta = h_\square$, this is the Brannen form. The circulant structure makes $S_3 \to \mathbb{Z}_3$ breaking manifest. $\square$

Proposition 2.10 (Cube identity)**.** $z_\square^3 = d_\square^{3/2}\, e^{iQ}$*. Equivalently,* $|z_\square^3| = 2\sqrt{2}$ *and* $\arg(z_\square^3) = 3h_\square = 2/3 = Q$*.*

Proof. $z^3 = (\sqrt{d})^3 e^{3ih} = d^{3/2}\, e^{3ih}$. Since $3h_\square = 2/3 = Q$, $\arg(z^3) = Q$. $\square$

Remark 2.11 (Self-consistency reformulation)*.* The cube identity transforms $\delta = h_\square$ into $\arg(z^3) = Q$, replacing an independent parameter identification with an internal consistency constraint of the WZW data.

Remark 2.12 (Connection to $J_3(\mathbb{O})$)*.* In the democratic basis, $(\sqrt{M})_{01} = (\mu/\sqrt{2})\, e^{i\delta}$. The WZW number encodes the off-diagonal Yukawa phase: $\arg(z_\square) = \delta = \arg(\sqrt{M})_{01}$.

3 The Hessian Derivation of $\delta = 2/9$

3.1 Setup

Let $\mathbb{R}^6 \subset \mathbb{R}^7$ be the subspace orthogonal to e_7. For an oriented 3-plane $W \in \mathrm{Gr}(3, \mathbb{R}^6)$ with frame (v_1, v_2, v_3), define [17]

$$f(W) = \frac{\varphi(v_1, v_2, v_3)}{\mathrm{vol}(v_1, v_2, v_3)}, \quad \mathrm{vol} = \sqrt{\det G}, \quad G_{ij} = \langle v_i, v_j \rangle. \tag{13}$$

The democratic point $V_0 = \mathrm{span}(e_1, e_2, e_3)$ has $f(V_0) = \varphi(e_1, e_2, e_3) = 1 \equiv \varphi_0$.

3.2 Perturbation expansion

The tangent space $T_{V_0}\mathrm{Gr}(3, \mathbb{R}^6) \cong \mathrm{Hom}(V_0, V_0^\perp) \cong M_3(\mathbb{R})$ parametrizes deformations $v_i(\varepsilon) = w_i + \varepsilon \sum_k A_{ik} u_k$, where $w_i = e_i$, $u_k = e_{k+3}$.

Lemma 3.1 (Levi-Civita)**.** *At V_0: (a) $\varphi(w_i, w_j, u_k) = 0$ for all i, j, k; (b) $\varphi(w_k, u_a, u_b) = -\varphi_0\, \varepsilon_{kab}$.*

Proof. Since $\varphi|_{\mathbb{R}^6} = \mathrm{Re}(\Omega)$ with $\Omega = dz_1 \wedge dz_2 \wedge dz_3$, only terms with an even number of u-type indices survive. Part (a): one u-index, so zero. Part (b): direct evaluation against (5). Both verified computationally for all 27 index combinations. □

Lemma 3.2 (Gram matrix). *$G_{ij}(\varepsilon) = \delta_{ij} + \varepsilon^2(AA^T)_{ij}$, so* $\mathrm{vol}(\varepsilon) = 1 + \frac{\varepsilon^2}{2}\|A\|^2 + O(\varepsilon^4)$.

Lemma 3.3 (Numerator). *$\varphi(v_1(\varepsilon), v_2(\varepsilon), v_3(\varepsilon)) = \varphi_0 + \varepsilon^2 N_2(A) + O(\varepsilon^3)$ with*

$$N_2(A) = \frac{\varphi_0}{2}\big(\mathrm{Tr}(A^2) - (\mathrm{Tr}\,A)^2\big). \tag{14}$$

Proof. The $O(\varepsilon)$ terms vanish by Lemma 3.1(a). The $O(\varepsilon^2)$ terms collect contributions where two frame vectors are perturbed. By Lemma 3.1(b) and $\sum_k \varepsilon_{kij}\varepsilon_{kab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$, one obtains (14). □

3.3 The Hessian formula

Theorem 3.4 (Hessian). *The Hessian of $f = \varphi/\mathrm{vol}$ on $\mathrm{Gr}(3, \mathbb{R}^6)$ at V_0 is*

$$H_f(A) = -\varphi_0\big((\mathrm{Tr}\,A)^2 + 2\|A_{\mathrm{anti}}\|^2\big), \tag{15}$$

where $A_{\mathrm{anti}} = \frac{1}{2}(A - A^T)$.

Proof. Combining Lemmas 3.2–3.3: $f(\varepsilon) = \varphi_0 + \varepsilon^2(N_2 - \varphi_0\|A\|^2/2) + O(\varepsilon^4)$. Decomposing $A = S + \Lambda$ into symmetric and antisymmetric parts: $\mathrm{Tr}(A^2) = \|S\|^2 - \|\Lambda\|^2$ and $\mathrm{Tr}(AA^T) = \|S\|^2 + \|\Lambda\|^2$. Hence $\mathrm{Tr}(A^2) - \mathrm{Tr}(AA^T) = -2\|\Lambda\|^2$, giving (15). □

Corollary 3.5 (Eigenvalue decomposition). *Under* $\mathrm{SO}(3)$, $M_3(\mathbb{R}) = \mathrm{Sym}^2_0(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3) \oplus \mathbb{R}\cdot I$ *with:*

Subspace	*dim*	*Eigenvalue*	*Interpretation*
$\mathrm{Sym}^2_0(\mathbb{R}^3)$	*5*	0	*SO(3) orbit (flat)*
$\Lambda^2(\mathbb{R}^3)$	*3*	$-2\varphi_0$	*mass splitting*
$\mathbb{R}\cdot I$	*1*	$-3\varphi_0$	*trace/scale*

3.4 The geometric ratio

Theorem 3.6 (Main geometric result). *The ratio of the mass-splitting eigenvalue to the normalized Laplacian is*

$$\delta_{\mathrm{geom}} = \frac{|\lambda_{\Lambda^2}|}{|\Delta f/\varphi_0|} = \frac{2}{9}. \tag{16}$$

Proof. From Corollary 3.5: $\Delta f/\varphi_0 = 5 \times 0 + 3 \times (-2) + 1 \times (-3) = -9$. Therefore $\delta_{\mathrm{geom}} = 2/9$. □

Observation 3.7 (Casimir coincidence). The geometric ratio 2/9 coincides with $C_2(\bar{3})/C_2(\mathrm{Sym}^3 3) = (4/3)/6 = 2/9$. More generally, $C_2(\mathrm{fund}_N)/C_2(\mathrm{Sym}^N N) = (N+1)/(2N^2)$, which equals 2/9 only for $N = 3$.

Remark 3.8. The Hessian trace on Λ^2 satisfies $|\mathrm{Tr}(H_f|_{\Lambda^2})|/\varphi_0 = 6 = C_2(\mathrm{Sym}^3 3)$. The factor of 2 in $\lambda_{\Lambda^2} = -2\varphi_0$ arises from the antisymmetrizer $\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$. This also appears in the Casimir formula via $C_2(\mathrm{Sym}^N N) = 2\dim\Lambda^2(\mathbb{R}^N)$ (Proposition 6.3), providing a suggestive—but not rigorously proven—link.

Remark 3.9. The decomposition $M_3(\mathbb{R}) = \mathrm{Sym}^2_0 \oplus \Lambda^2 \oplus \mathbb{R}\cdot I$ is an SO(3) decomposition. Relating the SO(3) Hessian eigenvalues to SU(3) Casimir values requires additional structure not provided here.

Confrontation with experiment. Using PDG 2024 pole masses for m_e and m_μ [4], and the HFLAV 2025 world average $m_\tau = 1776.96 \pm 0.09$ MeV [29] (which includes the Belle II measurement [30]):

$$\delta_{\text{exp}} = 0.22223, \quad \delta_{\text{pred}} = \frac{2}{9} = 0.22222\ldots \tag{17}$$

The deviation $|\Delta\delta|/\delta = 0.002\%$ is well within 1σ of the m_τ uncertainty. Setting $\delta = 2/9$ and fixing μ from m_μ:

Observable	Prediction ($\delta = 2/9$)	Data (HFLAV 2025)	Deviation
δ	$0.22222\ldots$	0.22223	0.002%
m_τ	1776.97 MeV	1776.96 ± 0.09 MeV	0.08σ
m_e	0.510994 MeV	0.510999 MeV	0.001%

The m_τ prediction agrees at 0.08σ with the HFLAV average, improved from 0.9σ with the PDG 2024 value (1776.86 ± 0.12 MeV). The trend of improved measurements has moved *toward* the prediction.

4 Conformal Field Theory Structure

4.1 The Master Identity

Theorem 4.1 (Master Identity). *For* SU(N) *at level* $k = N$*,*

$$C_2(\text{Sym}^N\Box) = k + h^\vee \iff N = 3. \tag{18}$$

Proof. $C_2(\text{Sym}^N\Box) = N(N-1)$ by direct computation. Setting $k = N$: $k + h^\vee = 2N$. The equation $N(N-1) = 2N$ reduces to $N = 3$. □

Corollary 4.2 (Universal Casimir–conformal identity). *At* $N = 3$*, for every integrable representation* R *of* $\text{SU}(3)_3$*:*

$$\frac{C_2(R)}{C_2(\text{Sym}^3 3)} = \frac{C_2(R)}{k + h^\vee} = h(R). \tag{19}$$

R	*Dynkin*	$C_2(R)$	$C_2(R)/6$	$h(R)$
$\Box$	$(1,0)$	4/3	2/9	2/9
$\bar{\Box}$	$(0,1)$	4/3	2/9	2/9
adj	$(1,1)$	3	1/2	1/2
$\text{Sym}^2\Box$	$(2,0)$	10/3	5/9	5/9
$\text{Sym}^3\Box$	$(3,0)$	6	1	1

4.2 Simple current structure

Proposition 4.3 (Simple current). $\mathrm{Sym}^3(\square) = (3,0)$ *of* $\mathrm{SU}(3)_3$ *is a* $\mathbb{Z}_3$ *simple current:* $d_{(3,0)} = 1$*,* $h_{(3,0)} = 1$*, generating the* $\mathbb{Z}_3$ *center via fusion:* $(3,0) \otimes (3,0) = (0,3)$*,* $(3,0) \otimes (0,3) = (0,0)$*.*

Proof. The quantum dimension $d_{(3,0)} = S_{(3,0),(0,0)}/S_{(0,0),(0,0)} = 1$ from the Kac–Peterson S-matrix [13]. Since $d = 1$, it is a simple current. Fusion verified via the Verlinde formula. □

Proposition 4.4 (Perturbative blindness). *The simple current deformation by* $(3,0)$ *has* β*-function* $\beta_n = 0$ *unless* $n \equiv 0 \pmod 3$*, with leading correction at* $O(\lambda^3)$*. The deformation acts non-perturbatively on local correlators, explaining the geometric Blindness Theorem 7.1 from the CFT perspective.*

Remark 4.5 (Democratic deformation). The trilinear decomposition $\square^{\otimes 3} = \mathrm{Sym}^3(\square) \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ shows why Sym^3 is blind to δ: its Clebsch–Gordan coefficients are generation-democratic. Mass splitting arises from the adjoint $\mathbf{8}$ channel ($h = 1/2$, relevant), which discriminates between generations via the d-symbol.

4.3 Q decomposition and OPE structure

Proposition 4.6 (Q decomposition). *The Koide quotient decomposes as*

$$Q = \frac{1}{N} + \frac{d_\square}{2N}, \tag{20}$$

where $1/N$ *is the democratic contribution (equal masses) and* $d_\square/(2N)$ *the splitting contribution. For* $N = 3$*:* $Q = 1/3 + 1/3 = 2/3$*.*

Proof. From Theorem 2.2: $Q = 1/3 + A^2/6$ with $A^2 = d_\square$. For general N at $k = N$: the democratic term is $1/N$ and the splitting term involves $d_\square = 1/\sin(\pi/(2N))$. □

Proposition 4.7 (OPE deficit). *The OPE deficit* $\Delta h_{\mathrm{OPE}} = h_{\mathrm{Sym}^2\square} - 2h_\square$ *satisfies*

$$\frac{d_\square}{2N} - \Delta h_{\mathrm{OPE}} = h_\square \iff N = 3. \tag{21}$$

Proof. For $\mathrm{SU}(N)_N$: $C_2(\mathrm{Sym}^2) = (N+2)(N-1)/N$, so $h_{\mathrm{Sym}^2} = (N+2)(N-1)/(2N^2)$. Then $\Delta h = (N-1)/(2N^2)$. The condition becomes $1/(2N\sin(\pi/(2N))) - (N-1)/(2N^2) = (N^2-1)/(4N^2)$. For $N = 3$: $\Delta h = 1/9$; $d_\square/(2N) = 1/3$; $1/3 - 1/9 = 2/9 = h_\square$. Verified numerically as unique for $N = 2, \ldots, 20$. □

Remark 4.8 (OPE interpretation). The deficit $\Delta h = 1/9$ measures the anomalous dimension of the leading $\square \times \square$ fusion channel. That $h_\square = d_\square/(2N) - \Delta h$ provides a 13th $N = 3$ condition.

Proposition 4.9 (Dimensional coincidence). *Define* $\delta_{\mathrm{OPE}}(N) = d_\square(\mathrm{SU}(N)_N)/(2N) - (h_{\mathrm{Sym}^2\square} - 2h_\square)$*. Then* $\delta_{\mathrm{OPE}}(N) = h_\square(\mathrm{SU}(N)_N)$ *if and only if* $N = 3$*.*

4.4 Crossing–Casimir coincidence

Consider the four-point function $\langle \square\square\bar{\square}\bar{\square} \rangle$ in $\mathrm{SU}(3)_3$. The s-channel conformal blocks from $\square \otimes \square = \bar{\square} \oplus \mathrm{Sym}^2(\square)$ are:

$$F_{\bar{\square}}(z) \sim z^{-2/9}(1 + \cdots), \quad F_{\mathrm{Sym}^2}(z) \sim z^{1/9}(1 + \cdots). \tag{22}$$

The exponent difference $h_{\mathrm{Sym}^2} - h_{\bar{\square}} = 1/3$ encodes mass-splitting information. At the $\mathbb{Z}_3$-symmetric crossing point $z = \omega = e^{2\pi i/3}$, define

$$\delta_X \equiv (h_{\mathrm{Sym}^2} - h_{\square}) \times \frac{2}{N}. \tag{23}$$

Theorem 4.10 (Crossing–Casimir coincidence)**.** *The three quantities*

$$\delta_C = \frac{C_2(\square)}{C_2(\mathrm{Sym}^N\square)} = \frac{N+1}{2N^2}, \tag{24}$$

$$\delta_X = \frac{(N-1)(N+3)}{2N^3}, \tag{25}$$

$$h(\square) = \frac{N^2-1}{4N^2} \tag{26}$$

all equal $2/9$ *simultaneously if and only if* $N = 3$.

Proof. Pairwise equalities: $\delta_C = h(\square)$: $(N+1)/(2N^2) = (N^2-1)/(4N^2)$ gives $N-1 = 2$, so $N = 3$. $\delta_C = \delta_X$: yields $N = 3$. $\delta_X = h(\square)$: $(N-3)(N+2) = 0$, giving $N = 3$. Two independent quadratics, each uniquely selecting $N = 3$. □

4.5 Knizhnik–Zamolodchikov exponent

The KZ equation [18] governs holomorphic dependence of WZW correlators:

$$\partial_{z_i}\Psi = \frac{1}{k+h^\vee}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i - z_j}\Psi, \tag{27}$$

where $\Omega_{ij} = \sum_a T_i^a \otimes T_j^a$ is the Casimir exchange operator.

Theorem 4.11 (KZ singlet exponent)**.** *The KZ equation for* $\square \times \bar{\square}$ *has local exponents* $\alpha_1 = -C_2(\square)/(k+h^\vee) = -h_\square$ *and* $\alpha_{\mathrm{adj}} = 1/(2N(k+N))$*. For* $\mathrm{SU}(3)_3$*:* $\alpha_1 = -2/9$ *and* $\alpha_{\mathrm{adj}} = 1/36$*.*

Proof. The Casimir exchange on $\square \otimes \bar{\square} = \mathbf{1} \oplus \mathrm{adj}$ has eigenvalues $\Omega|_{\mathbf{1}} = -C_2(\square)$ and $\Omega|_{\mathrm{adj}} = 1/(2N)$. Multiplying by $\kappa = 1/(k+h^\vee)$ gives the result. □

Corollary 4.12. *The singlet-channel propagator behaves as* $G_1(z,w) \propto (z-w)^{-h_\square}$*. Writing* $z - w = re^{i\phi}$*:* $G_1 \propto r^{-h_\square}e^{-ih_\square\phi}$*.*

Remark 4.13. If the compactification geometry arranges $\phi = 1$ radian between adjacent generation fixed points, the propagator phase is $-h_\square = -2/9$, matching the Brannen angle. The conjecture $\delta = h_\square$ translates into: $\arg(\Delta z_{\mathrm{internal}}) = 1$ radian. On the $\mathbb{Z}_3$-symmetric sphere with $z_k = \omega^k$, one gets $\arg((1-\omega)^{-h_\square}) = \pi/27 \neq 2/9$, verifying that KZ cannot produce $\delta = h_\square$ on standard symmetric configurations (Kill #63).

4.6 KZ–circulant incompatibility

Proposition 4.14 (KZ–circulant obstruction)**.** *The KZ equation determines the modulus of inter-generation correlators:* $|G_1(z,w)| \propto |z-w|^{-h_\square}$*, but is structurally incompatible with the circulant form* (12)*. The KZ equation produces power-law decay, while the circulant requires a phase* $\arg(z_\square) = h_\square$ *in the permutation channel.*

Proof. The KZ propagator $G_1 \propto (z-w)^{-h_\square}$ determines $|(\sqrt{M})_{ij}| \propto r^{-h_\square}$ from Casimir exchange, while the phase $\delta = h_\square$ must come from the C-field period $\int_{\Sigma_{ij}} C_3$. The two structures—power-law modulus from KZ and topological phase from C-field—are complementary, not redundant. □

Remark 4.15 (Modulus/phase separation)*.* This structural separation clarifies Kill #63: KZ governs the modulus, not the phase. The Brannen phase is literally the Yukawa coupling phase: $\delta = \arg(\sqrt{M})_{01}$, which in M-theory is a C-field period, not a KZ exponent.

Remark 4.16 (Refined conjecture)*.* The conjecture $\delta = h_\square$ decomposes into:

1. **Modulus:** $|(\sqrt{M})_{ij}|/\mu = 1/\sqrt{2}$, i.e., $A = \sqrt{2} = \sqrt{d_\square}$. Equivalent to $Q = 2/3$ (Theorem 2.2). *Proven.*
2. **Phase:** $\arg(\sqrt{M})_{01} = h_\square = 2/9$. The C-field identification $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3 = h_\square$. *Conjectured.*

Observation 4.17 (KZ–circulant as mechanism constraint)**.** The KZ–circulant incompatibility constitutes a fourth structural obstruction: any mechanism deriving $\delta = h_\square$ must operate on the *phase* of the Yukawa coupling (topological C-field datum), not its modulus (KZ/dynamical datum).

5 The Hessian–WZW Bridge

Proposition 5.1 (Hessian–WZW Bridge)**.** *The geometric ratio* $\delta_{\text{geom}} = |\lambda_{\Lambda^2}|/|\Delta f/\varphi_0| = 2/9$ *is rigorously computed from the Hessian of the* G_2 *3-form (Theorem 3.6). Writing the ratio formally as* $\delta_{\text{geom}}(N) = 2/N^2$*—where the numerator* 2 *from the antisymmetrizer is valid for all* N *and the denominator* N^2 *tracks the Laplacian scaling—and comparing with* $h_\square = (N^2-1)/(4N^2)$*, the uniqueness condition*

$$\delta_{\text{geom}}(N) = h_\square(\mathrm{SU}(N)_N) \iff N = 3 \tag{28}$$

is verified by direct computation.

Proof. The cross-ratios are:

$$\frac{|\lambda_{\Lambda^2}/\varphi_0|}{C_2(\square)} = \frac{4N}{N^2-1}, \qquad \frac{|\Delta f/\varphi_0|}{k+h^\vee} = \frac{N}{2}. \tag{29}$$

For $\delta_{\text{geom}} = h_\square$: $4N/(N^2-1) = N/2$, giving $N^2 - 1 = 8$, i.e., $N = 3$. At $N = 3$, both cross-ratios equal $3/2$.

N	$\lvert\lambda\rvert/C_2$	$N/2$	Equal?	δ_{geom}	$h_\square$	Match?
2	2.667	1.000	×	0.5000	0.1875	×
3	1.500	1.500	✓	0.2222	0.2222	✓
4	1.067	2.000	×	0.1250	0.2344	×
5	0.833	2.500	×	0.0800	0.2400	×

$\square$

Remark 5.2. The condition $4N/(N^2-1) = N/2$ requires $C_2(\square) \times N/2$ to match the antisymmetrizer coefficient 2. Geometrically: for $N = 3$, the Casimir of the fundamental times $N/2$ precisely matches the Hessian curvature.

Remark 5.3. The bridge condition $N^2 - 1 = 8$ is equivalent to $\dim \Lambda^2(\mathbb{R}^N) = N$ (Proposition 6.2), reflecting that Λ^2 has the same dimension as the fundamental only at $N = 3$.

Remark 5.4 (Absence of 2π). Both δ_{geom} and $h_\square$ are ratios of dimensionless quantities—a curvature ratio and a Casimir ratio. Neither involves 2π. This is why the bridge works without normalization mismatches.

Proposition 5.5 (Monodromy–Casimir matching)**.** *In* $\mathrm{SU}(N)_N$ *WZW theory, the simple current monodromy charge* $Q_J(\square) = 1/N$ *satisfies*

$$\frac{2}{N} = \frac{C_2(\square)}{C_2(\mathrm{Sym}^2\square) - C_2(\square)} \iff N = 3. \tag{30}$$

Proof. The RHS equals $(N+1)/(N+3)$. Setting $2/N = (N+1)/(N+3)$: $N^2 - N - 6 = (N-3)(N+2) = 0$, giving $N = 3$. $\square$

Remark 5.6. For $N = 3$: $2Q_J(\square) = 2/3 = Q$, connecting simple current monodromy to the Koide quotient—only at $N = 3$.

6 Generation Selection

Eighteen distinct mathematical conditions select $N = 3$:

Theorem 6.1 (Generation selection)**.** *The following conditions each have a unique solution* $N = 3$ *among positive integers* $N \geq 2$*:*

1. $\dim \Lambda^2(\mathbb{R}^N) = N$ *(Proposition 6.2);*
2. $h(\mathrm{Sym}^N\square) = 1$ *in* $\mathrm{SU}(N)_N$ *(marginality; Theorem 6.4);*
3. G_2 *exists in dimension* $2N+1$ *(*$G_2 \subset \mathrm{SO}(7)$*, unique for* $N = 3$*);*
4. $C_2(\mathrm{Sym}^N\square) = k + h^\vee$ *(Master Identity; Theorem 4.1);*
5. $\delta_C = h(\square)$ *(Casimir = conformal dim; Theorem 4.10);*
6. $\delta_C = \delta_X$ *(Casimir = crossing phase; Theorem 4.10);*
7. $C_2(\mathrm{fund}_N)/C_2(\mathrm{Sym}^N N) = 2/9$ *(uniquely for* $N = 3$*; Observation 3.7);*
8. $d_\square(\mathrm{SU}(N)_N) \in \mathbb{Q}$ *(quantum dimension rational; Proposition 6.5);*
9. $|\lambda_{\Lambda^2}|/\varphi_0 = d_\square$ *(Hessian eigenvalue = quantum dimension; Proposition 6.6);*
10. $\delta_{\text{geom}} = h_\square$ *(Hessian–WZW Bridge; Proposition 5.1);*
11. $2Q_J(\square) = C_2(\square)/[C_2(\mathrm{Sym}^2\square) - C_2(\square)]$ *(monodromy–Casimir; Proposition 5.5);*
12. $e_2(\sigma^*) = h_\square$ *(alcove–conformal coincidence; Theorem 6.12);*

13. $d_\square/(2N) - \Delta h_{\text{OPE}} = h_\square$ *(OPE–dimensional; Proposition 4.7);*

14. *OPE deficit selects* $N = 3$ *(Proposition 4.9);*

15. $\sigma_{(N-1)/N} = h_\square$ *(holonomy–conformal; Theorem 6.7);*

16. *Phase exclusion:* $\sigma_{1/3} = 1/9$ *fails (Proposition 6.8);*

17. *Cartan index* $j = N - 1$ *forced algebraically (Remark 6.9);*

18. T^c *self-consistency:* $c\,(h_\square - c/24) = h_\square$ *in* $\mathrm{SU}(N)_N$ *(Theorem 8.7).*

Proposition 6.2. $\dim \Lambda^2(\mathbb{R}^N) = \binom{N}{2} = N$ *if and only if* $N = 3$.

Proof. $N(N-1)/2 = N$ implies $N - 1 = 2$, hence $N = 3$. □

Proposition 6.3 (Structural identity)**.** *For all* $N \geq 2$*:* $C_2(\mathrm{Sym}^N N) = N(N-1) = 2\dim\Lambda^2(\mathbb{R}^N)$.

Proof. $C_2(\mathrm{Sym}^k(N)) = k(N+k)(N-1)/(2N)$ [15]. At $k = N$: $C_2 = N(N-1)$. □

Theorem 6.4 (Marginality)**.** *In* $\mathrm{SU}(N)$ *at level* $k = N$*,* $h(\mathrm{Sym}^N\square) = (N-1)/2$. *Marginality* $h = 1$ *gives* $N = 3$ *uniquely.*

Proof. $h = C_2(\mathrm{Sym}^N\square)/(k+h^\vee) = N(N-1)/(2N) = (N-1)/2$. Setting $= 1$: $N = 3$. □

Proposition 6.5 (Quantum rationality)**.** $d_\square(\mathrm{SU}(N)_N) \in \mathbb{Q}$ *if and only if* $N = 3$ *(among* $N \geq 2$*), where* $d_\square = 2$.

Proof. $d_\square = 1/\sin(\pi/(2N))$. By Niven's theorem [19]: $\sin(\pi/(2N)) \in \mathbb{Q}$ requires $\sin(\pi/(2N)) = 1/2$, giving $N = 3$. □

Proposition 6.6 (Hessian–quantum bridge)**.** $|\lambda_{\Lambda^2}|/\varphi_0 = d_\square(\mathrm{SU}(N)_N)$ *if and only if* $N = 3$.

Proof. $|\lambda_{\Lambda^2}|/\varphi_0 = 2$. By Proposition 6.5, $d_\square = 2$ only at $N = 3$. □

6.1 Holonomy–conformal selection

Theorem 6.7 (Holonomy–conformal selection)**.** *Let* A_{central} *be the central flat* $\mathrm{SU}(N)$ *connection on the lens space* $L(N,1) = S^3/\mathbb{Z}_N$*, with holonomy* $\mathrm{diag}(\omega, \omega^2, \ldots, \omega^{N-1})$ *where* $\omega = e^{2\pi i/N}$*. The holonomy parameters are* $\sigma_j = j/N$ *for* $j = 1, \ldots, N-1$*. Then*

$$\frac{\sigma_{N-1}}{N} = h_\square(\mathrm{SU}(N)_N) \iff N = 3. \tag{31}$$

Proof. $\sigma_{N-1}/N = (N-1)/N^2$. Setting equal to $h_\square = (N^2-1)/(4N^2)$: for $N \geq 2$, divide by $(N-1)/N^2$: $1 = (N+1)/4$, giving $N = 3$. At $N = 3$: $\sigma_2/3 = 2/9 = h_\square$. □

Proposition 6.8 (Phase exclusion)**.** *For* $\mathrm{SU}(3)_3$*,* $\sigma_{1/3} = 1/9$ *is excluded by: (1) algebraic:* $1/9 \neq 2/9 = h_\square$*; (2) empirical (Kill #68):* $\delta = 1/9$ *predicts* $m_\mu/m_e \approx 39.7$ *versus observed* 206.8 *(*$> 100\sigma$*).*

Remark 6.9. The selection of $j = N - 1$ (not $j = 1$) is algebraically forced by the holonomy–conformal identity, not empirically chosen.

Remark 6.10 (Holonomic derivation). Theorem 6.7 provides a conditional derivation of $\delta = 2/9$: if (H1) the relevant gauge connection is flat, (H2) the CS level is $k = N_{\text{gen}}$, (H3) the holonomy is central ($\mathbb{Z}_N$), and (H4) the Brannen phase equals a holonomy parameter divided by N, then $\delta = \sigma_{2/3} = 2/9$ is uniquely selected.

6.2 Alcove–conformal coincidence

Remark 6.11 (Weyl alcove geometry). The holonomy parameters $\sigma = (\sigma_1, \ldots, \sigma_{N-1})$ parametrize the Weyl alcove—the fundamental domain for the affine Weyl group. The $\mathbb{Z}_N$-symmetric centroid is $\sigma^* = (1/N, 2/N, \ldots, (N-1)/N)$.

Theorem 6.12 (Alcove–conformal coincidence)**.** *Let* $\sigma^* = (1/N, 2/N, \ldots, (N-1)/N)$ *be the* $\mathbb{Z}_N$*-symmetric centroid. The second elementary symmetric polynomial satisfies*

$$e_2(\sigma^*) = h_\Box(\mathrm{SU}(N)_N) \iff N = 3. \tag{32}$$

Proof. $e_2(\sigma^*) = \sum_{i<j} ij/N^2$. Using $\sum_{i<j} ij = \frac{1}{2}[(\sum i)^2 - \sum i^2]$ with $\sum_{i=1}^{N-1} i = N(N-1)/2$ and $\sum_{i=1}^{N-1} i^2 = (N-1)N(2N-1)/6$:

$$e_2(\sigma^*) = \frac{(N-1)(3N^2 - 7N + 2)}{24N}.$$

Setting equal to $h_\Box = (N-1)(N+1)/(4N^2)$ and dividing by $(N-1)$: cross-multiplying gives $3N^3 - 7N^2 - 4N - 6 = 0$, which factors as $(N-3)(3N^2+2N+2) = 0$. The quadratic has discriminant $4 - 24 = -20 < 0$, so $N = 3$ is the unique real solution. For $N = 3$: $e_2(1/3, 2/3) = 2/9 = h_\Box$. □

Remark 6.13 (Three-way coincidence). At $N = 3$: $\sigma_2/3 = e_2(\sigma^*) = h_\Box = 2/9$—a holonomy parameter, a symmetric function on the alcove, and a conformal dimension, all coinciding uniquely at $N = 3$.

6.3 Reduction theorem

Theorem 6.14 (Reduction of $N = 3$ conditions)**.** *Among integers* $N \geq 2$*, conditions (i) Master Identity, (ii) exact marginality, (iii)* $\dim \Lambda^2 = N$*, (iv) Casimir ratio* $= 2/9$*, (v) rational quantum dimension, (vi) Hessian–quantum bridge, (vii) Hessian–WZW bridge, (viii) monodromy–Casimir matching are all equivalent and unified by the single equation* $\sin(\pi/(2N)) = 1/(N-1)$*, with unique solution* $N = 3$*. The existence of* G_2 *provides the geometric context as a third, independent input.*

Proof. Conditions (i)–(iv), (vii)–(viii) are polynomial in N; all factor with root $N = 3$. Conditions (v)–(vi) are transcendental: $d_\Box = 1/\sin(\pi/(2N))$, and Niven's theorem forces $N = 3$. The unified equation $\sin(\pi/(2N)) = 1/(N-1)$ connects both classes. □

6.4 Self-consistency loop

Proposition 6.15 (Self-consistency selects $N = 3$)**.** *Given* $Q = 2/3$ *(empirical) and the Sumino* $\mathrm{SU}(N)_F$ *family gauge symmetry with* $A^2 = d_\Box$ *(WZW identification), the equation* $A^2(N) = d_\Box(N)$ *has a unique solution* $N = 3$*.*

Proof. The Koide quotient in the generalized $\mathbb{Z}_N$ Brannen parametrization gives $Q = 1/N + A^2/(2N)$. For $Q = 2/3$: $A^2 = 2N(2/3 - 1/N) = 4N/3 - 2$. The quantum dimension is $d_\Box(\mathrm{SU}(N)_N) = 1/\sin(\pi/(2N))$. Setting $4N/3 - 2 = 1/\sin(\pi/(2N))$: since the left side is linear in N while the right is transcendental, we verify numerically that equality holds only at $N = 3$, where $A^2 = 2 = 1/\sin(\pi/6) = d_\Box$. The uniqueness follows from Niven's theorem: $d_\Box \in \mathbb{Q}$ forces $\sin(\pi/(2N)) = 1/2$, hence $N = 3$. □

Remark 6.16 (Bootstrap does not select δ). The self-consistency loop $Q = 2/3 \to A^2 = d_\square \to N = 3 \to k_{\text{eff}} = 3$ uniquely determines the number of generations but is trivially closed for *any* δ once $N = 3$ is fixed. The Brannen phase is invisible to this argument: it selects the WZW model $\mathrm{SU}(3)_3$ but not the vacuum within it.

6.5 Chern–Simons Wilson loop identity

Observation 6.17 (CS Wilson loop reproduces Q)**.** In $\mathrm{SU}(3)_3$ Chern–Simons theory on T^2, quantized in the representation basis $\{|\lambda\rangle\}$ with S-matrix $S_{\lambda\alpha}$ diagonalizing the B-cycle holonomy, the fundamental Wilson loop expectation value in the $|\square\rangle$ state is

$$\langle\square|W_\square(B)|\square\rangle \;=\; \sum_\alpha |S_{\square,\alpha}|^2\,\chi_\square(\alpha) \;=\; \frac{d_\square}{N} \;=\; \frac{2}{3} \;=\; Q.$$

More generally, $\langle\lambda|W_\square|\lambda\rangle = d_\lambda/N$ for all states. The holonomy distribution in $|\square\rangle$ is *uniform*: $|S_{\square,\alpha}|^2 = 1/9$ for all nine non-adjoint flat connections, with $|S_{\square,\mathrm{adj}}|^2 = 0$. No particular holonomy value σ_0 is singled out (Kill #80).

Remark 6.18 (Relation to $Q = 1/3 + d_\square/6$). The identity $\langle W_\square\rangle_\square = d_\square/N = 2/3 = Q$ is the CS restatement of Theorem 2.2: the Koide quotient equals the normalized quantum dimension. This provides no new dynamical information—the CS path integral averages uniformly over all flat connections in the $|\square\rangle$ sector, reproducing Q as a representation-theoretic identity rather than selecting a specific vacuum.

7 Structural Obstructions

7.1 Calibrated blindness

Theorem 7.1 (Blindness)**.** *On the Brannen orbit* (2) *with $\varepsilon = \psi$ (Koide normalization), the elementary symmetric polynomials satisfy:*

$$e_1(\delta) = \theta_1 + \theta_2 + \theta_3 = 3\psi, \tag{33}$$

$$e_2(\delta) = \sum_{j<k} \theta_j\theta_k = 3\psi^2 - \tfrac{3}{2}\varepsilon^2, \tag{34}$$

$$e_3(\delta) = \theta_1\theta_2\theta_3 = \psi^3 - \tfrac{3}{2}\psi\varepsilon^2 + \tfrac{\varepsilon^3}{\sqrt{2}}\cos(3\delta). \tag{35}$$

In particular, e_1 and e_2 are independent of δ, and e_3 depends on δ only through $\cos(3\delta)$*.*

Proof. Using $\sum_k c_k = 0$, $\sum_{j<k} c_j c_k = -3/4$, and $\prod_k c_k = \frac{1}{4}\cos(3\delta)$, where $c_k = \cos(\delta + 2\pi k/3)$. □

Corollary 7.2. *The calibration function $f = \varphi/\mathrm{vol}$ on $\mathrm{Gr}(3,\mathbb{R}^6)$, restricted to the diagonal torus, evaluates to $f = \cos(\theta_1 + \theta_2 + \theta_3)$, independent of δ on the Brannen orbit. Every functional depending only on e_1 and e_2 is blind to δ. The determinant e_3 is the unique S_3-invariant that sees δ.*

Remark 7.3 (Scope). The blindness is exact on the Brannen orbit. It does not preclude mechanisms coupling to deformations off the orbit—the full Hessian analysis of §3 operates on the tangent space to $\mathrm{Gr}(3,\mathbb{R}^6)$, extending beyond the orbit.

7.2 CP and transcendence obstructions

Proposition 7.4 (CP obstruction). *Any $\mathbb{Z}_3$-symmetric, CP-conserving potential on the Brannen orbit has critical points satisfying either* $\sin(3\delta) = 0$ *or* $\cos(3\delta) \in \mathbb{Q}$. *Since* $\cos(2/3)$ *is transcendental,* $\delta = 2/9$ *cannot be a critical point.*

Proof. $\mathbb{Z}_3$ symmetry forces $V(\delta) = \sum_{n\geq 0} a_n \cos(3n\delta)$. Differentiating and factoring via Chebyshev: $V'(\delta) = -\sin(3\delta)\sum_{n\geq 1} 3n\, a_n U_{n-1}(\cos 3\delta)$. Roots require $\sin(3\delta) = 0$ or $\cos(3\delta)$ algebraic. □

Remark 7.5. This requires CP violation in the flavor sector—physically natural, since the SM already has CP violation through the CKM phase.

Proposition 7.6 (Transcendence). $\cos(2/3)$ *is transcendental.*

Proof. By Lindemann–Weierstrass: $e^{2i/3}, e^{-2i/3}, 1$ are $\overline{\mathbb{Q}}$-linearly independent. If $\cos(2/3)$ were algebraic, $e^{2i/3} + e^{-2i/3} - 2\cos(2/3)\cdot 1 = 0$ gives a contradiction. □

Remark 7.7. Since RCFT data are algebraic (cyclotomic for WZW models [14]), no RCFT mechanism can produce $\cos(2/3)$ exactly. However, the Spectral Selection Theorem bypasses this: it identifies $\delta = h_{\square}$ directly as a conformal dimension (rational), with the transcendental $\cos(2/3)$ arising as a consequence.

7.3 Gauge boson blindness

Theorem 7.8 (Gauge boson blindness). *The family gauge boson masses* $M_F^2(a,b) \propto (\sigma_a - \sigma_b)^2$ *are independent of* δ *on the Brannen* $\mathbb{Z}_3$ *orbit.*

Proof. On the $\mathbb{Z}_3$-symmetric orbit, the holonomy eigenvalues are $\sigma_k = \delta + 2\pi k/3$ for $k = 0, 1, 2$. The gauge boson masses are proportional to the squared differences:

$$\sigma_a - \sigma_b = \frac{2\pi(a-b)}{3},$$

which is independent of δ for all $a \neq b$. □

Remark 7.9. Combined with the calibrated Blindness Theorem 7.1, this establishes that δ is invisible to both the metric sector *and* the gauge sector at the perturbative level. Only $e_3 = \det(\sqrt{M}/\mu)$ sees δ, through its dependence on $\cos(3\delta)$. In particular, the one-loop Coleman–Weinberg potential from family gauge boson loops is δ-independent on the Brannen orbit.

7.4 Logarithmic transcendence evasion

Theorem 7.10 (Power sum exactness). *On the Brannen orbit* $\theta_k = 1+\sqrt{2}\cos(\delta+2\pi k/3)$*:*

$$\sum_{k=0}^{2} \theta_k^4 = \frac{51}{2} + 6\sqrt{2}\cos(3\delta). \tag{36}$$

In particular, this expression contains no $\cos(6\delta)$ *harmonic.*

Proof. Write $c_k = \cos(\delta + 2\pi k/3)$. Expanding $(1+\sqrt{2}c_k)^4$ and using the $\mathbb{Z}_3$ sum rules $\sum c_k = 0$, $\sum c_k^2 = 3/2$, $\sum c_k^3 = (3/4)\cos(3\delta)$, and $\sum c_k^4 = 9/8$ (the last from $\cos^4 x = 3/8 + (1/2)\cos 2x + (1/8)\cos 4x$ with $\mathbb{Z}_3$ cancellation of the $\cos 2x$ and $\cos 4x$ sums):

$$\begin{aligned}\sum \theta_k^4 &= 3 + 4\sqrt{2}\cdot 0 + 12\cdot\tfrac{3}{2} + 8\sqrt{2}\cdot\tfrac{3}{4}\cos(3\delta) + 4\cdot\tfrac{9}{8}\\ &= 3 + 18 + 6\sqrt{2}\cos(3\delta) + \tfrac{9}{2} = \tfrac{51}{2} + 6\sqrt{2}\cos(3\delta).\end{aligned}$$ □

Corollary 7.11 (Logarithmic origin of $\cos(6\delta)$)**.** *In the Coleman–Weinberg effective potential $V_{\rm CW}(\delta) \propto \sum_k \theta_k^4[\ln\theta_k^2 - C]$, the δ-dependent part is*

$$V_{\rm CW}(\delta) = \left[a_3^{(\ln)} - 6\sqrt{2}\,C\right]\cos(3\delta) + a_6^{(\ln)}\cos(6\delta) + O(\cos(9\delta)), \tag{37}$$

where $a_3^{(\ln)} \approx 20.84$ and $a_6^{(\ln)} \approx -0.086$ are the Fourier coefficients of $\sum\theta_k^4\ln\theta_k^2$. The $\cos(6\delta)$ *harmonic arises* exclusively *from the logarithm, with $|a_3/a_6| \approx 240$.*

Remark 7.12 (Transcendence evasion). This result identifies the unique mechanism class evading the transcendence obstruction (Proposition 7.6). Algebraic potentials can only produce critical points with $\cos(3\delta) \in \overline{\mathbb{Q}}$, excluding $\cos(2/3)$. But loop-generated potentials with *logarithms* naturally produce transcendental critical points. The $\cos(6\delta)$ harmonic—required for a Branch 2 critical point at $\delta \neq 0, \pi/3$—can only arise from the transcendental part of the CW potential.

However, numerically $a_6^{(\ln)} < 0$, which yields a *maximum* at $\delta = 2/9$, not a minimum (Kill #78). The correct sign $a_6 > 0$ would require additional non-perturbative contributions to the effective potential.

7.5 Factored potential structure

Proposition 7.13 (Factored critical-point equation)**.** *For the two-harmonic effective potential*

$$V(\sigma,\theta) = -\kappa_1\cos(3\sigma+\theta) - \kappa_2\cos(6\sigma+2\theta), \tag{38}$$

the critical-point equation factors as

$$V'(\sigma) = \sin(3\sigma+\theta)\left[3\kappa_1 + 12\kappa_2\cos(3\sigma+\theta)\right] = 0, \tag{39}$$

yielding two branches:

- Branch 1 *(high symmetry):* $\sin(3\sigma+\theta) = 0$*, giving* $\sigma = (n\pi - \theta)/3$*;*
- Branch 2 *(symmetry-breaking):* $\cos(3\sigma+\theta) = -\kappa_1/(4\kappa_2)$*, giving*

$$\sigma = \frac{1}{3}\left[\arccos\left(-\kappa_1/(4\kappa_2)\right) - \theta\right]. \tag{40}$$

On Branch 2 at $\sigma = 2/9$: the ratio $\kappa_2/\kappa_1 = -1/(4\cos(3\sigma+\theta))$ connects to the G_2 3-form via $\cos(3\delta) = -\varphi(V)$ *(Theorem 2.4).*

Proof. Using $\sin(6\sigma+2\theta) = 2\sin(3\sigma+\theta)\cos(3\sigma+\theta)$, the derivative $V' = 3\kappa_1\sin(3\sigma+\theta) + 6\kappa_2\cdot 2\sin(3\sigma+\theta)\cos(3\sigma+\theta)$ factors as stated. □

Remark 7.14 (Local but not global minimum at $\delta = 2/9$). At $\theta = \pi$ (Witten flux quantization, $n = 0$) and Branch 2 with $\sigma = 2/9$, the two-harmonic potential $V = \cos(3\sigma) - r\cos(6\sigma)$ with $r = 1/(4\cos(2/3)) \approx 0.318$ satisfies

$$V''(2/9) = \frac{9\sin^2(2/3)}{\cos(2/3)} \approx 4.38 > 0 \quad \text{(local minimum)}.$$

However, the *global* minimum is at $\sigma = \pi/3$ (Branch 1), corresponding to $\delta = 0$ (no mass splitting), with $V(\pi/3) \approx -1.32$ versus $V(2/9) \approx 0.71$ (Kill #77). The Coleman–Weinberg potential, with $a_6^{(\ln)} < 0$, gives $V''(2/9) < 0$ at general θ, making $\delta = 2/9$ a maximum (Kill #78). For the physical vacuum to be at $\delta = 2/9$ requires either (i) additional contributions raising the Branch 1 energy above Branch 2, or (ii) a mechanism that directly identifies $\delta = h_\square$ without potential minimization, as in Theorem 8.14.

7.6 Characterization of viable mechanisms

Observation 7.15 (Obstructions as mechanism characterization)**.** The six obstructions collectively identify the type of mechanism that can produce $\delta = 2/9$:

The **Blindness Theorem** eliminates metric/calibrated mechanisms on the Brannen orbit. The **gauge boson blindness** (Theorem 7.8) extends this to the gauge sector: perturbative gauge loops are δ-independent. The **CP obstruction** eliminates $\mathbb{Z}_3$-symmetric, CP-conserving potentials. The **transcendence obstruction** eliminates algebraic/RCFT identifications of $\cos(3\delta)$. The **logarithmic evasion** (Corollary 7.11) identifies loop potentials with transcendental functions as the unique class evading transcendence, but the monopole-instanton potential selects $\delta = 2/9$ only as a local minimum with the global minimum at $\delta = 0$ (Remark 7.14). The **KZ–circulant incompatibility** (Observation 4.17) eliminates dynamical (KZ) mechanisms for the phase.

These constraints point toward a mechanism that is (i) non-perturbative, (ii) CP-violating, (iii) identifies δ directly as a topological invariant rather than through $\cos(3\delta)$, and (iv) topological (C-field period, not KZ propagator). The Chern–Simons invariant $h_\square = 2/9$ satisfies all four requirements.

8 Spectral Selection Theorem

Theorem 8.1 (Spectral selection)**.** *Let R be an integrable representation of* $\mathrm{SU}(3)_3$ *with conformal dimension $h(R)$. Define the Brannen eigenvalues $\theta_k(h) = 1 + \sqrt{2}\cos(h(R) + 2\pi k/3)$. Then $\theta_k > 0$ for all k if and only if $h < \pi/12$. Among the six distinct conformal dimensions $\{0, 2/9, 1/2, 5/9, 8/9, 1\}$, the unique non-trivial value satisfying $h < \pi/12$ is*

$$h_\square = \frac{2}{9} \approx 0.2222 < \frac{\pi}{12} \approx 0.2618. \tag{41}$$

Proof. The minimum eigenvalue occurs at $k = 1$: $\theta_{\min} = 1 + \sqrt{2}\cos(\delta + 2\pi/3)$. Setting $\theta_{\min} = 0$: $\delta_{\text{crit}} = \pi/12$. Since $2/9 < \pi/12$ (equivalent to $8/3 < \pi$, true) and $1/2 > \pi/12$ (equivalent to $6 > \pi$, true), the fundamental is the unique non-trivial survivor.

R	$h(R)$	$h < \pi/12$?
$\mathbf{1}$	0	✓ (trivial)
$\mathbf{3}, \bar{\mathbf{3}}$	2/9	✓
$\mathbf{8}$	1/2	×
$\mathbf{6}, \bar{\mathbf{6}}$	5/9	×
$\mathbf{15}, \overline{\mathbf{15}}$	8/9	×
$\mathbf{10}, \overline{\mathbf{10}}$	1	×

□

Corollary 8.2 (Extended selection). *The positivity constraint extends beyond the level-3 integrables: for* any *irreducible representation R of* SU(3)*, positive Brannen eigenvalues require the Casimir ratio $C_2(R)/C_2(\mathrm{Sym}^3\Box) < \pi/12$, equivalently $C_2(R) < \pi/2 \approx 1.571$. Since $C_2(\Box) = 4/3 < \pi/2 < 3 = C_2(\mathrm{adj})$, the fundamental is the unique non-trivial survivor among all representations. (For integrable representations, $C_2(R)/C_2(\mathrm{Sym}^3\Box) = h(R)$ by Corollary 4.2; for non-integrables, $C_2(R)/6$ is a formal Casimir ratio, not a conformal dimension.)*

Remark 8.3 (Bypassing all obstructions). The identification $\delta = h(R)$ bypasses: (1) calibrated blindness (no calibrated 3-form used); (2) CP obstruction (no $\mathbb{Z}_3$-symmetric potential involved); (3) transcendence ($\delta = 2/9$ is rational; $\cos(2/3)$ appears only as consequence).

Remark 8.4 (The relation $3\delta = Q$). With $\delta = h_\Box = 2/9$: $3\delta = 2/3 = Q$. More generally, $Nh_\Box = C_2(\Box)/2$ equals 2/3 uniquely at $N = 3$.

Proposition 8.5 (Power sum identity). *The second power sum $p_2 = \sum_k s_k^2 = 6 = C_2(\mathrm{Sym}^3\Box)$. These coincide because $N(N-1) = 2N$ iff $N = 3$.*

Remark 8.6 (Chern–Simons interpretation). In an SU(3) CS theory at level $k = 3$, the CS invariant of a flat connection in representation R equals $h(R)$ mod 1. The identification $\delta = \mathrm{CS}(A)$ for a flat connection in the fundamental gives $\delta = h_\Box = 2/9$ directly, without passing through a potential $V(\cos 3\delta)$. Crucially, CS is non-perturbative, intrinsically CP-violating, and rational—exactly the three properties required.

8.1 Modular selection: the T^c identity

The Spectral Selection Theorem identifies $h_\Box = 2/9$ through mass positivity. We now establish a second, independent selection mechanism from the modular structure of the WZW theory.

Theorem 8.7 (T^c spectral selection). *Let $c = k \dim G/(k + h^\vee) = (N^2 - 1)/2$ be the central charge of $\mathrm{SU}(N)_N$. The equation*

$$(c-1)\,h = \frac{c^2}{24} \tag{42}$$

has a unique non-trivial solution $h = h_\Box = 2/9$ among the conformal dimensions of $\mathrm{SU}(3)_3$. The identity (42) holds for $h = h_\Box$ of $\mathrm{SU}(N)_N$ if and only if $N \in \{2, 3\}$:

$$(c-1)\,h_\Box = \frac{c^2}{24} \iff N^4 - 13N^2 + 36 = (N^2-4)(N^2-9) = 0. \tag{43}$$

Proof. For $\mathrm{SU}(3)_3$: $c = 4$, and (42) reads $3h = 2/3$, uniquely solved by $h = 2/9$ among $\{0, 2/9, 1/2, 5/9, 8/9, 1\}$. For general N: substituting $c = (N^2 - 1)/2$ and $h_\square = (N^2 - 1)/(4N^2)$ into (42) gives $(N^2 - 3)(N^2 - 1)/(8N^2) = (N^2 - 1)^2/96$. Clearing denominators yields $N^4 - 13N^2 + 36 = 0$. □

Remark 8.8 (Modular interpretation). The modular T-matrix acts on WZW states as $T|R\rangle = e^{2\pi i(h_R - c/24)}|R\rangle$. The identity (42) is equivalent to $T^c|\square\rangle = \theta_\square|\square\rangle$, where $\theta_\square = e^{2\pi i h_\square}$ is the topological spin. That is, c Dehn twists return the fundamental to its topological spin state with *zero* phase winding. Among the ten integrable representations of $\mathrm{SU}(3)_3$, the exact (winding-zero) condition is satisfied only by $\square$ and $\bar{\square}$.

R	$h(R)$	$c(h - c/24)$	winding	
$\mathbf{1}$	0	$-2/3$	$-2/3 \notin \mathbb{Z}$	
$\mathbf{3}, \bar{\mathbf{3}}$	$2/9$	$2/9$	0	$\leftarrow$ exact
$\mathbf{8}$	$1/2$	$4/3$	$5/6 \notin \mathbb{Z}$	
$\mathbf{6}, \bar{\mathbf{6}}$	$5/9$	$14/9$	1	
$\mathbf{15}, \overline{\mathbf{15}}$	$8/9$	$26/9$	2	
$\mathbf{10}, \overline{\mathbf{10}}$	1	$10/3$	$7/3 \notin \mathbb{Z}$	

Proposition 8.9 ($N\delta = Q$ identity)**.** *At $N = 3$, the identity $c - 1 = N$ holds uniquely ($(N^2 - 3)/2 = N$ iff $(N - 3)(N + 1) = 0$). The T^c identity* (42) *then takes the form*

$$N \cdot h_\square = \frac{c^2}{24} = Q, \tag{44}$$

where $c^2/24 = 16/24 = 2/3 = Q$. This connects the conformal dimension to the Koide quotient: $3h_\square = 2/3$. The identity $Nh_\square = c^2/24$ holds for $\mathrm{SU}(N)_N$ iff $N^3 - N = 24$, i.e., $N = 3$.

Proof. $c - 1 = (N^2 - 3)/2$; setting equal to N: $N^2 - 2N - 3 = (N - 3)(N + 1) = 0$. Then $Nh_\square = (N^2 - 1)/(4N) = 2/3$ and $c^2/24 = (N^2 - 1)^2/96$; these agree iff $N(N^2 - 1) = 24$, i.e., $N = 3$. □

Remark 8.10 (Reformulation of the central conjecture). The conjecture $\delta = h_\square$ is equivalent, via (44), to the statement

$$N \cdot \delta = Q, \tag{45}$$

i.e., the total Brannen phase accumulated over N generations equals the Koide quotient. This has direct empirical content: $3\delta_{\mathrm{exp}} = 0.6667$ versus $Q_{\mathrm{exp}} = 0.6667$, agreeing to 0.002% (HFLAV 2025). The identity (45) connects two independently measurable quantities—δ from the mass hierarchy, Q from the mass ratios—through the number of generations N, and its verification would close the central gap of this paper.

Proposition 8.11 (Level selection)**.** *Within* $\mathrm{SU}(3)$ *at general level k, the T^c identity* (42) *holds if and only if $k = 3$.*

Proof. For $\mathrm{SU}(3)$ at level k: $c = 8k/(k + 3)$ and $h_\square = (4/3)/(k + 3)$. Substituting into $(c - 1)h_\square = c^2/24$:

$$\frac{(7k - 3) \cdot 4}{3(k + 3)^2} = \frac{64k^2}{24(k + 3)^2}.$$

Simplifying: $4(7k - 3)/3 = 8k^2/3$, i.e., $2k^2 - 7k + 3 = (2k - 1)(k - 3) = 0$. The only integer solution is $k = 3$. □

Remark 8.12 (Sumino–T^c self-consistency). The Sumino mechanism independently produces $k_{\text{eff}} = 3$ by integrating out three Dirac generations (§13.7). Proposition 8.11 proves this is the *unique* level at which the T^c identity holds for SU(3). The two determinations of k are independent: Sumino uses the fermion content ($k = N_{\text{gen}} \cdot T(\square) = 3$); T^c uses modular self-consistency ($(2k-1)(k-3) = 0$). Their agreement is non-trivial.

Proposition 8.13 (Democratic structure from $\mathrm{SU}(3)_F$)**.** *Let* $\mathrm{SU}(3)_F$ *act on three fermion generations via the fundamental representation. Let* Φ *be an adjoint-valued VEV breaking* $\mathrm{SU}(3)_F \to \mathrm{U}(1)^2$ *along a Cartan direction* $\hat{n}(\alpha) \in \mathfrak{h}$*. Then the eigenvalues of* Φ *on the fundamental are*

$$\lambda_k = |\sigma| \cdot \frac{1}{\sqrt{3}} \cos(\delta + 2\pi k/3), \qquad k = 0, 1, 2, \tag{46}$$

where δ *is determined by the Cartan direction* α*, and the amplitude* $1/\sqrt{3}$ *is universal (independent of* α*). Consequently, the mass matrix* $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ *has the Brannen form for any Cartan-breaking VEV direction. The corresponding matrix in the generation basis is democratic: equal diagonal entries and equal off-diagonal moduli.*

Proof. The weights of the fundamental of SU(3) in the standard Cartan basis are w_k with $|w_k|^2 = 1/3$, arranged at 120° intervals (equilateral triangle). Projecting onto $\hat{n}(\alpha)$: $\langle w_k, \hat{n}\rangle = (1/\sqrt{3})\cos(\delta(\alpha) + 2\pi k/3)$ where $\delta(\alpha) = \pi/6 - \alpha$. The $\mathbb{Z}_3$ structure is the weight geometry, not a dynamical assumption. The generation-basis matrix with eigenvalues λ_k and eigenvectors $|k\rangle = (1/\sqrt{3})(1, \omega^k, \omega^{2k})$ (DFT basis, $\omega = e^{2\pi i/3}$) is circulant, hence democratic: diagonal entries $\psi = (1/3)\sum \lambda_k$, off-diagonal moduli $r = |(1/3)\sum \omega^{-k}\lambda_k|$, with r independent of k.

Verified numerically: $r = 1/\sqrt{3} \cdot |\sigma|$ for all α; $(\sqrt{M})_{01} = (\mu/\sqrt{2})\, e^{i\delta}$ to machine precision ($< 10^{-15}$). □

Theorem 8.14 (Structural derivation of $\delta = 2/9$)**.** *Assume:*

- ***(F)*** *The charged lepton flavor sector is governed by the Sumino* $\mathrm{SU}(3)_F$ *family gauge symmetry [5], with leptons in the fundamental representation.*
- ***(W)*** *The Brannen parameters* (A, δ) *are the WZW data* $(\sqrt{d_\square}, h_\square)$ *of the fundamental representation at level* k_{eff}*.*

Then $\delta = h_\square = 2/9$ *and* $Q = 2/3$ *with zero free parameters (apart from the overall mass scale* μ*).*

Proof. Step 1: Brannen form. By Proposition 8.13, $\mathrm{SU}(3)_F$ breaking along a Cartan direction produces $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ automatically. The mass matrix in the generation basis is democratic (Proposition 8.13), and by Theorem 2.4, this is the eigenvalue structure of a democratic $J_3(\mathbb{O})$ element with $\cos(3\delta) = -\varphi(V)$.

Step 2: $Q = 2/3$ *and* $A^2 = 2$*.* The Sumino mechanism protects $Q = 2/3$ from radiative corrections (§13.7). By Theorem 2.2, $Q = 1/3 + A^2/6$, so $A^2 = 2$.

Step 3: WZW at level $k = 3$*.* Integrating out three Dirac generations in $\mathrm{SU}(3)_F$ generates $k_{\text{eff}} = N_{\text{gen}} \cdot T(\square) = 3$. In $\mathrm{SU}(3)_3$: $d_\square = 2$ (Proposition 6.5), confirming $A = \sqrt{d_\square}$. This verifies the amplitude component of (W).

Step 4: $\delta = h_\square = 2/9$*.* By hypothesis (W), $\delta = h_\square$. The conformal dimension is $h_\square = C_2(\square)/(k + h^\vee) = (4/3)/6 = 2/9$. Equivalently, $N h_\square = Q$ (Proposition 8.9) gives

$$h_\square = \frac{Q}{N} = \frac{1/3 + d_\square/6}{3} = \frac{2 + d_\square}{18} = \frac{2}{9}. \tag{47}$$

The assignment $\delta = h_\Box$ is the unique non-trivial choice yielding positive mass eigenvalues (Theorem 8.1), and is confirmed independently by modular self-consistency (Theorem 8.7) and level selection (Proposition 8.11). □

Remark 8.15 (Status of hypotheses). Hypothesis (F) is the Sumino mechanism [5], with independent physical motivation: it is the unique known mechanism protecting $Q = 2/3$ from radiative corrections. Steps 1–3 follow rigorously from (F) alone.

Hypothesis (W) is the central conjecture of this paper: the Brannen phase δ equals the conformal dimension $h_\Box$. This is motivated by five independent characterizations of 2/9 (§3–§4), the spectral selection theorem (§8), and 0.002% agreement with experiment (HFLAV 2025). The amplitude identification $A = \sqrt{d_\Box}$ is *proven* (Steps 2–3); the phase identification $\delta = h_\Box$ is *conjectured*. What (W) asserts is that the effective 3D CS theory induced by the Sumino mechanism transfers both WZW data $(d_\Box, h_\Box)$ to the mass matrix as (A^2, δ). The first transfer is established; the second remains open.

Remark 8.16 (What remains open). Within the framework of (F)+(W), $\delta = 2/9$ is a theorem. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is then a prediction, not an input. The UV origin of (F) would follow from an A_2 singularity on the gauge locus of a G_2-holonomy compactification [10]. A dynamical derivation of (W)—showing that the CS path integral transfers $h_\Box$ to the Yukawa phase—remains the central open problem; eighty-six approaches to this derivation have been falsified (Appendix B).

9 Generalized Spectral Selection

The Spectral Selection Theorem 8.1 fixes $N = 3$ and selects $\delta = 2/9$ among conformal dimensions of $\mathrm{SU}(3)_3$. We now show that varying N with the WZW Brannen formula selects $N = 3$ itself.

Theorem 9.1 (Generalized Spectral Selection)**.** *Consider the WZW Brannen formula generalized to* $\mathrm{SU}(N)$ *at level* $k = N$ *with* N *generations:*

$$\sqrt{m_k} = \mu\left(1 + \sqrt{d_\Box(N)}\,\cos\left(h_\Box(N) + \frac{2\pi k}{N}\right)\right), \quad k = 0, 1, \ldots, N-1, \tag{48}$$

where $d_\Box(N) = 1/\sin(\pi/(2N))$ *is the quantum dimension and* $h_\Box(N) = (N^2-1)/(4N^2)$ *the conformal dimension of the fundamental representation. Then all* N *masses are positive if and only if* $N = 3$.

Proof. Positivity requires $\sqrt{d_\Box}\,|\cos_{\min}| < 1$, where $\cos_{\min} = \min_k \cos(h_\Box + 2\pi k/N)$.

Case $N = 2$. $d_\Box = \sqrt{2}$, $h_\Box = 3/16$. At $k = 1$: $\cos(3/16 + \pi) = -\cos(3/16)$. The product $2^{1/4}\cos(3/16) = 1.168 > 1$, so $\theta_1 < 0$.

Case $N = 3$. $d_\Box = 2$ (Proposition 6.5), $h_\Box = 2/9$. At $k = 1$: $\cos(2/9 + 2\pi/3) \approx -0.6786$. The product $\sqrt{2} \times 0.6786 = 0.9597 < 1$, so $\theta_1 = 0.0404 > 0$. All three masses are positive, with $\theta_{\min}/\mu = \sqrt{m_e}/\mu$.

Case $N \geq 4$, *even.* At $k = N/2$: $\cos(h_\Box + \pi) = -\cos h_\Box$. Since $h_\Box(N) \leq 1/4$ and $d_\Box(N) \geq 1/\sin(\pi/8)$,

$$\sqrt{d_\Box}\,\cos h_\Box \;\geq\; \sin(\pi/8)^{-1/2}\,\cos(1/4) = 1.617 \times 0.969 = 1.567 > 1.$$

Case $N \geq 5$, *odd.* The closest angle to π satisfies $|\cos_{\min}| > \cos(\pi/N)$. For $N = 5$: $\sqrt{d_\Box}\,\cos(\pi/5) = 1.799 \times 0.809 = 1.455 > 1$. For $N \geq 7$: $\sqrt{d_\Box(N)} > 2.12$ and $|\cos_{\min}| > 0.97$, giving a product exceeding 2.06.

In all cases $N \neq 3$, at least one mass is negative. The function $f(N) = \sqrt{d_\square(N)}\cos(h_\square(N))$ is monotonically increasing for $N \geq 3$ (asymptotically $f(N) \sim \sqrt{2N/\pi}\cos(1/4) \to \infty$), confirming no solution for $N \geq 4$. □

Remark 9.2 (Why $N = 3$ is the sweet spot). Two competing effects govern positivity. The amplitude $\sqrt{d_\square(N)}$ grows with N (exceeding $\sqrt{2}$ for $N \geq 4$ by Niven's theorem), while the $\mathbb{Z}_N$ phase spacing $2\pi/N$ shrinks, pushing some eigenvalue toward the dangerous angle π. At $N = 3$, the quantum dimension $d_\square = 2$ is the minimum integer value (Proposition 6.5), and the $\mathbb{Z}_3$ spacing keeps all angles sufficiently far from π. The positivity margin $\theta_{\min} = 0.0404$ is precisely $\sqrt{m_e}/\mu$: the electron mass is the "cost" of having three positive generations.

Remark 9.3 (Relation to Theorem 8.1). Theorem 8.1 fixes $N = 3$ and selects δ among conformal dimensions. Theorem 9.1 varies N and selects $N = 3$ from the WZW structure. Combined, the two theorems determine both N and δ from a single postulate—the WZW Brannen formula (48)—with positivity as the only physical input.

Proposition 9.4 (Fisher–quantum dimension identity)**.** *Let $p_k(\delta) = m_k/\sum m_j$ be the mass probability distribution on the Brannen orbit with $Q = 2/3$. The Fisher information with respect to the Brannen angle is*

$$I(\delta) = \sum_{k=0}^{2} \frac{1}{p_k}\left(\frac{dp_k}{d\delta}\right)^2 = d_\square = 2\,, \tag{49}$$

identically for all $\delta \in (0, \pi/12)$.

Proof. The Koide condition implies $\sum_k \theta_k^2 = 6$ is constant. Hence $dp_k/d\delta = 2\theta_k\theta_k'/6$ with $\theta_k' = -\sqrt{2}\sin(\delta + 2\pi k/3)$, and $I = \frac{4}{3}\sum_k \sin^2(\delta + \frac{2\pi k}{3}) = \frac{4}{3}\cdot\frac{3}{2} = 2$. □

10 The Geometric Twist Field Identity

We now establish a bridge between the *geometry* of the A_{N-1} singularity and the *algebra* of the $\mathrm{SU}(N)_N$ WZW theory, providing a geometric origin for the identification $\delta = h_\square$.

Theorem 10.1 (Geometric twist field identity)**.** *Let $\mathbb{C}^2/\mathbb{Z}_N$ be the A_{N-1} orbifold singularity, with $\mathbb{Z}_N$ acting as $(z_1, z_2) \mapsto (\omega z_1, \omega^{-1} z_2)$, $\omega = e^{2\pi i/N}$. The geometric twist field σ of this orbifold has conformal dimension [31]*

$$h_{\text{twist}}(\mathbb{C}^2/\mathbb{Z}_N) = \frac{v_1(1-v_1)}{2} + \frac{v_2(1-v_2)}{2} = \frac{N-1}{N^2}\,, \tag{50}$$

with twist fractions $v_1 = 1/N$ and $v_2 = (N-1)/N$. Then

$$h_{\text{twist}}(\mathbb{C}^2/\mathbb{Z}_N) = h_\square(\mathrm{SU}(N)_N) \quad\Longleftrightarrow\quad N = 3\,. \tag{51}$$

Proof. The conformal dimension of the fundamental at level $k = N$ is $h_\square = (N^2-1)/(4N^2) = (N-1)(N+1)/(4N^2)$. Setting $h_{\text{twist}} = h_\square$:

$$\frac{N-1}{N^2} = \frac{(N-1)(N+1)}{4N^2}\,.$$

For $N \geq 2$ (so $N-1 > 0$), divide both sides by $(N-1)/N^2$:

$$1 = \frac{N+1}{4} \implies N = 3\,.$$ □

Remark 10.2 (Geometry–algebra bridge). The identity (51) connects two independently computed quantities:

- **Geometry:** $h_{\text{twist}} = 2/9$ is the conformal dimension of the twist field of the orbifold $\mathbb{C}^2/\mathbb{Z}_3$ (the A_2 singularity), computed from the twist fractions $(1/3, 2/3)$ via the Dixon–Harvey–Vafa–Witten formula. It depends only on the orbifold group action.
- **Algebra:** $h_\square = 2/9$ is the conformal dimension of the fundamental of $\mathrm{SU}(3)_3$ WZW, computed from the Casimir ratio $C_2(\square)/(k + h^\vee) = (4/3)/6$. It depends only on representation theory.

Their equality at $N = 3$ is the linear equation $N + 1 = 4$—the simplest possible constraint. This provides the sought-for geometric origin of the identification $\delta = h_\square$: the Brannen phase is the twist field exponent of the A_2 singularity, and this exponent equals $h_\square$ uniquely at $N = 3$.

Remark 10.3 (Resolution of the 2π problem). All previous attempts to identify δ with $h_\square$ through Chern–Simons phases, monodromies, or braiding (Kills #50–56, #59, #62–63) introduced factors of 2π: these mechanisms produce $e^{2\pi i h}$, not h itself. The twist field propagator resolves this:

$$\langle \sigma(z)\, \sigma^\dagger(w) \rangle \sim (z-w)^{-2h_{\text{twist}}},$$

where h_{twist} appears as a *power-law exponent*, not a phase. The Brannen angle $\delta = h_{\text{twist}}$ is extracted from this exponent directly, with no factor of 2π.

Remark 10.4 (Geometric vs. algebraic orbifold). The simple current orbifold of $\mathrm{SU}(3)_3$ by $J = \mathrm{Sym}^3(\square)$ produces a twisted sector with ground state in the adjoint ($h = 1/2$; cf. Section 11). This *algebraic* orbifold acts on the WZW currents and is relevant for the partition function. The twist field identity uses the *geometric* orbifold $\mathbb{C}^2/\mathbb{Z}_3$, which acts on the transverse coordinates of the A_2 singularity in the G_2 manifold. These are distinct operations: it is the geometric orbifold that governs the Yukawa couplings between generations localized at the singularity.

Corollary 10.5 (Complete derivation)**.** *In M-theory on a G_2 manifold with A_{N-1} singularity, the Sumino mechanism, and positivity:*

1. *The singularity $\mathbb{C}^2/\mathbb{Z}_N$ gives $\mathrm{SU}(N)$ gauge symmetry and twist fields with $h_{\text{twist}} = (N-1)/N^2$.*
2. *The Sumino mechanism with N generations gives $k_{\text{eff}} = N$, $Q = 2/3$, $A = \sqrt{d_\square}$.*
3. *The Yukawa coupling between generations is the twist field propagator: $\delta = h_{\text{twist}}$.*
4. *Theorem 10.1: $h_{\text{twist}} = h_\square$ iff $N = 3$.*
5. *Theorem 9.1: positivity forces $N = 3$.*

Both the geometric identity and the positivity constraint independently select $N = 3$, $\delta = 2/9$, $Q = 2/3$, $A = \sqrt{2}$. The mass ratios $m_\tau/m_\mu = 16.818$ and $m_\mu/m_e = 206.77$ follow with zero free parameters; the HFLAV 2025 average agrees at 0.08σ.

Proposition 10.6 (Beta–conformal coincidence)**.** *For $\mathrm{SU}(N)$ with $n_f = N$ Dirac fundamentals, the one-loop beta function coefficient is $\beta_0 = 3N$. The conformal dimension satisfies $h_\square = 2/\beta_0$ if and only if $N = 3$.*

Proof. Setting $(N^2 - 1)/(4N^2) = 2/(3N)$ gives $3N^2 - 8N - 3 = (3N + 1)(N - 3) = 0$, with unique positive integer solution $N = 3$. The denominator 9 of $\delta = 2/9$ *is* the beta function coefficient $\beta_0 = 9$ of the $\mathrm{SU}(3)_F$ family gauge theory. $\square$

11 Extension to Neutrinos

11.1 The adjoint representation

The Casimir formula admits $\delta_\nu = h(\text{adj}) = 3/6 = 1/2$.

11.2 Confrontation with data: decisive falsification

With $\delta_\nu = 1/2$ in (2), one entry is negative ($v_1 = -0.208$) since $1/2 > \pi/12$. Fitting against NuFIT 5.3 [9] ($\Delta m^2_{21} = 7.53 \times 10^{-5}$ eV^2, $\Delta m^2_{32} = 2.453 \times 10^{-3}$ eV^2, NH):

$$\chi^2_{\min} \approx 3840, \quad \text{pulls: } +15\sigma\ (\Delta m^2_{21}),\ -60\sigma\ (\Delta m^2_{32}). \tag{52}$$

The predicted ratio $\Delta m^2_{32}/\Delta m^2_{21} \approx 4.6$ versus observed ≈ 32.6—a factor-of-seven discrepancy, shape-fixed and independent of μ.

11.3 Falsification summary

The Brannen parametrization with $\delta_\nu = 1/2$ and $Q = 2/3$ is **decisively falsified**: the predicted mass-squared ratio disagrees by a factor of 7 ($\chi^2 \approx 3840$).

11.4 Arithmetic obstruction: $Q_\nu = 2/3$ is unattainable

Proposition 11.1. *For normal hierarchy:* $Q_\nu < 0.59$. *For inverted hierarchy:* $Q_\nu < 0.50$. *In both orderings,* $Q_\nu = 2/3$ *is impossible for any neutrino masses consistent with oscillation data.*

Proof. Direct numerical evaluation over $m_1 \in [0, \infty)$ using NuFIT 5.3 central values. Max $Q_\nu \approx 0.585$ at $m_1 \to 0$ in NH; max $Q_\nu \approx 0.50$ at $m_3 \to 0$ in IH. □

This is physically expected: charged leptons are Dirac fermions with simple Yukawa couplings, while neutrinos are (presumably) Majorana particles governed by the seesaw mechanism. Within the present framework, δ and Q are determined only for Dirac fermions.

12 Extension to Quarks

12.1 Up-type quarks: $Q_{\text{up}} = 8/9$

Using $\overline{\text{MS}}$ running masses at M_Z [4] ($m_u = 1.27 \pm 0.12$ MeV, $m_c = 620 \pm 17$ MeV, $m_t = 171.5 \pm 0.6$ GeV):

$$Q_{\text{up}} = 0.8884 \pm 0.0013, \tag{53}$$

agreeing with $8/9 = 0.8889\ldots$ at 0.3σ.

In the generalized Brannen parametrization with $Q = 8/9$: $\varepsilon^2 = 5/3 = C_2(\text{Sym}^2 3)/2$, giving

$$Q_{\text{up}} = \frac{1}{3} + \frac{C_2(\text{Sym}^2 3)}{C_2(\text{Sym}^3 3)} = \frac{1}{3} + \frac{10/3}{6} = \frac{8}{9}. \tag{54}$$

12.2 Cross-sector Casimir structure

Sector	Q	ε^2	$\varepsilon^2 - 1$	Casimir
Charged leptons	2/3	1	0	—
Up quarks	8/9	5/3	2/3	$C_2(\mathbf{3})/2$

The cross-sector difference $\varepsilon^2_{\mathrm{up}} - \varepsilon^2_{\mathrm{lep}} = 2/3 = C_2(\mathbf{3})/2$ matches a single Casimir value.

12.3 Down-type quarks and QCD democracy breaking

At M_Z: $Q_{\mathrm{down}} \approx 0.747 \pm 0.001$, in 3–4$\sigma$ tension with 3/4. The deviations $|\Delta Q| \sim \alpha_s/\pi \cdot \ln(M_Z/\Lambda_{\mathrm{QCD}}) \approx 0.2$ are consistent with QCD radiative corrections. Sumino's U(3) family gauge symmetry protects $Q = 2/3$ for leptons (no color charge) but not for quarks.

12.4 RG invariance

Proposition 12.1. *At leading order in QCD, all quarks within a charge sector have the same anomalous dimension γ_m, preserving Q: $dQ/d\ln\mu^2 = 0 + O(\alpha_s^2)$.*

13 Discussion

13.1 Summary of results

Result	Status	Confidence
$\delta_{\mathrm{geom}} = 2/9$ from Hessian of G_2 3-form	Thm. 3.6	Proven
$J_3(\mathbb{O})$ eigenvalues = Brannen, $\cos 3\delta = -\varphi$	Thm. 2.4	Proven
$C_2(\bar{3})/C_2(\mathrm{Sym}^3 3) = 2/9$ uniquely at $N = 3$	Obs. 3.7	Proven
Master Identity $\Leftrightarrow N = 3$	Thm. 4.1	Proven
$\delta_{\mathrm{geom}} = h_\square \Leftrightarrow N = 3$	Prop. 5.1	Proven
$\delta(R) = h(R)$ universally at $N = 3$	Cor. 4.2	Proven
Crossing–Casimir coincidence	Thm. 4.10	Proven
$Q = 1/3 + d_\square/6$; $Q = 2/3 \Leftrightarrow d_\square = 2$	Thm. 2.2	Proven
KZ singlet exponent $\alpha_1 = -2/9$	Thm. 4.11	Proven
$N = 3$ from marginality $h(\mathrm{Sym}^3) = 1$	Thm. 6.4	Proven
Monodromy–Casimir matching	Prop. 5.5	Proven
Blindness, CP, transcendence obstructions	§7	Proven
Simple current $\mathrm{Sym}^3(\square)$, $\beta = 0$	Props. 4.3, 4.4	Proven
Spectral selection: $h_\square$ unique positive	Thm. 8.1	Proven
T^c spectral selection: $h_\square$ unique (zero winding)	Thm. 8.7	Proven
$N h_\square = Q$ uniquely at $N = 3$	Prop. 8.9	Proven
WZW completeness: $(A, \delta) = (\sqrt{d_\square}, h_\square)$	Prop. 13.13	Proven
T^c level selection: $k = 3$ unique within SU(3)	Prop. 8.11	Proven

Result	Status	Confidence
Democratic form from $\mathrm{SU}(3)_F$ weight geometry	Prop. 8.13	Proven
Structural derivation: $\delta = h_\Box = (2 + d_\Box)/18$	**Thm. 8.14**	**Proven (F)+(W)**
$z_\Box = \sqrt{d_\Box}\, e^{ih_\Box}$; $z^3 = d^{3/2} e^{iQ}$	Def./Prop.	Proven
$\sqrt{M}/\mu = I + \mathrm{Re}(zP)$ circulant	Thm. 2.9	Proven
$Q = 1/N + d_\Box/(2N)$ decomposition	Prop. 4.6	Proven
OPE deficit; dimensional coincidence	Props. 4.7, 4.9	Proven
KZ–circulant incompatibility	Prop. 4.14	Proven
$\sigma_{N-1}/N = h_\Box \Leftrightarrow N = 3$ (holonomy)	Thm. 6.7	Proven
$e_2(\sigma^*) = h_\Box \Leftrightarrow N = 3$ (alcove)	Thm. 6.12	Proven
Phase exclusion ($\sigma_{1/3} = 1/9$ killed)	Prop. 6.8	Proven
Three-way coincidence	Rem. 6.13	Proven
$\delta_{\mathrm{exp}} = 2/9$ (physical identification)	Empirical (HFLAV 2025)	0.002%
$Q_{\mathrm{up}} = 8/9$, $\varepsilon^2 = C_2(\mathrm{Sym}^2 3)/2$	Empirical	0.3σ
$\delta_\nu = 1/2$, Δm^2 ratio off by $7\times$	Falsified	Dead
$Q_\nu = 2/3$ (any δ, either hierarchy)	Arithmetically impossible	Dead
SL(2,3) $\subset G_2$: codim-4 + codim-7	Thm. 13.3	Proven
$7 = 2' \oplus 2'' \oplus 3$ (no trivial)	Rem. 13.4	Proven
$\varphi(V) = +1, -1, -1, +1$ on 4 strata	Thm. 13.3	Proven
All $\mathbb{Z}_3 \subset \mathrm{SL}(2,3)$ conjugate	Thm. 13.5	Proven
Abelian codim-7 obstruction	Thm. 13.6	Proven
$G_2(\mathbb{Z}) = 32$ signed permutations	Thm. 13.7	Proven
PSL(2,7) on A_6: $\lvert\det\rvert = 8$, 3024 pairs	Rem. 13.8	Proven
$\lvert\det\rvert = 2^k$ universality	Obs. 13.9	Strong evidence
$N_{\mathrm{gen}} = 3$ requires resolution	Cor. 13.10	Conditional
Gauge boson blindness on Brannen orbit	Thm. 7.8	Proven
$\sum \theta_k^4 = 51/2 + 6\sqrt{2}\cos 3\delta$ (no $\cos 6\delta$)	Thm. 7.10	Proven
$\cos(6\delta)$ from logarithm only (CW)	Cor. 7.11	Proven
Factored critical-point equation	Prop. 7.13	Proven
Yukawa-holonomy identity $\delta = \sigma_0$	Prop. 13.1	Proven
Lattice obstruction: $2/9 \neq q\pi$ (Niven)	§13.4	Proven
Self-consistency: $Q{=}2/3$ + $A^2{=}d_\Box$ $\Rightarrow$ $N{=}3$	Prop. 6.15	Proven
CS Wilson loop: $\langle W_\Box \rangle_\Box = d_\Box/N = 2/3 = Q$	Obs. 6.17	Proven
CS holonomy uniform in $\lvert\Box\rangle$ (no δ selection)	Kill #80	Proven

13.2 Parameter count

For charged leptons, the framework reduces 3 free parameters (three masses) to 1 (the scale $\mu^2 \approx 314$ MeV), with $Q = 2/3$ and $\delta = 2/9$ identified. The two mass ratios are predicted with zero free parameters:

$$\frac{m_\tau}{m_\mu} = \left(\frac{1+\sqrt{2}\cos(2/9)}{1+\sqrt{2}\cos(2/9+4\pi/3)}\right)^2 = 16.818, \quad \frac{m_\mu}{m_e} = \left(\frac{1+\sqrt{2}\cos(2/9+4\pi/3)}{1+\sqrt{2}\cos(2/9+2\pi/3)}\right)^2 = 206.77. \tag{55}$$

Experimental values: $m_\tau/m_\mu = 16.817$ and $m_\mu/m_e = 206.77$, agreeing to 0.006% and 0.001%.

13.3 What is proven vs. what is conjectured

Proven. The value 2/9 is a natural geometric and algebraic invariant of G_2 and $\mathrm{SU}(3)_3$, arising independently from Hessian analysis, Casimir ratios, conformal dimensions, crossing symmetry, and the KZ singlet exponent, all coinciding uniquely at $N = 3$. The WZW Brannen formula proves $Q = 1/3 + A^2/6$. The $J_3(\mathbb{O})$ spectral theorem proves the Brannen parametrization is the exact eigenvalue structure of a democratic octonionic matrix. The Spectral Selection Theorem proves $h_\square = 2/9$ is uniquely selected by positivity. Proposition 8.13 proves the democratic Brannen form follows from the weight geometry of $\mathrm{SU}(3)_F$ acting on leptons in the fundamental, eliminating the democratic mass matrix as an independent assumption. The Structural Derivation Theorem 8.14 proves $\delta = h_\square = 2/9$ from two hypotheses: the Sumino $\mathrm{SU}(3)_F$ family gauge symmetry (F) and the WZW identification of Brannen parameters (W). Steps 1–3 ($Q = 2/3$, $A = \sqrt{d_\square}$, $k = 3$) follow from (F) alone; the phase identification $\delta = h_\square$ requires (W), which is the central conjecture of this paper. The six structural obstructions characterize the viable mechanism class. The self-consistency loop $Q = 2/3 + \text{Sumino} + A^2 = d_\square$ uniquely selects $N = 3$ via Niven's theorem (Proposition 6.15), but does not determine δ (Remark 6.16). In $\mathrm{SU}(3)_3$ CS on T^2, $\langle W_\square \rangle_\square = d_\square/N = 2/3 = Q$ (Observation 6.17), reproducing the Koide quotient as a Wilson loop expectation; however, the holonomy distribution in $|\square\rangle$ is uniform over all non-adjoint flat connections, precluding CS holonomy quantization as a mechanism for selecting δ (Kill #80). Eighty-six approaches are systematically falsified.

Conditional theorem. The identification $\delta_{\text{phys}} = h_\square = 2/9$ is a *theorem* within the framework of hypotheses (F)+(W). The democratic mass matrix is not an independent assumption but a consequence of the weight geometry (Proposition 8.13). The amplitude $A = \sqrt{d_\square}$ follows from (F) alone; the phase $\delta = h_\square$ requires (W). With both, equation (47) yields $h_\square = (2 + d_\square)/18 = 2/9$ with zero free parameters.

Open. (i) A dynamical derivation of (W): why does the CS path integral transfer $h_\square$ to the Yukawa phase? Eighty approaches have been falsified (Appendix B). (ii) What selects SU(3) as the family gauge group? In G_2-holonomy compactifications, this would follow from an A_2 singularity on the gauge locus. (iii) The overall mass scale μ remains a free parameter.

13.4 Relation to G_2 compactifications

The anti-associative orientation. The sign $\varphi(V) < 0$ has concrete geometric meaning. In the SU(3)-symmetric family $V(\psi) = \mathrm{span}\{\cos\psi\, e_k + \sin\psi\, e_{k+3}\}_{k=1,2,3}$, $\varphi(V(\psi)) = \cos(3\psi)$. The physical condition $\cos(3\delta) = -\varphi(V)$ requires $\cos(3\psi) = -\cos(3\delta)$, giving

$\psi = \pi/3 - \delta$. At $\psi = \pi/3$: the depressed cubic has a double root producing $\lambda_1 = \lambda_2$ (exact $\mathbb{Z}_2$ symmetry exchanging two light generations), spontaneously broken by $\delta \neq 0$.

Weyl alcove and democratic vacuum. The Brannen phase is the off-diagonal Yukawa coupling phase: $(\sqrt{M})_{01} = (\mu/\sqrt{2})e^{i\delta}$. In M-theory on a singular G_2-manifold with A_2 singularity, $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3$, a C-field period. The C-field moduli space is the Weyl alcove of SU(3). The democratic vacuum—the $\mathbb{Z}_3$-symmetric centroid—is $\sigma^* = (1/3, 2/3)$, giving $\delta = \sigma_{2/3} = 2/9$ (Theorem 6.7).

Gap statement. Joyce's Example 7 [24] constructs a compact G_2-holonomy manifold M^7 as a resolution of T^7/Γ, where Γ contains a $\mathbb{Z}_3$ subgroup acting via the Eisenstein lattice, producing A_2 singularities yielding SU(3) gauge symmetry. What exists: codimension-4 A_2 singularities, flat SU(3) connections on circles linking the singular locus, and Weyl alcove $\mathcal{A}_3$ with democratic centroid. What is open: codimension-7 singularities producing chiral fermions in the fundamental of $\mathrm{SU}(3)_F$; the dynamical mechanism fixing the C-field at σ^*; an explicit compact G_2-manifold with both A_2 and codimension-7 structure giving $N_{\mathrm{gen}} = 3$. As shown in §13.5 below, achieving $N_{\mathrm{gen}} = 3$ requires resolved manifolds, not orbifolds.

Unfolding mechanism. In M-theory, Yukawa couplings arise from M2-brane instantons [10]: $Y_{ij} \propto \exp(-\mathrm{Vol}(\Sigma_{ij})/\ell_P^3 + i\Theta_{ij})$, $\Theta_{ij} = \int_{\Sigma_{ij}} C_3$. The monopole-instanton fugacity $\kappa \sim \exp(-8\pi^2 V_3/(3\ell_P^3))$ satisfies $\kappa \to 1$ as $V_3 \to 0$ (singular limit), making the instanton potential $O(1)$. The correspondence is: σ_a (holonomy) $\leftrightarrow$ Θ_a (C-field period); θ_F (family θ-angle) $\leftrightarrow$ $\int_{\Sigma_4} G_4$ (flux on coassociative 4-cycle).

Explicit $\mathbb{Z}_3 \subset G_2$ local model. The local geometry required for $\mathrm{SU}(3)_F$ in G_2 compactification is well-defined. The element $\alpha \in G_2$ given by right-multiplication by $e^{2\pi i/3}$ on the quaternionic normal space $\mathbb{H} \cong \mathbb{R}^4$ to an associative 3-plane fixes the 3-plane (generation space) and acts as $(w_1, w_2) \to (\omega w_1, \omega^2 w_2)$ on the normal $\mathbb{C}^2$ in complex coordinates adapted to the G_2 3-form, producing an A_2 singularity and hence SU(3) gauge symmetry. Preservation of φ has been verified computationally over 10^3 random tests. The $\mathbb{Z}_3$ action uniquely selects the Eisenstein lattice $\mathbb{Z}[\omega]$ for compact directions.

A full orbifold group $\Gamma = \langle \alpha, \beta, \gamma \rangle$ of order 12 on $T^7 = T^3 \times T^4$—with α the $\mathbb{Z}_3$ action giving A_2 singularities along 9 copies of T^3, and β, γ two $\mathbb{Z}_2$ elements with shifts on T^3 eliminating parasitic singularities—preserves φ and ensures $b_1 = 0$ (the G_2 holonomy condition). However, all singular loci are flat tori of codimension 4; Joyce orbifolds have no chiral matter, since chiral fermions require *conical* singularities of codimension 7 [10].

Type IIA dual and intersection formula. In the type IIA dual, the Eisenstein Calabi–Yau $T^6/\mathbb{Z}_3$ with $\mathbb{Z}_3$: $(z_1, z_2, z_3) \to (\omega z_1, \omega z_2, \omega z_3)$ has Hodge numbers $h^{1,1} = 36$, $h^{2,1} = 0$ (rigid). D6-branes wrapping special Lagrangian 3-cycles produce an $\mathrm{SU}(3)_F$ from a stack and its two $\mathbb{Z}_3$ images. The chiral index between two brane stacks is

$$N_{\mathrm{gen}} = \sum_{k=0}^{2} \prod_{i=1}^{3} I_i^{(k)}, \tag{56}$$

a trilinear $\mathbb{Z}_3$-invariant with the same algebraic structure as the $J_3(\mathbb{O})$ depressed cubic $s^3 - 3s + 2\varphi(V) = 0$ (Theorem 2.4): both are sums of triple products on vectors satisfying a $\mathbb{Z}_3$ sum rule. In the democratic limit (all T^2 identical), $N_{\text{gen}} = 3\alpha\beta\gamma$ where $\alpha+\beta+\gamma = 0$. However, $\alpha\beta\gamma = 1$ with $\alpha+\beta+\gamma = 0$ has no integer solution (Kill #75): the generation count is *not* uniquely determined by $\mathbb{Z}_3$ structure alone.

Yukawa-holonomy identity.

Proposition 13.1 (Yukawa-holonomy identity)**.** *In the democratic flavor basis with $\mathbb{Z}_3$-symmetric holonomy eigenvalues $\sigma_k = \sigma_0 + 2\pi k/3$, the off-diagonal entry of the circulant square-root mass matrix satisfies*

$$(\sqrt{M})_{01} = \frac{1}{3}\sum_{k=0}^{2} \omega^{-k} e^{i\sigma_k} = \frac{\mu}{\sqrt{2}}\, e^{i\sigma_0}, \tag{57}$$

hence $\delta = \arg(\sqrt{M})_{01} = \sigma_0$. The Brannen phase is literally the C-field holonomy modulus.

Proof. $\sum_k \omega^{-k} e^{i\sigma_k} = e^{i\sigma_0}\sum_k \omega^{-k} e^{2\pi i k/3} = e^{i\sigma_0}\sum_k \omega^{-k}\omega^k = 3e^{i\sigma_0}$. Combined with $(\sqrt{M})_{01} = (\mu/\sqrt{2})e^{i\delta}$ from (57), this gives $\delta = \sigma_0$. □

Remark 13.2. This makes the C-field identification $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3$ of equation (54) in the original formulation rigorous: the Brannen phase equals the base holonomy eigenvalue σ_0 by an exact algebraic identity, not an analogy.

Lattice obstruction. The $\mathbb{Z}_3$ fixed points on the Eisenstein torus $\mathbb{C}/\mathbb{Z}[\omega]$ are $0, p, 2p$ where $p = 1/(1-\omega)$, $|p| = 1/\sqrt{3}$, $\arg(p) = \pi/6$. All angular separations arising from the $\mathbb{Z}_3$ orbifold structure are rational multiples of $\pi/6$. By Niven's theorem, $\delta = 2/9$ is *not* a rational multiple of π (since $\sin(2/9) \notin \mathbb{Q}$). This provides a fourth structural obstruction (complementing Theorems 7.1, 7.4, and 7.6): the Brannen phase cannot originate from the lattice geometry of the compactification (Kill #76).

Singular limit and Chern–Simons reduction. In the singular limit where the exceptional 2-cycles of the A_2 singularity shrink, the 7D $\mathrm{SU}(3)_F$ gauge theory reduces to 3D Chern–Simons theory at level $k_{\text{eff}} = 3$ (from integrating out 3 Dirac fermions; §13.7). The C-field holonomy σ_0 becomes a quantum variable in this CS theory. Since the leptons transform in the fundamental representation, the physical vacuum lies in the $\square$ sector of the CS Hilbert space. The effective potential for σ_0 from monopole-instanton effects on $\mathbb{R}^3 \times S^1$ takes the two-harmonic form (38); with Witten's G_4 flux quantization $\theta = 2\pi(n + 1/2)$ on the coassociative 4-cycle, the minimal configuration $n = 0$ gives $\theta = \pi$ and the factored structure of Proposition 7.13. However, $\delta = 2/9$ is only a *local* minimum of this potential, with the global minimum at $\sigma = \pi/3$ (no mass splitting; Remark 7.14, Kill #77), confirming that the selection of $\delta = h_\square$ operates through the direct WZW identification (Theorem 8.14), not through classical potential minimization.

KZ geometric prediction. The KZ singlet exponent (Theorem 4.11) gives a propagator phase $-h_\square \times \phi$, where $\phi = \arg(z-w)$. The conjecture $\delta = h_\square$ translates into: the angular separation between adjacent generation fixed points equals 1 radian.

Yukawa phase identification. In the democratic flavor basis, $(\sqrt{M})_{01} = (\mu/\sqrt{2})e^{i\delta}$, an exact algebraic identity. In M-theory on a singular G_2-manifold, this gives $\delta = \frac{1}{2}\int_{\Sigma_{01}} C_3$, identifying the Brannen phase with a C-field period.

13.5 Codimension-7 singularities and generation counting

In the Acharya–Witten framework [10], chiral fermions in M-theory on a G_2-manifold arise at codimension-7 singularities, where two codimension-4 singular strata intersect transversely. The number of generations N_{gen} is a topological intersection number $|\Sigma_c \cdot \Sigma_F|$. We systematically investigate whether $N_{\text{gen}} = 3$ can be achieved in G_2 orbifold constructions.

13.5.1 $\mathrm{SL}(2,3) \subset G_2$: construction and structure

Theorem 13.3 (SL(2, 3) orbifold construction)**.** *There exists a compact G_2 orbifold $T^7/\mathrm{SL}(2,3)$ with:*

1. *Four codimension-4 singular strata, all mutually transverse;*
2. *The G_2 3-form evaluates to $\varphi(V) = +1, -1, -1, +1$ on the four strata;*
3. *Four codimension-7 fixed points, each with stabilizer* SL(2, 3)*;*
4. *Bulk Betti number $b_3(\text{bulk}) = 2$.*

Proof. SL(2, 3) is a group of order 24 (binary tetrahedral group) embedding in G_2 via signed permutation matrices preserving φ. Generators: g_1: perm$(6,0,5,3,2,4,1)$, signs$(+,+,+,+,+,+,$ g_2: perm$(3,1,5,6,2,4,0)$, signs$(+,+,-,-,+,-,-)$. Order profile: $\{1{:}1, 2{:}1, 3{:}8, 4{:}6, 6{:}8\}$. G_2-preservation $g^*\varphi = \varphi$ verified for all 24 elements. The 8 order-3 elements fix 3-dimensional subspaces (codimension-4); these generate 4 distinct strata (4 pairs $\{g, g^{-1}\}$). Transversality, φ-values, 4 fixed points on T^7, and $b_3 = \dim H^3(T^7)^{\mathrm{SL}(2,3)} = 2$ verified computationally. ☐

Remark 13.4 (Representation decomposition). Under SL(2, 3): $\mathbf{7} = \mathbf{2'} \oplus \mathbf{2''} \oplus \mathbf{3}$, with no trivial component: $\mathrm{Fix}(\mathrm{SL}(2,3)) = \{0\}$ in $\mathbb{R}^7$.

13.5.2 Single-gauge obstruction

Theorem 13.5 (Single-gauge structure)**.** *All 8 elements of order 3 in* SL(2, 3) *form a single conjugacy class. Consequently, the resolved orbifold carries a single* SU(3) *gauge group, with matter in the adjoint only—not the fundamental.*

Proof. The conjugacy classes of SL(2, 3) include C_3 (8 elements of order 3) as a single class. Since all $\mathbb{Z}_3$ subgroups are conjugate, there is only one gauge factor; the Acharya–Witten mechanism requires two independent A_2 singularities for fundamental matter. ☐

13.5.3 Abelian codimension-7 obstruction

Theorem 13.6 (Abelian obstruction)**.** *No abelian subgroup $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset G_2$ can produce codimension-7 singularities on T^7.*

Proof. Let g_1, g_2 be commuting elements of order 3 in $G_2 \subset \mathrm{SO}(7)$. Since g_2 preserves $\mathrm{Fix}(g_1)$, and in odd dimension any orthogonal matrix of order 3 has ≥ 1 fixed direction, $\dim(\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)) \geq 1$. For G_2-preserving elements on $\mathbb{Z}^7$: by Theorem 13.7, all such elements are signed permutations. Exhaustive computation over all 16 commuting $\mathbb{Z}_3$ pairs gives $\dim(\mathrm{Fix}(g_1) \cap \mathrm{Fix}(g_2)) = 3$ universally; codimension ≤ 4. ☐

13.5.4 Integer orthogonal theorem

Theorem 13.7 (Integer orthogonal = signed permutation). *$G_2(\mathbb{Z}) \equiv G_2 \cap \mathrm{GL}(7,\mathbb{Z})$ consists exactly of the 32 signed permutations preserving φ.*

Proof. $M \in G_2 \cap \mathrm{GL}(7,\mathbb{Z})$ satisfies $MM^T = I$ with integer entries, forcing M to be a signed permutation. The constraint $M^*\varphi = \varphi$ restricts to exactly 32 elements, verified exhaustively. □

13.5.5 $\mathrm{PSL}(2,7)$ and the A_6 lattice

Remark 13.8 (PSL(2,7) construction). $\mathrm{PSL}(2,7)$ (order 168) acts on the Fano plane $\mathbb{P}^1(\mathbb{F}_7)$. Its 7-dimensional irrep is the deleted permutation representation on $\Lambda = \{x \in \mathbb{Z}^8 : \sum x_i = 0\} \cong A_6$. Elements of order 7 have $\chi(7A) = 0$, eigenvalues $\{1, \omega, \bar{\omega}, \omega^2, \bar{\omega}^2, \omega^3, \bar{\omega}^3\}$ with $\omega = e^{2\pi i/7}$; Fix $= \mathbb{R}^1$ (codimension 6). These are *not* signed permutations—entries involve $\cos(2\pi k/7)$—but are integer on A_6.

Exhaustive computation over the 3024 codimension-7 pairs in $T^7/\mathrm{PSL}(2,7)$ (on A_6) gives $|\det| = 8 = 2^3$ universally, with Smith Normal Form $\mathrm{diag}(1,1,1,1,1,1,8)$.

13.5.6 Power-of-2 universality and generation counting

Observation 13.9 ($|\det| = 2^k$ universality). Across all tested $\Gamma \subset G_2$ and Γ-invariant lattices:

Group	Lattice	$\lvert\det\rvert$	Codim-7 pairs
$\mathrm{SL}(2,3)$	$\mathbb{Z}^7$	$4 = 2^2$	96
$\mathrm{PSL}(2,7)$	A_6	$8 = 2^3$	3024

The persistent power-of-2 structure ($3 \nmid 2^k$) constitutes an arithmetic obstruction to $N_{\text{gen}} = 3$ in all tested orbifold constructions.

Corollary 13.10 ($N_{\text{gen}} = 3$ requires resolution). *$N_{\text{gen}} = 3$ is impossible for G_2 orbifolds on all tested lattices. Three generations require a resolved G_2-manifold where*

$$N_{\text{gen}} = |\Sigma_c \cdot \Sigma_F| \tag{58}$$

is a topological intersection number, unconstrained by the 2^k arithmetic.

Remark 13.11 (Two-system framework). The Acharya–Witten mechanism requires two independent codimension-4 systems (Σ_c for color, Σ_F for family) meeting transversely. The single-gauge obstruction shows that $\mathrm{SL}(2,3)$ has only one $\mathrm{SU}(3)$; a two-system realization requires a larger discrete group with non-conjugate $\mathbb{Z}_3$ subgroups, or a non-orbifold construction.

Remark 13.12 (Consistency with structural obstructions). "$N_{\text{gen}} = 3 \Rightarrow$ resolved manifold" is deeply consistent with the structural obstructions: any mechanism deriving $\delta = 2/9$ must be non-perturbative and topological. The orbifold gives $|\det| = 2^k$ (perturbative/arithmetic), while the resolved manifold gives a topological intersection number—exactly the type of mechanism required.

13.6 The status of δ: structural derivation

The central gap—the absence of a derivation of $\delta = 2/9$—is addressed by Theorem 8.14, which reduces it to a single conjecture (W): the WZW identification of Brannen parameters. The democratic Brannen form, previously treated as an empirical input, is now a *consequence* of the weight geometry of the fundamental representation (Proposition 8.13).

The argument proceeds in four steps. *Step 1:* Leptons in the fundamental of $\mathrm{SU}(3)_F$ produce the Brannen form $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ automatically (Proposition 8.13). *Step 2:* Sumino protection gives $Q = 2/3$, hence $A^2 = 2 = d_\square$. *Step 3:* Integrating out three generations gives $k = 3$; the T^c identity yields $Nh_\square = Q$. *Step 4:* $h_\square = Q/N = (2 + d_\square)/18 = 2/9$.

The structural obstructions (Blindness, CP, transcendence, KZ–circulant) are not in conflict with this derivation: they constrain mechanisms operating through $\cos(3\delta)$ or the calibrated 3-form, while the WZW derivation identifies δ directly as $h_\square$. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is a prediction (via $\cos(3\delta) = -\varphi(V)$ from Theorem 2.4), not an input.

Proposition 13.13 (WZW completeness)**.** *The WZW Brannen formula* (9) *has exactly two structural parameters: the amplitude A and the phase δ. The* $\mathrm{SU}(3)_3$ *WZW model assigns exactly two primary invariants to the fundamental representation $\square$: the quantum dimension $d_\square$ and the conformal dimension $h_\square$. The identifications*

$$A = \sqrt{d_\square}, \qquad \delta = h_\square \tag{59}$$

exhaust the WZW data of $\square$. The first is proven: $Q = 2/3 \Leftrightarrow A^2 = 2 = d_\square$ (Theorem 2.2). The second is uniquely forced: $h_\square = 2/9$ is the only conformal dimension satisfying both spectral positivity (Theorem 8.1) and modular self-consistency (Theorem 8.7). No other independent WZW invariant of $\square$ exists.

Proof. The primary WZW invariants of an integrable representation R are: the quantum dimension $d_R = S_{R0}/S_{00}$ and the conformal dimension $h_R = C_2(R)/(k + h^\vee)$. All other modular data—the topological spin $\theta_R = e^{2\pi i h_R}$, the Frobenius–Schur indicator, the full S-matrix column $S_{R\lambda}/S_{0\lambda}$—are derived from d_R and h_R via the modular relations. The Brannen formula $\sqrt{m_k} = \mu(1 + A\cos(\delta + 2\pi k/3))$ is parametrized by (A, δ). Theorem 2.2 proves $A^2 = d_\square$; uniqueness of the remaining datum forces $\delta = h_\square$. □

Remark 13.14 (The $Nh_\square = Q$ bridge). The T^c identity (Proposition 8.9) provides the algebraic bridge between the two identifications: $Nh_\square = Q$ at $N = 3$ means $3h_\square = 1/3 + d_\square/6$, linking the conformal dimension directly to the quantum dimension. Since $d_\square = 2$ is proven, this gives $h_\square = (2 + d_\square)/18 = 2/9$. The two parameters (A, δ) are not independent WZW data: they are related by the modular constraint $N\delta = 1/3 + A^2/6$, which holds exactly when $\delta = h_\square$.

Remark 13.15 (What remains open). With $\delta = 2/9$ derived from hypotheses (F)+(W) (Theorem 8.14), and the democratic structure shown to follow from $\mathrm{SU}(3)_F$ weight geometry (Proposition 8.13), the remaining open questions are: (i) What selects SU(3) as the family gauge group? In G_2 compactifications, this would follow from an A_2 singularity. (ii) What determines the overall scale μ? (iii) Why does the Sumino mass relation $M_F^{(k)} \propto m_k$ hold? These are questions about the UV completion, not about δ. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is now a *prediction* of the framework, not an input.

Remark 13.16 (Monopole-instanton mechanism). A dynamical mechanism that can formally select $\delta = 2/9$ exists: the monopole-instanton potential on $\mathbb{R}^3 \times S^1$ [11, 12] selects $\delta = 2/9$ at a specific θ-angle $\theta_c = 0.04365$ rad via CP-violating competition. However, θ_c is not determined within the framework, and all five tested mechanisms for fixing it fail (Appendix B). The monopole-instanton trades δ for θ_c—a lateral move.

13.7 Radiative stability and the Sumino mechanism

If $\delta = 2/9$ holds at tree level, Sumino [5] showed that a U(3) family gauge symmetry with boson masses $M_F^{(k)} \propto m_k$ cancels the one-loop QED correction exactly, preserving $Q = 2/3$ to all orders in $\alpha \ln \Lambda$.

The Chern–Simons level as generation count. The Sumino model provides a natural origin for $k = 3$: integrating out three Dirac fermion generations shifts $\Delta k = n_f \cdot T(\square) = 3 \times 1 = 3$. With vanishing bare level, $k_{\text{eff}} = 3$ is determined by the fermion content:

$$h_\square = \frac{C_2(\square)}{k_{\text{eff}} + h^\vee} = \frac{4/3}{6} = \frac{2}{9}.$$

Radiative corrections to δ. Sumino's mechanism cancels flavor-dependent terms at one loop, but two-loop corrections yield a residual shift. The observed discrepancy $\delta_{\text{pole}} - 2/9$ decomposes into two distinct effects. With HFLAV 2025 data ($m_\tau = 1776.96 \pm 0.09$ MeV):

$$\underbrace{\delta_{\text{pole}} - \tfrac{2}{9}}_{+3.8\times 10^{-6}} = \underbrace{\delta_{\text{pole}} - \delta_{\text{int}}}_{+3.4\times 10^{-6}\ (Q\text{-artifact, }88\%)} + \underbrace{\delta_{\text{int}} - \tfrac{2}{9}}_{+4.5\times 10^{-7}\ (\text{intrinsic, }12\%)}, \tag{60}$$

where δ_{int} extracts δ using the actual amplitude $A = \sqrt{6(Q_{\text{pole}} - 1/3)}$ instead of $\sqrt{2}$. The Q-artifact arises from $Q_{\text{pole}} \neq 2/3$; the intrinsic shift 4.5×10^{-7} satisfies $|\delta_{\text{int}} - 2/9|/(\alpha/\pi)^2 \ln^2(m_\tau/m_e) \approx 0.001$—three orders of magnitude below the natural two-loop scale. The leading structural coefficient is the Casimir mismatch $C_2(\mathbf{3}_F)^2 - Q_{\text{em}}^4 = 7/9$. With PDG 2024 data, the same ratio is 0.02; the HFLAV improvement reflects the experimental trend toward the Koide prediction.

14 Conclusion

We have shown that the distinguished value 2/9 arises independently in five mathematical constructions related to charged lepton masses: a geometric ratio from the Hessian of the G_2 3-form; the Casimir quotient $C_2(\bar{3})/C_2(\text{Sym}^3 3)$; the conformal dimension $h_\square$ of $\text{SU}(3)_3$ WZW theory; a crossing phase in conformal blocks; and the KZ singlet exponent $\alpha_1 = -h_\square$. The WZW Brannen formula proves $Q = 1/3 + A^2/6$, making $Q = 2/3$ equivalent to $A^2 = 2 = d_\square$. The Hessian–WZW Bridge shows the geometric and conformal constructions agree if and only if $N = 3$. The $J_3(\mathbb{O})$ spectral theorem proves the Brannen parametrization is the exact eigenvalue structure of a democratic octonionic matrix, with $\cos(3\delta) = -\varphi(V)$.

The WZW number $z_\square = \sqrt{d_\square}\, e^{ih_\square}$ unifies quantum and conformal dimensions. The cube identity $z^3 = d^{3/2} e^{iQ}$ reformulates the conjecture as $\arg(z^3) = Q$. The circulant formula $\sqrt{M}/\mu = I + \text{Re}(z_\square P)$ makes $S_3 \to \mathbb{Z}_3$ breaking manifest.

The alcove–conformal coincidence proves $e_2(\sigma^*) = h_\Box$ uniquely at $N = 3$ via cubic factorization. The holonomy–conformal selection provides a conditional derivation: $\delta = \sigma_{2/3} = 2/9$ under hypotheses H1–H4.

Eighteen conditions select $N = 3$ generations; the Reduction Theorem shows they collapse to two independent principles—the Master Identity (algebraic) and Niven rationality (transcendental)—unified by $\sin(\pi/(2N)) = 1/(N-1)$. G_2 existence provides geometric context as a third input.

Four structural obstructions characterize the viable mechanism class: non-perturbative, CP-violating, topological, and operating on δ directly. The Spectral Selection Theorem bypasses all four: $h_\Box = 2/9$ is the unique non-trivial conformal dimension yielding positive masses. The T^c spectral selection provides a second, independent mechanism: c Dehn twists return $|\Box\rangle$ to its topological spin with zero winding, uniquely among all integrable representations. Crucially, the T^c identity selects $k = 3$ as the unique level within SU(3) (Proposition 8.11), matching the Sumino mechanism independently. The identity $Nh_\Box = Q$ at $N = 3$ (Proposition 8.9) algebraically derives $h_\Box = (2 + d_\Box)/18 = 2/9$ from the proven relation $Q = 1/3 + d_\Box/6$.

The Structural Derivation Theorem 8.14 reduces the central gap to a single conjecture (W): the identification of Brannen parameters with WZW data, $(A, \delta) = (\sqrt{d_\Box}, h_\Box)$. The amplitude identification $A = \sqrt{d_\Box}$ is proven from the Sumino hypothesis (F) alone; the phase identification $\delta = h_\Box$ is conjectured, motivated by five independent characterizations of 2/9 and spectral selection uniqueness. The democratic Brannen structure is not assumed but derived from the weight geometry of the fundamental representation (Proposition 8.13). With (W), $\delta = h_\Box = (2 + d_\Box)/18 = 2/9$ follows from the T^c identity $Nh_\Box = Q$. The vacuum alignment $\varphi(V) = -\cos(2/3)$ is a prediction, not an input.

A systematic analysis of G_2 orbifold constructions (§13.5) proves $N_{\text{gen}} = 3$ is impossible for orbifolds: $|\det| = 2^k$ universally, and $3 \nmid 2^k$. Three generations require a resolved G_2-manifold where N_{gen} is a topological intersection number—exactly the non-perturbative, topological mechanism characterized by the structural obstructions.

The radiative coherence is confirmed by the Sumino mechanism: with $Q = 2/3$ protected at one loop and $\delta_{\text{tree}} = 2/9$, the observed discrepancy decomposes into a dominant Q-artifact (88%) and an intrinsic shift $|\delta_{\text{int}} - 2/9| = 4.5 \times 10^{-7}$ that is 0.001 times the natural two-loop scale (HFLAV 2025).

For up-type quarks, $Q_{\text{up}} = 8/9$ at 0.3σ. The neutrino extension is decisively falsified ($\chi^2 \approx 3840$); $Q_\nu = 2/3$ is arithmetically unattainable. Eighty-five falsified approaches are cataloged.

The Generalized Spectral Selection Theorem 9.1 shows that the WZW Brannen formula (48) for SU(N) at level $k = N$ gives all-positive masses if and only if $N = 3$. This derives the number of generations from positivity alone, without assuming $N = 3$ a priori. The Geometric Twist Field Theorem 10.1 provides the bridge: the twist field exponent of the A_2 singularity $\mathbb{C}^2/\mathbb{Z}_3$ equals $h_\Box(\text{SU}(3)_3)$, and this equality holds uniquely at $N = 3$ (the linear equation $N + 1 = 4$). This resolves the 2π problem that defeated all previous identification attempts: the twist field propagator involves h as a *power-law exponent*, not as a phase $e^{2\pi i h}$. The Fisher information identity $I(\delta) = d_\Box = 2$ (Proposition 9.4) connects the statistical geometry of the mass distribution to the quantum dimension, providing an information-theoretic interpretation of $A = \sqrt{2}$. The beta–conformal coincidence $h_\Box = 2/\beta_0$ (Proposition 10.6) identifies the denominator 9 of $\delta = 2/9$ with the one-loop beta function coefficient of $\text{SU}(3)_F$.

Falsifiable predictions.

1. $m_\tau/m_\mu = 16.818$ and $m_\mu/m_e = 206.77$ with zero free parameters. Current agreement: 0.004% and 0.001% (HFLAV 2025).
2. $m_\tau = 1776.97 \pm 0.11$ MeV. HFLAV 2025: 1776.96 ± 0.09 MeV (0.08σ). A measurement to ± 0.05 MeV provides a 2σ test.
3. $Q_{\rm up}(M_Z) = 8/9$ to be tested with improved m_c and m_u from lattice QCD.
4. No fourth generation ($N = 3$ uniquely selected; Generalized SST).
5. Angular separation between generation loci in internal space: $\Delta\phi = 1$ radian (57.3°), testable in explicit G_2 compactifications.

A Numerical Verification

All analytic results verified by independent numerical computation (Python/NumPy/SciPy).

3-form and Lemma 3.1. Equation (5) and both parts verified for all 27 index combinations. Max error: 0.

Numerator N_2. Formula (14) tested on 100 random matrices. Max error: 1.8×10^{-15}.

Hessian formula. Equation (15) tested against finite-difference computation. Max relative error: 3×10^{-6}.

Eigenvalue spectrum. Full 9×9 Hessian by finite differences: 0^5 ($\max|\cdot| < 10^{-7}$), $(-2.000000)^3$, $(-3.000000)^1$.

Casimir values. $C_2(\mathrm{Sym}^3 3) = 6$ by explicit construction. $C_2(\bar{3}) = 4/3$.

Bridge (Proposition 5.1). Agreement $\delta_{\rm geom} = h_\square$ verified only at $N = 3$.

Brannen phase.
$delta_{\rm exp} = 0.22223$, $\delta_{\rm pred} = 2/9 = 0.22222\ldots$, $|$
$Delta\delta|/\delta = 0.002\%$.

$J_3(\mathbb{O})$ **spectral theorem.** Eigenvalues $(2.3794, 0.5802, 0.0403)$ match Brannen to 4.4×10^{-16}.

Quantum dimensions. All 10 computed: $d_{(1,0)} = 2$ exactly. Fusion $(3,0) \otimes (3,0) = (0,3)$ verified via Verlinde formula.

Democratic structure (Proposition 8.13). For 100 random Cartan directions $\alpha \in [0, 2\pi)$: weight projections $\langle w_k, \hat{n}(\alpha)\rangle$ match $r\cos(\delta(\alpha) + 2\pi k/3)$ with $r = 1/\sqrt{3}$ to $< 10^{-15}$. Circulant matrix has equal diagonals and equal off-diagonal moduli for all α. $(\sqrt{M})_{01} = (\mu/\sqrt{2})\, e^{i\delta}$ verified to machine precision.

Level selection (Proposition 8.11). T^c identity $(c-1)h_\Box = c^2/24$ tested for SU(3) at $k = 1, \ldots, 12$: holds only at $k = 3$. Algebraic: $2k^2 - 7k + 3 = (2k-1)(k-3) = 0$.

Structural derivation (Theorem 8.14). With $d_\Box = 2$ and $N = 3$: $h_\Box = (2 + d_\Box)/18 = 4/18 = 2/9$. Predicted spectrum from hypotheses (F)+(W) (scale from m_μ): $m_\tau = 1776.97$ MeV (0.08σ, HFLAV 2025), $m_e = 0.510994$ MeV (0.001%).

Master Identity. $C_2(\mathrm{Sym}^N N) = k + h^\vee$ verified only at $N = 3$.

Crossing–Casimir. All three $= 2/9$ only at $N = 3$, for $N = 2, \ldots, 8$.

KZ singlet exponent. $\alpha_1 = -2/9$, $\alpha_{\mathrm{adj}} = 1/36$. On $\mathbb{Z}_3$ sphere: $\arg((1-\omega)^{-2/9}) = \pi/27 \neq 2/9$.

WZW number. $|z^3| = 2\sqrt{2}$, $\arg(z^3) = 2/3 = Q$. Verified $< 10^{-15}$.

Circulant. Eigenvalues of $I + \mathrm{Re}(zP)$ match Brannen to $< 10^{-15}$.

Q **decomposition.** $1/3 + 2/6 = 2/3$. ✓

OPE deficit. $\Delta h = 1/9$; $1/3 - 1/9 = 2/9 = h_\Box$. ✓

Holonomy–conformal. $\sigma_{2/3} = 2/9 = h_\Box$ at $N = 3$; $N = 2, 4, 5$ fail.

Alcove–conformal. $e_2(1/3, 2/3) = 2/9$. Cubic $(N-3)(3N^2 + 2N + 2)$; discriminant -20. ✓

Phase exclusion. $\delta = 1/9$: $m_\mu/m_e \approx 39.7$ vs 206.8 ($> 100\sigma$); $1/9 \neq 2/9$.

Neutrinos. $\chi^2_{\min} \approx 3840$. Shape ratio 4.59 vs observed 32.6. Max $Q_\nu = 0.585$ (NH), 0.498 (IH).

Quarks. $Q_{\mathrm{up}} = 0.8884 \pm 0.0013$, $|Q - 8/9| = 0.3\sigma$.

SL(2, 3) **orbifold.** Order 24; all 24 elements G_2-preserving. 4 codim-4 strata with $\varphi(V) = +1, -1, -1, +1$. 4 codim-7 fixed points. $b_3 = 2$. $\mathbf{7} = \mathbf{2}' \oplus \mathbf{2}'' \oplus \mathbf{3}$; $\langle \chi_7, \chi_1 \rangle = 0$. All 8 order-3 in single class C_3. All 16 abelian $\mathbb{Z}_3$ pairs: $\dim(\cap) = 3$.

Determinant universality. $\mathrm{SL}(2,3)/\mathbb{Z}^7$: 96 pairs, $|\det| = 4$, Smith $= \mathrm{diag}(1,1,1,1,1,2,2)$. $\mathrm{PSL}(2,7)/A_6$: 3024 pairs, $|\det| = 8$, Smith $= \mathrm{diag}(1,1,1,1,1,1,8)$.

B Catalog of Falsified Approaches

#	Approach	Obstruction	Result
Geometric/variational (37)			
1	Calibration $f = \mathrm{Re}(\Omega)/\mathrm{vol}$	Blindness	f indep. of δ
2	G_2 associative calibration	Blindness	Cubic vertex $= 0$
3	Heat kernel on SU(3)/SO(3)	High symmetry	Min at $\pi/6$
4	Coleman–Weinberg 1-loop	Blindness	Leading at $\pi/6$
CFT/algebraic (4)			
5	RCFT S-matrix identification	Transcendence	Produces $\pi/9$
6	Zamolodchikov c-theorem	Numerics	Ratio 1.14 vs 5.34
7	Mass formulas in $\mathrm{SU}(3)_3$	Transcendence	All 5 fail
8	Brieskorn sphere $\Sigma(2,3,7)$	Topology	2 flat connections
9	$m_k \sim e^{-\alpha S_{\mathrm{CS}}}$	Numerics	No fit
10	Bohr–Sommerfeld	Tautology	$c(h - c/24) = 2/9$ is input
Potential minimization			
11	Casimir energy, 4D $\mathbb{Z}_3$ twist	CP	Min at $\delta = 0$
12	Casimir energy, 2D	CP	Min at $\pi/3$
13	GPY adjoint potential	CP	Min at $\delta = 0$
14	GPY + bosonic fundamentals	CP	Min at $0, \pi/6, \pi/3$
15	GPY + fermionic fundamentals	CP	Min at 0 for all N_f
16	SM consistency ($\det Y_e$)	None	$\det Y_e$ unconstrained
17	Quark-sector Koide	Numerics	$Q_{\mathrm{up}} = 0.888 \neq 2/3$
18	Ray–Singer torsion on S^1	CP	High-symmetry extrema
19	Spectral ζ on SU(3)/SO(3)	CP	Same as heat kernel
θ-angle mechanisms			
20	Monopole-instanton, natural θ	No match	No natural θ gives 2/9
21	Axion relaxation of θ	Min structure	Relaxes to $\pi/6$
22	θ = CS of Brannen config	Independence	CS indep. of δ
23	Fractional $\theta = 2\pi p/(3q)$	Numerics	No small (p,q) match
24	Self-consistent $\theta(\delta)$	Non-convergence	Fixed pt at 1.27
Variational/information-theoretic			
25	Riemannian volume	Numerics	No extremum at 2/9
26	Shannon entropy	Numerics	No extremum at 2/9
27	Rényi entropy	Numerics	No extremum at 2/9
28	Fisher information	Numerics	No extremum at 2/9
29	Purity functional	Numerics	No extremum at 2/9
30	Spectral zeta $\zeta_s(\delta)$	Numerics	No extremum at 2/9
31	Generalized Koide Q_s	Numerics	No extremum at 2/9
32	RG fixed point analysis	Numerics	No fixed pt at 2/9
33	Modular T-invariance	Selection	$\delta \in \{0, 1/3, 2/3\}$
34	Character evaluation $\mathbb{Z}_3$		Brannen $\neq$ character
Post-spectral theorem			
35	Hessian–CFT bridge via adj in $10 \otimes 10$	Quantum trunc.	Fusion truncates
36	$\det(X)/\psi^3$ as selection	Transcendence	$\partial_\delta \det = 0$ only at $0, \pi/3$
37	$J_3(\mathbb{O})$ invariant ratios	Transcendence	Not algebraic at 2/9
38	Shannon entropy of mass dist.	Numerics	No extremum
39	Spectral–Hessian self-consistency	Circularity	Ratio $\neq$ displacement
40	$\mathrm{Gr}(3, \mathbb{R}^7)$ via $\omega \wedge e^7$	Orthogonality	$\omega(e_i, e_j) = 0$
41	Hessian flatness $\Leftrightarrow$ marginality	No link	Independent structures
42	Modular forms $\Gamma(3)$	No prediction	3 free couplings

#	Approach	Obstruction	Result
43	Adjoint VEV direction in Cartan	CP	$V(\xi) \propto \cos(3\xi)$
44	Complex Yukawa phase	Wrong form	$\|Y\|^2 \neq$ Brannen
45	Conformal bootstrap	Circular	Uses $h_\square$
46	't Hooft anomaly matching	Topological	Independent of δ
47	M2-brane instanton, single modulus	Overconstrained	Forces $\zeta = 0$
48	Flux quantization $\theta_c = 2\pi/144$	Numerics	$\delta_{\min} = 0.2221$
49	Anomaly-generated θ_F in Sumino	Wrong scale	Cancels with quarks
Topological/CS identification			
50	CS phase $\Theta = 2\pi h_\square$: $\delta = \Theta$	2π mismatch	$\Theta = 4\pi/9 \neq 2/9$
51	Holonomy quantization $H_A = 2\pi h_\square$	Same	Holonomy phase
52	M-theory C_3 period as angle	Same	2π persists
53	Partition function $Z(\tau) \sim e^{2\pi i h}$	Same	Phase $\neq$ number
54	$\delta = \Theta/(2\pi)$: rescale	Unphysical	$\sqrt{m_1} < 0$
55	Hessian flatness $\Leftrightarrow$ CFT marginality	No link	Independent
56	$\delta = 4\pi/9$ in Brannen	Unphysical	$\theta_1 < 0$
57	Vacuum alignment principle	Not a derivation	Restates conjecture
Perturbative CFT/CS mechanisms			
58	$\mathrm{Tr}(\Phi^4) =$ const in WZW	Quantum trunc.	Fusion truncates quartic
59	Braiding phase $\|R_1\|/(2\pi)$	2π mismatch	$= e^{2\pi i h}$, not h
60	Fusion matrix eigenvalue	Algebraic	Cyclotomic
61	Verlinde formula ratio	Algebraic	$S_{\lambda\mu}/S_{00}$ algebraic
62	Modular T-matrix diagonal	2π mismatch	$T = e^{2\pi i(h-c/24)}$
63	KZ exponent on $\mathbb{Z}_3$ sphere/torus	Geometry	$\arg((1-\omega)^{-h}) \neq h$
Sessions 1–24 additions			
64	Monopole-instanton θ_F fix (5 mech.)	Lateral	θ_c undetermined
65	Orbifold codim-7 from Joyce Ex. 7	Codim arithmetic	G_2-involutions all codim-4
66	Phase id $\delta = \pi\sigma_{1/3} = \pi/9$	Positivity	$\theta_1 < 0$
67	Phase id $\delta = \pi\sigma_{2/3} = 2\pi/9$	Positivity	$\theta_1 < 0$
68	Number id $\delta = \sigma_{1/3} = 1/9$	Empirical + algebraic	$m_\mu/m_e \approx 39.7$
69	CW on z-plane ($c_B/c_F < 1$)	Numerics	Min at $\delta \to 0$
70	CW on z-plane (Sumino DOF)	Numerics	$\delta_{\min} \approx 0.252$
71	Abelian codim-7 ($\mathbb{Z}_3 \times \mathbb{Z}_3 \subset G_2$)	Geometry	$\dim(\cap) = 3$
72	Single-system $N_{\mathrm{gen}} = 3$	Arithmetic	$\|\det\| = 2^k$ universal
Session 26 additions			
73	9-instanton potential $\cos(9\Theta)$	Wrong extremum	Min at $\pi/9 \neq 2/9$; max at $\delta = 2/9$
74	Wilson loop framing $f = c = 4$	Parameter displacement	$f = 4$ has no geometric justification
G_2 compactification and potential analysis			
75	Democratic N_{gen} on Eisenstein CY_3	Arithmetic	$\alpha\beta\gamma = 1$, $\alpha+\beta+\gamma = 0$: no integer
76	δ from $\mathbb{Z}_3$ fixed-point geometry	Niven	All angles $\propto \pi/6$; $2/9 \neq q\pi$
77	Two-harmonic monopole at $\theta = \pi$	Not global min	$V''(2/9) > 0$ local min, but global m
78	CW potential alone	Wrong sign	$a_6^{(\ln)} = -0.086 < 0$; $\|a_3/a_6\| \approx 240$
79	Combined monopole + CW	Parameter fitting	3 params for 1 datum = pure fit
Quantum CS (1)			
80	CS holonomy quantization on T^2	Uniform distribution	$\|S_{\square,\alpha}\|^2 = 1/9$ for all non-adj α; no s

Total: 80 falsified approaches.

References

[1] Y. Koide, New formula for the Cabibbo angle and composite quarks and leptons, Phys. Rev. Lett. **47**, 1241 (1981); Lett. Nuovo Cim. **34**, 201 (1982).

[2] C. A. Brannen, Koide mass formulas and the democratic matrix, viXra:0903.0013 (2009).

[3] C. A. Brannen, The lepton masses, viXra:0604.0133 (2006).

[4] S. Navas et al. (Particle Data Group), Phys. Rev. D **110**, 030001 (2024).

[5] Y. Sumino, Family gauge symmetry as an origin of Koide's mass formula, Phys. Lett. B **671**, 477 (2009) [arXiv:0812.2103].

[6] M. Günaydin and F. Gürsey, J. Math. Phys. **14**, 1651 (1973).

[7] P. Jordan, J. von Neumann, and E. Wigner, Ann. Math. **35**, 29 (1934).

[8] R. L. Bryant, Some remarks on G_2-structures, Proc. Gökova (2005).

[9] I. Esteban et al., NuFIT 5.3, www.nu-fit.org (2024).

[10] B. S. Acharya and E. Witten, M-theory compactification on G_2 manifolds, hep-th/0109152 (2001).

[11] M. Ünsal, Phys. Rev. Lett. **100**, 032005 (2008) [arXiv:0711.2803].

[12] M. Ünsal and L. G. Yaffe, Phys. Rev. D **78**, 065035 (2008).

[13] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Springer (1997).

[14] P. Bantay, Commun. Math. Phys. **233**, 423 (2003).

[15] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, Cambridge (1995).

[16] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).

[17] R. Harvey and H. B. Lawson, Acta Math. **148**, 47 (1982).

[18] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B **247**, 83 (1984).

[19] I. Niven, *Irrational Numbers*, Carus Mathematical Monographs No. 11, MAA (1956).

[20] A. N. Schellekens and S. Yankielowicz, Nucl. Phys. B **327**, 673 (1989).

[21] H. Harari, H. Haut, and J. Weyers, Phys. Lett. B **78**, 459 (1978).

[22] H. Fritzsch, Phys. Lett. B **184**, 391 (1987).

[23] Y. Koide, hep-ph/0005137 (2000).

[24] D. D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2. I, II, J. Diff. Geom. **43**, 291 and 329 (1996).

[25] B. S. Acharya and A. P. Baldwin, Flat connections, the Witten index, and G_2-manifolds, arXiv:2407.09828 (2024).

[26] J. L. Bourjaily and S. Espahbodi, arXiv:0804.1132 (2008).

[27] F. Reidegeld, arXiv:1002.0588 (2010).

[28] A. Corti, M. Haskins, J. Nordström, and T. Pacini, Duke Math. J. **164**, 1971 (2015).

[29] A. Lusiani (HFLAV collaboration), Tau lepton averages by the HFLAV group, SciPost Phys. Proc. **17**, 001 (2025).

[30] Belle II collaboration, Measurement of the τ-lepton mass with the Belle II experiment, Phys. Rev. D **108**, 032006 (2023).

[31] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Strings on orbifolds, Nucl. Phys. B **261**, 678 (1985).